Basic concepts in mathematics: Diary

Nikita Kalinin, Diego Armando Sulca, Raphaël Fesler, Shuliang Gao

Lexicographic induction
(radicals + strokes)

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Guangdong Technion Israel Institute of Technology
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Contents

0 Preface

From a cup of coffee, a spoonful of coffee is poured into a cup of milk. Then a spoonful of the resulting mixture is poured back into the cup of coffee. Which is greater: the amount of milk in the cup of coffee or the amount of coffee in the cup of milk?

This course is intended for first-year students in mathematics and computer science; they have to learn how to read and write proofs.

It is not obvious at all why we need to prove something (professors demand it, but why?). The truth is that by proving we understand things better and discover new beauties (e.g., the formula $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + .. + n)^2$).

So, to reconcile the concept of proof with students, it is better to introduce proofs in questions whose answers are not obvious and are debatable. Examples logic puzzles, games (who has a winning strategy?), impossibility proofs (e.g., tiling a chessboard missing opposite corners with dominoes), and induction. These topics occupy the first few lectures.

How do you learn to prove? Let us use the metaphor "Mathematics is a language". When you learn a foreign language, you have different activities: listening, speaking, reading, writing. The same is true for mathematics. To study it, students should spend time reading and writing mathematics, listening to lectures, thinking about problems, and discussing ideas.

Also, mathematicians love mathematics because proofs are beautiful!

Solution to the coffee-milk problem:

Let's try to guess the answer. To do that, consider an extreme case (this is the first idea). Suppose there is just one spoonful of liquid in each cup. Then after pouring the coffee into the milk, we take the entire mixture back. The mixture will be uniform, so the amount of coffee and milk will be equal. Will it always be equal?

Since one spoonful was poured "there and back," the total volume of liquid in each cup did not change (this is the second idea).

Therefore (the third idea), the amount of coffee lost equals the amount of milk gained.

The volumes of coffee and milk in the cups can be different, you can pour the spoon back and forth ten times, you can even stir the mixture poorly — it doesn't matter: the amount of milk in the coffee will always equal the amount of coffee in the milk!

0.1 :: on collaboration and the use of AI

Collaboration. You are encouraged to discuss problems with classmates: compare approaches, explain ideas to one another, and ask for critique. Explaining your reasoning often reveals subtleties and gaps that you can then address. However, each student must write up their own solution independently, in their own words, after any discussion. Do not share written solutions or allow others to copy your work.

Attempt first. Before seeking help from classmates or AI tools, make a genuine attempt on each problem: write down an outline, partial calculations, or a strategy you tried (even if it failed). Learning to prove requires practice—like training for a sport—so expect to try, err, and revise.

AI: permitted uses. You may use AI tools to: (i) spot logical gaps or unclear steps in a solution you have already written, (ii) improve clarity and writing style, (iii) receive high-level hints about relevant definitions or theorems. When you do so, begin with your own draft (typed or a photo) and ask for feedback on that draft.

AI: not permitted uses (unless explicitly allowed). Do not ask AI to produce full solutions for graded assignments or to translate someone else's solution into "your own words." Do not paste problem statements and accept verbatim answers. Use AI as a reviewer, not as an author. If you are unsure whether a use is allowed, ask the instructor.

Reflection Logs. It is useful to reflect on how you use AI, so record the tool and its purpose in your diary, and then track how it affects your understanding and your ability to solve problems, write solutions, and read material without any outside help. Note that during the exams or midterms you cannot rely on anything except yourself.

Academic integrity. Copying another person's solution or submitting AI-generated solutions as your own is plagiarism. Violations will be handled under the university's academic integrity policy.

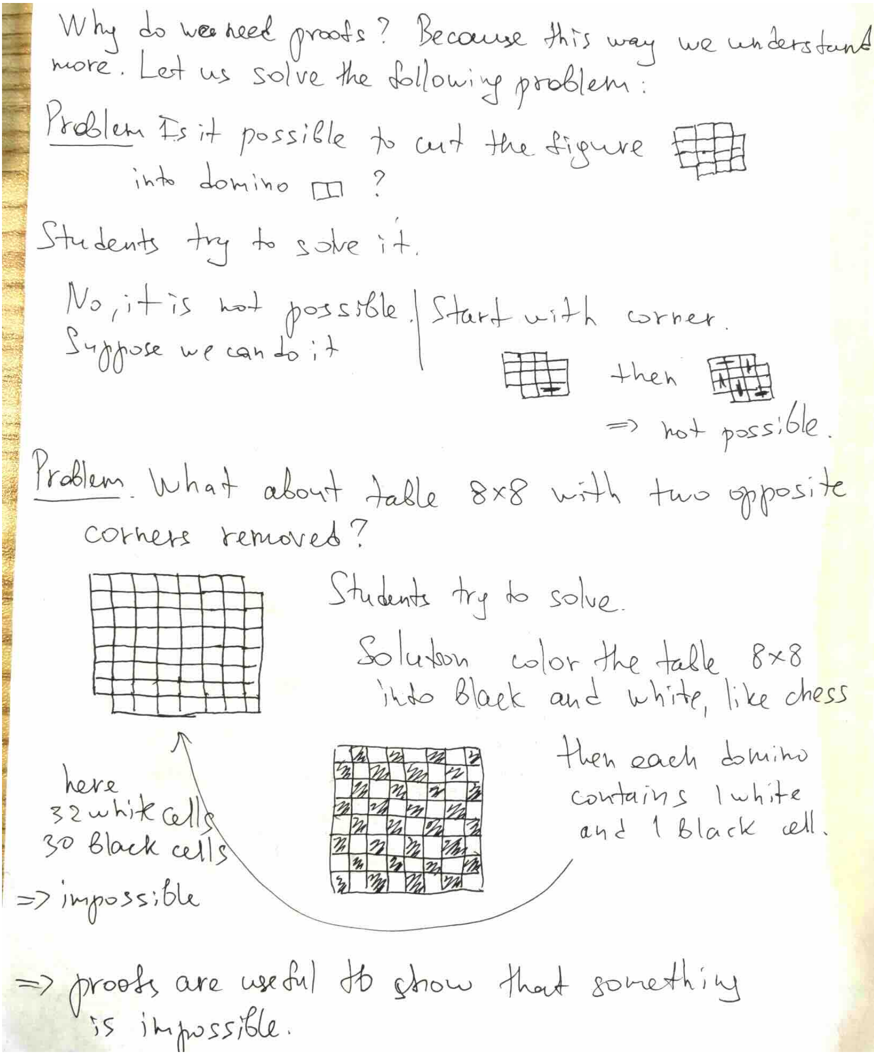
0.2 :: credits

The first chapter consists of edited AI-translated excerpts from the excellent book (in Russian) of Kanel-Belov and Kovaldgi "How to solve non-standard probelms" https://old.mccme.ru/free-books/olymp/KanKov.pdf

1 Methods of proving

1.1 :: parity, induction

1.1.1 :: :: parity



Why do we need proofs? Because this way we understand more. Let us solve the following problem:

Problem Is it possible to cut the figure into domino ?

Students try to solve it.

No, it is not possible. Suppose we can do it

Start with corner. then

$\Rightarrow$ not possible.

Problem. What about table 8×8 with two opposite corners removed?

Students try to solve.

Solution color the table 8×8 into black and white, like chess

then each domino contains 1 white and 1 black cell.

here 32 white cells 30 black cells $\Rightarrow$ impossible

$\Rightarrow$ proofs are useful to show that something is impossible.

Many problems become much easier once one notices that some quantity has a fixed *parity* (it is always even or always odd). Once a parity is fixed, any situation in which that quantity would have the opposite parity is impossible. Sometimes one has to construct this quantity, for example by considering the parity of a sum or product, by pairing objects up, by noticing an alternating pattern, or by colouring objects in two colours.

Example 1. *A grasshopper makes jumps of length 1 m along a straight line and eventually returns to its starting point. Show that it made an even number of jumps.*

Solution. If the grasshopper ends where it started then the number of jumps to the right must equal the number of jumps to the left. Consequently the total number of jumps is even.

Example 2. *Does there exist a closed broken line with seven segments that crosses each of its segments exactly once?*

Solution. Suppose such a broken line existed. Any two crossing segments can be paired. The number of segments must therefore be even, which contradicts the assumption that there are seven segments.

