EVALUATION OF LATTICE SUMS VIA TELESCOPING OVER TOPOGRAPHS

NIKITA KALININ

Abstract. Topographs, introduced by Conway in 1997, are infinite three-valent planar trees used to visualize the values of binary quadratic forms. In this work, we study series whose terms are indexed by the vertices of a topograph and show that they can be evaluated using telescoping sums whose terms correspond to the edges of a topograph.

Our technique provides new arithmetic proofs for modular graph function identities arising in string theory, yields alternative derivations of Hurwitz-style class number formulas in number theory, and serves as a unified framework for well-known Mordell-Tornheim series and Hata's series for the Euler constant γ.

Our theorems are of the following spirit: let us cut a topograph along an edge (called the *root*) in two parts, and then sum $\frac{1}{rst}$ (the reciprocal of the product of labels on regions adjacent to a vertex) over all vertices of one part. Then the sum is equal to an explicit expression depending only on the root and the discriminant of the topograph.

Keywords: modular graph functions, lattice sums, telescoping sums, binary quadratic forms, topographs, class number, digamma function, Euler's constant; AMS classification: 11E16, 11F67, 11M35

Contents

Guangdong Technion Israel Institute of Technology (GTIIT), 241 Daxue Road, Shantou, Guangdong Province 515603, P.R. China, Technion-Israel Institute of Technology, Haifa, 32000, Haifa district, Israel, nikaanspb@gmail.com

1. Introduction

The purpose of this article is to introduce and develop the *telescoping over topograph* method and to connect it to the works on class number formulas for quadratic number fields (a classical one [10] of Adolf Hurwitz and two recent ones: [8] of Duke, Imamoḡlu, Tóth and [13] by O'Sullivan) as well as to the modular graph functions in the low energy genus one expansion of type II string amplitudes due to d'Hoker, Green, Gürdoğan, Vanhove [6, 5].

The story begins with Hurwitz's 1905 fomula (2), which expresses the number of $SL(2,\mathbb{Z})$-classes of integer binary quadratic forms with negative discriminant as a certain infinite sum. After nearly a century of neglect, Duke–Imamoḡlu–Tóth established the corresponding positive-discriminant formula (5). O'Sullivan then recast these identities as vertex sums on Conway's topographs. In a parallel developement, Zagier computed the lattice sum (7) – essentially the keystone step in Hurwitz's argument – which later proved to be the prototypical example among modular graph functions in string theory, as studied by d'Hoker, Green, Gürdoğan, Vanhove, and others.

Our contribution is a unifying and elementary viewpoint: many of these results fall out from a simple telescoping geometric argument. Below we explain the context with more details, in historical order, starting from Hurwitz and moving forward.

1.1. **Formulas related to class number.**

For $v=(m,n)$, denote by $q=[A,B,C]$ a quadratic form

$$q(v)=Am^2+Bmn+Cn^2.$$

Two binary quadratic forms $[A,B,C]$ and $[A',B',C']$ with integer coefficients are called equivalent if there exist $a,b,c,d\in\mathbb{Z}$ with $ad-bc=1$ such that

$$A'm^2+B'mn+C'n^2=A(am+bn)^2+B(am+bn)(cm+dn)+C(cm+dn)^2.$$

That is,

$$[A,B,C]\sim[Aa^2+Bac+Cc^2,2Aab+B(ad+bc)+2Ccd,Ab^2+Bbd+Cd^2],$$

In particular,

$$[1,0,1]\sim[a^2+c^2,2ab+2cd,b^2+d^2]=[A,B,C],\ \text{and} \tag{1}$$

$$A+B+C=(a+b)^2+(c+d)^2 \text{ in this case.}$$

The *discriminant* D of a binary quadratic form $[A,B,C]$ is B^2-4AC and it is preserved by the above equivalence. An integer D is called a *fundamental* discriminant if either (a) D is square-free and $D\equiv 1 \pmod 4$, or (b) $D\equiv 0 \pmod 4$, $D/4$ is square-free and $D/4\equiv 2,3 \pmod 4$.

Denote by $h(D)$ the number of equivalence classes of forms with $\gcd(A,B,C)=1$ and discriminant D. Although formulas for $h(D)$ exist, it remains difficult to estimate its asymptotic behavior; for example, it is still an open problem to prove that there exist infinitely many values of $D>0$ for which $h(D)=1$.

In 1905, Adolf Hurwitz wrote a paper on an infinite series representation of the class number $h(D)$ in the positive-definite case.

Theorem 1 (Hurwitz,[10])**.** For $D<0$, a fundamental discriminant, all the terms in the following sum are positive and this sum converges to $h(D)$:

$$h(D)=\frac{\omega_D}{12\pi}|D|^{3/2}\sum_{\substack{A>0\\ B^2-4AC=D}}\frac{1}{A(A+B+C)C}, \tag{2}$$

where

$$\omega_D = \begin{cases} 1 \text{ for } D < -4, \\ 2 \text{ for } D = -4, \\ 3 \text{ for } D = -3. \end{cases} \tag{3}$$

Hurwitz's proof amounts to computing the area of a certain domain in two different ways, see Section 3.4 for details.

Example 1. For the quadratic form $q(v) = ||v||^2 = m^2 + n^2$, the discriminant $D = -4$, and $h(D) = 1$. Hurwitz's arguments [10, p. 20] lead to the formula

$$\sum_{\substack{a,b,c,d \in \mathbb{Z}_{\geq 0}, \\ ad-bc=1}} \frac{1}{(a^2+b^2)(c^2+d^2)((a+c)^2+(b+d)^2)} = \frac{\pi}{4}, \tag{4}$$

By the change of variables (1), it follows that

$$\frac{1}{A(A+B+C)C} = \frac{1}{(a^2+b^2)((a+c)^2+(b+d)^2)(c^2+d^2)},$$

thus, putting it all together and multiplying by the constant 3 coming from the three cyclic orders of the denominators of the terms, we get

$$h(-4) = 1 = \frac{2}{12\pi} \cdot 2^3 \cdot \frac{3\pi}{4} = \frac{\omega_D}{12\pi}|D|^{3/2} \sum_{\substack{A>0 \\ B^2-4AC=D}} \frac{1}{A(A+B+C)C}.$$

Hurwitz's result has been largely unnoticed for nearly a century, with the only exceptions of Dickson's *History of number theory* [7, p.167] in 1952 and Sczech's work [14] on Eisenstein cocycles for $GL_2\mathbb{Q}$ in 1992.

It has been revived in 2019, in the Duke-Imamoḡlu-Tóth paper [8], and a formula for the indefinite case ($D > 0$), similar to (2), was established:

Theorem 2 ([8], Theorem 3, p. 3997)**.** For $D > 0$, a fundamental discriminant,

$$h(D)\log\varepsilon_D = D^{1/2} \sum_{\substack{[A,B,C] \text{ reduced} \\ B^2-4AC=D}} \frac{1}{B} + d^{3/2} \sum_{\substack{A,C,A+B+C>0 \\ B^2-4AC=D}} \frac{1}{3(B+2A)B(B+2C)}. \tag{5}$$

Here ε_D, the fundamental unit, is defined as $\varepsilon_D := (t_D + u_D\sqrt{D})/2$ where (t_D, u_D) is the smallest solution to $t^2 - Du^2 = 4$ in positive integers.

In 1997, in his book *The Sensual (Quadratic) Form* [4], John H. Conway introduced topographs, a graphical tool for visualizing binary quadratic forms and their behavior over the integers. A topograph provides an intuitive and surprisingly powerful visual framework for understanding, for example, reduction algorithms.

A topograph for a binary quadratic form q is an infinite three-valent planar tree $\mathcal{T}$ with labels in the connected components (regions) of $\mathbb{R} \setminus \mathcal{T}$. To each region corresponds bijectively a pair $(v, -v)$ of primitive lattice vectors in $\mathbb{Z}^2$ (i.e., a point in $P\mathbb{Q}^2$), and the label on this region is the value $q(v) = q(-v)$ of the quadratic form q. At each vertex, the three adjacent regions correspond to three primitive vectors $v, w, v + w$, forming a basis of $\mathbb{Z}^2$. Thus, near each vertex of $\mathcal{T}$ the labels r, s, t on regions are exactly $q(v), q(w), q(v + w)$ for a certain basis (v, w) of $\mathbb{Z}^2$. Topographs are related to many objects in mathematics [3] and can be used to study variations of Markov triples; see a popular exposition [16].

Note that

$$A(A+B+C)C = q(\tbinom{1}{0})q(\tbinom{0}{1})q(\tbinom{1}{1}), \text{ for } v = \tbinom{1}{0}, w = \tbinom{0}{1}.$$

Thus (2) may be interpreted as the sum of

$$\frac{1}{q(\binom{1}{0})q(\binom{0}{1})q(\binom{1}{1})}$$

over all forms q of discrimanant D. Instead of applying the $SL(2,\mathbb{Z})$ action to a form, we may apply it to a basis, thus obtaining a sum like

$$\frac{1}{q(v)q(w)q(v+w)}$$

over a topograph.

