Basic concepts in mathematics: Diary

Nikita Kalinin, Diego Armando Sulca, Raphaël Fesler, Shuliang Gao

Lexicographic induction
(radicals + strokes)

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0 Preface

From a cup of coffee, a spoonful of coffee is poured into a cup of milk. Then a spoonful of the resulting mixture is poured back into the cup of coffee. Which is greater: the amount of milk in the cup of coffee or the amount of coffee in the cup of milk?

This course is intended for first-year students in mathematics and computer science; they have to learn how to read and write proofs.

It is not obvious at all why we need to prove something (professors demand it, but why?). The truth is that by proving we understand things better and discover new beauties (e.g., the formula $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + .. + n)^2$).

So, to reconcile the concept of proof with students, it is better to introduce proofs in questions whose answers are not obvious and are debatable. Examples logic puzzles, games (who has a winning strategy?), impossibility proofs (e.g., tiling a chessboard missing opposite corners with dominoes), and induction. These topics occupy the first few lectures.

How do you learn to prove? Let us use the metaphor "Mathematics is a language". When you learn a foreign language, you have different activities: listening, speaking, reading, writing. The same is true for mathematics. To study it, students should spend time reading and writing mathematics, listening to lectures, thinking about problems, and discussing ideas.

Also, mathematicians love mathematics because proofs are beautiful!

Solution to the coffee-milk problem:

Let's try to guess the answer. To do that, consider an extreme case (this is the first idea). Suppose there is just one spoonful of liquid in each cup. Then after pouring the coffee into the milk, we take the entire mixture back. The mixture will be uniform, so the amount of coffee and milk will be equal. Will it always be equal?

Since one spoonful was poured "there and back," the total volume of liquid in each cup did not change (this is the second idea).

Therefore (the third idea), the amount of coffee lost equals the amount of milk gained.

The volumes of coffee and milk in the cups can be different, you can pour the spoon back and forth ten times, you can even stir the mixture poorly — it doesn't matter: the amount of milk in the coffee will always equal the amount of coffee in the milk!

0.1 :: on collaboration and the use of AI

Collaboration. You are encouraged to discuss problems with classmates: compare approaches, explain ideas to one another, and ask for critique. Explaining your reasoning often reveals subtleties and gaps that you can then address. However, each student must write up their own solution independently, in their own words, after any discussion. Do not share written solutions or allow others to copy your work.

Attempt first. Before seeking help from classmates or AI tools, make a genuine attempt on each problem: write down an outline, partial calculations, or a strategy you tried (even if it failed). Learning to prove requires practice—like training for a sport—so expect to try, err, and revise.

AI: permitted uses. You may use AI tools to: (i) spot logical gaps or unclear steps in a solution you have already written, (ii) improve clarity and writing style, (iii) receive high-level hints about relevant definitions or theorems. When you do so, begin with your own draft (typed or a photo) and ask for feedback on that draft.

AI: not permitted uses (unless explicitly allowed). Do not ask AI to produce full solutions for graded assignments or to translate someone else's solution into "your own words." Do not paste problem statements and accept verbatim answers. Use AI as a reviewer, not as an author. If you are unsure whether a use is allowed, ask the instructor.

Reflection Logs. It is useful to reflect on how you use AI, so record the tool and its purpose in your diary, and then track how it affects your understanding and your ability to solve problems, write solutions, and read material without any outside help. Note that during the exams or midterms you cannot rely on anything except yourself.

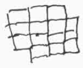
Academic integrity. Copying another person's solution or submitting AI-generated solutions as your own is plagiarism. Violations will be handled under the university's academic integrity policy.

1 Methods of proving

1.1 :: parity, induction

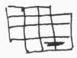
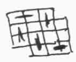
1.1.1 :: :: parity

Why do we need proofs? Because this way we understand more. Let us solve the following problem:

Problem Is it possible to cut the figure 
into domino ?

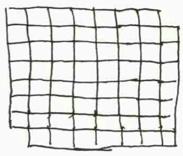
Students try to solve it.

