How to use this material

This document is a collection of tasks intended for 3 main uses: example solving in front of a black board, in-class example solving tasks, and homework.

For TAs

We propose the following way to use it.

- First, take 2-3 examples and solve it on the black board or on the projected computer screen. The goal is to familiarize the students with the particular problem-solving approach.
- Second, take 1 example and invite a student to solve it on the black board. Repeat with 1-2 students. The goals are to keep instructing the students on task solving, and also to create a sense of immediate assessment - students should feel motivated to try to understand the instructions during the class, as oppose to deferring the task.
- Third, give the whole class an assignment to solve 2-3 tasks by themselves during the class. Judge the number of tasks to solve as to limit the time to 10-15 minutes. After the time is up, ask a student to explain the solution at the black board.
- Avoid giving explicit homework to the students, but suggest them to try and solve the remaining exercises by themselves. Homework based on these exercises cannot be graded.

1 ODE to State Space

Given the following ODEs, transform them to state-space representation:

$$6\ddot{y} - 2\dot{y} + 1.5y = 0 \tag{1}$$

$$3\ddot{y} + 3\dot{y} + 5.5y = 1 \tag{2}$$

$$-\ddot{y} + 10\dot{y} - 20y = 2 \tag{3}$$

$$\dddot{y} - \ddot{y} + 10\dot{y} - 20y = 2 \tag{4}$$

$$10\dddot{y} + 5\ddot{y} - 2\dot{y} + 20y - 20 = 0 \tag{5}$$

$$\begin{cases} \dddot{y} - 2\ddot{y} + 5\dot{y} + 100y - 1 = 2 \\ \dot{z} + 5z = 10 \end{cases} \tag{6}$$

$$\begin{cases} -2\dddot{y} - 5\ddot{y} + 15\dot{y} + 2y - 10 = 0 \\ 5\ddot{z} + \dot{z} + 5z = 10 \end{cases} \tag{7}$$

$$\begin{cases} \dddot{y} + 3\ddot{y} + 7.5\dot{y} - 2\dot{z} + 2y - z - 10 = 0 \\ 5\ddot{z} + \dot{z} + 5z - y = 10 \end{cases} \tag{8}$$

$$\begin{cases} -2\,\dddot{y} + 2\ddot{y} + \dot{y} - \dot{z} + 12y + 8z = 6 \\ 25\ddot{z} + 16\dot{z} + 27\dot{y} + z + 4y = -6 \end{cases} \tag{9}$$

2 State Space to ODE

Given the following ODEs in State-Space form, transform them to a single ODE:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -7 & -5 \end{bmatrix} \mathbf{x} \tag{10}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 10 & 10 \end{bmatrix} \mathbf{x} \tag{11}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 4 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 5 \end{bmatrix} \tag{12}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -20 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix} \tag{13}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 10 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{14}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \mathbf{x} \tag{15}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 10 \end{bmatrix} \mathbf{x} \tag{16}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} \tag{17}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & -5 & 7 & 10 \end{bmatrix} \mathbf{x} \tag{18}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & -3 & 10 & 5 & 10 \end{bmatrix} \mathbf{x} \tag{19}$$

3 State Space to ODE (Advanced)

Given the following ODEs in State-Space form, transform them to a single ODE, and write explicit change of variables (linear transformations) that are needed to transform initial conditions of the state space form to the initial conditions of the single ODE, and change of variables (linear transformations) that are needed to transform the solution of the ODE to the solution of the equation in the state-space form:

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & 3 \\ -7 & -5 \end{bmatrix} \mathbf{x} \tag{20}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 1 \\ -10 & -20 \end{bmatrix} \mathbf{x} \tag{21}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & -10 \\ 2 & -5 \end{bmatrix} \mathbf{x} \tag{22}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & 1 & 2 \\ 3 & -10 & 2 \\ -1 & 0 & -10 \end{bmatrix} \mathbf{x} \tag{23}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 20 & 3 & 4 \\ 3 & 4.5 & 7 \\ 2 & 10 & -12 \end{bmatrix} \mathbf{x} \tag{24}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -25 & 10 & 7 \\ 3 & -3 & -5 \\ -9 & 7 & 5 \end{bmatrix} \mathbf{x} \tag{25}$$

4 Stability of simple ODEs

Check stability of the following ODEs:

$$\dot{\mathbf{x}} = \begin{bmatrix} -11 & 0 \\ 5 & -50 \end{bmatrix} \mathbf{x} \tag{26}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 4 \\ -1 & -20 \end{bmatrix} \mathbf{x} \tag{27}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -12 & 10 \\ 1.5 & -7 \end{bmatrix} \mathbf{x} \tag{28}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -50 & 0 & 2 \\ 0 & -30 & 20 \\ -1 & 0 & -30 \end{bmatrix} \mathbf{x} \tag{29}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 20 & 1 & 5 \\ 5 & 14.5 & 7 \\ 5 & -2 & -12 \end{bmatrix} \mathbf{x} \tag{30}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -25 & 10 & 2 \\ 4 & -30 & -5 \\ -4 & 7 & -2 \end{bmatrix} \mathbf{x} \tag{31}$$

$$2\ddot{y} + 7\dot{y} + 7y = 0 \tag{32}$$

$$-\ddot{y} + 9\dot{y} + 10y = 10 \tag{33}$$

$$\ddot{y} - 6\dot{y} + 2y = 2 \tag{34}$$

$$2\dddot{y} - \ddot{y} + 4\dot{y} + 8y = 2 \tag{35}$$

$$10\dddot{y} + 5\ddot{y} + 2\dot{y} + 30y + 15 = 0 \tag{36}$$

5 ODE to Transfer Functions

Given the following ODEs, transform them to transfer function (from u to y) representation:

$$-\ddot{y} - 10\dot{y} + 1.5y = u \tag{37}$$

$$2\ddot{y} + 10\dot{y} + 2y = 5u \tag{38}$$

$$\ddot{y} + 4.5\dot{y} - y = -u \tag{39}$$

$$\dddot{y} + 3\ddot{y} + 2\dot{y} + 8y = 2u \tag{40}$$

$$5\dddot{y} - 4\ddot{y} - 5\dot{y} - 3y = u \tag{41}$$

$$\begin{cases} 2\dddot{y} + 3\ddot{y} + 7\dot{y} + 12y - u = 0 \\ \dot{z} + 5z = u \end{cases} \tag{42}$$

$$\begin{cases} 3\dddot{y} + 9\ddot{y} + 2\dot{y} + 6y = u + v \\ 5\ddot{z} + \dot{z} + 5z = v \end{cases} \tag{43}$$

6 State-Space to Transfer Functions

Given the following ODEs in State-Space form, transform them to a transfer function:

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -7 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x} \end{cases} \tag{44}$$

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{cases} \tag{45}$$

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 10 \\ -3 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u \\ y = \begin{bmatrix} -1 & -1 \end{bmatrix} \mathbf{x} \end{cases} \tag{46}$$

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 8 & -20 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \mathbf{x} \end{cases} \tag{47}$$

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 4 & 11 & -20 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x} \end{cases} \tag{48}$$

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \mathbf{x} \end{cases} \tag{49}$$

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -3 & 2 & 10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} \mathbf{x} \end{cases} \tag{50}$$

$$\begin{cases} \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} -10 & -10 & 0 \end{bmatrix} \mathbf{x} \end{cases} \tag{51}$$