We will consider summations over all vertices V of a topograph, and we sum the reciprocal to the product $|rst|$ of labels on the adjacent regions to V or the product $|egf|$ of labels on the adjacent edges to V.

In 2024, O'Sullivan proposed in [13] a unifying approach to these class number series via topographs. In particular, (4) was rewritten in the language of topographs as

Theorem 3 ([13], Theorem 9.1)**.** Let $\mathcal{T}$ be any topograph of discriminant $D<0$. Then

$$|D|^{3/2} \sum_{r \succ^{s}_{t} \in \mathcal{T}} \frac{1}{|rst|} = 4\pi, \tag{6}$$

where we sum over all vertices of $\mathcal{T}$, each vertex contributing one term; here r,s,t denote the labels on regions adjacent to a given vertex of $\mathcal{T}$, as explained in Section 2.

Then, (2) follows because the RHS of (2) is essentially the sum over all vertices of all topographs of discrimant D, and the number of topographs of discriminant D is $h(D)$. In the same paper, the formula (5) for the class number for the indefinite case $D>0$ was rewritten in terms of topographs as

Theorem 4 ([13], Theorem 9.2)**.** Let $\mathcal{T}$ be any topograph of a non-square discriminant $D>0$. Define $\mathcal{T}_\star$ to be equal to $\mathcal{T}$ except that all the river edges are relabeled with $\sqrt{D}$. Then

$$D^{3/2} \sum_{{}^{f}_{e}\succ^{g} \in \mathcal{T}_\star} \frac{1}{|efg|} = 2\log \varepsilon_D,$$

where we sum over all vertices of $\mathcal{T}_\star$ modulo the river period (each vertex contributing one term, see [13] for details on the river and its period), and e,f,g are labels on the edges.

1.2. **String theory formulas.** There was an independent parallel story. Beyond number theory, similar lattice sums appear in the analysis of modular graph functions in string theory. In 2008, in an unpublished note [18], Zagier considers

$$D_{1,1,1}(z) = \sideset{}{'}\sum_{\omega_1+\omega_2+\omega_3=0} \frac{\mathrm{Im}(z)^3}{|\omega_1\omega_2\omega_3|^2}, \quad \omega_1,\omega_2,\omega_3 \in \mathbb{Z}z+\mathbb{Z},$$

where $\sum'$ denotes the summation over the fractions with non-zero denominators, and proves

$$D_{1,1,1}(z) = 2E(z,3) + \pi^3\zeta(3), \tag{7}$$

Zagier's proof involves analytic manipulations, partial telescoping and reduces the question to sums of $\frac{1}{(z+n)(z+m)}$, then to sums involving the real part of $\frac{1+q}{1-q}$ for $q = e^{2\pi i z}$. Finally, Zagier proves (7) up to a certain holomorphic, $SL(2,\mathbb{Z})$-invariant and small at infinity (hence identically zero) function.

Let us specialize $z = i$, so $\mathbb{Z}z+\mathbb{Z}$ becomes $\mathbb{Z}^2$. Then, one can rewrite $D_{1,1,1}(i)$ via the sum over ω_1, ω_2 that span parallelograms of area one and hence, up to standard transformations, (7) is equivalent to (4), see [11] for details.

In 2017, in [5], modular graph functions (MGF) were defined, appearing in the low-energy expansion of genus-one Type II superstring amplitudes. In perturbative type-II superstring theory, the genus-one four-graviton amplitude can be written as an integral over the torus moduli space of products of Green functions. The integrals turned into lattice sums that are modular invariant by construction. At weight two one obtains classical Eisenstein series. At weight three, the unique connected vacuum diagram is a "sunset" graph with two vertices joined by three propagators, and the corresponding lattice sum is precisely $D_{1,1,1}(z)$ defined above. In other words, Zagier's arithmetic identity (7) provides the first closed formula for a non-trivial modular graph function that appears in the low-energy expansion of superstring amplitudes.

A general modular graph function is constructed from a graph Γ. Assign to each edge $e \in \Gamma$ a variable $\frac{y}{|\omega_e|^2}$ and consider the sum of products $\prod\limits_{e\in G} \frac{y}{|\omega_e|^2}$ over all such tuples of $\omega_e \in \mathbb{Z}z + \mathbb{Z}$ such that the sum of incoming ω_e at each vertex is zero.

If Γ consist of two vertices and k edges between them, then

$$D_\Gamma(z) = \sideset{}{'}\sum_{\substack{\omega_1,\dots,\omega_k\in\Lambda\\ \omega_1+\cdots+\omega_k=0}} \frac{y^k}{\prod_{i=1}^k |\omega_i|^2}, \qquad \Lambda = \mathbb{Z}+\mathbb{Z}z,\ z = x+iy, \tag{8}$$

So we obtain the definition of $D_{1,1,1}$ when Γ is a graph with two vertices and three edges ($k = 3$). The study of modular graph functions has revealed a rich network of differential and algebraic relations between them, zeta-values, Eisenstein functions, etc., resulting in hundreds of articles. Our telescopic viewpoint furnishes a parallel, purely arithmetical derivation of some of those relations. For example, using (22) one can prove relation

$$D_{2,2,1} = \frac{2}{5}E_5 + \frac{\zeta(5)}{30}$$

for the graph Γ as above for $D_{1,1,1}$ but we subdivided two of its edges by additional vertex. However, for $D_{1,1,1,1}$ (two-vertex graph with four edges between them) we do not know how to use telescoping to prove a well-known identity

$$D_{1,1,1,1} = 24C_{2,1,1} + 3E_2^2 - 18E_4.$$

1.3. **New results.** In [11] the Zagier's formula (7) was obtained by a telescopic method altogether with (2). Then it became clear that the telescopic method allows one to derive the above Hurwitz-type formulas. The telescopic approach allows explicit evaluations of lattice sums by reducing global series to boundary contributions in topographs.

Definition 1. Let $\mathcal{T}$ be a topograph, and let E be an oriented edge of $\mathcal{T}$ labelled e_0, with adjacent regions r_0, t_0. Removing E separates $\mathcal{T}$ into two infinite components; let $\mathcal{T}'$ denote the component containing the target V_0 of E. The edge E is called the *root* of the subtree $\mathcal{T}'$. We call $\mathcal{T}'$ *the upper half of* $\mathcal{T}$ with respect to the root E.

Later we give a definition of an *admissible* upper half $\mathcal{T}'$ (Definition 6); in essence the admissibility means that all the sums that we consider converge and all terms make sense (no division by zero). The main theorems of this article are as follows.

Theorem 5. Let $\mathcal{T}$ be any topograph of discriminant $D < 0$. Let $\mathcal{T}'$ be the upper half of $\mathcal{T}$ with respect to a root E. Suppose $\mathcal{T}'$ is admissible. Then, for the appropriate branch of arcsin and arctan,

$$\sum_{r \succ^{s}_{t} \in \mathcal{T}'} \frac{1}{|rst|} = \frac{1}{D}\left(\frac{e_0}{r_0 t_0} - \frac{2\arcsin(\frac{e_0}{r_0 t_0}\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}}\right), \tag{9}$$

$$\sum_{{}^{f}_{e}\succ^{g} \in \mathcal{T}'} \frac{1}{|efg|} = \frac{1}{D}\left(\frac{\arctan(\frac{\sqrt{-D}}{e_0})}{\sqrt{-D}} - \frac{1}{e_0}\right), \tag{10}$$

where the summation is over all vertices of $\mathcal{T}'$, each contributing one term.

Theorem 6. Let $\mathcal{T}$ be any topograph of discriminant $D > 0$. Let $\mathcal{T}'$ be the upper half of $\mathcal{T}$ with respect to a root E. Suppose $\mathcal{T}'$ is admissible. Then

$$\sum_{r \succ^{s}_{t} \in \mathcal{T}'} \frac{1}{|rst|} = \frac{1}{D}\left(\frac{e_0}{r_0 t_0} - \frac{2\,\mathrm{arsinh}(\frac{e_0}{r_0 t_0}\cdot\frac{\sqrt{D}}{2})}{\sqrt{D}}\right), \tag{11}$$

$$\sum_{{}^{f}_{e}\succ^{g} \in \mathcal{T}'} \frac{1}{|efg|} = \frac{1}{D}\left(\frac{\mathrm{arctanh}(\frac{\sqrt{D}}{e_0})}{\sqrt{D}} - \frac{1}{e_0}\right), \tag{12}$$

where the summation is over all vertices of $\mathcal{T}'$, each contributing one term.

