Time Response Analysis, Design Specifications and Performance Indices

Time Response of First Order System

What is the time response of a control system?

Time response:
The time solution to differential equation.

What is the time response of a control system?

Time response consists of:

1. **Natural response**
2. **Forced response**

What is the time response of a control system?

Natural response is caused by the system itself. Also called **transient response** or homogenous solution

Forced response is caused by the input to the control system. Also called **steady-state response** or particular solution.

What is the time response of a control system?

Time response Plot

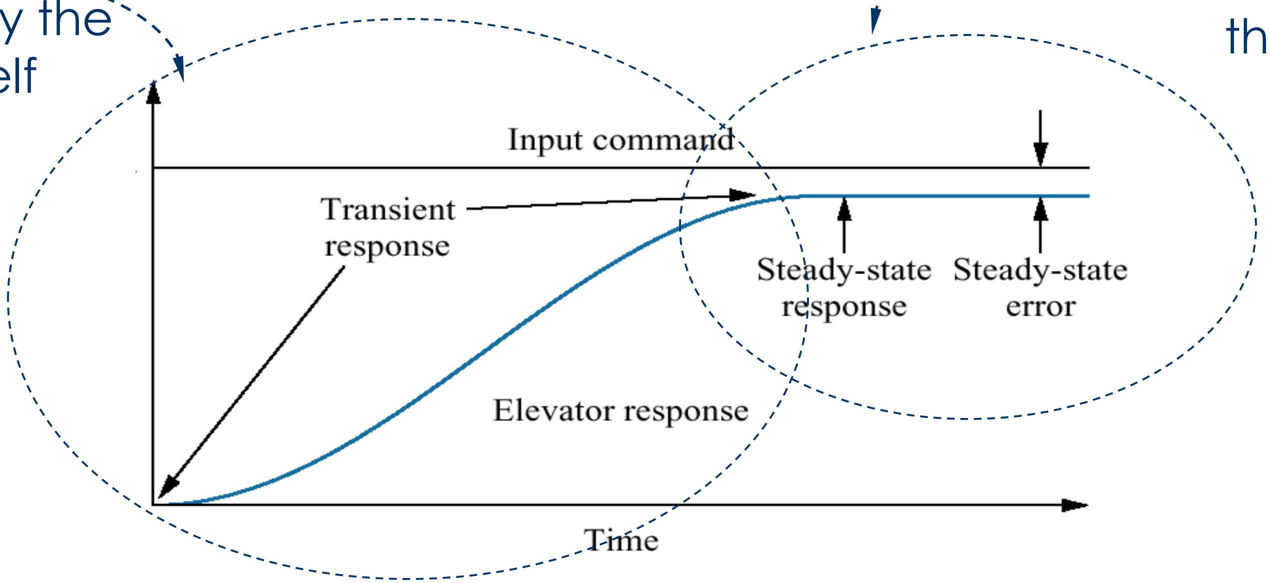
Time solution to differential equation

Natural response or Transient response

Caused by the system itself

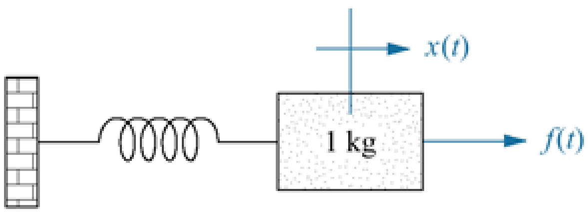
Steady-state response

Caused by the input



How to obtain time response?

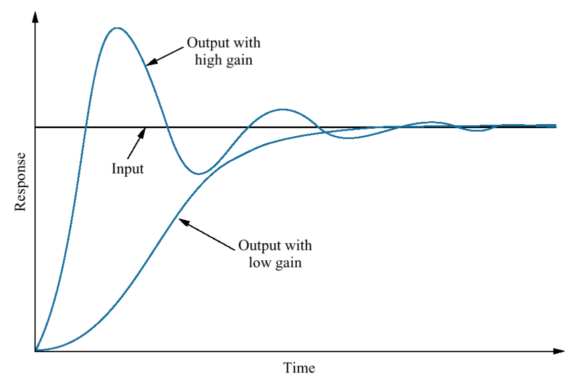
Schematic



Differential Equation

$$M\ddot{x} + Kx = f(t)$$

Time Response



Laplace Transform

$$MXs^2 + KX = F(s)$$

Transfer Function

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + K}$$

Why Transfer Function?

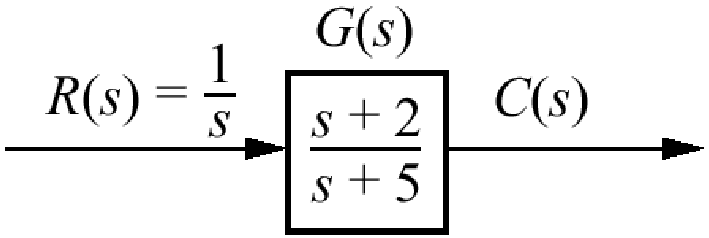
How to obtain time response?

Why use Transfer Function?

1. **Solution by inspection**
2. **Qualitative solution**

How to obtain time response from transfer function?

Given:



How to obtain the time response?

First, we need to understand the concept of **poles** and **zeros**

How to obtain time response from transfer function?

What are **poles** ?

First rule:

The values of Laplace transform variable, s, that cause the transfer function to become infinite.

Second rule:

Any roots of the denominator of the transfer function that are common to the roots of numerator

How to obtain time response from transfer function?

Transfer Function

$$G(s) = \frac{(s+3)}{(s+5)(s+2)(s+3)}$$

First rule:
When s = -5, or s = -2 or s = -3, then, G(s)=infinity! Therefore, the poles of the transfer function G(s) are -5, -2 and -3.

$$G(s) = \frac{\cancel{(s+3)}}{(s+5)(s+2)\cancel{(s+3)}}$$

Second rule:
Although the term (s+3) can be cancelled out, the value -3 is still the poles of the transfer function G(s).

How to obtain time response from transfer function?

What are **zeros** ?

First rule:

The values of Laplace transform variable, s, that cause the transfer function to become zero.

Second rule:

Any roots of the numerator of the transfer function that are common to roots of the denominator

How to obtain time response from transfer function?

Transfer Function

$$G(s) = \frac{(s+3)}{(s+5)(s+2)(s+3)}$$

First rule:
When s = -3, then, G(s)= 0.
Therefore, the zero of the transfer function G(s) is -3.

$$G(s) = \frac{\cancel{(s+3)}}{(s+5)(s+2)\cancel{(s+3)}}$$

Second rule:
Although the term (s+3) can be cancelled out, the value -3 is still the zero of the transfer function G(s).

How to obtain time response from transfer function?

Transfer Function

$$G(s) = \frac{(s+7)}{(s+1)(s+1)(s+3)}$$

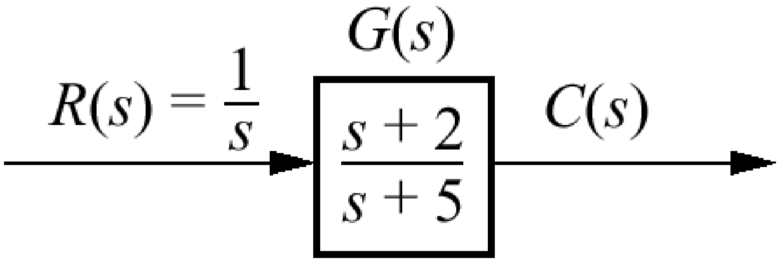
Zero = -7
Poles = -1, -1 and -3

$$G(s) = \frac{(s+13)(s+3)}{(s+11)(s+2)(s+9)}$$

Zero = -13 and -3
Poles = -11, -2 and -9

How to obtain time response from transfer function?

Given:

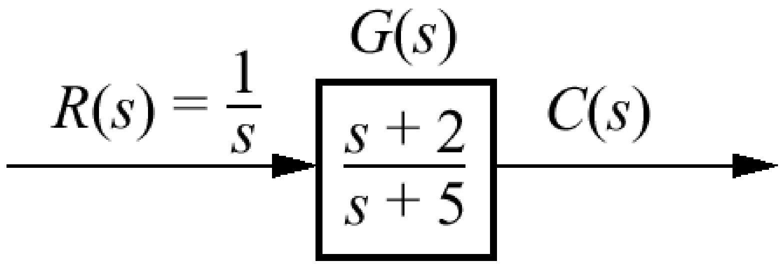


How to obtain the time response?

1. Find C(s)
2. Expand the transfer function using partial fraction expansion technique
3. Perform inverse Laplace transform

How to obtain time response from transfer function? –Find C(s)

Given:



$$G(s) = \frac{Output}{Input} = \frac{C(s)}{R(s)}$$

$$C(s) = G(s)R(s)$$

What is R(s)?

How to obtain time response from transfer function? –Find C(s)

What is R(s)?

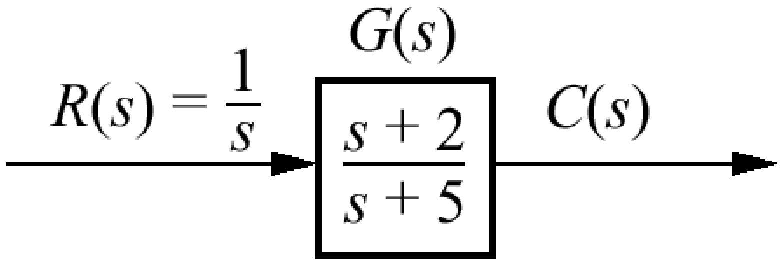
Step input: $r(t) = A$ Laplace Transform $R(s) = \frac{A}{s}$

Ramp input: $r(t) = At$ Laplace Transform $R(s) = \frac{A}{s^2}$

Sine input: $r(t) = \sin(\omega t)$ Laplace Transform $R(s) = \frac{A}{s^2 + \omega^2}$

How to obtain time response from transfer function? –Find C(s)

Given:



R(s) unit step input implies A = 1

$$C(s) = G(s)R(s)$$

$$C(s) = \frac{1}{s}\frac{(s+2)}{(s+5)}$$

How to obtain time response from transfer function? – Expand C(s)

Expand C(s) using Partial Fraction Expansion Technique

$$C(s) = \frac{1}{s}\frac{(s+2)}{(s+5)}$$

$$C(s) = \frac{A}{s} + \frac{B}{s+5} = \frac{(s+2)}{s(s+5)}$$

How to obtain time response from transfer function? – Expand C(s)

$$C(s) = \frac{A}{s} + \frac{B}{s+5} = \frac{(s+2)}{s(s+5)}$$

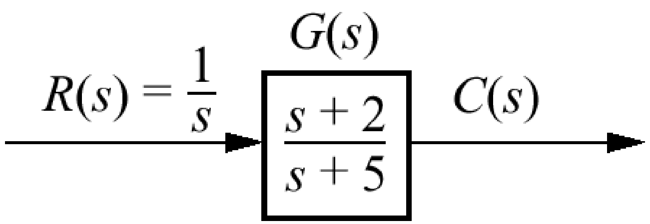
How to find A and B?

$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s\to 0} = \frac{2}{5}$$

$$B = \left.\frac{(s+2)}{(s)}\right|_{s\to -5} = \frac{3}{5}$$

$$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$$

How to obtain time response from transfer function? – Inverse Laplace Transform



$$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$$

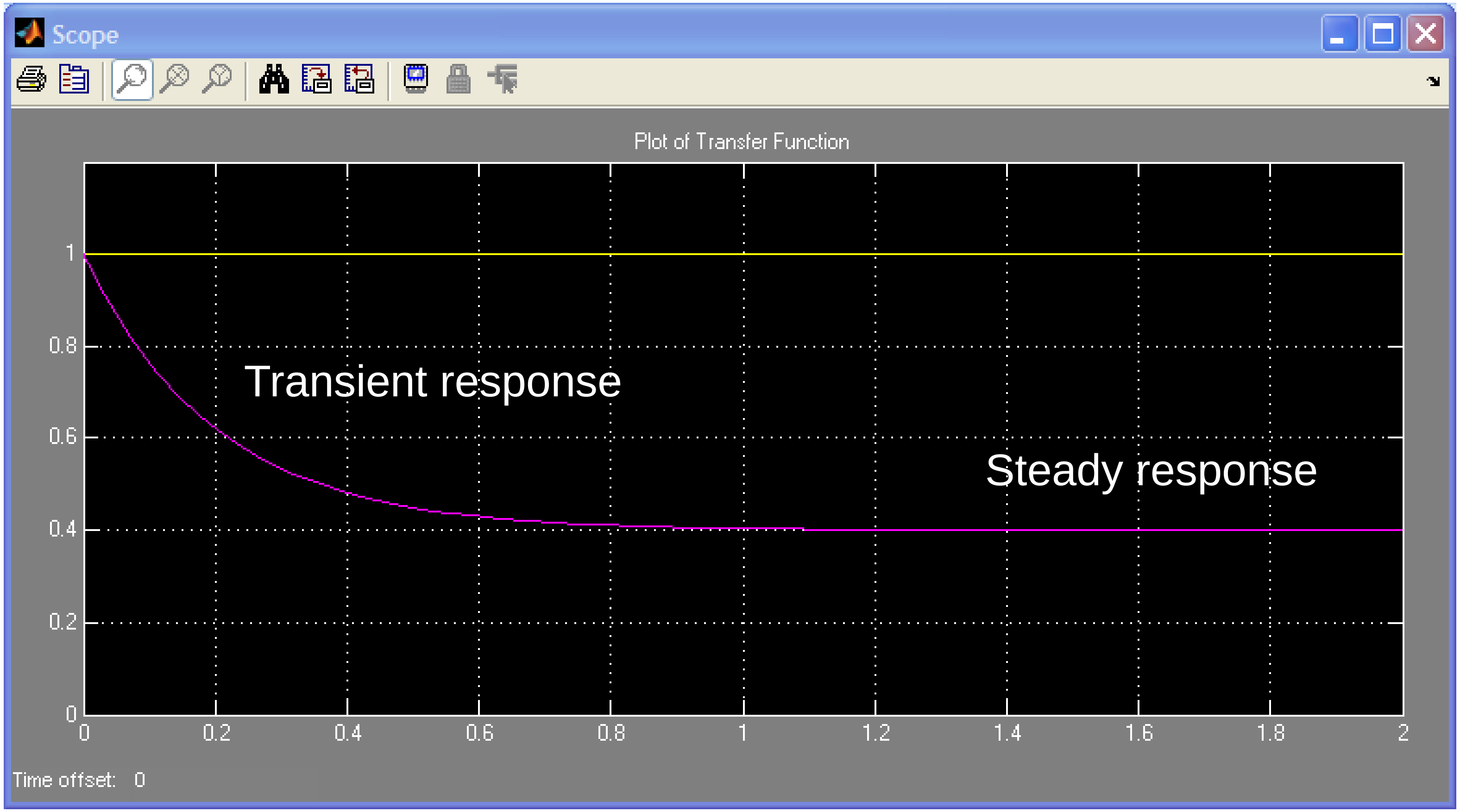
$\frac{2/5}{s}$ Inverse Laplace Transform $\frac{2}{5}$

$\frac{3/5}{s+5}$ Inverse Laplace Transform $\frac{3}{5}e^{-5t}$

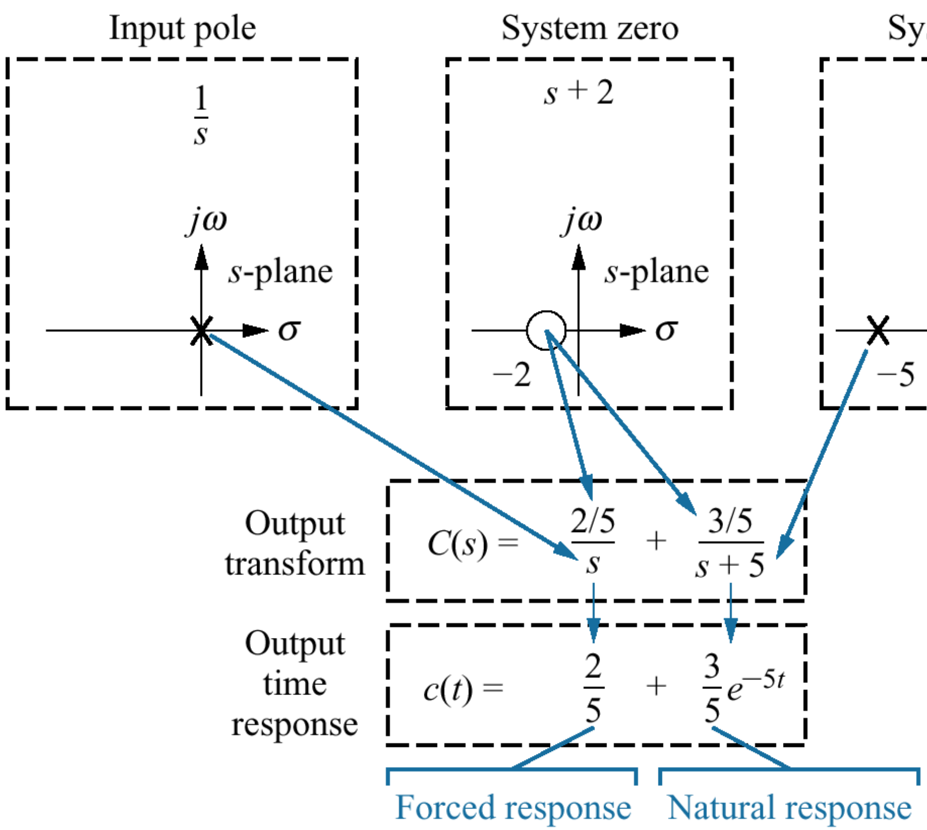
$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

Time response

What is the time response of a control system?



