*

$$\langle A, B\rangle = Tr(B^T A)$$

$$\overset{3\times 2}{B} = \left[\, \bar{b}_1 \,\middle|\, \bar{b}_2 \,\right] \qquad A = \left[\, \bar{a}_1 \,\middle|\, \bar{a}_2 \,\right]$$

1st column

2nd column

$$B^T A = \begin{bmatrix} \bar{b}_1^T \\ \bar{b}_2^T \end{bmatrix} \begin{bmatrix} \bar{a}_1 & \bar{a}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{b}_1^T \bar{a}_1 & \bar{b}_1^T \bar{a}_2 \\ \bar{b}_2^T \bar{a}_1 & \bar{b}_2^T \bar{a}_2 \end{bmatrix}$$

$$Tr(B^T A) = \underline{b}_1^T \underline{a}_1 + \underline{b}_2^T \underline{a}_2$$

$$= \begin{bmatrix} b_{11} & b_{21} & b_{31} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

$$+ \begin{bmatrix} b_{12} & b_{22} & b_{32} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

$$= \underbrace{b_{11} a_{11} + b_{21} a_{21} + b_{31} a_{31} + b_{12} a_{12} + b_{22} a_{22} + b_{32} a_{32}}_{\text{inner product}}$$

$A \in \mathbb{R}^{m \times n}$

$\Rightarrow$ Real $m \times n$ matrix

$B \in \mathbb{R}^{m \times n}$.

$$\mathrm{Tr}(B^T A) = \mathrm{Tr}\left(\begin{bmatrix} \bar{b}_1^T \\ \bar{b}_2^T \\ \vdots \\ \bar{b}_n^T \end{bmatrix} \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix} \right)$$

$$= \sum_{i=1}^{n} \bar{b}_i^T \bar{a}_i$$

n = number of columns.

$\bar{b}_i$ = i^{th} column of B

$\bar{a}_i$ = i^{th} column of A.

$$\|A\|^2 = \langle A, A \rangle$$

$$= \mathrm{Tr}(A^T A)$$

$$= \sum_{i=1}^{n} \bar{a}_i^T \bar{a}_i$$

$$= \sum_{i=1}^{n} \|\bar{a}_i\|^2$$

$$\|A\|_F^2 = \|\bar{a}_1\|^2 + \|\bar{a}_2\|^2 + \dots + \|\bar{a}_n\|^2$$

= sum of magnitude squared of all elements of A

Frobenius Norm

This is termed as Frobenius Norm

CAUCHY-SCHWARZ INEQUALITY:

Fundamental Property for inner product

$$\langle \bar{u}, \bar{v} \rangle^2 \leq \langle \bar{u}, \bar{u} \rangle \cdot \langle \bar{v}, \bar{v} \rangle$$

$$= \|\bar{u}\|^2 \cdot \|\bar{v}\|^2$$

$$\Rightarrow \boxed{|\langle \bar{u}, \bar{v} \rangle| \leq \|\bar{u}\| \cdot \|\bar{v}\|.}$$

Valid for any inner product

CS Inequality : Proof :

consider $\xleftarrow{\text{Parameter}}$

$$y(t) = \langle \bar{u} + t\bar{v}, \bar{u} + t\bar{v} \rangle \geqslant 0$$

$$= \langle u, \bar{u} + t\bar{v} \rangle + t\langle \bar{v}, \bar{u} + t\bar{v} \rangle$$

$$= \langle \bar{u}, \bar{u} \rangle + t\langle \bar{u}, \bar{v} \rangle + t\langle \bar{v}, \bar{u} \rangle + t^2\langle \bar{v}, \bar{v} \rangle$$

$$\underbrace{= \overbrace{\langle \bar{u}, \bar{u} \rangle}^{a} + t\,\overbrace{2\langle \bar{u}, \bar{v} \rangle}^{b} + t^2 \underbrace{\langle \bar{v}, \bar{v} \rangle}_{c.}}_{\text{Quadratic in } t} \geqslant 0$$

For all values of t

Holds true only when discriminant of quadratic ≤ 0

$$\Rightarrow \quad b^2 - 4ac \leq 0$$

$$\Rightarrow \quad 4\langle \bar{u}, \bar{v}\rangle^2 - 4\langle \bar{u}, \bar{u}\rangle \langle \bar{v}, \bar{v}\rangle \leq 0.$$

$$\Rightarrow \quad \langle \bar{u}, \bar{v}\rangle^2 \leq \langle \bar{u}, \bar{u}\rangle \langle \bar{v}, \bar{v}\rangle = \|\bar{u}\|^2 \, \|\bar{v}\|^2$$

$$\Rightarrow \quad \boxed{|\langle \bar{u}, \bar{v}\rangle| \leq \|\bar{u}\| \cdot \|\bar{v}\|}.$$

CS Inequality

$$|\bar{u}\cdot\bar{v}| \leq \|\bar{u}\|\cdot\|\bar{v}\|$$

Dot Product of 2 vectors.

$$\Rightarrow \quad -\|\bar{u}\|\,\|\bar{v}\| \leq \langle \bar{u}, \bar{v} \rangle \leq \|\bar{u}\|\cdot\|\bar{v}\|$$

$$\Rightarrow \quad -1 \leq \underbrace{\frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\|\cdot\|\bar{v}\|}}_{\cos\theta} \leq 1$$

$$\cos\theta = \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\|\cdot\|\bar{v}\|}$$

Angle between 2 vectors.

$\theta = 90^\circ$

$\Rightarrow \bar{u}, \bar{v}$ are perpendicular

$\Rightarrow \cos\theta = 0$

$\Rightarrow \boxed{\langle \bar{u}, \bar{v} \rangle = 0.}$

$\bar{u} \perp \bar{v}$ if $\langle \bar{u}, \bar{v} \rangle = 0.$

Valid for any general inner product.