We can also pass to the limit when $D \to 0$. We get

Theorem 7. Let $\mathcal{T}$ be any topograph of discriminant $D = 0$. Let $\mathcal{T}'$ be the upper half of $\mathcal{T}$ with respect to a root E. Suppose $\mathcal{T}'$ is admissible. Then

$$\sum_{r \succ^{s}_{t} \in \mathcal{T}'} \frac{1}{|rst|} = \left(\frac{e_0}{r_0 t_0}\right)^3 /24, \tag{13}$$

$$\sum_{{}^{f}_{e}\succ^{g} \in \mathcal{T}'} \frac{1}{|efg|} = \left(\frac{1}{e_0}\right)^3 /3. \tag{14}$$

where the summation is over all vertices of $\mathcal{T}'$, each contributing one term.

Corollary 1. As a direct corollary of Theorem 9.12 in [13] and formula (12), for non-square $D > 0$ we get

$$2\log\varepsilon_D = \sum \mathrm{arctanh}\,\frac{\sqrt{D}}{|e|},$$

where the sum involves only edges, adjacent to the vertices on the topograph's river, but not in the river (modulo river period), hence the sum is finite.

1.4. **Method of proof.**

Definition 2. Let $\mathcal{T}''$ be a connected subgraph of $\mathcal{T}'$, containing V_0, such that V_0 has degree 2 in $\mathcal{T}''$ and all other vertices have degree three or one, see Figure 1. An edge $E' \in \mathcal{T}''$ is called a *leaf* if it is not a root and is adjacent to a vertex of $\mathcal{T}''$ of degree one. The set of leaves of $\mathcal{T}''$ is called the *crown* of $\mathcal{T}''$, refer to Figure 1.

Fix an exhaustion $\{\mathcal{T}''_N\}_{N\geq 1}$ of $\mathcal{T}'$ by finite connected subgraphs as in Definition 2, all containing V_0, with crowns C_N consisting of the leaves of $\mathcal{T}''_N$, and with every edge of C_N at graph distance N from the root. At each interior vertex with region labels (r, s, t) and adjacent edge labels (e, f, g) (with appropriate directions) we use the basic telescoping identity

$$\frac{1}{rst} = \frac{1}{D}\left(\frac{e}{rt} - \frac{f}{rs} - \frac{g}{st}\right), \tag{15}$$

so that, upon summing (15) over all vertices of $\mathcal{T}''_N$, the interior contributions cancel pairwise and only (i) a root term and (ii) crown terms supported on C_N remain.

For the limits $N \to \infty$ we impose the smallness conditions on the crown:

$$\text{for (9),(11),(13):} \qquad \max_{E\in C_N} \left|\tfrac{e}{rt}\right| \longrightarrow 0, \qquad \sum_{E\in C_N} \left|\tfrac{e}{rt}\right|^3 \longrightarrow 0, \tag{16}$$

$$\text{for (10),(12),(14):} \qquad \max_{E\in C_N} \tfrac{1}{|e|} \longrightarrow 0, \qquad \sum_{E\in C_N} \tfrac{1}{|e|^3} \longrightarrow 0. \tag{17}$$

These ensure that when we replace arcsin(h)x or arctan(h)x by their first-order approximations ($\arcsin x = x + O(x^3)$, $\arctan x = x + O(x^3)$) on the crown, the total error over C_N tends to 0. With these estimates, the sum over $\mathcal{T}''_N$ equals the root term plus an $o(1)$ crown remainder, and hence converges to the claimed root contribution as $N \to \infty$.

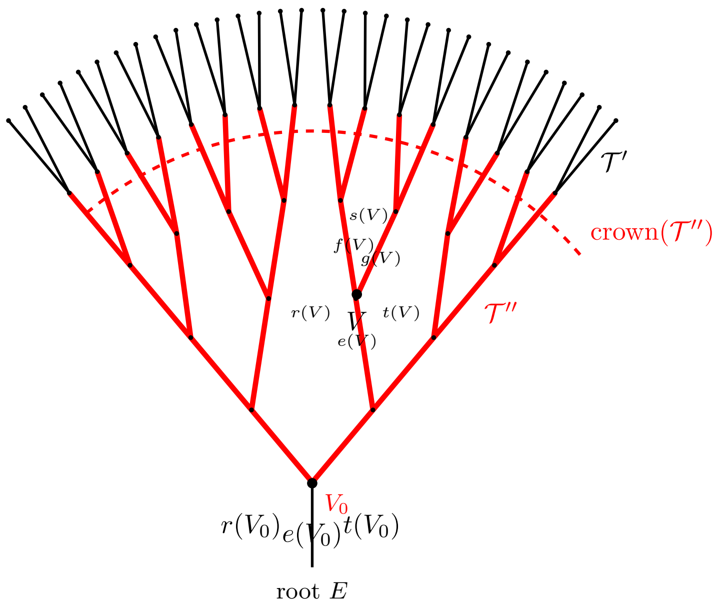

FIGURE 1. Illustration of $\mathcal{T}'$, an upper half of $\mathcal{T}$ with respect to a root E, a subtree $\mathcal{T}''_4$ (in red) with a crown C_4, consisting of leaves intersecting the dashed line. For each vertex V, we may consider the labels $r(V), s(V), t(V)$ on the adjacent regions and the labels $e(V), g(V), f(V)$ on the adjacent edges.

1.5. **Comparison with existing results.** Modular graph functions such as (8) are defined for positively definite quadratic forms. Our approach allows us to consider forms with any discriminant, but we cannot take the sum over all the vertices of the topograph, as this sum diverges for $D \geq 0$. One can consider sums with terms like $\frac{1}{|r^n s^m t^k|}$ for natural n, m, k; in [5] relations between such sums for various m, n, k were deduced. Naturally, we can write the same relations using identities from Section 2.1.

In [13] the sums of summands like $\frac{1}{|rst|}$ are considered for $D < 0$ and the sums with summands like $\frac{1}{|efg|}$ for $D > 0$. We can consider both types of sums for any $D \in \mathbb{R}$, but in the case where the sum over the topograph diverges, we should take only an appropriate half of the topograph, as stated in our theorems. Also, in both [8],[13], there appear sums with $\frac{1}{|r^2 s^2 t^2|}$ and bigger powers; in view of the above discussion about modular graph functions, there is a bunch of relations between them.

To the best of the author's knowledge, the formula for $\log \varepsilon_D$ as in Corollary 1 does not appear in the literature.

It would be nice to interpret our formulae as results of Eisenstein/Schech cocycles evaluation on modular symbols.

1.6. **Plan of the paper.** Section 2 defines topographs and establishes a bunch of useful identities for the labels near the vertices of topographs. Section 3.1 illustrates the main idea, how to get (4) via telescoping. In Section 3.2 we demonstrate how to obtain Mordell-Tornheim series using our approach. In Section 3.3, we show that Hata's series for the Euler constant is also a particular case of our construction. Section 3.4 highlights the geometric meaning of the summands and illustrates the duality between formulas including region labels and formulas including edge labels. Section 4 presents proofs in the general case.

2. Topographs

A topograph is a planar connected 3-valent tree $\mathcal{T}$ with labels on vertices, edges, and regions (connected components of $\mathbb{R}^2 \setminus \mathcal{T}$); all this information encodes the values of a binary quadratic form q and, at the same time, helps to navigate the set of forms $SL(2, \mathbb{Z})$-equivalent to q. Topographs were introduced by J.-H. Conway in [4] in 1997. Topographs provide a powerful geometric tool for visualizing the behavior of binary quadratic forms and understanding their equivalence classes.

Let us start with the graph structure.

Definition 3. A superbase is a triple $v_1, v_2, v_3 \in \mathbb{Z}^2, v_1 + v_2 + v_3 = 0$ and $\{v_1, v_2\}$ forms a basis of $\mathbb{Z}^2$. We consider superbases up to sign, i.e. triples $\{v_1, v_2, v_3\}$ and $\{-v_1, -v_2, -v_3\}$ are equal. Consider a graph $\mathcal{T}$ whose vertices represent all superbases. Let edges connect vertices of the form

$$\{v_1, v_2, -v_1 - v_2\} \text{ and } \{v_1, -v_2, -v_1 + v_2\}.$$

Thus each edge corresponds to four bases $\{\pm v_1, \pm v_2\}$ of $\mathbb{Z}^2$. Each vertex $\{v_1, v_2, v_3\}$ has degree three, with edges corresponding to $\{\pm v_1, \pm v_2\}, \{\pm v_1, \pm v_3\}, \{\pm v_2, \pm v_3\}$.

The graph $\mathcal{T}$ is connected and can be embedded in $\mathbb{R}^2$ without self-intersections; the resulting planar graph is called a topograph. Each region in the complement to the graph corresponds to a primitive vector $\pm v$, see Figure 2 for a local picture near a vertex and an edge.

