

---

# Basic concepts in mathematics: Diary

---

Nikita Kalinin, Diego Armando Sulca, Raphaël Fesler, Shuliang Gao

Lexicographic induction  
(radicals + strokes)

辤 (辵 ) ... 道道..達 ..  
邑 ... 邑 ... 邑 ..

<https://kilin-math.github.io/assets/teaching/diary.pdf>

Guangdong Technion Israel Institute of Technology

Shantou, China

January 4, 2026

# Contents

<b>0 Preface</b>	<b>2</b>
0.1 :: introduction (for teachers) . . . . .	3
0.2 :: how to study (for students) . . . . .	4
0.3 :: on collaboration and the use of AI . . . . .	6
<b>1 Methods of proving</b>	<b>7</b>
1.1 :: parity, induction . . . . .	7
1.2 :: pigeonhole principle, correspondence, invariants . . . . .	15
1.3 :: method of the extreme, counting in two ways, combinatorics . . . . .	25
1.4 :: feedback . . . . .	31
<b>2 Naive set theory</b>	<b>31</b>
2.1 :: notation, union, intersection, Venn diagrams . . . . .	31
2.2 :: functions, permutations, $S_n$ . . . . .	39
2.3 :: Cartesian products . . . . .	50
2.4 :: power set, Cantor's theorem, Russel's paradox . . . . .	56
2.5 :: infinite sequences, Thue-Morse sequence, Collatz conjecture . . . . .	71
2.6 :: countable and uncountable sets, Cantor–Bernstein theorem, restricted infinite product . . . . .	90
<b>3 Elementary number theory, equivalence relation, and real numbers</b>	<b>101</b>
3.1 :: remainders, congruences . . . . .	101
3.2 :: equivalence relations, partitions . . . . .	119
3.3 :: construction of natural, rational, real numbers . . . . .	135
<b>4 Order as a binary relation. Examples and lexicographic order</b>	<b>155</b>
4.1 :: orders, lexicographic order, lexicographic induction . . . . .	155
4.2 :: afterword for a course . . . . .	161
<b>5 :: Materials</b>	<b>162</b>
5.1 :: credits and what can be done differently . . . . .	162
5.2 Basic Graph Theory: Trees, Paths, and Leaves . . . . .	167
5.3 Corrections of students' homework . . . . .	168
5.4 Notes from tutorial 8 . . . . .	192
5.5 How to write . . . . .	198

# 0 Preface

**From a cup of coffee, a spoonful of coffee is poured into a cup of milk. Then a spoonful of the resulting mixture is poured back into the cup of coffee. Which is greater: the amount of milk in the cup of coffee or the amount of coffee in the cup of milk?**

This course is intended for first-year students in mathematics and computer science; they have to learn how to read and write proofs.

It is not obvious at all why we need to prove something (professors demand it, but why?). The truth is that **by proving we understand things better and discover new beauties** (e.g., the formula  $1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$ ).

So, to reconcile the concept of proof with students, it is better to introduce proofs in questions whose answers are not obvious and are debatable. Examples logic puzzles, games (who has a winning strategy?), impossibility proofs (e.g., tiling a chessboard missing opposite corners with dominoes), and induction. These topics occupy the first few lectures.

How do you learn to prove? Let us use the metaphor “Mathematics is a language”. When you learn a foreign language, you have different activities: listening, speaking, reading, writing. The same is true for mathematics. To study it, students should spend time reading and writing mathematics, listening to lectures, thinking about problems, and discussing ideas.

Also, mathematicians love mathematics because proofs are beautiful!

Solution to the coffee-milk problem:

*Let's try to guess the answer. To do that, consider an extreme case (this is the first idea). Suppose there is just one spoonful of liquid in each cup. Then after pouring the coffee into the milk, we take the entire mixture back. The mixture will be uniform, so the amount of coffee and milk will be equal. Will it always be equal?*

*Since one spoonful was poured “there and back,” the total volume of liquid in each cup did not change (this is the second idea).*

*Therefore (the third idea), the amount of coffee lost equals the amount of milk gained.*

The volumes of coffee and milk in the cups can be different, you can pour the spoon back and forth ten times, you can even stir the mixture poorly — it doesn't matter: the amount of milk in the coffee will always equal the amount of coffee in the milk!

## 0.1 :: introduction (for teachers)

This course is an introduction to the basic concepts of mathematics. By “basic concepts” we primarily mean **sequences**, **sets**, and **methods of proof**. The course is intended for first-year students in mathematics and computer science, many of whom encounter rigorous mathematical reasoning for the first time.

We believe that such a course should not be reduced to a collection of elementary definitions and easy exercises. On the contrary, it should occasionally confront students with genuinely nontrivial ideas. For example, proving that every permutation can be decomposed into transpositions is technically accessible at an early stage, yet conceptually deep. Similarly, asking students to compare the definition of a Cartesian product of two sets with that of an arbitrary (possibly infinite) family of sets pushes them toward abstraction (page 51). Another one: the bijection between partitions and equivalence relations on a given set (page 124).

At the same time, not every deep theorem is suitable for a first encounter (Cantor–Bernstein theorem). Nevertheless, their *applications*—for instance, showing that two sets have the same cardinality by constructing mutual injections—are both accessible and highly instructive.

A guiding principle throughout the course is that concepts should not be introduced in isolation. In particular, **equivalence relations** are not presented as a purely formal notion. Instead, they are immediately put to work: in the construction of arithmetic modulo  $p$ , in the passage from rational numbers to real numbers, and in other situations where identifying objects “up to an equivalence” is essential. This approach helps students understand *why* equivalence relations are introduced and how they naturally arise in mathematical practice.

We also deliberately address a skill that is often left implicit: how to *read* and *write* mathematical proofs (page 24). Rather than assuming that students will acquire this ability automatically, we treat it as an explicit learning objective.

To support this goal, students are given carefully chosen reading assignments (pages 14, 22, 49, 55, 115), including short texts and expository notes that discuss open problems or partially resolved questions. The purpose is not to master advanced material, but to help students see mathematics as a living discipline, where uncertainty, conjectures, and incomplete knowledge play an essential role.

Finally, several homework assignments are designed in an unconventional way: students are asked to analyze *incorrect* or incomplete proofs, to locate precisely where the reasoning fails, and to explain why the argument does not work (pages 30, 153).

## 0.2 :: how to study (for students)

Read this section after the course is finished.

Trace the ideas and methods that have appeared throughout the course. While reviewing the material, try to place each concept on a specific “shelf” in your mind and build your own internal classification. An object, definition, or problem may naturally belong to several categories at once; this overlap is not a defect but an essential feature of mathematical thinking. For example:

**Pigeonhole principle.** Introduced in a lecture (page 16); revisited in a homework assignment involving an incorrect solution (page 30); applied to divisibility rules (page 105); appeared in a midterm problem (page 67); more sophisticated examples were discussed in the (non-obligatory) reading (page 55).

**Induction.** Introduced in Lecture 1 (page 9) and used repeatedly in problems; supplemented by a homework reading on the formal structure of induction (page 22); reappeared in a workshop on the axiomatic construction of the natural numbers (Section 3.3.2, page 149); featured in a midterm problem (page 62); used to prove properties of the Fibonacci sequence (page 75); and culminated in a final “goodbye” problem (page 161).

**Fundamental theorem of arithmetic.** Formal statement using restricted products (page 51); proved using the method of the minimal counterexample (page 130, see also page 157), which is essentially equivalent to the method of the extreme (page 25) and can also be viewed as a form of induction, via the existence of a minimal element in every nonempty subset of the natural numbers (page 149).

**Examples of bad and good writing.** A short guideline is given on page 24, followed by further advice on clarity, structure, and logical flow on page 198.

**How to read texts/open problems.** A step-by-step analysis of a frog problem (page 14); examples illustrating how to compute and check claims while reading (page 115); general advice on reading mathematical texts (page 117); and an open problem on Egyptian fractions (page 116).

**Sequences.** We observed that the set of all infinite sequences with values in  $\mathbb{R}$  is the Cartesian product of countably many copies of  $\mathbb{R}$  (page 51); defined addition and multiplication of sequences (page 71); solved second-order recurrence relations (pages 72, 83, 84); studied the Fibonacci sequence (page 73); discussed the Collatz conjecture (page 86); introduced eventually constant sequences (page 100); and constructed real numbers as equivalence classes of Cauchy sequences (page 135).

**Divisibility by three.** Appeared in the problem about chameleons (page 18); numbers consisting of many ones (page 10); divisibility rules (page 103); applications of infinite descent for the equation  $x^2 + y^2 = 3z^2$  (page 110); and addition and

multiplication tables modulo 3 (page 122).

**Parity.** Chessboard coloring and related processes (page 7); the wonder-apple tree problem (page 17); and even and odd permutations (page 69).

**Binary and decimal representations.** Binary representation used in the definition of the Thue–Morse sequence (page 76); decimal representation in Cantor’s diagonal argument (page 96); recovery of binary digits from the parity of  $\lfloor n/2^k \rfloor$  (page 99); and the construction of decimal representations of real numbers defined via equivalence classes of Cauchy sequences (page 142).

**Cardinality.** The equality of cardinalities of  $[0, 1]^2$  and  $[0, 1]$  proved using Cantor–Bernstein (page 93); and by constructing explicit bijections (page 98).

Finally, lexicographic induction explains the cover picture for these notes.

## 0.3 :: on collaboration and the use of AI

**Collaboration.** You are encouraged to discuss problems with classmates: compare approaches, explain ideas to one another, and ask for critique. Explaining your reasoning often reveals subtleties and gaps that you can then address. However, each student must write up their own solution independently, in their own words, after any discussion. Do not share written solutions or allow others to copy your work.

**Attempt first.** Before seeking help from classmates or AI tools, make a genuine attempt on each problem: write down an outline, partial calculations, or a strategy you tried (even if it failed). Learning to prove requires practice—like training for a sport—so expect to try, err, and revise.

**AI: permitted uses.** You may use AI tools to: (i) spot logical gaps or unclear steps in a solution you have already written, (ii) improve clarity and writing style, (iii) receive high-level hints about relevant definitions or theorems. When you do so, begin with your own draft (typed or a photo) and ask for feedback on that draft.

**AI: not permitted uses (unless explicitly allowed).** Do not ask AI to produce full solutions for graded assignments or to translate someone else's solution into "your own words." Do not paste problem statements and accept verbatim answers. Use AI as a reviewer, not as an author. If you are unsure whether a use is allowed, ask the instructor.

**Reflection Logs.** It is useful to reflect on how you use AI, so record the tool and its purpose in your diary, and then track how it affects your understanding and your ability to solve problems, write solutions, and read material without any outside help. Note that during the exams or midterms you cannot rely on anything except yourself.

**Academic integrity.** Copying another person's solution or submitting AI-generated solutions as your own is plagiarism. Violations will be handled under the university's academic integrity policy.

# 1 Methods of proving

## 1.1 :: parity, induction

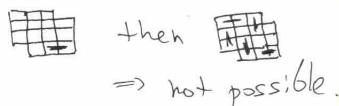
### :: :: parity

Why do we need proofs? Because this way we understand more. Let us solve the following problem:

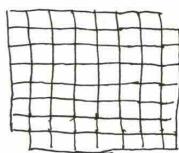
Problem Is it possible to cut the figure into domino  $\square$ ?

Students try to solve it.

No, it is not possible. Start with corner.  
Suppose we can do it



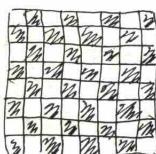
Problem. What about table  $8 \times 8$  with two opposite corners removed?



Students try to solve.

Solution color the table  $8 \times 8$  into black and white, like chess

here  
32 white cells  
30 black cells  
 $\Rightarrow$  impossible



then each domino contains 1 white and 1 black cell.

$\Rightarrow$  proofs are useful to show that something is impossible.

Many problems become much easier once one notices that some quantity has a fixed parity (it is always even or always odd). Once a parity is fixed, any situation in

which that quantity would have the opposite parity is impossible. Sometimes one has to construct this quantity, for example by considering the parity of a sum or product, by pairing objects up, by noticing an alternating pattern, or by colouring objects in two colours.

**Example 1.** *A grasshopper makes jumps of length 1 m along a straight line and eventually returns to its starting point. Show that it made an even number of jumps.*

*Solution.* If the grasshopper ends where it started then the number of jumps to the right must equal the number of jumps to the left. Consequently the total number of jumps is even.

**Example 2.** *Does there exist a closed broken line with seven segments that crosses each of its segments exactly once?*

*Solution.* Suppose such a broken line existed. Any two crossing segments can be paired. The number of segments must therefore be even, which contradicts the assumption that there are seven segments.

**Example 3.** *Martians may have any number of arms. One day all Martians joined hands in such a way that no free arms remained. Prove that the number of Martians with an odd number of arms is even.*

*Solution.* Call Martians with an even number of arms *even* and those with an odd number of arms *odd*. Since the hands form pairs, the total number of hands is even. The total number of hands of the even Martians is clearly even, so the total number of hands of the odd Martians must also be even. But each odd Martian contributes an odd number of hands, so there must be an even number of them.

Such pictures are called *graphs*, they consist of *vertices* (representing Martians) and *edges* (representing a handshaking pair of hands). The number of edges incident to a vertex (i.e. the number of hands of a Martian) is called the *degree* (or *valency*) of a vertex. See Section 5.2.

We proved that in each graph the number of vertices of odd degree is even.

## :: :: induction

How to compute  $1+2+3+\dots+100$ ?

Easy:   $= \boxed{100}^{100} = 100 \cdot 100$   
 $\Rightarrow 1+2+\dots+100 = \frac{100(100)}{2}$

The same method works for each natural  $n$

$$1+2+\dots+n = \frac{n(n+1)}{2} = S(n) \quad S(n) \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 1 & 3 & 6 & 10 \\ \hline \end{array} \dots$$

Let us find  $1^3+2^3+3^3+\dots+n^3$

try:  $n=1 \quad 1^3=1$

$n=2 \quad 1^3+2^3=9$

$n=3 \quad 1^3+2^3+3^3=36$

$n=4 \Rightarrow 1^3+2^3+3^3+4^3=36+64=100$

Conjecture..  $1^3+2^3+\dots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$  from here

But how to prove it? May be it is a coincidence.

It is true for  $n=1, 2, 3, 4$ .

Let us prove that if it is true for  $n=k$  then it is true for  $n=k+1$ .

Indeed  $1^3+2^3+\dots+(k+1)^3 = 1^3+2^3+\dots+k^3+(k+1)^3 =$

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = \dots = \left(\frac{(k+1)(k+2)}{2}\right)^3$$

then it is true for all natural  $n$ .

The method of mathematical induction is used to prove statements of the form "For every natural number  $n$  a certain property holds." Such a statement can be viewed as an infinite chain of assertions: "For  $n=1$  the property holds", "For  $n=2$  the property holds", and so on. The first assertion in the chain is called the base (or the foundation) of the induction and is usually easy to check. One then proves the induction step: "If the assertion with number  $n$  is true, then the assertion with number  $n+1$  is true." Sometimes one needs a stronger form of the induction step: "If all assertions with numbers from 1 to  $n$  are true, then the assertion with number

$n + 1$  is true." There is also the technique of *inductive descent*, in which one proves that if an assertion with number  $n$  (with  $n > 1$ ) can be reduced to one or several assertions with smaller numbers and the first assertion is true, then all assertions are true.

If both the base and the induction step have been proved, then all assertions in the chain hold; this is the principle of mathematical induction.

**Example 1.** Prove that the number consisting of 243 consecutive ones is divisible by 243.

*Solution.* Notice that  $243 = 3^5$ . We shall prove a more general statement: the number consisting of  $3^n$  consecutive ones is divisible by  $3^n$ . For  $n = 1$  the assertion says that 111 is divisible by 3, which is true. Suppose the number consisting of  $3^{n-1}$  ones is divisible by  $3^{n-1}$ . Write

$$\underbrace{11 \cdots 1}_{3^n \text{ times}} = \underbrace{11 \cdots 1}_{3^{n-1} \text{ times}} \times \underbrace{10 \cdots 010 \cdots 01}_{\text{a block containing only one non zero digit}} .$$

It is not hard to check that the second factor on the right-hand side is divisible by 3. Multiplying a multiple of  $3^{n-1}$  by a multiple of 3 yields a multiple of  $3^n$ . Therefore the number of  $3^n$  ones is divisible by  $3^n$ , completing the induction.

**Example 2.** Several lines and circles are drawn in the plane. Prove that the regions into which the plane is divided can be coloured in two colours so that adjacent regions (sharing a segment or an arc) are coloured differently.

*Solution.* First erase all the lines and circles, remembering where they were. Colour the entire plane one colour. Then restore the boundaries one by one, recolouring the regions they divide. When adding a line, recolour in the opposite colour all regions on one side of it and leave unchanged those on the other side. When adding a circle, recolour all regions lying inside it and leave unchanged those outside. In this way each time you add a boundary the recoloured regions lie on one side only. Consequently any two neighbouring regions (sharing part of a boundary) always have different colours.

**Example 3.** Prove that if  $x + \frac{1}{x}$  is an integer, then  $x^n + \frac{1}{x^n}$  is an integer for all  $n \geq 0$ .

*Solution.* Set  $T_n = x^n + \frac{1}{x^n}$ . Note that  $T_0 = 2$  and  $T_1 = x + 1/x$  are integers. Observe that

$$T_n T_1 = (x^n + \frac{1}{x^n})(x + \frac{1}{x}) = x^{n+1} + \frac{1}{x^{n+1}} + x^{n-1} + \frac{1}{x^{n-1}} = T_{n+1} + T_{n-1}.$$

Thus  $T_{n+1} = T_n T_1 - T_{n-1}$ . By induction on  $n$  this recurrence shows that all  $T_n$  are integers.

**Example 4. (if time permits)** Five robbers have obtained a sack of gold sand. They wish to divide it so that each robber is sure he received at least one fifth of the gold. They have no measuring instruments, but each can judge by eye the amount of a pile of sand. Opinions about the size of the piles may differ. How can they divide the loot?

*Solution (First method).* First two robbers divide the sand between themselves: one divides the sack into two piles that he believes equal, and the other chooses his pile. Each of these two divides his share into four equal (to his mind) parts, and the third robber takes one part from each. Now these three each divide their share into three parts and the fourth robber takes one part from each. Finally these four divide their shares into two parts and the fifth robber takes one part from each. Each robber can check that the portion he receives is at least one fifth according to his judgment.

*Solution (Second method).* Find the most modest robber and give him his portion first. To do so, ask the first robber to separate what he believes to be  $1/5$  of the sack. Ask the second robber whether the separated part is larger than  $1/5$ : if he thinks it is larger, have him reduce it to what he considers  $1/5$ ; if he thinks it is not larger, ask the third robber, and so on. When someone finally agrees that the separated part is exactly  $1/5$ , give that part to the last person who modified it. Among the remaining robbers find the most modest of those and repeat. In the end every robber receives a portion he believes is at least  $1/5$  of the original amount.

### :: :: problems for tutorial

1. You have coins of 3 HKD and 5 HKD. Prove that any number of HKD greater than seven can be exchanged for coins of 3 and 5 HKD.
2. Several lines divide the plane into regions. Each line grows hair on one side. Prove that there is a region all of whose boundaries have hair “outside”. <sup>1</sup>

---

<sup>1</sup>Comment: Induct on the number of lines. Removing one line yields a configuration where the claim holds; then restore the line and pick an appropriate subregion on the hatched side.

