The starting point

Let

$$A = \{(x, y) \mid x, y \in \mathbb{Z}^2_{\geq 0}, \det(x \;\; y) = 1\},$$

i.e., the set of pairs of lattice vectors in the first quadrant that span lattice parallelograms of the oriented area one. Then,

Theorem

$$4 \sum_A \frac{1}{|x|^2 \cdot |y|^2 \cdot |x+y|^2} = \pi.$$

We will discuss several proofs, it would be nice if someone finds more proofs.

The goal of the lectures is to interpret this formula in several contexts and reveal connections to other subjects.

Conway's topographs

A topograph is a planar connected 3-valent tree with labels on vertices, edges, and regions.

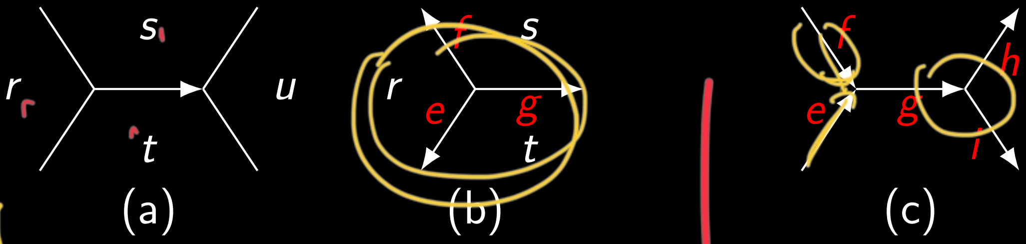


Figure: Topographs locally

Labels on regions satisfy $\boxed{r + u = 2(s + t)}$. So, $r, s + t, u$ form an arithmetic progression with difference $g := s + t - r$. Note: $e + g = 2t$. Orient the edge such that the label e on it is ≥ 0.

$D := -ef - fg - eg$ is the same for all vertices, and is called the discriminant of the topograph.

Indeed, on figure c) we have $g - e = 2t = i - g$ and

+eg +gf +ef ih+ig+hg

$$(g - e)(g - f) = (i - g)(h - g) = g^2 - eg - gf + ef = g^2 + ih - ig - hg$$

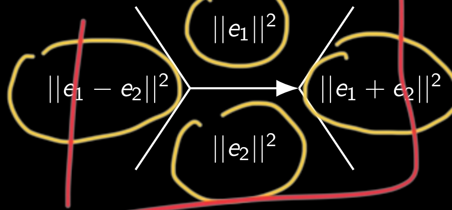
How this is related to the quadratic forms?

$q(e_1)$ $q(e_1-e_2)$ $q(e_2)$ $q(e_1+e_2)$

Let e_1, e_2 be a basis of $\mathbb{Z}^2$.

Then put

$$s = ||e_1||^2, t = ||e_2||^2, r = ||e_1 - e_2||^2, u = ||e_1 + e_2||^2.$$



We obtain the so-called parallelogramm law:

$$||e_1 - e_2||^2 + ||e_1 + e_2||^2 = 2(||e_1||^2 + ||e_2||^2).$$

The same holds for any bilinear form $B(x, y), x = (x_1, x_2), y = (y_1, y_2)$,

$$B(x, y) = ax_1y_1 + b(x_1y_2 + x_2y_1) + cx_2y_2,$$
$$q(x) = B(x, x).$$

Then,

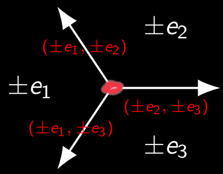
$$q(x + y) + q(x - y) = 2(q(x) + q(y)).$$

Labels of topographs

$e_1, e_2, -(e_1+e_2)$ $(e_1, -e_2, e_2-e_1)$

$(e_1, e_2, e_3) = (-e_1, -e_2, -e_3)$

Here is another definition of a topolgraph.
Consider a graph, all of whose vertices are marked by superbases (i.e. triples $e_1, e_2, e_3 \in \mathbb{Z}^2$, $e_1 + e_2 + e_3 = 0$ and (e_1, e_2) forms a basis of $\mathbb{Z}^2$) up to sign.

