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HW 1

## Homework 1

**1. In class we discussed the timescales of life, across different orders of magnitude. Here, describe with reasoning what you think are the different possible length scales of life. Across how many length scales does life exist? Use rough estimates to within an order of magnitude, and give examples across the range.**

I think the different possible length scales of life are subatomic, atomic/molecular, microscopic, organs, organism, ecosystem, and global. Thus, life exists across 7 length scales, ranging from  $\sim 10^{-18}$  to  $\sim 10^7$  m:

1. In the  $10^{-18}$  m range are particles like protons and electrons.
2. In the  $\sim 10^{-10}$  m range are atoms and molecules, like DNA and proteins.
3. In the  $\sim 10^{-6}$  m range are cells and their organelles.
4. In the  $\sim 10^{-3}$  m range are tissues and organs, like the heart, brain, etc.
5. In the 0.1 to 10m range are organisms like bacteria and humans.
6. In the  $10^3$  to  $10^7$  m range are ecosystems, like forests or deserts.
7. In the  $10^6$  to  $10^7$  m range is the biosphere.

**2. The Malthusian growth equation can be written as:  $dN/dt = \alpha N$ , where  $N$  is the number of people and  $\alpha$  is a constant parameter.**

**a. Solve this equation to find  $N(t)$ , the population size as a function of time. Initially, at  $t=0$ , the population size is 10.**

$N$  = number of people,  $\alpha$  = constant parameter

$$\frac{dN}{dt} = \alpha N$$

$$dN = \alpha N \cdot dt$$

$$\frac{dN}{N} = \alpha \cdot dt$$

$$\int \frac{1}{N} dN = \int \alpha \cdot dt$$

$$\ln(N) = \alpha t + c$$

$$e^{\ln(N)} = e^{\alpha t + c} = e^{\alpha t} \cdot e^c$$

$$N(t) = Ke^{\alpha t}$$

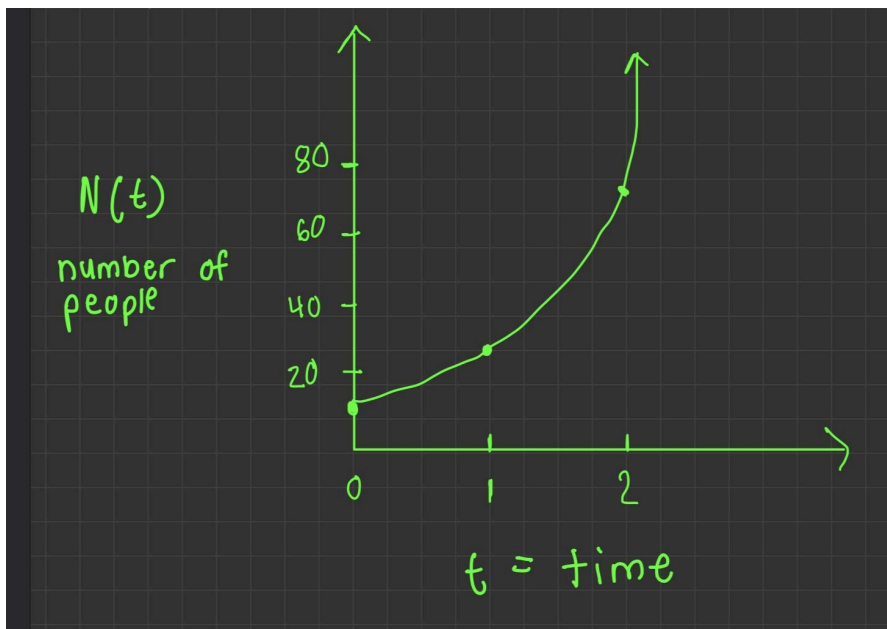
When  $t = 0$ ,  $N = 10$ , so:  $10 = Ke^0 \rightarrow 10 = K$

$$N(t) = 10e^{\alpha t}$$

**b. If  $\alpha=1$ , sketch the first three time points ( $t=0,1,2$ ) for  $N(t)$ . You can approximate  $e$  as 2.7.**

$$N(t) = 10 \cdot 2.7^t$$

t	$10(2.7)$	N(t)
0	$10(2.7^0)$	10
1	$10(2.7^1)$	27
2	$10(2.7^2)$	72.9



**c. If food becomes limited for this population, write down a new differential equation for the rate of change of the population. Describe its terms.**

Original Equation:  $\frac{dN}{dt} = \alpha N$

If food becomes limited for this population:

$$\frac{dN}{dt} = \alpha N \cdot \left(1 - \frac{N}{K}\right)$$

where:

$\frac{dN}{dt}$  is the rate of change of the population overtime

$\alpha$  is a constant parameter

$N$  is the number of people

$K$  is the carrying capacity or maximum population size the environment can support if food becomes limited for the population

$\left(1 - \frac{N}{K}\right)$  is the limiting factor that reduces the growth rate as the population approaches the carrying capacity

### **3. General knowledge of ODE modeling.**

- **In words: define what is a steady state.**

A steady state is a condition in which the system's variables do not change overtime. In the context of the population model, a steady state is when the population remains constant. The population does not increase or decrease when in a steady state.

- **Mathematically: define what is a steady state.**

A steady state is a point in a system where all time derivatives are equal to 0. In context of the population model, a steady state is when  $\frac{dx}{dt} = 0$ .

- **In words: define what it means if a steady state is stable.**

A steady state is stable if when the system is slightly disturbed, it will naturally return to that state overtime.