3. From a  $128 \times 128$  square one unit square was removed. Prove that the remaining shape can be tiled with L shaped trominoes consisting of three unit squares.
4. For every natural  $k$  prove the inequality  $2^k > k$ .
5. Prove the Cauchy–Schwarz inequality in the form

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

where  $x_1, \dots, x_n$  are non negative numbers.<sup>2</sup>

### :: :: problems for workshop

1. Can one break 25 HKD into ten coins of denominations 1, 3 and 5 HKD?<sup>3</sup>
2. Nine gears are arranged in a circle, each meshing with the next. The first meshes with the second, the second with the third,  $\dots$ , the ninth with the first. Can they all rotate at the same time? What happens if there are  $n$  gears?<sup>4</sup>
3. A row contains 100 towers. You may interchange any two towers that have exactly one tower between them. Is it possible in this way to reverse the entire order of the towers?<sup>5</sup>
4. Six numbers 1, 2, 3, 4, 5, 6 lie on the table. You are allowed to add 1 to any two of them. Can all the numbers eventually be made equal?<sup>6</sup>

---

<sup>2</sup>Comment: A common proof uses induction on  $n$  by first proving the case where  $n$  is a power of two and then reducing the general case to the nearest lower power of two. This problem invites the reader to explore that technique. See details in Section 5.1, page 164.

<sup>3</sup>Comment: Let  $x$ ,  $y$  and  $z$  be the numbers of 1–, 3– and 5–HKD coins. Then  $x + 3y + 5z = 25$  and  $x + y + z = 10$ . Subtracting yields  $2y + 4z = 15$ , which has no integer solutions. Therefore it is impossible.

<sup>4</sup>Comment: An odd number of meshed gears arranged in a cycle cannot all turn, because each contact reverses the sense of rotation. For an even number of gears a consistent rotation is possible.

<sup>5</sup>Comment: Label the positions 1,  $\dots$ , 100. A permitted move swaps the towers in positions  $i$  and  $i + 2$ , both of which have the same parity (both odd or both even). Consequently each tower always occupies squares of the same parity as its starting position. In the reversed arrangement the tower originally at position 1 would have to move to position 100 and hence to a square of opposite parity. This is impossible, so a complete reversal cannot be achieved.

<sup>6</sup>Comment: Each move increases the sum of the numbers by 2. If eventually all six numbers were equal to  $k$ , then  $6k = 1 + 2 + 3 + 4 + 5 + 6 + 2t = 21 + 2t$ . The left side is even, while  $21 + 2t$  is odd for all  $t$ . Hence the numbers can never all become equal.

5. All dominoes from the standard set are laid out in a single chain according to the usual rule (neighbouring halves show the same number). One end of the chain has a five. What can be at the other end?<sup>7</sup>
6. Can a line that does not pass through any vertex of an 11-gon intersect all of its sides?<sup>8</sup>
7. On a table stand seven overturned cups. You may simultaneously turn over any two cups. Is it possible to end up with all the cups upright?<sup>9</sup>

---

<sup>7</sup>Comment: In the usual “double six” set the tiles are the pairs  $(i, j)$  with  $1 \leq i \leq j \leq 6$ . Each number  $1, \dots, 6$  occurs exactly six times. Consider the graph whose vertices are the numbers and whose edges correspond to dominoes. In this graph every vertex has even degree, so the tiles can be arranged in a closed Eulerian circuit. Cutting this circuit yields an open chain whose two ends are identical. Hence if one end displays a five, the other end must also display a five.

<sup>8</sup>Comment: In any polygon the number of intersections of a straight line with the sides is even (because the line goes inside the polygon, then outside, then inside,... finally it goes outside). Since 11 is odd, no such line exists.

<sup>9</sup>Comment: The parity of the number of overturned cups changes by 0 or 2 at each move. Starting with seven (odd) and wanting to end with zero (even) is impossible.

## 1.1.1 :: :: 1st homework, read and write

### Read:

Paul Zeitz - The Art and Craft of Mathematical Problem Solving, pp. 13-15.

#### The Frog Problem

- The frog problem is a classic Russian math circle problem.
- Three frogs are situated at 3 of the corners of a square. Every minute, 1 frog is chosen to leap over another chosen frog, so if you drew a line from the starting position to the ending position of the leaper, the leapee is at the exact midpoint.
- Will a frog ever occupy the vertex of the square that was originally unoccupied?
- How can we effectively investigate this problem?
- Graph paper allows us to attach numbers to the positions of the frogs. Once we have numbers, we can employ arithmetical and algebraic methods. Thus, place the frogs at  $(0, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ . The question now is, can a frog ever reach  $(1, 0)$ ?
- Thinking about the appropriate venue for investigation is an essential starting strategy for any problem.
- Another investigative idea: Use colored pencils to keep track of individual frogs. This adds information, as it allows us to keep track of 1 frog at a time. Color the  $(1, 1)$ ,  $(0, 1)$ , and  $(0, 0)$  frogs red, blue, and green, respectively.
- Notice, by experimenting, that the red frog only seems to hit certain points, forming a larger (2-unit) grid.
- Some of the coordinates that the red frog hits are  $(1, 1)$ ,  $(1, 3)$ ,  $(1, -1)$ ,  $(-1, 1)$ ,  $(-1, -1)$ , and  $(-1, -3)$ . They are all odd numbers!
- Likewise, the blue frog only hits certain points on a 2-unit grid, including  $(0, 1)$ ,  $(2, 1)$ ,  $(4, 1)$ , and  $(0, -1)$ ; these are all of the form  $(\text{even}, \text{odd})$ .

- Likewise, the green frog only hits  $(\text{even}, \text{even})$  points.
- On the other hand, the missing southeast vertex was  $(1, 0)$ , which has the form  $(\text{odd}, \text{even})$ . It seems as though it is impossible, but how can we formulate this in an airtight way?
- It is often very profitable to contemplate parity (oddness and evenness).
- The essential reason for this is that a parity focus reduces a problem from possibly infinitely many states to just 2.
- Parity involves the number 2. Where in this problem do we see this number? In doubling, because of the symmetry of the way the frogs leap. When the leaper jumps over the leapee, she adds twice the horizontal displacement to her original horizontal coordinate. The same holds for vertical coordinates.
- So when a frog jumps, its coordinates change by even numbers!
- For example, suppose the red  $(1, 1)$  frog jumps over the green frog at  $(0, 0)$ . The horizontal and vertical displacements to the leapee are both  $-1$  (since it is moving left and down), so the final change in coordinates will be  $-2$ . The horizontal coordinate will be  $1 + -2 = -1$ , and the vertical will also be  $-1$ .
- Suppose now that the red frog jumps over the blue frog, which is  $(0, 1)$ . The horizontal displacement is  $+1$ , and the vertical displacement to the target is  $+2$ . So the new horizontal coordinate will be  $-1$  (the starting value)  $+ 2 \times 1 = +1$ , and the new vertical coordinate will be  $-1$  (the starting value)  $+ 2 \times 2 = 3$ . Thus the red frog jumps from  $(-1, -1)$  to  $(1, 3)$ .

- In general, when a frog jumps, we will take its starting  $x$ -coordinate and add twice the horizontal displacement to its target. Likewise, we take its starting  $y$ -coordinate and add twice the vertical displacement to the target. These displacements may be positive, negative, or zero.
- In other words, you take the starting coordinates and add even numbers to them. But when you add an even number to something, its parity does not change!
- So the  $(\text{odd}, \text{odd})$  frog—the red frog—is destined to stay at  $(\text{odd}, \text{odd})$  coordinates, no matter what.

#### Suggested Reading

Polya, *How to Solve It*.  
Zeitz, *The Art and Craft of Problem Solving*, chap. 2.

#### Questions to Consider

- Write the numbers from 1 to 10 in a row and place either a minus or a plus sign between the numbers. Is it possible to get an answer of zero?
- A group of jealous professors is locked up in a room. There is nothing else in the room but pencils and 1 tiny scrap of paper per person. The professors want to determine their average (mean, not median) salary so that each can gloat or grieve over his or her personal situation compared to the others. However, they are secretive people and do not want to give away salary information to anyone else. Can they determine the average salary in such a way that no professor can discover any fact about the salary of anyone but herself? For example, even facts such as “one professor earns less than \$90,000” are not allowed.

### Write:

**Problem 1.** Write the full solution (with all details) of the problem that we cannot cut a square  $4 \times 4$  without opposite corners into domino. Write the solution where we use case-by-case strategy, without using coloring.

**Problem 2.** Write the solution of the problem that we cannot cut a square  $8 \times 8$  without opposite corners into domino (you can use coloring).

**Problem 3.** Show that you can cut a square in  $n$  squares for each  $n > 6$ .

**Problem 4.** The numbers  $1, 2, \dots, 101$  are written on a blackboard. You are allowed to erase any two numbers and write their difference in their place. After repeating this operation 100 times only one number remains. Prove that this number cannot be zero.

## 1.2 :: pigeonhole principle, correspondence, invariants

From a hundred rabbits, you can never assemble a horse;  
a hundred suspicions do not constitute a proof.

*Crime and Punishment*, F. Dostoevsky

### :: :: proof by contradiction

One of the most widely used techniques in elementary mathematics is the *proof by contradiction*. The general strategy is to assume that the statement to be proved is false and then show that this assumption leads to an impossibility. Having reached a contradiction, one concludes that the original statement must in fact be true.

**Example 1.** *Prove that there are infinitely many prime numbers.*

*Solution.* Suppose the contrary, namely that there are only finitely many primes. List them as  $p_1, p_2, \dots, p_n$ . Consider the number

$$N = p_1 p_2 \cdots p_n + 1.$$

By construction  $N$  is not divisible by any of the primes  $p_i$ . Therefore  $N$  has no prime divisors at all, which contradicts the fundamental fact that every integer greater than 1 has at least one prime divisor. Hence there cannot be only finitely many primes.

**Example 2.** *Five boys found nine mushrooms. Prove that at least two of them must have found the same number of mushrooms.*

*Solution.* Assume that the boys all found different numbers of mushrooms. Order them by increasing number of mushrooms: the first boy picked at least 0 mushrooms, the second at least 1, the third at least 2, the fourth at least 3 and the fifth at least 4. Altogether they would have picked at least  $0 + 1 + 2 + 3 + 4 = 10$  mushrooms, contradicting the fact that there were only nine mushrooms. Thus at least two boys must have collected the same number of mushrooms.

**Example 3.** *Prove that there does not exist a tetrahedron (triangular pyramid) in which each edge is adjacent to an obtuse angle of one of its faces.*

*Solution.* Suppose such a tetrahedron exists. In any triangle the side opposite an obtuse angle is the largest side. Therefore each edge of the tetrahedron must be strictly shorter than some other edge that is adjacent to the obtuse angle. Since

the number of edges in a tetrahedron is finite, this strict inequality cannot cycle indefinitely; the assumed configuration leads to an infinite descending chain of lengths, which is impossible. Hence no such tetrahedron exists.

**Example 4.** *Prove that  $\log_2 3$  is an irrational number.*

*Solution.* Assume otherwise and write  $\log_2 3 = \frac{p}{q}$  with  $p$  and  $q$  positive integers. Then  $2^{p/q} = 3$ , i.e.  $2^p = 3^q$ . The left side is even while the right side is odd. This contradiction shows that  $\log_2 3$  is irrational.

### :: :: pigeonhole principle

In its simplest form the pigeonhole principle says that if ten rabbits sit in nine boxes, then some box must contain at least two rabbits. A more general formulation is: “If  $n$  rabbits sit in  $k$  boxes, then there exists a box containing at least  $\lceil n/k \rceil$  rabbits and a box containing at most  $\lfloor n/k \rfloor$  rabbits.” Do not be alert by fractional rabbits—if  $10/9$  rabbits must sit in a box, then in fact at least two do.

**Proof of the principle.** Assume that each of the  $k$  boxes contains strictly fewer than  $n/k$  rabbits. Then altogether there are fewer than  $(n/k)k = n$  rabbits, which contradicts the assumption that there are  $n$  rabbits. This simple argument illustrates why similar reasoning occurs throughout combinatorics.

The pigeonhole principle may seem obvious, but in order to apply it one must sometimes think carefully about what plays the role of “rabbits” and what plays the role of “boxes”. For example, if each element of a set  $A$  corresponds to exactly one element of a set  $B$ , then one may call the elements of  $A$  rabbits and those of  $B$  boxes.

The principle also has continuous versions. For instance: “If  $n$  rabbits eat  $m$  kg of grass, then some rabbit ate at least  $m/n$  kg and some rabbit ate at most  $m/n$  kg.” In this formulation the rabbits play the role of boxes for the grass, while the grass plays the role of rabbits sitting in boxes.

**Example 1.** *There are 400 students in a school. Prove that at least two of them were born on the same day of the year.*

*Solution.* There are 366 days in a year. Think of the students as rabbits and the days of the year as boxes. Then some box must contain at least  $\lceil 400/366 \rceil = 2$  students, i.e. two students share a birthday.

**Example 2.** *The ocean covers more than half of the Earth's surface. Prove that somewhere in the ocean there are two antipodal points (diametrically opposite points on the globe).*

*Solution.* Reflect the ocean through the centre of the Earth. The union of the ocean and its reflection covers more than the entire surface of the Earth, so there is a point that lies in both. Such a point and its antipode are both in the ocean.

**Example 3.** *Sixty five schoolchildren came to an exam. They were given three tests. On each test the score was one of 2, 3, 4 or 5. Must there be two pupils who got the same grades on all three tests?*

*Solution.* There are  $4^3 = 64$  possible triples of grades. Since 65 pupils take part, by the pigeonhole principle at least two of them must have identical triples of grades.

**Example 4.** *Prove that among any five people there are two who have the same number of acquaintances among these five people (a person may be unacquainted with everyone).<sup>10</sup>*

### :: :: invariants

*Invariant* — a quantity which does not change as a result of certain operations (for example, cutting and rearranging parts of a figure does not change the total area). If an invariant distinguishes two configurations, then it is impossible to pass from one to the other. As an invariant one can use *parity* or a *coloring*. In problems about the sum of digits, the remainders upon division by 3 or 9 are often used.

*Semi-invariant* — a quantity that changes only in one direction (that is, it can only increase or only decrease). The concept of a semi-invariant is often used when proving that a process terminates.

**Example 1.** A wonder-apple tree grows both bananas and pineapples. In one move one is allowed to pick two fruits from it. If two bananas or two pineapples are picked, then one more pineapple grows; if one banana and one pineapple are picked, then one banana grows. In the end there remains one fruit. Which fruit is it, if it is known how many bananas and pineapples were there initially?

---

<sup>10</sup>Comment: The possible numbers of acquaintances are 0,1,2,3,4. If 0 is among them, then 4 is not possible, so we have only 4 possibilities for 5 people. The same is true when there is no person with 0 acquaintances.

*Solution.* The parity of the number of bananas does not change, therefore, if the number of bananas was even, the remaining fruit is a pineapple, and if it was odd, then it is a banana.<sup>11</sup>

**Example 2.** In one cell of a  $4 \times 4$  square table there is a minus sign, and in all the other cells there are pluses. It is allowed to change at once the signs in all cells of any one row or any one column. Prove that, no matter how many such changes of signs we perform, it is impossible to obtain a table consisting entirely of pluses.

*Solution.* Let us replace the sign “+” by the number 1 and the sign “−” by  $-1$ . Note that the *product of all numbers in the table* does not change when the signs in all numbers of any one row or column are flipped. In the initial position this product is  $-1$ , and in the table consisting entirely of pluses it is  $+1$ , which proves the impossibility of transition.<sup>12</sup>

**Example 3.** On a straight line there stand two chips: on the left a red one, on the right a blue one. It is allowed to perform either of two operations: insert two chips of the same color next to each other (between any two chips or at the edge), or remove a pair of neighboring chips of the same color (between which there are no other chips). Is it possible, by using such operations, to leave exactly two chips on the line, a blue one on the left and a red one on the right?

*Solution.* Consider the number of pairs of chips of different colors (not necessarily neighboring), where the left chip is red. Note that the parity of this number does not change. In the initial configuration this number is odd (equal to 1), but in the desired configuration it is even (0). Therefore it is impossible to reach the desired configuration.<sup>13</sup>

**Example 4.** On the island of Grey-Brown-Raspberry live chameleons: 13 grey, 15 brown, and 17 raspberry-colored ones. If two chameleons of different colors meet, then both change their color to the third one. Can it happen that at some moment all the chameleons on the island become of the same color?

*Hint.* Let us denote by  $B$ ,  $G$ , and  $M$  the numbers of brown, grey, and raspberry chameleons, respectively. Prove that the remainders upon division by 3 of the

---

<sup>11</sup>Comment: The invariant is the parity of the number of bananas. The text implicitly assumes that the process continues until one fruit remains.

<sup>12</sup>Comment: The invariant here is the product of all entries in the table, equal to  $-1$  at the start.

<sup>13</sup>Comment: The invariant is the parity of the number of red-blue pairs in that order.

differences  $B - G$ ,  $G - M$ , and  $M - B$  do not change.<sup>14</sup>

**Example 6.** Is it possible to cut a round disc into several parts and assemble from them a square? (The cuts are pieces of straight lines and arcs of the circle.)

*Solution.* Consider the invariant: the difference of the sums of the lengths of concave and convex boundary arcs of all the pieces. This quantity does not change when a piece is cut into two, nor when two pieces are joined together. For a unit round disc this invariant equals  $2\pi$ , and for a square it equals 0. Therefore “squaring the circle” is impossible.<sup>15</sup>

### :: :: problems for tutorial

1. Is it possible to connect five cities by roads so that each city is connected with exactly three others?<sup>16</sup>
2. Prove that there does not exist a polyhedron whose number of faces is odd and such that each face has an odd number of vertices.<sup>17</sup>
3. In each cell of an  $m \times k$  rectangular array there is a number. The sum of the numbers in each row and in each column is 1. Prove that  $m = k$ .<sup>18</sup>
4. A class contains 25 students. It is known that among any three students there are two who are friends. Prove that there is a student who has at least 12 friends.<sup>19</sup>

---

<sup>14</sup>Comment: The invariant consists of the three residues modulo 3 of the color differences; since they cannot all become zero simultaneously, a single color for all chameleons is impossible.

<sup>15</sup>Comment: The invariant compares total curvature (arc excess vs. deficit) of the boundary. Since it is preserved under cutting and joining, the transition from circle to square is impossible.

<sup>16</sup>Comment: In graph-theoretic terms this asks for a five vertex 3-regular graph. The sum of degrees would be 15, which contradicts the handshaking lemma stating that the sum of degrees must be even.

<sup>17</sup>Comment: If  $F$  is the number of faces and each face has an odd number of edges, then the sum of the number of all edges in of faces is odd; however this sum also equals  $2E$  (since each edge belongs to two faces), which is even. Therefore  $F$  must be even.

<sup>18</sup>Comment: Summing by rows gives a total of  $m$ , while summing by columns gives a total of  $k$ . Therefore  $m = k$ .

<sup>19</sup>Comment: In a graph on 25 vertices without an independent set of size 3 the average degree is more than  $\frac{24}{2}$ ; hence some vertex has degree at least 12.

5. A committee of 60 people held 40 meetings. Exactly 10 committee members attended each meeting. Prove that some two members met at committee meetings at least twice. <sup>20</sup>

### :: :: problems for workshop

1. Is there a convex polygon with more than three acute angles? <sup>21</sup>
2. Prove that there are infinitely many prime numbers of the forms
  - (a)  $4k + 3$ , <sup>22</sup>
  - (b)  $3k + 2$ , <sup>23</sup>
  - (c)  $6k + 5$ . <sup>24</sup>
3. Prove that if  $(m - 1)! + 1$  is divisible by  $m$ , then  $m$  is a prime number. <sup>25</sup>
4. In a class of 30 pupils a test was written. Nikita made 13 mistakes and every other pupil made strictly fewer. Prove that there are three pupils who made the same number of mistakes. <sup>26</sup>
5. The Earth has more than six billion inhabitants and no person older than 150 years exists. Prove that there are two people on Earth who were born at exactly the same second. <sup>27</sup>

---

<sup>20</sup>Comment: Double count the pairs (meeting, {two participants}). There are  $40 \times \binom{10}{2}$  such pairs, but there are only  $\binom{60}{2}$  pairs of committee members. By the pigeonhole principle some pair must occur at least twice.