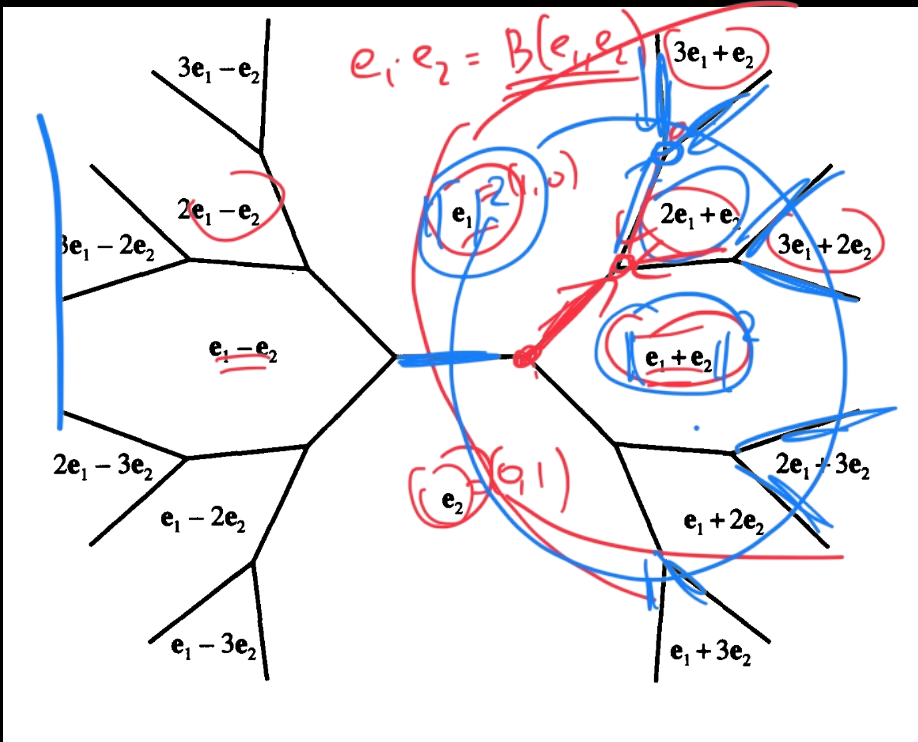


Edges are labeled by bases (e_1, e_2) up to sign.

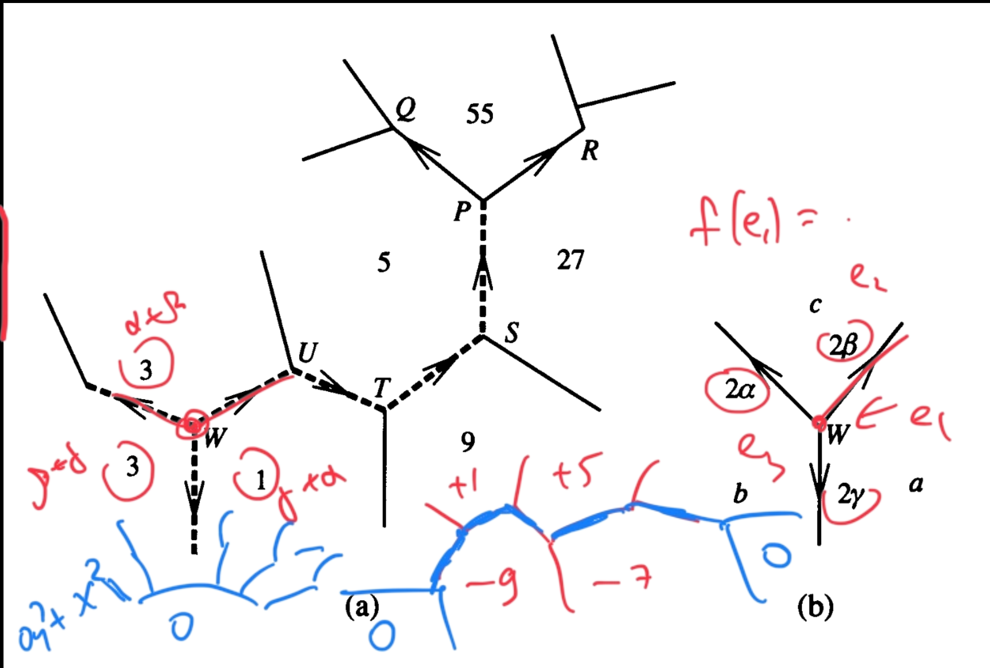
To get the labels from the previous slide:
Given $q(x), B(x, y)$ write $q(e_i)$ on the region labelled by e_i.