No, it is not possible. Suppose we can do it | Start with corner.

 then 

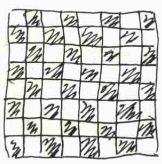
$\Rightarrow$ not possible.

Problem. What about table 8×8 with two opposite corners removed?



Students try to solve.

Solution color the table 8×8 into black and white, like chess



then each domino contains 1 white and 1 black cell.

here 32 white cells 30 black cells
$\Rightarrow$ impossible

$\Rightarrow$ proofs are useful to show that something is impossible.

Many problems become much easier once one notices that some quantity has a fixed *parity* (it is always even or always odd). Once a parity is fixed, any situation in which that quantity would have the opposite parity is impossible. Sometimes one has to construct this quantity, for example by considering the parity of a sum or product, by pairing objects up, by noticing an alternating pattern, or by colouring objects in two colours.

Example 1. *A grasshopper makes jumps of length 1 m along a straight line and eventually returns to its starting point. Show that it made an even number of jumps.*

Solution. If the grasshopper ends where it started then the number of jumps to the right must equal the number of jumps to the left. Consequently the total number of jumps is even.

Example 2. *Does there exist a closed broken line with seven segments that crosses each of its segments exactly once?*

Solution. Suppose such a broken line existed. Any two crossing segments can be paired. The number of segments must therefore be even, which contradicts the assumption that there are seven segments.

Example 3. *Martians may have any number of arms. One day all Martians joined hands in such a way that no free arms remained. Prove that the number of Martians with an odd number of arms is even.*

Solution. Call Martians with an even number of arms *even* and those with an odd number of arms *odd*. Since the hands form pairs, the total number of hands is even. The total number of hands of the even Martians is clearly even, so the total number of hands of the odd Martians must also be even. But each odd Martian contributes an odd number of hands, so there must be an even number of them.

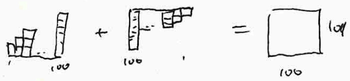
Such pictures are called *graphs*, they consist of *vertices* and *edges*. The number of edges incident to a vertex is called the *degree* (or *valency*) of a vertex.

We proved that in each graph the number of vertices of odd degree is even.

1.1.2 :: :: induction

How to compute $1+2+3+\dots+100$?

Easy:



$= 100\cdot 101$

$\Rightarrow 1+2+\dots+100 = \frac{100(101)}{2}$

The same method works for each natural n

$1+2+\dots+n = \frac{n(n+1)}{2} = S(n)$

n	1	2	3	4	...
$S(n)$	1	3	6	10	...

Let us find $1^3+2^3+3^3+\dots+n^3$

try: $n=1$ $\quad 1^3=1$

$n=2$ $\quad 1^3+2^3=9$

$n=3$ $\quad 1^3+2^3+3^3=36$

$n=4$ $\quad \Rightarrow 1^3+2^3+3^3+4^3=36+64=100$

Conjecture: $1^3+\dots+n^3=\left(\frac{n(n+1)}{2}\right)^2$ from here

But how to prove it? May be it is a coincidence.

It is true for $n=1,2,3,4$.

Let us prove that if it is true for $n=k$ then it is true for $n=k+1$.

Indeed $1^3+2^3+\dots(k+1)^3 = 1^3+\dots+k^3+(k+1)^3 =$
$\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 = \dots \left(\frac{(k+1)(k+2)}{2}\right)^2$

then it is true for all natural n.

The method of mathematical induction is used to prove statements of the form "For every natural number n a certain property holds." Such a statement can be viewed as an infinite chain of assertions: "For $n=1$ the property holds", "For $n=2$ the property holds", and so on. The first assertion in the chain is called the base (or the foundation) of the induction and is usually easy to check. One then proves the induction step: "If the assertion with number n is true, then the assertion with number $n+1$ is true." Sometimes one needs a stronger form of the induction step: "If all assertions with numbers from 1 to n are true, then the assertion with number

$n+1$ is true." There is also the technique of *inductive descent*, in which one proves that if an assertion with number n (with $n > 1$) can be reduced to one or several assertions with smaller numbers and the first assertion is true, then all assertions are true.

