Mathematical Modeling of Control Systems

Control system Design Procedure

1. Modeling of control systems
2. Analysis of control systems
3. Design controller for control system

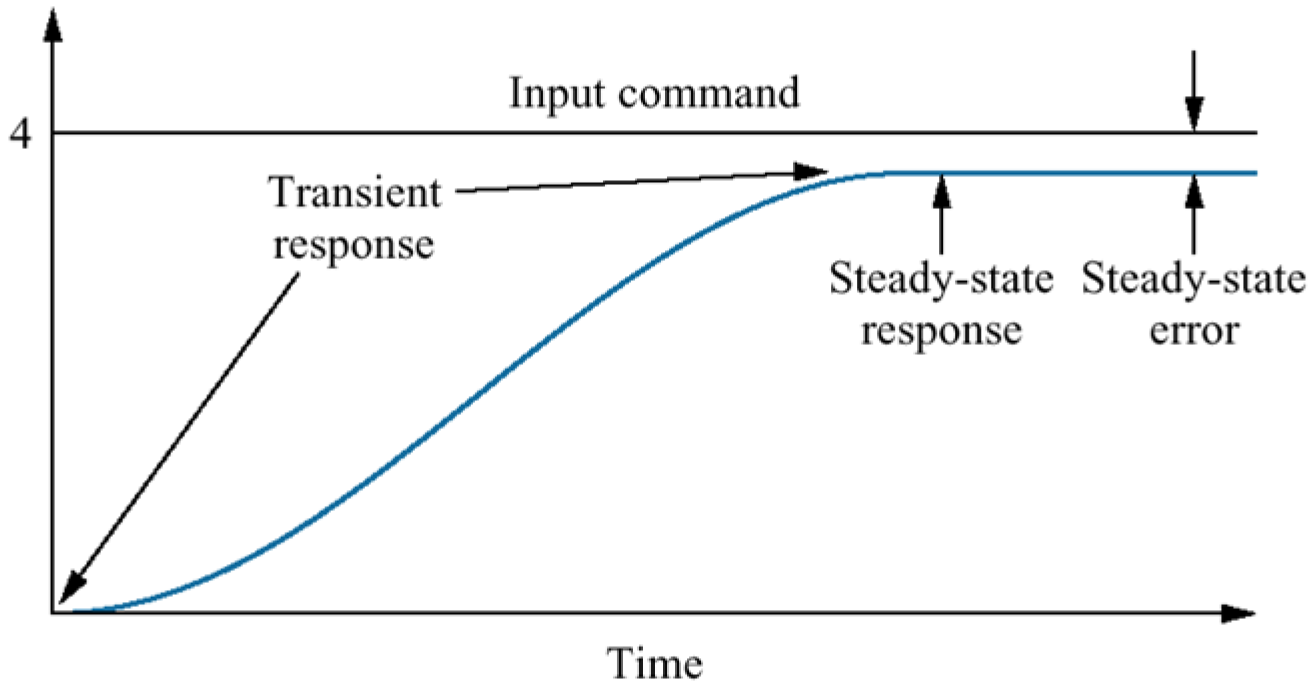
Control system modeling

1. Differential equation
2. Transfer function
3. State-space

Control system analysis

1. Transient response
2. Steady-state Response
3. Stability

Control system analysis



Control system design

1. Time Domain
2. Frequency Domain
3. State-space Analysis

What is Laplace Transform?

A technique to solve differential equation

Transforming time domain function to frequency domain function

Laplace Transform Definition

$$F(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$$

Why use Laplace Transform?

Solving differential equation is easy that is through algebra. No need to carry out differentiation or integration.

Laplace Transform Table?

Item no.	*f(t)*	*F(s)*
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace Transform Theorem?

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s)-f(0-)$	Differentiation theorem

Laplace Transform Theorem?

8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$= s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right]$	$= \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty)$	$= \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+)$	$= \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

Example of Laplace Transform technique?

Integral Approach

$$\frac{dy}{dt} = a \quad \text{zero initial condition}$$

$$y(t) = \int_0^t a\,dt = \left[at\right]_0^t = at$$

Example of Laplace Transform technique?

Laplace Transform Approach

$$\frac{dy}{dt} = a \quad \text{with zero initial condition}$$

Taking Laplace Transform

$$L\left[\frac{dy}{dt}\right] = L[a]$$

$$sY(s) = \frac{a}{s}$$

$$Y(s) = \frac{a}{s^2}$$

Taking Inverse Laplace Transform

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{a}{s^2}\right]$$

$$y(t) = at$$

What if Initial Condition is not zero?

Example of Laplace Transform technique?

Find the Laplace Transform of $f(t) = Ae^{-at}u(t)$

Solve using Laplace Transform definition

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(s+a)t}\,dt$$

$$F(s) = \left[-\frac{A}{s+a}e^{-(s+a)t}\right]_0^\infty = \frac{A}{s+a}$$

Example of Laplace Transform technique?

Find the Laplace Transform of $f(t) = Ae^{-at}u(t)$

Solve using Laplace Transform Table

$$f(t) = Ae^{-at}u(t)$$

$$f(t) = e^{-at}u(t) \rightarrow F(s) = \frac{1}{s+a}$$

By linearity theorem

$$L[kf(t)] = kF(s)$$

Therefore

$$f(t) = Ae^{-at}u(t) \rightarrow F(s) = \frac{A}{s+a}$$

What is Transfer Function?

Frequency domain mathematical model that separates input from output

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \longrightarrow \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

What is Transfer Function?

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

$$sC(s) + 2C(s) = R(s)$$

$$\frac{C(s)}{R(s)} = \frac{1}{s+2}$$

Converting differential equation to transfer function

$$\frac{d^3c(t)}{dt^3}+3\frac{d^2c(t)}{dt^2}+7\frac{dc(t)}{dt}+5c(t)=\frac{dr^2(t)}{dt^2}+4\frac{dr(t)}{dt}+3r(t)$$

$$s^3C(s)+3s^2C(s)+7sC(s)+5C(s)=s^2R(s)+4sR(s)+3R(s)$$

$$C(s)(s^3+3s^2+7s+5)=R(s)(s^2+4s+3)$$

$$\frac{C(s)}{R(s)}=\frac{s^2+4s+3}{s^3+3s^2+7s+5}$$

Converting transfer function to differential equation

$$G(s) = \frac{2s+1}{s^2+6s+2} = \frac{C(s)}{R(s)}$$

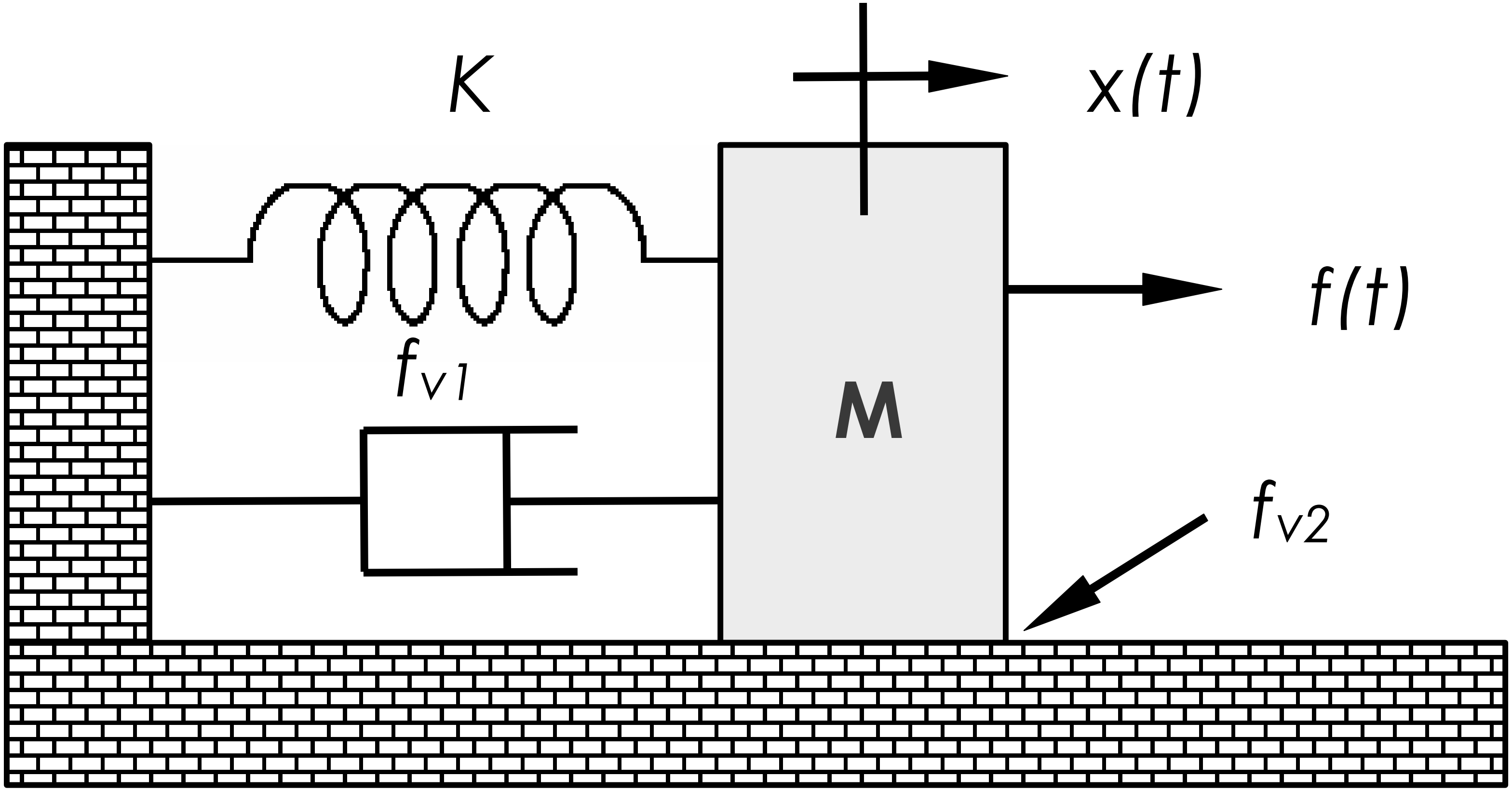
$$C(s)[s^2+6s+2] = R(s)[2s+1]$$

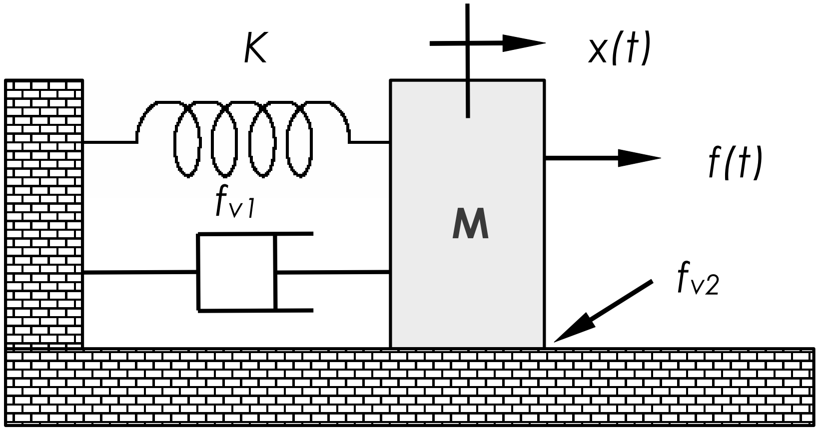
$$C(s)s^2 + C(s)6s + 2C(s) = R(s)2s + R(s)$$

$$\frac{d^2c(t)}{dt^2} + 6\frac{dc(t)}{dt} + 2c(t) = 2\frac{dr(t)}{dt} + r(t)$$

Transfer Function of Translational Mechanical System

Example 1

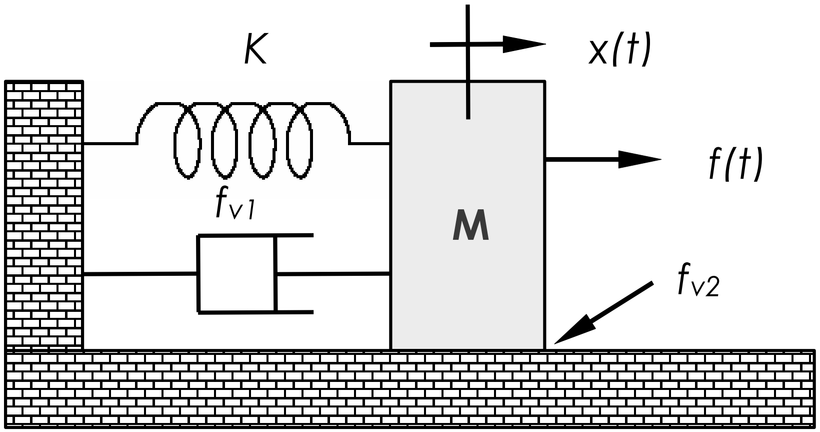




Modeling Steps:

1. Decide the input and the output
2. The free body diagram of the mass M
3. The frequency-domain representation of the forces
4. The transfer function

Step 1: Decide input and output



Input variable:

Applied force $f(t)$

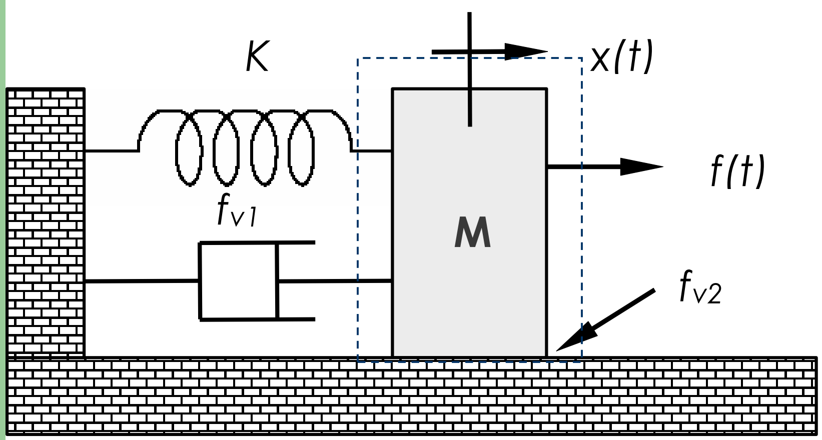
Output variable:

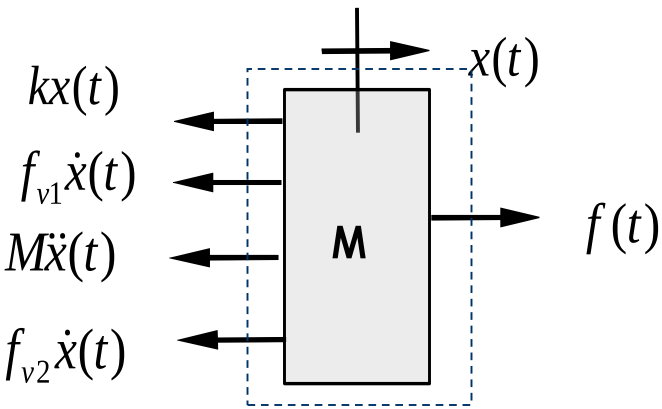
Mass position $x(t)$

Mass velocity $\dot{x}(t)$

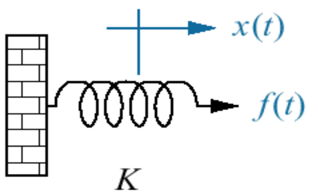
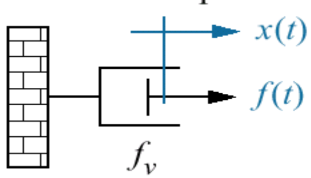
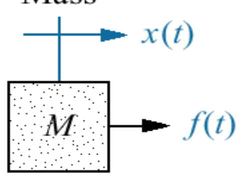
Mass acceleration $\ddot{x}(t)$

Step 2: The free body diagram



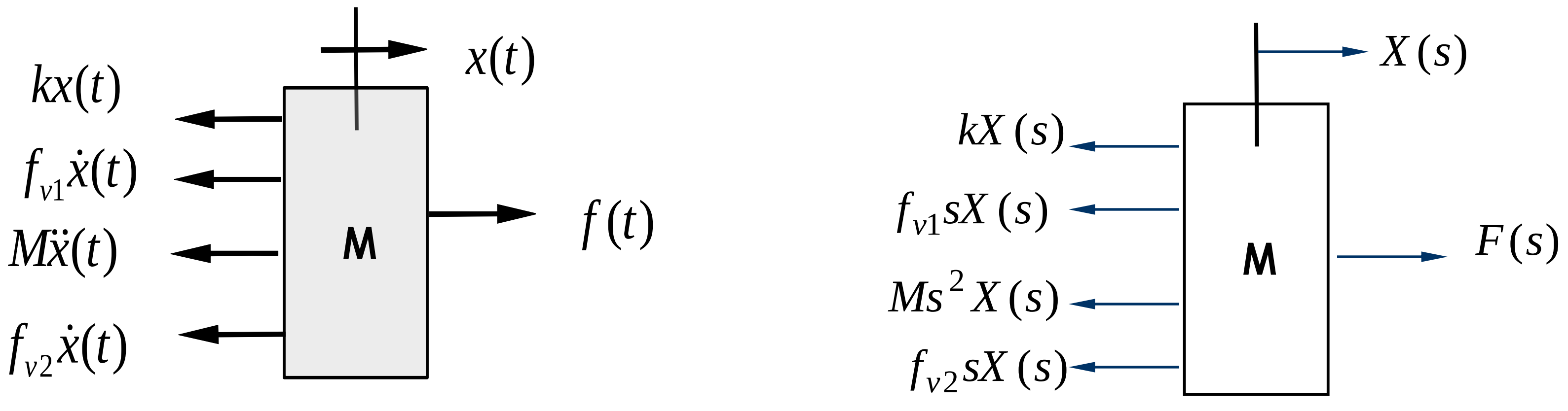


Step 2: The frequency response representation

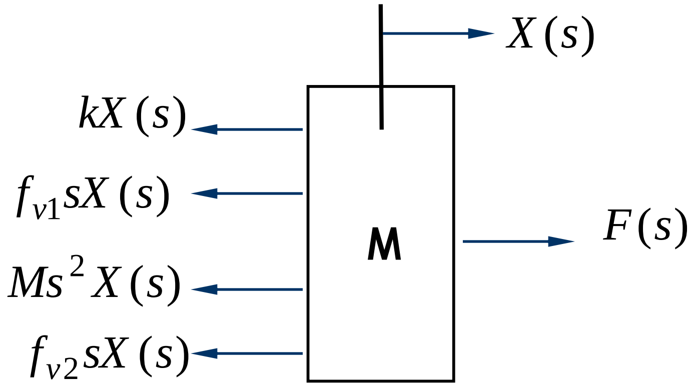
Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

"Force is mapped to Voltage and displacement to current, Impedance is voltage/current"