Labels on edges are $|2B(\pm e_i, \pm e_j)|$ where we can choose signs arbitrarily; indeed

$$||x||^2 + ||y||^2 - ||x + y||^2 = -2(x \cdot y).$$



from Conway's "The sensual quadratic form" book.



a) an example how to use the topograph: for the positive definite form there is a unique "well".

b) Sellings formula ($\alpha, \beta, \gamma \geq 0$):

$f(m_1 e_1 + m_2 e_2 + m_3 e_3) = \alpha(m_2 - m_3)^2 + \beta(m_1 - m_3)^2 + \gamma(m_1 - m_2)^2$

Proof of the identity from the beginning

Recall that we consider

$$A = \{(x,y) \mid x,y \in \mathbb{Z}^2_{\geq 0}, \det(x \;\; y) = 1\}, \quad 2\sum_A \frac{1}{|x|^2 \cdot |y|^2 \cdot |x+y|^2} = \pi/2.$$

(a,b)

$ad - bc = 1$

Proof, (N.K.).

Define $F(x,y) = \frac{x \cdot y}{|x|^2 \cdot |y|^2}$, $F : (\mathbb{Z}^2)^2 \to \mathbb{R}$. Then,

$$F(x,y) - F(x+y,y) - F(x,x+y) = \frac{-2\det(x \;\; y)^2}{|x|^2 \cdot |y|^2 \cdot |x+y|^2}.$$

$x, y \in A_n$

$F\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) = 0$

Let $A_n = \{x \in \mathbb{Z}^2_{\geq 0} \cap [0,n]^2\}$. We telescope

$$F(x,y) - F(x+y,y) - F(x,x+y)$$

over $\{x, y \in A_n, \det(x \;\; y) = 1\}$ obtaining the sum of $-F(x+y,y) - F(x,x+y)$ over $B_n = \{x, y \in A_n, \det(x \;\; y) = 1, x+y \notin A_n\}$.
The latter sum tends to $-\pi/2$ since the area of the parallelogram spanned by x, y is 1, so $\frac{x \cdot y}{|x^2| \cdot |y|^2}$ is the angle between x and y up to second order terms, and the set of angles at the origin of the parallelograms in B_n partition the angle $\pi/2$ of the first quadrant. □

Topograph interpretation

$\frac{1}{rst}$

$F(x,y) = \frac{x \cdot y}{|x|^2 |y|^2}$

So, A corresponds to the part of the topograph growing from the base $e_1 = (1,0), e_2 = (0,1)$, we sum up the terms $\frac{1}{r^2s^2t^2}$ over all the vertices.

Idea: present each value $\cancel{\frac{1}{r^2s^2t^2}}$ $1/rst$ as a sum of terms corresponding to three incoming edges. Terms corresponding to the opposite orientations of the same edge must cancel each other.
One orientation of an edge $(\pm e_1, \pm e_2)$ corresponds to (e_1, e_2), the opposite orientation corresponds to $(e_1, -e_2)$. Let

$$F(e_1, e_2) = \frac{e_1 \cdot e_2}{|e_1|^2 |e_2|^2}.$$

$= + F(e_1, -e_2)$

F is symmetric and has opposite signs for the opposite orientations of an egde.
Thus we can telescope!