<sup>21</sup>Comment: The answer is no. In a convex  $n$ -gon the sum of the interior angles is  $180(n - 2)$  degrees. If all  $k$  angles are acute then their sum is less than  $90k$ , forcing  $180(n - 2) < 90k + 180(n - k)$ ; from this one sees that  $k \leq 3$ .

<sup>22</sup>Comment: Consider the number  $4p_1p_2 \cdots p_n + 3$  where the  $p_i$  run over all primes of the form  $4k + 3$ .

<sup>23</sup>Comment: The same idea works by considering  $3p_1p_2 \cdots p_n + 2$ .

<sup>24</sup>Comment: The argument is analogous; take  $6p_1p_2 \cdots p_n + 5$ .

<sup>25</sup>Comment: This is a one direction form of Wilson's theorem. The converse is well known: if  $m$  is prime then  $(m - 1)! \equiv -1 \pmod{m}$ .

<sup>26</sup>Comment: Excluding Nikita leaves 29 pupils making between 0 and 12 mistakes. There are 13 possible values and 29 pupils, so by the pigeonhole principle some value is taken by at least three pupils.

<sup>27</sup>Comment: The number of seconds in 150 years is approximately  $150 \times 365 \times 24 \times 60 \times 60$ , which is less than six billion. Therefore two of the more than six billion birth instants must coincide.

6. Twelve lines are drawn in the plane. Prove that some two of them form an angle of at most  $15^\circ$ .<sup>28</sup>
7. A drawer contains socks: 10 black, 10 blue and 10 white. What is the smallest number of socks one must draw without looking in order to guarantee that among the drawn socks there are
- two of the same colour,<sup>29</sup>
  - two of different colours;<sup>30</sup>
  - two black socks?<sup>31</sup>
8. Thirty six stones were mined in a quarry. Their masses form an arithmetic progression  $490, 495, 500, \dots, 665$  kg. Is it possible to transport these stones using seven trucks, each of capacity 3 tonnes (3000 kg)?<sup>32</sup>

---

<sup>28</sup>Comment: Partition the  $180^\circ$  around a point into 12 intervals of length  $15^\circ$ . By the pigeonhole principle two of the directions of lines must lie in the same interval.

<sup>29</sup>Comment: Taking 4 socks guarantees at least two of the same colour, since 3 colours and 4 socks force a repetition.

<sup>30</sup>Comment: One could draw all 10 socks of one colour without yet having two colours. The eleventh sock guarantees at least one sock of a different colour.

<sup>31</sup>Comment: In the worst case one draws all 20 non black socks before drawing a black one. The next sock (the 22nd) must then be a second black.

<sup>32</sup>Comment: The total mass of the stones is  $36 \times 577.5 = 20790$  kg. Seven trucks can carry 21000 kg, so sufficient capacity exists. But....

## :: :: 2nd homework

### Read:

Kenneth H. Rosen, Discrete Mathematics and Its Applications, p.313, 320-321.  
If you don't understand certain notation, just skip it for now (or find the explanation in the first chapter of this book), we will cover it later.

**PRINCIPLE OF MATHEMATICAL INDUCTION** To prove that  $P(n)$  is true for all positive integers  $n$ , where  $P(n)$  is a propositional function, we complete two steps:

**BASIC STEP:** We verify that  $P(1)$  is true.

**INDUCTIVE STEP:** We show that the conditional statement  $P(k) \rightarrow P(k+1)$  is true for all positive integers  $k$ .

To complete the inductive step of a proof using the principle of mathematical induction, we assume that  $P(k)$  is true for an arbitrary positive integer  $k$  and show that under this assumption,  $P(k+1)$  must also be true. The assumption that  $P(k)$  is true is called the **inductive hypothesis**. Once we have both steps to prove mathematical induction, we have shown that  $P(n)$  is true for all positive integers, that is, we have shown that  $\forall n P(n)$  is true where the quantification is over the set of positive integers. In the inductive step, we show that  $\forall k (P(k) \rightarrow P(k+1))$  is true, where again, the domain is the set of positive integers.

Expressed as a rule of inference, this proof technique can be stated as

$$(P(1) \wedge \forall k (P(k) \rightarrow P(k+1))) \rightarrow \forall n P(n),$$

when the domain is the set of positive integers. Because mathematical induction is such an important technique, it is worthwhile to explain in detail the steps of a proof using this technique. The first thing we do to prove that  $P(n)$  is true for all positive integers  $n$  is to show that  $P(1)$  is true. This is called the **basis step**. Then we move to the **inductive step**, where we show that if  $P(k)$  is true, then  $P(k+1)$  must also be true. There we must show that  $P(k) \rightarrow P(k+1)$  is true for every positive integer  $k$ . To prove that this conditional statement is true for every positive integer  $k$ , we need to show that  $P(k+1)$  cannot be false when  $P(k)$  is true. This can be accomplished by assuming that  $P(k+1)$  is true and showing that under this hypothesis  $P(k+1)$  must also be true.

**Remark:** In a proof by mathematical induction it is *not* assumed that  $P(k)$  is true for all positive integers! It is only shown that *if it is assumed that  $P(k)$  is true, then  $P(k+1)$  is also true*. Thus, a proof by mathematical induction is not a case of begging the question, or circular reasoning.

When we use mathematical induction to prove a theorem, we first show that  $P(1)$  is true. Then we know that  $P(2)$  is true, because  $P(1)$  implies  $P(2)$ . Further, we know that  $P(3)$  is true, because  $P(2)$  implies  $P(3)$ . Continuing along these lines, we see that  $P(n)$  is true for every positive integer  $n$ .

### Write:

1. A snail crawls in the plane at a constant speed and every 15 minutes turns  $90^\circ$  (sometimes left, sometimes right). Prove that it can return to its starting point only after an integer number of hours.
2. In the language of a certain tribe there are only two sounds (written here as **w** and **y**). Two words are considered the same if one can be obtained from the other by a sequence of operations of two types: deleting a consecutive occurrence of **wy** or **yyww** and inserting **yw** at any position. Do the words **ywy** and **wyw** represent the same word?
3. Three grasshoppers sit at three vertices of a square in the plane. They play a game of leapfrog: one grasshopper jumps to the point symmetric to another grasshopper across the third (a point  $A$  is symmetric to a point  $B$  with respect

**EXAMPLE 7** An Inequality for Harmonic Numbers The harmonic numbers  $H_j$ ,  $j = 1, 2, 3, \dots$ , are defined by

$$H_1 = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j}.$$

For instance,

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever  $n$  is a nonnegative integer.

5.1 Mathematical Induction 321

**Solution:** To carry out the proof, let  $P(n)$  be the proposition that  $H_{2^n} \geq 1 + \frac{n}{2}$ .

**BASIC STEP:**  $P(0)$  is true, because  $H_0 = H_1 = 1 \geq 1 + \frac{0}{2}$ .

**INDUCTIVE STEP:** The inductive hypothesis is the statement that  $P(k)$  is true, that is,  $H_{2^k} \geq 1 + \frac{k}{2}$ , where  $k$  is an arbitrary nonnegative integer. We must show that if  $P(k)$  is true, then  $P(k+1)$ , which states that  $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$ , is also true. So, assuming the inductive hypothesis, it follows that

$$\begin{aligned} H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}} && \text{by the definition of harmonic number} \\ &= H_{2^k} + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}} && \text{by the definition of } 2^{\text{th}} \text{ harmonic number} \\ &\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} && \text{by the inductive hypothesis} \\ &\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} && \text{because there are } 2^k \text{ terms} \\ &\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} && \text{each } \geq \frac{1}{2^{k+1}} \\ &= 1 + \frac{k+1}{2}. && \text{canceling a common factor of } 2^k \text{ is second term.} \end{aligned}$$

This establishes the inductive step of the proof.

We have completed the basis step and the inductive step. Thus, by mathematical induction  $P(n)$  is true for all nonnegative integers  $n$ . That is, the inequality  $H_{2^n} \geq 1 + \frac{n}{2}$  for the harmonic numbers holds for all nonnegative integers  $n$ .

**Remark:** The inequality established here shows that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \dots$$

is a divergent infinite series. This is an important example in the study of infinite series.

to a point  $C$  if  $C$  is the middle of the interval  $AB$ , here  $A, B, C$  are points on the plane.) Can any grasshopper ever land on the fourth vertex of the square?

4. There is an island populated by knights and liars. Knights always tell truth, liars always lie. A stranger meets three local persons and asks everybody: "how many knights are among you three?". The first answered: no one. The second answered: one. What is the answer of the third person?
5. Automate can shred a piece of paper on 4 or 6 pieces. What number of pieces can be reached from one sheet?
6. Prove that among any 52 integers there exist two whose sum or difference is divisible by 100.

**Just for fun, not a part of the homework: use AI, ask them to solve the following problem, then try to understand their solution, then check, if it is at all correct:**

*Four identical jars are filled to three quarters with paint, each containing a different colour. It is possible to pour any portion of the contents of one jar into another. Is it possible to perform a finite sequence of pourings so that each jar contains an identical mixture of the four colours?*

Try to solve this problem by yourself.

Last, but not least. Read the next page for a short advice on how to write solutions with a good and bad example.

## Example of bad and good writing

Five boys found nine mushrooms. Prove that at least two of them must have found the same number of mushrooms.

### Bad solution

It is not possible that they found different numbers of mushroom because in this case they will find at least 10 mushrooms<sup>33</sup>, because in the worst<sup>34</sup> case the first<sup>35</sup> found 0, the second found 1 mushroom, etc,  $0 + 1 + 2 + 3 + 4 = 10$ .

### Good solution

Assume that the boys all found different numbers of mushrooms. Order them by increasing number of mushrooms. Then, the first boy picked at least 0 mushrooms, the second at least 1, the third at least 2, the fourth at least 3 and the fifth at least 4. Altogether they would have picked at least  $0 + 1 + 2 + 3 + 4 = 10$  mushrooms, contradicting the fact that there were only nine mushrooms. Thus at least two boys must have collected the same number of mushrooms.

In a good solution each step is small and incremental, it might be

- an assumption (the first phrase states the proof strategy 'by contradiction'),
- an action ('Order them...'),
- introducing a notation ('Denote ... by ... '>,
- a computation,
- or a conclusion derived from the preceding steps.

Use single-action sentences: 'Assume this. Count that. Then A. Then B.' A solution is good (or formal) if it is easy to follow its logical structure and easy to check each step separately (assume that a reader has a limited amount of memory).

Instead of 'A, because of B', write 'B, therefore A'.

**Exercise: look at solutions of the problems across these notes and evaluate their quality, try to cut them into such elementary steps.**

<sup>33</sup>Here the reader asks 'why'? It is better if, whenever it is possible, each next step immediately follows from the previous one, and explanations are given BEFORE a statement.

<sup>34</sup>Where is the definition of the worst case? And even if you give such a definition, then you need to show that the case 0,1,2,3,4 is indeed the worst case.

<sup>35</sup>What if the first found 3 and the second 0? Then you need to consider all these cases or convince the reader that these cases are all the same, i.e. as 'ordering' step in the good solution.

## 1.3 :: method of the extreme, counting in two ways, combinatorics

### Method of the Extreme

Special or extreme objects often serve as a cornerstone of a solution. For example, one might look for the largest number, the nearest point, a corner point, a degenerate circle or a limiting case. It is therefore useful to consider such special objects.

In problems solvable by the method of the extreme one often uses the method of the minimal counterexample: assume that the statement is false. Then there exists a counterexample minimal in some sense. If one can show that this counterexample can be made smaller, one obtains the desired contradiction.

**Example 1.** *The plane is cut by  $N \geq 3$  lines in general position. Prove that to each line there is a triangle adjacent to that line.*

*Solution.* Choose a line and consider all intersection points of the other lines. Among these points pick the one that is closest to the chosen line. The two lines passing through this closest point intersect the chosen line and form with it a triangle. No other lines can intersect the interior of this triangle (otherwise one would find a closer intersection point), so the triangle is indeed adjacent to the chosen line.

**Example 2.** *Prove that in any convex polyhedron there are two faces with the same number of edges.*

*Solution.* Consider the face  $G$  with the largest number  $n$  of edges. Each edge of  $G$  is adjacent to another face; there are  $n$  faces adjacent to  $G$ . The number of edges of each adjacent face lies between 3 and  $n - 1$ , giving only  $n - 2$  possible values. Since there are more adjacent faces than possible values, by the pigeonhole principle two of them must have the same number of edges.

**Example 3.** *In each cell of a chessboard a number is written. It turns out that every number equals the arithmetic mean of the numbers in the neighbouring cells (by side). Prove that all the numbers are equal.*

*Solution.* Let  $M$  be the maximum of all the numbers. Since  $M$  equals the average of its neighbours, all neighbouring numbers must also equal  $M$ . Proceeding by connectivity of the board shows that all numbers are equal to  $M$ .

**Example 4.** *From a point inside a convex polygon perpendiculars are dropped to its sides or to their extensions. Prove that at least one of the perpendiculars falls on a*

side of the polygon.

*Solution.* Consider the perpendicular whose foot on the boundary of the polygon is closest to the original point. If that foot lay on the extension of a side, one could shift slightly in the direction of the polygon and produce a shorter perpendicular, a contradiction. Hence at least one perpendicular must land on an actual side.

**Example 5.** *Prove that the number*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is never an integer. (Here  $n \geq 2$ .)

*Solution.* Look at the denominators of the summands and identify the largest power of 2 appearing there. When the terms are written with a common denominator, the numerator will be odd, while the denominator is a multiple of that power of 2; hence the result cannot be an integer.

## Counting in Two Ways

When setting up equations one often expresses some quantity—such as an area, a distance or a time—in two different ways. Sometimes one estimates a quantity in two different ways and obtains either an inequality or values with different parity. This idea is closely related to the notion of an invariant and is often a source of contradictions (see also the section on proof by contradiction).

**Example 1.** *Can one arrange numbers in a  $5 \times 5$  square table so that the sum of the numbers in each row is positive and in each column is negative?*

*Solution.* Suppose such an arrangement were possible. If one sums all the numbers by rows, the total sum is positive. If one sums all the numbers by columns, the total sum is negative. This contradiction shows that no such arrangement exists.

**Example 2.** *In a class there are 27 pupils. Each boy is friends with four girls and each girl is friends with five boys. How many boys and how many girls are there?*

*Solution.* Let  $B$  be the number of boys and  $G$  the number of girls. Count the total number of friendships in two ways. On the one hand, each boy is friends with four girls, giving  $4B$  friendships. On the other hand, each girl is friends with five boys, giving  $5G$ . Thus  $4B = 5G$ . Since  $B + G = 27$ , simple algebra yields  $B = 15$  and  $G = 12$ .

**Example 3.** Find the sum of the geometric progression

$$S_n = 1 + 3 + 9 + \cdots + 3^n.$$

*Solution.* Observe that, given  $S_n$ , the next sum  $S_{n+1}$  can be obtained in two ways: either add  $3^{n+1}$  to  $S_n$  or multiply all terms of  $S_n$  by 3 and then add 1. Hence  $S_n + 3^{n+1} = 3S_n + 1$ , which simplifies to  $S_n = \frac{3^{n+1}-1}{2}$ .

**Example 4.** Can all faces of a convex polyhedron have six or more sides?

*Solution.* No. Estimate in two ways the average of all the angles of all the faces. On the one hand, the average interior angle of an  $n$ -gon with  $n \geq 6$  is at least  $120^\circ$ . On the other hand, at each vertex of the polyhedron at least three faces meet and the sum of the angles meeting at a vertex is strictly less than  $360^\circ$ . Therefore the average angle at each vertex is strictly less than  $120^\circ$ . The contradiction shows that such a polyhedron cannot exist.

**Example 5.** A binomial coefficient  $\binom{n}{k}$  (pronounced ‘ $n$  choose  $k$ ’) is the number of ways to choose  $k$  different numbers from the set  $1, 2, \dots, n$  of  $n$  numbers. Using the above definition of the binomial coefficients, prove (without any formulas) that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

①

① ①

① ② ①

① ③ ③ ①

① ④ ⑥ ④ ①

The picture on the left is called the Pascal triangle. Left and right sides are filled with ones. Then we fill rows one by one. Each interior entry equals the sum of the two entries above it: specifically, the entry in row  $n$ , position  $k$  is  $\binom{n}{k}$ .

**Problem 1.** Prove the **Newton binomial formula**:  $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ , for example  $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$ .

## :: :: problems for tutorial

Before this tutorial read Section 5.2.

1. A football fan drew the positions of the players on a football field at the beginning of the first and the second halves. It turned out that some players had exchanged places, while the others remained in their positions. At the same time, the distance between any two players did not increase. Prove that all these distances in fact remained the same.<sup>36</sup>
2. A traveller leaves his home city  $A$  and goes to the city  $B$  in his country that is farthest from  $A$ . From that city he goes to the city  $C$  farthest from his current city  $B$ , and so on. Prove that if the city  $C$  does not coincide with  $A$ , then the traveller will never return home. (Distances between cities are assumed to be distinct.)<sup>37</sup>
3. One of the heads of a hundred headed dragon wants to arrange his heads so that each head lies strictly between two others. Can he do it? (You may regard the heads as points in space.)<sup>38</sup>
4. Show (hint: by induction) that a connected graph with  $n$  vertices and without cycles has exactly  $n - 1$  edge.

---

<sup>36</sup>Comment: An invariant argument: since no pairwise distance increases, the total sum of distances acts as a semi-invariant.

<sup>37</sup>Comment: Each step strictly increases the distance to the farthest city left behind. If one could return home, one would form a cycle of increasing distances, which is impossible.

<sup>38</sup>Comment: It is impossible for every point in a finite set to lie between two others on a straight line, because the extreme points on the convex hull of the configuration cannot be between two other points. Start solution by: consider the head with biggest  $z$ -coordinate. Among them consider the head with the biggest  $y$ -coordinate, among them...

## :: :: problems for workshop

**Exercise 1.** In cafe, you can order green, red, or black soup and green, red, or black tea. How many different orders you can make if it is prohibited to order soup and tea of the same color?

**Exercise 2.** Five persons  $A, B, C, D, E$  visit a cafe every day and always seat on the same table with five chairs (chairs are numbered: 1, 2, 3, 4, 5.). In how many different ways they can seat on these five chairs?<sup>39</sup>

**Exercise 3.** In cafe, the cook has a) 4, b) 8 ingredients. A dish is made of three different ingredients (their order is not important). How many different dishes the cook can cook? <sup>40</sup>

**Exercise 4.** How many four-digit numbers contain 7?

**Exercise 5.** In cafe, the cook has 9 ingredients. A dish is made of three different ingredients (their order is important). How many different dishes the cook can cook if the 3rd and 4th ingredients are not allowed to use together?