What is time response of a control system?



Input pole generates force response

System pole generates natural response

System pole generates natural response in the form of e^{-at}.

Both zero and pole generates the amplitude of time response

What is time response of a control system? An Example

$$R(s) = \frac{1}{s} \longrightarrow \boxed{\frac{(s+3)}{(s+2)(s+4)(s+5)}} \longrightarrow C(s)$$

What are poles and zeros?

Poles = -2, -4 and -5

Zero = -3

By inspection:

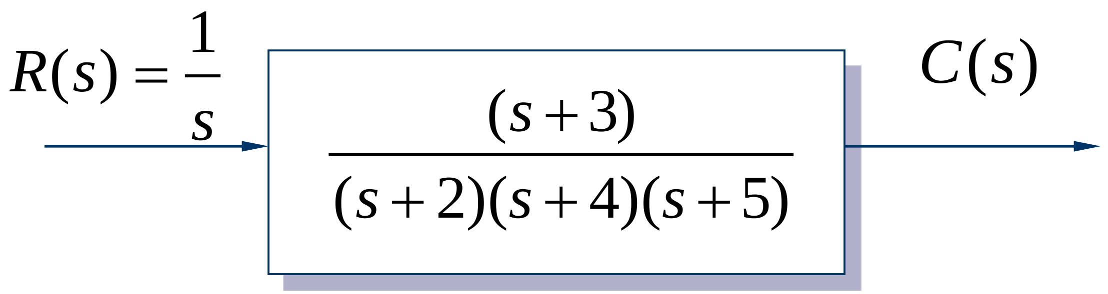
$$C(s) \equiv \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+5)}$$

Influence of poles on time response

Force response

Natural response

What is time response of a control system? An Example



By inspection:

$$C(s) \equiv \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+5)}$$

Inverse Laplace Transform

$$c(t) \equiv K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}$$

Time response – solution to differential equation

What is time response of a control system? An Example

$$C(s) \equiv \frac{K_1}{s} + \frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+5)}$$

$$c(t) \equiv K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}$$

How to evaluate K1, K2, K3 and K4?

$$C(s) = \frac{(s+3)}{s(s+2)(s+4)(s+5)}$$

$$sC(s) = K_1 = \left.\frac{s(s+3)}{s(s+2)(s+4)(s+5)}\right|_{s\to 0} = \left.\frac{(s+3)}{(s+2)(s+4)(s+5)}\right|_{s\to 0} = \frac{3}{11}$$

What is time response of a control system? An Example

$$sC(s)=K_1=\left.\frac{s(s+3)}{s(s+2)(s+4)(s+5)}\right|_{s\to 0}=\left.\frac{(s+3)}{(s+2)(s+4)(s+5)}\right|_{s\to 0}=\frac{0+3}{(0+2)(0+4)(0+5)}=\frac{3}{40}$$

$$(s+2)C(s)=K_2=\left.\frac{(s+2)(s+3)}{s(s+2)(s+4)(s+5)}\right|_{s\to -2}=\left.\frac{(s+3)}{s(s+4)(s+5)}\right|_{s\to -2}=\frac{(-2+3)}{-2(-2+4)(-2+5)}=\frac{1}{-12}$$

$$(s+4)C(s)=K_1=\left.\frac{(s+4)(s+3)}{s(s+2)(s+4)(s+5)}\right|_{s\to -4}=\left.\frac{(s+3)}{s(s+2)(s+5)}\right|_{s\to -4}=\frac{(-4+3)}{(-4)(-4+2)(-4+5)}=\frac{-1}{8}$$

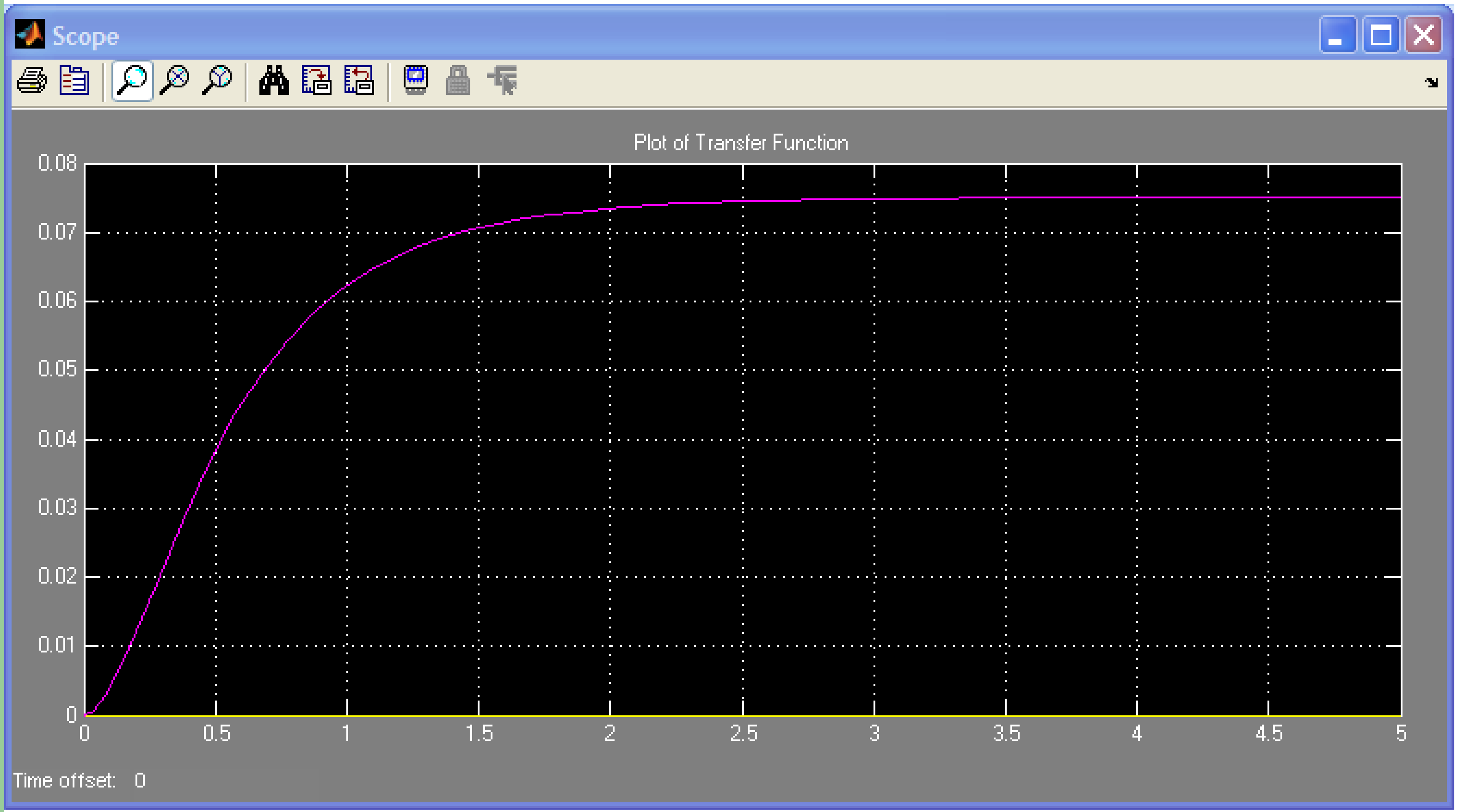
$$(s+5)C(s)=K_1=\left.\frac{(s+5)(s+3)}{s(s+2)(s+4)(s+5)}\right|_{s\to -5}=\left.\frac{(s+3)}{s(s+2)(s+4)}\right|_{s\to -5}=\frac{(-5+3)}{-5(-5+2)(-5+4)}=\frac{-2}{-15}$$

What is time response of a control system? An Example

$$c(t) \equiv K_1 + K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}$$

$$c(t) \equiv \frac{3}{40} - \frac{1}{12} e^{-2t} - \frac{1}{8}{}_{3}\, e^{-4t} + \frac{2}{15} e^{-5t}$$

What is time response of a control system? An Example



What is the order of a control system?

The highest order of differential equations

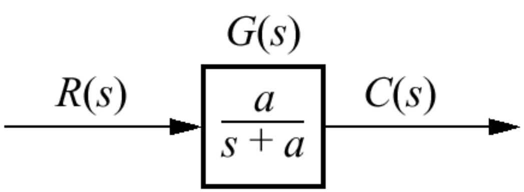
The number of poles

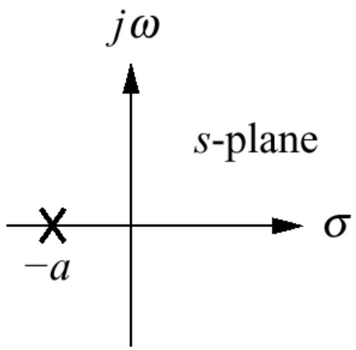
What is the order of a control system?

First Order System

What is the first order of a control system?

One Pole





Time Response

$$c(t) = 1 - e^{-at}$$

Solution Using Inspection

$$C(s) = \frac{a}{s(s+a)}$$

$$K_1 = sC(s) = \left.\frac{sa}{s(s+a)}\right|_{s\to 0} = \left.\frac{a}{(s+a)}\right|_{s\to 0} = \frac{a}{a} = 1$$

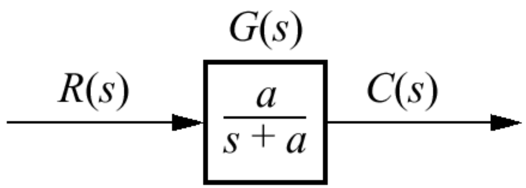
$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+a)}$$

$$K_2 = (s+a)C(s) = \left.\frac{(s+a)a}{s(s+a)}\right|_{s\to -a} = \left.\frac{a}{s}\right|_{s\to -a} = \frac{a}{-a} = -1$$

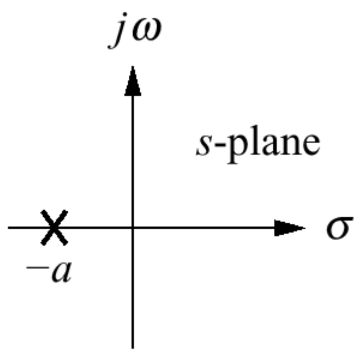
$$c(t) = K_1 + K_2 e^{-at}$$

What is the first order of a control system?

Transfer Function



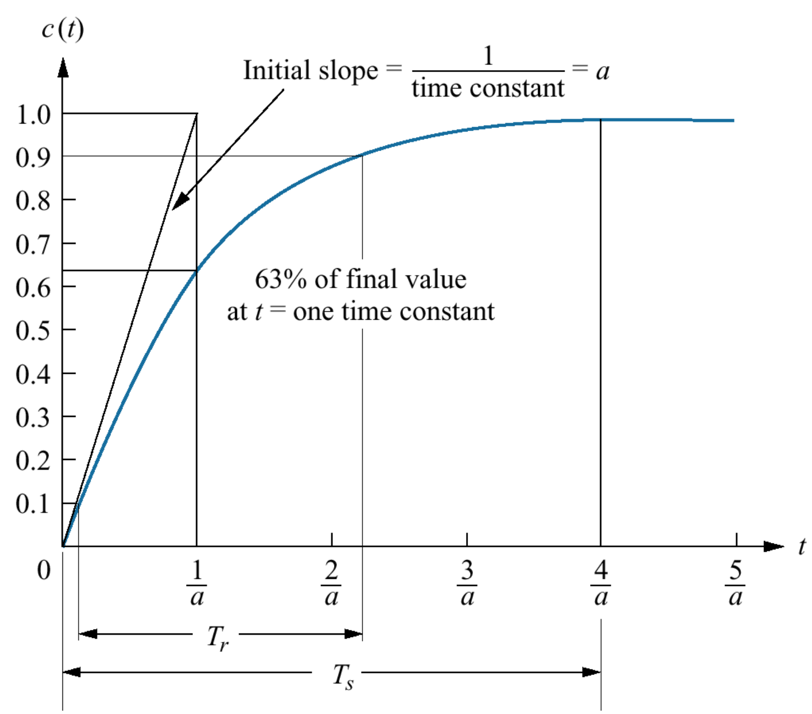
Pole Location



Time response

$$c(t) = 1 - e^{-at}$$

Time Response Plot

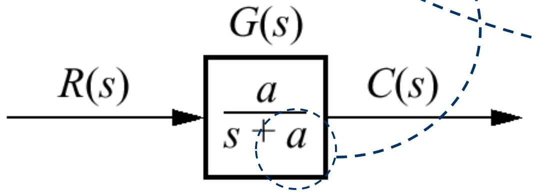


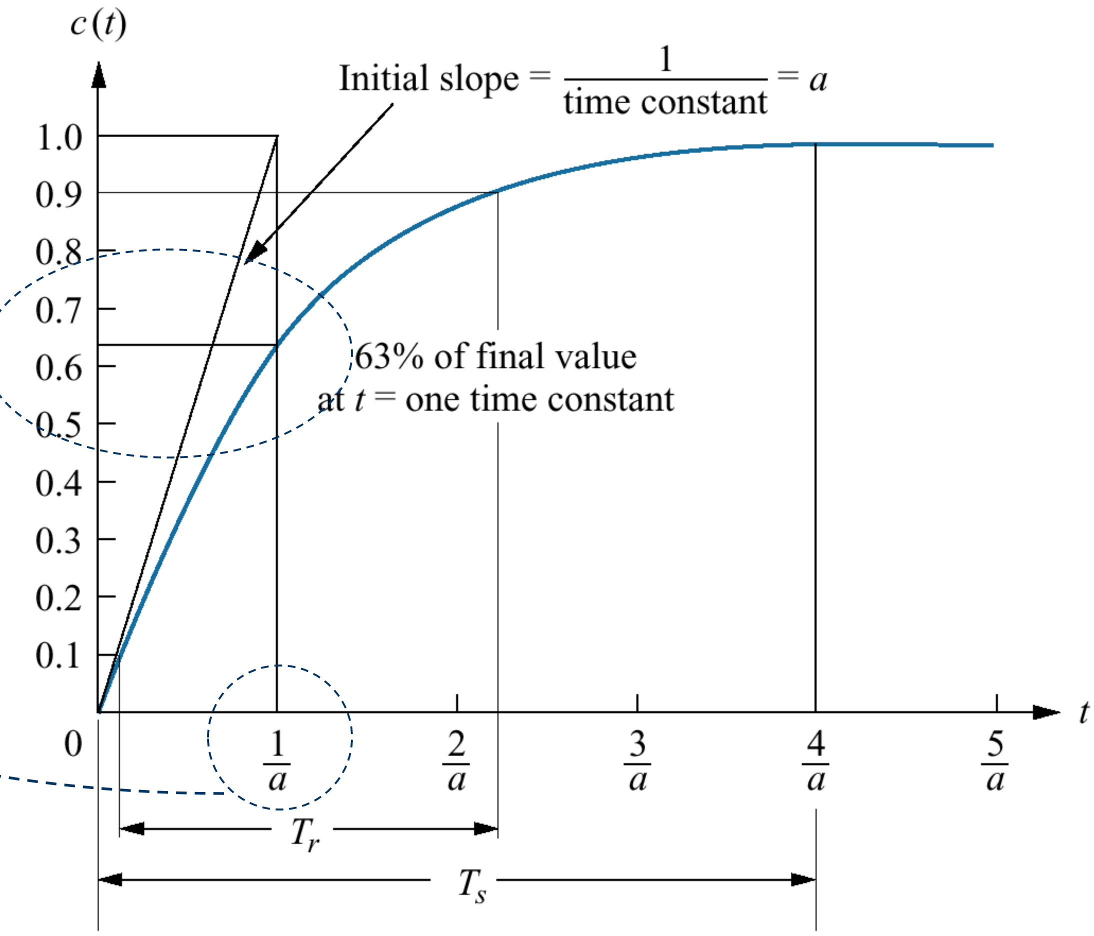
What is the first order of a control system? – Performance Parameters