Example 3. *Martians may have any number of arms. One day all Martians joined hands in such a way that no free arms remained. Prove that the number of Martians with an odd number of arms is even.*

Solution. Call Martians with an even number of arms *even* and those with an odd number of arms *odd*. Since the hands form pairs, the total number of hands is even. The total number of hands of the even Martians is clearly even, so the total number of hands of the odd Martians must also be even. But each odd Martian contributes an odd number of hands, so there must be an even number of them.

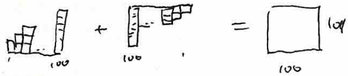
Such pictures are called *graphs*, they consist of *vertices* (representing Martians) and *edges* (representing a handshaking pair of hands). The number of edges incident to a vertex (i.e. the number of hands of a Martian) is called the *degree* (or *valency*) of a vertex. See Section 5.

We proved that in each graph the number of vertices of odd degree is even.

1.1.2 :: :: induction

How to compute $1+2+3+\dots+100$?

Easy:



$= 100 \cdot 101$

$\Rightarrow 1+2+\dots+100 = \frac{100(101)}{2}$

The same method works for each natural n

$1+2+\dots+n = \frac{n(n+1)}{2} = S(n)$

	1	2	3	4	…
$S(n)$	1	3	6	10	…

Let us find $1^3+2^3+3^3+\dots+n^3$

try: $n=1$ $\quad 1^3=1$

$n=2$ $\quad 1^3+2^3=9$

$n=3$ $\quad 1^3+2^3+3^3=36$

$n=4$ $\Rightarrow 1^3+2^3+3^3+4^3=36+64=100$

Conjecture: $1^3+\dots+n^3=\left(\frac{n(n+1)}{2}\right)^2$ from here

But how to prove it? May be it is a coincidence.

It is true for $n=1,2,3,4$.

Let us prove that if it is true for $n=k$ then it is true for $n=k+1$.

Indeed $1^3+2^3+\dots(k+1)^3 = 1^3+\dots+k^3+(k+1)^3 =$

$\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 = \dots \left(\frac{(k+1)(k+2)}{2}\right)^2$

then it is true for all natural n.

The method of mathematical induction is used to prove statements of the form "For every natural number n a certain property holds." Such a statement can be viewed as an infinite chain of assertions: "For $n=1$ the property holds", "For $n=2$ the property holds", and so on. The first assertion in the chain is called the base (or the foundation) of the induction and is usually easy to check. One then proves the induction step: "If the assertion with number n is true, then the assertion with number $n+1$ is true." Sometimes one needs a stronger form of the induction step: "If all assertions with numbers from 1 to n are true, then the assertion with number

$n+1$ is true." There is also the technique of *inductive descent*, in which one proves that if an assertion with number n (with $n > 1$) can be reduced to one or several assertions with smaller numbers and the first assertion is true, then all assertions are true.

If both the base and the induction step have been proved, then all assertions in the chain hold; this is the principle of mathematical induction.

Example 1. *Prove that the number consisting of 243 consecutive ones is divisible by 243.*

Solution. Notice that $243 = 3^5$. We shall prove a more general statement: the number consisting of 3^n consecutive ones is divisible by 3^n. For $n = 1$ the assertion says that 111 is divisible by 3, which is true. Suppose the number consisting of 3^{n-1} ones is divisible by 3^{n-1}. Write

$$\underbrace{11\cdots 1}_{3^n \text{ times}} = \underbrace{11\cdots 1}_{3^{n-1} \text{ times}} \times \underbrace{10\cdots 010\cdots 01}_{\text{a block containing only one non zero digit}}.$$

It is not hard to check that the second factor on the right–hand side is divisible by 3. Multiplying a multiple of 3^{n-1} by a multiple of 3 yields a multiple of 3^n. Therefore the number of 3^n ones is divisible by 3^n, completing the induction.

Example 2. *Several lines and circles are drawn in the plane. Prove that the regions into which the plane is divided can be coloured in two colours so that adjacent regions (sharing a segment or an arc) are coloured differently.*

Solution. First erase all the lines and circles, remembering where they were. Colour the entire plane one colour. Then restore the boundaries one by one, recolouring the regions they divide. When adding a line, recolour in the opposite colour all regions on one side of it and leave unchanged those on the other side. When adding a circle, recolour all regions lying inside it and leave unchanged those outside. In this way each time you add a boundary the recoloured regions lie on one side only. Consequently any two neighbouring regions (sharing part of a boundary) always have different colours.

Example 3. *Prove that if $x + \frac{1}{x}$ is an integer, then $x^n + \frac{1}{x^n}$ is an integer for all $n \geq 0$.*

Solution. Set $T_n = x^n + \frac{1}{x^n}$. Note that $T_0 = 2$ and $T_1 = x + 1/x$ are integers. Observe that

$$T_n T_1 = (x^n + \tfrac{1}{x^n})(x + \tfrac{1}{x}) = x^{n+1} + \tfrac{1}{x^{n+1}} + x^{n-1} + \tfrac{1}{x^{n-1}} = T_{n+1} + T_{n-1}.$$

Thus $T_{n+1} = T_n T_1 - T_{n-1}$. By induction on n this recurrence shows that all T_n are integers.

Example 4. (if time permits) *Five robbers have obtained a sack of gold sand. They wish to divide it so that each robber is sure he received at least one fifth of the gold. They have no measuring instruments, but each can judge by eye the amount of a pile of sand. Opinions about the size of the piles may differ. How can they divide the loot?*

Solution (First method). First two robbers divide the sand between themselves: one divides the sack into two piles that he believes equal, and the other chooses his pile. Each of these two divides his share into four equal (to his mind) parts, and the third robber takes one part from each. Now these three each divide their share into three parts and the fourth robber takes one part from each. Finally these four divide their shares into two parts and the fifth robber takes one part from each. Each robber can check that the portion he receives is at least one fifth according to his judgment.

Solution (Second method). Find the most modest robber and give him his portion first. To do so, ask the first robber to separate what he believes to be $1/5$ of the sack. Ask the second robber whether the separated part is larger than $1/5$: if he thinks it is larger, have him reduce it to what he considers $1/5$; if he thinks it is not larger, ask the third robber, and so on. When someone finally agrees that the separated part is exactly $1/5$, give that part to the last person who modified it. Among the remaining robbers find the most modest of those and repeat. In the end every robber receives a portion he believes is at least $1/5$ of the original amount.

1.1.3 :: :: problems for tutorial

1. You have coins of 3 HKD and 5 HKD. Prove that any number of HKD greater than seven can be exchanged for coins of 3 and 5 HKD.

2. Several lines divide the plane into regions. Each line grow hair on one side. Prove that there is a region all of whose boundaries have hair “outside”. [1]

[1] Comment: Induct on the number of lines. Removing one line yields a configuration where the claim holds; then restore the line and pick an appropriate subregion on the hatched side.

3. From a 128×128 square one unit square was removed. Prove that the remaining shape can be tiled with L shaped trominoes consisting of three unit squares.

4. For every natural k prove the inequality $2^k > k$.

5. Prove the Cauchy–Schwarz inequality in the form

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

where $x_1, \ldots, x_n$ are non negative numbers. [2]

1.1.4 :: :: problems for workshop

1. Can one break 25 HKD into ten coins of denominations 1, 3 and 5 HKD?[3]

2. Nine gears are arranged in a circle, each meshing with the next. The first meshes with the second, the second with the third, ..., the ninth with the first. Can they all rotate at the same time? What happens if there are n gears?[4]

3. A row contains 100 towers. You may interchange any two towers that have exactly one tower between them. Is it possible in this way to reverse the entire order of the towers?[5]

4. Six numbers $1, 2, 3, 4, 5, 6$ lie on the table. You are allowed to add 1 to any two of them. Can all the numbers eventually be made equal?[6]

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[2] Comment: A common proof uses induction on n by first proving the case where n is a power of two and then reducing the general case to the nearest lower power of two. This problem invites the reader to explore that technique. See details in Section 5, page 52.

[3] Comment: Let x, y and z be the numbers of 1–, 3– and 5–HKD coins. Then $x + 3y + 5z = 25$ and $x + y + z = 10$. Subtracting yields $2y + 4z = 15$, which has no integer solutions. Therefore it is impossible.

[4] Comment: An odd number of meshed gears arranged in a cycle cannot all turn, because each contact reverses the sense of rotation. For an even number of gears a consistent rotation is possible.

