*

POSITIVE SEMIDEFINITE & POSITIVE DEFINITE MATRICES

(PSD) (PD)

Square matrices.

Consider square matrix A

If $\bar{x}^T A \bar{x} \geq 0$ for all $\bar{x}$

Then A is PSD

For Real vectors/matrices.

$\bar{x}^T A \bar{x} > 0$ For all $\bar{x}$

Then A is PD

For complex vectors/matrices.

$\bar{x}^H A \bar{x} \geq 0$ For all $\bar{x}$

$\Rightarrow$ PSD

$\bar{x}^T A \bar{x} > 0$ For all $\bar{x}$

$\Rightarrow$ PD.

Ex:

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$

$$\bar{x}^T A \bar{x}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \left(2x_1^2 + 18x_2^2 + 12x_1x_2\right)$$

$$= \underline{\left(2x_1 + 3x_2\right)^2} \geq 0$$

$2x_1 + 3x_2 = 0$

if $x_1 = -\frac{3}{2}x_2$

Hence A is PSD. since $\bar{x}^T A \bar{x} \geq 0$ for all $\bar{x}$

Property of PSD, PD matrices:

Eigenvalues λ_i

If A is PD, then $\lambda_i(A) > 0$

If A is PSD, then $\lambda_i(A) \geq 0$

Ex:

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$

$$|A - \lambda I|$$

$$= \left| \begin{bmatrix} 2-\lambda & 6 \\ 6 & 18-\lambda \end{bmatrix} \right|$$

$$\Rightarrow (2-\lambda)(18-\lambda) - 36 = 0.$$

Characteristic Equation

$$\Rightarrow 36 - 20\lambda + \lambda^2 - 36 = 0$$

$$\Rightarrow \lambda^2 - 20\lambda = 0.$$

$$\Rightarrow \lambda^2 = 20\lambda$$

$$\Rightarrow \boxed{\lambda = 0, 20}$$

$$\lambda_1 = 0.$$

$\Rightarrow$ Matrix is PSD.

For a symmetric matrix A, if eigenvalues $\lambda_i(A) \geq 0$ then A is PSD.

if $\lambda_i(A) > 0$ the A is PD.

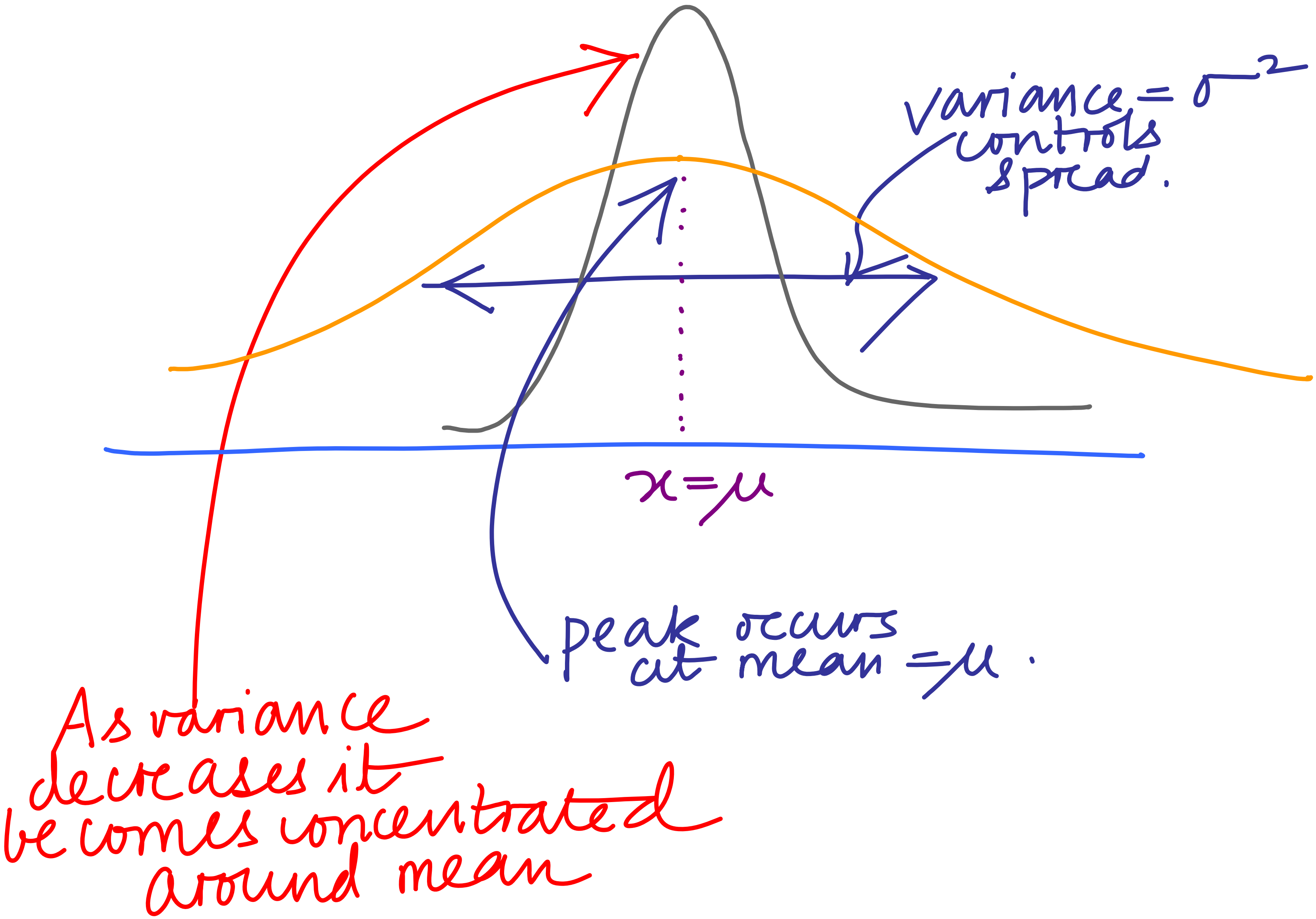
Gaussian RV:

X is Gaussian Random Variable with mean $= \mu$ var $= \sigma^2$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

Probability Density Function (PDF) of Gaussian RV


variance $= \sigma^2$ controls spread.
$x = \mu$
peak occurs at mean $= \mu$.
As variance decreases it becomes concentrated around mean

$$\tilde{X} = \frac{X-\mu}{\sigma}$$

Gaussian RV

$E\{\tilde{X}\} = 0$

$E\{\tilde{X}^2\} = 1$

Gaussian RV with mean $= 0$, var $= 1$

Termed as Standard Normal Random Variable

$$f_{\tilde{X}}(\tilde{x}) \doteq \frac{1}{\sqrt{2\pi}} e^{-\tilde{x}^2/2}$$

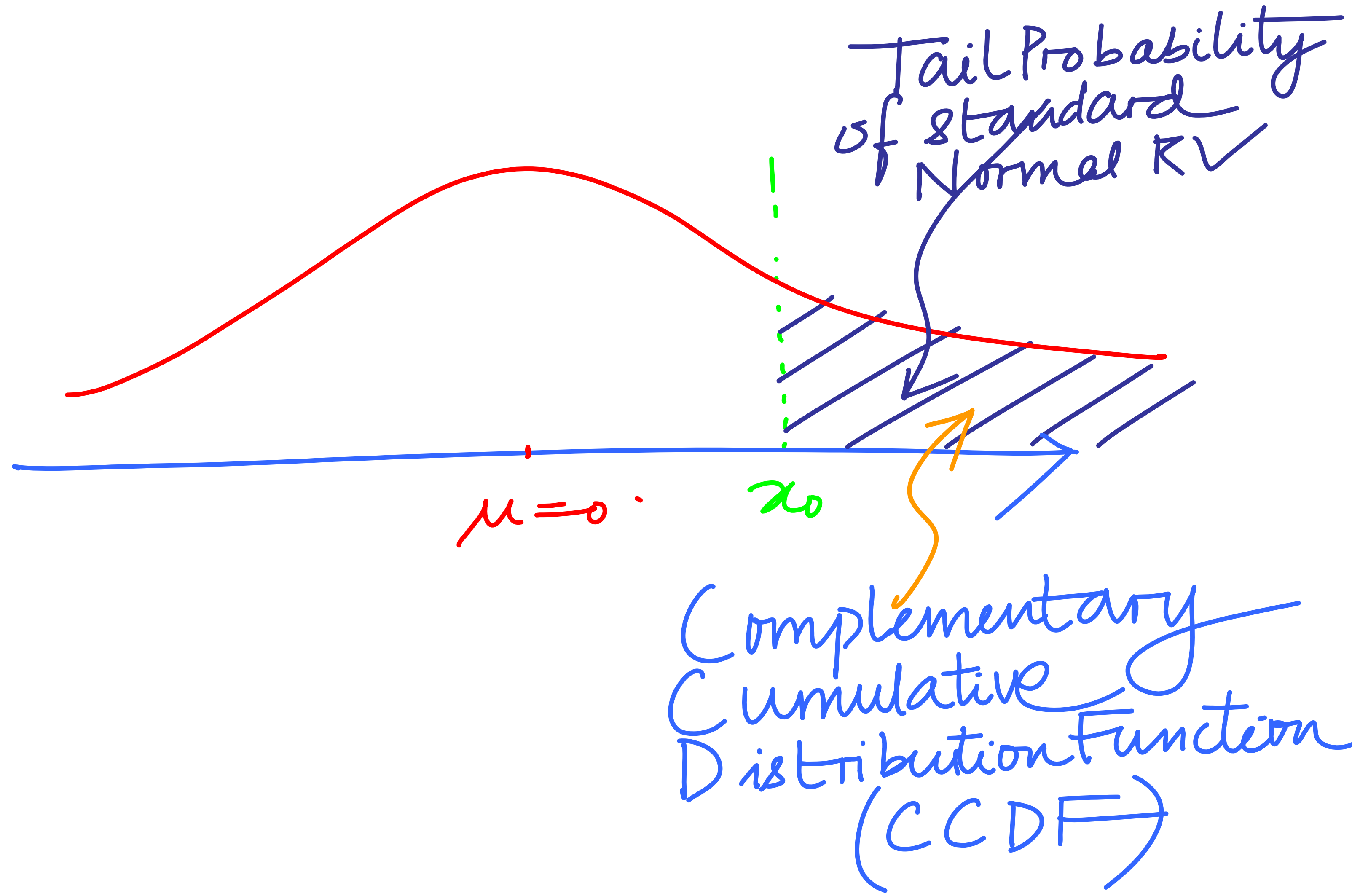
PDF of Standard Normal.