If both the base and the induction step have been proved, then all assertions in the chain hold; this is the principle of mathematical induction.

Example 1. *Prove that the number consisting of 243 consecutive ones is divisible by 243.*

Solution. Notice that $243 = 3^5$. We shall prove a more general statement: the number consisting of 3^n consecutive ones is divisible by 3^n. For $n = 1$ the assertion says that 111 is divisible by 3, which is true. Suppose the number consisting of 3^{n-1} ones is divisible by 3^{n-1}. Write

$$\underbrace{11\cdots1}_{3^n \text{ times}} = \underbrace{11\cdots1}_{3^{n-1} \text{ times}} \times \underbrace{10\cdots010\cdots01}_{\text{a block containing only one non zero digit}}.$$

It is not hard to check that the second factor on the right–hand side is divisible by 3. Multiplying a multiple of 3^{n-1} by a multiple of 3 yields a multiple of 3^n. Therefore the number of 3^n ones is divisible by 3^n, completing the induction.

Example 2. *Several lines and circles are drawn in the plane. Prove that the regions into which the plane is divided can be coloured in two colours so that adjacent regions (sharing a segment or an arc) are coloured differently.*

Solution. First erase all the lines and circles, remembering where they were. Colour the entire plane one colour. Then restore the boundaries one by one, recolouring the regions they divide. When adding a line, recolour in the opposite colour all regions on one side of it and leave unchanged those on the other side. When adding a circle, recolour all regions lying inside it and leave unchanged those outside. In this way each time you add a boundary the recoloured regions lie on one side only. Consequently any two neighbouring regions (sharing part of a boundary) always have different colours.

Example 3. *Prove that if $x + \frac{1}{x}$ is an integer, then $x^n + \frac{1}{x^n}$ is an integer for all $n \geq 0$.*

Solution. Set $T_n = x^n + \frac{1}{x^n}$. Note that $T_0 = 2$ and $T_1 = x + 1/x$ are integers. Observe that

$$T_n T_1 = (x^n + \tfrac{1}{x^n})(x + \tfrac{1}{x}) = x^{n+1} + \tfrac{1}{x^{n+1}} + x^{n-1} + \tfrac{1}{x^{n-1}} = T_{n+1} + T_{n-1}.$$

Thus $T_{n+1} = T_n T_1 - T_{n-1}$. By induction on n this recurrence shows that all T_n are integers.

Example 4. (if time permits) *Five robbers have obtained a sack of gold sand. They wish to divide it so that each robber is sure he received at least one fifth of the gold. They have no measuring instruments, but each can judge by eye the amount of a pile of sand. Opinions about the size of the piles may differ. How can they divide the loot?*

Solution (First method). First two robbers divide the sand between themselves: one divides the sack into two piles that he believes equal, and the other chooses his pile. Each of these two divides his share into four equal (to his mind) parts, and the third robber takes one part from each. Now these three each divide their share into three parts and the fourth robber takes one part from each. Finally these four divide their shares into two parts and the fifth robber takes one part from each. Each robber can check that the portion he receives is at least one fifth according to his judgment.

Solution (Second method). Find the most modest robber and give him his portion first. To do so, ask the first robber to separate what he believes to be $1/5$ of the sack. Ask the second robber whether the separated part is larger than $1/5$: if he thinks it is larger, have him reduce it to what he considers $1/5$; if he thinks it is not larger, ask the third robber, and so on. When someone finally agrees that the separated part is exactly $1/5$, give that part to the last person who modified it. Among the remaining robbers find the most modest of those and repeat. In the end every robber receives a portion he believes is at least $1/5$ of the original amount.

1.1.3 :: :: problems for tutorial

1. You have coins of 3 HKD and 5 HKD. Prove that any number of HKD greater than seven can be exchanged for coins of 3 and 5 HKD.

2. Several lines divide the plane into regions. Each line grow hair on one side. Prove that there is a region all of whose boundaries have hair "outside".