**Problem 2.** Find the coefficient behind  $x^3$  in  $(1+x)^{10}(1+2x)^7$ .

Solution: we can choose  $x$  from three parenthesis of the first type  $(1+x)$  and 1 from all parenthesis of the second type  $(1+2x)$ . We can do this in  $10 \times 9 \times 8$  cases, but since it is not important in which order we choose the first type parenthesis, we divide by 6. So, this gives  $\binom{10}{3}$ . We also can choose  $x$  from two parenthesis of the first type and from one of the second type, this gives  $2 \times \binom{10}{2} \binom{7}{1}$ , etc. The answer<sup>41</sup>:

$$\binom{10}{3} + 2 \cdot \binom{10}{2} \binom{7}{1} + 4 \cdot \binom{10}{1} \binom{7}{2} + 8 \cdot \binom{10}{1} \binom{7}{3}.$$

**Exercise 6.** Find the coefficient behind  $x^4$  in  $(1+x)^{10}(1+2x)^7$ .

---

<sup>39</sup>Recall the notation  $n!$

<sup>40</sup>For the case a) one can list all dishes, then explain that it is  $\frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1}$ . Introduce the notation  $\binom{n}{k}$ , provide the formula for it, tell that the proof is exactly the same as in b).

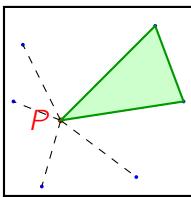
<sup>41</sup>You do not need to compute it numerically, the below answer is fine.

### 1.3.1 :: :: 3rd homework: find a mistake

#### Read:

*Problem: In a unit square, 101 points are thrown in such a way that no three of them lie on one straight line. Prove that there exists a triangle with vertices at these points whose area does not exceed  $\frac{1}{100}$ .*

Solution: Select a point  $P$  among 101 points and connect it with all the others — we obtain 100 segments. Choose a direction (say, clockwise) and successively connect the ends of these segments — we obtain 100 non-overlapping triangles whose total area does not exceed 1 (the total area of the square). By the pigeonhole principle, there exists at least one triangle among them whose area does not exceed  $\frac{1}{100}$ .



#### Write:

- Find the mistake in the above solution, explain it in your homework.
- One hundred numbers are arranged on a circle. Each number is equal to the arithmetic mean of its two neighbours. Prove that all the numbers are equal.
- Prove that if a graph with  $n$  vertices has only  $n - 2$  edges, then it is disconnected.
- Using the principle of mathematical induction, prove that

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1) \cdot (2n+1) = \frac{n(4n^2 + 6n - 1)}{3}.$$

- Each student in the class took part in at least one of two hikes. In each hike, the proportion of boys did not exceed  $2/5$ . Prove that in the whole class, the proportion of boys does not exceed  $4/7$ .
- On a plane,  $N$  points are given. Some pairs of points are connected by segments. If two segments with different endpoints intersect, they may be replaced by two other non-intersecting segments with the same endpoints. Can this process continue indefinitely?

## 1.4 :: feedback

During the lectures we skipped some of the problems listed above and occasionally postponed others to the next class. For the easier problems we paused and gave students time to work on them.

After the first three weeks we ran a survey. Most of the respondents said that the material is interesting—difficult, but manageable. A few students (mainly those with a background in school math competitions) said the material was easy.

Several students said they especially liked the domino-and-chessboard problem. Many of them use AI tools to understand the wording of the problems and homework, and to clarify the lecture content (also, English is an issue). At the same time, many students struggle with what a *formal solution* is and how to write it properly. They do, however, enjoy the story-style settings of the problems (frogs, Martians, etc.).

Some students complained that we do not provide a template or model for writing up solutions.

## 2 Naive set theory

### 2.1 :: notation, union, intersection, Venn diagrams

Definition of a set, Euler-Venn diagrams, symbols  $\cap, \cup, \forall, \exists, \wedge, \vee, \subset, \in, \notin, \Rightarrow, \Leftrightarrow, \emptyset, A \setminus B, \neg$  (negation), XOR, characteristic function (addition on the image is xor multiplication is AND, the same as multiplication and addition for even and odd numbers) constructions such as

$$\mathbb{Q}_+ = \{q \in \mathbb{Q} | q > 0\}, \sum_{i=1..n} f(i), \sum_{a \in A} f(a).$$

When does  $A \setminus B = B \setminus A$ ?

**Problem 1.** Translate into the plain English the following.

Def.  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous  $\Leftrightarrow$

$\Leftrightarrow \forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0$ , such that  $\forall x' \in [x - \delta, x + \delta]$  we have  $|f(x') - f(x)| < \varepsilon$ .

Write a definition of a prime number without using words and using only mathematical notation.

Prove that for any three sets the inclusion (and give an example of three sets for which this inclusion is strict.)

$$(A \setminus B) \setminus C \subset A \setminus (B \setminus C)$$

**Solution.** The left-hand side of (i) consists of those  $x \in A$  that do not belong to  $B$  and do not belong to  $C$ , is contained in  $A \setminus B$ . The right-hand side consists of those  $x \in A$  that do not belong to  $B \setminus C$  and, consequently, contains  $A \setminus B$ .

We can also express this in another way: the left-hand side consists of those  $x$  for which

$$x \in A \wedge x \notin B \wedge x \notin C,$$

while the right-hand side consists of those  $x$  for which

$$(x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C).$$

Thus, to give an example of three sets for which a strict inclusion occurs here, it suffices to take any three sets for which  $A \cap C$  is not contained in  $B$ , i.e.  $A = C = A \cap C = \{x\}$ ,  $B = \emptyset$ .

The meaning of introducing operations  $1 + 1 = 0 + 0 = 0, 0 + 1 = 1 + 0 = 1, 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, 1 \cdot 1 = 1$  on the set  $\{0, 1\}$  consists in the fact that now we can express Boolean operations on subsets of a set  $X$  in terms of operations on their characteristic functions.

**Problem.** Prove that for any two subsets  $A, B \subset X$  the following equalities hold:

1.  $\chi_{A \cap B} = \chi_A \chi_B$ ,
2.  $\chi_{A \Delta B} = \chi_A + \chi_B$ ,
3.  $\chi_{A \cup B} = \chi_A + \chi_B + \chi_A \chi_B$ ,
4.  $\chi_{A \setminus B} = \chi_A + \chi_A \chi_B$ .

The results of this problem serve as yet another argument in favor of considering  $\cap$  as the *product* of sets, and  $\Delta$  — rather than  $\cup$  — as the *sum* of sets.

**Problems on Intervals of the Real Line.** *What is the intersection of two intervals on the real line?*

**Answer.** Let  $(a, b)$  with  $a < b$  and  $(c, d)$  with  $c < d$  be two intervals. If  $b \leq c$  or  $d \leq a$ , their intersection is empty. If  $b > c$  and  $d > a$ , then the intersection of these intervals is the interval

$$(\max(a, c), \min(b, d)).$$

Consider two intervals of the real line  $(a, b)$ ,  $a < b$ , and  $(c, d)$ ,  $c < d$ . When is their union again an interval? What is it equal to in this case?

**Answer.** The union is an interval if and only if  $(a, b)$  and  $(c, d)$  intersect, that is, when  $c < b$  and  $a < d$ . In this case their union equals

$$(\min(a, c), \max(b, d)).$$

If three intervals of the real line have a common point, then at least one of them is contained in the union of the other two.

**Solution.** Let  $I_i = (a_i, b_i)$ ,  $a_i < b_i$ , where  $i = 1, 2, 3$ , be the three intervals. By the condition, there exists a point  $x \in \mathbb{R}$  such that  $a_i < x < b_i$  for all  $i = 1, 2, 3$ . Applying twice the result of the previous problem, we obtain

$$I_1 \cup I_2 \cup I_3 = (\min(a_1, a_2, a_3), \max(b_1, b_2, b_3)).$$

Take an index  $i$  such that  $\min(a_1, a_2, a_3) = a_i$ . Next, take some  $j \neq i$  such that  $\max(b_1, b_2, b_3) = b_j$ , or, if no such  $j$  exists (that is, if  $b_i > b_j$  for all  $j \neq i$ ), take any  $j \neq i$ . Then

$$I_i \cup I_j = (a_i, b_j) = I_1 \cup I_2 \cup I_3,$$

so if  $h$  is an index such that  $\{i, j, h\} = \{1, 2, 3\}$ , then  $I_h \subset I_i \cup I_j$ . □

## :: :: problems for tutorial

1. Is it true that

$$(A \setminus B) \setminus C = (A \setminus C) \setminus B?$$

2. Is it true that

$$(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)?$$

3. Prove the so-called **four sets identity**:

$$(A \cap B) \cup (C \cap D) = (A \cup C) \cap (A \cup D) \cap (B \cup C) \cap (B \cup D)$$

4. Express operations  $\cup$  and  $\setminus$  through the operations  $\cap$  and  $\Delta$ .

5. Express the operations  $\cap$  and  $\setminus$  through the operations  $\cup$  and  $\Delta$ .

In fact, Boolean operations can be expressed through any two of them.

**Expression through  $\cap$  and  $\Delta$ .** The operations  $\cup$  and  $\setminus$  are expressed through the operations  $\cap$  and  $\Delta$  as follows:

$$A \cup B = (A \Delta B) \Delta (A \cap B), \quad A \setminus B = A \Delta (A \cap B).$$

**Expression through  $\cup$  and  $\Delta$ .** The operations  $\cap$  and  $\setminus$  are expressed through the operations  $\cup$  and  $\Delta$ . Since  $\setminus$  is already expressed through  $\Delta$  and  $\cap$ , it suffices to express  $\cap$  through  $\cup$  and  $\Delta$ . Such an expression is given by the following formula:

$$A \cap B = (A \cup B) \Delta (A \Delta B).$$

see also Section 5.1

## 2.1.1 :: :: problems for workshop: formal statements

**Exercise 7.** Some of these expressions are grammatically or logically incorrect. Identify them and explain what is the fault. (In what follows,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a real function and  $A, B, C \subset \mathbb{R}$ .)

$\{1 + 1\}$	$\{3\} \setminus \{\{3\}\}$	$1 + 1 \Rightarrow 2$
$\{1, 2\} \Leftrightarrow \{2, 1\}$	$\sqrt{2} \notin \mathbb{Q}$	$\mathbb{Z} \setminus (\mathbb{Z} \setminus \mathbb{N})$
$\mathbb{Z} \Rightarrow \mathbb{Q}$	$( x  \in \mathbb{Z}) \Rightarrow ( x  \in \mathbb{Q})$	$(x \in A) \cup (x \in B)$
$(3 < 1) \Rightarrow \emptyset$	$A \leq (A \setminus B)$	$f(A) \in \{f(A)\}$
$(A \subset B) \cap C$	$A \subset (B \cap C)$	$A \subset B \subset A$
$(2, 4, 6, \dots) \subset (1, 2, 3, \dots)$	$\{A, \mathbb{Z}\}$	$\{\emptyset\} \cap \emptyset$
$f(1) \in \{2, 3\}$	$f(\{1, 2\}) \in \mathbb{N}$	$f(\mathbb{Q}) \subset \mathbb{Q}$
$\{x \in \mathbb{N} : -x\}$	$\{-x : x \in \mathbb{N}\}$	$\{x : x \Leftrightarrow 2\}$
$\{x \in \mathbb{Z} : x \notin \mathbb{Z}\}$	$\{\{x :  x  < 2\}\}$	$\{x \in \mathbb{Q} : 1 = 0\}$
$\{x \in \mathbb{Q} : x^2 \notin \mathbb{Z}\}$	$\{\{f(x)\} : x \in \mathbb{Q}\}$	$\{x : f(x) \in \mathbb{Q}\}$

**Exercise 8.** The following expressions define sets. Turn words into symbols.

1. The set of negative odd integers:
2. The set of natural numbers with three decimal digits:
3. The set of rational numbers which are the ratio of odd integers:
4. The set of rational numbers between 3 and  $\pi$ :
5. The set of real numbers at distance  $\frac{1}{4}$  from an integer:
6. The complement of the unit circle in the Cartesian plane:
7. The set of lines tangent to the unit circle:

**Exercise 9.** Draw the Euler-Venn diagrams for  $(A \cap B) \setminus C$  and  $(A \setminus C) \cap (A \setminus B)$ .

**Exercise 10.** Draw the Euler-Venn diagram for  $((A \cap B) \cap (C \cup D)) \cup (A \setminus C)$ .

Answers for tutorial:

$$\{ n \in \mathbb{Z} : n < 0 \text{ and } n \text{ is odd} \} = \{ n \in \mathbb{Z} : n < 0, \ n = 2k + 1, \ k \in \mathbb{Z} \}.$$

$$\{ n \in \mathbb{N} : 10^2 \leq n < 10^3 \}.$$

$$\left\{ \frac{p}{q} \in \mathbb{Q} : p, q \in \mathbb{Z}, \ p, q \text{ odd}, \ q \neq 0 \right\}.$$

$$\{ q \in \mathbb{Q} : 3 < q < \pi \}.$$

$$\{ x \in \mathbb{R} : |x - n| = \frac{1}{4} \text{ for some } n \in \mathbb{Z} \}.$$

$$\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1 \}.$$

$$\{ ax + by = 1 : a^2 + b^2 = 1 \}.$$

## :: :: 4th homework

1. A straight (finite) path in a park is completely illuminated by several lamps, each of which lights up a certain segment. Prove that it is possible to turn off some of the lamps so that the path remains completely illuminated, but no part of it is illuminated by three lamps at once.
2. Is it possible to cover an equilateral triangle with two smaller equilateral triangles?
3. Prove that the sum of the valencies (aka (also known as) the degrees) of the vertices of a graph is twice the number of the edges. (This is called **the handshaking lemma**)
4. Prove that every positive integer can be represented as a sum of different powers of two.
5. Draw the Euler-Wenn diagramm for  $((A \setminus B) \cap (C \cup (D \setminus A))) \cup (D \cap B \cap A)$ .
6. Let  $I$  be an index set, and  $\{A_i\}_{i \in I}$  be a family of sets. Prove de Morgan laws without words, i.e. show that every element from the left set belongs to the right set and vice versa. It is prohibited to write words:

$$\left( \bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c$$

$$\left( \bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c$$

where  $A^c$  is the complement to  $A$  in the “universal set”.

## 7. The Meticulous Job Applicant

A company requires that ideal candidates must satisfy **all** of the following conditions: speak English, have programming experience, possess a university degree, have at least 2 years of work experience. Describe in words the set of candidates who are **NOT** ideal for the position.

## Solution

Let  $E$ : candidates who speak English,  $P$ : candidates with programming experience,  $D$ : candidates with a university degree,  $W$ : candidates with  $\geq 2$  years of work experience

The set of ideal candidates is:

$$I = E \cap P \cap D \cap W$$

Candidates who are **NOT** ideal form the complement:

$$I^c = (E \cap P \cap D \cap W)^c$$

Applying De Morgan's Law:

$$I^c = E^c \cup P^c \cup D^c \cup W^c$$

**Verbal Interpretation:** A candidate is **not ideal** if they lack **at least one** of the required qualifications.

## 2.2 :: functions, permutations, $S_n$

A function  $f : A \rightarrow B$  (pronounced: “a function  $f$  from  $A$  to  $B$ ”, “a map  $f$  from  $A$  to  $B$ ”, “a mapping from  $A$  to  $B$ ”) is a rule that assigns to each element  $a \in A$  an element  $f(a) \in B$ .

A function  $f$  is called an *injection* (we also say:  $f$  is *injective*, or  $f$  is *one-to-one*) if it sends different elements to different elements, i.e.

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

A function  $f$  is called a *surjection* (we also say:  $f$  is *surjective*, or  $f$  is *onto*) if for each  $b \in B$  there exists an element  $a \in A$  (not necessarily unique) such that

$$f(a) = b.$$

A function  $f$  is called a *bijection* (we also say:  $f$  is *bijective*, or  $f$  is a *one-to-one correspondence*) if  $f$  is both injective and surjective.

### Examples.

- (*Inclusion map.*) If  $A \subset B$ , then there exists a map  $\iota : A \rightarrow B$ , called the *inclusion*, defined by

$$\iota(a) = a \quad \text{for all } a \in A.$$

- (*Identity map.*) The *identity map*  $\text{id}_A : A \rightarrow A$  is defined by

$$\text{id}_A(a) = a \quad \text{for all } a \in A.$$

- (*Restriction map.*) Let  $f : A \rightarrow B$  and let  $X \subset A$ . Then the *restriction* of  $f$  to  $X$  is the map

$$f|_X : X \rightarrow B, \quad f|_X(x) = f(x) \quad \text{for all } x \in X.$$

This notation is useful to distinguish  $f$  and  $f|_X$ , because they are defined on different sets:  $f$  is defined on  $A$ , while  $f|_X$  is defined on  $X$ .

The *composition*  $g \circ f$  of two functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$  is the function

$$g \circ f : A \rightarrow C, \quad (g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A.$$

Let  $f : A \rightarrow B$  and  $b \in B$ . The *preimage* (or *inverse image*) of  $b$  is the set

$$f^{-1}(\{b\}) = \{a \in A \mid f(a) = b\}.$$

Often we write simply  $f^{-1}(b)$  for this set. This set may be empty.

More generally, for  $Y \subset B$  we define the preimage of  $Y$  as

$$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\} = \bigcup_{b \in Y} f^{-1}(\{b\}).$$

The set  $f(A) = \{f(a) \mid a \in A\} \subset B$  is called the *image* of  $A$  under  $f$ .

**Problem 1.** Let  $f : A \rightarrow B$  be a function and let  $Y \subset B$ . Show that

$$f^{-1}(Y) = f^{-1}(Y \cap f(A)).$$

Conclude that the equality

$$Y = f(f^{-1}(Y))$$

holds if and only if  $Y \subset f(A)$ .

Solution. Recall that for  $Z \subset B$ ,

$$f^{-1}(Z) = \{a \in A \mid f(a) \in Z\}.$$

First we show the equality of sets.

$\subset$ : Let  $a \in f^{-1}(Y)$ . Then  $f(a) \in Y$ . But of course  $f(a) \in f(A)$  for every  $a \in A$ . Hence  $f(a) \in Y \cap f(A)$ . Thus  $a \in f^{-1}(Y \cap f(A))$ . This proves

$$f^{-1}(Y) \subset f^{-1}(Y \cap f(A)).$$

$\supseteq$ : Let  $a \in f^{-1}(Y \cap f(A))$ . Then  $f(a) \in Y \cap f(A)$ , so in particular  $f(a) \in Y$ . Hence  $a \in f^{-1}(Y)$ . Thus

$$f^{-1}(Y \cap f(A)) \subset f^{-1}(Y).$$

Combining the two inclusions we get

$$f^{-1}(Y) = f^{-1}(Y \cap f(A)).$$

For the second statement, recall that for any function  $f$  and set  $Y \subset A$ , we always have

$$f(f^{-1}(Y)) \subset Y,$$

because  $f^{-1}(Y)$  consists exactly of those elements mapped into  $Y$ .

Now suppose  $Y \subset f(A)$ . Then every  $y \in Y$  is equal to  $f(a)$  for some  $a \in A$ . Since  $y \in Y$ , we have  $a \in f^{-1}(Y)$ , and thus  $y = f(a) \in f(f^{-1}(Y))$ . Hence

$$Y \subset f(f^{-1}(Y)).$$

Together with the always-true inclusion  $f(f^{-1}(Y)) \subset Y$ , this gives

$$Y = f(f^{-1}(Y)).$$

Conversely, if  $Y = f(f^{-1}(Y))$ , then every  $y \in Y$  is of the form  $y = f(a)$  for some  $a \in f^{-1}(Y) \subset A$ . Hence  $y \in f(A)$ , so  $Y \subset f(A)$ .