Step 2: The frequency response representation free body diagram



Step 3: Frequency response representation



$$\underbrace{Ms^2X(s)+(f_{v1}+f_{v2})sX(s)+kX(s)}_{\text{Output}}=\underbrace{F(s)}_{\text{Input}}$$

Step 4: Transfer Function

$$Ms^2X(s)+(f_{v1}+f_{v2})sX(s)+kX(s)=F(s)$$

If the position of the mass is the interested output: $x(t)\rightarrow X(s)$

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+(f_{v1}+f_{v2})s+k}$$

Step 4: Transfer Function

$$Ms^2X(s)+(f_{v1}+f_{v2})sX(s)+kX(s)=F(s)$$

If the velocity of the mass is the interested output: $\dot{x}(t)\rightarrow sX(s)$

$$G(s)=\frac{X(s)s}{F(s)}=\frac{1}{Ms+(f_{v1}+f_{v2})+k\frac{1}{s}}$$

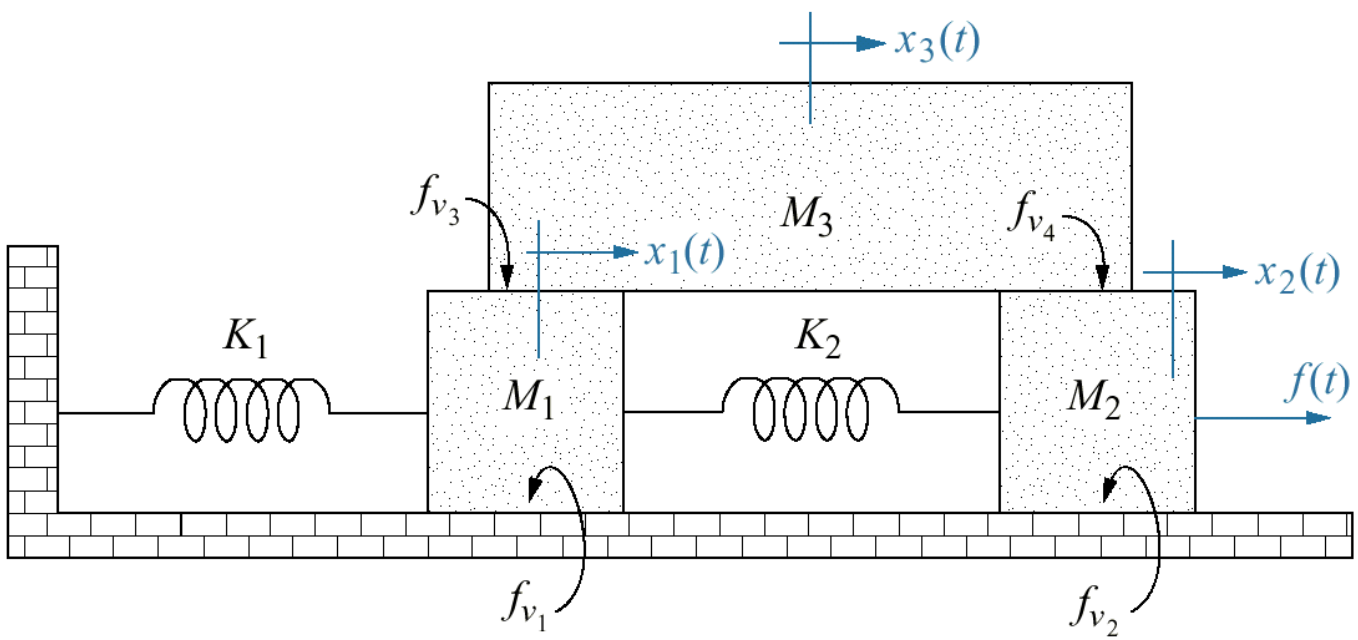
Step 4: Transfer Function

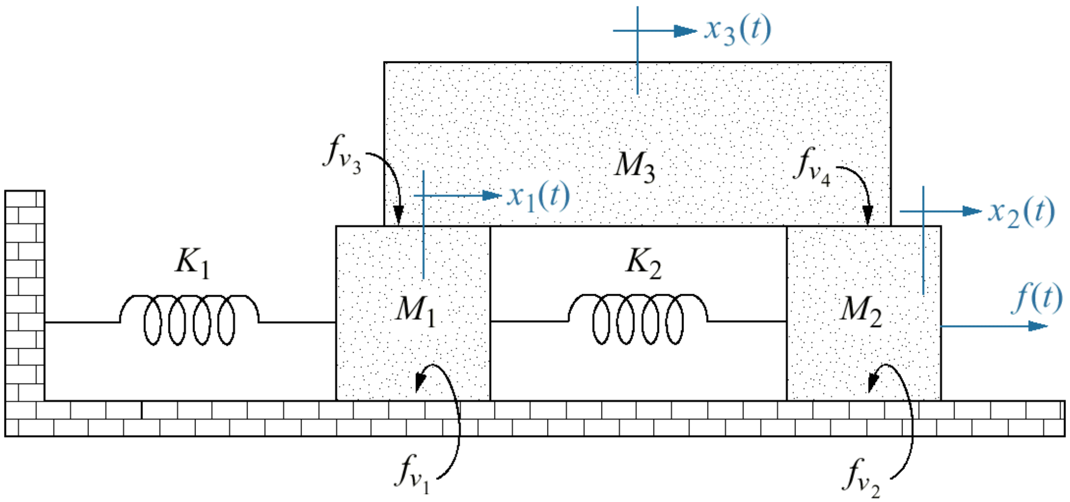
$$Ms^2X(s)+(f_{v1}+f_{v2})sX(s)+kX(s)=F(s)$$

If the acceleration of the mass is the interested variable: $\ddot{x}(t) \rightarrow X(s)s^2$

$$G(s)=\frac{X(s)s^2}{F(s)}=\frac{1}{M+(f_{v1}+f_{v2})\frac{1}{s}+k\frac{1}{s^2}}$$

Example 2

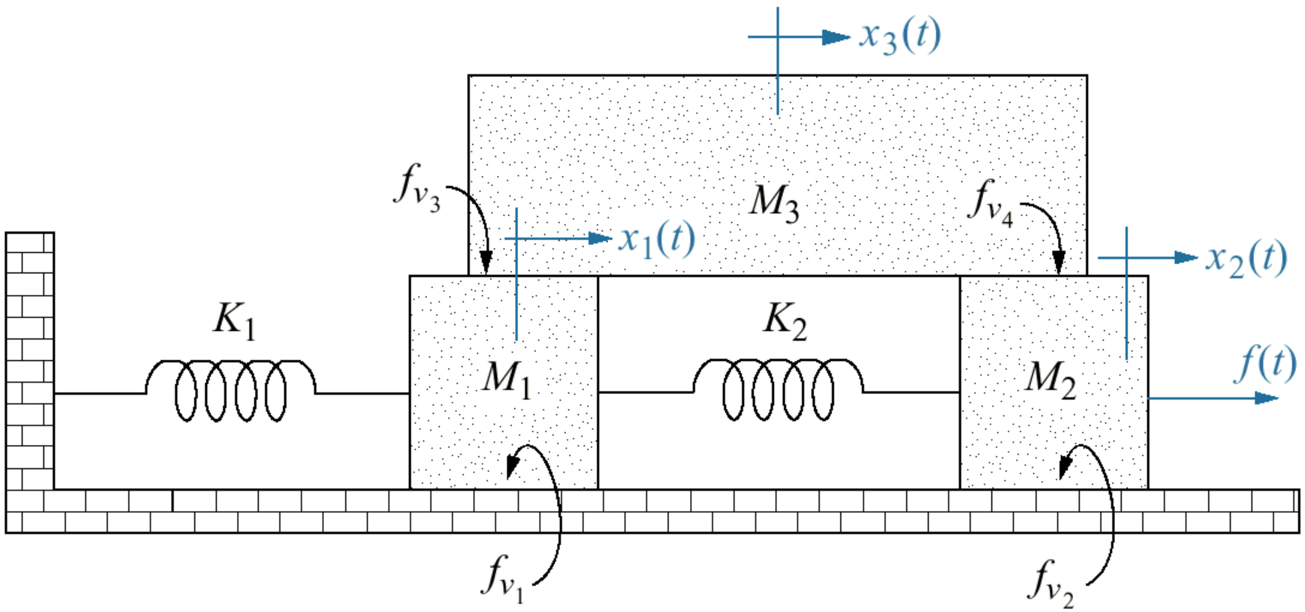




Modeling Steps:

1. Decide the input and the output
2. The free body diagram of the mass M
3. The frequency-domain representation of the forces
4. The transfer function

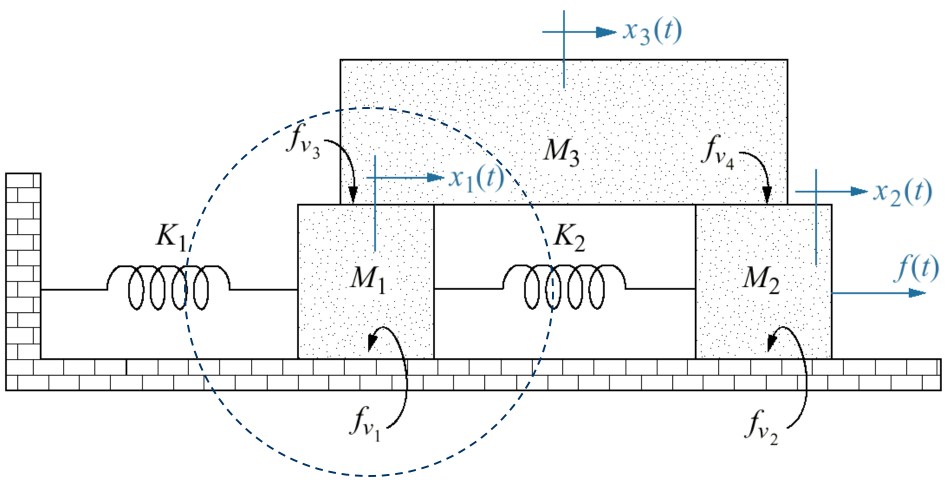
Step 1: Input and Output variables

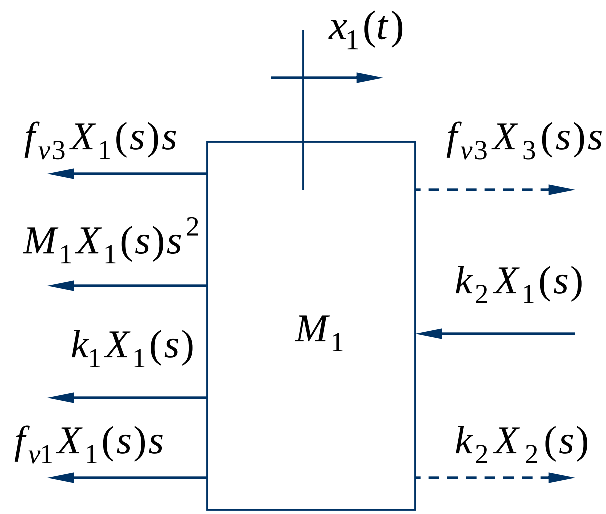


Input variable: $f(t)$

Output variable: Position, velocity or acceleration of the mass

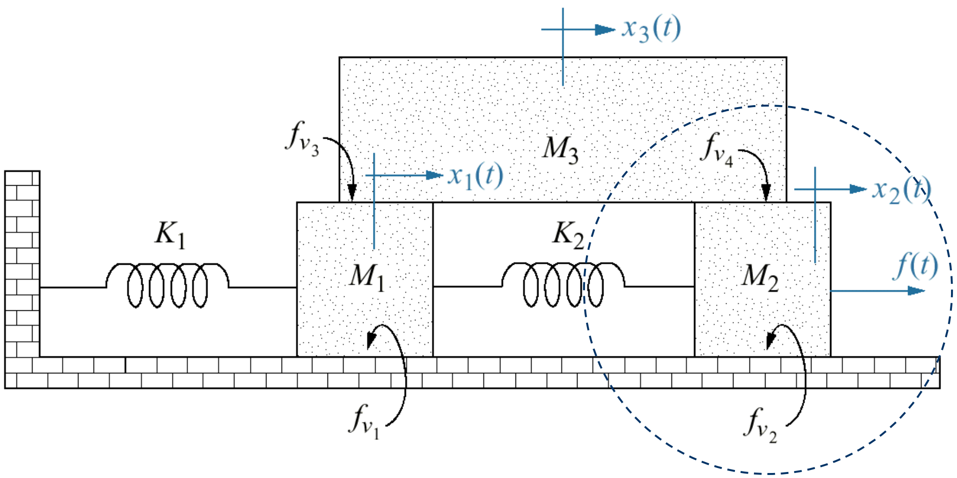
Step 2,3 and 4: The free body diagram, the frequency response representation and the transfer function equation – Mass M1

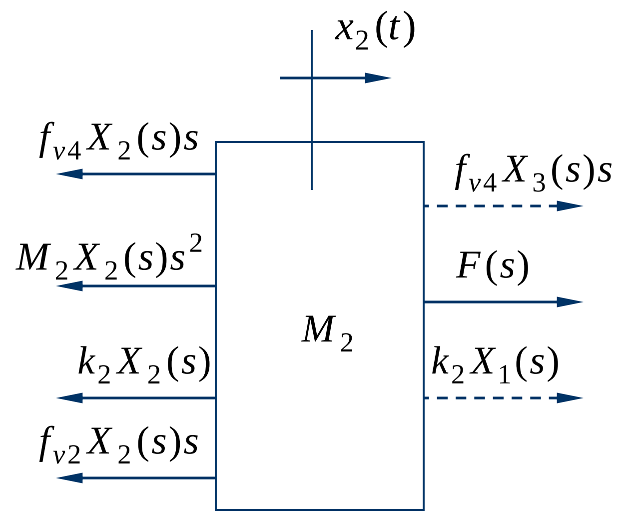




$$\overbrace{\left(M_1s^2 + f_{v1}s + f_{v3}s + k_1 + k_2\right)X_1(s)}^{\text{Terms related to } x_1} - \overbrace{f_{v_3}X_3(s)s - k_2X_2(s)}^{\text{Terms not related to } x_1} = 0$$

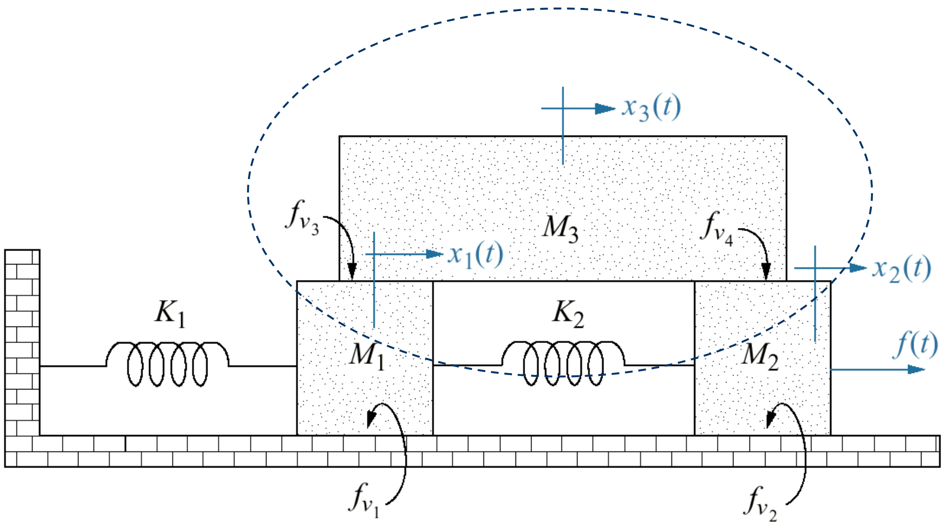
Step 2,3 and 4: The free body diagram, the frequency response represantation and the transfer function equation – Mass M2

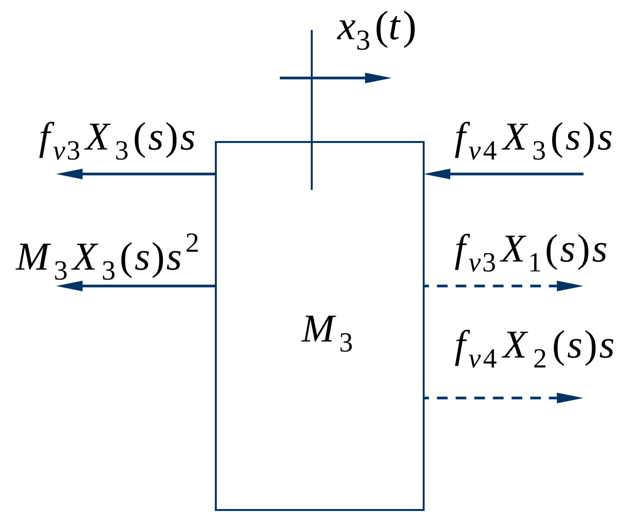




$$\overbrace{\left(M_2s^2 + f_{v2}s + f_{v4}s + k_2\right)X_2(s)}^{\text{Terms related to } x_2} - \overbrace{f_{v_4}X_3(s)s - k_2X_1(s)}^{\text{Terms not related to } x_2} = F(s)$$

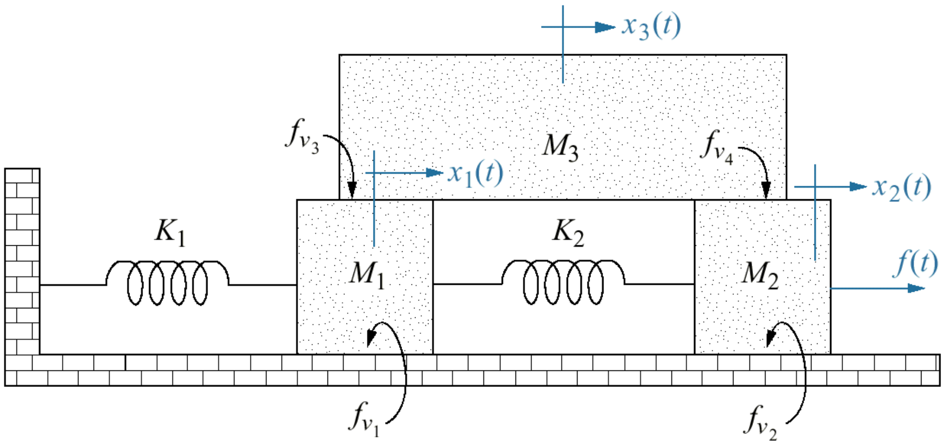
Step 2,3 and 4: The free body diagram, the frequency response represantation and the transfer function equation – Mass M3





$$\overbrace{\left(M_3 s^2 + f_{v3}s + f_{v4}s\right)X_3(s)}^{\text{Terms related to } x_3} - \overbrace{f_{v3}X_1(s)s - f_{v4}X_2(s)s}^{\text{Terms not related to } x_3} = 0$$

