* MATHEMATICAL PRELIMINARIES:

VECTORS:

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Vector

column vector $n \times 1$

n Dimensional vector

$$x_1, x_2, \ldots, x_n \in \mathbb{R}$$

Real numbers.

$$\Rightarrow \bar{x} \in \mathbb{R}^n$$

n Dimensional. Real vectors.

$$\bar{x}^T = [x_1, x_2, \dots x_n]$$

Row vector

$1 \times n$

$$\bar{x}^T \bar{x} = [x_1 \; x_2 \dots x_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1^2 + x_2^2 + \dots + x_n^2$$

$$= \|\bar{x}\|_2^2$$

l_2 norm = Default

$$\|\bar{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

length of vector

COMPLEX VECTORS

$x_1, x_2, \ldots, x_n \in \mathbb{C}$

Complex Numbers.

$$\Rightarrow \bar{x} \in \mathbb{C}^n$$

n dimensional.
complex vectors.

$$\bar{x}^H = \begin{bmatrix} x_1^* & x_2^* & \cdots & x_n^* \end{bmatrix}$$

Row vector + complex conjugate of elements

$$\bar{x}^H \bar{x} = \begin{bmatrix} x_1^* & x_2^* & \cdots & x_n^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$$

$$= \|\bar{x}\|_2^2$$

$$\|\bar{x}\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

General Definition
For real & complex
vectors.

$$\tilde{x} = \frac{\bar{x}}{\|\bar{x}\|}$$

unit-Norm
vector.

$$\tilde{x}^H \cdot \tilde{x} = \frac{\bar{x}^H}{\|\bar{x}\|} \cdot \frac{\bar{x}}{\|\bar{x}\|}$$

$$= \frac{\|\bar{x}\|^2}{\|\bar{x}\|^2} = 1$$

$$\Rightarrow \|\tilde{x}\|^2 = 1$$

$$\Rightarrow \boxed{\|\tilde{x}\| = 1}$$

Ex: $\bar{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$

$$\|\bar{x}\| = \sqrt{1 + 1 + \cdots + 1}$$

$$= \sqrt{n}$$

$$\|\bar{x}\|^2 = n$$

$$\tilde{x} = \frac{\bar{x}}{\|\bar{x}\|} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

unit Norm vector

MATRICES:

$A = m \times n$ matrix

$\Rightarrow$ m rows
n columns.

$$A = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \right.$$

n columns.

m rows.

a_{ij} = element in i^{th} row & j^{th} column

If $m = n$
Then, A = Square matrix

$$\underbrace{\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_m}_{m \text{ vectors.}}$$

Linear Independence (LI)

Linearly independent if there do NOT exist $c_1, c_2, \ldots c_m$ (NOT all zero) such that

$$\underbrace{c_1\overline{w}_1 + c_2\overline{w}_2 + \cdots + c_m\overline{w}_m}_{\text{Linear Combination}} = 0$$

Linear Dependence:

Linearly dependent if there exist $c_1, c_2, \dots, c_m$ NOT all zero, such that

$$\underbrace{c_1\overline{w}_1 + c_2\overline{w}_2 + \dots + c_m\overline{w}_m}_{\text{Linear combination}} = 0$$

Ex:

$$\overline{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \overline{w}_2 = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$$

$$2 \cdot \overline{w}_1 + 1 \cdot \overline{w}_2 = 0$$

$\Rightarrow \overline{w}_1, \overline{w}_2$ are Linearly Dependent

$$\bar{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \bar{w}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\bar{w}_1, \bar{w}_2$ are linearly independent

$\Rightarrow$ There do NOT exist c_1, c_2 (NOT both zero) such that

$$c_1 \bar{w}_1 + c_2 \bar{w}_2 = 0$$

$$A = [\underbrace{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n}_{n \text{ columns}}]$$

$$= \begin{bmatrix} \tilde{a}_1^T \\ \tilde{a}_2^T \\ \vdots \\ \tilde{a}_m^T \end{bmatrix}$$

m rows.

Column rank of A

$=$ maximum number of linearly independent columns of A

Row rank of A

$=$ maximum number of linearly independent rows of A

$$\text{Row rank}(A) = \text{column rank}(A) = \text{rank}(A)$$

$$\boxed{\operatorname{rank}(A) \leq \min\{m, n\}}$$

rows. # columns

rank ≤ minimum of number of rows & columns of the matrix.