Thus

$$Y = f(f^{-1}(Y)) \iff Y \subset f(A).$$

## Problems on images, preimages, and inverses of maps

Let  $f: X \rightarrow Y$  be a map. For a subset  $A \subset X$  write

$$f(A) := \{f(x) : x \in A\} \subset Y,$$

and for  $C \subset Y$  write the full preimage

$$f^{-1}(C) := \{x \in X : f(x) \in C\} \subset X.$$

**Problem 2** (When does  $f(A \cap B) = f(A) \cap f(B)$  hold for all  $A, B \subset X$ ?). For arbitrary subsets  $A, B \subset X$  we always have

$$f(A \cap B) \subset f(A) \cap f(B).$$

1. Prove the inclusion above.
2. Find a necessary and sufficient condition on  $f$  such that the reverse inclusion also holds for every pair of subsets  $A, B \subset X$ , i.e.

$$f(A \cap B) = f(A) \cap f(B) \quad \text{for all } A, B \subset X.$$

**Hint 1.** To prove necessity, assume  $f$  is not injective, so  $x \neq x'$  but  $f(x) = f(x')$ . Try  $A = \{x\}$ ,  $B = \{x'\}$ . To prove sufficiency, start with  $y \in f(A) \cap f(B)$  and use injectivity to show  $y \in f(A \cap B)$ .

**Problem 3** (Intersection with a full preimage). Let  $A \subset X$  and  $C \subset Y$ . Prove that

$$f(A \cap f^{-1}(C)) = f(A) \cap C.$$

**Hint 2.** Prove two inclusions.

- If  $y \in f(A \cap f^{-1}(C))$ , then  $y = f(x)$  for some  $x \in A$  with  $f(x) \in C$ . Conclude  $y \in f(A) \cap C$ .
- If  $y \in f(A) \cap C$ , choose  $x \in A$  with  $f(x) = y$ . Since  $y \in C$ , we get  $x \in f^{-1}(C)$ . Hence  $x \in A \cap f^{-1}(C)$ , so  $y \in f(A \cap f^{-1}(C))$ .

**Problem 4** (Bijections and inverse maps). A map  $f: X \rightarrow Y$  is called *bijective* if it is both injective and surjective.

1. Suppose  $f$  has an inverse map  $g: Y \rightarrow X$ , i.e.

$$g \circ f = \text{id}_X, \quad f \circ g = \text{id}_Y.$$

Prove that  $f$  is bijective.

2. Conversely, suppose  $f$  is bijective. Define a map  $g: Y \rightarrow X$  by: for each  $y \in Y$ , let  $g(y)$  be the unique  $x \in X$  with  $f(x) = y$ . Prove that  $g$  is well-defined and that it is an inverse to  $f$ .

**Hint 3.** For (1): use  $g \circ f = \text{id}_X$  to prove injectivity, and  $f \circ g = \text{id}_Y$  to prove surjectivity. For (2): surjectivity gives existence of  $x$ , injectivity gives uniqueness.

**Remark 1.** The equivalence “ $f$  is bijective  $\iff f$  has an inverse” is used frequently in proofs. If it is not stated explicitly early, later arguments can become unnecessarily complicated.

**Problem 5.** Prove that if  $f^{\circ n} = \text{id}_X$  for some positive integer  $n$ , then  $f$  is a bijection.

Solution. Assume  $f^{\circ n} = \text{id}_X$ , i.e.

$$\underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}} = \text{id}_X.$$

**Injectivity.** Suppose  $f(x_1) = f(x_2)$ . Apply  $f^{\circ(n-1)}$  to both sides:

$$f^{\circ(n-1)}(f(x_1)) = f^{\circ(n-1)}(f(x_2)).$$

The left-hand side is  $f^{\circ n}(x_1) = \text{id}_X(x_1) = x_1$ , and similarly the right-hand side is  $x_2$ . Hence  $x_1 = x_2$ . So  $f$  is injective.

**Surjectivity.** Let  $y \in X$  be arbitrary. We must find  $x \in X$  such that  $f(x) = y$ . Consider

$$x := f^{\circ(n-1)}(y).$$

Then

$$f(x) = f(f^{\circ(n-1)}(y)) = f^{\circ n}(y) = \text{id}_X(y) = y.$$

Thus every  $y \in X$  has a preimage, so  $f$  is surjective.

Since  $f$  is both injective and surjective, it is bijective.

## Permutations

Now let  $A$  be a finite set with  $n$  elements, i.e.  $|A| = n$ . (We denote the number of elements in a finite set  $A$  by  $|A|$ .)

All bijections  $f : A \rightarrow A$  are called *permutations* of  $A$ . One may think of such a map as permuting the ordered list  $1, 2, 3, \dots, n$ .

## Examples of permutations

Let us work with the set

$$A = \{1, 2, 3, 4\}.$$

## Some permutations in two-line notation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, \quad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}.$$

This means, for example, that  $\sigma(1) = 2$ ,  $\sigma(2) = 4$ ,  $\sigma(3) = 1$ ,  $\sigma(4) = 3$ .

## Composition of permutations

Given two permutations  $\sigma, \tau : A \rightarrow A$ , their composition  $\tau \circ \sigma$  is the permutation defined by

$$(\tau \circ \sigma)(i) = \tau(\sigma(i)) \quad \text{for all } i \in A.$$

For our examples above, let us compute  $\tau \circ \sigma$ .

First compute  $\sigma(i)$ , then apply  $\tau$ :

$$(\tau \circ \sigma)(1) = \tau(\sigma(1)) = \tau(2) = 2,$$

$$(\tau \circ \sigma)(2) = \tau(\sigma(2)) = \tau(4) = 1,$$

$$(\tau \circ \sigma)(3) = \tau(\sigma(3)) = \tau(1) = 3,$$

$$(\tau \circ \sigma)(4) = \tau(\sigma(4)) = \tau(3) = 4.$$

So

$$\tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}.$$

In cycle notation this is

$$\tau \circ \sigma = (1\ 2),$$

because it swaps 1 and 2 and fixes 3, 4.

## Decomposition into cycles: example 1

Take

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

To write  $\sigma$  in cycle notation, start from 1:

$$1 \mapsto 2, \quad 2 \mapsto 4, \quad 4 \mapsto 3, \quad 3 \mapsto 1.$$

So we return to 1, and we get one 4-cycle:

$$\sigma = (1\ 2\ 4\ 3).$$

A 4-cycle can be written as a product of **transpositions**:

$$(1\ 2\ 4\ 3) = (1\ 3)(1\ 4)(1\ 2).$$

**Exercise.** Any 4-cycle  $(a\ b\ c\ d)$  can be written as  $(a\ d)(a\ c)(a\ b)$ .

## Decomposition into cycles: example 2

Take

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

Start from 1:

$$1 \mapsto 3, \quad 3 \mapsto 4, \quad 4 \mapsto 1,$$

so we get the cycle  $(1\ 3\ 4)$ . What about 2? We have  $\tau(2) = 2$ , so 2 is a fixed point. Thus

$$\tau = (1\ 3\ 4),$$

with 2 omitted in cycle notation. As a product of transpositions,

$$(1\ 3\ 4) = (1\ 4)(1\ 3).$$

## Product of two permutations and its cycle form

We have already computed

$$\tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} = (1\ 2).$$

This is already a transposition (a 2-cycle), so its decomposition into transpositions is just itself:

$$\tau \circ \sigma = (1\ 2).$$

If we reverse the order and compute  $\sigma \circ \tau$ , we get in general a different permutation:

$$(\sigma \circ \tau)(i) = \sigma(\tau(i)).$$

Let us compute it explicitly:

$$\begin{aligned} (\sigma \circ \tau)(1) &= \sigma(\tau(1)) = \sigma(3) = 1, \\ (\sigma \circ \tau)(2) &= \sigma(\tau(2)) = \sigma(2) = 4, \\ (\sigma \circ \tau)(3) &= \sigma(\tau(3)) = \sigma(4) = 3, \\ (\sigma \circ \tau)(4) &= \sigma(\tau(4)) = \sigma(1) = 2. \end{aligned}$$

So

$$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}.$$

In cycle notation this is

$$\sigma \circ \tau = (2\ 4),$$

again a single transposition.

Thus, in these examples we saw:

- how to write a permutation in cycle notation;
- how to express a cycle as a product of transpositions;
- how to compose two permutations and then write the result in cycle form.

The set of all bijections of  $\{1, 2, \dots, n\}$  is denoted by  $S_n$ . As seen above, we can compose its elements, and one can check that  $S_n$  possesses the algebraic structure of a *group*, it is called the *symmetric group of degree n*. The symmetric group  $S_n$  is one of the most important groups in mathematics.

Theorem. Every permutation can be written as a product of cycles. Proof: follow the cycles.

Remark.  $(a_1 a_2 \dots a_k)(b_1 b_2 \dots b_n) = (b_1 b_2 \dots b_n)(a_1 a_2 \dots a_k)$  if  $a_i \neq b_j$  for all  $i, j$ .

Theorem. Every permutation can be written as a product of transpositions.

Proof. Use the previous theorem and then  $(1 a_1 a_2 \dots a_k) = (1 a_k)(1 a_{k-1}) \dots (1 a_1)$ .

Remark:

$$(12)(23) = (123) \neq (132) = (23)(12), (12)(23)(12) = (23)(12)(23) = (13)$$

So if we denote  $(ij) = \sigma_{ij}$  then  $\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}$  but  $\sigma_{12}\sigma_{23} \neq \sigma_{23}\sigma_{12}$ . So the group  $S_n$  is not commutative but has another curious relations between its elements!

## :: :: problems for tutorial

1. Prove the following, using the Euler-Venn diagrams, and then only writing symbols, without any words and pictures.
  - a)  $(A \Delta B) \setminus C = (A \cup C) \Delta (B \cup C)$ ;
  - b)  $A \setminus (B \Delta C) = (A \setminus (B \cup C)) \cup (A \cap B \cap C)$ .
2. Let  $f : X \rightarrow Y$ . Prove that for any subset  $A \subset X$  and any subset  $B \subset Y$  the inclusions hold:
  - a)  $A \subset f^{-1}(f(A))$ , (the requirement that this turn into equality singles out the class of injective mappings)
  - b)  $B \supseteq f(f^{-1}(B))$ . (the requirement that this turn into equality singles the class of surjective mappings.)
  - c) Give examples showing that these inclusions can be strict.
3. Prove that if for a function  $f : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$  one has  $f^{-1} = 1/f$ , then  $(f \circ f)(x) = 1/x$  for every  $x \in \mathbb{R}_{>0}$ . In particular,  $f \circ f \circ f \circ f = \text{id}$ .
4. (if time permits, if not, it goes to the homework) Compute the  $n$ -th iterate of the mapping  $x \mapsto \frac{x}{\sqrt{1+x^2}}$ .

## 2.2.1 :: :: problems for workshop: cycle decomposition

**Exercise 11.** Let  $A = \{1, 2, 3, 4\}$ ,  $B = \{1, 2, 3, 4, 5\}$ . Let

$$\begin{array}{ll} f : A \rightarrow B \text{ is given by} & \begin{array}{l} 1 \rightarrow 3 \\ 2 \rightarrow 4 \\ 3 \rightarrow 2 \\ 4 \rightarrow 5 \end{array} \quad g : B \rightarrow A \text{ is given by} & \begin{array}{l} 1 \rightarrow 2 \\ 2 \rightarrow 1 \\ 3 \rightarrow 2 \\ 4 \rightarrow 3 \\ 5 \rightarrow 4 \end{array} \end{array}$$

Find the compositions  $f \circ g : B \rightarrow B$ ,  $g \circ f : A \rightarrow A$ . Are they injections, surjections, or bijections? Find  $f^{-1}$ ,  $g^{-1}$ ,  $g^{-1} \circ f^{-1}$ ,  $(f \circ g)^{-1}$ .

It turns out that the complete preimage behaves very nicely with respect to set-theoretic Boolean operations.

**Problem.** Let  $f : X \rightarrow Y$ . Prove that for any two subsets  $A, B \subset Y$  the following equalities hold (use only symbols, no words or pictures):

1.  $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ ,
2.  $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ ,
3.  $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$ .

The behavior of the image is somewhat more complicated: some of the equalities listed above remain valid, but others turn into inclusions, which, generally speaking, can be strict.

**Problem.** Let  $f : X \rightarrow Y$ . Prove that for any two subsets  $A, B \subset X$  the following relations hold (use only symbols, no words or pictures):

1.  $f(A \cup B) = f(A) \cup f(B)$ ,
2.  $f(A \cap B) \subset f(A) \cap f(B)$ ,
3.  $f(A \setminus B) \supseteq f(A) \setminus f(B)$ .

Give examples showing that the inclusions in points 2) and 3) can be strict, draw the corresponding pictures alike Euler-Venn diagramms.

## :: :: 5th homework

### Read:

**Problem.** Thirty boots stand in a row: 15 left and 15 right. Prove that among some ten consecutive boots there are equally many left and right ones.

**Solution.** Let  $L_i$  denote the number of left boots among the ten boots occupying positions  $i, i+1, \dots, i+9$ . Then

$$L_1 + L_{11} + L_{21} = 15,$$

because these three disjoint blocks of ten boots together contain all 15 left boots.

If one of the numbers  $L_1, L_{11}, L_{21}$  equals 5, then we have already found the required block of ten boots with 5 left and 5 right.

Otherwise, one of these numbers is  $> 5$  and another is  $< 5$ . When we shift the window  $i, i+1, \dots, i+9$  by one position, the number of left boots in it can change by at most 1, i.e.

$$|L_{i+1} - L_i| \leq 1.$$

Therefore, when we go from the block whose value is  $> 5$  to the block whose value is  $< 5$ , we must pass through a block with value exactly 5. Thus, for some  $k$  we have  $L_k = 5$ , and the corresponding ten consecutive boots contain an equal number of left and right boots.

### Write:

1. Find  $|A|$  (the number of elements in the set  $A$ ), where  $A$  is the set of natural numbers from a) 1 to 30, b) 1 to 100 which are divisible by at least one number from the set  $\{3, 5, 7, 9\}$ .
2. Let  $A, B, C, D$  be four finite sets such that  $A \cap B = C \cap D = \emptyset$ . How to compute  $|A \cup B \cup C \cup D|$  using numbers  $|A|, |B|, |C|, |D|, |A \cap C|, |A \cap D|, |B \cap C|, |B \cap D|$ ? Hint: draw the Euler-Venn diagramm and then for each region write a variable expressing the number of elements there. Thus you may express  $|X|$  for all sets  $X$  mentioned above.
3. Find the presentation of  $\tau$  as a) a product of cycles b) as a product of transpositions

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}.$$

## 2.3 :: Cartesian products

People frequently write  $X := Y$  ("is defined to be") when the object on the left is defined by the expression on the right.

**Binary product.** The Cartesian product of sets  $A$  and  $B$  is the set of ordered pairs where the first element is from  $A$  and the second is from  $B$ , i.e.

$$A \times B := \{(a, b) \mid a \in A, b \in B\}.$$

*Example.* If  $A = \{1, 2\}$  and  $B = \{a, b, c\}$ , then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

If  $A, B$  are finite sets, then  $|A \times B| = |A| \cdot |B|$ . We abbreviate  $A^2 = A \times A$ ; thus  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$  is the plane, and in general the space  $\mathbb{R}^n$  of dimension  $n$  consists of points, each of them having  $n$  coordinates

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R} \text{ } (1 \leq i \leq n)\}.$$

**Graphs and functions.** For a function  $f: A \rightarrow B$  its *graph* is

$$\Gamma_f := \{(a, f(a)) \mid a \in A\} \subset A \times B = \bigcup_{a \in A} \{(a, f(a))\}.$$

One says " $a$  runs over  $A$ ". Not every subset of  $A \times B$  is a graph: for each  $a \in A$  there must be *exactly one* pair in the subset whose first coordinate is  $a$ .

**Definition 1.** A function  $f: A \rightarrow B$  is a subset  $\Gamma \subset A \times B$  such that for every  $a \in A$  there exists a unique  $b \in B$  with  $(a, b) \in \Gamma$ . We then write  $b = f(a)$  and  $\Gamma = \Gamma_f$ .

*Examples.* The identity  $\text{id}_A: A \rightarrow A$  has graph  $\Gamma_{\text{id}_A} = \{(a, a) \mid a \in A\} \subset A \times A$  (the diagonal).

**Restriction.** If  $X \subset A$  and  $f: A \rightarrow B$ , the restriction  $f|_X: X \rightarrow B$  has graph

$$\Gamma_{f|_X} := \Gamma_f \cap (X \times B).$$

**Projections.** The product  $A \times B$  comes with projections

$$\pi_A: A \times B \rightarrow A, \quad \pi_A(a, b) = a, \quad \pi_B: A \times B \rightarrow B, \quad \pi_B(a, b) = b.$$

*Exercise.* Express  $\pi_A, \pi_B$  using the graph definition of a function.

**Finite sequences. Products of families.** A finite sequence  $(a_1, \dots, a_{100})$  of integers is an element of  $\mathbb{Z}^{100}$ .

Is it possible to define a direct product of an infinite family of sets? For example, we want to define an infinite sequence, what is the set of all infinite sequences  $(a_1, a_2, \dots)$  on integer numbers?

We will give another definition of a direct product of two sets:

$$A_1 \times A_2 :=$$

the set of all functions  $f : \{1, 2\} \rightarrow A_1 \cup A_2$  such that  $f(1) \in A_1, f(2) \in A_2$

Then we can do the same for any family of sets! Let  $\{A_i\}_{i \in I}$  be any family of sets parametrised by  $I$ . Define

The direct (Cartesian) product of this family is

$$\prod_{i \in I} A_i := \left\{ f : I \rightarrow \bigcup_{i \in I} A_i \mid f(i) \in A_i \text{ for all } i \in I \right\},$$

i.e. the set of *choice functions* picking one element from each  $A_i$ . (For two sets this agrees with  $A_1 \times A_2$  via  $f \mapsto (f(1), f(2))$ .)

Now, a sequence of real numbers is a function  $\mathbb{N} \rightarrow \mathbb{R}$ . And the set of all such sequences is  $\prod_{n \in \mathbb{N}} A_n$  where all  $A_n = \mathbb{R}$ .

**Restricted product over subsets of  $\mathbb{Z}$  with basepoint 0.** Let  $\{S_i\}_{i \in I}$  be a family of subsets  $S_i \subset \mathbb{Z}_{\geq 0}$  such that  $0 \in S_i$  for all  $i$ . Define the (restricted) direct product with zero–basepoint by

$$\prod_{i \in I}^* S_i := \left\{ (s_i)_{i \in I} \in \prod_{i \in I} S_i \mid s_i = 0 \text{ for all but finitely many } i \right\}.$$

**Application: Fundamental Theorem of Arithmetic.** Let  $I = \mathcal{P}$  be the set of prime numbers and take  $S_p = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\} \subset \mathbb{Z}$  for each  $p \in \mathcal{P}$ . For  $n \in \mathbb{Z}_{\geq 1}$  write  $v_p(n)$  for the exponent of  $p$  in the prime factorization of  $n$ . So,

$$n = 2^{v_2(n)} 3^{v_3(n)} 5^{v_5(n)} \dots$$

Then

$$\Phi : \mathbb{Z}_{\geq 1} \longrightarrow \prod_{p \in \mathcal{P}}^* \mathbb{Z}_{\geq 0}, \quad \Phi(n) = (v_p(n))_{p \in \mathcal{P}},$$

is a bijection with inverse

$$(e_p)_{p \in \mathcal{P}} \longmapsto \prod_{p \in \mathcal{P}} p^{e_p}.$$

Moreover, it behaves as the logarithm:

$$\Phi(nm) = \Phi(n) + \Phi(m) \quad (\text{componentwise addition}).$$

Thus the Fundamental Theorem of Arithmetic can be expressed as the identification

$$(\mathbb{Z}_{\geq 1}, \cdot) \cong \left( \prod_{p \in \mathcal{P}}^* \mathbb{Z}_{\geq 0}, + \right),$$

i.e. positive integers under multiplication correspond to finitely supported (i.e. only a finite number of non-zeros) exponent families  $(e_p)$  with values in  $\mathbb{Z}_{\geq 0}$  and basepoint 0.