We label the regions of the topograph with numbers.

Example 2. Let us label the region corresponding to $\pm v$ by $||v||^2 = m^2 + n^2$ where $v = (m, n)$. Recall the *parallelogram law*: for any v_1, v_2,

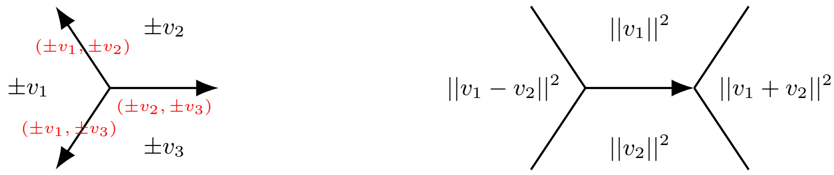

FIGURE 2. Left: local picture near a vertex corresponding to a superbase $\{v_1, v_2, v_3\}$. Right: local picture near an edge with labels corresponding to the quadratic form $q(v) = ||v||^2$.

$$||v_1 - v_2||^2 + ||v_1 + v_2||^2 = 2(||v_1||^2 + ||v_2||^2).$$

Given a binary quadratic form q, label the region corresponding to $\pm v$ by $q(v)$.

Note that $q(v_1 - v_2) + q(v_1 + v_2) = 2(q(v_1) + q(v_2))$. Using this identity, one can recover all the values of a quadratic form on vectors in $\mathbb{Z}^2$ knowing values of q on vectors in a superbase.

Conversely, if we label all the regions of the topograph, such that near every edge as in Figure 3, a), labels satisfy

$$r + u = 2(s + t), \tag{18}$$

these labels determine a unique quadratic form q. Indeed, if

$$q(v_1, v_2) = av_1^2 + bv_1v_2 + cv_2^2$$

then a, b, c can be found from $q(\binom{1}{0}) = a, q(\binom{0}{1}) = b, q(\binom{-1}{-1}) = a + b + c$.

Note that $r, s+t, u$ form an arithmetic progression with difference $g := s + t - r$, so the oriented edge pointing from r to u receives the label g, see Figure 3, a,b), which changes sign if the orientation of the edge is reversed.

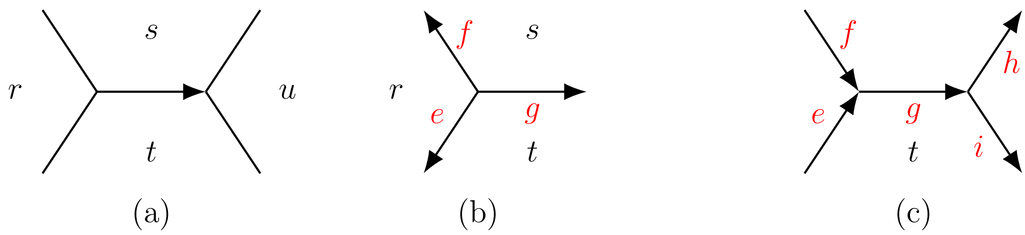

FIGURE 3. Topographs locally.

Similarly define $e = r + t - s, f = r + s - t$, see Figure 3. Then $e + g = 2t$.

Definition 4. The number $D := -ef - fg - eg$, where e, f, g are the oriented edge labels near a vertex(as in Figure 3 b)), is called the discriminant of the topograph.

It is a straightforward computation to verify that D is independent of the choice of vertex. Indeed, on figure c) we have $g - e = 2t = i - g$, and hence

$$(g - e)(g - f) = (i - g)(h - g) = g^2 - eg - gf + ef = g^2 + ih - ig - hg,$$

thus $-eg - gf + ef = ih - ig - hg$, with the orientations as in Figure c).

Given a quadratic form q with an associated bilinear form $B(x, y)$, the label on the edge $\{\pm e_i, \pm e_j\}$ is $|2B(\pm e_i, \pm e_j)|$, with sign depending on the orientation of the edge; for example, for the standard dot product $B(x, y) = x \cdot y$ we have

$$s + t - r = ||x||^2 + ||y||^2 - ||x - y||^2 = 2(x \cdot y).$$

Remark 1. Note that identity $fg = e\cdot(-f)+e\cdot(-g)-D$ (with appropriate choice of directions of edges), for $D = -4$ and after scaling all variables by $\sqrt{-D}/2$, is equivalent to

$$\cot(X)\cot(Y) = \cot(X)\cot(X+Y) + \cot(Y)\cot(X+Y) + 1$$

which is discussed in [17] in connections to various telescoping identities related to $SL(2,\mathbb{Z})$.

2.1. **Useful identities in a topograph.** In this section, we state the identities that form the basis of the telescoping and cancellation argument, as they demonstrate that certain first-order approximations are additive, making their telescoping trivial. Let us recall the following identity:

$$\frac{g}{st}+\frac{f}{rs}+\frac{e}{rt}=\frac{gr+ft+es}{rst}=\frac{g(f+e)+f(e+g)+e(f+g)}{2rst}=\frac{-D}{rst} \tag{19}$$

$$\frac{1}{e}+\frac{1}{f}+\frac{1}{g}=\frac{ef+fg+ge}{efg}=\frac{-D}{efg}. \tag{20}$$

$$\frac{s}{fg}+\frac{r}{ef}+\frac{t}{eg}=\frac{gr+ft+es}{efg}=\frac{g(f+e)+f(e+g)+e(f+g)}{2efg}=\frac{-D}{efg} \tag{21}$$

$$\frac{g}{s^2t^2}+\frac{f}{r^2s^2}+\frac{e}{r^2t^2}=-\frac{6}{rst}-\frac{D(r+s+t)}{r^2s^2t^2} \tag{22}$$

$$\frac{s}{f^2g^2}+\frac{r}{e^2f^2}+\frac{t}{e^2g^2}=-\frac{3}{2efg}-\frac{D(e+f+g)}{2e^2f^2g^2} \tag{23}$$

Interestingly, the combinatorial structure of a topograph also encodes trigonometric and hyperbolic relations, depending on the sign of the discriminant.

Lemma 1. For $D<0$, we have

$$\arcsin(\frac{e}{rt}\cdot\frac{\sqrt{-D}}{2})+\arcsin(\frac{f}{rs}\cdot\frac{\sqrt{-D}}{2})+\arcsin(\frac{g}{st}\cdot\frac{\sqrt{-D}}{2})=0, \tag{24}$$

$$\arctan(\frac{\sqrt{-D}}{e})+\arctan(\frac{\sqrt{-D}}{f})+\arctan(\frac{\sqrt{-D}}{g})=0 \tag{25}$$

for the appropriate branches of arcsin and arctan. If the sum of any two terms does not exceed π in absolute value, then the formula holds for the principal values of arcsin and arctan (that belong to $(-\pi/2,\pi/2)$).

For $D>0$ we have

$$\operatorname{arsinh}(\frac{e}{rt}\cdot\frac{\sqrt{D}}{2})+\operatorname{arsinh}(\frac{f}{rs}\cdot\frac{\sqrt{D}}{2})+\operatorname{arsinh}(\frac{g}{st}\cdot\frac{\sqrt{D}}{2})=0 \tag{26}$$

$$\operatorname{arctanh}(\frac{\sqrt{D}}{e})+\operatorname{arctanh}(\frac{\sqrt{D}}{f})+\operatorname{arctanh}(\frac{\sqrt{D}}{g})=0. \tag{27}$$

Proof. To prove the first identity, note that $-\arcsin A = \arcsin B + \arcsin C$ (for appropriate branches of arcsin) if and only if $-A = B\sqrt{1-C^2}+C\sqrt{1-B^2}$, then we square, and then square again; then all terms cancel. The same proof works for arsinh. For arctan and arctanh these formulas follow from (20) and the identities

$$\arctan(x_1)+\arctan(x_2)+\arctan(x_3)=\arctan\left(\frac{x_1+x_2+x_3-x_1x_2x_3}{1-x_1x_2-x_2x_3-x_3x_1}\right)$$

$$\operatorname{arctanh}(x_1)+\operatorname{arctanh}(x_2)+\operatorname{arctanh}(x_3)=\operatorname{artanh}\left(\frac{x_1+x_2+x_3+x_1x_2x_3}{1+x_1x_2+x_2x_3+x_3x_1}\right)$$

provided the denominators are not zero and $|x_i| < 1$ in the formula for arctanh. $\square$

3. Examples and computations

This section is actually the core of the paper. In Section 3.1 we discuss a telescopic proof of (4), thus highlighting the main idea in the simplest case. In Section 3.2 we obtain Mordell-Torhheim series $\sum\limits_{n,m\geq 1} \frac{1}{n^2m^2(n+m)^2}$ taking a limit by a parameter. In Section 3.3 we derive Hata's series for the Euler constant, using a form $q(n,m) = nm$. In Section 3.4, we explain the geometric meaning of the summand and illustrate the duality between formulas including region labels and formulas including edge labels.