Other quadratic forms

$q(a,b) = a^2+2b^2$

What if we want to sum up, for examlpe

$$\sum_A \frac{1}{(a^2+2b^2)\cdot(c^2+2d^2)\cdot((a+c)^2+2(b+d)^2)} = ?$$

Just consider the vectors $\boxed{ae_1+\sqrt{2}be_2}$. Then

$$F((a,b),(c,d)) - F((a+c,b+d),(c,d)) - F((a,b),(a+c,b+d)) =$$

$$= \frac{-2(\sqrt{2})^2}{(a^2+2b^2)\cdot(c^2+2d^2)\cdot((a+c)^2+2(b+d)^2)}.$$

$\frac{1}{|x||y|} = \sin\alpha$

And $\boxed{\frac{x\cdot y}{|x^2|\cdot|y|^2}} = \alpha/\sqrt{2}$ where α is the angle between x and y, in this case.
So,

$(a,b)=(1,0) \quad (c,d)=(0,1)$

$$\sum_A \frac{1}{(a^2+t^2b^2)\cdot(c^2+t^2d^2)\cdot((a+c)^2+t^2(b+d)^2)} = \frac{-0+\frac{\pi}{2}/t}{2t^2}.$$

$\frac{1}{4}$

I want to put $t=0$ or $t=i$.

$(a,b) = 1$

Restrict this sum for the cone generated by $(1,0),(1,t)$ Then

$$\sum \frac{1}{(a^2+t^2b^2)\cdot(c^2+t^2d^2)\cdot((a+c)^2+t^2(b+d)^2)} = \frac{-\frac{(1,0)\cdot(1,t)}{1\cdot(1+t^2)}+\frac{\arctan t}{t}}{2t^2}.$$

When $t=0$:

$$\frac{\arctan t}{t} = 1 - \frac{t^2}{3} + \frac{t^4}{5} - \cdots$$

$$\sum_{a,c\geq 1,\gcd(a,c)=1} \frac{1}{a^2\cdot c^2\cdot(a+c)^2} = \frac{1}{3}.$$

When $t=i/2$:

$$\sum \frac{1}{(4a^2-b^2)\cdot(4c^2-d^2)\cdot(4(a+c)^2-(b+d)^2)} = \frac{1}{64}\frac{-4/3+\ln 3}{2(i/2)^2}.$$

here we sum over all $a,b,c,d\in\mathbb{Z}_{\geq 0}$, $ad-bc=1$, $a\geq b$, $c\geq d$.

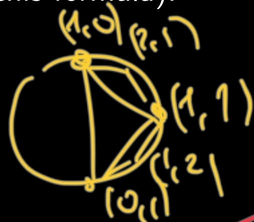
.... also we can expand the formula above in t, this gives values of sums like

$$\sum \frac{b^{2k}d^{2n}(b+d)^{2m}}{a^{2k+2}c^{2n+2}(a+c)^{2m+2}}$$

$$\frac{1}{a^2c^2(a+c)^2}$$

Geometric way to prove the formula

This is due to A. Hurwitz (1905, a neglected article) and (D. Speyer, 2023 answering my question on mathoverflow about possible other proofs of this formula):



Parametrize the right half of the unit circle by

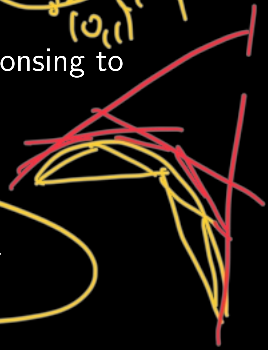
$$(a, b) \to \left(\frac{2ab}{a^2 + b^2}, \frac{a^2 - b^2}{a^2 + b^2} \right).$$

Then the area of the triangle with vertices on the circle, corresponsing to

$$(a, b), (c, d), (a + c, b + d)$$

is equal to

$$\frac{2}{(a^2 + b^2) \cdot (c^2 + d^2) \cdot ((a + c)^2 + (b + d)^2)}$$



Hence the sum is equal to $\pi/4$.