[5] Comment: Label the positions $1, \ldots, 100$. A permitted move swaps the towers in positions i and $i + 2$, both of which have the same parity (both odd or both even). Consequently each tower always occupies squares of the same parity as its starting position. In the reversed arrangement the tower originally at position 1 would have to move to position 100 and hence to a square of opposite parity. This is impossible, so a complete reversal cannot be achieved.

[6] Comment: Each move increases the sum of the numbers by 2. If eventually all six numbers were equal to k, then $6k = 1 + 2 + 3 + 4 + 5 + 6 + 2t = 21 + 2t$. The left side is even, while $21 + 2t$ is odd for all t. Hence the numbers can never all become equal.

5. All dominoes from the standard set are laid out in a single chain according to the usual rule (neighbouring halves show the same number). One end of the chain has a five. What can be at the other end?[7]

6. Can a line that does not pass through any vertex of an 11–gon intersect all of its sides?[8]

7. On a table stand seven overturned cups. You may simultaneously turn over any two cups. Is it possible to end up with all the cups upright?[9]

[7] Comment: In the usual "double six" set the tiles are the pairs (i, j) with $1 \leq i \leq j \leq 6$. Each number $1, \dots, 6$ occurs exactly six times. Consider the graph whose vertices are the numbers and whose edges correspond to dominoes. In this graph every vertex has even degree, so the tiles can be arranged in a closed Eulerian circuit. Cutting this circuit yields an open chain whose two ends are identical. Hence if one end displays a five, the other end must also display a five.

[8] Comment: In any polygon the number of intersections of a straight line with the sides is even (becasue the line goes inside the polygon, then outside, then inside,... finally it goes outside). Since 11 is odd, no such line exists.

[9] Comment: The parity of the number of overturned cups changes by 0 or 2 at each move. Starting with seven (odd) and wanting to end with zero (even) is impossible.

1.1.5 :: :: 1st homework

Read:

Paul Zeitz - The Art and Craft of Mathematical Problem Solving, pp. 13-15.

The Frog Problem

- The frog problem is a classic Russian math circle problem.
- Three frogs are situated at 3 of the corners of a square. Every minute, 1 frog is chosen to leap over another chosen frog, so that if you drew a line from the starting position to the ending position of the leaper, the leapee is at the exact midpoint.
- Will a frog ever occupy the vertex of the square that was originally unoccupied?
- How can we effectively investigate this problem?
- Graph paper allows us to attach numbers to the positions of the frogs. Once we have numbers, we can employ arithmetical and algebraic methods. Thus, place the frogs at (0, 0), (0, 1), and (1, 1). The question now is, can a frog ever reach (1, 0)?
- Thinking about the appropriate venue for investigation is an essential starting strategy for any problem.
- Another investigative idea: Use colored pencils to keep track of individual frogs. This adds information, as it allows us to keep track of 1 frog at a time. Color the (1, 1), (0, 1), and (0, 0) frogs red, blue, and green, respectively.
- Notice, by experimenting, that the red frog only seems to hit certain points, forming a larger (2-unit) grid.
- Some of the coordinates that the red frog hits are (1, 1), (1, 3), (1, −1), (−1, 1), (−1, −1), and (−1, −3). They are all odd numbers!
- Likewise, the blue frog only hits certain points on a 2-unit grid, including (0, 1), (2, 1), (4, 1), and (0, −1); these are all of the form (even, odd).
- Likewise, the green frog only hits (even, even) points.
- On the other hand, the missing southeast vertex was (1, 0), which has the form (odd, even). It seems as though it is impossible, but how can we formulate this in an airtight way?
- It is often very profitable to contemplate parity (oddness and evenness).
- The essential reason for this is that a parity focus reduces a problem from possibly infinitely many states to just 2.
- Parity involves the number 2. Where in this problem do we see this number? In doubling, because of the symmetry of the way the frogs leap. When the leaper jumps over the leapee, she adds twice the horizontal displacement to her original horizontal coordinate. The same holds for vertical coordinates.
- So when a frog jumps, its coordinates change by even numbers!
- For example, suppose the red (1, 1) frog jumps over the green frog at (0, 0). The horizontal and vertical displacements to the leapee are both −1 (since it is moving left and down), so the final change in coordinates will be −2. The horizontal coordinate will be $1 + -2 = -1$, and the vertical will also be −1.
- Suppose now that the red frog jumps over the blue frog, which is (0, 1). The horizontal displacement is +1, and the vertical displacement to the target is +2. So the new horizontal coordinate will be −1 (the starting value) $+ 2 \times 1 = +1$, and the new vertical coordinate will be −1 (the starting value) $+ 2 \times 2 = 3$. Thus the red frog jumps from (−1, −1) to (1, 3).
- In general, when a frog jumps, we will take its starting x-coordinate and add twice the horizontal displacement to its target. Likewise, we take its starting y-coordinate and add twice the vertical displacement to the target. These displacements may be positive, negative, or zero.
- In other words, you take the starting coordinates and add even numbers to them. But when you add an even number to something, its parity does not change!
- So the (odd, odd) frog—the red frog—is destined to stay at (odd, odd) coordinates, no matter what.

Suggested Reading

Polya, *How to Solve It.*

Zeitz, *The Art and Craft of Problem Solving*, chap. 2.

Questions to Consider

1. Write the numbers from 1 to 10 in a row and place either a minus or a plus sign between the numbers. Is it possible to get an answer of zero?
2. A group of jealous professors is locked up in a room. There is nothing else in the room but pencils and 1 tiny scrap of paper per person. The professors want to determine their average (mean, not median) salary so that each can gloat or grieve over his or her personal situation compared to the others. However, they are secretive people and do not want to give away salary information to anyone else. Can they determine the average salary in such a way that no professor can discover any fact about the salary of anyone but herself? For example, even facts such as "one professor earns less than \$90,000" are not allowed.

Write:

Problem 1. Write the full solution (with all details) of the problem that we cannot cut a square 4×4 without opposite corners into domino. Write the solution where we use case-by-case strategy, without using coloring.

Problem 2. Write the solution of the problem that we cannot cut a square 8×8 without opposite corners into domino (you can use coloring).

Problem 3. Show that you can cut a square in n squares for each $n > 6$.

Problem 4. The numbers $1, 2, \ldots, 101$ are written on a blackboard. You are allowed to erase any two numbers and write their difference in their place. After repeating this operation 100 times only one number remains. Prove that this number cannot be zero.

1.2 :: pigeonhole principle, correspondence, invariants

From a hundred rabbits, you can never assemble a horse;
a hundred suspicions do not consitute a proof.

Crime and Punishment, F. Dostoevsky

1.2.1 :: :: proof by contradiction

One of the most widely used techniques in elementary mathematics is the *proof by contradiction*. The general strategy is to assume that the statement to be proved is false and then show that this assumption leads to an impossibility. Having reached a contradiction, one concludes that the original statement must in fact be true.

Example 1. *Prove that there are infinitely many prime numbers.*

Solution. Suppose the contrary, namely that there are only finitely many primes. List them as $p_1, p_2, \ldots, p_n$. Consider the number

$$N = p_1\, p_2 \cdots p_n + 1.$$

By construction N is not divisible by any of the primes p_i. Therefore N has no prime divisors at all, which contradicts the fundamental fact that every integer greater than 1 has at least one prime divisor. Hence there cannot be only finitely many primes.

Example 2. *Five boys found nine mushrooms. Prove that at least two of them must have found the same number of mushrooms.*

Solution. Assume that the boys all found different numbers of mushrooms. Order them by increasing number of mushrooms: the first boy picked at least 0 mushrooms, the second at least 1, the third at least 2, the fourth at least 3 and the fifth at least 4. Altogether they would have picked at least $0 + 1 + 2 + 3 + 4 = 10$ mushrooms, contradicting the fact that there were only nine mushrooms. Thus at least two boys must have collected the same number of mushrooms.

Example 3. *Prove that there does not exist a tetrahedron (triangular pyramid) in which each edge is adjacent to an obtuse angle of one of its faces.*

Solution. Suppose such a tetrahedron exists. In any triangle the side opposite an obtuse angle is the largest side. Therefore each edge of the tetrahedron must be strictly shorter than some other edge that is adjacent to the obtuse angle. Since the number of edges in a tetrahedron is finite, this strict inequality cannot cycle indefinitely; the assumed configuration leads to an infinite descending chain of lengths, which is impossible. Hence no such tetrahedron exists.