Time constant

63% of Final value

$$T_c = \frac{1}{a}$$





What is the first order of a control system? – Performance Parameters

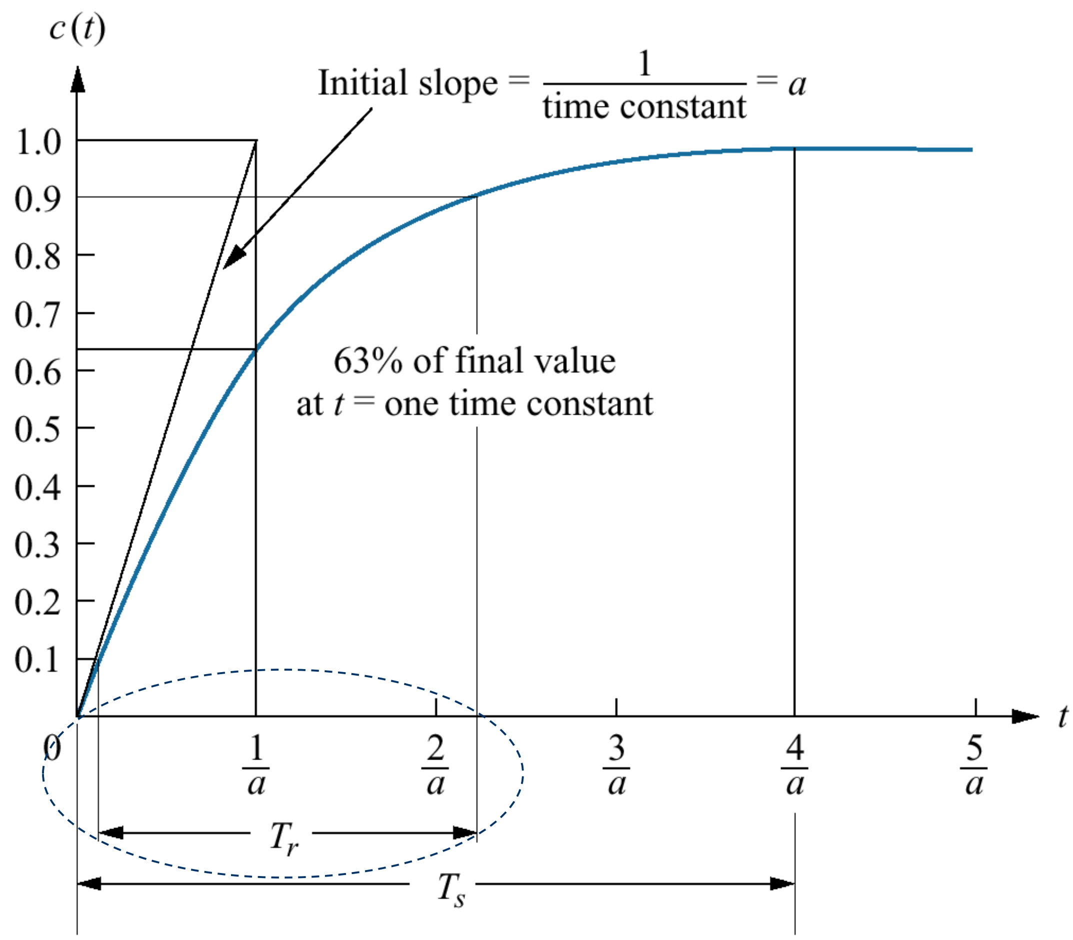
Rise Time

Time to rise from 0.1 to 0.9 of final value

$$T_r = T_{90\%} - T_{10\%}$$

$$T_r = \frac{2.31}{a} - \frac{0.11}{a}$$

$$T_r = \frac{2.2}{a}$$

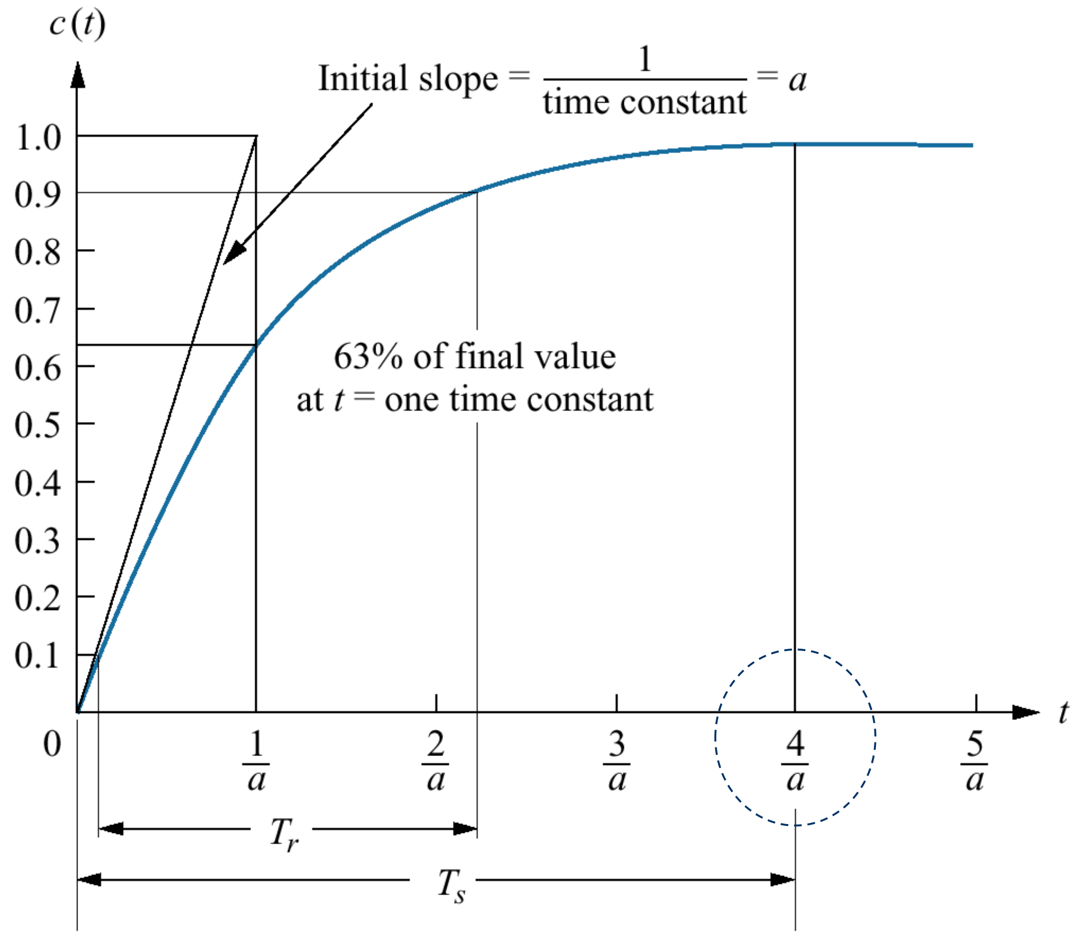


What is the first order of a control system? – Performance Parameters

Settling Time

Time to reach 2% of final value

$$T_s = \frac{4}{a}$$



What is the first order of a control system? – Performance Parameters

Time constant

63% of Final value

$$T_c = \frac{1}{a}$$

Rise Time

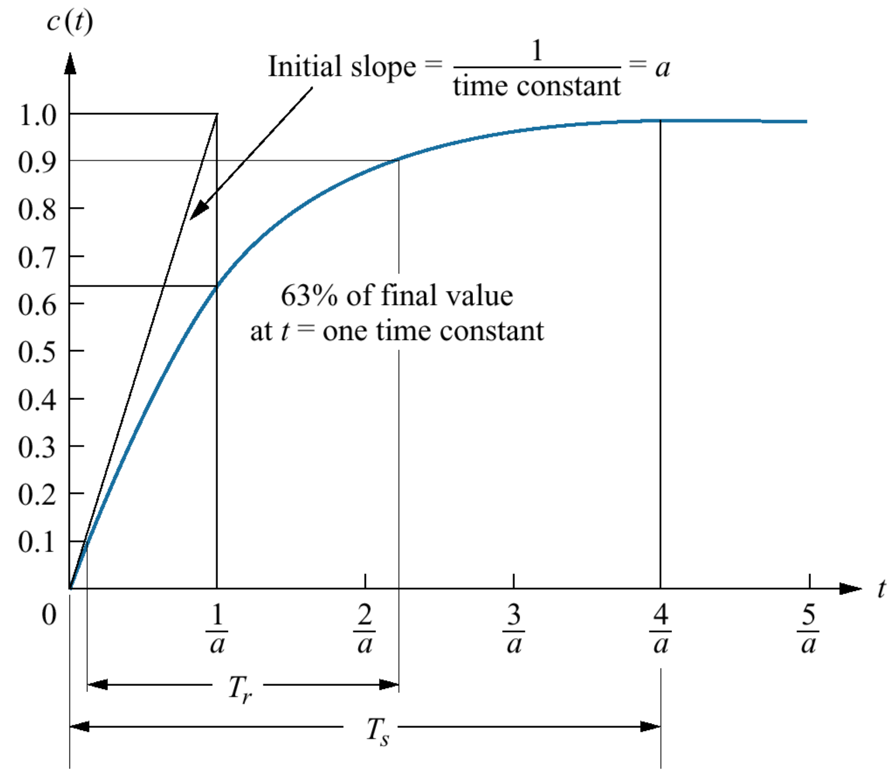
Time to rise from 0.1 to 0.9 of final value

$$T_r = \frac{2.2}{a}$$

Settling Time

Time to reach 2% of final value

$$T_s = \frac{4}{a}$$



What is the first order of a control system? – An Example

$$G(s) = \frac{50}{s+50} \qquad R(s) = \frac{1}{s}$$

First Order because one pole: pole = a = -50

By inspection the solution of unit step input is:

$$c(t) = 1 - e^{-50t}$$

Time constant

$$T_c = \frac{1}{a} = \frac{1}{50}\,\text{sec}$$

Rise Time

$$T_r = \frac{2.2}{a} = \frac{2.2}{50}\,\text{sec}$$

Settling Time

$$T_s = \frac{4}{a} = \frac{4}{50}\,\text{sec}$$

What is the first order of a control system? – An Example

$$G(s) = \frac{200}{s+50} \qquad R(s) = \frac{1}{s}$$

But the numerator (200) is not equal to "a" (50).

$$G(s) = 4\frac{50}{s+50}$$

By inspection the solution of unit step input is:

$$c(t) = 4(1 - e^{-50t})$$

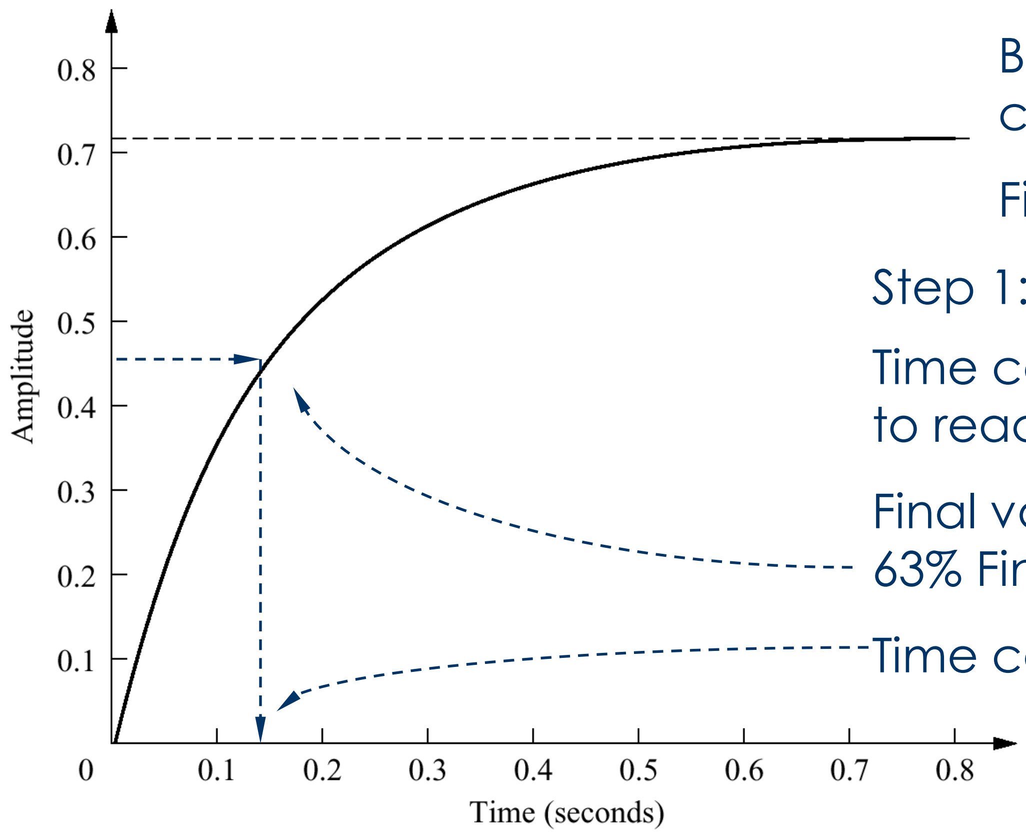
In general:

$$G(s) = \frac{K}{s+a} \qquad c(t) = \frac{K}{a}(1 - e^{-at})$$

Notice that time constant, rise time and settling time are still the same. The performances only depend on the pole.

$$T_c = \frac{1}{a} \qquad T_s = \frac{4}{a} \qquad T_r = \frac{2.2}{a}$$

What is the first order of a control system? – An Example



By experiment we get this curve!!

Find transfer function?