3.1. **Hurwitz series.** Let $\det(x,y)$ denote the determinant of 2×2 matrix with columns $x, y \in \mathbb{Z}^2$. Let

$$A = \left\{(x,y) \mid x,y \in \mathbb{Z}^2_{\geq 0}, \det(x,y) = 1\right\},$$

i.e., the set of pairs of lattice vectors $x = (a,b), y = (c,d)$ in the first quadrant that span lattice parallelograms of oriented area one, i.e. $ad - bc = 1$.

We prove the Hurwitz result (4) via a telescopic method.

Theorem 8.

$$4\sum_A \frac{1}{|x|^2|y|^2|x+y|^2} = \pi.$$

Proof. Define $F(x,y) = \frac{2x\cdot y}{|x|^2|y|^2}$, $F : (\mathbb{Z}^2)^2 \to \mathbb{R}$. An explicit computation shows that

$$F(x,y) - F(x+y,y) - F(x,x+y) = \frac{-4\det(x,y)^2}{|x|^2|y|^2|x+y|^2}. \tag{28}$$

Consider the sum of the expressions

$$F(x,y) - F(x+y,y) - F(x,x+y)$$

over the set $\{(x,y)|x,y \in \mathbb{Z}^2_{\geq 0} \cap [0,n]^2, \det(x,y) = 1\}$. By cancelling identical terms with opposite signs we get $F(\binom{1}{0},\binom{0}{1})$ together with the sum of $-F(x+y,y) - F(x,x+y)$ over the set

$$\left\{(x,y)|x,y \in \mathbb{Z}^2_{\geq 0} \cap [0,n]^2, \det(x,y) = 1, x+y \notin \mathbb{Z}^2_{\geq 0} \cap [0,n]^2\right\}.$$

Note that each element (x,y) of the above set represents a parallelogram spanned by x and y. All these parallelograms have an area of 1, and their angles at the origin partition the angle $\pi/2$ of the first quadrant. Next, $\frac{2x\cdot y}{|x|^2\cdot|y|^2}$ is 2α up to third order terms, where α is the angle between x and y. Indeed, $\sin\alpha\cdot|x||y| = 1$

$$\frac{2x\cdot y}{|x|^2|y|^2} = 2\cos\alpha\sin\alpha = \sin 2\alpha = 2\alpha - \frac{(2\alpha)^3}{3!} + \dots$$

Also, for $\sum_{i=1}^n \alpha_i = \pi/2$ we have that

$$|2\sum_{i=1}^n \sin\alpha_i\cos\alpha_i - \pi| \leq \frac{8\pi}{3}\max_{i=1..N}|\alpha_i|^2 \to 0 \text{ as } \max_{i=1..N}|\alpha_i| \to 0.$$

Thus, since $F(\binom{1}{0},\binom{0}{1}) = 0$, as $n\to\infty$,

$$F(\binom{1}{0},\binom{0}{1}) + \sum(-F(x+y,y) - F(x,x+y)) \to -\pi.$$

Finally, we multiply by -4 from (28) to get the desired formula. $\square$

In terms of topographs, we consider the formula (9)

$$\sum_{r \succ^{s}_{t} \in \mathcal{T}'} \frac{1}{|rst|} = \frac{1}{D}\left(\frac{e_0}{r_0 t_0} - \frac{2\arcsin(\frac{e_0}{r_0 t_0}\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}}\right),$$

for the edge $E = \{\pm\binom{1}{0}, \pm\binom{0}{1}\}$ and the positive-definite quadratic form $q(n,m) = n^2 + m^2$ with $D = -4$, $e_0 = 0, r_0 = t_0 = 1$.

To prove the formula we telescope

$$\frac{e}{rt} = 2\frac{x \cdot y}{|x|^2|y|^2}.$$

The $\frac{1}{D}$ in the formula is -4 in (28). The only surviving term $\frac{e_0}{r_0 t_0}$ is zero in this case.

Each term at the crown, up to third order, is

$$\frac{2\arcsin(\frac{e_0}{r_0 t_0}\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}} = \arcsin(\frac{e_0}{r_0 t_0}).$$

So the geometric meaning of the terms that we telescope is the angle between vectors. Note that to obtain the telescoping relation, we may consider other quadratic forms, not necessarily $q(v) = ||v||^2$. This leads to telescoping identitites over topographs, i.e. formulas 9,10,11,12,13,14.

3.2. **Mordell-Tornheim series.** Let us pick $\mu > 0$ and compute the following sum:

$$\sum_{\mu} = \sum_{\substack{a \geq b \\ c \geq d \\ ad - bc = 1}} \frac{2\mu^2}{(a^2 + \mu^2 b^2)(c^2 + \mu^2 d^2)\left((a+c)^2 + \mu^2(b+d)^2\right)}.$$

This is equivalent to computing

$$\sum_{\mu} = \sum \frac{2|\det(x,y)|^2}{|x|^2|y|^2|x+y|^2},$$

where

$$x, y \in \mathbb{Z}_{\geq 0}\cdot\binom{1}{0} + \mathbb{Z}_{\geq 0}\cdot\binom{1}{\mu},$$

Thus, we can use the same method and the same function F as in the proof of Theorem 8.

Performing telescoping, we get

$$\sum_{\mu} = -\left(F\left(\binom{1}{0}, \binom{1}{\mu}\right) - \sum_{(u,v)\in\text{crown}} F(u,v)\right),$$

since the determinant of the matrix formed by these basis vectors is μ, refer to (28). Therefore, we proved

Theorem 9.

$$\sum_{\substack{a \geq b \\ c \geq d \\ ad - bc = 1}} \frac{1}{(a^2 + \mu^2 b^2)(c^2 + \mu^2 d^2)\left((a+c)^2 + \mu^2(b+d)^2\right)} = \frac{1}{2\mu^2}\left(\frac{\arctan\mu}{\mu} - \frac{1}{1+\mu^2}\right).$$

Taking the limit as $\mu \to 0$ yields $\sum_\mu /2\mu^2 \to 1/3$. Indeed,

$$\sum_{\substack{a,c\geq 1\\ \gcd(a,c)=1}} \frac{1}{a^2c^2(a+c)^2} = \frac{1}{3}, \tag{29}$$

because for each such a pair (a,c) there exists a unique pair $(b,d) \in \mathbb{Z}^2_{\geq 0}$ such that $ad - bc = 1, b \leq a, d \leq c$. This identity (29) is well-known [15, 9].

Since on both sides of equality we have analytic functions, we may substitute $\mu = \frac{i}{2}$. Then,

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}} \frac{1}{(4a^2-b^2)(4c^2-d^2)\left(4(a+c)^2-(b+d)^2\right)} =$$

$$= \frac{1}{64}\cdot\left(\frac{\arctan\frac{i}{2}}{\frac{i}{2}} - \frac{1}{1+(\frac{i}{2})^2}\right)\Big/\left(2\cdot\left(\tfrac{i}{2}\right)^2\right) = \frac{\frac{4}{3}-\ln 3}{32}.$$

Remark 2. Since

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}} \frac{1}{(a^2+\mu^2b^2)(c^2+\mu^2d^2)\left((a+c)^2+\mu^2(b+d)^2\right)} = \frac{1}{2\mu^2}\left(\frac{\arctan\mu}{\mu} - \frac{1}{1+\mu^2}\right) =$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{k+1}{2k+3}\mu^{2k} = \frac{1}{3} - \frac{2}{5}\mu^2 + \frac{3}{7}\mu^4 - \ldots.$$

we can derive the identities by looking for other coefficients, for example

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}} \frac{1}{a^2c^2(a+c)^2}\left(\frac{b^2}{a^2} + \frac{d^2}{c^2} + \frac{(b+d)^2}{(a+c)^2}\right) = \frac{2}{5}\ .$$

3.3. **Hata's series.** Let us examine a series for γ, due to Hata Masayoshi.

Definition 5. Let $\mathcal{F}$ denote the set of ordered pairs of fractions $\left(\frac{a}{b}, \frac{c}{d}\right)$ in lowest terms such that: $0 \leq \frac{a}{b} < \frac{c}{d} \leq 1$, $ad - bc = -1$. Thus $\mathcal{F}$ is the set of pairs of consecutive Farey fractions. Let $\mathcal{F}^* = \left\{\left(\frac{a}{b}, \frac{c}{d}\right) = \left(\frac{0}{1}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$.

Masayoshi Hata proved the following theorem.