### 2.3.1 :: :: problems for tutorial: projections and Lebesgue's mistake

1. For a function  $f : A \rightarrow B$  and  $Y \subset B$  guess what is  $\pi_A(\Gamma_f \cap (A \times Y))$
2. Guess what is  $\pi_B(\Gamma_f \cap (X \times B))$  for  $X \subset A$ .
3. For functions  $f : A \rightarrow B, g : B \rightarrow C$  express  $\Gamma_{g \circ f}$  using set operations and  $\Gamma_f, \Gamma_g$ .

**Projections and countable intersections** Let  $\pi_A : A \times B \rightarrow A$  be the projection  $\pi_A(a, b) = a$ .

1. Prove that for any family  $\{E_i\}_{i \in I} \subset A \times B$ ,

$$\pi_A \left( \bigcup_{i \in I} E_i \right) = \bigcup_{i \in I} \pi_A(E_i).$$

2. Prove that for any family  $\{E_i\}_{i \in I} \subset A \times B$ ,

$$\pi_A \left( \bigcap_{i \in I} E_i \right) \subset \bigcap_{i \in I} \pi_A(E_i),$$

and show that equality need not hold.

3. Give an explicit counterexample in  $\mathbb{R}^2$ : define a decreasing sequence of sets

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots \subset \mathbb{R}^2$$

such that  $\bigcap_{n \geq 1} E_n = \emptyset$  but

$$\bigcap_{n \geq 1} \pi_x(E_n) = \mathbb{R}.$$

4. (*Comment*) Parts (2)–(3) show that projection does not commute with countable intersections. A famous Lebesgue's paper said that they commute without proof; apparently, he considered it obvious. A counterexample to this "obvious" statement was discovered by Mikhail Suslin in 1916, which led Suslin and Luzin to the creation of the theory of analytic sets.

### 2.3.2 :: :: problems for workshop: DNF/CNF

#### Implication, DNF/CNF, and the set-theoretic analogue

(a) **Two variables.** Show that the truth table for  $A \Rightarrow B$  coincides with

$$(\neg A \wedge \neg B) \vee (A \wedge B) \vee (\neg A \wedge B) \quad \text{and also with} \quad \neg A \vee B.$$

(b) **Three variables (same idea, two representations).** Consider the three-argument Boolean function

$$F(A, B, C) := (A \Rightarrow B) \wedge (B \Rightarrow C).$$

1. Compute its truth table.
2. Show that  $F$  has the *conjunctive normal form (CNF)*

$$F \equiv (\neg A \vee B) \wedge (\neg B \vee C).$$

3. Show that  $F$  has the *disjunctive normal form (DNF)*

$$F \equiv (\neg A \wedge \neg B) \vee (\neg A \wedge C) \vee (B \wedge C).$$

4. Conclude that the two formulas in (b.2) and (b.3) are logically equivalent by comparing with your truth table from (b.1).

(c) **Sets (same two representations).** Let  $X, Y, Z$  be subsets of a fixed universe  $U$ , and write complements as  $X^c = U \setminus X$ . Translate the Boolean equivalence from part (b) into a set identity:

$$(X^c \cup Y) \cap (Y^c \cup Z) = (X^c \cap Y^c) \cup (X^c \cap Z) \cup (Y \cap Z).$$

Prove this equality either by element-chasing (membership tables) or by distributive laws for  $\cup, \cap$ .

# :: :: 6th homework

**Read:** Miklós Laczkovich - Conjecture and Proof, pp. 49-52.

## The Pigeonhole Principle

Suppose that several objects are distributed among some boxes (for example, letters among pigeonholes) such that the number of the objects is greater than the number of the boxes. Then there must be a box that contains at least two objects. This simple observation is called the *pigeonhole principle*, and is used frequently in proofs of existence. In this section we shall give three examples. The first one is combinatorial, and deals with *Sidon sequences*.

A sequence  $a_1 < \dots < a_k$  of positive integers is called a *Sidon sequence* if the numbers  $a_i + a_j$  ( $1 \leq i \leq j \leq k$ ) are all different. Now the problem is the following: what is the length of the longest possible Sidon sequence satisfying  $a_k \leq n$ ?

Consider, for example,  $n = 100$ . If we want to find a long Sidon sequence, we may try the "greedy algorithm" that chooses at each step the first number that does not violate the Sidon property. In this way we obtain the numbers 1, 2, 4, 8, 13, 21, 31, 45, 66, 81, 97. But this is not the longest Sidon sequence up to 100; for example 1, 3, 7, 25, 30, 41, 44, 56, 69, 76, 77, 86 is a longer one. (This shows that the greedy algorithm need not give the best result.)

Let  $s(n)$  denote the length of the longest Sidon sequence satisfying  $a_k \leq n$ . Then we have  $s(100) \geq 12$  and it can be shown by a computer search that, in fact,  $s(100) = 12$ . The value of  $s(n)$  is not known in general. An upper bound can be obtained by using the pigeonhole principle as follows.

Let  $a_1 < \dots < a_k$  be a Sidon sequence with  $a_k \leq n$ . There are  $\binom{k}{2}$  pairs of indices  $(i, j)$  satisfying  $1 \leq i < j \leq k$ . Since  $2 \leq a_i + a_j \leq 2n$  for every such pair, it follows that  $\binom{k}{2} + k \leq 2n - 1$ . (Here the objects are the numbers  $a_i + a_j$  and the boxes are the possible values  $2, \dots, 2n$ . If

## next pages:

Indeed, by a basic theorem of number theory (the so-called Wilson theorem),  $p$  divides  $(p-1)!$ . If  $p = 4k+1$ , then it is easy to check that  $(2k)!^2$  and  $(2k+1)(2k+2) \cdots (4k)$  give the same remainder when divided by  $4k+1$ . Therefore  $p$  divides  $(2k)!^2 + 1$  and then, by the previous theorem,  $p$  is the sum of two squares.

For example, 1997 is a prime of the form  $4k+1$  and, accordingly, it is the sum of two squares:  $1997 = 34^2 + 29^2$ .

The next application of the pigeonhole principle deals with approximation by rationals. Every number can be approximated by rational numbers with an arbitrary small error. For example, if we want to approximate the number  $\alpha$  with a error less than, say,  $10^{-10}$  then we can proceed as follows: we find an integer  $p$  such that  $p/10^8 \leq \alpha < (p+1)/10^8$ ; then the error  $|\alpha - (p/10^8)|$  will be smaller than  $10^{-6}$ . Similarly, for every  $q$  there is a  $p$  such that  $|\alpha - (p/q)| < 1/q$ .

In general the error  $|\alpha - (p/q)|$  is not much smaller than  $1/q$ . For example, if  $\alpha = 1/3$  and if  $q$  is not divisible by 3, then for every  $p$  we have  $|\alpha - (p/q)| = |q - 3p|/(3q) \geq 1/(3q)$ . Or, if the decimal expansion of  $\alpha$  does not contain the digits 0 and 9, then  $|\alpha - (p/q)| \geq 1/(10q)$  whenever  $q$  is a power of 10 (why?). However, using other denominators, we can find much more effective approximations.

Consider for example  $\alpha = \sqrt{2}$ . In the fifth proof of the irrationality of  $\sqrt{2}$  we remarked that  $(\sqrt{2}-1)^n = a_n + b_n\sqrt{2}$ , where  $a_n, b_n$  are integers. Since  $(\sqrt{2}-1)^n \rightarrow 0$ ,  $|b_n| \rightarrow \infty$  (why?). It follows from the binomial theorem that  $(-\sqrt{2}-1)^n = c_n - d_n\sqrt{2}$ , and thus

$$(a_n + b_n\sqrt{2})(a_n - b_n\sqrt{2}) = (-1)^n.$$

Let  $p_n = |a_n|$ ,  $q_n = |b_n|$ . Then

$$|p_n + q_n\sqrt{2}| \cdot |p_n - q_n\sqrt{2}| = 1,$$

from which  $q_n|p_n - q_n\sqrt{2}| < 1$ , and

$$\left| \sqrt{2} - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

That is,  $|\sqrt{2} - (p/q)| < 1/q^2$  holds for infinitely many rationals  $p/q$ . Next we show that every irrational number has this property.

For every irrational  $\alpha$  there are infinitely many rationals  $\frac{p}{q}$  with

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

$\binom{n}{2} + k > 2n - 1$ , that is, the number of objects is greater than the number of boxes, then there would be two objects in the same box, contradicting the Sidon property. This gives  $k(k-1) + 2k \leq 4n - 2$ ,  $k^2 < 4n$ ,  $k < 2\sqrt{n}$  and thus  $s(n) < 2\sqrt{n}$ .

A better estimate is obtained if we also observe that the numbers  $a_j - a_i$  ( $i < j$ ) are different. Since  $1 \leq a_j - a_i \leq n-1$  for every  $i < j$ , the pigeonhole principle gives  $\binom{n}{2} \leq n-1$  and  $(k-1)^2 < 2(n-1)$ ;  $\binom{n}{2} \leq 2(n-1) < 2n$ ; hence  $s(n) < \sqrt{2} \cdot \sqrt{n} + 1$ .

The best known upper bound is  $\sqrt{n} + \sqrt[4]{n} + 1$ . Paul Erdős conjectured that  $s(n) - \sqrt{n}$  is bounded. Erdős considered this problem so important that he offered \$1000 for the proof or disproof of this statement.

The next application is the following theorem: *If  $a$  and  $b$  are coprime integers, then every positive divisor of  $a^2 + b^2$  is the sum of two squares.*

For example, the positive divisors of  $10001 = 10^4 + 1^2$  are 1, 73, 137, and 10001 itself. Each of these numbers is the sum of two squares:  $1 = 1^2 + 0^2$ ,  $73 = 8^2 + 3^2$ , and  $137 = 11^2 + 4^2$ .

To prove the general statement, let  $n$  be a positive divisor of  $a^2 + b^2$ . We may assume that  $n$  is not a perfect square. We shall prove that there are integers  $x, y$ , at least one of which is nonzero, such that  $n$  divides  $x^2 + y^2$ , and  $|x|, |y| \leq \sqrt{n}$ . The last condition implies  $x^2 + y^2 \leq 2\sqrt{n}^2 < 2n$ . Since  $x^2 + y^2$  is a nonzero multiple of  $n$ , this gives  $x^2 + y^2 = n$ , which is to be proven.

Since  $\gcd(a, b) = 1$ , it follows that  $\gcd(n, b) = 1$  (why?). Hence,  $n$  is a divisor of  $x^2 + y^2$  if and only if  $n$  divides

$$b^2x^2 + b^2y^2 = b^2(x^2 + y^2) = (bx - ay)(bx + ay) + (a^2 + b^2)y^2.$$

Therefore  $|n| \cdot x^2 + y^2$  will be satisfied if  $|n| \cdot bx - ay$ .

Now consider the numbers  $bx - ay$ , where  $0 \leq x, y \leq \sqrt{n}$ . There are  $(\lfloor \sqrt{n} \rfloor + 1)^2 = n$  such numbers. However, the possible remainders of these numbers when divided by  $n$  are  $0, 1, \dots, n-1$ , and thus, by the pigeonhole principle, there are distinct pairs  $(x_1, y_1)$  and  $(x_2, y_2)$  such that  $bx_1 - ay_1$  and  $bx_2 - ay_2$  give the same remainder. Let  $x = x_1 - x_2$  and  $y = y_1 - y_2$ . Then  $bx - ay = (bx_1 - ay_1) - (bx_2 - ay_2)$  is divisible by  $n$ . Also,  $|x| \leq \max(x_1, x_2) \leq \lfloor \sqrt{n} \rfloor$ , and similarly,  $|y| \leq \lfloor \sqrt{n} \rfloor$ . Finally,  $x$  and  $y$  cannot be both zero, since the pairs  $(x_1, y_1)$  and  $(x_2, y_2)$  were distinct. This completes the proof.

As an important application, we obtain the following theorem of Fermat: *every prime of the form  $4k+1$  is the sum of two squares.*

**Proof.** Let  $n$  be a natural number and consider the numbers  $0, \{o\}, \{2o\}, \dots, \{nx\}$ , where  $\{z\}$  denotes the fractional part of  $z$ ; that is  $\{z\} = z - [z]$ .

These numbers define  $n+1$  points distributed among the  $n$  intervals  $[\frac{i-1}{n}, \frac{i}{n}]$  ( $i = 1, \dots, n$ ). There must be an interval which contains at least two of these points, and therefore there are numbers  $0 \leq k < m \leq n$  such that  $\{m\alpha\} - \{k\alpha\} < \frac{1}{n}$ . Let  $q_n = m - k$  and  $p_n = \{m\alpha\} - \{k\alpha\}$ . Then  $0 < q_n \leq n$  and  $|p_n - (p_n/q_n)| < 1/n \leq 1/q_n$ , from which  $|\alpha - (p_n/q_n)| < 1/q_n^2$ .

In order to complete the proof we have to check that there are infinitely many different numbers among the quotients  $p_n/q_n$ . This is an immediate consequence of  $0 < |q_n\alpha - p_n| < 1/n$ .

## Exercises

- Prove that for every odd integer  $n$  there is a positive integer  $i$  such that  $|n|^2 - 1$ .
- Let  $a_1, a_2, \dots, a_{100}$  be a sequence of integers. Prove that there is a subsequence  $a_{i_1}, a_{i_2}, \dots, a_{i_k}$  such that 100 divides  $a_{i_1} + a_{i_2} + \dots + a_{i_k}$ .
- Let  $a_1, a_2, \dots, a_{51}$  be integers with  $1 \leq a_i \leq 100$  ( $i = 1, \dots, 51$ ). Prove that  $a_i | a_j$  holds for some  $i \neq j$ . (H)
- The Fibonacci sequence  $u_n$  is defined as follows. Let  $u_0 = 0$ ,  $u_1 = 1$ , and  $u_n = u_{n-1} + u_{n-2}$  ( $n = 2, 3, \dots$ ). Prove that there is an  $n > 0$  such that  $1000|u_n$ .
- Let  $p_n, q_n$  be as in Exercise 1.3, and let  $a_n, b_n$  be integers such that  $(\sqrt{2}-1)^n = a_n + b_n\sqrt{2}$ . Prove that  $a_n = p_n$  and  $b_n = -q_n$ .
- Let  $r_1 = 1$ ,  $s_1 = 0$ , and  $r_{n+1} = 2r_n + 3s_n$ ,  $s_{n+1} = r_n + 2s_n$  ( $n = 1, 2, \dots$ ). Prove that  $|\sqrt{3} - (r_n/s_n)| < 1/s_n^2$ . (H)
- Let  $D$  be a positive integer, and let  $n$  be a positive divisor of  $a^2 + Db^2$ , where  $a$  and  $b$  are coprime integers. Prove that there are integers  $i, x, y$  such that  $1 \leq i \leq D$  and  $i \cdot n = x^2 + Dy^2$ .
- Prove that if  $\gcd(a, b) = 1$ , then every positive divisor of  $a^2 + 2b^2$  is of the form  $x^2 + 2y^2$  with integer  $x, y$ . Check the result on the divisors of  $10002 = 100^2 + 2 \cdot 1^2$ .
- Prove that if  $\gcd(a, b) = 1$ , then every positive and odd divisor of  $a^2 + 3b^2$  is of the form  $x^2 + 3y^2$  with integer  $x, y$ . Check the result on the divisors of  $10003 = 100^2 + 3 \cdot 1^2$ .

## 2.4 :: power set, Cantor's theorem, Russel's paradox

Let  $A$  be a set. By  $2^A$  we denote the set of all subsets of  $A$ , it is called the **power set** for  $A$ . Also it is frequently denoted as  $\mathbb{P}(A)$  (the power set for  $A$ ). Remember that  $2^A$  is a set, it is not a number.

Example:

$$A = \{1, x, B\},$$

$$2^A = \{\emptyset, \{1\}, \{x\}, \{B\}, \{1, x\}, \{1, B\}, \{x, B\}, \{1, x, B\}\}$$

**Theorem:**  $2^A$  can be identified with the set of all functions  $A \rightarrow \{0, 1\}$ .

Proof. Indeed, each subset  $Y \subset A$  corresponds to the characteristic function of this subset,  $\chi_Y : A \rightarrow \{0, 1\}$  and vice versa.

The set of all functions  $f : A \rightarrow B$  is denoted by  $B^A$ . What is  $\emptyset^A, A^\emptyset$ ?

Later in the Set theory course the number 2 will be **defined** as  $\{0, 1\}$ . This explains the notation  $2^A$  for the set of maps  $A \rightarrow \{0, 1\}$ , i.e. the power set of  $A$ .

**Problem 1.** If  $|A| < \infty$  (i.e.  $A$  is finite), then  $|2^A| = 2^{|A|}$ .

Indeed, for each element in  $A$ , we have two choices: include or not include it to a subset. So,  $2^{|A|}$  possibilities.

**Problem 2.** For sets  $A, B$  with  $A \cap B = \emptyset$ , prove that there is a natural bijection  $2^A \times 2^B \cong 2^{A \cup B}$ .

Proof. We construct a **bijection** between  $2^A \times 2^B$  and  $2^{A \cup B}$ . The first set is the set of pairs  $(a, b)$  where  $a \subset A, b \subset B$ . Given such a pair, consider the set  $f(a, b) = a \cup b \subset A \cup B$ , note that  $a \cup b$  is an element of  $2^{A \cup B}$ , so we constructed a map  $f : 2^A \times 2^B \rightarrow 2^{A \cup B}$ . The map  $g : 2^{A \cup B} \rightarrow 2^A \times 2^B$  in other direction is constructed as follows: let  $c \in 2^{A \cup B}$ , construct a pair  $(c \cap A, c \cap B) \in 2^A \times 2^B$ , this gives a map  $g : 2^{A \cup B} \rightarrow 2^A \times 2^B$ . Then check that  $f \circ g$  and  $g \circ f$  are both identity maps, so  $f, g$  are both bijections.

**Theorem 1.** For any sets  $A$  and  $B$ , there is a natural bijection

$$(2^A)^B \cong 2^{A \times B}.$$

In words: the set of all functions  $B \rightarrow 2^A$  is the same as the set of all subsets of the Cartesian product  $A \times B$ .

*Proof.* An element of  $(2^A)^B$  is a function

$$f : B \longrightarrow 2^A,$$

so for each  $b \in B$  we have a subset  $f(b) \subset A$ .

Define a subset of  $A \times B$  by

$$S_f := \{(a, b) \in A \times B \mid a \in f(b)\}.$$

This gives a map

$$\Phi : (2^A)^B \longrightarrow 2^{A \times B}, \quad f \mapsto S_f.$$

Conversely, if  $S \subset A \times B$ , define a function

$$f_S : B \longrightarrow 2^A, \quad f_S(b) := \{a \in A \mid (a, b) \in S\}.$$

This gives a map

$$\Psi : 2^{A \times B} \longrightarrow (2^A)^B, \quad S \mapsto f_S.$$

It is immediate from the definitions that

$$\Psi(\Phi(f)) = f \quad \text{and} \quad \Phi(\Psi(S)) = S.$$

Thus  $\Phi$  and  $\Psi$  are inverse bijections, proving the claim.  $\square$

**Interpretation for**  $A = \{a_1, a_2, a_3\}$ ,  $B = \{b_1, b_2, b_3, b_4\}$

Elements of  $A \times B$  are the 12 pairs

$$(a_i, b_j), \quad 1 \leq i \leq 3, 1 \leq j \leq 4.$$

A subset  $S \subset A \times B$  can be visualised as selecting some of these pairs. For each fixed  $b_j \in B$ , the *fiber*

$$S_{b_j} = \{a_i \in A : (a_i, b_j) \in S\}$$

is a subset of  $A$ . Therefore, from such an  $S$  we obtain a function

$$f_S : B \longrightarrow 2^A, \quad f_S(b_j) = S_{b_j}.$$

Conversely, given any function

$$f : B \rightarrow 2^A,$$

for each  $b_j$  we choose a subset  $f(b_j) \subset A$ . These four subsets determine exactly which of the pairs  $(a_i, b_j)$  belong to the corresponding subset of  $A \times B$ :

$$(a_i, b_j) \in S_f \iff a_i \in f(b_j).$$

Thus specifying a subset of the grid

$$A \times B = \{a_1, a_2, a_3\} \times \{b_1, b_2, b_3, b_4\}$$

is equivalent to specifying, for each column  $b_j$ , a subset of  $A$ . This is exactly a function  $B \rightarrow 2^A$ .

