UUM504E Structural Dynamics - Homework 1

Spring Term 2022-2023

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Due: April 16, 2023 @Ninova 23:00

Question-1

An automobile is modeled as a 1000-kg mass supported by a spring of stiffness $k = 480,000$ N/m. When it oscillates it does so with a maximum deflection of 10 cm. When loaded with passengers, the mass increases to as much as 1200 kg. Calculate the change in frequency, velocity amplitude, and acceleration amplitude if the maximum deflection remains 10 cm. (Note: M1 and M2 are changed.)

Solution: Q1

In this problem, the m_1 is 1000 kg and m_2 is 1200 kg. The spring stiffness (k) is given as 480000 N/m. Also, the maximum deflection (x_{max}) is 10 cm when the system oscillates.

First, calculate the frequencies (in rad/s) and the frequency changes for both two masses:

$$\omega_{n1} = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{480,000}{1000}} = 21.91 \text{ rad/s}$$
$$\omega_{n2} = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{480,000}{1200}} = 20.00 \text{ rad/s}$$
$$\Delta\omega = 20.00 - 21.91 = -1.91 \text{ rad/s}$$
$$\Delta f = \frac{\Delta\omega}{2\pi} = \frac{|-1.91|}{2\pi} = \mathbf{0.3038} \text{ Hz}$$

Then, using these frequency values, we can calculate the velocity and acceleration amplitudes:

$$v_1 = A\omega_{n1} = 10 \text{ cm} \times 21.91\text{rad/s} = 219.09 \text{ cm/s}$$
$$v_2 = A\omega_{n2} = 10 \text{ cm} \times 20.00\text{rad/s} = 200.00 \text{ cm/s}$$
$$\Delta v = 200.00 - 219.09 = -\mathbf{19.09} \text{ cm/s} \text{ or } -\mathbf{0.1909} \text{ m/s}$$
$$a_1 = A{\omega_{n1}}^2 = 10 \text{ cm} \times (21.91\text{rad/s})^2 = 4800 \text{ cm/s}^2$$
$$a_2 = A{\omega_{n2}}^2 = 10 \text{ cm} \times (20.00\text{rad/s})^2 = 4000 \text{ cm/s}^2$$
$$\Delta a = 4000 - 4800 = -\mathbf{800} \text{ cm/s}^2 \text{ or } -\mathbf{8.00} \text{ m/s}^2$$

Question - 2

The spring mass system of 180 kg mass, stiffness of 1800 N/m and damping coefficient of 250 Ns/m is given an initial velocity of 12 mm/s and an initial displacement of -6 mm. Calculate the form of the response and plot it for as long as it takes to die out. How long does it take to die out?

Solution: Q2

In this problem, we first need to calculate the frequency values using the stiffness and mass information:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1800}{180}} = 3.1623 \text{ rad/s}$$

then, the damping term becomes:

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{200}{2\sqrt{1800 * 180}} = 0.2196$$

We can understand that the system is underdamped since the damping term is less than 1 ($0 < \zeta < 1$). Therefore, the damped natural frequency can be determined as:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.1623\sqrt{1-0.2196^2} = 3.0851 \text{ rad/s}$$

To calculate the response of this underdamped system, we can use the following equations and determine the coefficients of the displacement equation:

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \quad \phi = \tan^{-1}\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}$$

$$A = \sqrt{\frac{((12/1000) + 0.2196 * 3.1623(-6/1000))^2 + ((-6/1000)3.0851)^2}{3.0851^2}} = 0.006515$$
$$\phi = \tan^{-1}\frac{(-6/1000)3.0851}{(12/1000) + 0.2196 * 3.1623(-6/1000)} = -1.170466$$

The displacement equation is:

$$x(t) = A(\sin w_d t + \phi)e^{-\zeta w_n t} = (0.006515)(\sin 3.0851 * t - 1.170466)e^{-0.21961*3.1623*t}$$

where t is the time taken from 0 to 12 for this problem.

Finally, the response graph of the underdamped system is illustrated in Fig. 1. This figure shows that the system response disappears after the time of **8 (s)**.