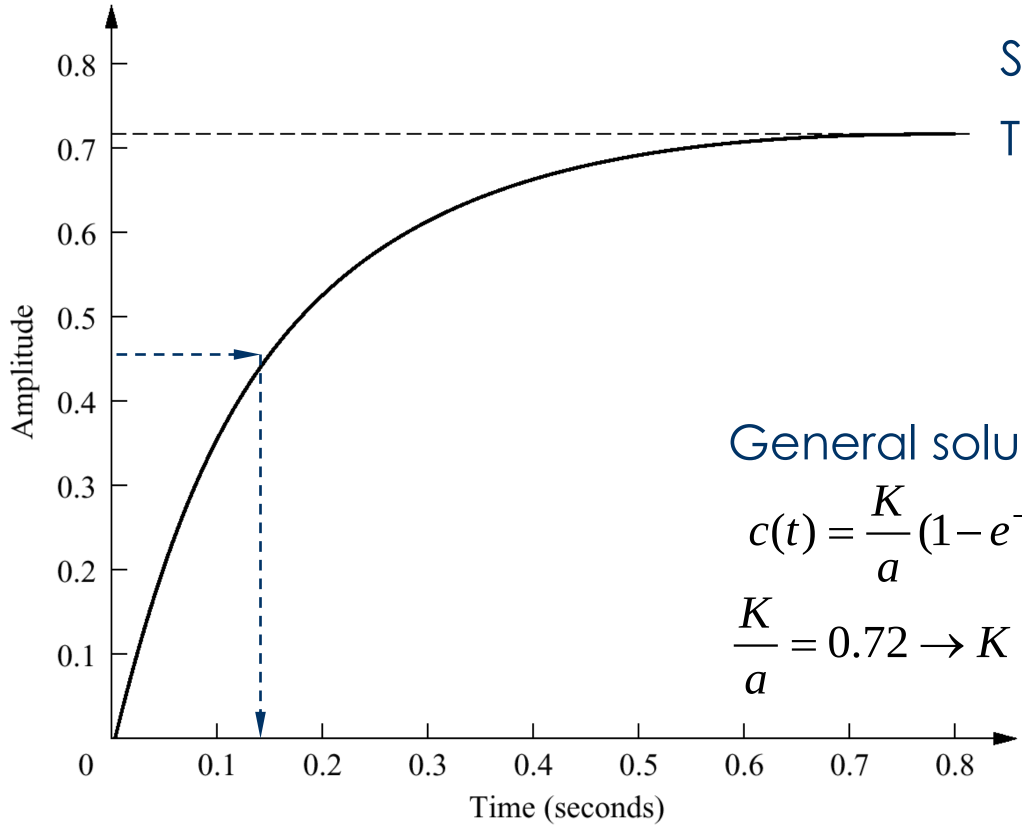
Step 1: Find the pole

Time constant is the time to reach 63% of final value

Final value = 0.72
63% Final value = 0.63X0.72=0.45

Time constant = 0.13 sec

What is the first order of a control system? – An Example



Step 1: Find the pole

Time constant = 0.13 sec

$$T_c = \frac{1}{a}$$

$$a = \frac{1}{T_c} = \frac{1}{0.13} = 7.7$$

General solution

$$c(t) = \frac{K}{a}(1 - e^{-at})$$

$$\frac{K}{a} = 0.72 \rightarrow K = 0.72a = 0.72(7.7) = 5.54$$

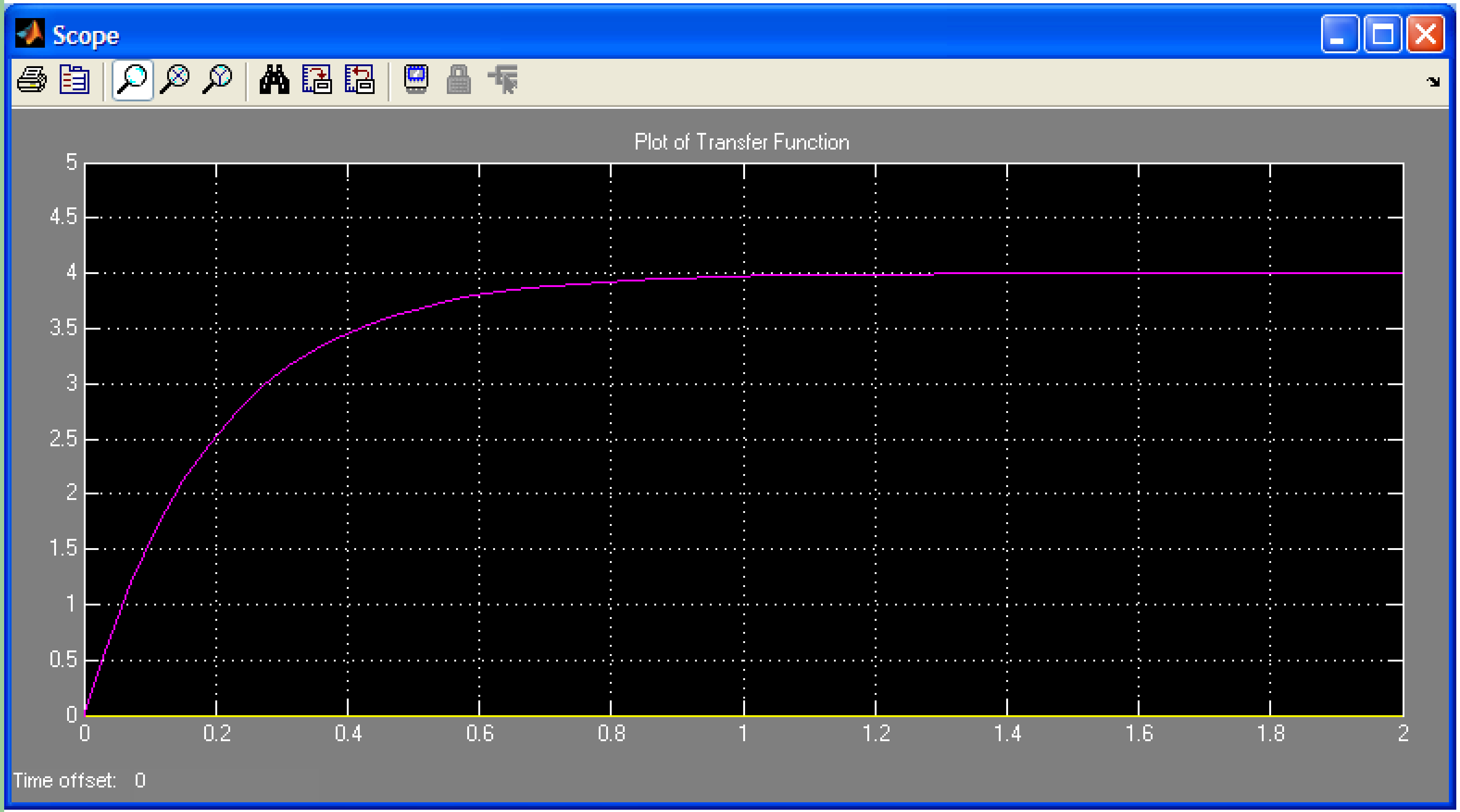
$$G(s) = \frac{K}{s+a} = \frac{5.54}{s+7.7}$$

What is the first order of a control system? – An Example

$$G(s) = \frac{20}{s+5}$$

What are the time constant, rise time, settling time and steady-state value?

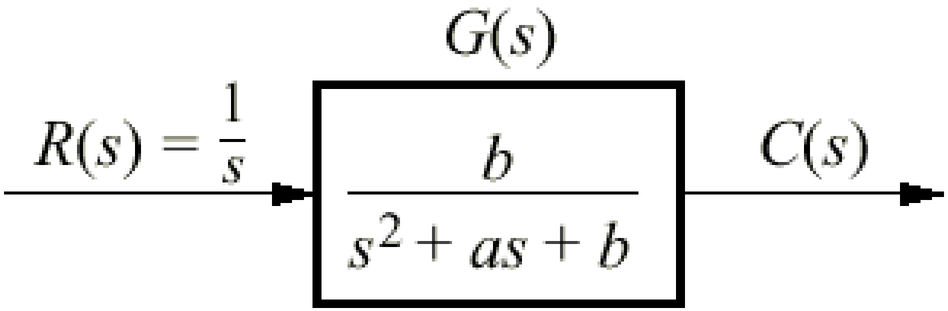
What is the first order of a control system? – An Example



General Response of Second Order System

Time response of a second order control system?

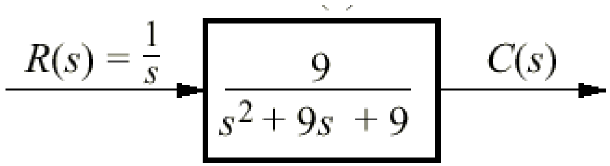
Second Order System
System with two poles



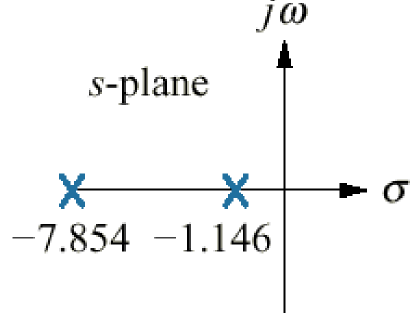
General

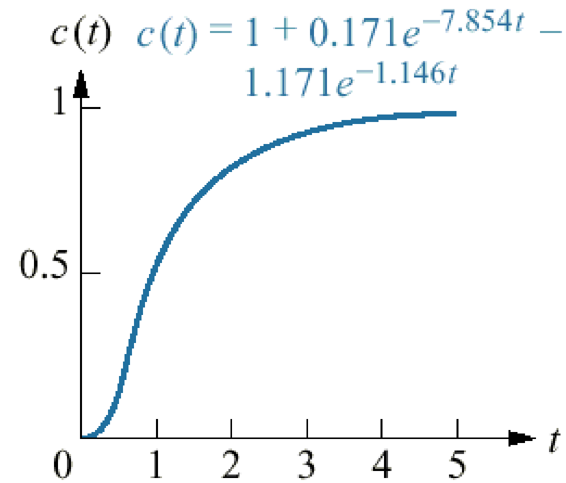
Time response of a second order control system?

Over-damped Time Response



Overdamped





$$\sigma_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sigma_{1,2} = \frac{-9 \pm \sqrt{9^2 - 4(1)(9)}}{2(1)} = \frac{-9 \pm \sqrt{45}}{2}$$

$$\sigma_1 = -7.854$$

$$\sigma_2 = -1.146$$

General solution

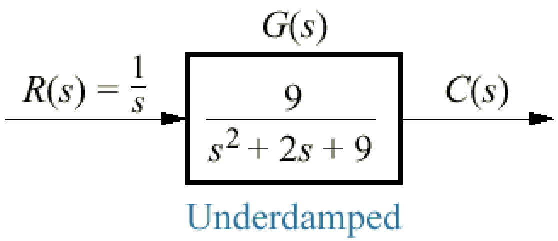
$$C(s) = \frac{9}{s(s+\sigma_1)(s+\sigma_2)} = \frac{K_1}{s} + \frac{K_2}{s+\sigma_1} + \frac{K_3}{s+\sigma_2}$$

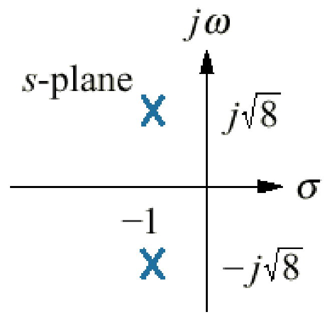
$$c(t) = K_1 - K_2 e^{-\sigma_1 t} - K_3 e^{-\sigma_2 t}$$

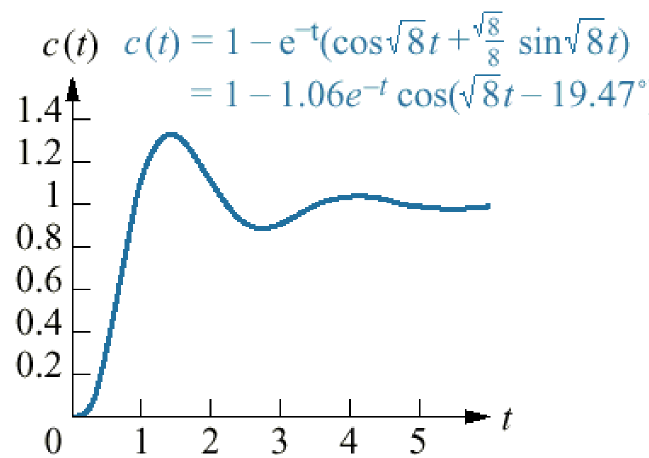
Check this!

Time response of a second order control system?

Under-damped Time Response







$$\sigma_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sigma_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4(1)(9)}}{2(1)} = \frac{-2 \pm j\sqrt{32}}{2}$$

$$\sigma_1 = -1 + j\sqrt{8} \qquad \sigma_{1,2} = -1 \pm j\sqrt{8}$$

$$\sigma_2 = -1 - j\sqrt{8} \qquad \sigma_{1,2} = -\sigma_d \pm j\omega_n$$

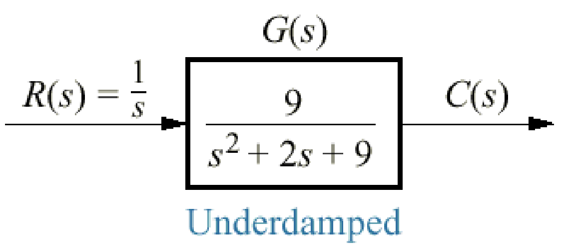
General solution

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d - \phi)$$

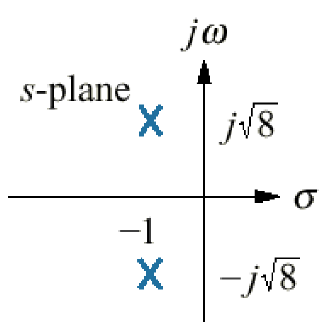
Check this!

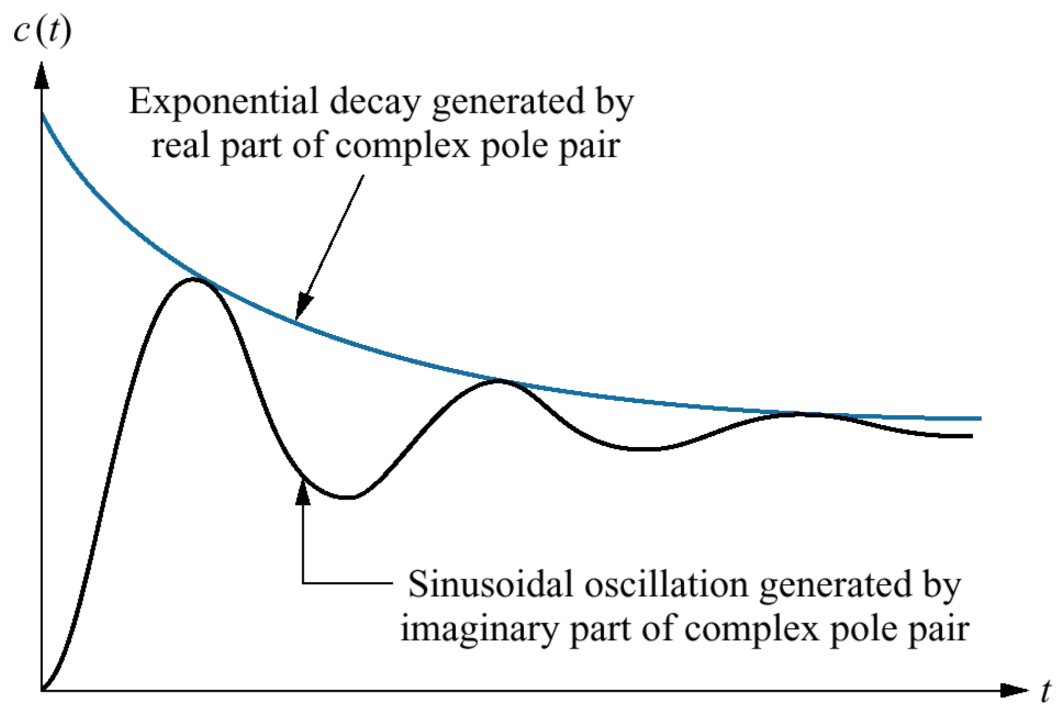
Time response of a second order control system?

Under-damped Time Response



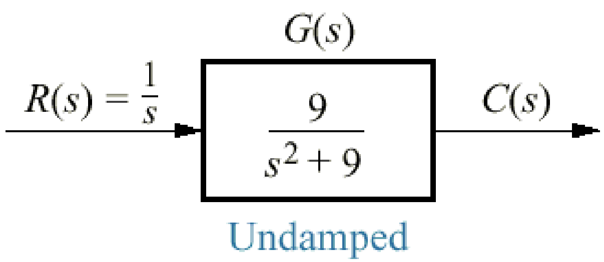
$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d - \phi)$$

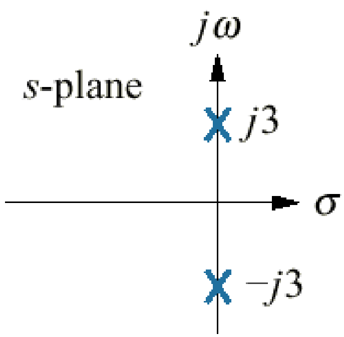


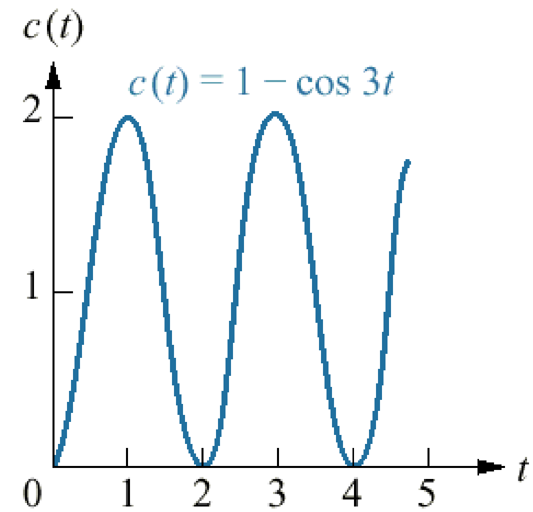


Time response of a second order control system?

Un-damped Time Response







$$\sigma_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sigma_{1,2} = \frac{-0 \pm \sqrt{0^2 - 4(1)(9)}}{2(1)} = \frac{\pm j\sqrt{36}}{2}$$

$$\sigma_1 = +j3$$

$$\sigma_2 = -j3$$

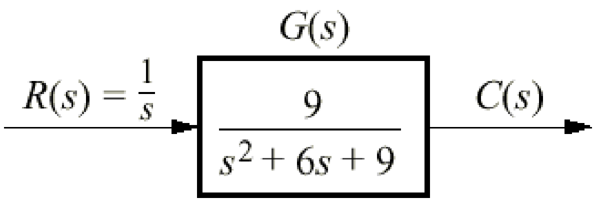
General solution

$$c(t) = A\cos(\omega - \phi)$$

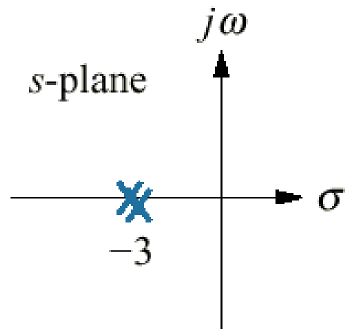
Check this!

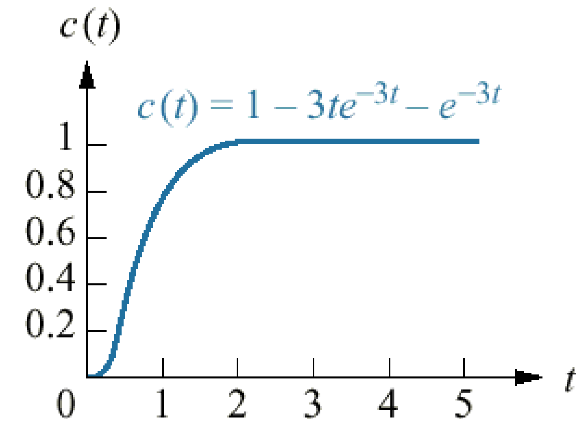
Time response of a second order control system?

Critically-damped Time Response



Critically damped





$$\sigma_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sigma_{1,2} = \frac{-6 \pm \sqrt{6^2 - 4(1)(9)}}{2(1)} = \frac{-6}{2}$$

$$\sigma_{1,2} = -3$$

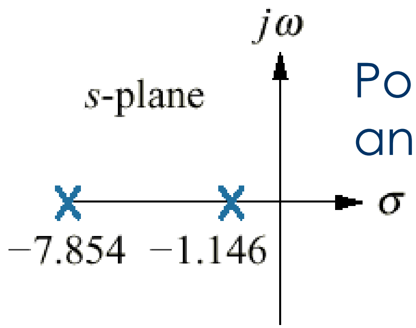
General solution

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

Check this!