Example 4. *Prove that* $\log_2 3$ *is an irrational number.*

Solution. Assume otherwise and write $\log_2 3 = \frac{p}{q}$ with p and q positive integers. Then $2^{p/q} = 3$, i.e. $2^p = 3^q$. The left side is even while the right side is odd. This contradiction shows that $\log_2 3$ is irrational.

1.2.2 :: :: pigeonhole principle

In its simplest form the pigeonhole principle says that if ten rabbits sit in nine boxes, then some box must contain at least two rabbits. A more general formulation is: "If n rabbits sit in k boxes, then there exists a box containing at least $\lceil n/k \rceil$ rabbits and a box containing at most $\lfloor n/k \rfloor$ rabbits." Do not be put off by fractional rabbits—if $10/9$ rabbits must sit in a box, then in fact at least two do.

Proof of the principle. Assume that each of the k boxes contains strictly fewer than n/k rabbits. Then altogether there are fewer than $(n/k)\,k = n$ rabbits, which contradicts the assumption that there are n rabbits. This simple argument illustrates why similar reasoning occurs throughout combinatorics.

The pigeonhole principle may seem obvious, but in order to apply it one must sometimes think carefully about what plays the role of "rabbits" and what plays the role of "boxes". For example, if each element of a set A corresponds to exactly one element of a set B, then one may call the elements of A rabbits and those of B boxes.

The principle also has continuous versions. For instance: "If n rabbits eat m kg of grass, then some rabbit ate at least m/n kg and some rabbit ate at most m/n kg." In this formulation the rabbits play the role of boxes for the grass, while the grass plays the role of rabbits sitting in boxes.

Example 1. *There are 400 students in a school. Prove that at least two of them were born on the same day of the year.*

Solution. There are 366 days in a year. Think of the students as rabbits and the days of the year as boxes. Then some box must contain at least $\lceil 400/366 \rceil = 2$ students, i.e. two students share a birthday.

Example 2. *The ocean covers more than half of the Earth's surface. Prove that somewhere in the ocean there are two antipodal points (diametrically opposite points on the globe).*

Solution. Reflect the ocean through the centre of the Earth. The union of the ocean and its reflection covers more than the entire surface of the Earth, so there is a point that lies in both. Such a point and its antipode are both in the ocean.

Example 3. *Sixty five schoolchildren came to an exam. They were given three tests. On each test the score was one of* 2, 3, 4 *or* 5*. Must there be two pupils who got the same grades on all three tests?*

Solution. There are $4^3 = 64$ possible triples of grades. Since 65 pupils take part, by the pigeonhole principle at least two of them must have identical triples of grades.

Example 4. *Prove that among any five people there are two who have the same number of acquaintances among these five people (a person may be unacquainted with everyone).*[10]

1.2.3 :: :: invariants

Invariant — a quantity which does not change as a result of certain operations (for example, cutting and rearranging parts of a figure does not change the total area). If an invariant distinguishes two configurations, then it is impossible to pass from one to the other. As an invariant one can use *parity* or a *coloring*. In problems about the sum of digits, the remainders upon division by 3 or 9 are often used.

Semi-invariant — a quantity that changes only in one direction (that is, it can only increase or only decrease). The concept of a semi-invariant is often used when proving that a process terminates.

Example 1. A wonder-apple tree grows both bananas and pineapples. In one move one is allowed to pick two fruits from it. If two bananas or two pineapples are picked, then one more pineapple grows; if one banana and one pineapple are picked, then one banana grows. In the end there remains one fruit. Which fruit is it, if it is known how many bananas and pineapples were there initially?

[10] Comment: The possible numbers of acquaintances are 0,1,2,3,4. If 0 is among them, then 4 is not possible, so we have only 4 possibilities for 5 people. The same is true when there is no person with 0 acquaintances.

Solution. The parity of the number of bananas does not change, therefore, if the number of bananas was even, the remaining fruit is a pineapple, and if it was odd, then it is a banana.[11]

Example 2. In one cell of a 4×4 square table there is a minus sign, and in all the other cells there are pluses. It is allowed to change at once the signs in all cells of any one row or any one column. Prove that, no matter how many such changes of signs we perform, it is impossible to obtain a table consisting entirely of pluses.

Solution. Let us replace the sign "$+$" by the number 1 and the sign "$-$" by -1. Note that the *product of all numbers in the table* does not change when the signs in all numbers of any one row or column are flipped. In the initial position this product is -1, and in the table consisting entirely of pluses it is $+1$, which proves the impossibility of transition.[12]

Example 3. On a straight line there stand two chips: on the left a red one, on the right a blue one. It is allowed to perform either of two operations: insert two chips of the same color next to each other (between any two chips or at the edge), or remove a pair of neighboring chips of the same color (between which there are no other chips). Is it possible, by using such operations, to leave exactly two chips on the line, a blue one on the left and a red one on the right?

Solution. Consider the number of pairs of chips of different colors (not necessarily neighboring), where the left chip is red. Note that the parity of this number does not change. In the initial configuration this number is odd (equal to 1), but in the desired configuration it is even (0). Therefore it is impossible to reach the desired configuration.[13]

Example 4. On the island of Grey-Brown-Raspberry live chameleons: 13 grey, 15 brown, and 17 raspberry-colored ones. If two chameleons of different colors meet, then both change their color to the third one. Can it happen that at some moment all the chameleons on the island become of the same color?

Hint. Let us denote by B, G, and M the numbers of brown, grey, and raspberry chameleons, respectively. Prove that the remainders upon division by 3 of the

[11] Comment: The invariant is the parity of the number of bananas. The text implicitly assumes that the process continues until one fruit remains.

[12] Comment: The invariant here is the product of all entries in the table, equal to -1 at the start.

[13] Comment: The invariant is the parity of the number of red–blue pairs in that order.

differences $B - G$, $G - M$, and $M - B$ do not change.[14]

Example 6. Is it possible to cut a round disc into several parts and assemble from them a square? (The cuts are pieces of straight lines and arcs of the circle.)

Solution. Consider the invariant: the difference of the sums of the lengths of concave and convex boundary arcs of all the pieces. This quantity does not change when a piece is cut into two, nor when two pieces are joined together. For a unit round disc this invariant equals 2π, and for a square it equals 0. Therefore "squaring the circle" is impossible.[15]

1.2.4 :: :: problems for tutorial

1. Is it possible to connect five cities by roads so that each city is connected with exactly three others? [16]

2. Prove that there does not exist a polyhedron whose number of faces is odd and such that each face has an odd number of vertices. [17]

3. In each cell of an $m \times k$ rectangular array there is a number. The sum of the numbers in each row and in each column is 1. Prove that $m = k$. [18]

4. A class contains 25 students. It is known that among any three students there are two who are friends. Prove that there is a student who has at least 12 friends. [19]

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[14] Comment: The invariant consists of the three residues modulo 3 of the color differences; since they cannot all become zero simultaneously, a single color for all chameleons is impossible.

[15] Comment: The invariant compares total curvature (arc excess vs. deficit) of the boundary. Since it is preserved under cutting and joining, the transition from circle to square is impossible.

[16] Comment: In graph–theoretic terms this asks for a five vertex 3–regular graph. The sum of degrees would be 15, which contradicts the handshaking lemma stating that the sum of degrees must be even.

[17] Comment: If F is the number of faces and each face has an odd number of edges, then the sum of the number of all edges in of faces is odd; however this sum also equals $2E$ (since each edge belongs to two faces), which is even. Therefore F must be even.

[18] Comment: Summing by rows gives a total of m, while summing by columns gives a total of k. Therefore $m = k$.

[19] Comment: In a graph on 25 vertices without an independent set of size 3 the average degree is more than $\frac{24}{2}$; hence some vertex has degree at least 12.

5. A committee of 60 people held 40 meetings. Exactly 10 committee members attended each meeting. Prove that some two members met at committee meetings at least twice. [20]

1.2.5 :: :: problems for workshop

1. Is there a convex polygon with more than three acute angles? [21]

2. Prove that there are infinitely many prime numbers of the forms

   (a) $4k+3$, [22]

   (b) $3k+2$, [23]

   (c) $6k+5$. [24]

3. Prove that if $(m-1)!+1$ is divisible by m, then m is a prime number. [25]

4. In a class of 30 pupils a test was written. Nikita made 13 mistakes and every other pupil made strictly fewer. Prove that there are three pupils who made the same number of mistakes. [26]

5. The Earth has more than six billion inhabitants and no person older than 150 years exists. Prove that there are two people on Earth who were born at exactly the same second. [27]

---

[20] Comment: Double count the pairs (meeting, {two participants}). There are $40 \times \binom{10}{2}$ such pairs, but there are only $\binom{60}{2}$ pairs of committee members. By the pigeonhole principle some pair must occur at least twice.