Step 2,3 and 4: The free body diagram, the frequency response represantation and the transfer function equation – Mass M1, M2 and M3



$$\left(K_1 + K_2 + f_{v3}s + f_{v1}s + M_1 s^2\right) X_1(s) \quad \left(-K_2\right) X_2(s) \quad \left(-f_{v3}s\right) X_3(s) = 0$$

$$\left(-K_2\right) X_1(s) \quad \left(K_2 + f_{v2}s + f_{v4}s + M_2 s^2\right) X_2(s) \quad \left(-f_{v4}s\right) X_3(s) = F(s)$$

$$\left(-f_{v3}s\right) X_1(s) \quad \left(-f_{v4}s\right) X_2(s) \quad \left(f_{v3}s + f_{v4}s + M_3 s^2\right) X_3(s) = 0$$

Step 4: Transfer Function

$$\begin{bmatrix} s^2M_1 + s(f_{v1} + f_{v3}) + (k_1 + k_2) & -k_2 & -sf_{v3} \\ -k_2 & s^2M_2 + s(f_{v2} + f_{v4}) + k_2 & -sf_{v4} \\ -sf_{v3} & -sf_{v4} & s^2M_3 + s(f_{v3} + f_{v4}) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}$$

Assume all parameters equal to 1

$$\begin{bmatrix} s^2 + 2s + 2 & -1 & -s \\ -1 & s^2 + 2s + 1 & -s \\ -s & -s & s^2 + 2s \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}$$

Step 4: Transfer Function

$$\begin{bmatrix} s^2+2s+2 & -1 & -s \\ -1 & s^2+2s+1 & -s \\ -s & -s & s^2+2s \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}$$

If we are interested to control the position of the mass M_2, then solve for X_2. Cramer's rule:

$$X_2 = \frac{\begin{vmatrix} s^2+2s+2 & 0 & -s \\ -1 & F & -s \\ -s & 0 & s^2+2s \end{vmatrix}}{\begin{vmatrix} s^2+2s+2 & -1 & -s \\ -1 & s^2+2s+1 & -s \\ -s & -s & s^2+2s \end{vmatrix}}$$

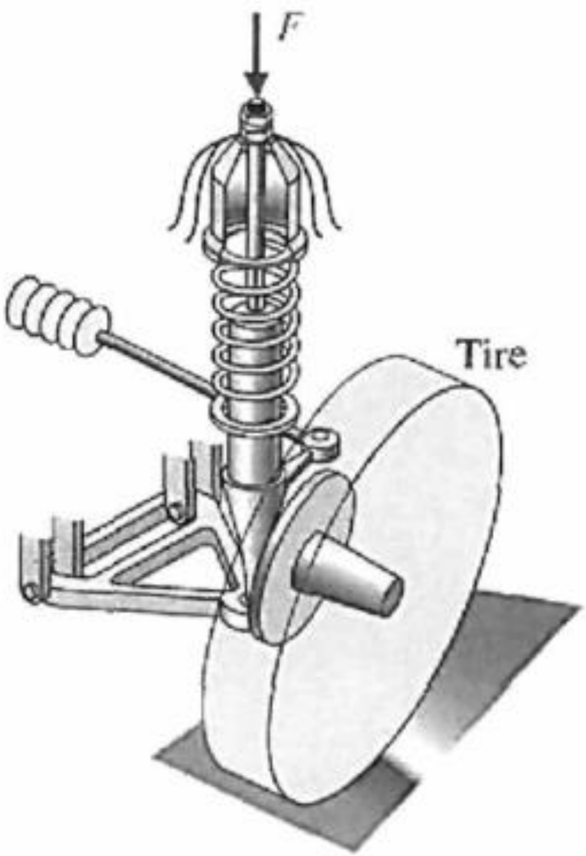
Step 4: Transfer Function

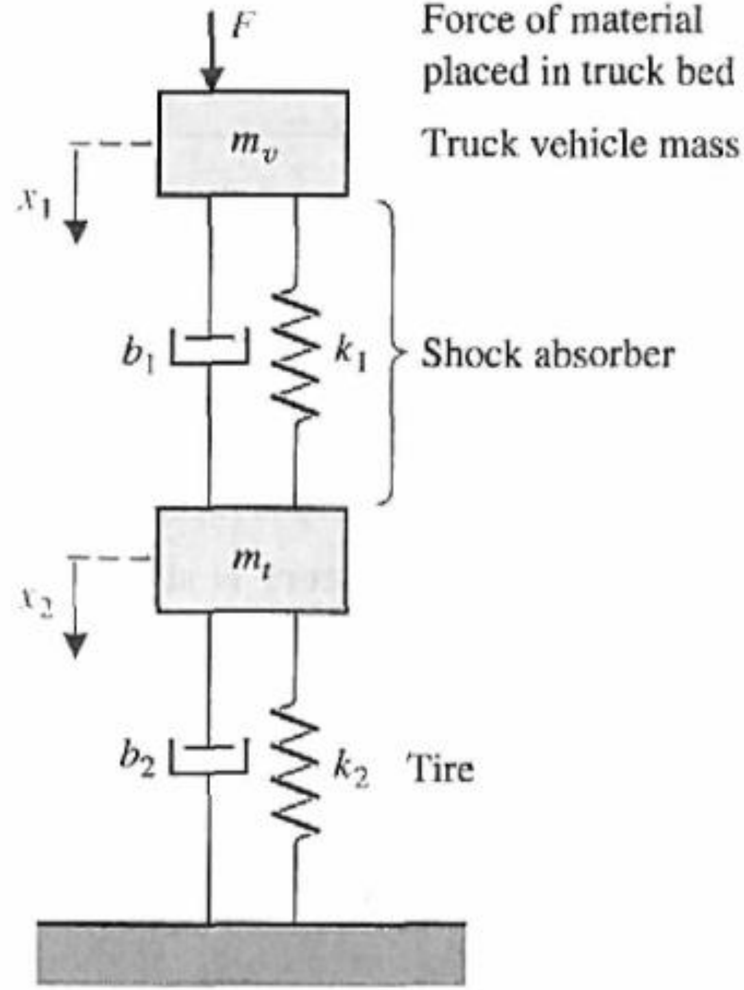
Using MATLAB

$$X_2 = \frac{\begin{vmatrix} s^2+2s+2 & 0 & -s \\ -1 & F & -s \\ -s & 0 & s^2+2s \end{vmatrix}}{\begin{vmatrix} s^2+2s+2 & -1 & -s \\ -1 & s^2+2s+1 & -s \\ -s & -s & s^2+2s \end{vmatrix}} = \frac{F(4s^3+16s^2+20s+16)}{s^5+6s^4+13s^3+16s^2+8s+2}$$

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{4s^3+16s^2+20s+16}{s^5+6s^4+13s^3+16s^2+8s+2}$$

P2.46 A load added to a truck results in a force F on the support spring, and the tire flexes as shown in Figure P2.46(a). The model for the tire movement is shown in Figure P2.46(b). Determine the transfer function $X_1(s)/F(s)$.

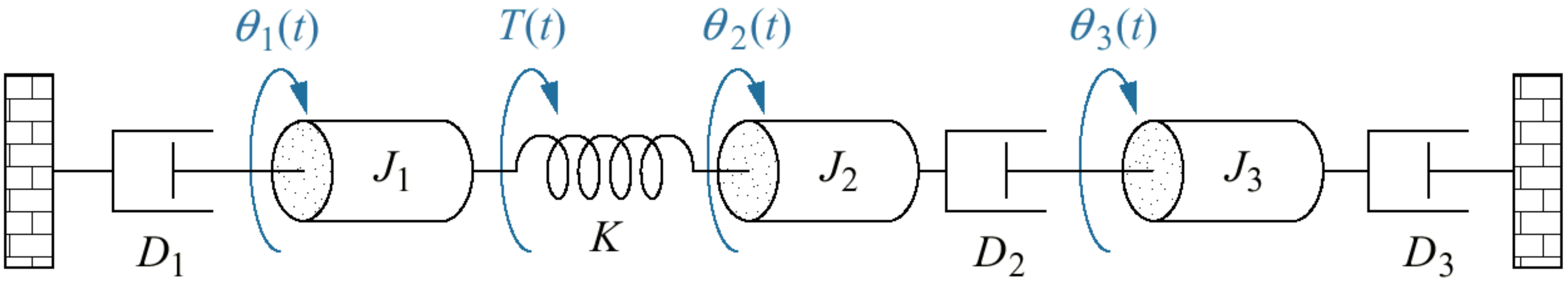


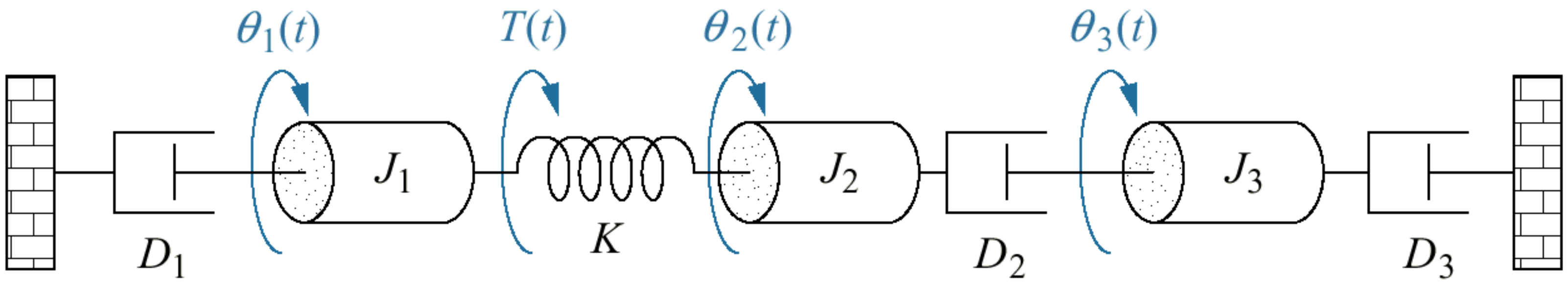


Transfer Function of Rotational Mechanical System without Gearing

5 September 2018

Example 1

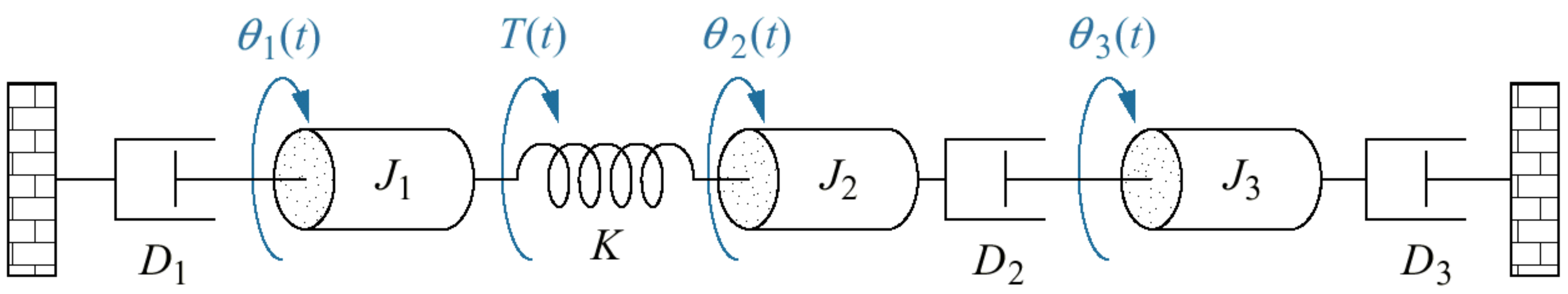




Modeling Steps:

1. Decide input and the output
2. Draw the free body diagram of the masses
3. Convert to the frequency-domain representations
4. Create transfer function

Step 1: Decide Input and Output



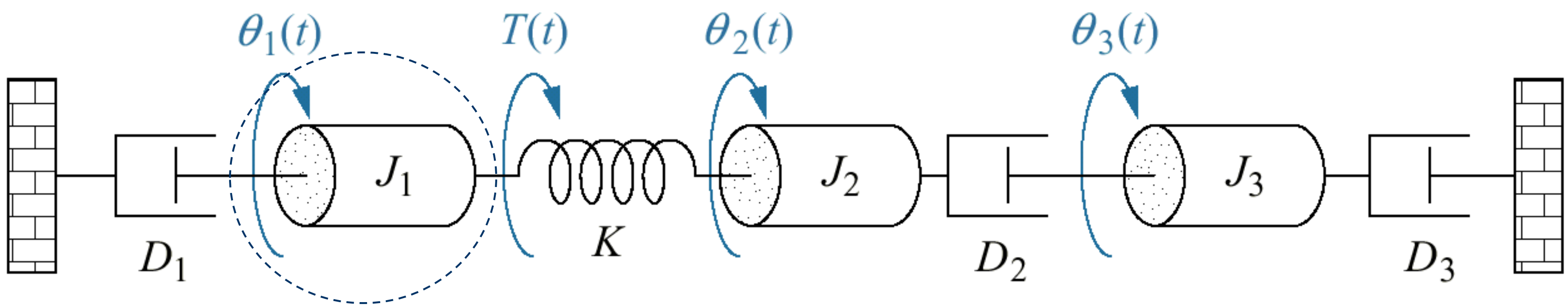
Input variable:

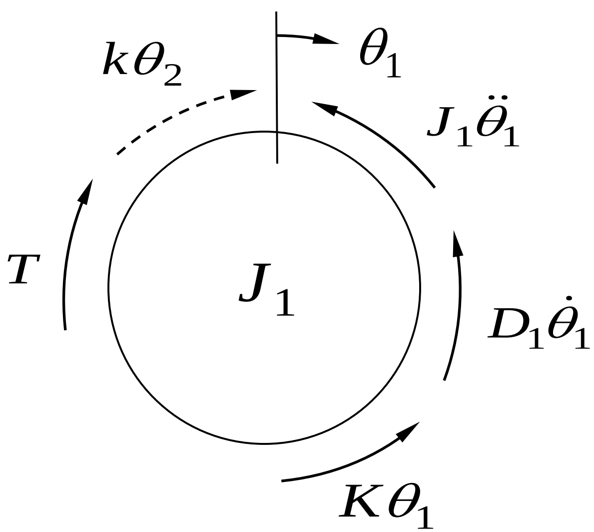
$T(t)$

Output variable:

Any angular position, velocity or acceleration of the rotational masses

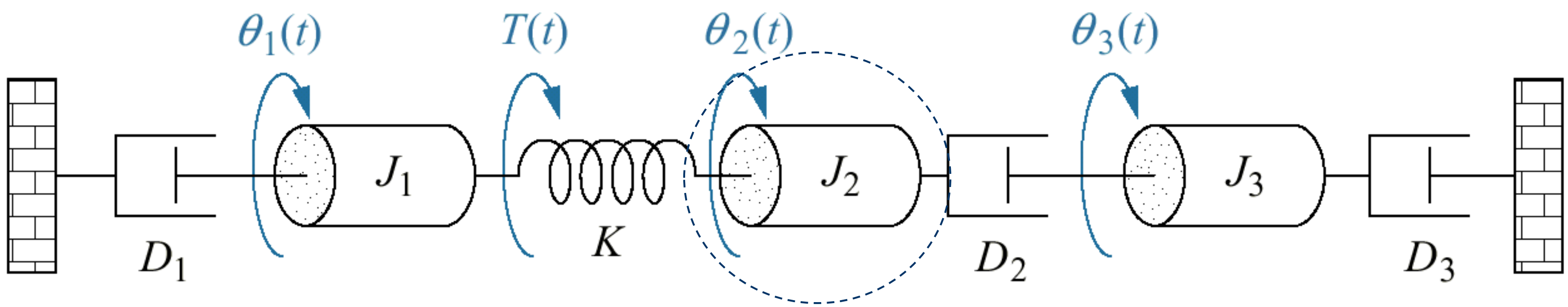
Step 2: The free body diagram *J_1*
Step 3: The frequency response representation

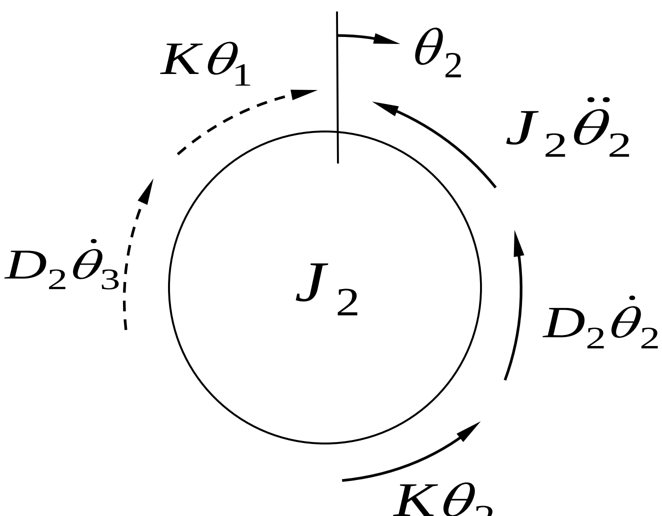




$$\overbrace{J_1\ddot{\theta}_1 + D_1\dot{\theta}_1 + K\theta_1}^{\text{Terms related to } \theta_1} - K\theta_2 = T$$

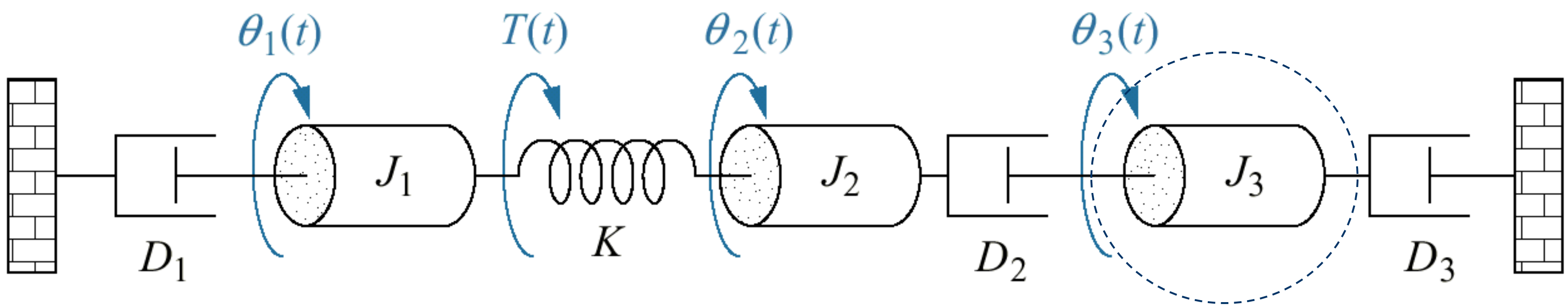
Step 2: The free body diagram *J2*
Step 3: The frequency response representation

