*

INNER PRODUCT SPACE:

Inner product of a real vector space V is an assignment of a real number

$$\langle \bar{u}, \bar{v} \rangle$$

Inner product

For any 2 vectors $\bar{u}$, $\bar{v}$

INNER PRODUCT SATISFIES Following properties:

1. Linearity: $\langle a\bar{u} + b\bar{v}, \bar{w} \rangle$

$$= a\langle \bar{u}, \bar{w} \rangle + b\langle \bar{v}, \bar{w} \rangle$$

Linearity property.

2. SYMMETRIC PROPERTY:

$$\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$$

3. POSITIVE SEMI-DEFINITE PROPERTY :

For any $\bar{u} \in V$

$$\langle \bar{u}, \bar{u} \rangle \geq 0$$

$\langle \bar{u}, \bar{u} \rangle = 0$ if and only if $\bar{u} = 0$.

Ex: DOT-PRODUCT :

2 vectors.

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

n Dimensional Real vectors.

$\in \mathbb{R}^n$

Euclidean n-space.

$$\langle \bar{u}, \bar{v} \rangle = \bar{u}^T \bar{v}$$

$$= [u_1 \; u_2 \; \cdots \; u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n .$$

DOT Product between 2 vectors:

$\bar{u} \cdot \bar{v}$

DOT Product is an Inner Product:

1. LINEARITY:

$$\langle a\bar{u} + b\bar{v}, \bar{w} \rangle$$

$$= (a\bar{u} + b\bar{v})^T \bar{w}$$

$$= a \cdot \bar{u}^T \bar{w} + b \cdot \bar{v}^T \bar{w}$$

$$= a \langle \bar{u}, \bar{w} \rangle + b \langle \bar{v}, \bar{w} \rangle$$

2. SYMMETRY:

$$\langle \bar{u}, \bar{v} \rangle = \bar{u}^T \bar{v}$$
$$= \bar{v}^T \bar{u}$$
$$= v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$$
$$= \langle \bar{v}, \bar{u} \rangle$$

3. POSITIVE SEMI DEFINITE:

$$\langle \bar{u}, \bar{u} \rangle = \bar{u}^T \bar{u}$$
$$= u_1^2 + u_2^2 + \cdots + u_n^2 = \|\bar{u}\|_2^2$$
$$\geq 0.$$

$= 0$ if and only if $u_1 = u_2 = \cdots = u_n = 0$
$\bar{u} = 0.$

DOT PRODUCT is an INNER PRODUCT

Standard inner product on $\mathbb{R}^n$

n Dimensional space of Real vectors.

Ex:

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$$

$$\langle \bar{x}, \bar{y} \rangle = 2x_1 y_1 - x_1 y_2 - x_2 y_1 + 5 x_2 y_2$$

Show this is inner product

Linearity:

$$a\bar{x} + b\tilde{x}$$
$$= a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

$$\langle a\bar{x} + b\tilde{x}, \bar{y} \rangle$$
$$= 2(ax_1 + b\tilde{x}_1)y_1 - (ax_1 + b\tilde{x}_1)y_2 - (ax_2 + b\tilde{x}_2)y_1 + 5(ax_2 + b\tilde{x}_2)y_2$$
$$= \underline{a\langle \bar{x}, \bar{y} \rangle + b\langle \tilde{x}, \bar{y} \rangle}$$

$\Rightarrow$ Linear

SYMMETRY:

$$\langle \bar{x}, \bar{y} \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$$

$$= 2y_1x_1 - y_1x_2 - y_2x_1 + 5y_2x_2$$

$$\langle \bar{x}, \bar{y} \rangle = \langle \bar{y}, \bar{x} \rangle$$

$\Rightarrow$ Symmetry

POSITIVE SEMI-DEFINITE:

$$\langle \bar{x}, \bar{x} \rangle = 2x_1^2 - 2x_1x_2 + 5x_2^2$$

$$= (x_1^2 + x_2^2 + 2x_1x_2) + (x_1^2 + 4x_2^2 - 4x_1x_2)$$

$$= (x_1 + x_2)^2 + (x_1 - 2x_2)^2 \geq 0$$

$\Rightarrow$ PSD Property

$= 0$ only if

$$\left.\begin{array}{l} x_1 + x_2 = 0 \\ x_1 - 2x_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = x_2 \\ = 0 \end{array}$$

$$\Rightarrow \bar{x} = 0.$$

$$\langle \bar{x}, \bar{x} \rangle \geq 0$$

$= 0$ only if $\bar{x} = 0.$

$$\langle \bar{x}, \bar{y} \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$$

Inner Product

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \bar{x}^T A \bar{y}$$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$$

It can be seen that A is PD

symmetric $A = A^T$

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 2-\lambda & -1 \\ -1 & 5-\lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow (2-\lambda)(5-\lambda) - 1 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 10 - 1 = 0$$

$$\Rightarrow \quad \lambda^2 - 7\lambda + 9 = 0.$$

$$\Rightarrow \quad \lambda = \frac{7 \pm \sqrt{13}}{2} > 0$$

Eigenvalues.

> 0

Symmetric + EV > 0

$\Rightarrow$ A is PD

Positive Definite Matrix

$\langle \bar{x}, \bar{y} \rangle = \bar{x}^T A \bar{y}$ where A is a symmetric PD matrix

is a inner product

NORM: "Norm" can be defined using inner product

$$\|\bar{u}\|^2 = \langle \bar{u}, \bar{u} \rangle$$

$$\Rightarrow \|\bar{u}\| = \sqrt{\langle \bar{u}, \bar{u} \rangle}$$

Unit - Norm vector

$$\hat{u} = \frac{\bar{u}}{\|\bar{u}\|} = \frac{\bar{u}}{\sqrt{\langle \bar{u}, \bar{u} \rangle}}$$

Normalization

$x \in \mathbb{R}^n$

$\langle \bar{x}, \bar{x} \rangle$ Standard inner product

$$= x_1^2 + x_2^2 + \cdots + x_n^2$$

$$= \|\bar{x}\|_2^2$$

$$\|\bar{x}\| = \sqrt{\bar{x}^T A \bar{x}}$$

For previous example.

OTHER EXAMPLES OF INNER PRODUCTS.

$\bar{u}^T \bar{v} = \langle \bar{u}, \bar{v} \rangle =$ inner product

$C[a,b]$ continuous function on $[a,b]$

$f, g \in C[a,b]$

$$\langle f, g \rangle = \int_a^b f(x)\,g(x)\,dx$$

inner product for functions f, g.

$$\|f\|^2 = \langle f, f \rangle$$

$$= \int_a^b f^2(t)\,dt$$

Energy of signal in interval $[a, b]$

$m \times n$ matrices.

Ex: $m = 3$ $n = 2$

$\Rightarrow$ 3×2 matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$\langle A, B \rangle = \mathrm{Tr}(B^T A)$$

'Trace' of Square matrix = sum of Diagonal elements

Inner product.