[21] Comment: The answer is no. In a convex n–gon the sum of the interior angles is $180(n-2)$ degrees. If all k angles are acute then their sum is less than $90k$, forcing $180(n-2) < 90k + 180(n-k)$; from this one sees that $k \leq 3$.

[22] Comment: Consider the number $4p_1p_2\cdots p_n+3$ where the p_i run over all primes of the form $4k+3$.

[23] Comment: The same idea works by considering $3p_1p_2\cdots p_n+2$.

[24] Comment: The argument is analogous; take $6p_1p_2\cdots p_n+5$.

[25] Comment: This is a one direction form of Wilson's theorem. The converse is well known: if m is prime then $(m-1)! \equiv -1 \pmod{m}$.

[26] Comment: Excluding Nikita leaves 29 pupils making between 0 and 12 mistakes. There are 13 possible values and 29 pupils, so by the pigeonhole principle some value is taken by at least three pupils.

[27] Comment: The number of seconds in 150 years is approximately $150 \times 365 \times 24 \times 60 \times 60$, which is less than six billion. Therefore two of the more than six billion birth instants must coincide.

6. Twelve lines are drawn in the plane. Prove that some two of them form an angle of at most 15°. [28]

7. A drawer contains socks: 10 black, 10 blue and 10 white. What is the smallest number of socks one must draw without looking in order to guarantee that among the drawn socks there are

   (a) two of the same colour;[29]

   (b) two of different colours;[30]

   (c) two black socks?[31]

8. Thirty six stones were mined in a quarry. Their masses form an arithmetic progression $490, 495, 500, \ldots, 665$ kg. Is it possible to transport these stones using seven trucks, each of capacity 3 tonnes (3000 kg)?[32]

---

[28] Comment: Partition the 180° around a point into 12 intervals of length 15°. By the pigeonhole principle two of the directions of lines must lie in the same interval.

[29] Comment: Taking 4 socks guarantees at least two of the same colour, since 3 colours and 4 socks force a repetition.

[30] Comment: One could draw all 10 socks of one colour without yet having two colours. The eleventh sock guarantees at least one sock of a different colour.

[31] Comment: In the worst case one draws all 20 non black socks before drawing a black one. The next sock (the 22nd) must then be a second black.

[32] Comment: The total mass of the stones is $36 \times 577.5 = 20790$ kg. Seven trucks can carry 21000 kg, so sufficient capacity exists. But....

1.2.6 :: :: 2nd homework

Read:

Kenneth H. Rosen, Discrete Mathematics and Its Applications, p.313, 320-321. If you don't understand certain notation, just skip it for now (or find the explanation in the first chapter of this book), we will cover it later.

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the **inductive hypothesis**. Once we complete both steps in a proof by mathematical induction, we have shown that $P(n)$ is true for all positive integers, that is, we have shown that $\forall n P(n)$ is true where the quantification is over the set of positive integers. In the inductive step, we show that $\forall k (P(k) \to P(k+1))$ is true, where again, the domain is the set of positive integers.

Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \land \forall k (P(k) \to P(k+1))) \to \forall n P(n),$$

when the domain is the set of positive integers. Because mathematical induction is such an important technique, it is worthwhile to explain in detail the steps of a proof using this technique. The first thing we do to prove that $P(n)$ is true for all positive integers n is to show that $P(1)$ is true. This amounts to showing that the particular statement obtained when n is replaced by 1 in $P(n)$ is true. Then we must show that $P(k) \to P(k+1)$ is true for every positive integer k. To prove that this conditional statement is true for every positive integer k, we need to show that $P(k+1)$ cannot be false when $P(k)$ is true. This can be accomplished by assuming that $P(k)$ is true and showing that *under this hypothesis* $P(k+1)$ must also be true.

Remark: In a proof by mathematical induction it is *not* assumed that $P(k)$ is true for all positive integers! It is only shown that *if it is assumed* that $P(k)$ is true, then $P(k+1)$ is also true. Thus, a proof by mathematical induction is not a case of begging the question, or circular reasoning.

When we use mathematical induction to prove a theorem, we first show that $P(1)$ is true. Then we know that $P(2)$ is true, because $P(1)$ implies $P(2)$. Further, we know that $P(3)$ is true, because $P(2)$ implies $P(3)$. Continuing along these lines, we see that $P(n)$ is true for every positive integer n.

EXAMPLE 7 **An Inequality for Harmonic Numbers** The **harmonic numbers** H_j, $j = 1, 2, 3, \ldots$, are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

For instance,

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a nonnegative integer.

5.1 Mathematical Induction **321**

Solution: To carry out the proof, let $P(n)$ be the proposition that $H_{2^n} \geq 1 + \frac{n}{2}$.

BASIS STEP: $P(0)$ is true, because $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $H_{2^k} \geq 1 + \frac{k}{2}$, where k is an arbitrary nonnegative integer. We must show that if $P(k)$ is true, then $P(k+1)$, which states that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$, is also true. So, assuming the inductive hypothesis, it follows that

$$\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of harmonic number} \\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of } 2^k\text{th harmonic number} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the inductive hypothesis} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} && \text{because there are } 2^k \text{ terms each} \geq 1/2^{k+1} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} && \text{canceling a common factor of } 2^k \text{ in second term} \\
&= 1 + \frac{k+1}{2}.
\end{aligned}$$

This establishes the inductive step of the proof.

We have completed the basis step and the inductive step. Thus, by mathematical induction $P(n)$ is true for all nonnegative integers n. That is, the inequality $H_{2^n} \geq 1 + \frac{n}{2}$ for the harmonic numbers holds for all nonnegative integers n. ◀

Remark: The inequality established here shows that the **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is a divergent infinite series. This is an important example in the study of infinite series.

Write:

1. A snail crawls in the plane at a constant speed and every 15 minutes turns 90° (sometimes left, sometimes right). Prove that it can return to its starting point only after an integer number of hours.

2. In the language of a certain tribe there are only two sounds (written here as **w** and **y**). Two words are considered the same if one can be obtained from the other by a sequence of operations of two types: deleting a consecutive occurrence of **wy** or **yyww** and inserting **yw** at any position. Do the words **ywy** and **wyw** represent the same word?

3. Three grasshoppers sit at three vertices of a square in the plane. They play a game of leapfrog: one grasshopper jumps to the point symmetric to another grasshopper across the third (a point A is symmetric to a point B with respect

to a point C if C is the middle of the interval AB, here A, B, C are points on the plane.) Can any grasshopper ever land on the fourth vertex of the square?

4. There is an island populated by knights and liars. Knights always tell truth, liars always lie. A stranger meets three local persons and asks everybody: “how many knights are among you three?”. The first answered: no one. The second answered: one. What is the answer of the third person?

5. Automate can shred a piece of paper on 4 or 6 pieces. What number of pieces can be reached from one sheet?

6. Prove that among any 52 integers there exist two whose sum or difference is divisible by 100.

Just for fun, not a part of the homework: use AI, ask them to solve the following problem, then try to understand their solution, then check, if it is at all correct:

Four identical jars are filled to three quarters with paint, each containing a different colour. It is possible to pour any portion of the contents of one jar into another. Is it possible to perform a finite sequence of pourings so that each jar contains an identical mixture of the four colours?

Try to solve this problem by yourself.

Last, but not least. Read the next page for a short advice on how to write solutions with a good and bad example.

Example of bad and good writing

Five boys found nine mushrooms. Prove that at least two of them must have found the same number of mushrooms.

Bad solution	Good solution
It is not possible that they found different numbers of mushroom because in this case they will find at least 10 mushrooms[33], because in the worst[34] case the first[35] found 0, the second found 1 mushroom, etc, $0 + 1 + 2 + 3 + 4 = 10$.	Assume that the boys all found different numbers of mushrooms. Order them by increasing number of mushrooms. Then, the first boy picked at least 0 mushrooms, the second at least 1, the third at least 2, the fourth at least 3 and the fifth at least 4. Altogether they would have picked at least $0 + 1 + 2 + 3 + 4 = 10$ mushrooms, contradicting the fact that there were only nine mushrooms. Thus at least two boys must have collected the same number of mushrooms.