### Cantor's theorem: no injection $2^A \rightarrow A$

**Lemma 1.** Let  $A$  and  $B$  be sets. Then the following are equivalent:

1. There exists an injection  $A \rightarrow B$ .
2. There exists a surjection  $B \rightarrow A$ .

In particular:

- If no injection  $A \rightarrow B$  exists, then no surjection  $B \rightarrow A$  exists.
- If no surjection  $B \rightarrow A$  exists, then no injection  $A \rightarrow B$  exists.

*Proof.* (1)  $\Rightarrow$  (2): Assume  $f : A \rightarrow B$  is injective. Then  $f$  identifies  $A$  with the subset  $f(A) \subset B$ . Let  $g : B \rightarrow A$  be defined by

$$g(b) = \begin{cases} a, & \text{if } b = f(a) \text{ for some } a \in A, \\ a_0, & \text{if } b \notin f(A), \end{cases}$$

where  $a_0 \in A$  is fixed.

Every  $a \in A$  has  $f(a) \in B$  with  $g(f(a)) = a$ , so  $g$  is surjective.

(2)  $\Rightarrow$  (1): Assume  $g : B \rightarrow A$  is surjective. For each  $a \in A$  choose one  $b \in B$  with  $g(b) = a$ . Define  $f : A \rightarrow B$  by sending each  $a$  to one of these chosen  $b$ 's.

If  $f(a) = f(a') = b$ , then applying  $g$  gives

$$a = g(b) = a',$$

so  $f$  is injective.

The two “in particular” statements follow immediately: if (1) fails then (2) cannot hold, and if (2) fails then (1) cannot hold.  $\square$

**Theorem 2** (Cantor). For every set  $A$  there is no injective map

$$f : 2^A \longrightarrow A.$$

Equivalently,  $|2^A| > |A|$  for every set  $A$ .

We first give a proof in the finite case, using an explicit construction, and then a more general argument that works for arbitrary sets.

**Lemma 2** (Finite case). Let  $A$  be a finite set. Then there is no injective map  $f : 2^A \rightarrow A$ .

*Proof.* Assume, for contradiction, that such an injective map  $f : 2^A \rightarrow A$  exists.

Define inductively

$$X_0 := \emptyset, \quad x_1 := f(X_0), \quad X_1 := \{x_1\}.$$

Having defined  $X_n \subset A$  and  $x_{n+1} = f(X_n)$ , put

$$X_{n+1} := X_n \cup \{x_{n+1}\}.$$

*Claim:* For every  $n \geq 0$  we have  $x_{n+1} \notin X_n$ , and hence all elements  $x_1, x_2, x_3, \dots$  are pairwise distinct.

Indeed, suppose  $x_{n+1} \in X_n$ . Then  $x_{n+1} = x_j$  for some  $1 \leq j \leq n$ . But by construction

$$x_{n+1} = f(X_n) \quad \text{and} \quad x_j = f(X_{j-1}).$$

Hence

$$f(X_n) = f(X_{j-1}).$$

Since  $j - 1 < n$ , we have  $X_{j-1} \subsetneq X_n$ , so  $X_{j-1} \neq X_n$ . This contradicts the injectivity of  $f$ .

Thus each step produces a new element  $x_{n+1} \notin \{x_1, \dots, x_n\}$ , so  $A$  contains infinitely many distinct elements  $x_1, x_2, \dots$ , which is impossible because  $A$  is finite.

Therefore no injective map  $2^A \rightarrow A$  can exist when  $A$  is finite.  $\square$

For arbitrary sets we use a slightly different (and classical) argument. It is usually stated in terms of *surjections*  $A \rightarrow 2^A$ , but the two forms are equivalent.

**Lemma 3** (Cantor's theorem in surjective form). For every set  $A$  there is no surjective map

$$g : A \longrightarrow 2^A.$$

*Proof.* Assume, for contradiction, that  $g : A \rightarrow 2^A$  is surjective.

Define a special subset  $B \subset A$  by

$$B := \{ a \in A \mid a \notin g(a) \}.$$

Since  $B \subset A$ , it is an element of  $2^A$ . By surjectivity of  $g$  there exists some  $b \in A$  such that

$$g(b) = B.$$

Now ask: does  $b$  belong to  $B$ ?

*Case 1:* Suppose  $b \in B$ . By the definition of  $B$  this means  $b \notin g(b)$ . But  $g(b) = B$ , so this says  $b \notin B$ , a contradiction.

*Case 2:* Suppose  $b \notin B$ . Then, by the definition of  $B$ , we must have  $b \in g(b)$ . Since  $g(b) = B$ , this says  $b \in B$ , again a contradiction.

In both cases we reach a contradiction, so no such surjection  $g$  can exist (this idea is sometimes called “The Cantor diagonal argument”).  $\square$

Exercise: rewrite this proof to show that for each set  $A$  there exists no injective map  $2^A \rightarrow A$ .

**Corollary 1.** For every set  $A$  there is no injective map  $f : 2^A \rightarrow A$ .

**Remark 2.** The finite proof above can be viewed as a concrete version of the fact that  $|2^A| > |A|$ : assuming an injection  $2^A \rightarrow A$  lets us construct an infinite sequence of distinct elements inside a finite set. The general diagonal argument shows that this phenomenon persists for *all* sets  $A$ , finite or infinite.

## Russell’s Paradox

**Theorem 3** (Russell’s paradox). There is no set  $R$  such that

$$R = \{ x \mid x \notin x \}.$$

In other words, the collection of all sets that do not contain themselves cannot be a set.

*Proof.* Assume, for contradiction, that such a set  $R$  exists.

Ask whether  $R \in R$ .

*Case 1:* Suppose  $R \in R$ . By the definition of  $R$ , a set belongs to  $R$  precisely when it does *not* belong to itself. Hence from  $R \in R$  we must conclude  $R \notin R$ , a contradiction.

*Case 2:* Suppose  $R \notin R$ . Then by the defining condition of  $R$ , any set that does not belong to itself *must* be in  $R$ . Thus  $R \in R$ , again a contradiction.

Both possibilities lead to contradiction, so such a set  $R$  cannot exist.  $\square$

**Remark 3.** Russell's paradox shows that naive "comprehension" (forming a set of all objects satisfying a property  $P(x)$ ) must be restricted. Modern set theory avoids the paradox by allowing only subsets of existing sets (separation), not arbitrary collections.

## Countable and uncountable sets

**Definition 2** (Countable, denumerable, uncountable). A set  $A$  is called *countable* if there exists an injective map  $A \rightarrow \mathbb{N}$ .

A set  $A$  is called *denumerable* (or *countably infinite*) if there exists a bijection

$$f : \mathbb{N} \longrightarrow A.$$

A set is called *uncountable* if it is not countable.

**Example 1.** The sets  $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{Q}$  are denumerable. In particular,  $\mathbb{N}$  is denumerable via the identity map  $n \mapsto n$ .

We now show that uncountable sets actually exist.

**Theorem 4.** The power set  $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$  is uncountable.

*Proof.* Suppose, for contradiction, that  $2^{\mathbb{N}}$  is countable. Then there exists an injective map

$$f : 2^{\mathbb{N}} \longrightarrow \mathbb{N}.$$

But this contradicts Cantor's theorem (proved above), which states that there is *no* injective map  $2^A \rightarrow A$  for any set  $A$ , in particular for  $A = \mathbb{N}$ .

Hence  $2^{\mathbb{N}}$  cannot be countable; it is uncountable.  $\square$

**Corollary 2.** There exists at least one uncountable set (namely  $2^{\mathbb{N}}$ ).

## 2.4.1 :: :: tutorial/midterm solutions and common mistakes

In what follows we suppose that the two following propositions are true:

**Proposition 1.** The teaching team of Basic Concepts of Maths wants its student to do good in the course, including midterm, exams etc...

**Proposition 2.** Nikita sent an email claiming that in the midterm there will be 5 exercises covering: **induction**, **pigeonhole principle**, **counting in two ways**, **proving of inclusion** of two sets in a formal way (without words), presentation of a given permutation as a product of cycles and as a product of transpositions.

Combining the two above propositions we get the following corollary.

**Corollary 3.** There is a bijection between the questions of the midterm and the topics of the proposition (2)

We will try to find the bijection between exercises and topics!

---

**Exercise 1.** We draw an arrow on each side and each diagonal of a polygon. Show that there exists a vertex  $Z$  of this polygon such that any other vertex is reachable from  $Z$  if we are allowed to move along the arrows.

**Solution.** Here proving of inclusion of two sets in a formal way (without words), presentation of a given permutation as a product of cycles and as a product of transpositions don't make sense. Pigeonhole doesn't either (what would we put in boxes?). Same for counting in two ways, what should we count? We are left with **induction**.

Label the vertices of the polygon by  $1, 2, \dots, n$ . On each side and each diagonal we draw exactly one arrow between its two endpoints, so between any two distinct vertices there is exactly one directed edge (one arrow). Thus this is a directed graph<sup>42</sup> on  $n$  vertices.

We want to prove: **There exists a vertex  $Z$  such that every other vertex is reachable from  $Z$  by following arrows.**

We prove this by induction on  $n$ .

*Base case  $n = 3$ .* This is a direct verification. Draw a triangle with all possible directions of arrows and we are done. A remark, if we consider graphs and not polygons, we can even reduce to the case where  $n = 1$ .

---

<sup>42</sup>called a *tournament* in graph theory

*Induction hypothesis.* Assume that in every polygon where diagonals are drawn as well as arrows with  $n$  vertices, there exists a vertex from which all other vertices are reachable along directed paths.

*Induction step: from  $n$  to  $n + 1$ .* Consider a such a polygon  $T$  on  $n + 1$  vertices. Pick one vertex and call it  $x$ , and remove it temporarily, as well as all the arrows/edges joining it. The remaining  $n$  vertices still form a polygon with the needed properties. By the induction hypothesis, there exists a vertex  $Z$  among these  $n$  vertices such that every other one of these  $n$  vertices is reachable from  $Z$  by a directed path.

Now look at the direction of the arrow between  $Z$  and  $x$ . There are two cases:

- **Case 1:**  $Z \rightarrow x$ .

Then  $Z$  already has directed paths to all the other  $n - 1$  vertices (by the induction hypothesis), and there is a direct arrow from  $Z$  to  $x$ . Therefore, every vertex (including  $x$ ) is reachable from  $Z$ . So in this case  $Z$  is the vertex we need.

- **Case 2:**  $x \rightarrow Z$ .

Then from  $x$  we can go directly to  $Z$ , and from  $Z$  we can reach every other vertex among the remaining  $n - 1$  vertices (again by the induction hypothesis). Thus from  $x$  there is a directed path to every other vertex:

$$x \rightarrow Z \rightarrow \dots \rightarrow v$$

for any vertex  $v \neq x$ . So in this case  $x$  is the vertex we need.

In either case, we have found a vertex (either  $Z$  or  $x$ ) from which all other vertices of the  $(n + 1)$ -vertex are reachable by following arrows.

This completes the induction, and hence in any orientation of all sides and diagonals of the polygon there exists a vertex  $Z$  from which every other vertex is reachable.

□

**Common mistakes:** Many students supposed that they can choose the directions of the arrows. **We cannot** as we don't know these directions, they can be any. Some student wanted to split a vertex, it is not clear at all what this means. Students also tried to prove it by contradiction: Suppose there is no such vertex  $Z$ . Maybe it is possible to do so, but many details are needed to be taken cared of which was not done. Many times the induction was done on the **edges**, no! What we need is an induction on the **verticies**.

**Exercise 2.** Is it possible to divide a convex 17-gon into 14 triangles ?

**Solution.** We have already used the induction in the previous exercise. We are left with pigeonhole principle, counting in two ways, proving of inclusion of two sets in a formal way (without words), presentation of a given permutation as a product of cycles and as a product of transpositions. It is more or less clear that proving of inclusion of two sets in a formal way (without words), presentation of a given permutation as a product of cycles and as a product of transpositions is not what we need. We are left with pigeonhole or counting in two ways. The **counting in two ways** make more sense here.

Suppose a convex 17-gon is covered exactly by  $N$  triangles. Each triangle has angle sum  $180^\circ$ , so the sum of the angles of all triangles is

$$N \cdot 180^\circ.$$

We compute the same quantity in another way by grouping angles at each point where triangles meet.

Let

- $n = 17$  be the number of original vertices of the polygon,
- $b$  be the number of additional *boundary* vertices created by subdividing edges,
- $k$  be the number of *interior* vertices inside the polygon.

At each original vertex of the convex polygon, the sum of angles of all triangles meeting there equals the interior angle of the polygon at that vertex. Thus the total contribution from the original vertices is

$$(n - 2) \cdot 180^\circ = (17 - 2) \cdot 180^\circ = 15 \cdot 180^\circ.$$

At each additional boundary vertex, the polygon boundary is straight, so the region inside the polygon forms a half-plane. The triangles that meet at such a point fill exactly that half-plane, so the sum of their angles there is

$$180^\circ.$$

Thus the  $b$  boundary subdivision points contribute  $b \cdot 180^\circ$ .

At each interior vertex, the triangles surround the point completely and fill a full circle, so the sum of the triangle angles at such a point is

$$360^\circ.$$

Hence the  $k$  interior points contribute  $k \cdot 360^\circ$ .

Summing all contributions, we obtain another expression for the total sum of triangle angles:

$$N \cdot 180^\circ = 15 \cdot 180^\circ + b \cdot 180^\circ + k \cdot 360^\circ.$$

Divide both sides by  $180^\circ$ :

$$N = 15 + b + 2k.$$

Since  $b \geq 0$  and  $k \geq 0$ , we conclude that

$$N \geq 15.$$

Therefore a covering of a convex 17-gon by only 14 triangles is impossible.  $\square$

**Common mistakes:** A lot of students understood that they should provide an explicit division of the 17-gon and show that this division gives 15 triangles. No, the exercise should be read as follows, I give you 14 triangles, with absolutely no details about these triangle, is it possible with these 14 triangles to divide our 17-gon ? So we are not asking you to divide the 17-gon by taking all the diagonals starting from one vertex and prove that there are actually 15 triangles, or using other diagonals and show that we get again 15 triangles. These are fixed division that you constructed, and we are not interested by this. Other students thought about using counting in two ways, but doing it with the edges. Again the idea was with a fixed division coming from diagonals, so not what we need. Students also tried to use induction here, but again, this gave a fixed division of the 17-gon. Finally some students understood that it was about angles. But they forgot to speak about angles on the edges of the 17-gon, or angles inside the 17-gon, but this was already very good!

**Exercise 3.** Let  $A, B, C, D$  be sets. Prove or disprove that

$$(A \Delta C) \cap (B \Delta D) \subset (A \cup B) \Delta (C \cup D).$$

**Solution.** This is clearly about inclusion of sets. The claim is false. Indeed by taking  $A = D = \{1\}$  and  $C = B = \{\emptyset\}$  we get

$$(A \Delta C) \cap (B \Delta D) = \{1\} \tag{1}$$

whereas

$$(A \cup B) \Delta (C \cup D) = \{\emptyset\} \quad (2)$$

showing that the claim is false.  $\square$

**Common mistakes:** You need to remember this for the future, this is very important: *If you think that a statement is true, then you prove it. If you think that a claim is false then giving a counterexample is enough!* Many students tried to prove that the claim is false. You should not! It was almost impossible to understand the logic of what was written.

**Exercise 4.** Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 1 & 2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 3 & 4 \end{pmatrix}.$$

Present  $\tau \circ \sigma$  as a product of cycles and as a product of transpositions.

**Solution.** We can all agree that this is about permutations.

By definition,

$$(\tau \circ \sigma)(i) = \tau(\sigma(i)).$$

Compute this for  $i = 1, \dots, 6$ .

$$(\tau \circ \sigma)(1) = \tau(\sigma(1)) = \tau(6) = 4, \quad (3)$$

$$(\tau \circ \sigma)(2) = \tau(\sigma(2)) = \tau(4) = 5, \quad (4)$$

$$(\tau \circ \sigma)(3) = \tau(\sigma(3)) = \tau(5) = 3, \quad (5)$$

$$(\tau \circ \sigma)(4) = \tau(\sigma(4)) = \tau(3) = 1, \quad (6)$$

$$(\tau \circ \sigma)(5) = \tau(\sigma(5)) = \tau(1) = 2, \quad (7)$$

$$(\tau \circ \sigma)(6) = \tau(\sigma(6)) = \tau(2) = 6. \quad (8)$$

So in two-line notation,

$$\tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 3 & 1 & 2 & 6 \end{pmatrix}.$$

Now write as disjoint cycles: follow the images.

$$1 \mapsto 4 \mapsto 1 \implies (1 \ 4),$$

$$2 \mapsto 5 \mapsto 2 \implies (2 \ 5),$$

$$3 \mapsto 3, \quad 6 \mapsto 6,$$

so 3 and 6 are fixed points.

Thus

$$\tau \circ \sigma = (1\ 4)(2\ 5)(3)(6),$$

where we usually omit the 1-cycles (3) and (6).

As a product of transpositions, we already have such a decomposition, since a 2-cycle is a transposition. Therefore

$$\tau \circ \sigma = (1\ 4)(2\ 5)(3)(6)$$

is the required expression as a product of cycles (disjoint cycles) and also as a product of transpositions. □

**Common mistakes:** Not much to say here, be careful with the computations. We do not write things like (3, 3) or (6, 6), this should be just (3) or (6).

**Exercise 5.** During the last academic year, Raphael Laoshi solved at least one problem in mathematics every day. However, in any week (any 7 consecutive days) he solved no more than 12 problems. Prove that there exist some consecutive days during which he solved exactly 20 problems.

**Solution.** We are left with the last exercise. And we only have the pigeonhole left. Thus this exercise is about pigeonhole. This exercise is **HARD!** The hardest from the midterm. Because the problem is hard the solution will be very wordy to explain what we are doing.

**Proof.** Let  $a_i$  be the number of problems solved on day  $i$ . We know two things:

- Every day he solves at least one problem, so  $a_i \geq 1$ .
- In any 7 consecutive days he solves at most 12 problems.

**Step 1: Look at the first 21 days.** Split these 21 days into three blocks of 7 days each:

$$\{1, \dots, 7\}, \quad \{8, \dots, 14\}, \quad \{15, \dots, 21\}.$$

Each block has at most 12 problems, so the total for all 21 days is at most 36. Since he solves at least one problem every day, the total is at least 21. Thus the number of problems solved in the first 21 days lies between 21 and 36.

## Step 2: Prefix sums.

Let

$$S_m = a_1 + a_2 + \cdots + a_m$$

be the total number of problems solved up to day  $m$  (and set  $S_0 = 0$ ). Then

$$0 = S_0 < S_1 < S_2 < \cdots < S_{21} \leq 36.$$

So we now have 22 different numbers between 0 and 36.