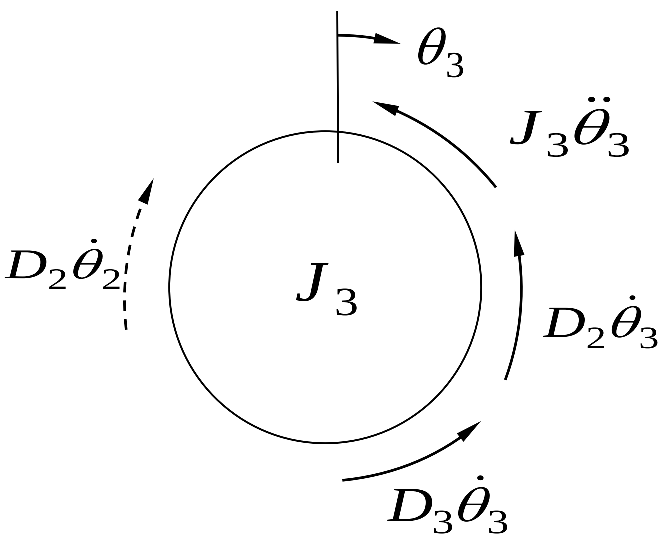




$$\overbrace{J_2\ddot{\theta}_2 + D_2\dot{\theta}_2 + K\theta_2}^{\text{Terms related to } \theta_2} - D_2\dot{\theta}_3 - K\theta_1 = 0$$

Step 2: The free body diagram *J3*
Step 3: The frequency response representation





$$\overbrace{J_3\ddot{\theta}_3 + D_2\dot{\theta}_3 + D_3\dot{\theta}_3}^{\text{Terms related to } \theta_3} - D_2\dot{\theta}_2 = 0$$

Step 4: Transfer Function

Inertial 1: $s^2 J_1\theta_1(s) + sD_1\theta_1(s) + K\theta_1(s) - K\theta_2(s) = T$

Inertial 2: $s^2 J_2\theta_2(s) + sD_2\theta_2(s) + K\theta_2(s) - sD_2\theta_3(s) - K\theta_1(s) = 0$

Inertial 3: $s^2 J_3\theta_3(s) + sD_2\theta_3(s) + sD_3\theta_3(s) - sD_2\theta_2(s) = 0$

Put the equations in matrix form

$$\begin{bmatrix} s^2 J_1 + sD_1 + K & -K & 0 \\ -K & s^2 J_2 + sD_2 + K & -sD_2 \\ 0 & -sD_2 & s^2 J_3 + s(D_2 + D_3) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix}$$

Step 4: Transfer Function

$$\begin{bmatrix} s^2J_1 + sD_1 + K & -K & 0 \\ -K & s^2J_2 + sD_2 + K & -sD_2 \\ 0 & -sD_2 & s^2J_3 + s(D_2 + D_3) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix}$$

If we are interested to control the position of the mass J_3, then solve for θ_3.
Cramer's rule:

$$\theta_3 = \frac{\begin{vmatrix} s^2J_1 + sD_1 + K & -K & T \\ -K & s^2J_2 + sD_2 + K & 0 \\ 0 & -sD_2 & 0 \end{vmatrix}}{\begin{vmatrix} s^2J_1 + sD_1 + K & -K & 0 \\ -K & s^2J_2 + sD_2 + K & -sD_2 \\ 0 & -sD_2 & s^2J_3 + s(D_2 + D_3) \end{vmatrix}}$$

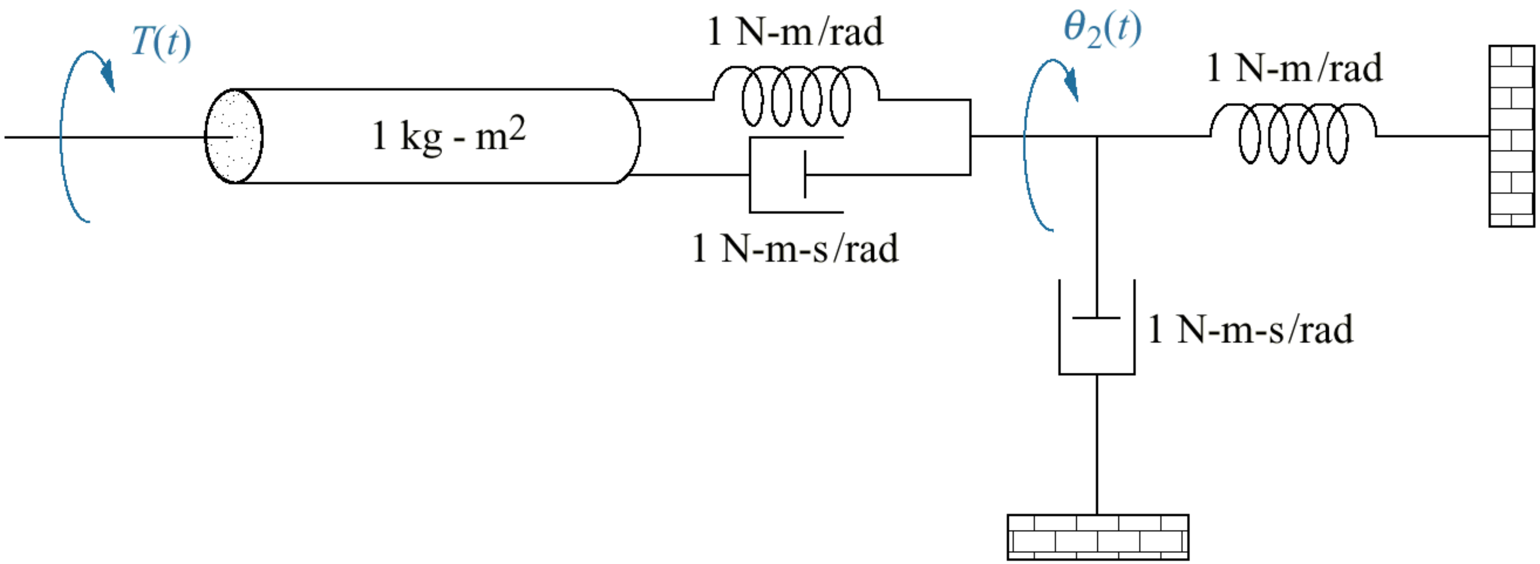
Step 4: Transfer Function

Using MATLAB

$$\theta_3(s) = \frac{\begin{vmatrix} s^2J_1 + sD_1 + K & -K & T \\ -K & s^2J_2 + sD_2 + K & 0 \\ 0 & -sD_2 & 0 \end{vmatrix}}{\begin{vmatrix} s^2J_1 + sD_1 + K & -K & 0 \\ -K & s^2J_2 + sD_2 + K & -sD_2 \\ 0 & -sD_2 & s^2J_3 + s(D_2 + D_3) \end{vmatrix}}$$

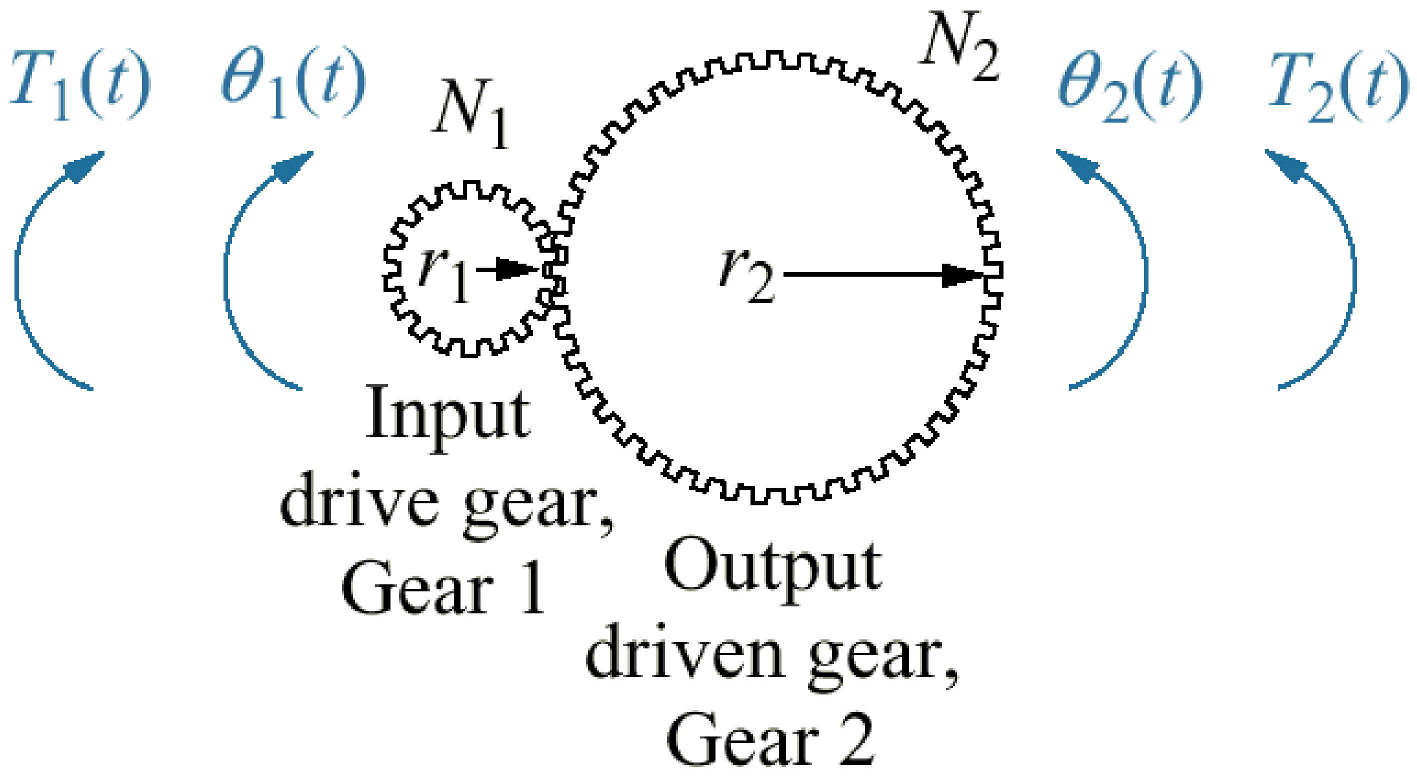
Example 2

Find the transfer function between $T(t)$ and $\theta_2(t)$

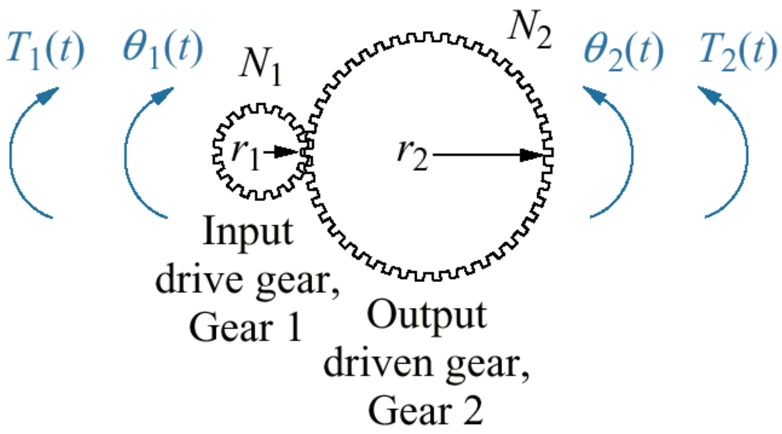


Transfer Function of Rotational Mechanical System with Gearing

Basic of Gearing System



Gearing System – Position relationship

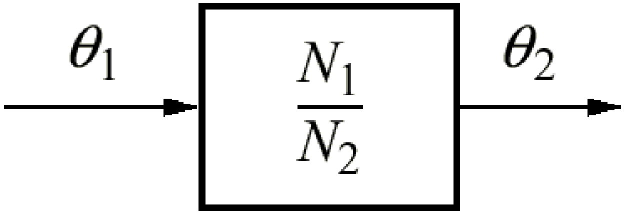


1. Distance travel by Gear 1 must equal distance travel by Gear 2

$$r_1\theta_1 = r_2\theta_2$$

$$\frac{r_1}{r_2} = \frac{\theta_2}{\theta_1}$$

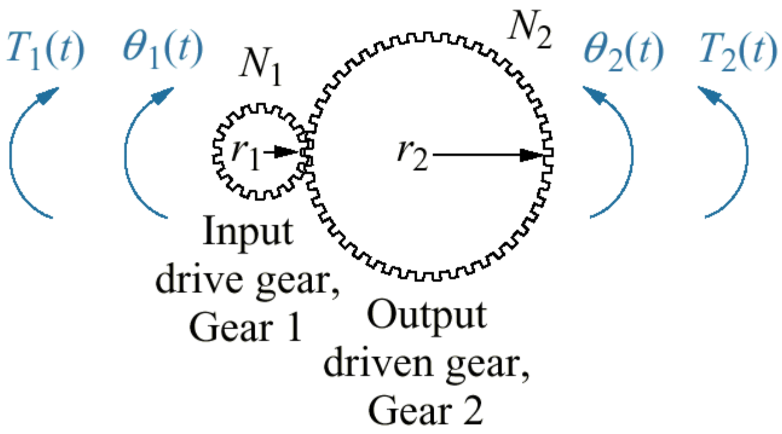
Transfer Function



2. Ratio of radius between Gear 1 and Gear 2 is equal to ratio of number of teeth between Gear 1 and Gear 2

$$\frac{N_1}{N_2} = \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1}$$

Gearing System – Torque relationship

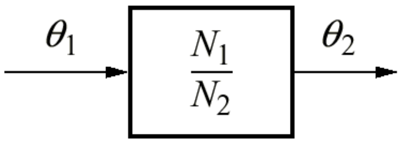


1. Assume work generated by Gear 1 is equal to work consumed by Gear 2

$$W_1 = W_2$$

$$T_1\theta_1 = T_2\theta_2$$

2. From previous result

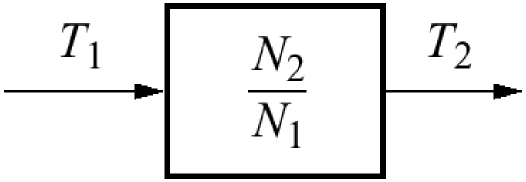


$$T_1\theta_1 = T_2\theta_2$$

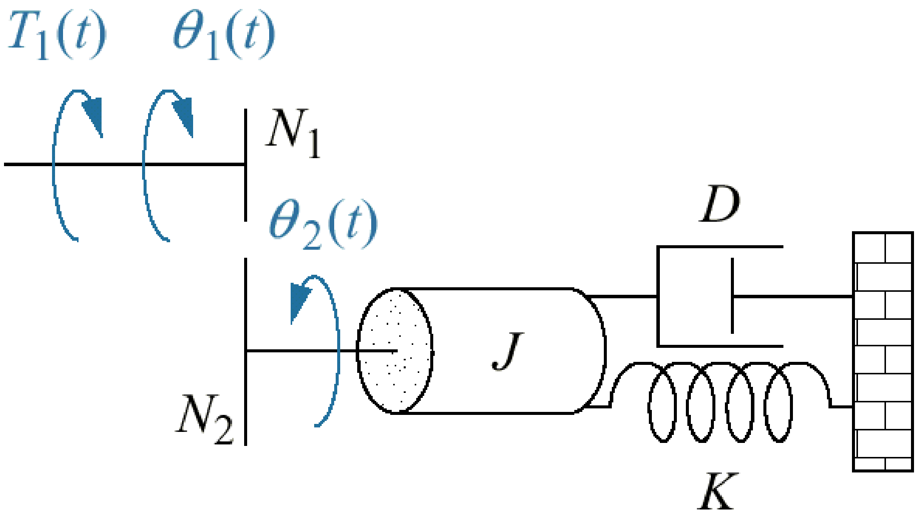
$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1}$$

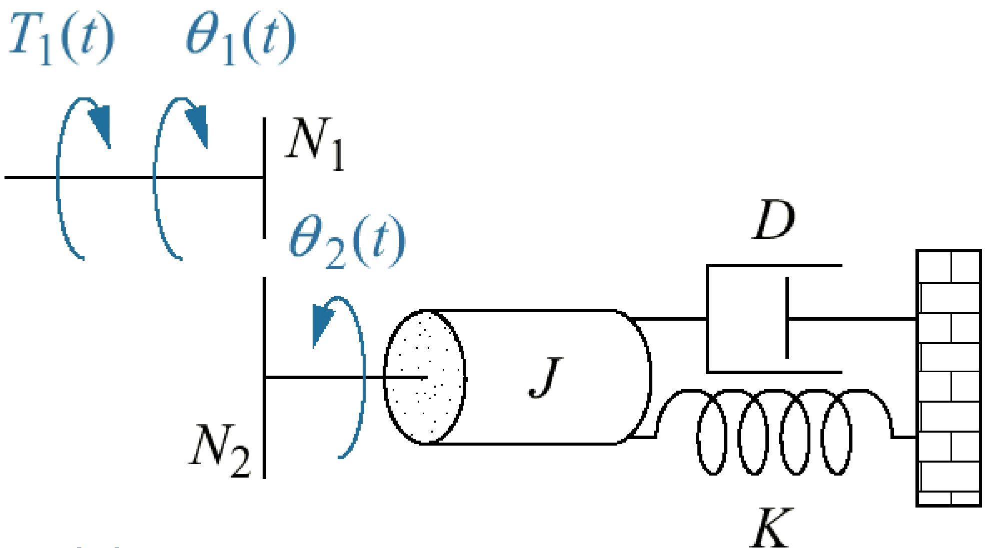
$$\frac{T_1}{T_2} = \frac{N_1}{N_2}$$

Transfer Function



Example 1





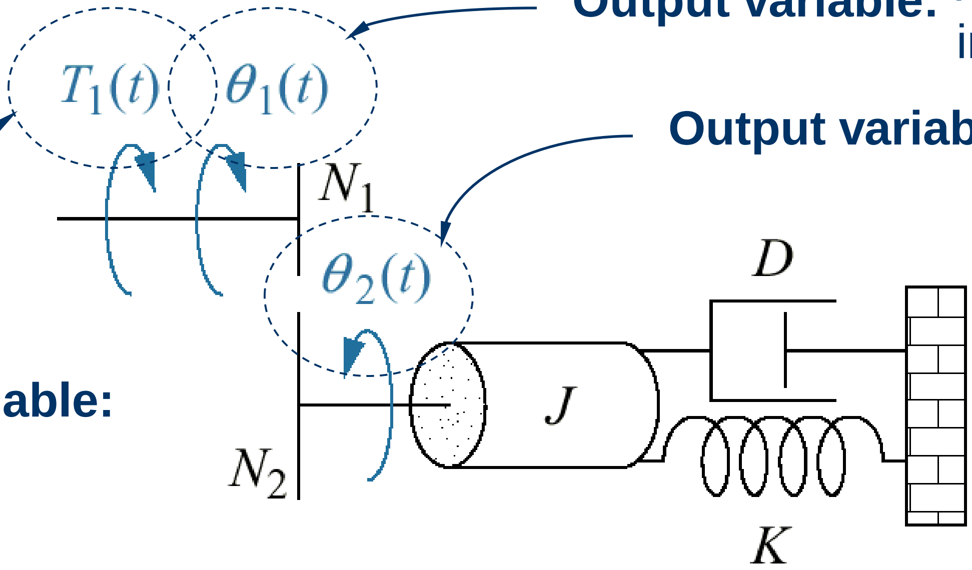
Modeling Steps:

1. Decide the input and the output
2. Draw free body diagram of the inertia
3. Convert time function to frequency-domain
4. Obtain the transfer function

Step 1: Input and Output Variables

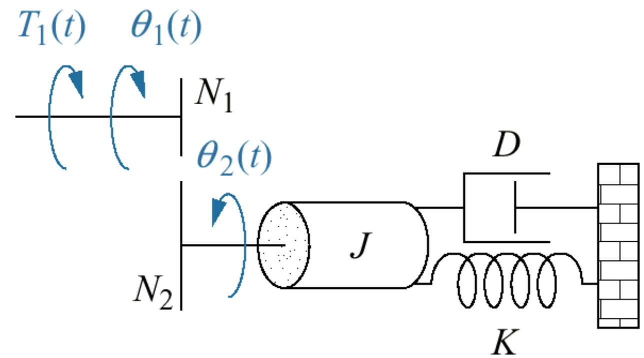
Output variable: Angular position of input shaft $\theta_1(t)$

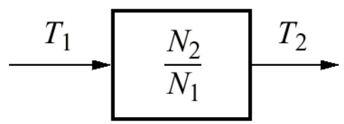
Output variable: Angular position of input shaft $\theta_2(t)$

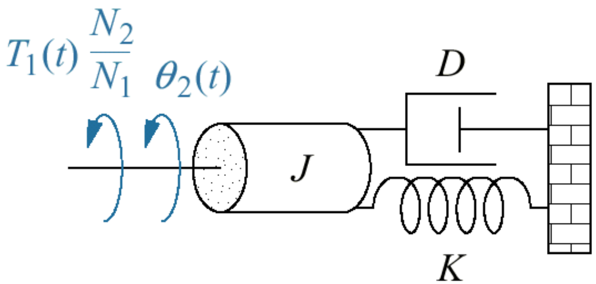


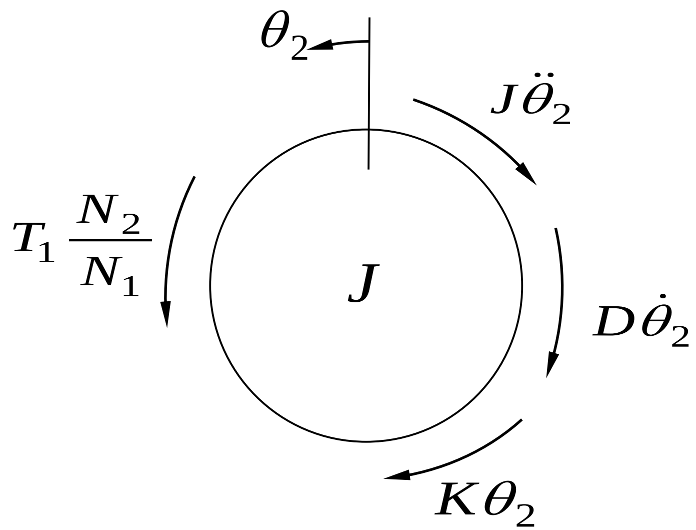
Input variable:

Step 2: The free body diagram of *J*
Step 3: The differential equation of J









$$J\ddot{\theta}_2 + D\dot{\theta}_2 + K\theta_2 = T_1\frac{N_2}{N_1}$$

$$(Js^2 + Ds + K)\theta_2 = T_1\frac{N_2}{N_1}$$

Step 4: Transfer Function – $\theta_2(t)$ as output

Inertial J: $$J\ddot{\theta}_2 + D\dot{\theta}_2 + K\theta_2 = T_1 \frac{N_2}{N_1}$$

By Laplace Transform

Inertial J: $$s^2 J\theta_2(s) + sD\theta_2(s) + K\theta_2(s) = T_1 \frac{N_2}{N_1}$$

$$\frac{\theta_2(s)}{T_1(s)} = \frac{N_2}{N_1(Js^2 + Ds + K)}$$

Step 4: Transfer Function – *θ₁(t)* as output

Inertial J: $$J\ddot{\theta}_2 + D\dot{\theta}_2 + K\theta_2 = T_1\frac{N_2}{N_1}$$

Gear system relationship

$$\theta_1 \longrightarrow \boxed{\frac{N_1}{N_2}} \longrightarrow \theta_2$$

$$\theta_2 = \theta_1\frac{N_1}{N_2}$$

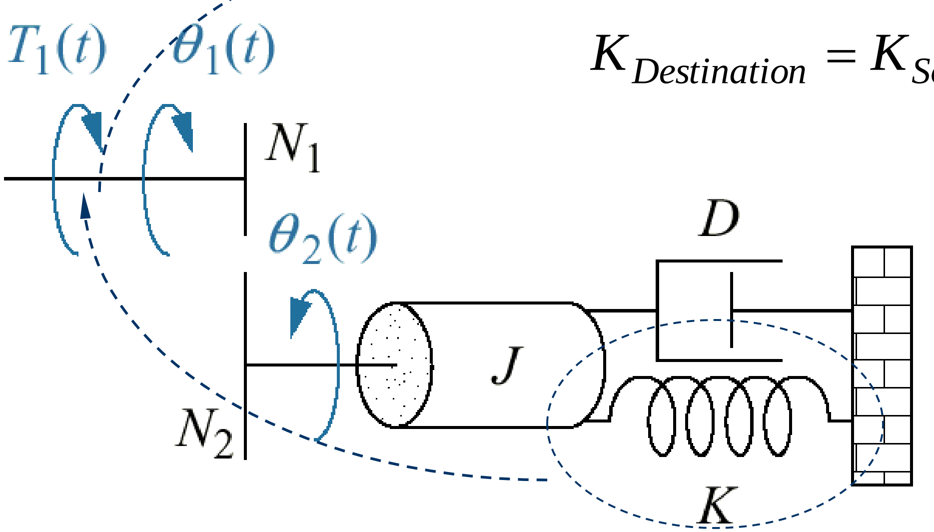
$$(s^2J + sD + K)\theta_2(s) = T_1\frac{N_2}{N_1} \qquad (s^2J + sD + K)\theta_1(s)\frac{N_1}{N_2} = T_1\frac{N_2}{N_1}$$

$$\frac{\theta_1(s)}{T_1(s)} = \frac{1}{\left(\left(\frac{N_1}{N_2}\right)^2 Js^2 + \left(\frac{N_1}{N_2}\right)^2 Ds + \left(\frac{N_1}{N_2}\right)^2 K\right)}$$

$$\frac{\theta_1(s)}{T_1(s)} = \frac{1}{\left(\left(\frac{N_1}{N_2}\right)^2 Js^2 + \left(\frac{N_1}{N_2}\right)^2 Ds + \left(\frac{N_1}{N_2}\right)^2 K\right)}$$

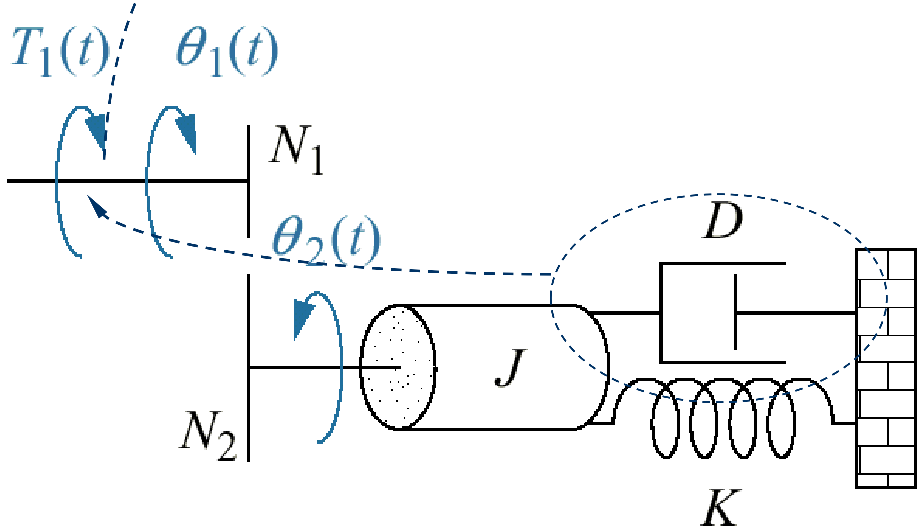
Gearing system causes impedance transfer

$$K_{Destination} = K_{Source}\left[\frac{N_{Destination}}{N_{Source}}\right]^2$$

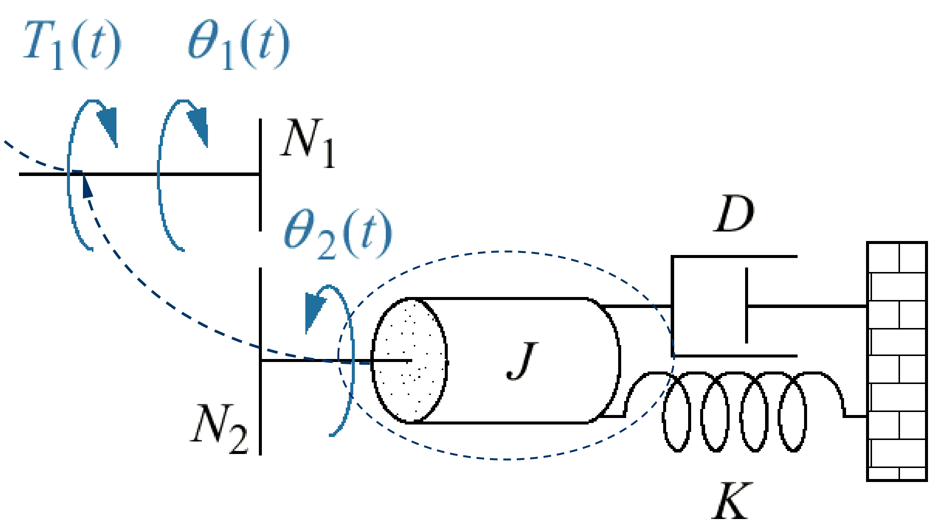


$$K_{S1} = K_{S2}\left[\frac{N_1}{N_2}\right]^2$$

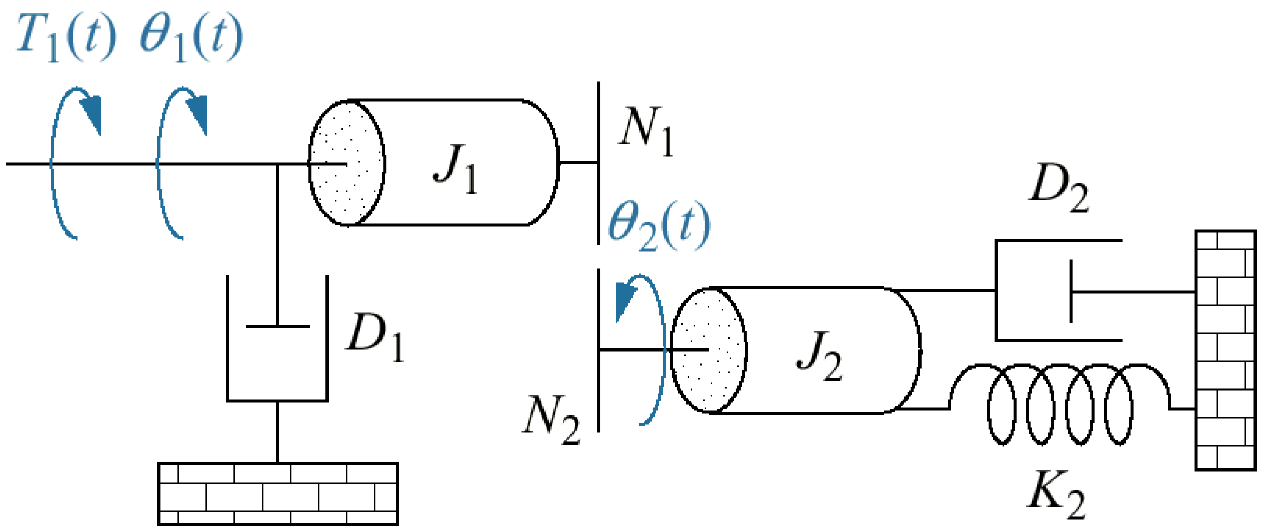
$$\frac{\theta_1(s)}{T_1(s)} = \frac{1}{\left(\left(\frac{N_1}{N_2}\right)^2 Js^2 + \left(\frac{N_1}{N_2}\right)^2 Ds + \left(\frac{N_1}{N_2}\right)^2 K\right)}$$



$$\frac{\theta_1(s)}{T_1(s)} = \frac{1}{\left(\left(\frac{N_1}{N_2}\right)^2 Js^2 + \left(\frac{N_1}{N_2}\right)^2 Ds + \left(\frac{N_1}{N_2}\right)^2 K\right)}$$

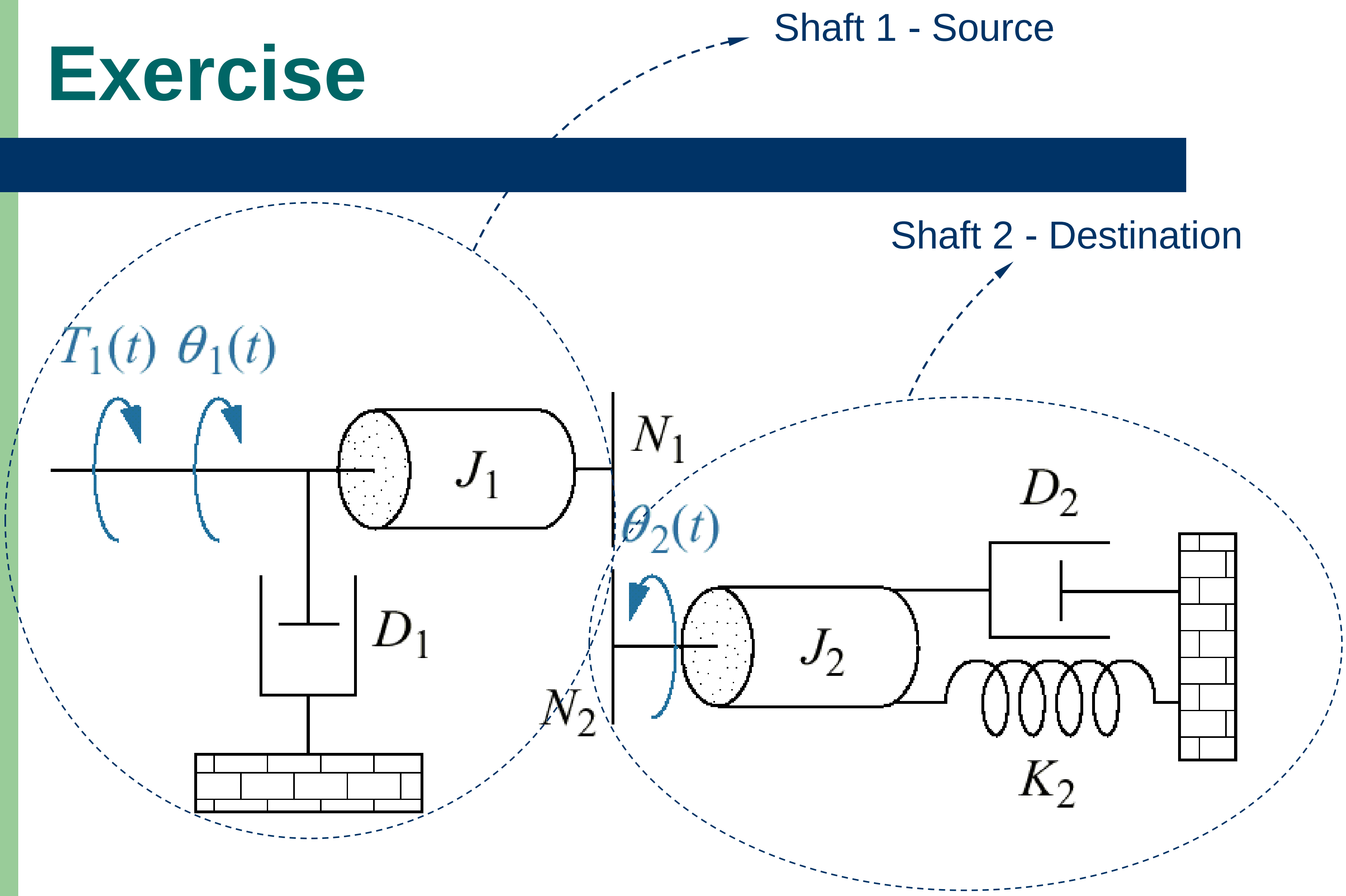


Exercise



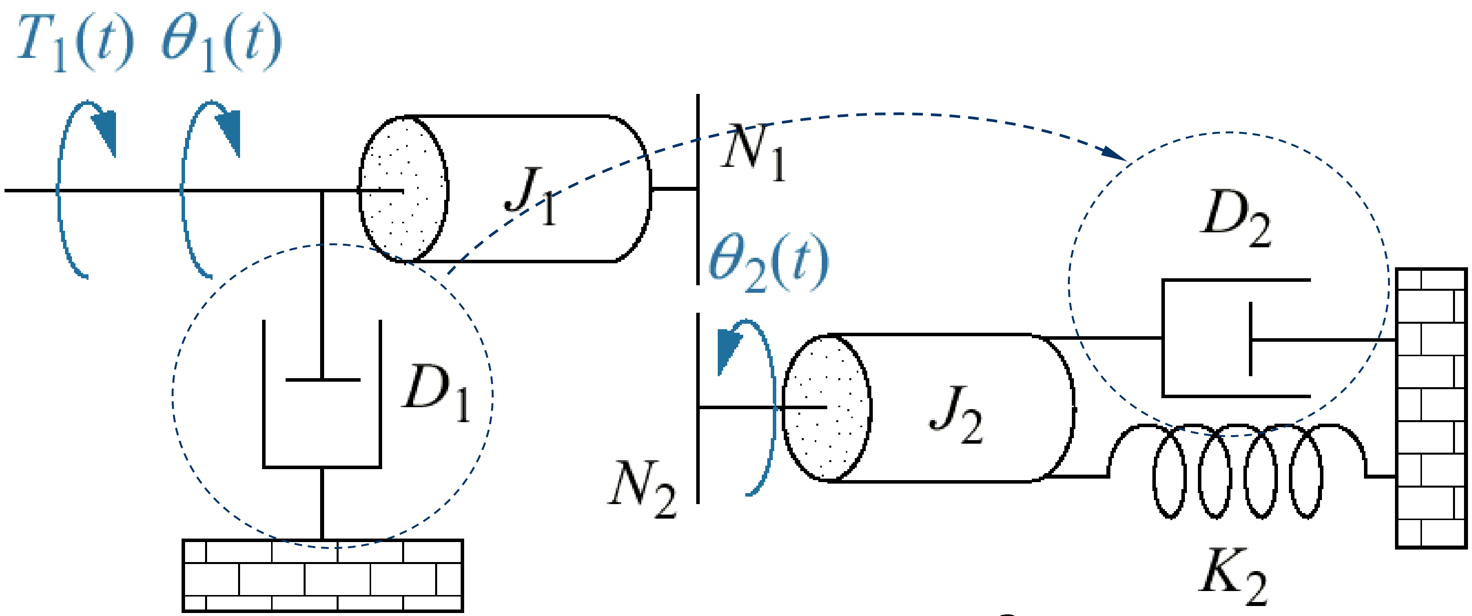
Find Transfer Function? $$\frac{\theta_2(s)}{T_1(s)}$$