Theorem 10 ([9])**.** In the above notation

$$\gamma = \frac{1}{2} + \frac{1}{2}\sum_{\left(\frac{a}{b},\frac{c}{d}\right)\in\mathcal{F}\setminus\mathcal{F}^*} \frac{1}{abcd(a+c)(b+d)}.$$

Hata's proof is very nice. He studies presentations of functions in a certain Schauder basis associated with pairs of consecutive Farey fractions. Then, using an identity of Parseval-type for the function $\psi(x) = x\{\frac{1}{x}\}(1 - \{\frac{1}{x}\})$ he arrives to the above theorem.

Note that

$$\frac{1}{abcd(a+c)(b+d)} = \frac{1}{q(x)q(y)q(x+y)} = \frac{1}{rst}$$

for the quadratic form $q(v) = mn$, for $v = (m,n)$. Thus, this formula can be deduced using our telescopic method.

Lemma 2. For $a, b, c, d \geq 0$ with $ad - bc = -1$ one has

$$\operatorname{arsinh}\left(\frac{ad+bc}{2abcd}\right) = \log\left(\frac{ad}{bc}\right).$$

Proof. By a direct check, we see that

$$\sqrt{1+\left(\frac{(ad+bc)}{2abcd}\right)^2}=\frac{(ad+bc)^2+1}{4abcd},$$

then,

$$\operatorname{arsinh}\left(\frac{ad+bc}{2abcd}\right)=\log(\frac{(ad+bc)}{2abcd}+\sqrt{1+\left(\frac{(ad+bc)}{2abcd}\right)^2})=$$

$$\log(\frac{(ad+bc)^2+2(ad+bc)+1}{4abcd})=\log(\frac{(ad)^2}{abcd})=\log(\frac{ad}{bc}).$$

□

Thus, we see that

$$\operatorname{arsinh}\left(\frac{ad+bc}{2abcd}\right)=\operatorname{arsinh}\left(\frac{a(b+d)+b(a+c}{2ab(a+c)(b+d)}\right)+\operatorname{arsinh}\left(\frac{(a+c)d+(b+d)c}{2(a+c)(b+d)cd}\right).$$

In order to find $\sum\frac{1}{rst}$ one take a telescopic sum of $\frac{e}{rt}-\frac{g}{st}-\frac{h}{rs}$ which, in our case, becomes (since $e=q(x+y)-q(x)-q(y)$)

$$F(\binom{a}{b},\binom{c}{d})=\frac{ad+bc}{abcd}\sim 2\operatorname{arsinh}\left(\frac{ad+bc}{2abcd}\right),$$

if it is small.

Therefore

$$\sum_{(\frac{a}{b},\frac{c}{d})\in\mathcal{F}\setminus\mathcal{F}^*}\frac{1}{abcd(a+c)(b+d)}=\sum_{n=1}^{\infty}(F(\binom{1}{n},\binom{1}{n+1})-2\operatorname{arsinh}\left(\frac{1\cdot(n+1)+n\cdot 1}{2n(n+1)}\right))=$$

$$=2\lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n)-1=2\gamma-1.$$

Using the topograph relation (22) and the same strategy of proof we obtain

Theorem 11. In the above notation

$$\sum_{(\frac{a}{b},\frac{c}{d})\in\mathcal{F}\setminus\mathcal{F}^*}\frac{ab+cd+(a+c)(b+d)}{(abcd(a+c)(b+d))^2}=7-12\gamma.$$

3.4. **Duality: inside and outside the circle.** The telescopic identities have a natural geometric interpretation, revealing a duality between sums of the products of reciprocals to region labels and sums of the products of reciprocals to edge labels of the topograph. The product of the region labels at a vertex corresponds to the area of the inscribed triangle. In contrast, the product of the edge labels gives the area of the triangle formed by the tangent lines at the corresponding points, cf. Legendre duality in [12].

To relate the Farey tessellation to points on a unit circle, we use the rational parametrization of $x^2+y^2=1$ by

$$f:\binom{a}{b}\to\left(\frac{2ab}{a^2+b^2},\frac{a^2-b^2}{a^2+b^2}\right).$$

Note that $f(\binom{1}{0}) = (0,1), f(\binom{0}{1}) = (0,-1)$. By a direct calculation, the area of the triangle with vertices $f(\binom{a}{b}), f(\binom{c}{d}), f(\binom{a+c}{b+d})$ is equal to

$$\frac{2|ad-bc|^3}{(a^2+b^2)\cdot(c^2+d^2)\cdot((a+c)^2+(b+d)^2)} = \frac{2}{rst}. \tag{30}$$

Here r, s, t denote the values of the quadratic form $q(n,m)$ on a superbase vectors $\left\{\binom{a}{b}, \binom{c}{d}, -\binom{a+c}{b+d}\right\}$, the notation that we use for the topograph's labels on regions.

Theorem 12 ([10])**.**

$$\sum_{\substack{a,b,c,d\in\mathbb{Z}_{\geq 0}\\ ad-bc=1}} \frac{1}{(a^2+b^2)(c^2+d^2)((a+c)^2+(b+d)^2)} = \frac{\pi}{4}.$$

Proof. The area of the right half of the unit disc is $\pi/2$ and the triangles with vertices

$$f(\binom{a}{b}), f(\binom{c}{d}), f(\binom{a+c}{b+d}), a, b, c, d \geq 0, ad-bc-1$$

tile it completely. So we divide (30) by two and sum over all tuples (a,b,c,d) with $a, b, c, d \geq 0, ad-bc-1$. □

This reasoning is due to the original article of A. Hurwitz [10]. Hurwitz's proof works for any positive-definite binary quadratic form q (the above case corresponds to $q(v) = ||v||^2, v \in \mathbb{Z}^2$) and consists of using a rational parametrization of a quadric curve to cut its interior into triangles corresponding to consecutive Farey fractions $(\frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d})$, and then the areas of triangles are proportional to

$$(q(a,b)\cdot q(c,d)\cdot q(a+c,b+d))^{-2}.$$

Alternatively, consider the tangent lines $l_{a,b}$ to the unit circle at points $f(\binom{a}{b})$. The area of the triangle formed by these lines $l_{a,b}, l_{c,d}, l_{a+c,b+d}$ is

$$\frac{|ad-bc|^3}{(ac+bd)(a(a+c)+b(b+d))((a+c)c+(b+d)d)} = \frac{8}{efg}.$$

Note that the dot products in the denominators are exactly e, f, g on the edges in the topograph. Since these triangles tile the domain between the circle and the tangents at $(0,1)$ and $(1,0)$, we can evaluate the sum $\sum \frac{1}{efg}$, namely:

Theorem 13.

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}} \frac{1}{(ac+bd)(a(a+c)+b(b+d))((a+c)c+(b+d)d)} = \frac{1-\pi/4}{8}.$$

Remark 3. This identity may be generalized by deforming the lattice as in Section 3.2. Consider the vectors $a\binom{1}{0} + b\binom{1}{\mu}$ and draw the tangent lines at points $f(\binom{a}{\mu b})$. Then the following sum is equal to the area of a part between of the unit circle and tangents to it at $(0,1)$ and $\frac{2\mu}{\mu^2+1}, \frac{\mu^2-1}{\mu^2+1}$, namely

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}} \frac{\mu^3}{(ac+\mu^2 bd)(a(a+c)+\mu^2 b(b+d))((a+c)c+\mu^2(b+d)d)} = \mu - \arctan\mu.$$

Dividing by μ^3 and substituting $\mu = 0$ we get (29) one more time:

$$\sum_{\gcd(a,c)=1} \frac{1}{(ac)(a(a+c))((a+c)c)} = \frac{1}{3}.$$

Investigating the next coefficient in the Taylor series of arctan, we get

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}} \frac{1}{a^2c^2(a+c)^2}\left(\frac{bd}{ac}+\frac{b(b+d)}{a(a+c)}+\frac{(b+d)d}{(a+c)c}\right)=\frac{1}{5}.$$

We may also plug $\mu = i/2$, getting

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}} \frac{1}{(4ac-bd)(4a(a+c)-b(b+d))(4(a+c)c-(b+d)d)} =$$

$$= \frac{1}{64(i/2)^3}(i/2-\arctan i/2) = \frac{\ln 3 - 1}{16}.$$

3.5. **Non-square positive discriminants: proof of Corollary 1.** We treat the case of a non-square discriminant $D > 0$. Throughout, $\mathcal{T}$ denotes a topograph of discriminant D, with edge labels (e, f, g) and adjacent region labels (r, s, t) at a vertex as in Section 2.