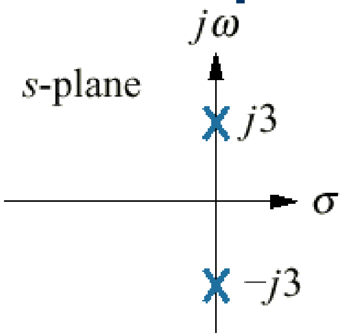
Time response of second order control system? - Summary

Over-damped



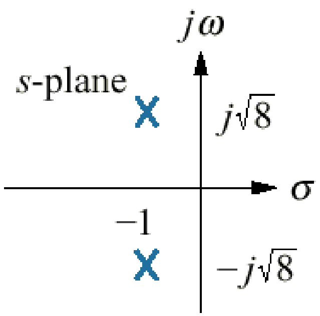
Poles are real and distinct

Un-damped



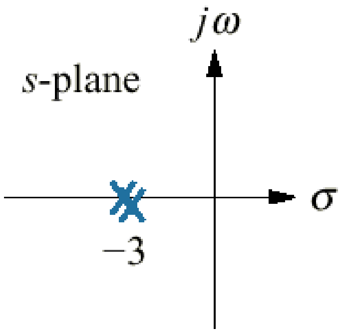
Poles are imaginary

Under-damped



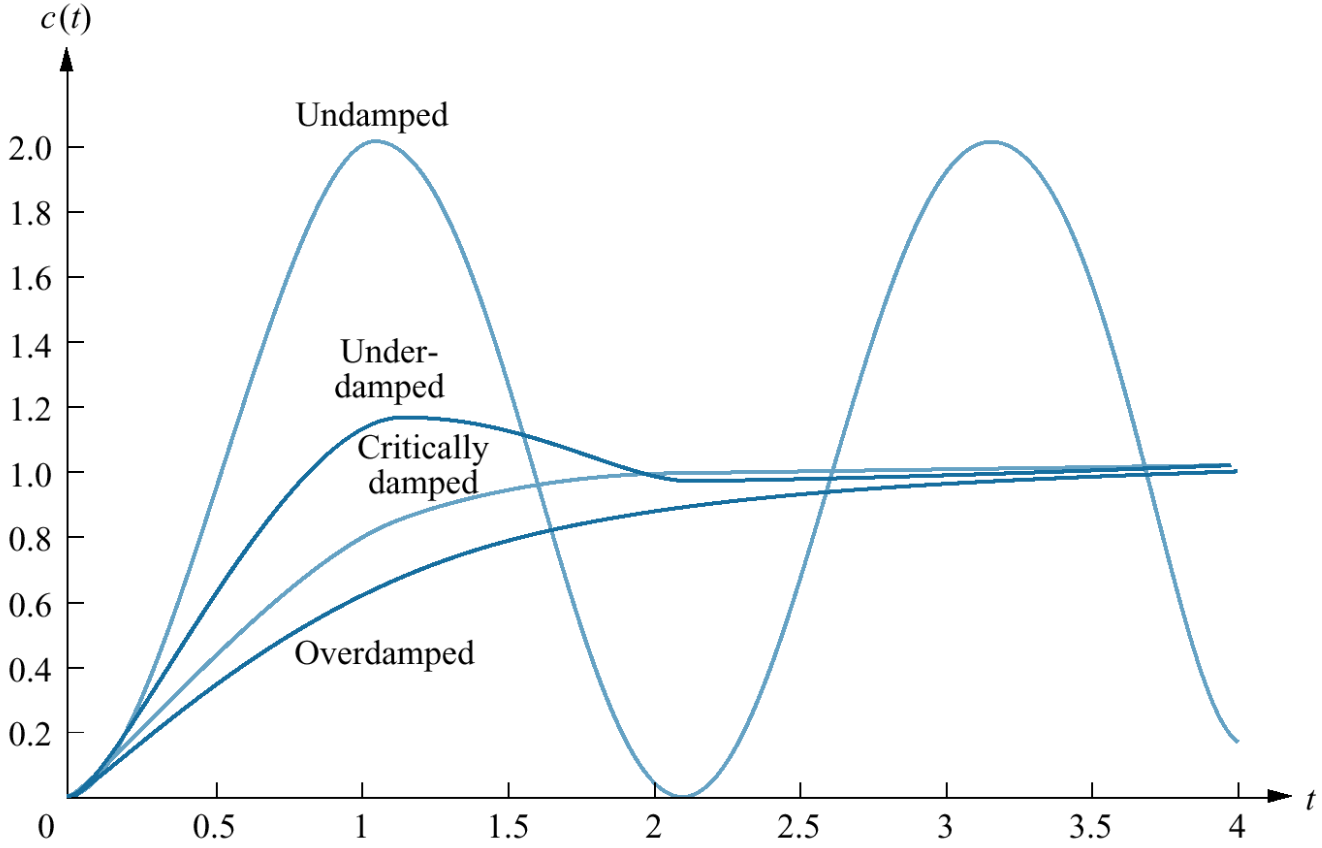
Poles are real and imaginary

Critically-damped



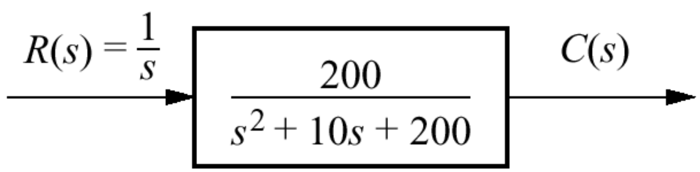
Poles are real and repeated

Time response of second order control system? - Summary



Time response of second order control system? - Example

Check this!



General solution of under-damped

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d - \phi)$$

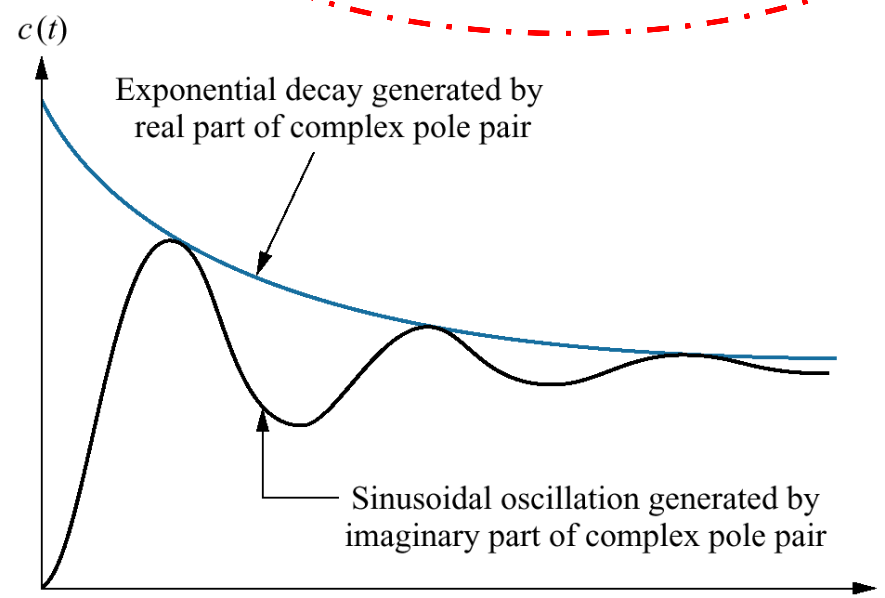
$$c(t) = Ae^{-5t}\cos(13.23 - \phi).$$

$$\sigma_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sigma_{1,2} = \frac{-10 \pm \sqrt{10^2 - 4(1)(200)}}{2(1)}$$

$$\sigma_{1,2} = \frac{-10 \pm j\sqrt{4(175)}}{2}$$

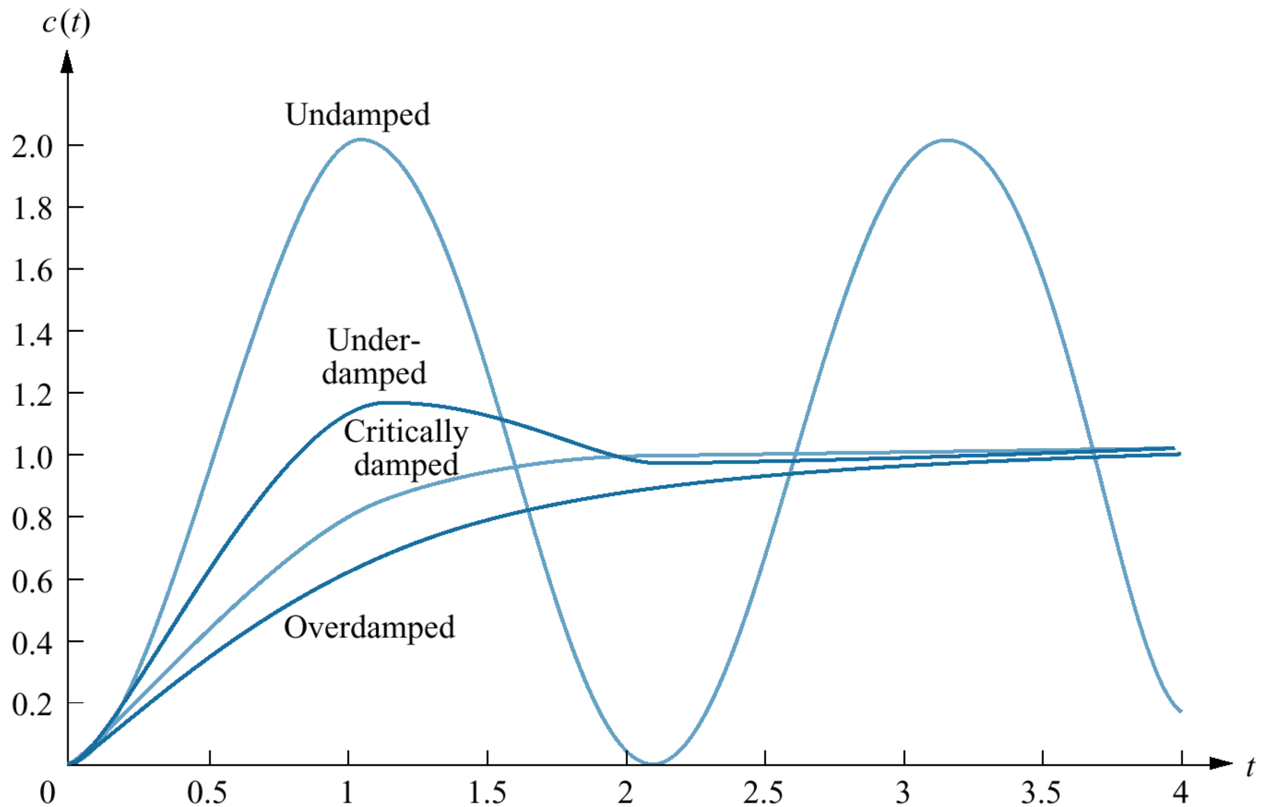
$$\sigma_1,2 = -5 \pm j13.23$$



The poles are real and imaginary. Therefore UNDER-DAMPED.

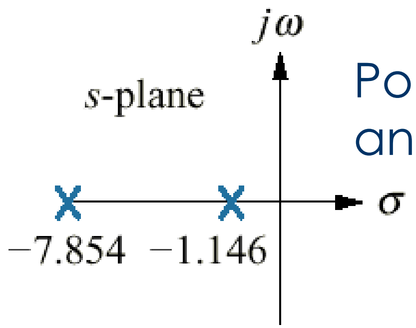
Time Response of Second Order System using Damping Ratio and Natural Frequency

Time response of second order control system? - Summary



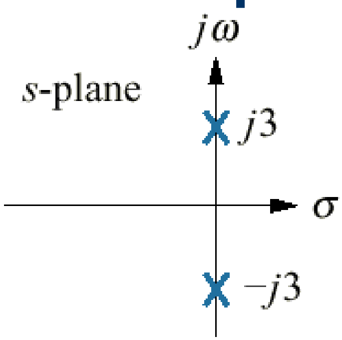
Time response of second order control system? - Summary

Over-damped



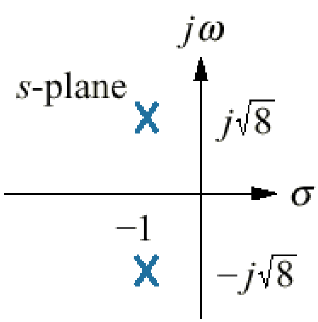
Poles are real and distinct

Un-damped



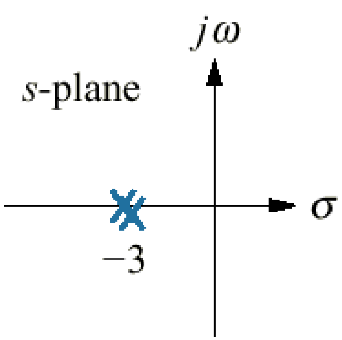
Poles are imaginary

Under-damped



Poles are real and imaginary

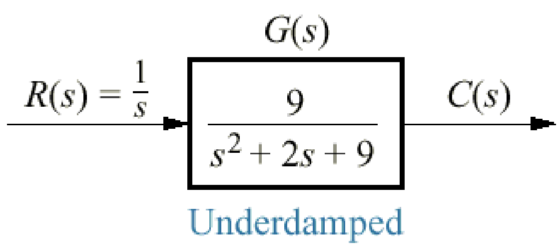
Critically-damped



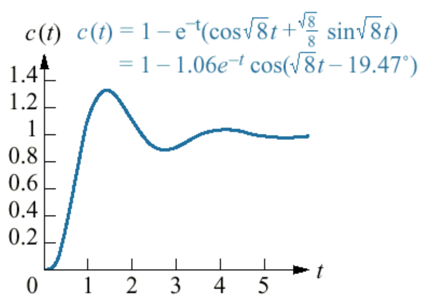
Poles are real and repeated

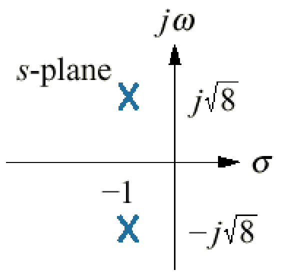
Time response of second order control system?

So far, we based on the poles of the second order system



$$\sigma_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$





Time response of second order control system?

We are going to learn two quantities that will help us analyze second order system

Natural Frequency

$$\omega_n$$

Damping Ratio

$$\xi$$

Time response of second order control system? Natural Frequency

Natural Frequency ω_n

Natural frequency of a second order system is the frequency of oscillation of the system without damping

Time response of second order control system? Natural frequency

$$G(s) = \frac{b}{s^2 + as + b}$$

Natural frequency of a second order system is the frequency of oscillation of the system without damping

$$G(s) = \frac{b}{s^2 + \cancel{as} + b}$$

$$G(s) = \frac{b}{s^2 + b}$$

Poles:

$$s^2 + b = 0$$

$$s = \pm\sqrt{-b}$$

$$s = \pm j\sqrt{b}$$

$$s = \pm j\omega_n$$

Therefore,

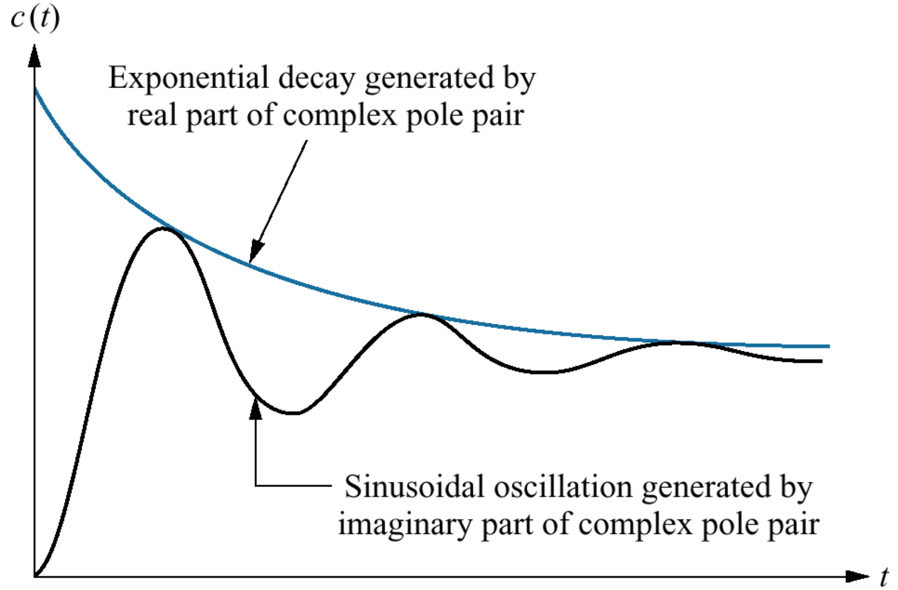
$$\omega_n = \sqrt{b}$$

$$b = \omega_n^2$$

Time response of second order control system? Damping Ratio

Damping Ratio ξ

$$\xi = \frac{\text{Exponential Decay Frequency}}{\text{Natural Frequency (rad/sec)}}$$



Time response of second order control system? Damping Ratio

$$G(s) = \frac{b}{s^2 + as + b}$$

Damping Ratio

$$\xi = \frac{a}{2\omega_n}$$

Time response of second order control system?

General Form

$$G(s) = \frac{b}{s^2 + as + b} \qquad G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Time response of second order control system?

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Find damping ratio and the natural frequency?