In a good solution each step is small and incremental, it might be

- an assumption (the first phrase states the proof strategy 'by contradiction'),
- an action ('Order them...'),
- introducing a notation ('Denote ... by ... '),
- a computation,
- or a conclusion derived from the preceeding steps.

Use single-action sentences: 'Assume this. Count that. Then A. Then B.' A solution is good (or formal) if it is easy to follow its logical structure and easy to check each step separately (assume that a reader has a limited amount of memory).

Instead of *'A, because of B'*, write *'B, therefore A'*.

Exercise: look at solutions of the problems across these notes and evaluate their quality, try to cut them into such elementary steps.

[33] Here the reader asks 'why'? It is better if, whenever it is possible, each next step immediately follows from the previous one, and explanations are given BEFORE a statement.

[34] Where is the definition of the worst case? And even if you give such a definition, then you need to show that the case 0,1,2,3,4 is indeed the worst case.

[35] What if the first found 3 and the second 0? Then you need to consider all these cases or convince the reader that these cases are all the same, i.e. as 'ordering' step in the good solution.

1.3 :: method of the extreme, counting in two ways, combinatorics

1.3.1 Method of the Extreme

Special or extreme objects often serve as a cornerstone of a solution. For example, one might look for the largest number, the nearest point, a corner point, a degenerate circle or a limiting case. It is therefore useful to consider such special objects.

In problems solvable by the method of the extreme one often uses the method of the minimal counterexample: assume that the statement is false. Then there exists a counterexample minimal in some sense. If one can show that this counterexample can be made smaller, one obtains the desired contradiction.

Example 1. *The plane is cut by $N \geq 3$ lines in general position. Prove that to each line there is a triangle adjacent to that line.*

Solution. Choose a line and consider all intersection points of the other lines. Among these points pick the one that is closest to the chosen line. The two lines passing through this closest point intersect the chosen line and form with it a triangle. No other lines can intersect the interior of this triangle (otherwise one would find a closer intersection point), so the triangle is indeed adjacent to the chosen line.

Example 2. *Prove that in any convex polyhedron there are two faces with the same number of edges.*

Solution. Consider the face G with the largest number n of edges. Each edge of G is adjacent to another face; there are n faces adjacent to G. The number of edges of each adjacent face lies between 3 and $n-1$, giving only $n-2$ possible values. Since there are more adjacent faces than possible values, by the pigeonhole principle two of them must have the same number of edges.

Example 3. *In each cell of a chessboard a number is written. It turns out that every number equals the arithmetic mean of the numbers in the neighbouring cells (by side). Prove that all the numbers are equal.*

Solution. Let M be the maximum of all the numbers. Since M equals the average of its neighbours, all neighbouring numbers must also equal M. Proceeding by connectivity of the board shows that all numbers are equal to M.

Example 4. *From a point inside a convex polygon perpendiculars are dropped to its sides or to their extensions. Prove that at least one of the perpendiculars falls on a* *side of the polygon.*

Solution. Consider the perpendicular whose foot on the boundary of the polygon is closest to the original point. If that foot lay on the extension of a side, one could shift slightly in the direction of the polygon and produce a shorter perpendicular, a contradiction. Hence at least one perpendicular must land on an actual side.

Example 5. *Prove that the number*

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n}$$

is never an integer. (Here $n \geq 2$.)

Solution. Look at the denominators of the summands and identify the largest power of 2 appearing there. When the terms are written with a common denominator, the numerator will be odd, while the denominator is a multiple of that power of 2; hence the result cannot be an integer.

1.3.2 Counting in Two Ways

When setting up equations one often expresses some quantity—such as an area, a distance or a time—in two different ways. Sometimes one estimates a quantity in two different ways and obtains either an inequality or values with different parity. This idea is closely related to the notion of an invariant and is often a source of contradictions (see also the section on proof by contradiction).

Example 1. *Can one arrange numbers in a* 5×5 *square table so that the sum of the numbers in each row is positive and in each column is negative?*

Solution. Suppose such an arrangement were possible. If one sums all the numbers by rows, the total sum is positive. If one sums all the numbers by columns, the total sum is negative. This contradiction shows that no such arrangement exists.

Example 2. *In a class there are 27 pupils. Each boy is friends with four girls and each girl is friends with five boys. How many boys and how many girls are there?*

Solution. Let B be the number of boys and G the number of girls. Count the total number of friendships in two ways. On the one hand, each boy is friends with four girls, giving $4B$ friendships. On the other hand, each girl is friends with five boys, giving $5G$. Thus $4B = 5G$. Since $B + G = 27$, simple algebra yields $B = 15$ and $G = 12$.

Example 3. *Find the sum of the geometric progression*

$$S_n = 1 + 3 + 9 + \cdots + 3^n.$$

Solution. Observe that, given S_n, the next sum S_{n+1} can be obtained in two ways: either add 3^{n+1} to S_n or multiply all terms of S_n by 3 and then add 1. Hence $S_n + 3^{n+1} = 3S_n + 1$, which simplifies to $S_n = \frac{3^{n+1}-1}{2}$.

Example 4. *Can all faces of a convex polyhedron have six or more sides?*

Solution. No. Estimate in two ways the average of all the angles of all the faces. On the one hand, the average interior angle of an n–gon with $n \geq 6$ is at least 120°. On the other hand, at each vertex of the polyhedron at least three faces meet and the sum of the angles meeting at a vertex is strictly less than 360°. Therefore the average angle at each vertex is strictly less than 120°. The contradiction shows that such a polyhedron cannot exist.

Example 5. A *binomial coefficient* $\binom{n}{k}$ (pronounced '*n choose k*') is the number of ways to choose k different numbers from the set $1, 2, \ldots, n$ of n numbers. Using the above definition of the binomial coefficients, prove (without any formulas) that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The picture on the left is called the Pascal triangle. Left and right sides are filled with ones. Then we fill rows one by one. Each interior entry equals the sum of the two entries above it: specifically, the entry in row n, position k is $\binom{n}{k}$.

Problem 5. Prove the **Newton binomial formula:** $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, for example $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$.

1.3.3 :: :: problems for tutorial

Before this tutorial read Section 5.

1. A football fan drew the positions of the players on a football field at the beginning of the first and the second halves. It turned out that some players had exchanged places, while the others remained in their positions. At the same time, the distance between any two players did not increase. Prove that all these distances in fact remained the same.[36]

2. A traveller leaves his home city A and goes to the city B in his country that is farthest from A. From that city he goes to the city C farthest from his current city B, and so on. Prove that if the city C does not coincide with A, then the traveller will never return home. (Distances between cities are assumed to be distinct.)[37]

3. One of the heads of a hundred headed dragon wants to arrange his heads so that each head lies strictly between two others. Can he do it? (You may regard the heads as points in space.)[38]

4. Show (hint: by induction) that a connected graph with n vertices and without cycles has exactly $n-1$ edge.

[36] Comment: An invariant argument: since no pairwise distance increases, the total sum of distances acts as a semi-invariant.

[37] Comment: Each step strictly increases the distance to the farthest city left behind. If one could return home, one would form a cycle of increasing distances, which is impossible.

[38] Comment: It is impossible for every point in a finite set to lie between two others on a straight line, because the extreme points on the convex hull of the configuration cannot be between two other points. Start solution by: consider the head with biggest z-coordinate. Among them consider the head with the biggest y-coordinate, among them...

1.3.4 :: :: problems for workshop

Exercise 1. In cafe, you can order green, red, or black soup and green, red, or black tea. How many different orders you can make if it is prohibited to order soup and tea of the same color?

Exercise 2. Five persons A, B, C, D, E visit a cafe every day and always seat on the same table with five chairs (chairs are numbered: $1, 2, 3, 4, 5$.). In how many different ways they can seat on these five chairs?[39]

Exercise 3. In cafe, the cook has a) 4, b) 8 ingredients. A dish is made of three different ingredients (their order is not important). How many different dishes the cook can cook? [40]

Exercise 4. How many four-digit numbers contain 7?

Exercise 5. In cafe, the cook has 9 ingredients. A dish is made of three different ingredients (their order is important). How many different dishes the cook can cook if the 3rd and 4th ingredients are not allowed to use together?

Problem 6. Find the coefficient behind x^3 in $(1+x)^{10}(1+2x)^7$.