For $D > 0$ there is a unique periodic bi-infinite *river* (a bi-infinite path) on $\mathcal{T}$ which separates regions with labels of different signs. Following O'Sullivan, define the modified topograph $\mathcal{T}^\star$ by keeping T unchanged except that every *river edge* is relabeled by $\sqrt{D}$; the region labels off the river remain unchanged. One then sums *edge* contributions over $\mathcal{T}^\star$ modulo the river's fundamental period. In O'Sullivan's formulation this yields the class number series for $D > 0$ in terms of edge labels [13, Thm. 9.2]:

$$D^{3/2} \sum_{(e,f,g)\in\mathcal{T}^\star/\mathrm{period}} \frac{1}{|efg|} \;=\; 2\log\varepsilon_D,$$

with the sum taken over one river period of $\mathcal{T}^\star$ (one term per vertex).

Now, take any edge E with a source on the river and the target outside of the river, as a root for $\mathcal{T}'$, a half of $\mathcal{T}$. As it is stated in (12)

$$\sum_{\overset{f\ \ g}{\underset{e}{\succ}} \in \mathcal{T}'} \frac{1}{|efg|} = \frac{1}{D}\left(\frac{\operatorname{arctanh}(\frac{\sqrt{D}}{e})}{\sqrt{D}} - \frac{1}{e}\right),$$

where e is the label on E.

Summing over all the vertices in the river period (recall that the labels on the river edges are $\sqrt{D}$) and all such edges we get

$$2\log\varepsilon_D = D^{3/2}\sum_{(e,f,g)\in\mathcal{T}^\star/\mathrm{period}}\frac{1}{|efg|} = \sum\frac{1}{e}+\sum(\operatorname{arctanh}\frac{\sqrt{D}}{|e|}-\frac{1}{e}) = \sum\operatorname{arctanh}\frac{\sqrt{D}}{|e|},$$

where the sum involves only edges E with labels e, adjacent to the vertices on the topograph's river, but not in the river (modulo river period), hence the sum is finite.

3.6. **Class number for square discriminants.** In the case of a *square* discriminant $D = m^2$, the usual topograph features a *river* between two regions (*lakes*) with zero labels, so the vertex identity used in our telescoping arguments appears to degenerate.

Define an auxiliarly function

$$W_1(x) = 2\int_0^\infty \Re\left(\frac{y}{(y^2+1)(e^{\pi(y+2ix)}-1)}\right)dy.$$

It is immediate that $W_1(x)$ is 1-periodic and even.

Theorem 14. [1] For each real x with $\frac{1}{2} \pm x \notin \{0, -1, -2, \dots\}$,

$$2W_1(x) + \log 4 + \psi\left(\tfrac{1}{2}+x\right) + \psi\left(\tfrac{3}{2}-x\right) = 0. \tag{31}$$

In particular, (31) holds for all rational numbers $x = r/m \in (0, 1)$.

Theorem 15 (O'Sullivan, Thm. 9.4)**.** Let $\mathcal{T}$ be any topograph of square discriminant $D = m^2 > 1$. Form the modified topograph $\mathcal{T}^\star$ by keeping $\mathcal{T}$ unchanged except that every river edge directed rightwards is relabeled by $\sqrt{D} = m$. Let r and s be the congruence classes modulo m of the region labels adjacent to a lake. Then

$$m^3 \sum_{\substack{(e,f,g)\in T^\star \\ \text{vertex not on a lake}}} \frac{1}{|efg|} \;+\; W_1\Big(\frac{r}{m}\Big) \;+\; W_1\Big(\frac{s}{m}\Big) \;=\; 2\log\bigg(\frac{m}{2\ \gcd(m,r)}\bigg).$$

(Each vertex contributes one term in the sum.) For $D = m = 1$, the identity remains valid after adding 2 to the right-hand side.

Let $\gcd(m, r) = 1$. Near the lake with nearby label equal to $r \pmod m$, the labels on edges are

$$e_i = 2r + (2k+1)m, k = 0, 1, \dots, l_i = 2r - (2k+1)m, k = 1, 2, \dots$$

Thus, using (12) we evaluate the sum of $\frac{1}{|efg|}$ as

$$\sum(\operatorname{arctanh}\frac{m}{|e_i|} - \frac{1}{e_i}) + \sum(\operatorname{arctanh}\frac{m}{|l_i|} - \frac{1}{l_i}).$$

Thus, after simplification, we obtain

$$2\log\bigg(\frac{m}{2\ \gcd(m,r)}\bigg) - (W_1\Big(\frac{r}{m}\Big) \;+\; W_1\Big(\frac{s}{m}\Big)) = m^3 \sum_{\substack{(e,f,g)\in T^\star \\ \text{vertex not on a lake}}} \frac{1}{|efg|} =$$

$$= -\frac{1}{2}(2\gamma + \log\frac{r(m-r)}{m^2} + \psi(\frac{r}{m}+\frac{1}{2}) + \psi(\frac{3}{2}-\frac{r}{m}) + 2\gamma + \log\frac{s(m-s)}{m^2} + \psi(\frac{s}{m}+\frac{1}{2}) + \psi(\frac{3}{2}-\frac{s}{m})) + \sum_{river} \operatorname{arctanh}\frac{m}{|e|}.$$

Using (31) we obtain

$$\sum_{river} \operatorname{arctanh}\frac{m}{|e|} = 2\gamma - \log 4 + \frac{1}{2}\log(r(m-r)s(m-s)).$$

Unfortunately, it is hard to tell the labels on the edges adjacent to the river, so we stop here.

Then, recall the class–number identity (9.29) (for $m > 1$) is

$$h(m^2)\ \log\Big(\frac{m}{2}\Big) = \sum_{\substack{b^2-4ac=m^2 \\ \gcd(a,b,c)=1 \\ a,c,\, a+b+c>0}} \frac{m^3}{3\, b(b+2a)(b+2c)} + \sum_{\substack{[a,b,c]\ Z\text{-reduced} \\ b^2-4ac=m^2 \\ \gcd(a,b,c)=1}} \frac{m}{b} + \sum_{\substack{1\le r<m \\ (r,m)=1}} W_1\Big(\frac{r}{m}\Big). \tag{32}$$

Here the first sum ranges over primitive triples $[a,b,c]\in\mathbb{Z}^3$ of discriminant m^2 with the sign condition $a, c, a+b+c>0$; the second sum runs over primitive $\mathbb{Z}$-reduced representatives $[a,b,c]$ (in the paper's sense of reduction).

From (31) and the Gauss multiplication theorem for ψ, one finds

$$\sum_{\substack{1\le r<m\\(r,m)=1}} W_1\Big(\frac{r}{m}\Big) = \varphi(m)\log\Big(\frac{m}{2}\Big) + \varphi(m)\sum_{p\mid m}\frac{\log p}{p-1} - m\sum_{d\mid m}\frac{\mu(d)}{d}\,\psi\Big(\frac{m}{2d}\Big). \tag{33}$$

It holds that $h(m^2)=\varphi(m)$ for $m>1$. Substituting (33) into (32) and replacing $h(m^2)$ by $\varphi(m)$, the terms $\varphi(m)\log(\frac{m}{2})$ cancel and we obtain:

Proposition 1. For every integer $m>1$,

(34)

$$\sum_{\substack{b^2-4ac=m^2\\ \gcd(a,b,c)=1\\ a,c,\,a+b+c>0}}\frac{m^3}{3\,b(b+2a)(b+2c)} + \sum_{\substack{[a,b,c]\ \mathbb{Z}\text{-reduced}\\ b^2-4ac=m^2\\ \gcd(a,b,c)=1}}\frac{m}{b} = m\sum_{d\mid m}\frac{\mu(d)}{d}\,\psi\Big(\frac{m}{2d}\Big) - \varphi(m)\sum_{p\mid m}\frac{\log p}{p-1}.$$

In case $m=p$ a prime odd number, we get

$$\sum_{\substack{b^2-4ac=p^2\\ \gcd(a,b,c)=1\\ a,c,\,a+b+c>0}}\frac{p^3}{3\,b(b+2a)(b+2c)} + \sum_{\substack{[a,b,c]\ \mathbb{Z}\text{-reduced}\\ b^2-4ac=p^2\\ \gcd(a,b,c)=1}}\frac{p}{b} = p(\psi(\frac{p}{2})-\frac{1}{p}\psi(\frac{1}{2}))-\log p =$$

$$p(-\gamma-2\log 2+\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k-1/2}-\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{k})-(-2\log 2-\gamma)-\log p.$$

4. Proofs

We now generalize the telescoping argument of Section 3.1 to prove Theorems 5,6 for arbitrary discriminant D.

Our arguments rely on the trigonometric identities derived in Section 2.1. To proceed, we need the following lemma, in the notation presented in Figure 4.