3. From a 128×128 square one unit square was removed. Prove that the remaining shape can be tiled with L shaped trominoes consisting of three unit squares.

4. For every natural k prove the inequality $2^k > k$.

5. Prove the Cauchy–Schwarz inequality in the form

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

where $x_1, \ldots, x_n$ are non negative numbers. [1]

1.1.4 :: :: problems for workshop

1. Can one break 25 HKD into ten coins of denominations 1, 3 and 5 HKD?[2]

2. Nine gears are arranged in a circle, each meshing with the next. The first meshes with the second, the second with the third, ..., the ninth with the first. Can they all rotate at the same time? What happens if there are n gears?[3]

3. A row contains 100 towers. You may interchange any two towers that have exactly one tower between them. Is it possible in this way to reverse the entire order of the towers?[4]

4. Six numbers $1, 2, 3, 4, 5, 6$ lie on the table. You are allowed to add 1 to any two of them. Can all the numbers eventually be made equal?[5]

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[1] Comment: A common proof uses induction on n by first proving the case where n is a power of two and then reducing the general case to the nearest lower power of two. This problem invites the reader to explore that technique.

[2] Comment: Let x, y and z be the numbers of 1–, 3– and 5–HKD coins. Then $x + 3y + 5z = 25$ and $x + y + z = 10$. Subtracting yields $2y + 4z = 15$, which has no integer solutions. Therefore it is impossible.

[3] Comment: An odd number of meshed gears arranged in a cycle cannot all turn, because each contact reverses the sense of rotation. For an even number of gears a consistent rotation is possible.

[4] Comment: Label the positions $1, \ldots, 100$. A permitted move swaps the towers in positions i and $i + 2$, both of which have the same parity (both odd or both even). Consequently each tower always occupies squares of the same parity as its starting position. In the reversed arrangement the tower originally at position 1 would have to move to position 100 and hence to a square of opposite parity. This is impossible, so a complete reversal cannot be achieved.

[5] Comment: Each move increases the sum of the numbers by 2. If eventually all six numbers were equal to k, then $6k = 1 + 2 + 3 + 4 + 5 + 6 + 2t = 21 + 2t$. The left side is even, while $21 + 2t$ is odd for all t. Hence the numbers can never all become equal.

5. All dominoes from the standard set are laid out in a single chain according to the usual rule (neighbouring halves show the same number). One end of the chain has a five. What can be at the other end?[6]

6. Can a line that does not pass through any vertex of an 11–gon intersect all of its sides?[7]

7. On a table stand seven overturned cups. You may simultaneously turn over any two cups. Is it possible to end up with all the cups upright?[8]

[6]Comment: In the usual "double six" set the tiles are the pairs (i, j) with $1 \leq i \leq j \leq 6$. Each number $1, \ldots, 6$ occurs exactly six times. Consider the graph whose vertices are the numbers and whose edges correspond to dominoes. In this graph every vertex has even degree, so the tiles can be arranged in a closed Eulerian circuit. Cutting this circuit yields an open chain whose two ends are identical. Hence if one end displays a five, the other end must also display a five.

[7]Comment: In any polygon the number of intersections of a straight line with the sides is even (becasue the line goes inside the polygon, then outside, then inside,... finally it goes outside). Since 11 is odd, no such line exists.

[8]Comment: The parity of the number of overturned cups changes by 0 or 2 at each move. Starting with seven (odd) and wanting to end with zero (even) is impossible.

1.1.5 :: :: 1st homework

Read:

Paul Zeitz - The Art and Craft of Mathematical Problem Solving, pp. 13-15.