Natural Frequency

$$\omega_n^2 = 36$$
$$\omega_n = 6$$

Damping Ratio

$$2\xi\omega_n = 4.2$$
$$\xi = \frac{4.2}{2(6)} = 0.35$$

Time response of second order control system?

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Find poles in general form?

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2}$$

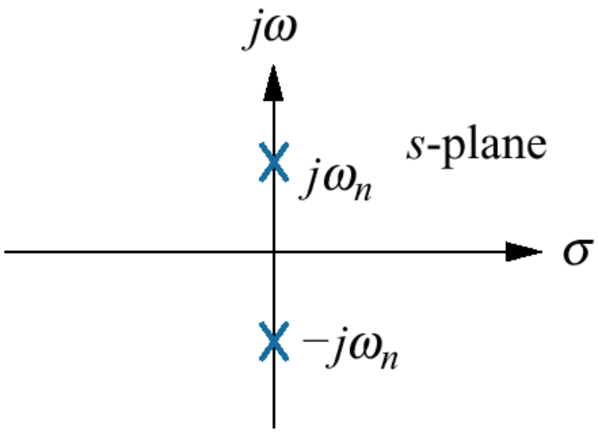
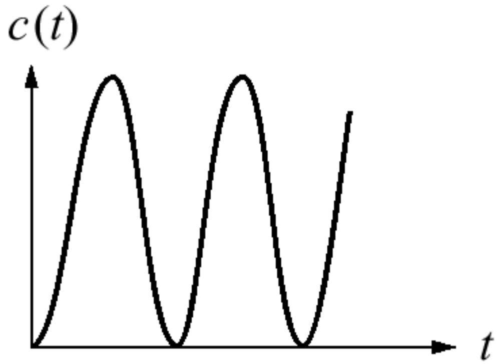
$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Time response of second order control system?

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Case 1: $\xi = 0$

$$s_{1,2} = \pm j\omega_n$$

Poles	Step response
	

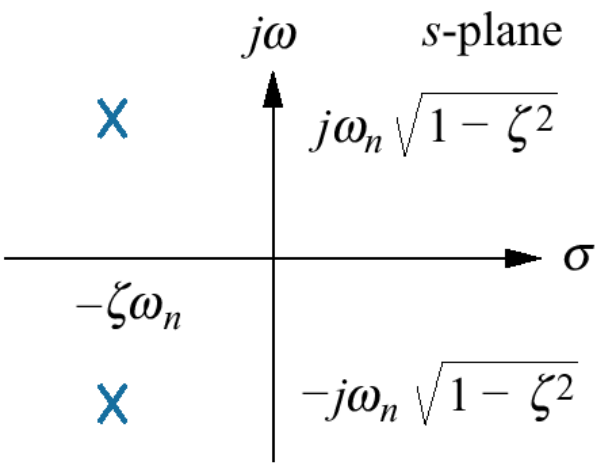
Undamped

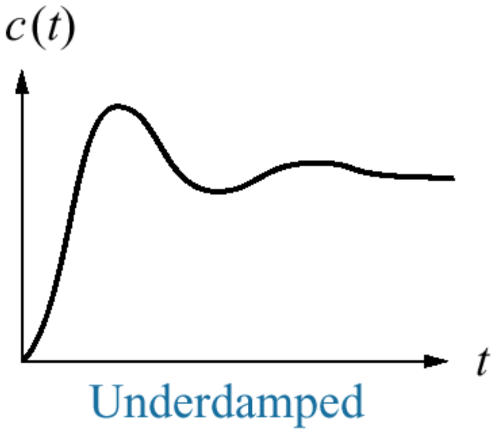
Time response of second order control system?

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Case 2: $0 < \xi < 1$

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}$$



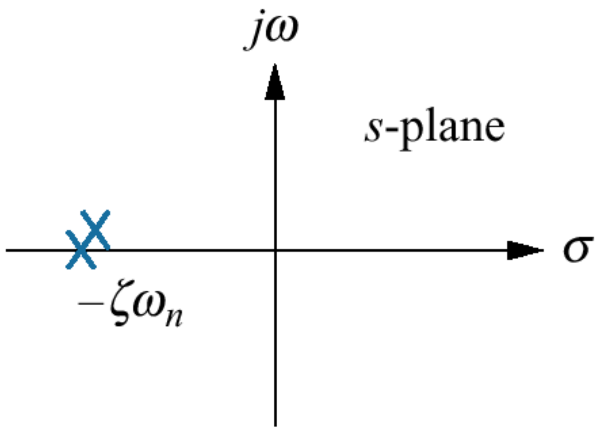


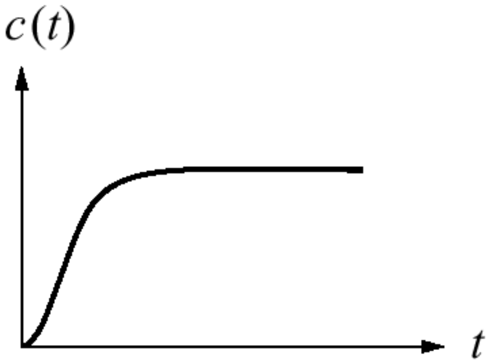
Time response of second order control system?

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Case 3: $\xi = 1$

$$s_{1,2} = -\xi\omega_n$$

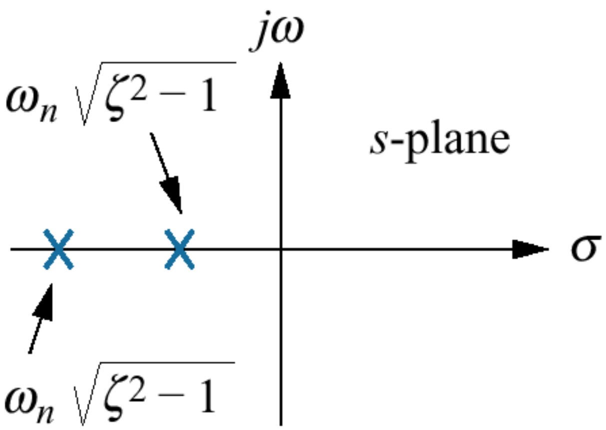


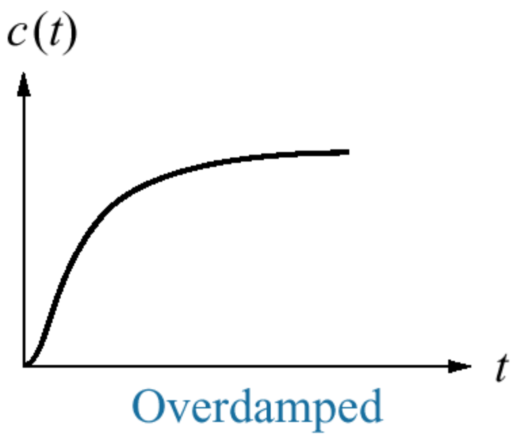


Time response of second order control system?

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Case 4: $\xi > 1$ $s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$





Time response of second order control system?

THE POINT IS…IF WE KNOW

WE KNOW THE TYPE OF RESPONSE

$$\xi$$

$\xi > 1$	OVERDAMPED
$\xi = 1$	CRITICALLY DAMPED
$0 < \xi < 1$	UNDER DAMPED
$\xi = 0$	UNDAMPED

WHERE DO WE GET

$$\xi$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Time response of second order control system? Example

$$G(s)=\frac{400}{s^2+12s+400}$$

$$\omega_n=\sqrt{400}=20$$

$$\xi=\frac{12}{2(20)}=0.3$$

UNDERDAMPED

$$G(s)=\frac{900}{s^2+90s+900}$$

$$\omega_n=\sqrt{900}=30$$

$$\xi=\frac{90}{2(30)}=1.5$$

OVER-DAMPED

$$G(s)=\frac{225}{s^2+30s+225}$$

$$\omega_n=\sqrt{225}=15$$

$$\xi=\frac{30}{2(15)}=1$$

CRITICALLY-DAMPED

$$G(s)=\frac{625}{s^2+625}$$

$$\omega_n=\sqrt{625}=25$$

$$\xi=\frac{0}{2(25)}=0$$

UN-DAMPED

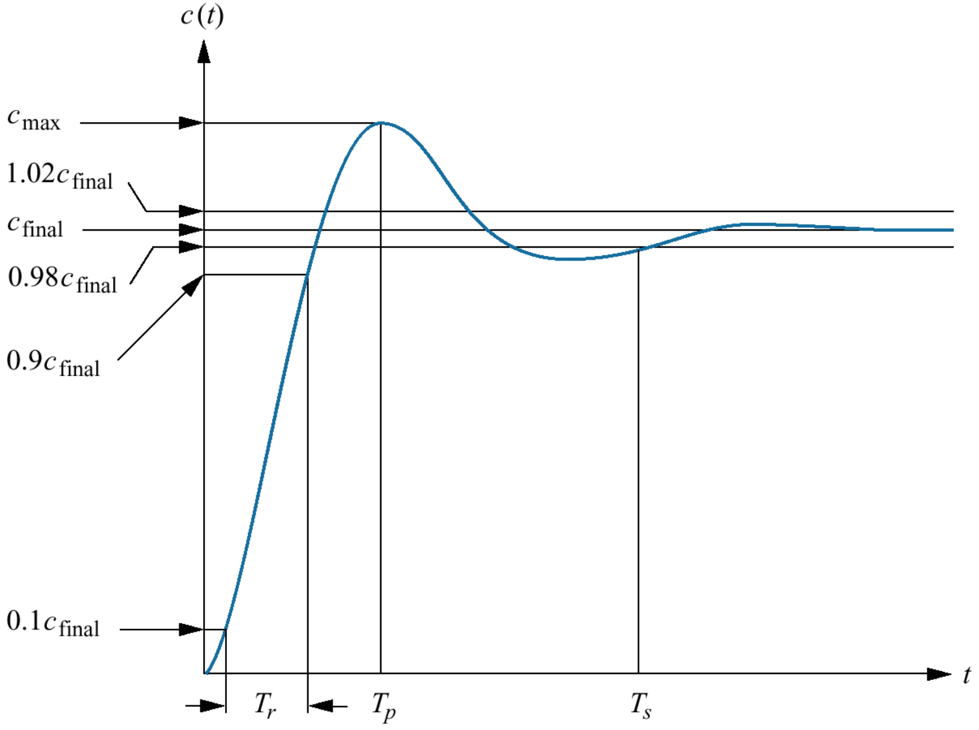
Time response of second order control system? Special Case: Performance of under-damped System

Under-damped:

$$0 < \xi < 1$$

Four performance parameters:

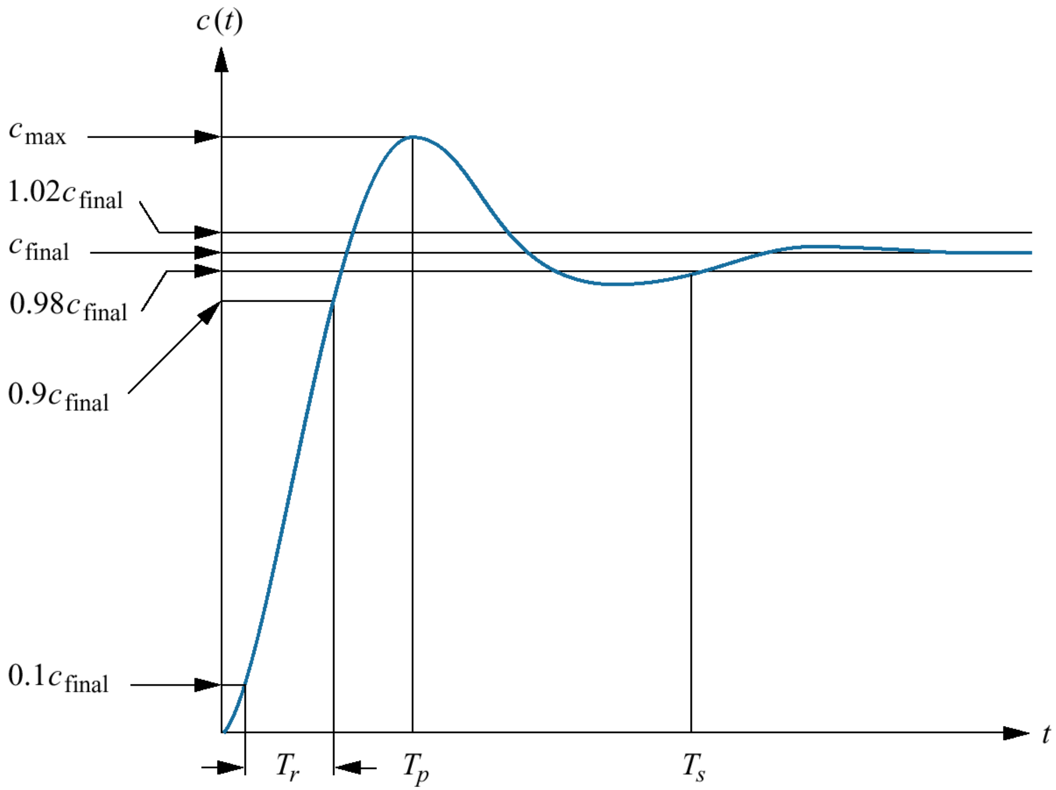
1. Rise time
2. Peak time
3. Percent overshoot
4. Settling time



Time response of second order control system? Special Case: Performance of under-damped System

Rise Time

The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value for overdamped system and 0 to 1 of the final value for underdamped system

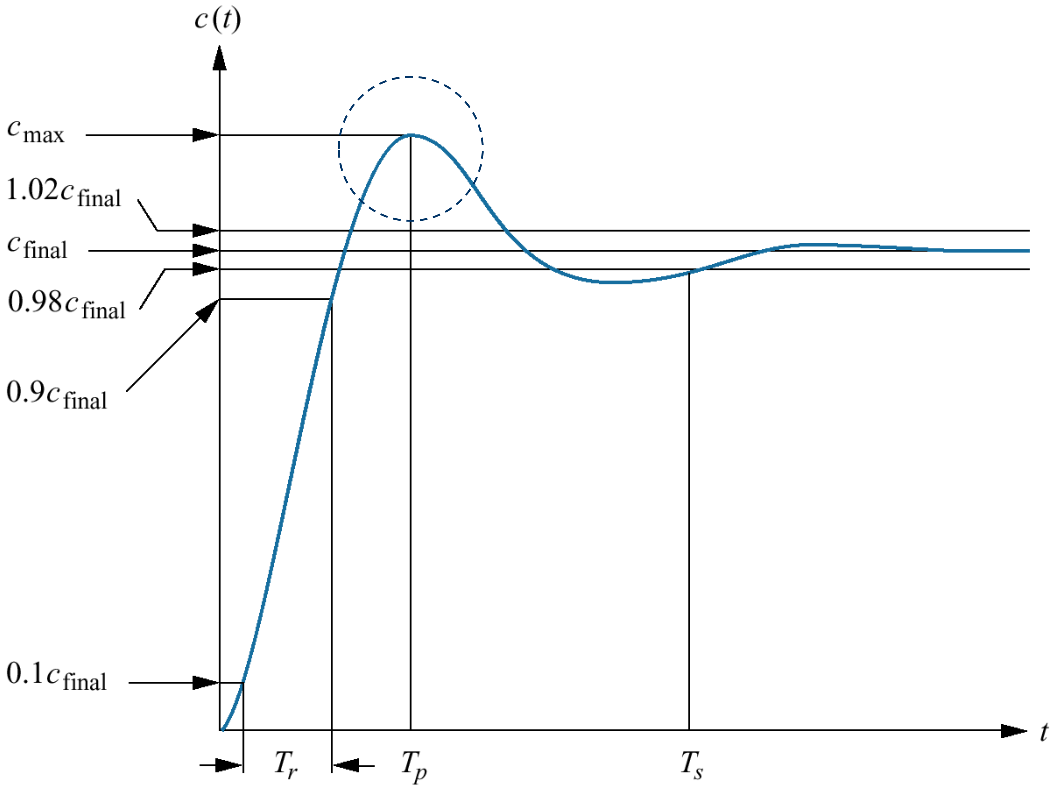


Time response of second order control system? Special Case: Performance of under-damped System