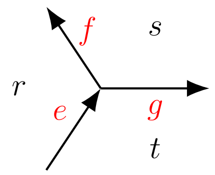

Figure 4. Topographs locally

Lemma 3. Consider a topograph with discriminant $D<0$ and its vertex with labels as in Figure 4. If the right-hand side of the following formula lies in $(-\pi/2,\pi/2)$, we have

$$\arcsin(\frac{e}{rt}\cdot\frac{\sqrt{-D}}{2}) = \arcsin(\frac{f}{rs}\cdot\frac{\sqrt{-D}}{2}) + \arcsin(\frac{g}{st}\cdot\frac{\sqrt{-D}}{2})$$

$$\arctan(\frac{\sqrt{-D}}{e}) = \arctan(\frac{\sqrt{-D}}{f}) + \arctan(\frac{\sqrt{-D}}{g})$$

For a topograph with discriminant $D>0$, we have

$$\operatorname{arsinh}(\frac{e}{rt}\cdot\frac{\sqrt{D}}{2}) = \operatorname{arsinh}(\frac{f}{rs}\cdot\frac{\sqrt{D}}{2}) + \operatorname{arsinh}(\frac{g}{st}\cdot\frac{\sqrt{D}}{2})$$

$$\operatorname{arctanh}(\frac{\sqrt{D}}{e}) = \operatorname{arctanh}(\frac{\sqrt{D}}{f}) + \operatorname{arctanh}(\frac{\sqrt{D}}{g})$$

Proof. Follows immediately from Lemma 1 since we only change the orientation of the edge with label e. □

Before the proofs of the main theorem we can give the definition of admissibility of a half of a topograph.

Definition 6 (Admissible upper half)**.** Let $\mathcal{T}$ be a topograph of discriminant $D \in \mathbb{R}$, let E be an oriented root with adjacent region labels (r_0, t_0) and edge label e_0, and let $\mathcal{T}'$ be the component in the target direction of E. We call $\mathcal{T}'$ *admissible* if:

(A1) (No poles) For every vertex $V \in \mathcal{T}'$, all region labels $r(V), s(V), t(V)$ are nonzero.

(A2) (Decay along rays) Along any infinite ray in T' starting at the root vertex V_0, we have

$$\frac{e(V)}{r(V)\,t(V)} \longrightarrow 0 \qquad \text{and} \qquad \frac{1}{|e(V)|} \longrightarrow 0.$$

(For $D > 0$ this means T' contains no river edges.)

(A3) (Summable error on crowns) For any exhaustion $T''_N \subset T'$ by finite connected subtrees with crown C_N at graph distance N from V_0,

$$\sum_{V \in C_N} \Big(\frac{e(V)}{r(V)\,t(V)}\Big)^3 \xrightarrow[N\to\infty]{} 0 \quad \text{and} \quad \sum_{E \in C_N} \frac{1}{|e(E)|^3} \xrightarrow[N\to\infty]{} 0.$$

Outline of the method. Fix an oriented root edge with root vertex V_0 and an admissible half $T' \subset T$. Choose an exhaustion $\{T''_N\}_{N\ge 1}$ of T' by finite connected subgraphs containing V_0, whose leaves form the crown C_N at graph distance N from V_0. At each degree–3 vertex with region labels (r, s, t) and edge labels (e, f, g) we use a local identity that admits an addition law:

- for the *region* versions (negative/positive D), the arcsine/arsinh identity applied to $\frac{e}{rt}, \frac{f}{rs}, \frac{g}{st}$;
- for the *edge* versions, the arctan/artanh identity applied to $\frac{1}{e}, \frac{1}{f}, \frac{1}{g}$ under $ef + fg + ge = -D$.

Summing the local identity over all degree–3 vertices of T''_N makes interior contributions cancel pairwise (telescoping on the tree). Only a fixed *root term* (depending on the two root edges) and the *crown* sum over C_N remain. On the crown we linearize the special function using

$$\arcsin / \arctan / \operatorname{arsinh} / \operatorname{artanh}(x) = x + O(x^3),$$

and impose the corresponding smallness conditions:

$$\max_{E\in C_N} \left|\tfrac{e}{rt}\right| \to 0, \quad \sum_{E\in C_N} \left|\tfrac{e}{rt}\right|^3 \to 0 \quad \text{for (9),(11)}, \qquad \max_{E\in C_N} \tfrac{1}{|e|} \to 0, \quad \sum_{E\in C_N} \tfrac{1}{|e|^3} \to 0 \quad \text{for (10),(12)}.$$

These bounds make the total cubic error on C_N go to 0, so the crown sum equals its linearization $+\, o(1)$. Letting $N \to \infty$ yields a *half–graph limit* equal to the root term. We repeat the construction on the complementary half of T and add the two limits; choosing the continuous branch of the inverse trigonometric/hyperbolic function across the root produces the expected π (or hyperbolic) jump, giving the constants. For $D > 0$ we either choose crowns that avoid river vertices (no zero region labels) or work on the regularized topograph $T^\star$ where river labels are replaced as prescribed; the same estimates apply.

Proof of Theorem 5, part (9). In the notation of Figure 4, the identity (19) becomes

$$\frac{1}{D}\left(\frac{e}{rt}-\frac{f}{rs}-\frac{g}{st}\right)=\frac{1}{rst}. \tag{35}$$

This is immediate generalisation of (28) for $q(v)=||v||^2$, because $F(x,y)$ in (28) is $\frac{e}{rt}$ and $D=-4$. Summing (35) over the vertices of $V\in\mathcal{T}''$ of degree bigger than one, we see that all the intermediate terms cancel, and only the terms corresponding to the root and the edges in the crown survive, so we have

$$\sum_{V\in\mathcal{T}''}\frac{1}{rst}=\frac{1}{D}\left(\frac{e}{rt}(\text{root})-\sum_{V\in\text{crown}(\mathcal{T}'')}\frac{e}{rt}(V)\right).$$

Now, if $\frac{e}{rt}(V)\to 0$ as the distance from V to the root grows, then

$$\sum_{V\in\text{crown}(\mathcal{T}'')}\frac{e}{rt}(V)-\sum_{V\in\text{crown}(\mathcal{T}'')}\frac{\arcsin(\frac{e}{rt}(V)\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}/2}=0$$

when the distance between the crown and the root tends to infinity. Now, it follows from Lemma 1 that

$$\sum_{V\in\text{crown}(\mathcal{T}'')}\frac{\arcsin(\frac{e}{rt}(V)\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}/2}=\frac{\arcsin(\frac{e}{rt}(\text{root})\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}/2}$$

and this finishes the proof. $\square$

The proof of Theorem 6 part (11) is identical; the only difference is that we use formulas for arsinh.

So, when we telescope $\frac{1}{rst}$ over both branches $\mathcal{T}',\mathcal{T}\setminus\mathcal{T}'$ of a topograph, we get

$$\frac{1}{D}\left(\frac{e_0}{r_0t_0}-\frac{2\arcsin(\frac{e_0}{r_0t_0}\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}}\right)+\frac{1}{D}\left(\frac{-e_0}{r_0t_0}-\frac{2\arcsin(\frac{-e_0}{r_0t_0}\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}}\right)=\frac{2\cdot 2\pi}{(-D)^{3/2}},$$

i.e., exactly (9), and then it implies the Hurwitz class number formula.

Telescoping $\frac{1}{efg}$ (labels on edges) over a topograph gives the class number formula for $D>0$.

Proof of Theorem 5, part (10). Here we will telescope $\frac{s}{fg}$ and use the telescoping identity (21).

Denote the edges of the region s by $f=f_0,f_1,\ldots$ on the left and $g=g_0,g_1,\ldots$ on the right. Note that $f_{k+1}=f_k+2s, g_{k+1}=g_k+2s$ and

$$\sum_{k=0}^{\infty}\frac{s}{f_kf_{k+1}}=\frac{s}{2s}\sum_{k=0}^{\infty}\left(\frac{1}{f_k}-\frac{1}{f_{k+1}}\right)=\frac{1}{2f_0}.$$

Therefore in the sum of $\frac{s}{fg}$ we have $\frac{1}{2e_0}$ twice, all intermediate terms cancel because $\frac{1}{2f_0}+\frac{1}{2g_0}-\frac{s}{f_0g_0}=0$ for $s=\frac{f_0+g_0}{2}$. The sum of the terms at the crown is $-\sum\frac{1}{e}$ over all e in the crown. Then we use the approximation

$$\frac{\sqrt{-D}}{e}\sim\arctan\left(\frac{\sqrt{-D}}{e}\right)$$

at the edges of the crown and Lemma 3 to conclude the proof of the theorem. $\square$

The second part of Theorem 6 is proven similarly.
Theorem 7 is proven by taking the limit $D \to 0$.

5. Acknowledgements

In 2024, I discovered a simple telescopic proof [11] of (2) without knowing all the above story about topographs that I learned recently, thanks to users of MathOverflow [2]. I thank Wadim Zudilin, Mikhail Shkolnikov, and Ernesto Lupercio for discussions.

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