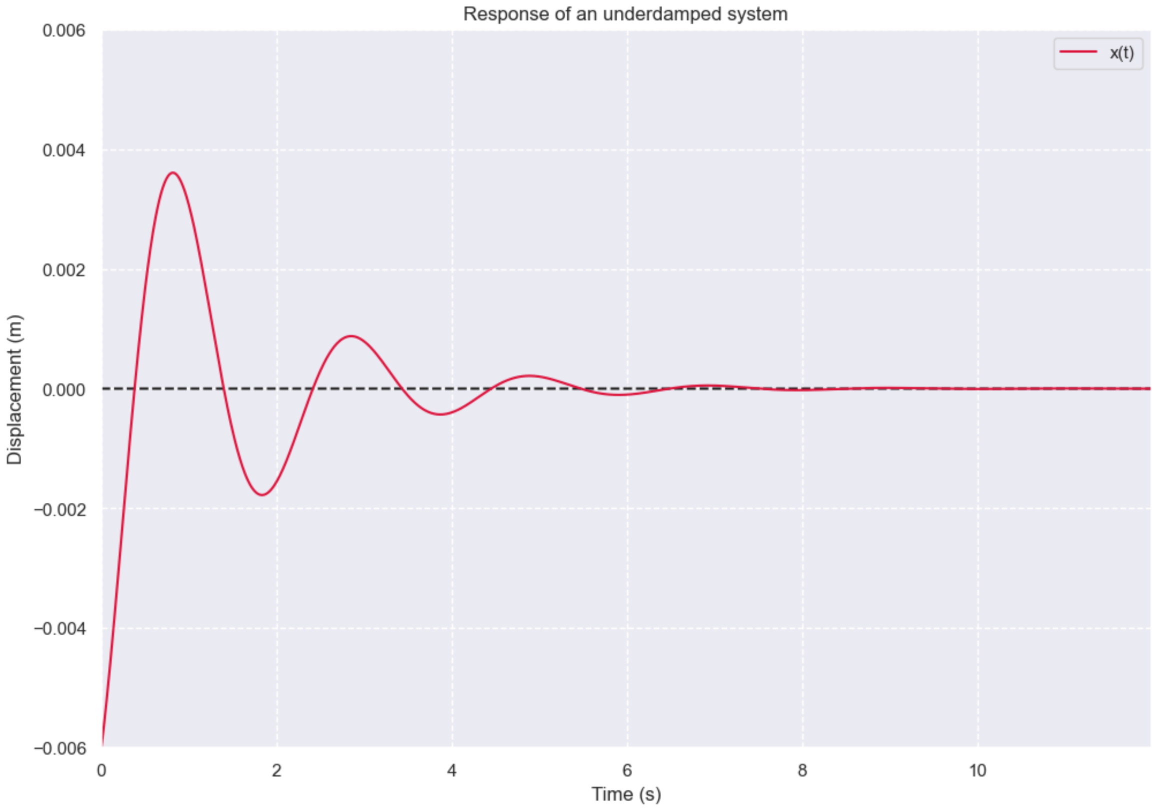

Figure 1: Response of the underdamped system.

Question - 3

Consider the disk in Fig. 2 connected to two springs. Use the energy method to calculate the system's natural frequency of oscillation for small angles $\theta(t)$.

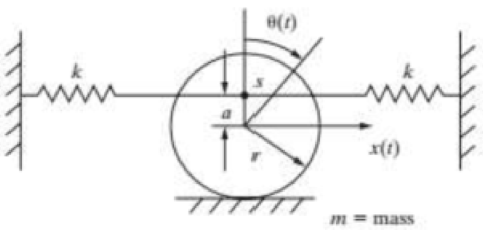

Figure 2: Question - 3: The rolling disk system.

Solution: Q3

In this problem, the rotation θ and the translation x are determined by $x = r\boldsymbol{\theta}$. Thus, $x = r\boldsymbol{\theta}, \dot{x} = r\dot{\boldsymbol{\theta}}$ and $T_{\text{rot}} = \frac{1}{2}J_o\dot{\boldsymbol{\theta}}^2$ where $J_o = \frac{1}{2}mr^2$.

$$\begin{aligned}
\text{Kinetic Energy (T):} \quad T_{\text{rot}} &= \frac{1}{2}J_o\dot{\boldsymbol{\theta}}^2 = \frac{1}{2}\left(\frac{mr^2}{2}\right)\boldsymbol{\theta}^2 = \frac{1}{4}mr^2\dot{\boldsymbol{\theta}}^2 \\
T_{\text{trans}} &= \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mr^2\dot{\boldsymbol{\theta}}^2 \\
T = T_{\text{rot}} + T_{\text{trans}} &= \frac{1}{4}mr^2\dot{\boldsymbol{\theta}}^2 + \frac{1}{2}mr^2\dot{\boldsymbol{\theta}}^2 = \frac{3}{4}mr^2\dot{\boldsymbol{\theta}}^2
\end{aligned}$$

$$\text{Potential Energy (U):} \quad U = 2\left(\frac{1}{2}k[(a+r)\boldsymbol{\theta}]^2\right) = k(a+r)^2\boldsymbol{\theta}^2$$