Peak Time

The time required to reach the first maximum peak

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

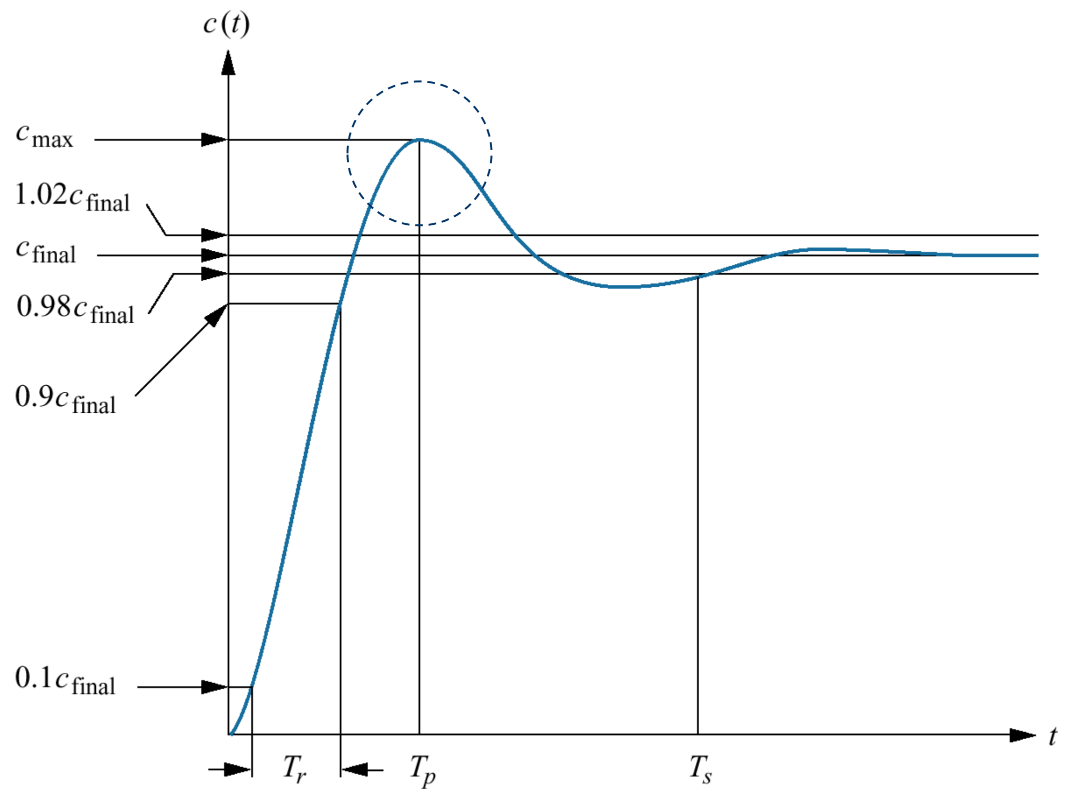


Time response of second order control system? Special Case: Performance of under-damped System

Percent Overshoot

The amount that the waveform overshoots the steady-state

$$\%OS = 100e^{\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right)}$$

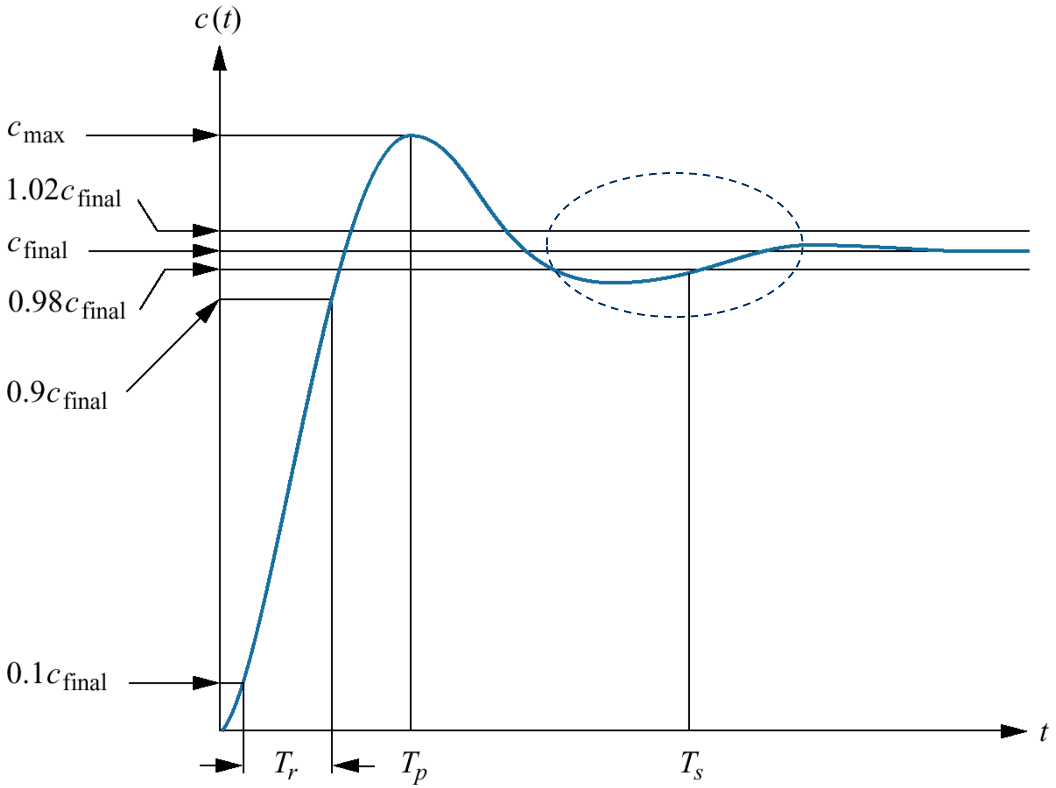


Time response of second order control system? Special Case: Performance of under-damped System

Settling Time

The time required for the transient damped oscillations to reach and stay within 2% of the steady-state value

$$T_s = \frac{4}{\xi\omega_n}$$



Time response of second order control system? Special Case: Performance of under-damped System – Example

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

Find: $\omega_n, \xi, T_s, T_p, T_r, \%OS$

$$\omega_n = \sqrt{361} = 19$$

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}} = \frac{\pi}{19\sqrt{1-0.421^2}} = 0.182$$

$$\xi = \frac{16}{2(19)} = 0.421$$

$$T_s = \frac{4}{\xi \omega_n} = \frac{4}{0.421(19)} = 0.5$$

$$\%OS = 100e^{\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right)} = 100e^{\left(\frac{-0.421\pi}{\sqrt{1-0.421^2}}\right)} = 23.3\%$$

Time response of second order control system? Special Case: Performance of under-damped System – Example

$$G(s) = \frac{361}{s^2 + 16s + 361}$$

Find: T_r

$$\omega_n = 19$$

$$\xi = 0.421$$

$$T_r \omega_n = 1.501$$

$$T_r = \frac{1.501}{19} = 0.079$$

Time response of second order control system? –Lessons

What lessons?

$\xi > 1$ OVERDAMPED

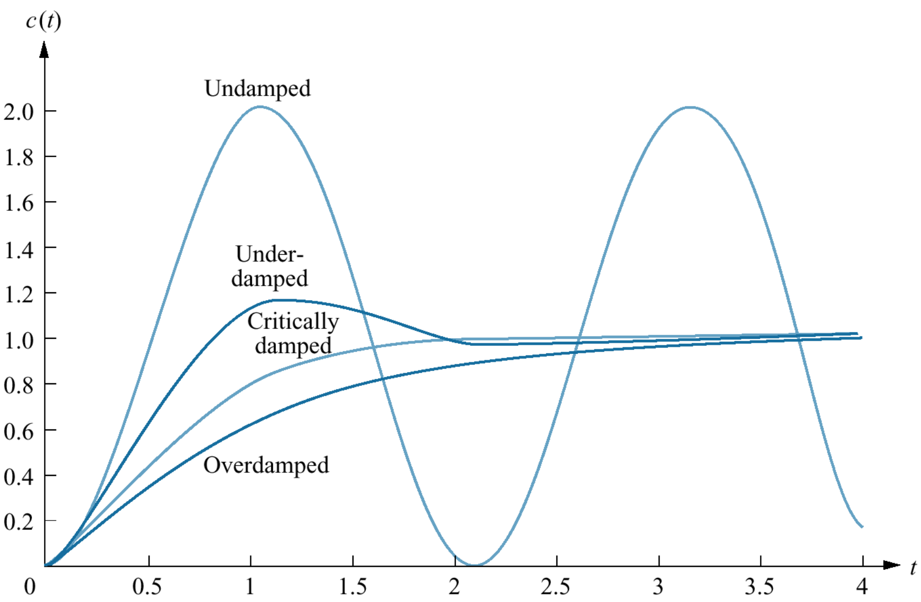
$\xi = 1$ CRITICALLY DAMPED

$0 < \xi < 1$ UNDER DAMPED

$\xi = 0$ UNDAMPED

WHERE DO WE GET ξ

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$



Time response of second order control system?

Performances of second order **under-damped** control system are determined by two quantities called natural frequency and damping ratio

Four performance parameters:

1. Rise time
2. Peak time
3. Percent overshoot
4. Settling time

$$T_r = \frac{f(\xi)}{\omega_n} \qquad T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$\%OS = 100e^{-\pi\xi / \sqrt{1-\xi^2}} \qquad T_s = \frac{4}{\xi\omega_n}$$

Time response of second order control system?

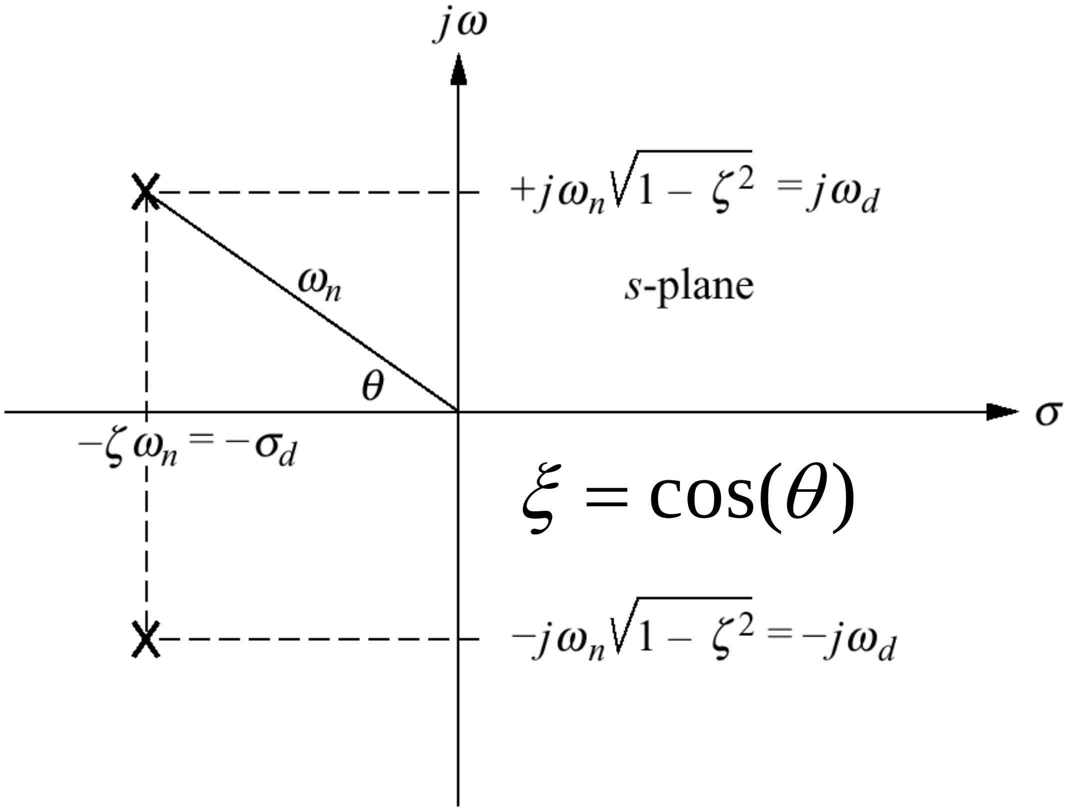
$$G(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\sigma_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

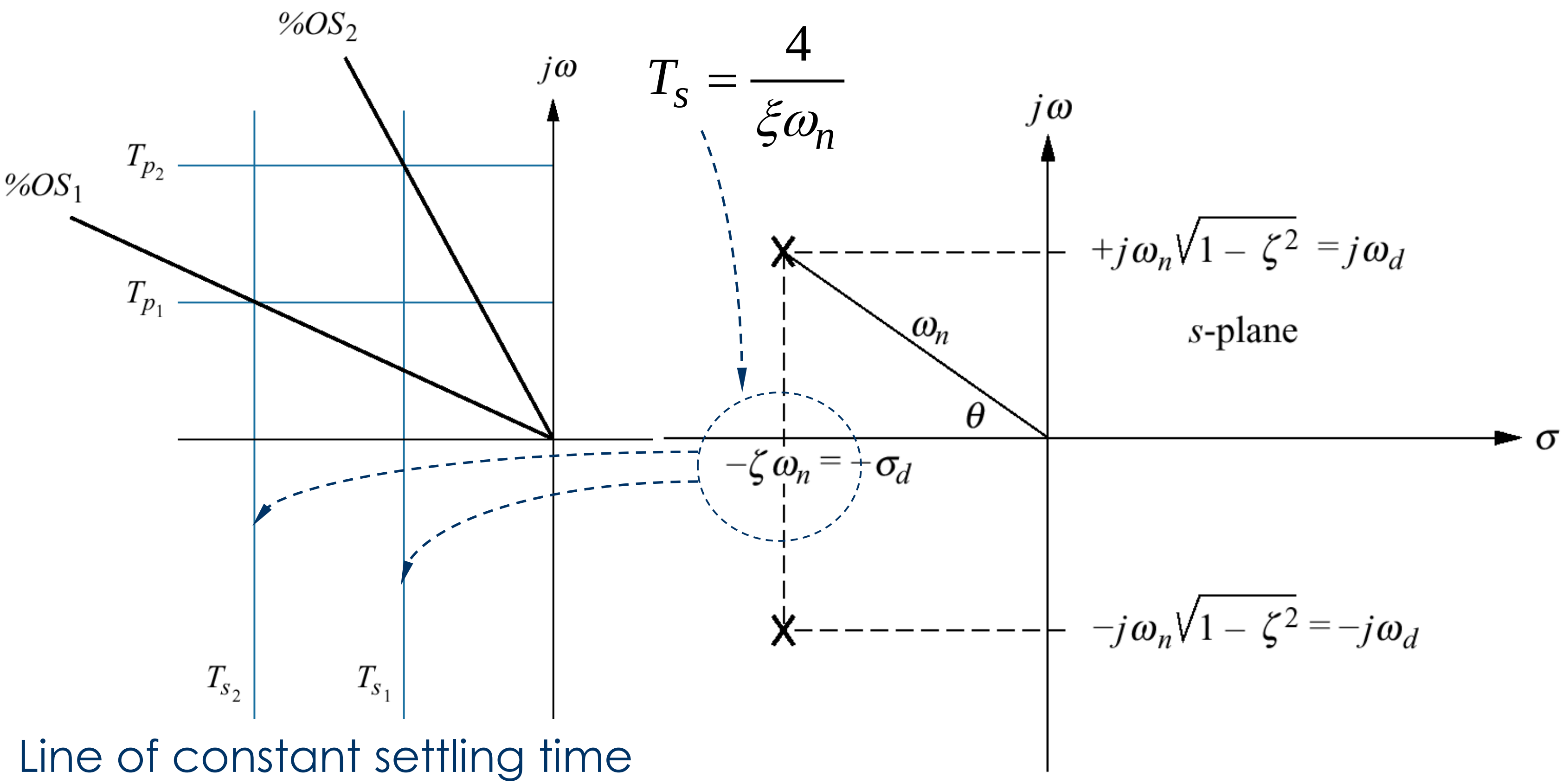
For under-damped system

$$s_{1,2}=-\xi\omega_n\pm j\omega_n\sqrt{1-\xi^2}$$

$$s_{1,2}=-\sigma_d\pm j\omega_d$$

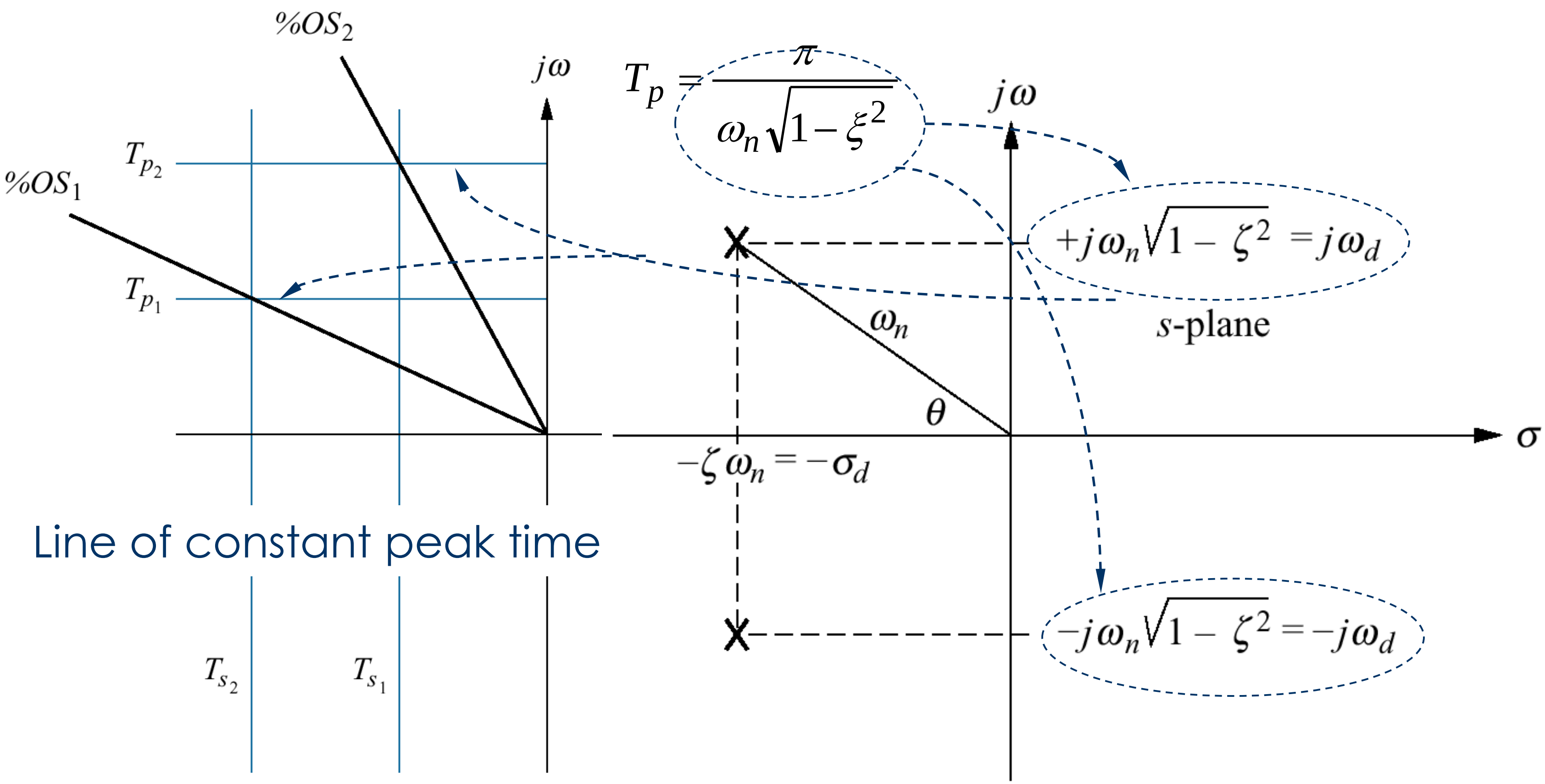


Time response of second order control system?