Solution: we can choose x from three parenthesis of the first type $(1+x)$ and 1 from all parenthesis of the second type $(1+2x)$. We can do this in $10 \times 9 \times 8$ cases, but since it is not important in which order we choose the first type parenthesis, we divide by 6. So, this gives $\binom{10}{3}$. We also can choose x from two parenthesis of the first type and from one of the second type, this gives $2 \times \binom{10}{2}\binom{7}{1}$, etc. The answer[41]:

$$\binom{10}{3} + 2 \cdot \binom{10}{2}\binom{7}{1} + 4 \cdot \binom{10}{1}\binom{7}{2} + 8 \cdot \binom{10}{1}\binom{7}{3}.$$

Exercise 6. Find the coefficient behind x^4 in $(1+x)^{10}(1+2x)^7$.

---

[39] Recall the notation $n!$

[40] For the case a) one can list all dishes, then explain that it is $\frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1}$. Introduce the notation $\binom{n}{k}$, provide the formula for it, tell that the proof is exactly the same as in b).

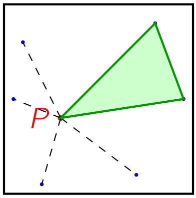
[41] You do not need to compute it numerically, the below answer is fine.

1.3.5 :: :: 3rd homework

Read:

Problem: In a unit square, 101 points are thrown in such a way that no three of them lie on one straight line. Prove that there exists a triangle with vertices at these points whose area does not exceed $\frac{1}{100}$.

Solution: Select a point P among 101 points and connect it with all the others — we obtain 100 segments. Choose a direction (say, clockwise) and successively connect the ends of these segments — we obtain 100 non-overlapping triangles whose total area does not exceed 1 (the total area of the square). By the pigeonhole principle, there exists at least one triangle among them whose area does not exceed $\frac{1}{100}$.



Write:

1. Find the mistake in the above solution, explain it in your homework.

2. One hundred numbers are arranged on a circle. Each number is equal to the arithmetic mean of its two neighbours. Prove that all the numbers are equal.

3. Prove that if a graph with n vertices has only $n-2$ edges, then it is disconnected.

4. Using the principle of mathematical induction, prove that

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1) \cdot (2n+1) = \frac{n(4n^2+6n-1)}{3}.$$

5. Each student in the class took part in at least one of two hikes. In each hike, the proportion of boys did not exceed $2/5$. Prove that in the whole class, the proportion of boys does not exceed $4/7$.

6. On a plane, N points are given. Some pairs of points are connected by segments. If two segments intersect, they may be replaced by two other non-intersecting segments with the same endpoints. Can this process continue indefinitely?

1.4 :: feedback

Lectures: in lectures we skip some of the problems presented above and sometimes postpone problems to the next lecture. For easy problems we stopped and gave time for students to try to solve them.

Tutorials:

Workshops:

Homeworks:

2 Naive set theory

2.1 :: notation, union, intersection, Venn diagrams

Definition of a set, Euler-Venn diagrams, symbols $\cap, \cup, \forall, \exists, \wedge, \vee, \subset, \in, \notin, \Rightarrow, \Leftrightarrow, \varnothing, A \setminus B$, $\neg$ (negation), XOR, characteristic function (addition on the image is xor multiplications is AND, the same as multipication and addition for even and odd numbers) constructions such as

$$\mathbb{Q}_+ = \{q \in \mathbb{Q} | q > 0\}, \sum_{i=1..n} f(i), \sum_{a \in A} f(a).$$

When does $A \setminus B = B \setminus A$?

Problem 7. Translate into the plain English the following.

Def. $f : \mathbb{R} \to \mathbb{R}$ is continuous $\Leftrightarrow$

$$\Leftrightarrow \forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \text{ such that } \forall x' \in [x - \delta, x + \delta] \text{ we have } |f(x') - f(x)| < \varepsilon.$$

Write a definition of a prime number without using words and using only mathematical notation.

1. Prove that for any three sets the inclusion (and give an example of three sets for which this inclusion is strict.)
2. $$(i)(A \setminus B) \setminus C \subseteq A \setminus (B \setminus C)$$
3. (ii) $(A \triangle B) \setminus C = (A \cup C) \triangle (B \cup C)$;
4. (iii) $A \setminus (B \triangle C) = (A \setminus (B \cup C)) \cup (A \cap B \cap C)$.

   holds.

Solution. The left-hand side of (i) consists of those $x \in A$ that do not belong to B and do not belong to C, is contained in $A \setminus B$. The right-hand side consists of those $x \in A$ that do not belong to $B \setminus C$ and, consequently, contains $A \setminus B$.

We can also express this in another way: the left-hand side consists of those x for which

$$x \in A \ \wedge \ x \notin B \ \wedge \ x \notin C,$$

while the right-hand side consists of those x for which

$$(x \in A \ \wedge \ x \notin B) \ \vee \ (x \in A \ \wedge \ x \in C).$$

Thus, to give an example of three sets for which a strict inclusion occurs here, it suffices to take any three sets for which $A \cap C$ is not contained in B, i.e. $A = C = A \cap C = \{x\}, B = \varnothing$.

The meaning of introducing operations $1+1 = 0+0 = 0, 0+1 = 1+0 = 1, 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, 1 \cdot 1 = 1$ on the set $\{0,1\}$ consists in the fact that now we can express Boolean operations on subsets of a set X in terms of operations on their characteristic functions.

Problem. Prove that for any two subsets $A, B \subseteq X$ the following equalities hold:

1. $\chi_{A \cap B} = \chi_A \chi_B$,
2. $\chi_{A \triangle B} = \chi_A + \chi_B$,
3. $\chi_{A \cup B} = \chi_A + \chi_B + \chi_A \chi_B$,
4. $\chi_{A \setminus B} = \chi_A + \chi_A \chi_B$.

The results of this problem serve as yet another argument in favor of considering $\cap$ as the *product* of sets, and $\triangle$ — rather than $\cup$ — as the *sum* of sets.

Problems on Intervals of the Real Line. *What is the intersection of two intervals on the real line?*

Answer. Let (a, b) with $a < b$ and (c, d) with $c < d$ be two intervals. If $b \leq c$ or $d \leq a$, their intersection is empty. If $b > c$ and $d > a$, then the intersection of these intervals is the interval

$$(\max(a, c), \min(b, d)).$$

Consider two intervals of the real line (a, b), $a < b$, and (c, d), $c < d$. When is their union again an interval? What is it equal to in this case?

Answer. The union is an interval if and only if (a, b) and (c, d) intersect, that is, when $c < b$ and $a < d$. In this case their union equals

$$(\min(a, c), \max(b, d)).$$

If three intervals of the real line have a common point, then at least one of them is contained in the union of the other two.

Solution. Let $I_i = (a_i, b_i)$, $a_i < b_i$, where $i = 1, 2, 3$, be the three intervals. By the condition, there exists a point $x \in \mathbb{R}$ such that $a_i < x < b_i$ for all $i = 1, 2, 3$. Applying twice the result of the previous problem, we obtain

$$I_1 \cup I_2 \cup I_3 = (\min(a_1, a_2, a_3), \max(b_1, b_2, b_3)).$$

Take an index i such that $\min(a_1, a_2, a_3) = a_i$. Next, take some $j \neq i$ such that $\max(b_1, b_2, b_3) = b_j$, or, if no such j exists (that is, if $b_i > b_j$ for all $j \neq i$), take any $j \neq i$. Then

$$I_i \cup I_j = (a_i, b_j) = I_1 \cup I_2 \cup I_3,$$

so if h is an index such that $\{i, j, h\} = \{1, 2, 3\}$, then $I_h \subset I_i \cup I_j$. $\square$

2.1.1 :: :: problems for tutorial

1. Is it true that
$$(A \setminus B) \setminus C = (A \setminus C) \setminus B?$$

2. Is it true that
$$(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)?$$

3. Prove the so-called **four sets identity:**
$$(A \cap B) \cup (C \cap D) = (A \cup C) \cap (A \cup D) \cap (B \cup C) \cap (B \cup D)$$

4. Express operations $\cup$ and $\setminus$ through the operations $\cap$ and $\triangle$

5. Express the operations $\cap$ and $\setminus$ through the operations $\cup$ and $\triangle$.

In fact, Boolean operations can be expressed through any two of them.