The Frog Problem

- The frog problem is a classic Russian math circle problem.
- Three frogs are situated at 3 of the corners of a square. Every minute, 1 frog is chosen to leap over another chosen frog, so that if you drew a line from the starting position to the ending position of the leaper, the leapee is at the exact midpoint.
- Will a frog ever occupy the vertex of the square that was originally unoccupied?
- How can we effectively investigate this problem?
- Graph paper allows us to attach numbers to the positions of the frogs. Once we have numbers, we can employ arithmetical and algebraic methods. Thus, place the frogs at (0, 0), (0, 1), and (1, 1). The question now is, can a frog ever reach (1, 0)?
- Thinking about the appropriate venue for investigation is an essential starting strategy for any problem.
- Another investigative idea: Use colored pencils to keep track of individual frogs. This adds information, as it allows us to keep track of 1 frog at a time. Color the (1, 1), (0, 1), and (0, 0) frogs red, blue, and green, respectively.
- Notice, by experimenting, that the red frog only seems to hit certain points, forming a larger (2-unit) grid.
- Some of the coordinates that the red frog hits are (1, 1), (1, 3), (1, −1), (−1, 1), (−1, −1), and (−1, −3). They are all odd numbers!
- Likewise, the blue frog only hits certain points on a 2-unit grid, including (0, 1), (2, 1), (4, 1), and (0, −1); these are all of the form (even, odd).

- Likewise, the green frog only hits (even, even) points.
- On the other hand, the missing southeast vertex was (1, 0), which has the form (odd, even). It seems as though it is impossible, but how can we formulate this in an airtight way?
- It is often very profitable to contemplate parity (oddness and evenness).
- The essential reason for this is that a parity focus reduces a problem from possibly infinitely many states to just 2.
- Parity involves the number 2. Where in this problem do we see this number? In doubling, because of the symmetry of the way the frogs leap. When the leaper jumps over the leapee, she adds twice the horizontal displacement to her original horizontal coordinate. The same holds for vertical coordinates.
- So when a frog jumps, its coordinates change by even numbers!
- For example, suppose the red (1, 1) frog jumps over the green frog at (0, 0). The horizontal and vertical displacements to the leapee are both −1 (since it is moving left and down), so the final change in coordinates will be −2. The horizontal coordinate will be $1 + -2 = -1$, and the vertical will also be −1.
- Suppose now that the red frog jumps over the blue frog, which is (0, 1). The horizontal displacement is +1, and the vertical displacement to the target is +2. So the new horizontal coordinate will be −1 (the starting value) $+ 2 \times 1 = +1$, and the new vertical coordinate will be −1 (the starting value) $+ 2 \times 2 = 3$. Thus the red frog jumps from (−1, −1) to (1, 3).

- In general, when a frog jumps, we will take its starting x-coordinate and add twice the horizontal displacement to its target. Likewise, we take its starting y-coordinate and add twice the vertical displacement to the target. These displacements may be positive, negative, or zero.
- In other words, you take the starting coordinates and add even numbers to them. But when you add an even number to something, its parity does not change!
- So the (odd, odd) frog—the red frog—is destined to stay at (odd, odd) coordinates, no matter what.

Suggested Reading

Polya, *How to Solve It.*

Zeitz, *The Art and Craft of Problem Solving*, chap. 2.

Questions to Consider

1. Write the numbers from 1 to 10 in a row and place either a minus or a plus sign between the numbers. Is it possible to get an answer of zero?
2. A group of jealous professors is locked up in a room. There is nothing else in the room but pencils and 1 tiny scrap of paper per person. The professors want to determine their average (mean, not median) salary so that each can gloat or grieve over his or her personal situation compared to the others. However, they are secretive people and do not want to give away salary information to anyone else. Can they determine the average salary in such a way that no professor can discover any fact about the salary of anyone but herself? For example, even facts such as "one professor earns less than $90,000" are not allowed.

Write:

Problem 1. Write the full solution (with all details) of the problem that we cannot cut a square 4×4 without opposite corners into domino. Write the solution where we use case-by-case strategy, without using coloring.

Problem 2. Write the solution of the problem that we cannot cut a square 8×8 without opposite corners into domino (you can use coloring).

Problem 3. Show that you can cut a square in n squares for each $n > 6$.

Problem 4. The numbers $1, 2, \ldots, 101$ are written on a blackboard. You are allowed to erase any two numbers and write their difference in their place. After repeating this operation 100 times only one number remains. Prove that this number cannot be zero.