Using conservation of energy:

$$\begin{aligned}
&T + U = \text{ Constant} \\
&\frac{d}{dt}(T+U) = 0 \\
&\frac{d}{dt}\left(\frac{3}{4}mr^2\dot{\boldsymbol{\theta}}^2 + k(a+r)^2\boldsymbol{\theta}^2\right) = 0 \\
&\frac{3}{4}mr^2(2\dot{\boldsymbol{\theta}}\ddot{\boldsymbol{\theta}}) + k(a+r)^2(2\dot{\boldsymbol{\theta}}\boldsymbol{\theta}) = 0 \\
&\frac{3}{2}mr^2\ddot{\boldsymbol{\theta}} + 2k(a+r)^2\boldsymbol{\theta} = 0
\end{aligned}$$

This expression is in form of $m_{coeff}\ddot{x} + k_{coeff}x = 0$ and recall that the natural frequency can be obtained as $w_n = \sqrt{\frac{k_{coeff}}{m_{coeff}}}$ from this equation. Accordingly, solving the last expression to yield the natural frequency w_n:

$$\begin{aligned}
&\omega_n = \sqrt{\frac{k_{coeff}}{m_{coeff}}} = \sqrt{\frac{2k(a+r)^2}{\frac{3}{2}mr^2}} \\
&\omega_n = 2\frac{a+r}{r}\sqrt{\frac{k}{3m}} \quad \text{rad/s} \quad \textbf{[Ans.]}
\end{aligned}$$

Question - 4

Consider the system in Fig. 3, write the equation of motion, and calculate the response assuming (a) that the system is initially at rest, and (b) that the system has an initial displacement of 0.04 m.

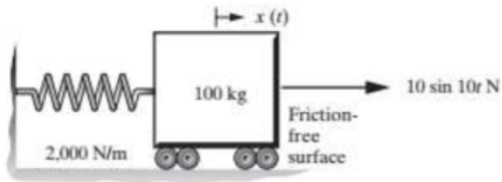

Figure 3: Question - 4: The system.

Solution: Q4

The equation of motion is

$$m\ddot{x} + kx = 10\sin 10t$$

First, determine the general solution for

$$m\ddot{x}(t) + kx(t) = F_0 \sin\omega t \quad \text{or} \quad \ddot{x} + \omega_n^2 x = f_0 \sin\omega t \quad \text{where} \quad f_0 = \frac{F_0}{m}$$

Using the undetermined coefficients method, the particular solution for sinusoidal becomes

$$-\omega^2 X \sin\omega t + \omega_n^2 X \sin\omega t = f_0 \sin\omega t \quad \text{where} \quad x_p(t) = X\sin\omega t$$

From this equation, we can get the solution of X:

$$X = \frac{f_0}{\omega_n^2 - \omega^2} \quad \text{and } X_p(t) \text{ becomes} \quad \frac{f_0}{\omega_n^2 - \omega^2}\sin\omega t$$

Furthermore, we can write the particular solutions with a higher order of derivatives: $\dot{x}_p(t) = \omega X \cos\omega t$ and $\ddot{x}_p(t) = -\omega^2 X \sin\omega t$. Putting the particular solutions in the general solution equation yields:

$$x(t) = A_1 \sin\omega_n t + A_2 \cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin\omega t$$

Then, obtain the coefficients (A_1 and A_2) in terms of the given initial conditions x_0 and v_0

$$x(0) = x_0 = A_2 \quad \text{and} \quad \dot{x}(t) = \omega_n A_1 + \frac{\omega f_0}{\omega_n^2 - \omega^2} = v_0$$

This gives

$$A_1 = \frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \frac{f_0}{\omega_n^2 - \omega^2} \quad \text{and} \quad A_2 = x_0$$

The total solution for a sinusoidal harmonic input can be written as

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2} \right) \sin\omega_n t + x_0 \cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin\omega t \quad \textbf{(*)}$$

