On representations of Gaussian integers by primes and powers of $1 + i$

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Conjecture. Let $z = a + bi \in \mathbb{Z}[i]$ with $a + b$ odd. Then z admits a representation of the form

$$z = \pi + \varepsilon(1 + i)^k,$$

where π is a Gaussian prime, $\varepsilon \in \{\pm 1, \pm i\}$ is a unit, and $k \geq 0$ is an integer. No counterexamples were found within the tested range ($a^2 + b^2 < 10^{11}$).

Possible strategy to disprove. As in [1, 2] we could try to construct for each unit $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = 1, -1, i, -i$, a covering system of moduli $\{m_{ij}\}_{j=1}^4$ together with pairwise disjoint sets of distinct Gaussian primes $\{p_{ij}\}_{j=1}^4$, such that

$$(1 + i)^{m_{ij}} \equiv 1 \pmod{p_{ij}}$$

for all relevant indices. One would then attempt to combine these congruences to force $\pi = z - \varepsilon_j(1 + i)^k$ to be divisible by a Gaussian prime from the corresponding set $\{p_{ij}\}$, for each $j = 1, 2, 3, 4$, depending on the residue class of k. No such covering systems satisfying the required disjointness and congruence conditions were yet found.

References

[1] P. Erdos. On integers of the form $2^k + p$ and some related problems. *Summa Brasil. Math*, 2:113–123, 1950.

[2] Z.-W. Sun. On integers not of the form $\pm p^a \pm q^b$. *Proceedings of the American Mathematical Society*, 128(4):997–1002, 2000.

*The most recent version of this file is on `https://kilin-math.github.io/assets/numbers/primes.pdf`