Exercise



Shaft 1 - Source
Shaft 2 - Destination
$T_1(t)$ $\theta_1(t)$
J_1
N_1
D_1
$\theta_2(t)$
N_2
J_2
D_2
K_2

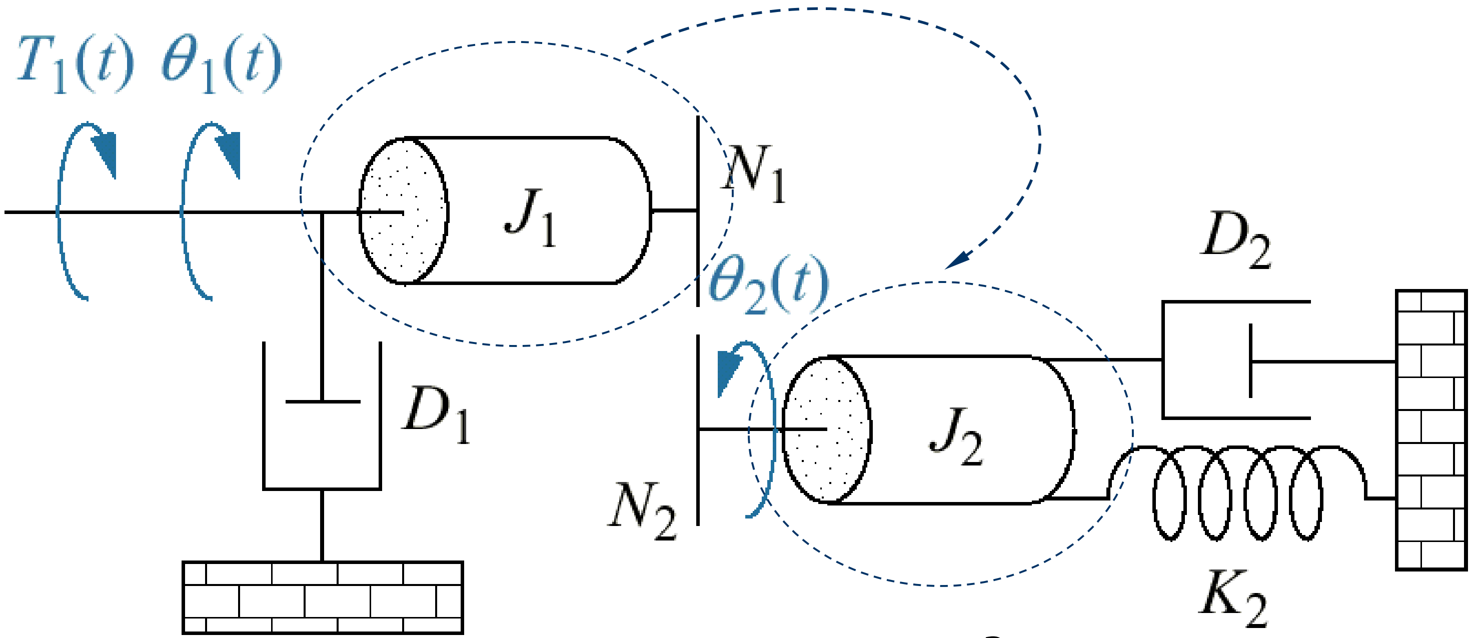
Exercise



$$D_e = D_2 + D_1\left(\frac{N_2}{N_1}\right)^2$$

N_2 because destination is shaft 2

Exercise

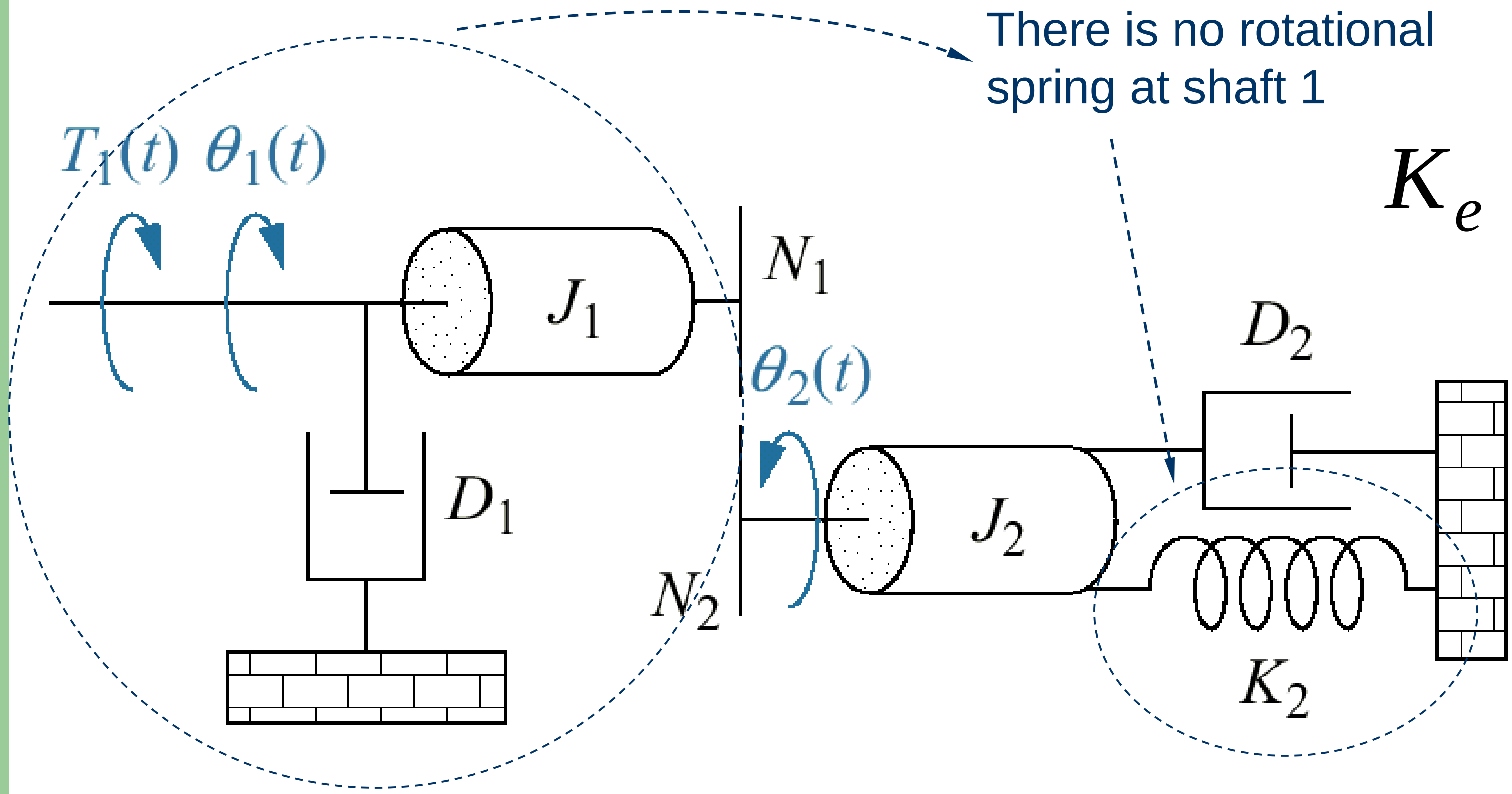


$$J_e = J_2 + J_1\left(\frac{N_2}{N_1}\right)^2$$

N_2 because destination is shaft 2

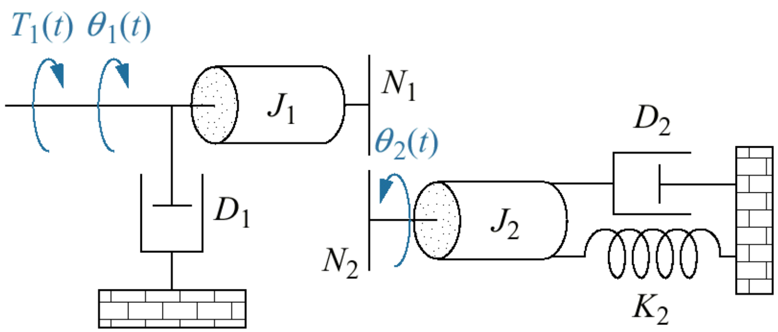
Exercise

There is no rotational spring at shaft 1

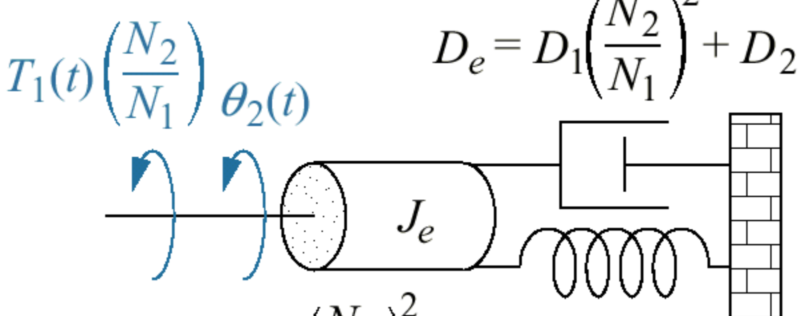


$$K_e = K_2$$

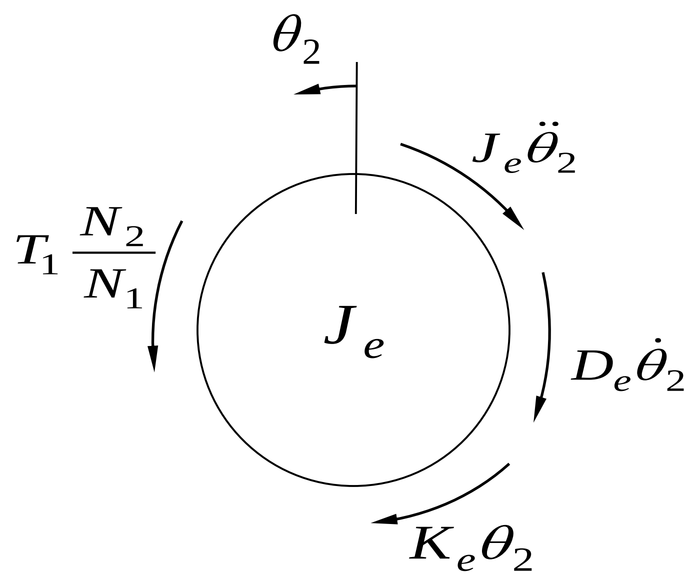
Exercise



$$D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2$$



$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2 \qquad K_e = K_2$$



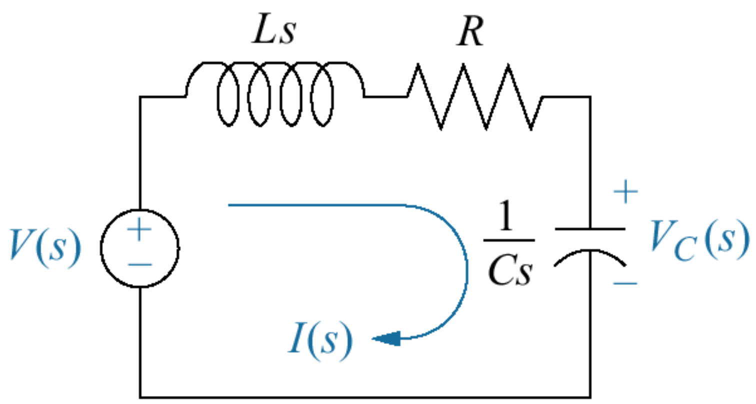
$$J_e\ddot{\theta}_2 + D_e\dot{\theta}_2 + K_e\theta_2 = T_1\frac{N_2}{N_1}$$

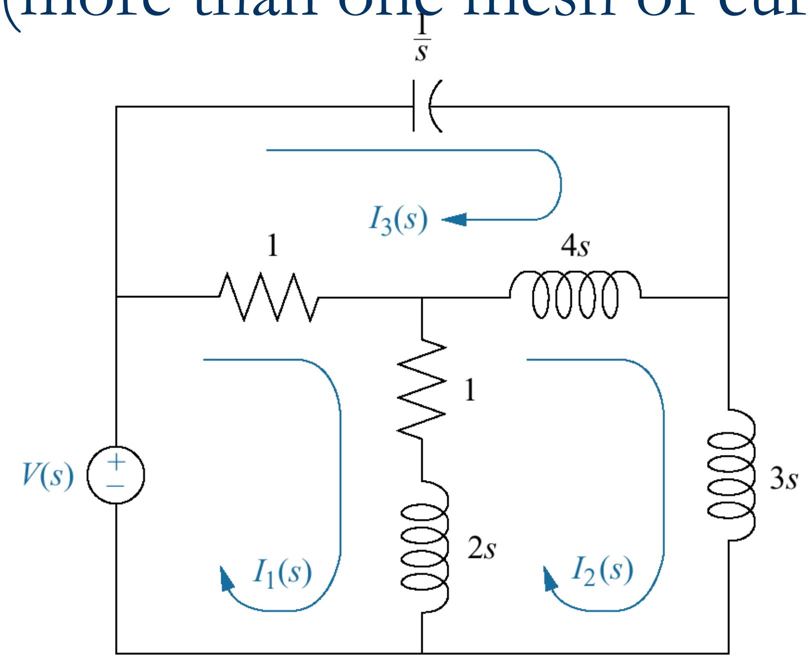
Transfer Function of Passive Electrical Networks

What is the meaning of single loop and multi-loop electrical network?

Single loop network
(one mesh or current)

Multi-loop electrical network
(more than one mesh or current)



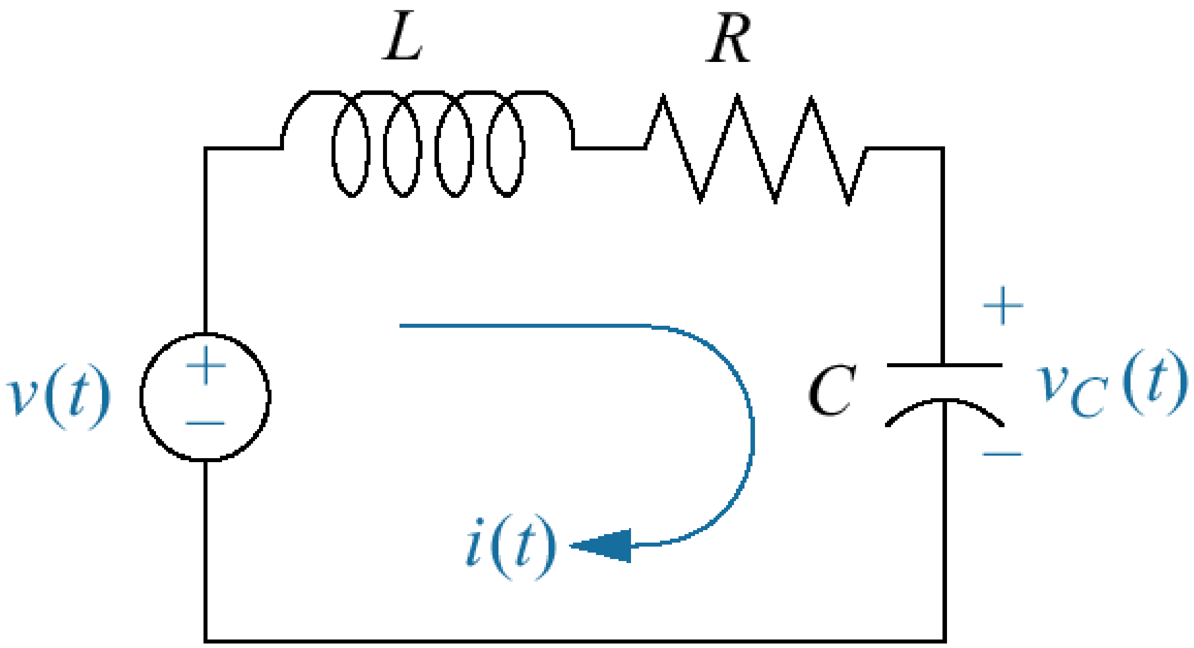


How to creating transfer function for a passive electrical system

Following four Steps:

1. Decide input and output
2. Convert each component representation to frequency domain representation
3. Obtain relationship between voltage and current (Ohm's law)
4. Obtain transfer function between output and input

Creating transfer function for a single-loop passive electrical system



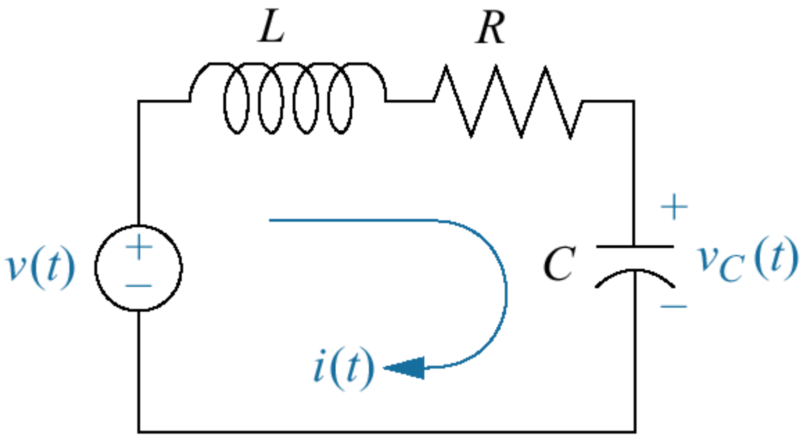
Step 1: Decide input and output

Input:

Supply voltage v(t)

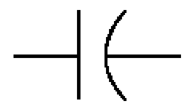
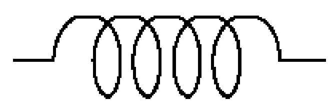
Output:

Capacitor voltage $v_c(t)$



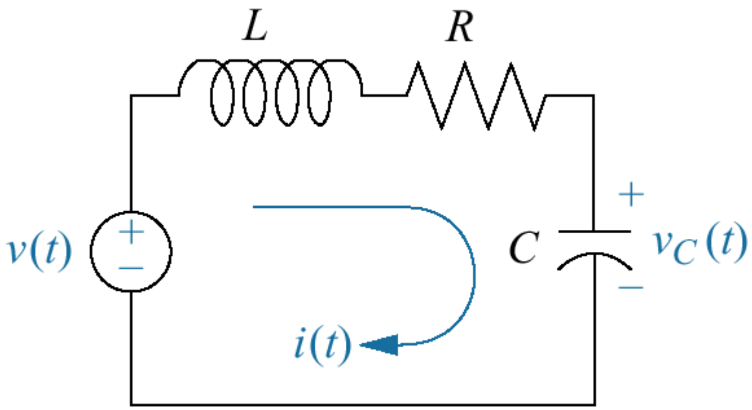
Step 2: Convert into frequency domain

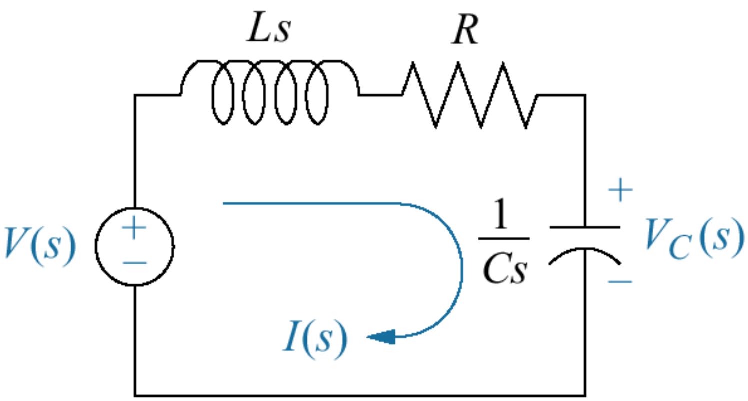
Table Conversion between Time Domain and Frequency Domain

 Capacitor	$\frac{1}{Cs}$
 Resistor	R
 Inductor	Ls

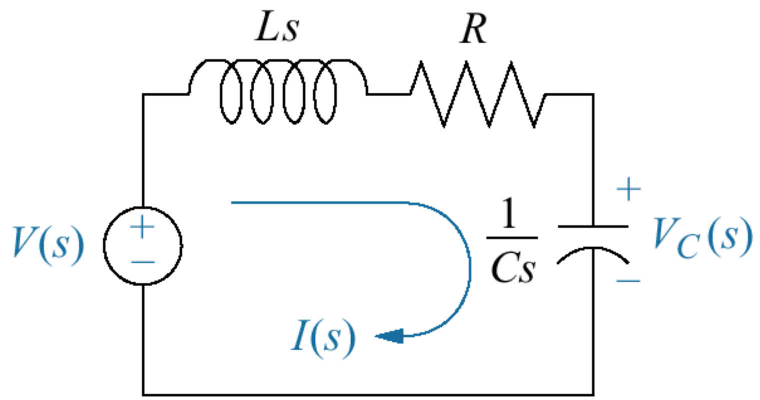
Step 2: Convert into frequency domain

Time Domain → **Frequency domain**





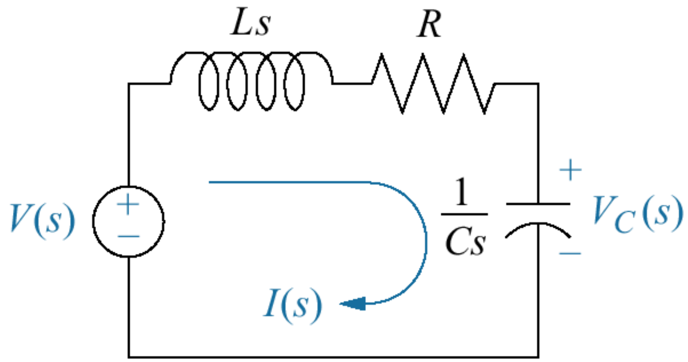
Step 3: Obtain relationship between voltage and current (Ohm's Law)



$$V(s) = (Ls + R + \frac{1}{Cs})I(s)$$

Step 4: Obtain transfer function between output and input

Frequency domain



$$Input = V(s)$$

$$Output = V_c(s)$$

$$V_c(s) = \frac{1}{Cs} I(s)$$

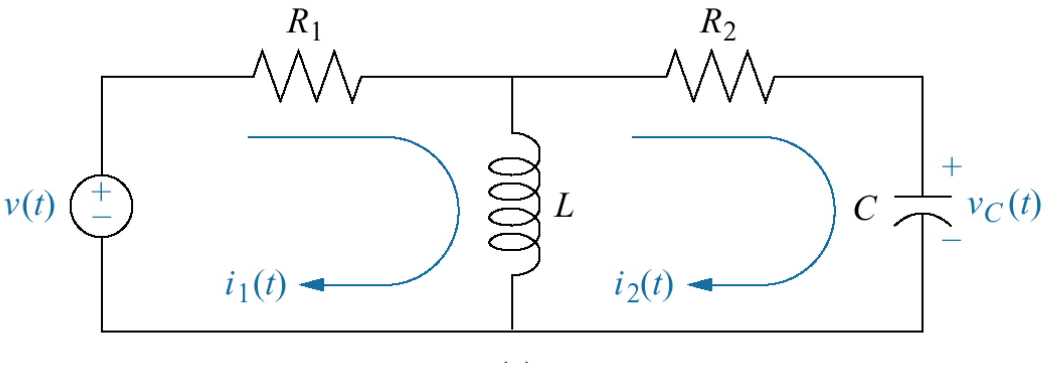
$$V(s) = (Ls + R + \frac{1}{Cs}) Cs V_c(s)$$

$$G(s) = \frac{V_c(s)}{V(s)} = \frac{output}{input} = \frac{1}{CLs^2 + CRs + 1}$$

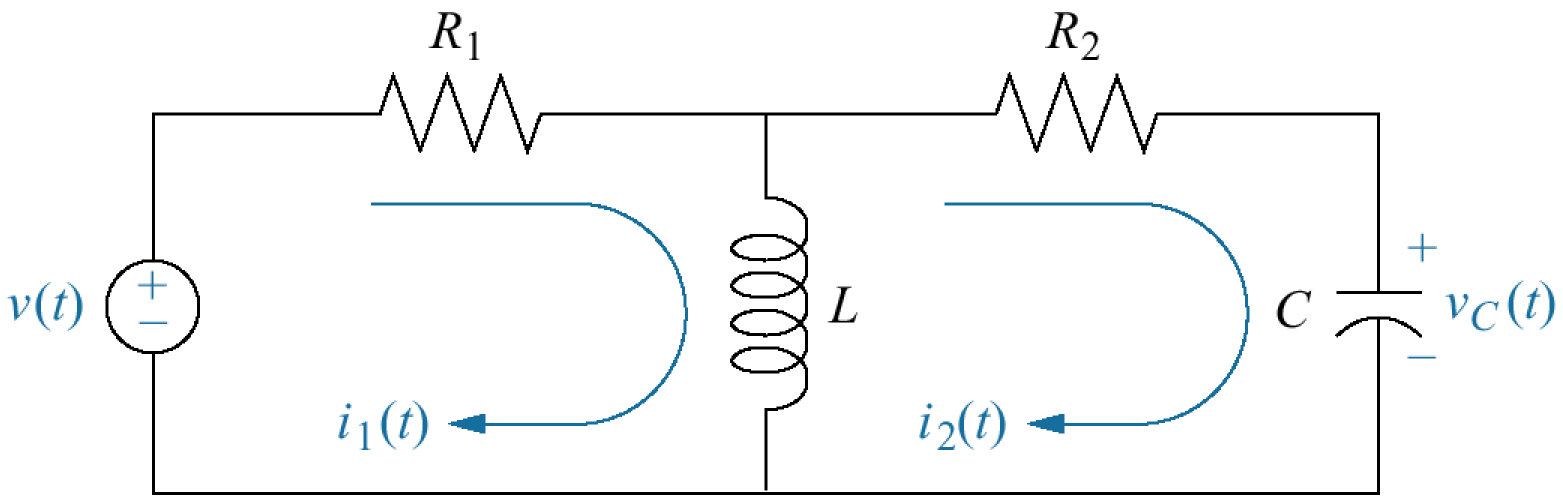
Creating transfer function of two-loop electrical network

Steps:

1. Decide input and output
2. Convert to frequency domain
3. Obtain relationship between voltage and current (Ohm's law)
4. Obtain transfer function between output and input



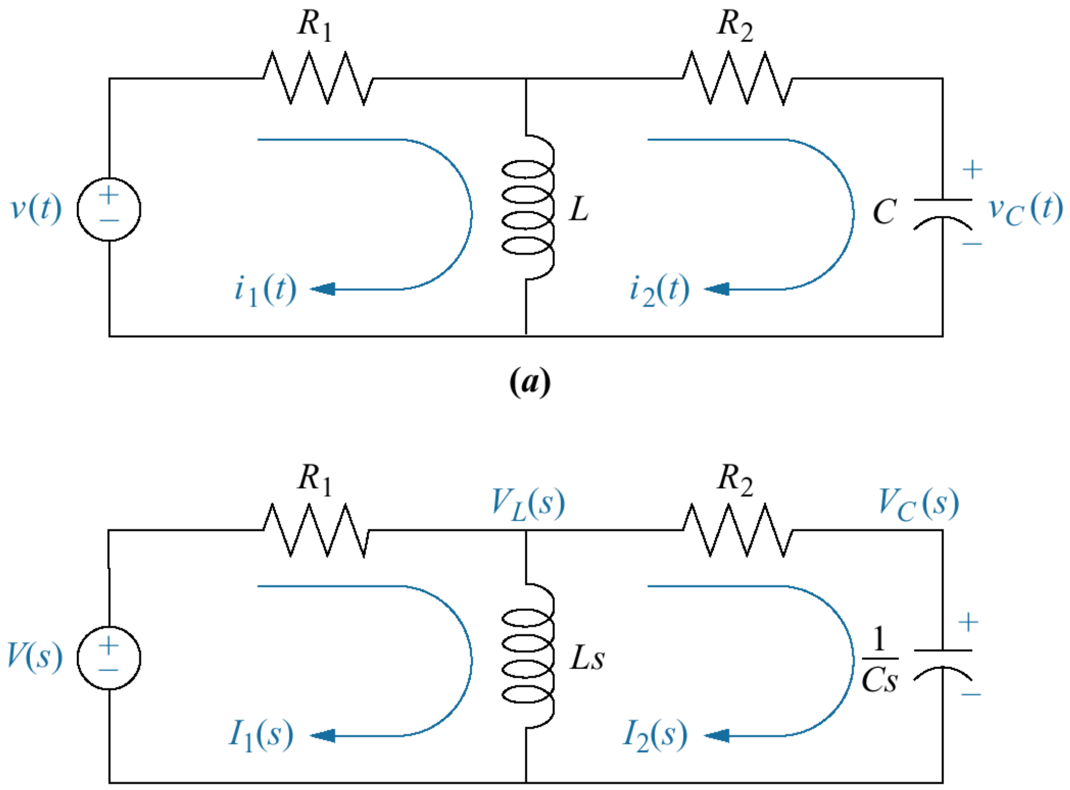
Step 1: Decide input and output



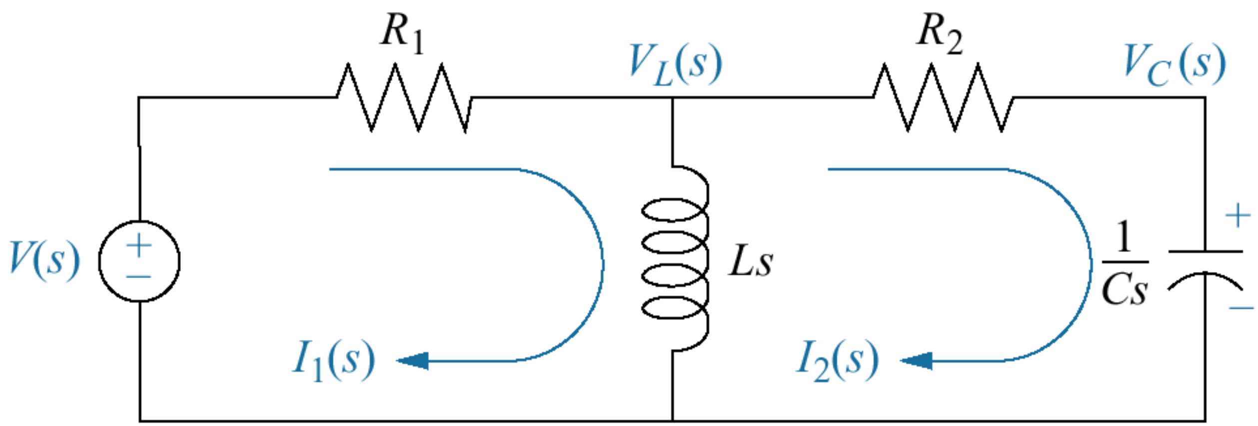
Input $v(t)$

Output $i_2(t)$

Step 2: Convert to Frequency domain



Step 3: Relationship between voltage and current



1. Loop by loop analysis
2. Two equations since there is three loops

Loop 1:

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

Loop 2:

$$R_2 I_2(s) + \frac{1}{Cs} I_2(s) + LsI_2(s) - LsI_1(s) = 0$$

Step 4: Obtain transfer function

$$R_1 I_1(s) + LsI_1(s) - LsI_2(s) = V(s)$$

$$R_2 I_2(s) + \frac{1}{Cs} I_2(s) + LsI_2(s) - LsI_1(s) = 0$$

$$\begin{bmatrix} R_1 + Ls & -Ls \\ -Ls & R_2 + \frac{1}{Cs} + Ls \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \end{bmatrix}$$

Step 4: Obtain transfer function

Let

$$R_1 = 1; R_2 = 1; L = 1; C = 1$$

$$\begin{bmatrix} 1+s & -s \\ -s & \dfrac{s^2+s+1}{s} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \end{bmatrix}$$

Step 4: Obtain transfer function

$$I_1(s) = \frac{\begin{vmatrix} V(s) & -s \\ 0 & \dfrac{s^2+s+1}{s} \end{vmatrix}}{\begin{vmatrix} 1+s & -s \\ -s & \dfrac{s^2+s+1}{s} \end{vmatrix}} \qquad I_2(s) = \frac{\begin{vmatrix} 1+s & V(s) \\ -s & 0 \end{vmatrix}}{\begin{vmatrix} 1+s & -s \\ -s & \dfrac{s^2+s+1}{s} \end{vmatrix}}$$

Step 4: Obtain transfer function

$$I_2(s) = \frac{\begin{vmatrix} 1+s & V(s) \\ -s & 0 \end{vmatrix}}{\begin{vmatrix} 1+s & -s \\ -s & \dfrac{s^2+s+1}{s} \end{vmatrix}} = \frac{V(s)s}{\dfrac{(1+s)(s^2+s+1)}{s} + s^2}$$

$$I_2(s) = \frac{s^2 V(s)}{s^3 + (s+1)(s^2+s+1)}$$

Step 4: Obtain transfer function

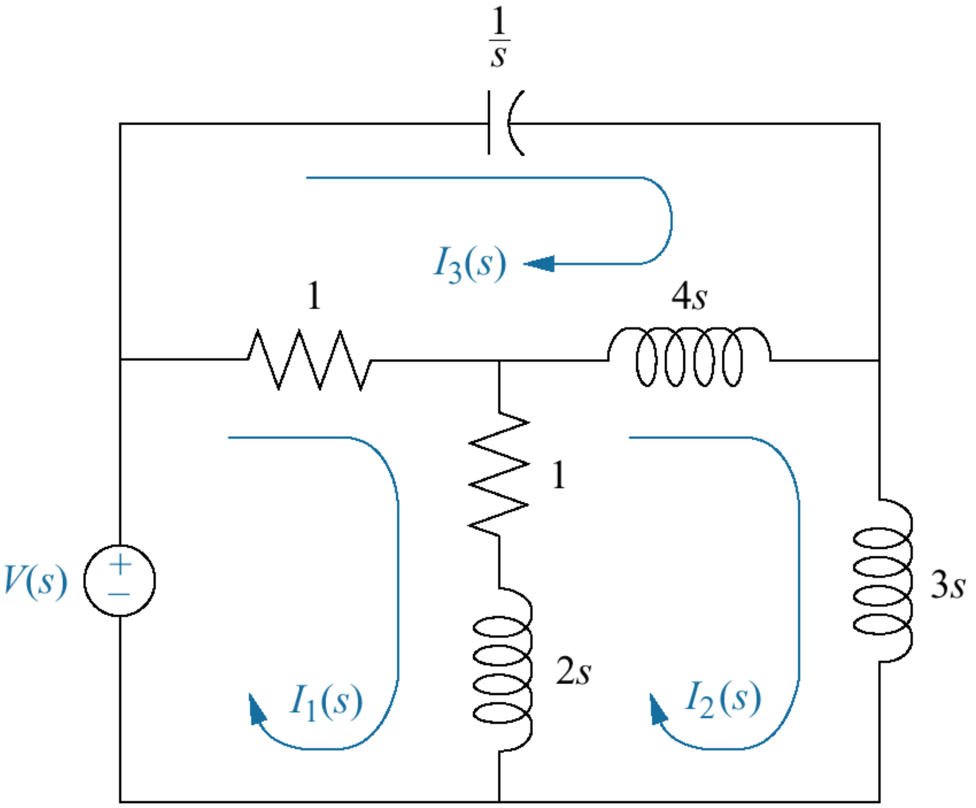
$$I_2(s) = \frac{s^2 V(s)}{s^3 + (s+1)(s^2+s+1)}$$

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{\dfrac{s^2 V(s)}{s^3 + (s+1)(s^2+s+1)}}{V(s)} = \frac{s^2}{s^3 + (s+1)(s^2+s+1)}$$

Creating transfer function of multi-loop electrical network

Steps:

1. Decide input and output
2. Convert to frequency domain
3. Obtain relationship between voltage and current (Ohm's law)
4. Obtain transfer function between output and input