Given: $k = 2000$ N/m, $m = 100$ kg, $\omega = 10$ rad/s,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{100}} = 4.472\text{rad/s} \quad f_0 = \frac{F_0}{m} = \frac{10}{100} = 0.1 \text{ N/kg}$$

a) @ rest: $x_0 = 0$ m, $v_0 = 0$ m/s. Using the total solution (*****) that we derived previously:

$$\begin{aligned} x(t) &= \left(0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{\sqrt{20}^2 - 10^2} \right) \sin\sqrt{20}t + 0 + \frac{0.1}{\sqrt{20}^2 - 10^2} \sin 10t \\ &= 2.795 \times 10^{-3} \sin 4.472t - 1.25 \times 10^{-3} \sin 10t \quad \textbf{[Ans.]} \end{aligned}$$

b) @ initial condition of $x_0 = 0.04$ m, $v_0 = 0$ m/s. Similarly, using the total solution (*****) that we derived previously:

$$\begin{aligned} x(t) &= \left(0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{\sqrt{20}^2 - 10^2} \right) \sin\sqrt{20}t + 0.04 \cos\sqrt{20}t + \frac{0.1}{\sqrt{20}^2 - 10^2} \sin 10t \\ &= 0.002795 \sin 4.472t + 0.04 \cos 4.472t - 0.00125 \sin 10t \quad \textbf{[Ans.]} \end{aligned}$$

Question - 5

Compute the response of a shaft-and-disk system to an applied moment of

$$M = 10 \sin 312t$$

as indicated in Fig. 4. Assume that the shaft is initially at rest (zero initial conditions) and $J = 0.6$ kg m^2, the shear modulus is $G = 8.2x10^{10}$ N/m^2, the shaft is 1 m long of diameter 5 cm, and made of steel. Assume the damping ratio of steel is $\zeta = 0.01$.

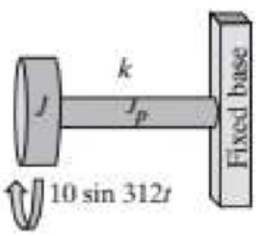

Figure 4: Question - 5: The system.

Solution: Q5

Given input parameters:

$$J = 0.6 \text{ kg m}^2 \quad G = 8.2 * 10^{10} \text{ N/m}^2 \quad L = 1 \text{ m} \quad d = 0.05 \text{ m} \quad a = \frac{\pi * d^2}{4} = 0.0019635 \text{ m}^2$$
$$M = 10 \sin 312t \quad \zeta = 0.01 \quad J_p = \pi * \left(\frac{0.05}{32}\right)^4 = 6.1359 * 10^{-7} \text{ kg m}^2$$

where J is the moment of inertia of the shaft, G is the modulus of rigidity, L is the length of the shaft, d is the diameter of the shaft, a is the area, and J_p is the polar moment of inertia.

The natural frequency and damped natural frequency of the system is

$$\omega_n = \sqrt{\left(\frac{k}{J}\right)} = \sqrt{\left(\frac{GJ_p}{LJ}\right)} = \sqrt{\frac{8.2 * 10^{10} * 6.135910^{-7}}{1}} = 289.5818 \quad \text{rad/s}$$
$$\omega_d = \omega_n \times \sqrt{(1.0 - 0.01^2)} = 289.5673 \quad \text{rad/s}$$

Apply the moments to the equation of motion:

$$0.6\bar{\theta}(t) + \frac{GJ_p}{l}\theta(t) = 10\sin(312t)$$
$$\ddot{\theta}(t) + \frac{GJ_p}{0.6l}\theta(t) = \mathbf{16.67}\sin(\mathbf{312}t)$$

from this equation, we can derive that the mass-normalized force is ($f_0 = 16.67$) and the driving frequency is ($\omega = 312$ rad/s).

The general response in the underdamped case has the form [1]:

$$\theta(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\cos(\omega t - \theta)$$

where forced response:

$$\phi = \tan^{-1}\frac{\omega_d(x_0 - X\cos\theta)}{v_0 + (x_0 - X\cos\theta)\zeta\omega_n - \omega X\sin\theta}, A = \frac{x_0 - X\cos\theta}{\sin\phi},$$
$$\theta = \tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} \text{ and } X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

Therefore, the magnitude and the phase of the steady state response are from window 2.3 [1]:

$$X = \frac{f_0}{\sqrt{\left((\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right)}} = \frac{16.67}{\sqrt{\left((289.5818^2 - 312^2)^2 + (2 * 0.01 * 289.5818 * 312)^2\right)}} = 0.001225 \text{ m}$$
$$\theta = \tan^{-1}\left(\frac{(2\zeta\omega_n\omega)}{((\omega_n^2 - \omega^2))}\right) = \tan^{-1}\left(\frac{(2 * 0.01 * 289.5818 * 312)}{((289.5818^2 - 312^2))}\right) = -0.133193 \text{ rad/s}$$