- **Mathematically: define the conditions under which a steady state is stable.**

A steady state is stable if small deviations from that steady state decay to 0, thus returning to the steady state overtime. We can analyze stability using the Jacobian matrix and checking the eigenvalues. If all eigenvalues have negative real parts, the steady state is stable.

4. For the following ODE model, where  $x$  is a population of stem cells:  $x' = 0.3x(1 - \frac{x}{10})$

- Define the meaning of each term in words.

$x'$  is the rate of change of the population of stem cells with respect to time

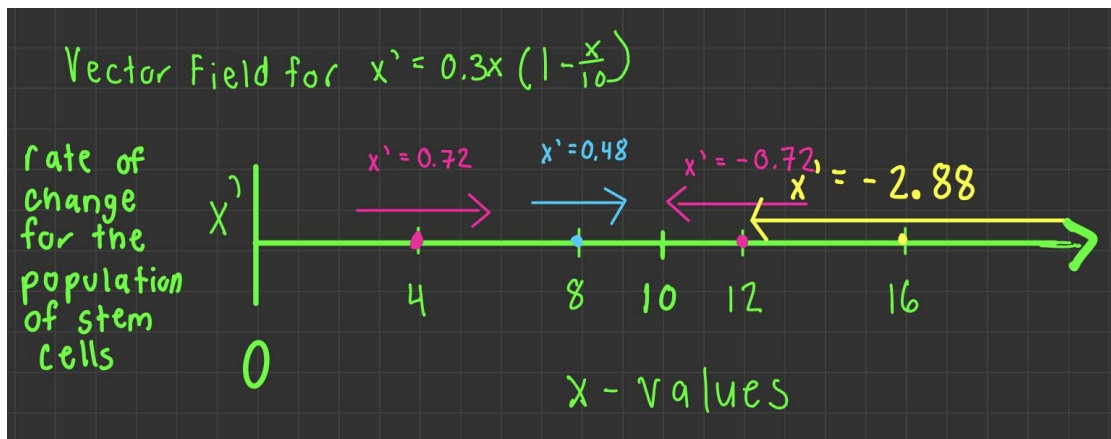
$0.3x$  is the growth rate of the population, where 0.3 is a constant parameter and  $x$  is the current population of stem cells

$(1 - \frac{x}{10})$  is the limiting factor for population growth, where  $\frac{x}{10}$  is the proportion of the population relative to its carrying capacity.

- Sketch the vector field of the model. Show your working and illustrate the magnitude of  $x'$  for at least four values of  $x$ .

$x$	$x'$
0	0
4	0.72
8	0.48
10	0
12	-0.72
16	-2.88

From 0 to 10,  $x'$  is positive and the arrows point to the right. At 10,  $x' = 0$  so there are no arrows. From beyond 10,  $x'$  is negative and the arrows point to the left.



**5. Cats ( $c$ ) catch and eat mice ( $m$ ). Assume that the cats and mice follow Lotka-Volterra type dynamics with the terms as given below. Sketch the phase plane for this system. Show your working for individual vectors in the  $(m, c)$  plane.**

$$\frac{dm}{dt} = m' = 5m - 0.1mc$$

$$\frac{dc}{dt} = c' = 0.05mc - 2c$$

**Find equilibrium points:**

$$\frac{dm}{dt} = 0 = 5m - 0.1mc$$

$$0 = m(5 - 0.1c)$$

If  $m = 0$  or  $c = 50$ , then  $dm/dt = 0$ .

$$\frac{dc}{dt} = 0 = 0.05mc - 2c$$

$$0 = c(0.05m - 2)$$

If  $c = 0$  or  $m = 40$ , then  $dc/dt = 0$ .

Therefore the equilibrium points are  $(0, 0)$  and  $(40, 50)$ . The first  $(0, 0)$  is when both species are extinct. The second is the coexistence equilibrium.

**Find vectors for phase plane:**

Mice- horizontal axis, cats-vertical axis

When  $m > 40$  and  $c > 50$  (ex.  $m = 50, c = 60$ )

$$dm/dt = 5(50) - 0.1(50)(60) = -50 \rightarrow \text{mice decrease}$$

$$dc/dt = 0.05(50)(60) - 2(60) = 30 \rightarrow \text{cats increase}$$

Direction: left up

When  $m < 40$  and  $c < 50$  (ex.  $m = 30, c = 40$ )

$$dm/dt = 5(30) - 0.1(30)(40) = 30 \rightarrow \text{mice increase}$$

$$dc/dt = 0.05(30)(40) - 2(40) = -20 \rightarrow \text{cats decrease}$$

Direction: right down

When  $m < 40$  and  $c > 50$  (ex.  $m = 30, c = 60$ )

$$dm/dt = 5(30) - 0.1(30)(60) = -30 \rightarrow \text{mice decrease}$$

$$dc/dt = 0.05(30)(60) - 2(60) = -30 \rightarrow \text{cats decrease}$$

Direction: left down

When  $m > 40$  and  $c < 50$  (ex.  $m = 50, c = 40$ )

$$dm/dt = 5(50) - 0.1(50)(40) = 50 \rightarrow \text{mice increase}$$

$$dc/dt = 0.05(50)(40) - 2(40) = 50 \rightarrow \text{cats increase}$$

Direction: right up

Point	$m'$	$c'$	Vector Direction
(50,60)	-	+	left-up ↖
(30,40)	+	-	right-down ↘
(30,60)	-	-	left-down ↙
(50,40)	+	+	right-up ↗

asymptotes:  
 $m=0$   
 $c=0$