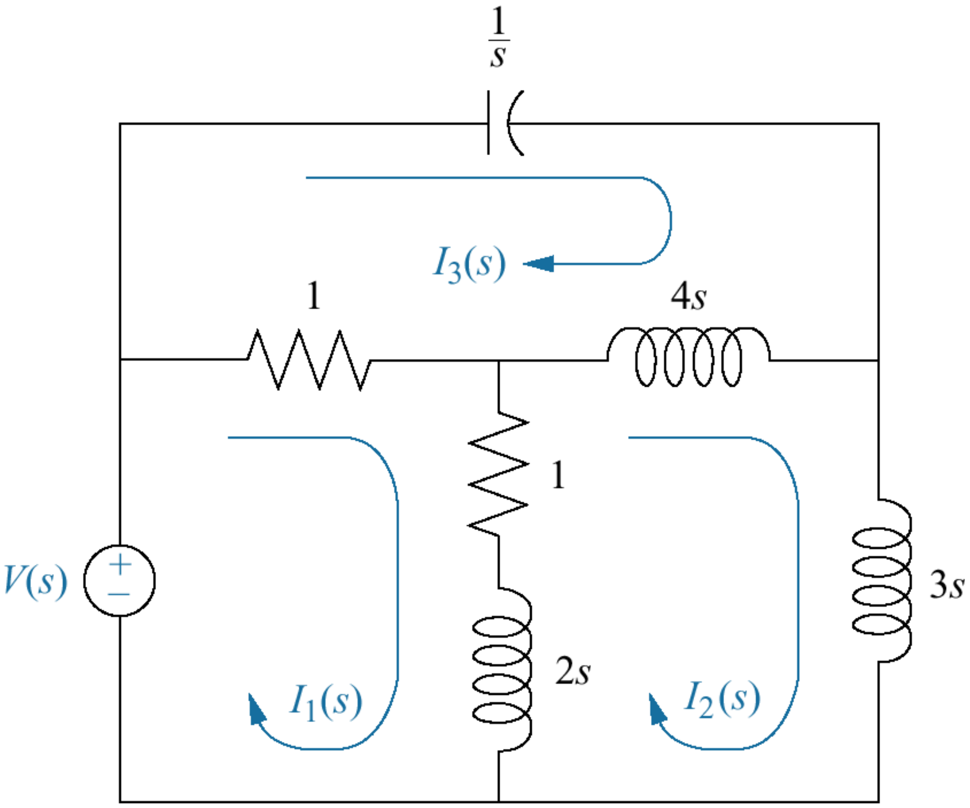
Step 1: Decide input and output

Input:

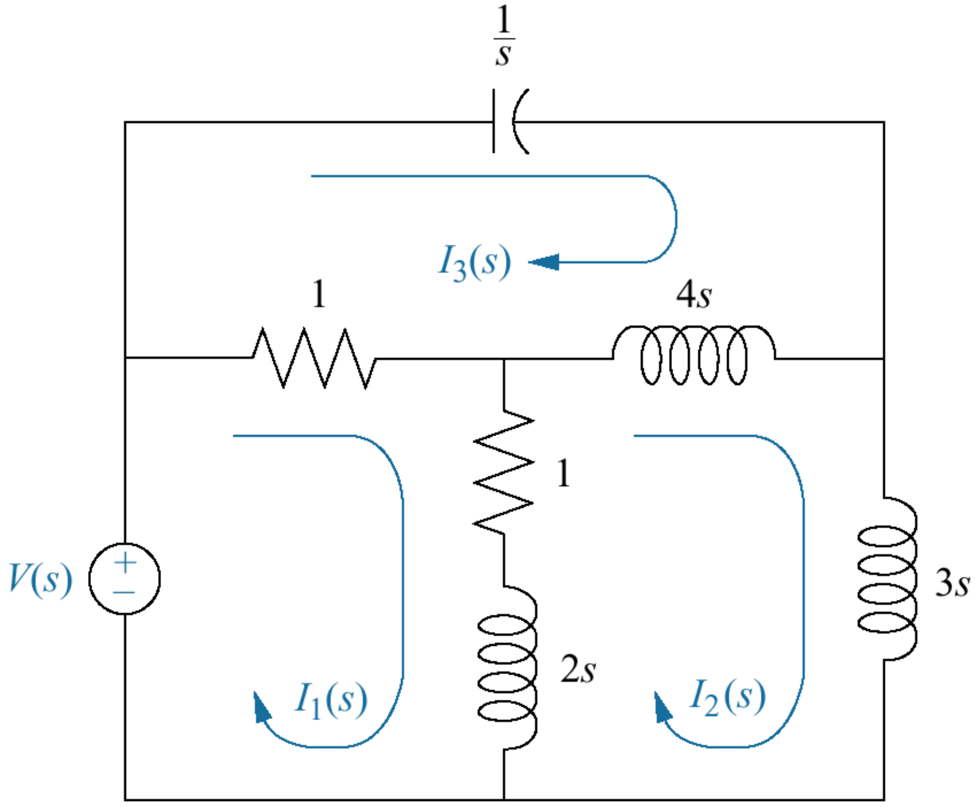
Supply voltage V(s)

Output:

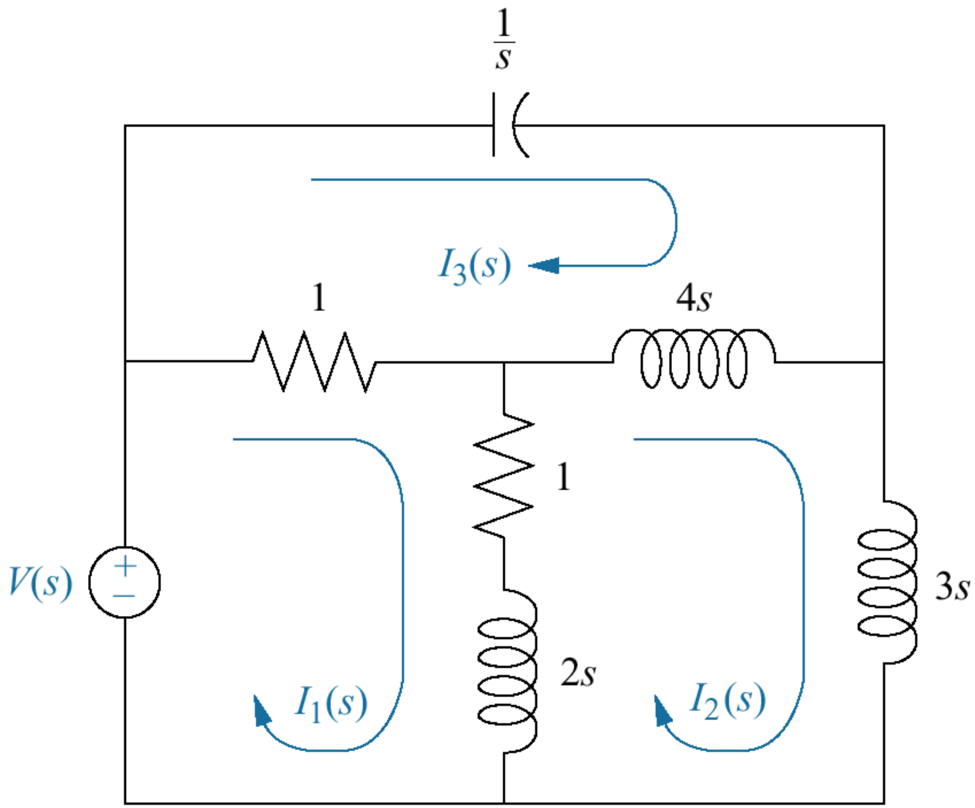
Capacitor voltage $V_c(s)$



Step 2: Convert into frequency domain

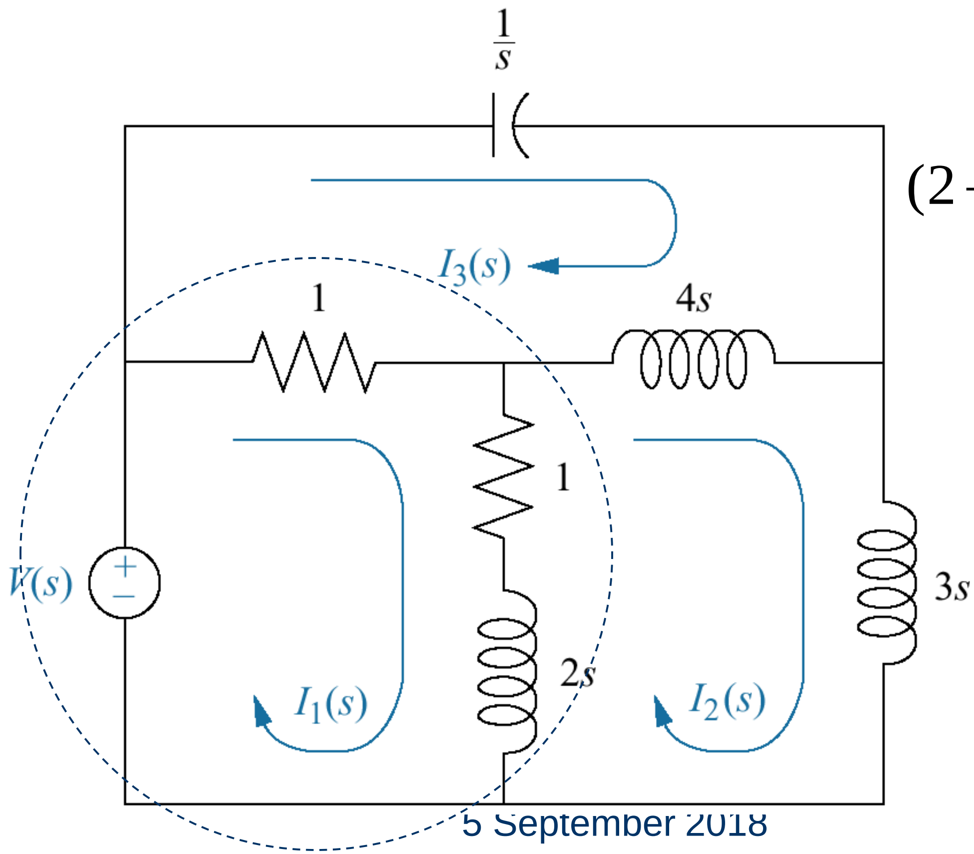


Step 3: Obtain relationship between voltage and current (Ohm's Law)



1. Loop by loop analysis
2. Three equations since there is three loops

Step 3: Obtain relationship between voltage and current (Ohm's Law)



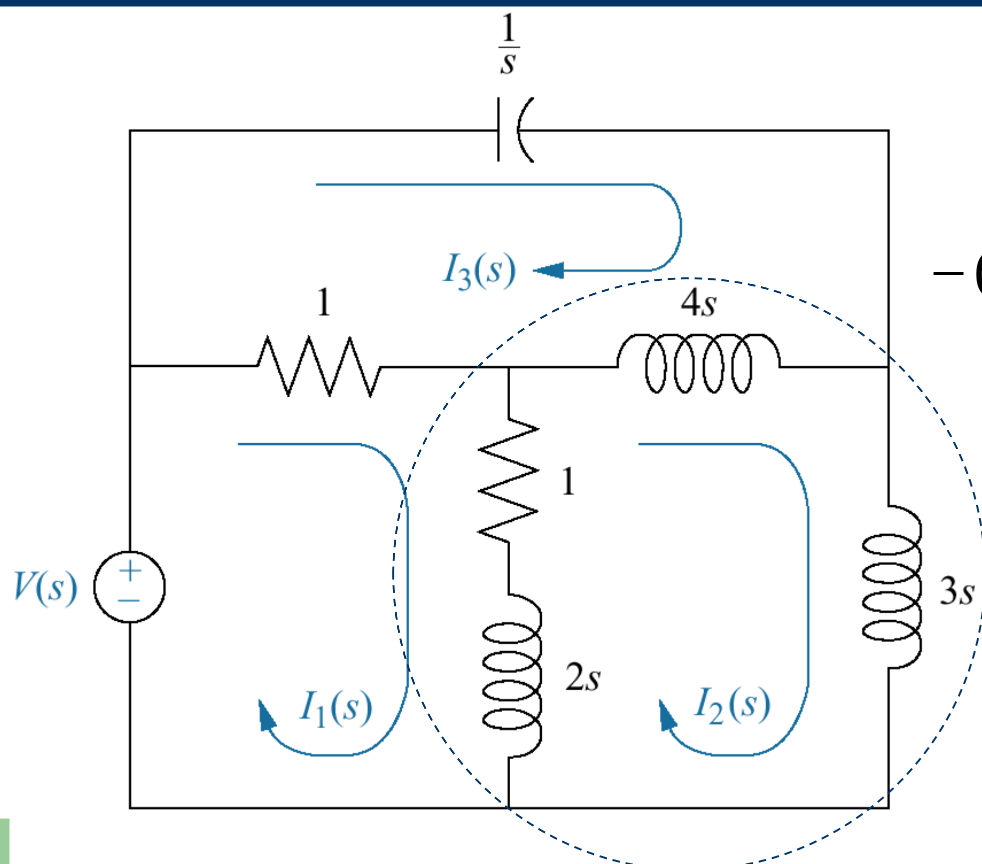
Loop 1:

$$(2+2s)I_1-(1+2s)I_2-(1)I_3=V(s)$$

Method

1. Sum impedances around loop
2. Minus impedances share with other loops

Step 3: Obtain relationship between voltage and current (Ohm's Law)



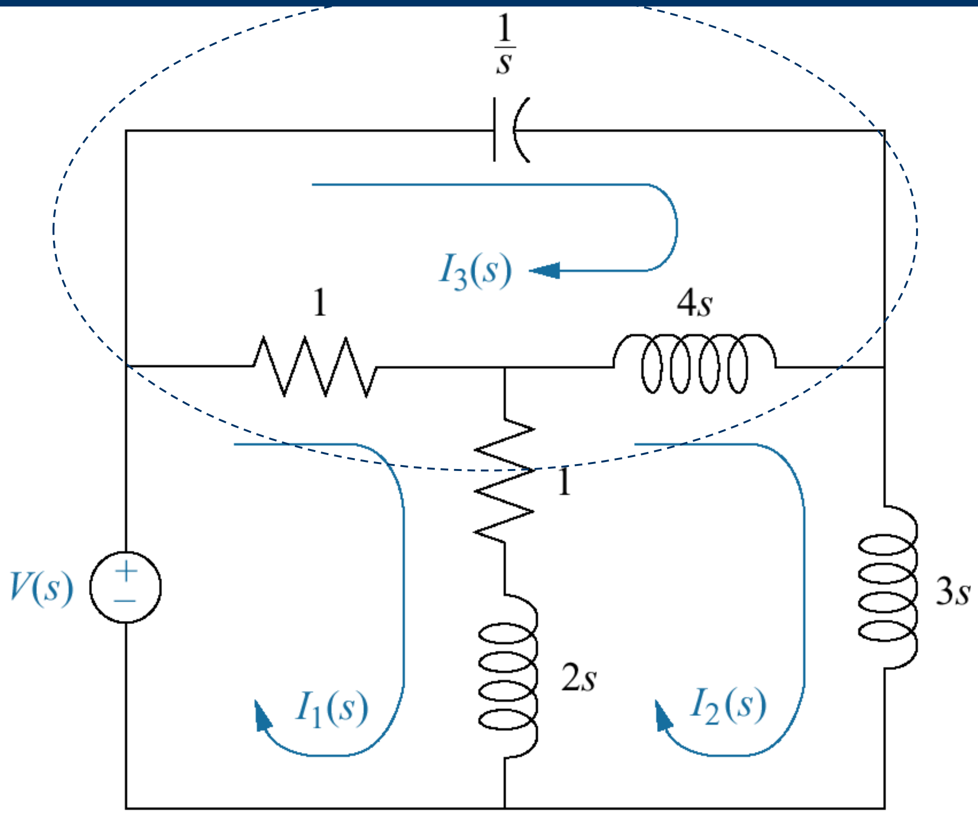
Loop 2:

$$-(1+2s)I_1 + (1+9s)I_2 - (4s)I_3 = 0$$

Method

1. Sum impedances around loop
2. Minus impedances share with other loops

Step 3: Obtain relationship between voltage and current (Ohm’s Law)



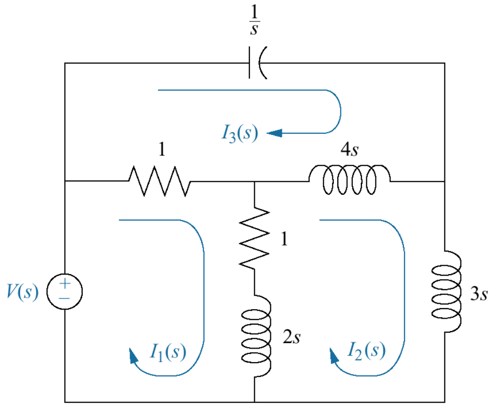
Loop 3:

$$-(1)I_1 - (4s)I_2 + (4s + \frac{1}{s} + 1)I_3 = 0$$

Method

1. Sum impedances around loop
2. Minus impedances share with other loops

Step 3: Obtain relationship between voltage and current (Ohm's Law)



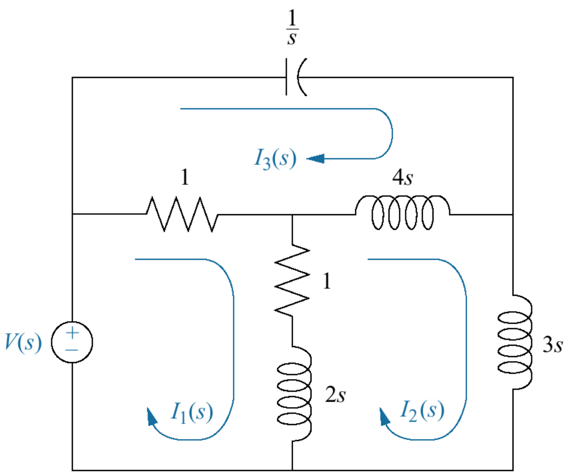
$$(2+2s)I_1-(1+2s)I_2-(1)I_3=V(s)$$

$$-(1+2s)I_1+(1+9s)I_2-(4s)I_3=0$$

$$-(1)I_1-(4s)I_2+(4s+\frac{1}{s}+1)I_3=0$$

$$\begin{bmatrix} (2s+2) & -(2s+1) & -1 \\ -(2s+1) & (9s+1) & -4s \\ -1 & -4s & \dfrac{(4s^2+s+1)}{s} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}=\begin{bmatrix} V(s) \\ 0 \\ 0 \end{bmatrix}$$

Step 4: Obtain transfer function between output and input



$$Input = V(s)$$

$$Output = V_c(s)$$

$$V_c(s) = \frac{1}{s} I_3(s)$$

Need to solve this simultaneous equations for I_3

$$\begin{bmatrix} (2s+2) & -(2s+1) & -1 \\ -(2s+1) & (9s+1) & -4s \\ -1 & -4s & \dfrac{(4s^2+s+1)}{s} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \\ 0 \end{bmatrix}$$

Step 4: Obtain transfer function between output and input

$$\begin{bmatrix} (2s+2) & -(2s+1) & -1 \\ -(2s+1) & (9s+1) & -4s \\ -1 & -4s & \dfrac{(4s^2+s+1)}{s} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \\ 0 \end{bmatrix}$$

$$I_s(s) = \frac{\begin{vmatrix} (2s+2) & -(2s+1) & V(s) \\ -(2s+1) & (9s+1) & 0 \\ -1 & -4s & 0 \end{vmatrix}}{\begin{vmatrix} (2s+2) & -(2s+1) & -1 \\ -(2s+1) & (9s+1) & -4s \\ -1 & -4s & \dfrac{(4s^2+s+1)}{s} \end{vmatrix}}$$