The phase of the transient is (zero initial conditions: $x_0 = 0$ and $v_0 = 0$)

$$\phi = \tan^{-1}\frac{\omega_d(x_0 - X\cos\theta)}{v_0 + (x_0 - X\cos\theta)\zeta\omega_n - \omega X\sin\theta}$$
$$= \tan^{-1}\frac{289.5673(0 - 0.001225\cos(-0.133193))}{0 + (0 - 0.001225 * \cos(-0.133193))\, 0.01 * 289.5818 - 312 * 0.001225 * \sin(-0.133193)} = -1.437231 \text{ rad/s}$$

The amplitude of the transient is

$$A = \frac{(x_0 - X\cos\theta)}{\sin\phi} = \frac{(0 - 0.001225 * \cos(0.133193))}{\sin(1.437231)} = 0.001225 \text{ m}$$

The total response is

$$\theta(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\cos(\omega t - \theta)$$
$$\theta(t) = 0.001249e^{-0.01289.58t}\sin(289.57t - 1.4732) + 0.0012249\cos(312t - 0.13319) \quad \textbf{[Ans.]}$$

Also, we can visualize the total response of the system from the time 0 to 2 (s) in Fig. 5. As the damping term is less than 1 ($0 < \zeta = 0.01 < 1$) the response of the system is underdamped and it disappears within a time.

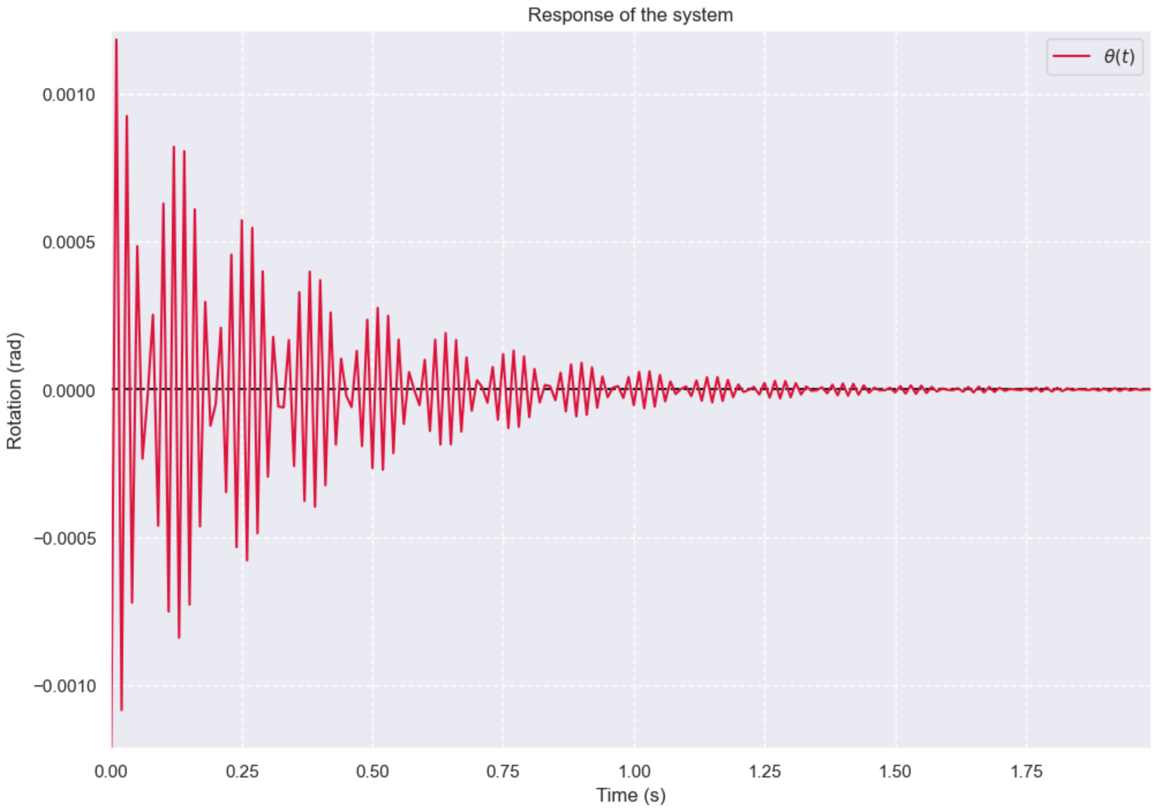

Figure 5: Response of the system.

References

[1] D.J. Inman. *Engineering Vibration*. Prentice Hall, 2001.