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INNER PRODUCT & PROPERTIES

Cauch-Schwarz inequality

$$\langle \bar{u}, \bar{v} \rangle \leq \|\bar{u}\| . \|\bar{v}\| .$$

$$\langle \bar{u}+t\bar{v}, \bar{u}+t\bar{v} \rangle \geq 0$$

$$\Rightarrow \quad b^2 - 4ac \leq 0$$

$$\Rightarrow \quad 4\langle \bar{u}, \bar{v} \rangle^2 - 4 \langle \bar{u}, \bar{u} \rangle \langle \bar{v}, \bar{v} \rangle \leq 0$$

If $b^2 - 4ac = 0$

$\Rightarrow$ Quadratic equation in t has a unique root $\tilde{t}$

$\Rightarrow \langle \bar{u} + \tilde{t}\bar{v}, \bar{u} + \tilde{t}\bar{v} \rangle = 0$

$\bar{u} + \tilde{t}\bar{v} = 0$

$\Rightarrow$

$\Rightarrow \bar{u} = -\tilde{t}\bar{v}$

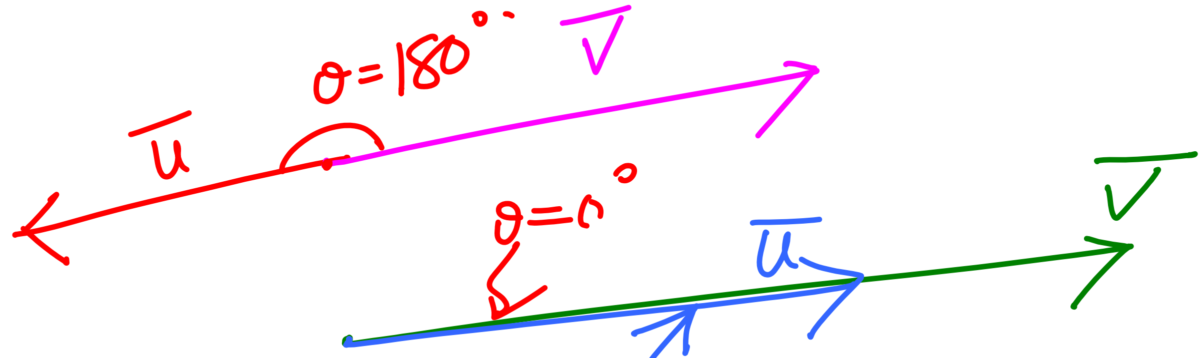
$= k\bar{v}$

$\Rightarrow b^2 = 4ac$

$\Rightarrow \langle \bar{u}, \bar{v} \rangle^2 = \|\bar{u}\|^2 \cdot \|\bar{v}\|^2$

$\Rightarrow |\langle \bar{u}, \bar{v} \rangle| = \|\bar{u}\| \cdot \|\bar{v}\|$.

only if there exists a constant k for which $\boxed{\bar{u} = k\bar{v}}$



vector $\bar{u}$ must lie along $\bar{v}$

or $\bar{u}$ = scaled version of $\bar{v}$

$$\theta = 0^\circ \text{ or } 180^\circ$$

$$|\langle \bar{u}, \bar{v} \rangle| = \|\bar{u}\| \cdot \|\bar{v}\|.$$

PROPERTIES OF NORM:

#1. $\|\bar{v}\| \geq 0.$

$$\|\bar{v}\|^2 = \langle \bar{v}, \bar{v} \rangle \geq 0$$

$$\|\bar{v}\| = \sqrt{\langle \bar{v}, \bar{v} \rangle} \geq 0$$

$\|\bar{v}\| = 0$ iff $\langle \bar{v}, \bar{v} \rangle = 0$

$$\Rightarrow \bar{v} = 0$$

#2. $\|c\bar{v}\| = |c| \cdot \|\bar{v}\|.$

$$\|c\bar{v}\|^2 = \langle c\bar{v}, c\bar{v} \rangle = c \langle \bar{v}, c\bar{v} \rangle$$

$$= c^2 \langle \bar{v}, \bar{v} \rangle$$

$$= c^2 \|\bar{v}\|^2$$

$$\Rightarrow \boxed{\|c\bar{v}\| = |c| \cdot \|\bar{v}\|}$$

#3. TRIANGLE INEQUALITY:

$$\|\bar{u}+\bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$$

$$\|\bar{u}+\bar{v}\|^2 = \langle \bar{u}+\bar{v}, \bar{u}+\bar{v} \rangle$$

$$= \langle \bar{u}, \bar{u}+\bar{v} \rangle + \langle \bar{v}, \bar{u}+\bar{v} \rangle$$

$$= \langle \bar{u}, \bar{u} \rangle + \langle \bar{u}, \bar{v} \rangle + \langle \bar{v}, \bar{u} \rangle + \langle \bar{v}, \bar{v} \rangle$$

$$= \|\bar{u}\|^2 + 2\langle \bar{u}, \bar{v} \rangle + \|\bar{v}\|^2$$

$$\leq \|\bar{u}\|^2 + 2\|\bar{u}\| \cdot \|\bar{v}\| + \|\bar{v}\|^2$$

$$= \left(\|\bar{u}\| + \|\bar{v}\|\right)^2$$

$$\Rightarrow \boxed{\|\bar{u}+\bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|}$$

EXAMPLES:

#1. Rank of matrix via Gaussian Elimination + Echelon Form

$$X = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & 2 & 2 \end{bmatrix}$$

Pivot

Perform Gaussian Elimination

$R_2 - 2R_1$.

$$\equiv \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ -4 & 2 & 2 \end{bmatrix}$$

Reduced element below pivot to zero.

Perform $R_3 + 2R_1$

$$X \equiv \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

Pivot

entries below each column in pivot $= 0$.

$R_3 - 4R_1$.

$$\equiv \left. \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} \right\} \text{3 non zero rows.}$$

Pivots

Below each pivot = column of zeros.

upper Triangular matrix

= "ECHELON" Form of matrix

$$\boxed{\text{Number of non-zero rows} = \text{Rank of matrix}}$$

$$X = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 1 \end{bmatrix}$$

$$R_2 - 2R_1$$

Pivot

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 6 & 1 \end{bmatrix}$$

$$R_3 - 3R_1$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

zeros:

$R_3 + \frac{2}{3} R_2$

$$\equiv \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Num of nonzero rows = 2

Pivots

All zero row

Echelon Form of matrix.

$\Rightarrow$ Rank of matrix $= 2$

Obtained via Gaussian Elimination

Each pivot lies to right of pivot in row above