*

EIGENVECTORS & EIGENVALUES:

For square matrix A
$\bar{x}$ is an eigenvector if

$$\boxed{A\bar{x} = \lambda \bar{x}}$$

Eigenvalue Eigenvector

$$\Rightarrow \quad A\bar{x} = \lambda I \bar{x}$$

$$\Rightarrow \quad A\bar{x} - \lambda I \bar{x} = 0$$

$$\Rightarrow \quad \underline{(A - \lambda I)}\,\bar{x} = 0$$

singular matrix

$$\Rightarrow \quad |A - \lambda I| = 0$$

Determinant

Gives characteristic polynomial of A

roots of characteristic polynomial = Eigenvalues of A

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

2x2 Square matrix

Find Eigenvalues & Eigenvectors of A

$A - \lambda I$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow (1-\lambda)(-1-\lambda) - 1 = 0$$

$$\Rightarrow -(1-\lambda)(1+\lambda) = 1$$

$$\Rightarrow \lambda^2 - 1 = 1$$

$$\Rightarrow \boxed{\lambda = \pm\sqrt{2}}$$

Eigenvalues of A

Eigenvector:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bar{x} = \sqrt{2}\, \bar{x}$$
$$= \sqrt{2}\, I\, \bar{x}$$

$$\Rightarrow \begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow (1-\sqrt{2})x_1 + x_2 = 0$$
$$x_1 - (1+\sqrt{2})x_2 = 0$$
$$\times (1-\sqrt{2})$$
$$\Rightarrow (1-\sqrt{2})x_1 + x_2 = 0$$

These 2 equations are same

set $x_1 = 1$

$\Rightarrow x_2 = -(1-\sqrt{2})$

$$\boxed{\bar{x} = \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}}$$

one of the eigenvectors of matrix A

<u>Check:</u>

$A\bar{x}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$$

$\bar{x}$

$$= \begin{bmatrix} \sqrt{2} \\ 2-\sqrt{2} \end{bmatrix}$$

$$= \sqrt{2}\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$$

$$= \lambda \bar{x}.$$

$\lambda = \sqrt{2}$

verifies that $\sqrt{2}$ = Eigen value

$\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$ = Eigenvector

Similarly, for eigenvalue $-\sqrt{2}$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bar{x} = -\sqrt{2}\,\bar{x}$$

$$\Rightarrow \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \sqrt{2} I \right) \bar{x} = 0$$

$$\Rightarrow \begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow (1+\sqrt{2})x_1 + x_2 = 0 .$$

Set $x_1 = 1$

$$x_2 = -(1+\sqrt{2})$$

Eigenvector

$$\boxed{\bar{x} = \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix}}$$

Other eigenvector corresponding to eigenvalue $-\sqrt{2}$

SYMMETRIC AND HERMITIAN MATRICES:

$$A \in \mathbb{R}^{n \times n}$$

Symmetric if $\boxed{A = A^T}$

$$\Rightarrow a_{ij} = a_{ji} \quad \text{For all } i, j$$

Hermitian if $\boxed{A = A^H}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \end{bmatrix}$$

$$A^H = \begin{bmatrix} a_{11}^* & a_{21}^* & \cdots \\ a_{12}^* & a_{22}^* & \\ \vdots & & \end{bmatrix}$$

Transpose + conjugate

$$\boxed{A = A^H \Rightarrow a_{ij} = a_{ji}^*}$$

1. Eigenvalues of Hermitian & symmetric matrices are REAL

2. Eigenvectors corresponding to DISTINCT Eigenvalues are ORTHOGONAL

$\Rightarrow$ $\bar{v}_1$, $\bar{v}_2$ are eigenvectors corresponding to λ_1, λ_2

$$\Rightarrow \boxed{\bar{v}_1^H \cdot \bar{v}_2 = 0}$$

$\bar{v}_1$, $\bar{v}_2$ are ORTHOGONAL

Ex:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

symmetric matrix

$A = A^T$

Eigenvalues $= \pm\sqrt{2}$

$=$ REAL

$$ev = \underbrace{\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}}_{\bar{v}_1} \quad \underbrace{\begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix}}_{\bar{v}_2}$$

$$\bar{v}_1^T \cdot \bar{v}_2$$

$$= \begin{bmatrix} 1 & \sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix}$$

$$= 1 - (2-1) = 0$$

$$\Rightarrow \bar{v}_1^H \cdot \bar{v}_2 = 0$$

ORTHOGONAL