Expression through $\cap$ and $\triangle$. The operations $\cup$ and $\setminus$ are expressed through the operations $\cap$ and $\triangle$ as follows:
$$A \cup B = (A \triangle B) \triangle (A \cap B), \qquad A \setminus B = A \triangle (A \cap B).$$

Expression through $\cup$ and $\triangle$. The operations $\cap$ and $\setminus$ are expressed through the operations $\cup$ and $\triangle$. Since $\setminus$ is already expressed through $\triangle$ and $\cap$, it suffices to express $\cap$ through $\cup$ and $\triangle$. Such an expression is given by the following formula:
$$A \cap B = (A \cup B) \triangle (A \triangle B).$$

2.1.2 :: :: problems for workshop

Exercise 7. Some of these expressions are grammatically or logically incorrect. Identify them and explain what is the fault. (In what follows, $f : \mathbb{R} \to \mathbb{R}$ is a real function and $A, B, C \subset \mathbb{R}$.)

$\{1+1\}$	$\{3\} \setminus \{\{3\}\}$	$1+1 \Rightarrow 2$
$\{1,2\} \Leftrightarrow \{2,1\}$	$\sqrt{2} \notin \mathbb{Q}$	$\mathbb{Z} \setminus (\mathbb{Z} \setminus \mathbb{N})$
$\mathbb{Z} \Rightarrow \mathbb{Q}$	$(\lvert x\rvert \in \mathbb{Z}) \Rightarrow (\lvert x\rvert \in \mathbb{Q})$	$(x \in A) \cup (x \in B)$
$(3 < 1) \Rightarrow \varnothing$	$A \leq (A \setminus B)$	$f(A) \in \{f(A)\}$
$(A \subset B) \cap C$	$A \subset (B \cap C)$	$A \subset B \subset A$
$(2,4,6,\ldots) \subset (1,2,3,\ldots)$	$\{A, \mathbb{Z}\}$	$\{\varnothing\} \cap \varnothing$
$f(1) \in \{2,3\}$	$f(\{1,2\}) \in \mathbb{N}$	$f(\mathbb{Q}) \subset \mathbb{Q}$
$\{x \in \mathbb{N} : -x\}$	$\{-x : x \in \mathbb{N}\}$	$\{x : x \Leftrightarrow 2\}$
$\{x \in \mathbb{Z} : x \notin \mathbb{Z}\}$	$\{\{x : \lvert x\rvert < 2\}\}$	$\{x \in \mathbb{Q} : 1 = 0\}$
$\{x \in \mathbb{Q} : x^2 \notin \mathbb{Z}\}$	$\{\{f(x)\} : x \in \mathbb{Q}\}$	$\{x : f(x) \in \mathbb{Q}\}$

Exercise 8. The following expressions define sets. Turn words into symbols.

1. The set of negative odd integers:
2. The set of natural numbers with three decimal digits:
3. The set of rational numbers which are the ratio of odd integers:
4. The set of rational numbers between 3 and π:
5. The set of real numbers at distance $\frac{1}{4}$ from an integer:
6. The complement of the unit circle in the Cartesian plane:
7. The set of lines tangent to the unit circle:

Exercise 9. Draw the Euler-Venn diagrams for $(A \cap B) \setminus C$ and $(A \setminus C) \cap (A \setminus B)$.

Exercise 10. Draw the Euler-Venn diagram for $\big((A \cap B) \cap (C \cup D)\big) \cup (A \setminus C)$.

Answers for tutorial:

$$\{\, n \in \mathbb{Z} : n < 0 \text{ and } n \text{ is odd} \,\} = \{\, n \in \mathbb{Z} : n < 0,\ \ n = 2k+1,\ \ k \in \mathbb{Z} \,\}.$$

$$\{\, n \in \mathbb{N} : 10^2 \leq n < 10^3 \,\}.$$

$$\left\{\, \frac{p}{q} \in \mathbb{Q} : p, q \in \mathbb{Z},\ \ p, q \text{ odd},\ \ q \neq 0 \,\right\}.$$

$$\{\, q \in \mathbb{Q} : 3 < q < \pi \,\}.$$

$$\{\, x \in \mathbb{R} : |x - n| = \tfrac{1}{4} \text{ for some } n \in \mathbb{Z} \,\}.$$

$$\{\, (x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1 \,\}.$$

$$\{\, ax + by = 1 : a^2 + b^2 = 1 \,\}.$$

2.1.3 :: :: 4th homework

1. A straight (finite) path in a park is completely illuminated by several lamps, each of which lights up a certain segment. Prove that it is possible to turn off some of the lamps so that the path remains completely illuminated, but no part of it is illuminated by three lamps at once.

2. Is it possible to cover an equilateral triangle with two smaller equilateral triangles?

3. Prove that the sum of the valencies (aka (also known as) the degrees) of the vertices of a graph is twice the number of the edges. (This is called **the handshaking lemma**)

4. Prove that every positive integer can be represented as a sum of different powers of two.

5. Draw the Euler-Wenn diagramm for $((A \setminus B) \cap (C \cup (D \setminus A))) \cup (D \cap B \cap A)$.

6. Let I be an index set, and $\{A_i\}_{i \in I}$ be a family of sets. Prove de Morgan laws without words, i.e. show that every element from the left set belongs to the right set and vice versa. It is prohibited to write words:

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$$

$$\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$$

   where A^c is the complement to A in the "universal set".

7. **The Meticulous Job Applicant**

   A company requires that ideal candidates must satisfy **all** of the following conditions: speak English, have programming experience, possess a university degree, have at least 2 years of work experience. Describe in words the set of candidates who are **NOT** ideal for the position.

3 :: Materials

Comments to the arithmetic/geometri mean inequality. Notes from tutorial 1

In the tutorial, we proved that if $n = 2^k$ for some positive integer k, and $x_1, \dots, x_{2^k}$ are positive numbers, then the inequality

$$(x_1 \cdots x_{2^k})^{\frac{1}{2^k}} \leq \frac{x_1 + \cdots + x_{2^k}}{2^k} \tag{1}$$

holds. In particular, equality occurs if and only if

$$x_1 = x_2 = \cdots = x_{2^k}.$$

We are left to show the case when n is not a power of 2, that is, we need to fill the gaps between powers of 2.

Let $x_1, \dots, x_n$ be arbitrary positive numbers, and choose k such that $n \leq 2^k$. Define

$$\alpha_i := \begin{cases} x_i, & i \leq n, \\ A, & n+1 \leq i \leq 2^k, \end{cases} \quad \text{where} \quad A := \frac{x_1 + \cdots + x_n}{n}.$$

We apply inequality (1) to

$$(\alpha_1 \alpha_2 \cdots \alpha_{2^k})^{\frac{1}{2^k}},$$

which yields (remark that $nA = x_1 + \cdots + x_n$)

$$\left(x_1 \cdots x_n \, A^{2^k - n}\right)^{\frac{1}{2^k}} \leq \frac{x_1 + \cdots + x_n + (2^k - n)A}{2^k} = A.$$

We obtain

$$(x_1 \cdots x_n)^{\frac{1}{2^k}} \leq A^{1 - \frac{2^k - n}{2^k}} = A^{\frac{n}{2^k}}.$$

Finally, raising both sides to the power $\frac{2^k}{n}$ gives

$$(x_1 \cdots x_n)^{\frac{1}{n}} \leq A = \frac{x_1 + \cdots + x_n}{n}.$$

This completes the proof.

Written homework

Students should write a text. To learn writing there should be feedback, and examples how to write, bad and good, with explanations, what is good and what is bad.

Then we have a text to read with explanations, how to write and how not to write this solution.

Basic Graph Theory: Trees, Paths, and Leaves

Graph. A *graph* $G = (V, E)$ consists of a set of *vertices* V and a set of *edges* $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$.

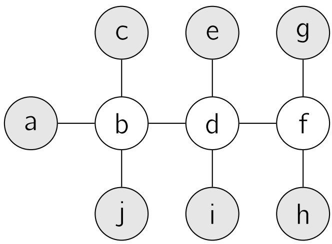
Path. A *path* is a sequence of distinct vertices $v_1, \ldots, v_k$ such that each consecutive pair (v_i, v_{i+1}) is an edge; its length is $k - 1$.

Tree. A *tree* is a connected graph with no cycles (acyclic). We will prove in tutorial that a tree on n vertices has exactly $n - 1$ edges.

A question before the tutorial

A *leaf* is a vertex of degree 1.
Why every tree has a leaf ?

Example tree with leaves highlighted



In the picture above, the light vertices a, c, e, g, h, i, j are *leaves* (each has degree 1). Vertices b, d, f have degree ≥ 2.