Line of constant settling time

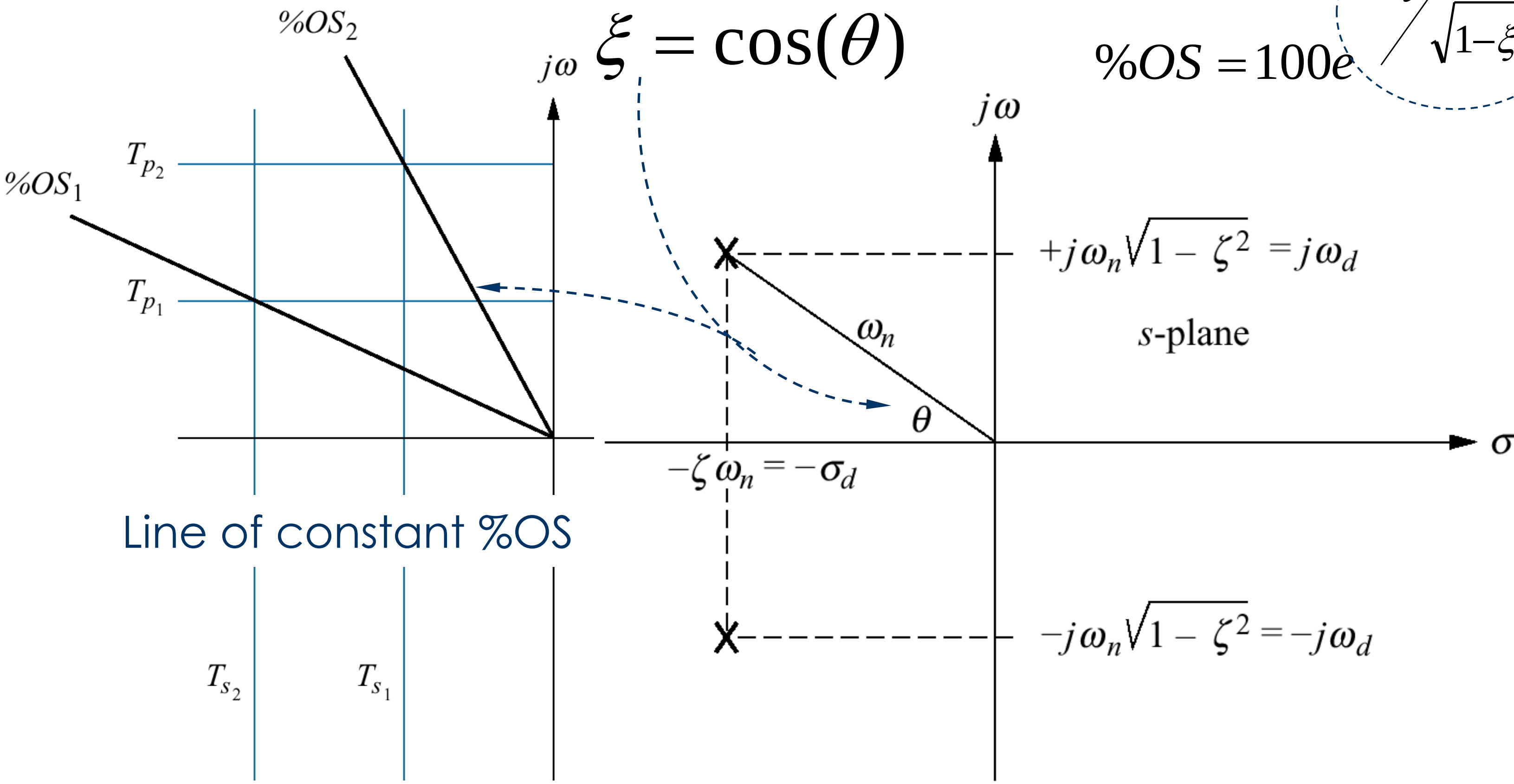
Time response of second order control system?



Time response of second order control system?

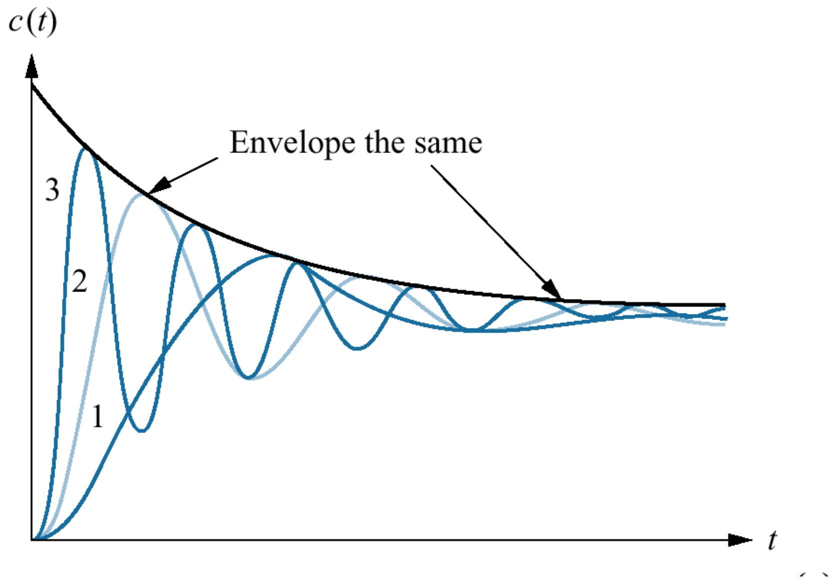
$$\xi = \cos(\theta)$$

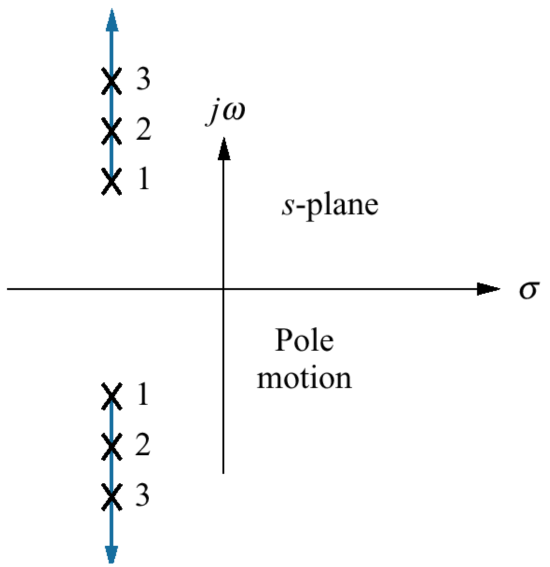
$$\%OS = 100e^{-\pi\xi / \sqrt{1-\xi^2}}$$



Line of constant %OS

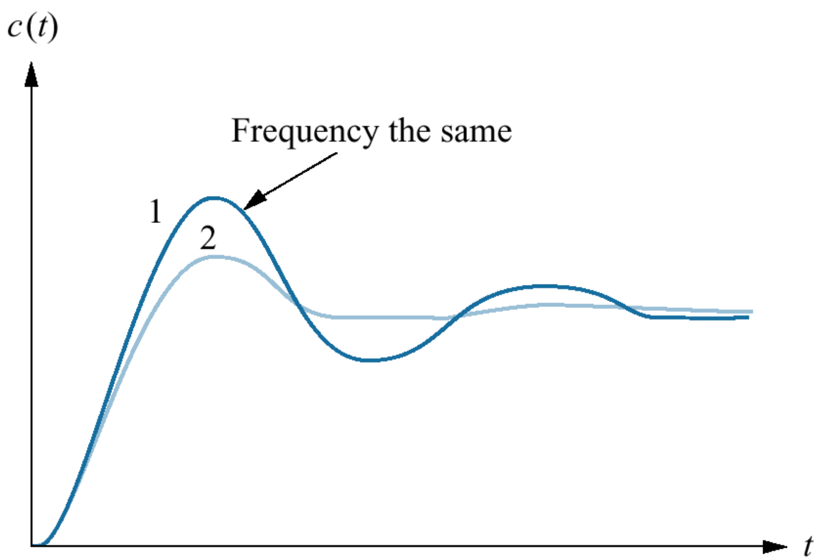
Time response of second order control system?

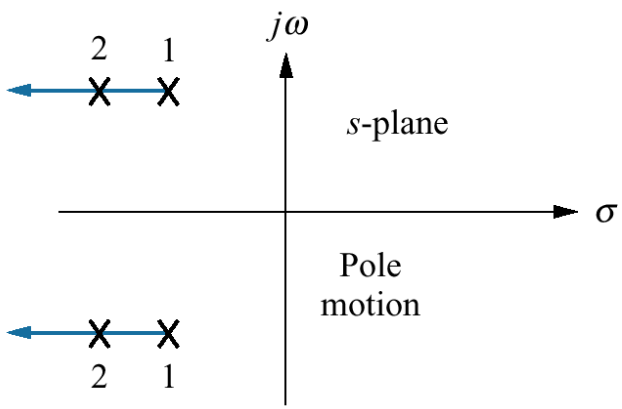




Constant Settling Time

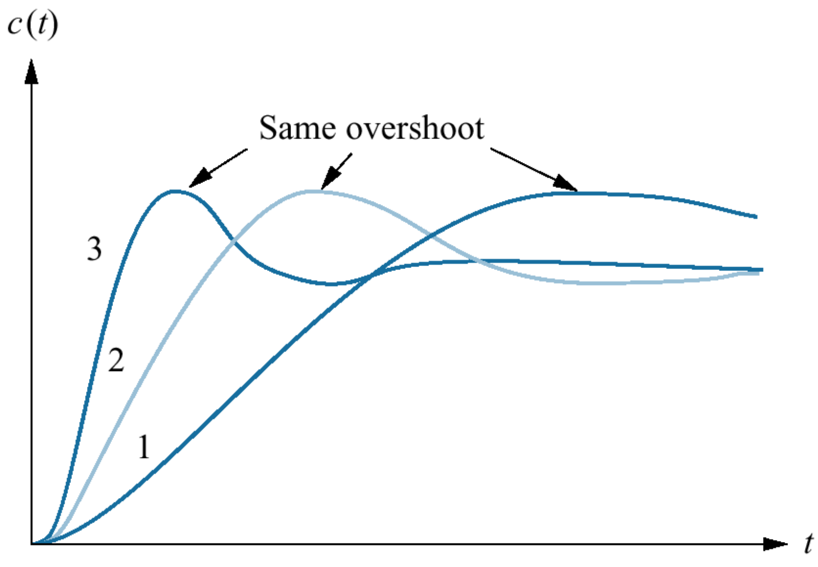
Time response of second order control system? – Lessons

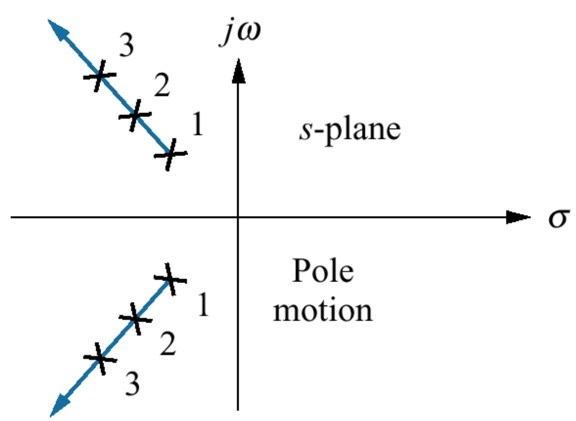




Constant Peak Time

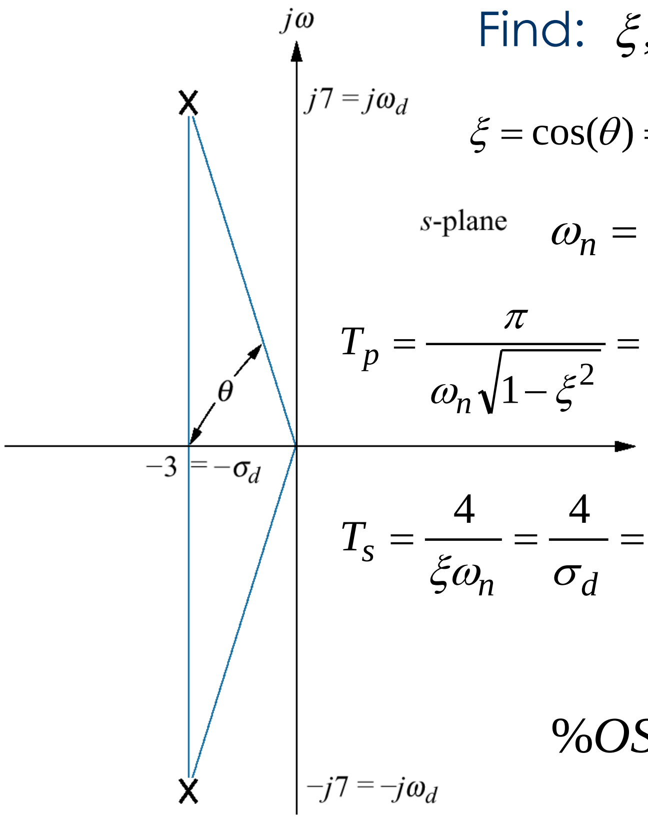
Time response of second order control system? – Lessons





Constant Percent Overshoot

Time response of second order control system? – Lessons



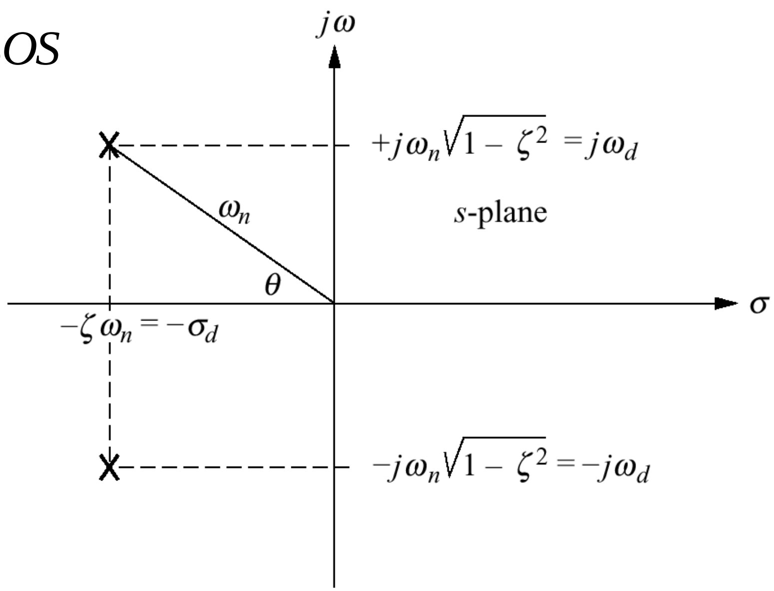
Find: ξ, ω_n, T_p, T_s and $\%OS$

$$\xi = \cos(\theta) = \tan^{-1}(\frac{7}{3}) = 0.394$$

$$\omega_n = \sqrt{3^2 + 7^2} = 7.616$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449$$

$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333$$



$$\%OS = 100e^{-\pi\xi / \sqrt{1-\xi^2}} = 100e^{-\pi(0.394) / \sqrt{1-(0.394)^2}} = 26\%$$