$$\Pr(\tilde{X} \geq x_0) = Q(x_0).$$

Gaussian Q-function

$$= \Pr(\tilde{X} \in [x_0, \infty))$$

$$= \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\tilde{x}^2/2} \, d\tilde{x}$$


Tail Probability of standard Normal RV
$\mu = 0$
x_0
Complementary Cumulative Distribution Function (CCDF)

Multivariate Gaussian RV:

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Gaussian RV

$$E\{\bar{x}\} = \begin{bmatrix} E\{x_1\} \\ \vdots \\ E\{x_n\} \end{bmatrix}$$

$$= \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} = \bar{\mu}$$

Mean vector

$$R = E\{(\bar{x}-\bar{\mu})(\bar{x}-\bar{\mu})^T\}$$

Covariance matrix
$n \times n$ matrix

$$\bar{X} \sim \mathcal{N}(\bar{\mu}, R)$$

$$f_{\bar{X}}(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n |R|}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu})^T R^{-1} (\bar{x}-\bar{\mu})}$$

PDF of multivariate Gaussian Random vector
mean $= \bar{\mu}$
covariance matrix $= R$

Consider multivariate Gaussian with

$$E\{\bar{x}\} = \bar{\mu} = 0$$

$$\underline{E\{x_i x_j\}} = \begin{cases} 0 & \text{if } i \neq j \\ \sigma^2 & \text{if } i = j \end{cases}$$

uncorrelated Gaussian RVs.

$$E\left((\bar{x}-\bar{\mu})(\bar{x}-\bar{\mu})^T\right)$$

$$= E\left\{\bar{x}\,\bar{x}^T\right\}$$

$$= E\left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \right\}$$

$$= E\left\{ \begin{bmatrix} x_1^2 & x_1 x_2 & \\ x_2 x_1 & x_2^2 & \\ & & \ddots \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \sigma^2 & 0 & \cdots \\ 0 & \sigma^2 & \\ \vdots & & \ddots & \\ & & & \sigma^2 \end{bmatrix}$$

$$= \sigma^2 I$$

$$R = \sigma^2 I$$

covariance matrix

$$|R| = (\sigma^2)^n = \sigma^{2n}$$

$$f_{\bar{X}}(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n \sigma^{2n}}} \cdot e^{-\frac{1}{2}\bar{x}^T \frac{I}{\sigma^2} \cdot \bar{x}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2} \cdot \bar{x}^T \bar{x}}$$

$$\bar{x}^T \bar{x} = \|\bar{x}\|^2$$

$$= x_1^2 + x_2^2 + \cdots + x_n^2$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\|\bar{x}\|^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2}$$

$$= \prod_{i=1}^{n} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x_i^2}{2\sigma^2}}}$$

Product
$i=1$ to n

individual
Gaussian PDFs
of X_i mean $=0$
var $=\sigma^2$

$\Longrightarrow$ Gaussian RVs are INDENDENT

For Gaussian RV

Uncorrelated $\Longrightarrow$ Independent.

NOT true for any general RV.