**Step 3: What we are looking for.** If we want to find some consecutive days whose total is exactly 20, we are trying to find days  $p+1$  through  $q$  such that

$$a_{p+1} + \cdots + a_q = 20.$$

But the left side is just a difference of prefix sums:

$$S_q - S_p = 20.$$

So our real goal is this:

*Find two prefix sums that differ by exactly 20.*

**Step 4: Remainders and the pigeonhole.** To find such a pair, take all the prefix sums  $S_0, \dots, S_{21}$ . Look at the remainders of these numbers when we divide them by 20. Since all possible remainders are  $\{0, 1, 2, \dots, 19\}$ , by the pigeonhole principle we must have  $S_p, S_q$  with the same remainder. We may assume  $q > p$ . Then 20 divides  $1 \leq S_q - S_p \leq 36$ . Thus  $S_q - S_p = 20$ .

Thus the days  $p+1$  through  $q$  have total

$$S_q - S_p = a_{p+1} + \cdots + a_q = 20.$$

**Conclusion.** There must be a block of consecutive days during which he solved exactly 20 problems. □

**Common mistakes:** There was a lot of hand-waving proofs, that were actually not proofs. On the other hand this exercise is hard, so nothing to worry about if you couldn't find the solution. Very few students mentioned that we should use the pigeonhole principle here.

## 2.4.2 :: :: problems for workshop: Even and Odd Permutations

Let  $S_n$  be the set of all permutations of  $\{1, \dots, n\}$ . A *transposition* is a permutation that swaps two elements and fixes all others, it is written as  $(i\ j)$ .

- **[Definition of the parity via transpositions]** Every permutation  $\sigma \in S_n$  can be written as a product of transpositions:

$$\sigma = \tau_1 \tau_2 \cdots \tau_k.$$

We say that  $\sigma$  is *even* if it can be written as a product of an even number of transpositions;  $\sigma$  is *odd* if it can be written as a product of an odd number of transpositions. This notion is well-defined (see the problems below).

- **[Definition of the parity via inversions]** For  $\sigma \in S_n$ , define an *inversion* to be a pair  $(i, j)$  with  $1 \leq i < j \leq n$  and  $\sigma(i) > \sigma(j)$ . Let  $\text{inv}(\sigma)$  be the number of inversions. (cf. Problem on page 18)

Define  $\varepsilon(\sigma) = (-1)^{\text{inv}(\sigma)}$ . Show that  $\varepsilon(\text{id}) = 1$ . Show that  $\varepsilon(\sigma \circ (i\ j)) = -\varepsilon(\sigma)$ . Conclude that  $\varepsilon(\sigma) = \text{sgn}(\sigma)$ .

- **[All permutations in  $S_3$ ]**

- List all 6 permutations in  $S_3$  in cycle notation.
- Decompose each into transpositions and determine whether it is even or odd.
- Check using the number of inversions.

- **[Computations in  $S_5$ ]** Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}.$$

Compute  $\text{inv}(\sigma)$ .

- **[Sign of a cycle]** Let  $c = (1\ 2\ \dots\ k)$  be a  $k$ -cycle.

- Show that

$$(1\ 2\ \dots\ k) = (1\ k)(1\ k-1)\cdots(1\ 3)(1\ 2).$$

- Deduce that  $\text{sgn}(c) = (-1)^{k-1}$ .
- Conclude that  $\text{sgn}$  of any permutation in disjoint cycle form is the product of the signs of its cycles.

**Lemma 4.** Let  $\sigma \in S_n$  and let  $\tau = (ij)$  with  $i < j$ ,  $\sigma' = \tau\sigma$ . Then

$$\text{inv}(\sigma') = \text{inv}(\tau\sigma) \equiv \text{inv}(\sigma) + 1 \pmod{2}.$$

*Proof.* Recall that  $\text{inv}(\sigma) = \#\{(p, q) \mid p < q, \sigma(p) > \sigma(q)\}$ .

(1) For indices less than  $i$  and bigger than  $j$  the inversion count for  $\sigma$  and  $\sigma'$  does not change at all.

(2) Middle indices  $i < k < j$ . For each  $k$  with  $i < k < j$  look at triples  $(i, k, j)$ . Before the transposition we compare  $\sigma(i), \sigma(k), \sigma(j)$ ; after it,  $\sigma(i)$  and  $\sigma(j)$  swap places, and  $\sigma(k)$  stays the same. Note that the parity for the number of inversions in two pairs  $\sigma(i)\sigma(k)$  and  $\sigma(k)\sigma(j)$  does not change when we swap  $\sigma(i)$  and  $\sigma(j)$ . Indeed, without loss of generality we may assume that  $i = 1, k = 2, j = 3$  and  $\sigma \in S_3$  and so check all six possible choices for  $\sigma$  to verify the statement.

(3) The pair  $(i, j)$ . Here we compare  $\sigma(i)$  with  $\sigma(j)$ , while in  $\sigma'$  we compare  $\sigma'(i) = \sigma(j)$  with  $\sigma'(j) = \sigma(i)$ . So the pair  $(\sigma(i), \sigma(j))$  changes from inversion to non-inversion or vice versa, contributing  $+1$  modulo 2.

Therefore

$$\text{inv}(\tau\sigma) \equiv \text{inv}(\sigma) + 1 \pmod{2}. \quad \square$$

## :: :: 7th homework

- Let

$$\sigma = (1\ 7\ 5\ 3)(2\ 6)(4\ 9\ 8) \in S_9.$$

- Compute  $\text{sgn}(\sigma)$  using the fact that a  $k$ -cycle has sign  $(-1)^{k-1}$ .
- Verify your answer by expressing  $\sigma$  as a product of transpositions.
- Verify again by computing the number of inversions of  $\sigma$ .

- Let

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 7 & 3 & 2 & 8 & 4 & 6 \end{pmatrix} \in S_8.$$

- Rewrite  $\tau$  in disjoint cycle notation.
- Compute  $\text{sgn}(\tau)$  using the formula for the sign of a cycle.
- Check your answer by computing the parity of the number of inversions of  $\tau$ .

• On the boundary of a convex polygon there grow hairs, in outside direction. Someone draws several non-intersection diagonals in the polygon, each such diagonal has hairs on one side. These diagonals dissect polygon into parts. Show that among these parts there exists a polygon with hairs outside. [Hint: induction]

## 2.5 :: infinite sequences, Thue-Morse sequence, Collatz conjecture

**Definition 3** (Finite sequences). Let  $A$  be a set and let  $n \in \mathbb{N}$ . A *finite sequence of length  $n$*  with values in  $A$  is a function

$$a : \{1, 2, \dots, n\} \longrightarrow A.$$

We write such a sequence as

$$(a(1), a(2), \dots, a(n)),$$

or, using subscript notation,

$$(a_1, a_2, \dots, a_n).$$

**Definition 4** (Infinite sequences). An *infinite sequence* with values in  $A$  is a function

$$a : \mathbb{N} \longrightarrow A.$$

We usually denote it by

$$a(i), \quad a_i, \quad \text{or} \quad (a_i)_{i \in \mathbb{N}},$$

all of which refer to the value of the sequence at index  $i$ .

**Remark 4.** Thus a sequence is simply a function with domain either  $\{1, \dots, n\}$  (finite case) or  $\mathbb{N}$  (infinite case).

### Componentwise operations

Let  $A$  be a set equipped with an operation. Typical example is  $A = \mathbb{Q}$  or  $\mathbb{R}$  or  $\mathbb{C}$ , with ordinary addition or multiplication.

**Definition 5** (Componentwise sum). Let  $(a_i)_{i \in I}$  and  $(b_i)_{i \in I}$  be two sequences in a set  $A$  equipped with a binary operation  $+$ . Their *componentwise sum* is the sequence

$$(a_i + b_i)_{i \in I}, \quad \text{defined by} \quad (a_i + b_i)(i) = a_i + b_i.$$

**Definition 6** (Componentwise product). If  $A$  has a multiplication  $\cdot$ , then the *componentwise product* of two sequences  $(a_i)$  and  $(b_i)$  is the sequence

$$(a_i \cdot b_i)_{i \in I}, \quad \text{defined by} \quad (a_i \cdot b_i)(i) = a_i \cdot b_i.$$

These operations are defined both for finite sequences and infinite sequences. They simply operate index-by-index. We also have the usual properties such as

$$a(b + c) = ab + ac, ab = ba, a(bc) = (ab)c, \text{etc.}$$

## Second-order linear recurrences

**Definition 7** (Second-order linear recurrence). Let  $A$  be a set equipped with addition and scalar multiplication (e.g.  $A = \mathbb{R}$  or  $\mathbb{C}$ ). A sequence  $(u_n)_{n \geq 0}$  satisfies a *second-order linear recurrence* if there exist constants  $c_1, c_2$  such that

$$u_{n+2} = c_1 u_{n+1} + c_2 u_n \quad \text{for all } n \geq 0.$$

**Proposition 3** (Two initial values determine the sequence). Let  $(u_n)$  satisfy

$$u_{n+2} = c_1 u_{n+1} + c_2 u_n.$$

Then the entire sequence is uniquely determined by the two initial values  $u_0$  and  $u_1$ .

*Proof.* Knowing  $u_0$  and  $u_1$ , the recurrence gives

$$u_2 = c_1 u_1 + c_2 u_0.$$

Once  $u_2$  is known, the recurrence gives  $u_3$ , then  $u_4$ , and so on. Thus every term is determined inductively and uniquely.  $\square$

## Characteristic polynomial

To such a recurrence we associate the *characteristic polynomial*

$$P(z) = z^2 - c_1 z - c_2.$$

**Proposition 4** (Distinct roots give the general formula). Suppose the characteristic polynomial

$$z^2 - c_1 z - c_2$$

has two distinct roots  $z_1$  and  $z_2$  (in  $\mathbb{C}$ ). Then every solution of the recurrence has the form

$$u_n = Az_1^n + Bz_2^n$$

for some constants  $A, B$  (determined by  $u_0$  and  $u_1$ ).

Moreover, for any given pair  $(u_0, u_1)$  there exists a unique pair  $(A, B)$  producing a sequence whose first two terms are exactly  $u_0$  and  $u_1$ .

*Proof.* Since  $z_1$  and  $z_2$  satisfy the polynomial equation  $z^2 - c_1 z - c_2 = 0$ , we have

$$z_1^{n+2} = c_1 z_1^{n+1} + c_2 z_1^n, \quad z_2^{n+2} = c_1 z_2^{n+1} + c_2 z_2^n.$$

Thus the sequences  $(z_1^n)$  and  $(z_2^n)$  both satisfy the recurrence.

Any linear combination

$$u_n = Az_1^n + Bz_2^n$$

also satisfies the recurrence (because the recurrence is linear).

To match initial conditions, solve the system

$$\begin{cases} u_0 = A + B, \\ u_1 = Az_1 + Bz_2. \end{cases}$$

Since  $z_1 \neq z_2$ , this  $2 \times 2$  system has a unique solution

$$A = \frac{u_1 - u_0 z_2}{z_1 - z_2}, \quad B = \frac{u_0 z_1 - u_1}{z_1 - z_2}.$$

Thus the coefficients are uniquely determined, and therefore the entire sequence is uniquely determined.  $\square$

**Remark 5.** The phrase “then it is automatic” refers to the fact that once the coefficients  $A$  and  $B$  have been chosen to make  $u_0$  and  $u_1$  correct, the formula

$$u_n = Az_1^n + Bz_2^n$$

automatically satisfies the recurrence for all  $n \geq 0$ , so no further verification is required.

### Example: the Fibonacci sequence

The Fibonacci sequence  $(F_n)_{n \geq 0}$  is defined by the second-order recurrence

$$F_{n+2} = F_{n+1} + F_n \quad \text{with} \quad F_0 = 0, F_1 = 1.$$

Thus it is an example of a recurrence of the form

$$u_{n+2} = c_1 u_{n+1} + c_2 u_n, \quad c_1 = 1, c_2 = 1.$$

**Characteristic polynomial.** The associated characteristic polynomial is

$$z^2 - z - 1 = 0.$$

Its two distinct roots are

$$z_1 = \frac{1 + \sqrt{5}}{2} = \varphi, \quad z_2 = \frac{1 - \sqrt{5}}{2} = \psi,$$

often called the golden ratio and its conjugate.

**Explicit formula (“Binet’s formula”).** Since the roots are distinct, the general solution is

$$F_n = Az_1^n + Bz_2^n.$$

The constants  $A$  and  $B$  are determined from the initial conditions.

From  $F_0 = 0$ :

$$0 = A + B \quad \Rightarrow \quad B = -A.$$

From  $F_1 = 1$ :

$$1 = Az_1 + Bz_2 = Az_1 - Az_2 = A(z_1 - z_2).$$

Since

$$z_1 - z_2 = \frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2} = \sqrt{5},$$

we obtain

$$A = \frac{1}{\sqrt{5}}, \quad B = -\frac{1}{\sqrt{5}}.$$

Hence the closed form is

$$F_n = \frac{1}{\sqrt{5}} (\varphi^n - \psi^n).$$

**Why this works automatically.** Because the formula

$$u_n = Az_1^n + Bz_2^n$$

satisfies the recurrence whenever  $z_1$  and  $z_2$  are roots of the characteristic polynomial, once the constants  $A$  and  $B$  are chosen to match  $u_0$  and  $u_1$ , the whole sequence automatically satisfies

$$u_{n+2} = u_{n+1} + u_n \quad \text{for all } n.$$

## First values.

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \dots$$

Fibonacci numbers thus give the simplest and most famous example of a second-order linear recurrence with two distinct characteristic roots.

## Cassini's identity and other inductive properties

The Fibonacci sequence satisfies many remarkable identities that can be proved by simple induction once the recurrence

$$F_{n+2} = F_{n+1} + F_n$$

is known.

**Theorem 5** (Cassini's identity). For every integer  $n \geq 1$ ,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

*Proof.* We use induction on  $n$ .

*Base case:* For  $n = 1$ ,

$$F_2F_0 - F_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1.$$

*Induction step:* Assume the identity holds for  $n$ :

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

Using the recurrence  $F_{n+2} = F_{n+1} + F_n$ , compute:

$$\begin{aligned} F_{n+2}F_n - F_{n+1}^2 &= (F_{n+1} + F_n)F_n - F_{n+1}^2 \\ &= F_{n+1}F_n + F_n^2 - F_{n+1}^2 \\ &= -(F_{n+1}^2 - F_{n+1}F_n - F_n^2) \\ &= -(F_{n+1}(F_{n+1} - F_n) - F_n^2). \end{aligned}$$

A cleaner way: rewrite the LHS using the recurrence on the induction hypothesis:

$$\begin{aligned} F_{n+2}F_n - F_{n+1}^2 &= (F_{n+1} + F_n)F_n - F_{n+1}^2 \\ &= -(F_{n+1}^2 - F_{n+1}F_n - F_n^2) \\ &= -(F_{n+1}F_{n-1} - F_n^2) \quad (\text{by the recurrence}) \\ &= -(-1)^n = (-1)^{n+1}. \end{aligned}$$

Thus the identity holds for  $n + 1$ . By induction it holds for all  $n$ . □

**Theorem 6** (Sum of the first  $n$  Fibonacci numbers). For all  $n \geq 0$ ,

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

*Proof.* We again use induction.

*Base case:* For  $n = 0$ ,

$$F_0 = 0 = F_2 - 1 = 1 - 1.$$

*Induction step:* Assume the formula holds for  $n$ :

$$\sum_{k=0}^n F_k = F_{n+2} - 1.$$

Then for  $n + 1$ ,

$$\sum_{k=0}^{n+1} F_k = \left( \sum_{k=0}^n F_k \right) + F_{n+1} = (F_{n+2} - 1) + F_{n+1} = F_{n+3} - 1,$$

using  $F_{n+3} = F_{n+2} + F_{n+1}$ . This completes the induction. □

**Remark 6.** Both identities illustrate a common phenomenon: many global properties of the Fibonacci sequence reduce to short calculations once the recurrence relation is known. The recurrence allows inductive proofs to extend “local” relations to all  $n$ .

## The Thue–Morse sequence

### Binary representations

Every nonnegative integer can be written uniquely in base 2.

**Definition 8** (Binary representation). For each integer  $n \geq 0$  there exist unique digits  $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_k \in \{0, 1\}$  such that

$$n = \varepsilon_0 2^0 + \varepsilon_1 2^1 + \cdots + \varepsilon_k 2^k \quad \text{and} \quad \varepsilon_k = 1 \text{ if } n > 0.$$

We write

$$n = (\varepsilon_k \varepsilon_{k-1} \dots \varepsilon_1 \varepsilon_0)_2$$

and call this the *binary representation* of  $n$ .

The *binary digit sum* of  $n$  is

$$s_2(n) := \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_k.$$

## Example 2.

$$5 = (101)_2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0, \quad s_2(5) = 1 + 0 + 1 = 2;$$
$$13 = (1101)_2 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0, \quad s_2(13) = 1 + 1 + 0 + 1 = 3.$$

We now give four equivalent ways to define the Thue–Morse sequence.

**Definition 9** (Thue–Morse sequence). The *Thue–Morse sequence* is the infinite 0–1 sequence  $(t_n)_{n \geq 0}$  which can be described in the following equivalent ways.

- (a) **Recursion in base 2.**

$$t_0 = 0, \quad t_{2n} = t_n, \quad t_{2n+1} = 1 - t_n \quad \text{for all } n \geq 0.$$

- (b) **Parity of the binary digit sum.** For each  $n \geq 0$ , write  $n$  in binary and let  $s_2(n)$  be the sum of its binary digits. Then

$$t_n \equiv s_2(n) \pmod{2},$$

i.e.  $t_n = 0$  if the number of 1's in the binary expansion of  $n$  is even, and  $t_n = 1$  if it is odd.

- (c) **Fixed point of a substitution.** Consider the substitution

$$\mu : 0 \mapsto 01, \quad 1 \mapsto 10.$$

Starting from the one-letter word  $w_0 = 0$ , define

$$w_{k+1} := \mu(w_k) \quad (k \geq 0).$$

where

$$\mu(\varepsilon_k \varepsilon_{k-1} \dots \varepsilon_1 \varepsilon_0) = (\mu(\varepsilon_k) \mu(\varepsilon_{k-1}) \dots \mu(\varepsilon_1) \mu(\varepsilon_0))$$

Thus

$$w_0 = 0, \quad w_1 = 01, \quad w_2 = 0110, \quad w_3 = 01101001, \quad \dots$$

Each  $w_k$  is a prefix of  $w_{k+1}$ , so there is a unique infinite word  $w = 01101001\dots$  having all  $w_k$  as initial segments. Then  $t_n$  is the  $n$ -th symbol of this word.

- (d) **Recursive block construction.** For a finite word  $u = u_0 u_1 \dots u_{m-1}$  over  $\{0, 1\}$ , let  $\bar{u}$  be the word obtained by flipping all bits, i.e. replacing  $0 \leftrightarrow 1$ :

$$\bar{u} := (1 - u_0)(1 - u_1) \dots (1 - u_{m-1}).$$

Define a sequence of finite words  $(u_k)_{k \geq 0}$  by

$$u_0 := 0, \quad u_{k+1} := u_k \bar{u}_k \quad (k \geq 0).$$

So

$$u_0 = 0, \quad u_1 = 01, \quad u_2 = 0110, \quad u_3 = 01101001, \quad \dots$$

Again, each  $u_k$  is a prefix of  $u_{k+1}$ , so there is a unique infinite word having all  $u_k$  as prefixes, and  $(t_n)_{n \geq 0}$  is defined to be its sequence of symbols.

The first few terms are

$$(t_n)_{n \geq 0} = 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \dots$$

**Proposition 5.** Descriptions (a), (b), (c), and (d) in Definition 9 all define the same sequence  $(t_n)_{n \geq 0}$ .

*Proof.* We prove a chain of equivalences.

**(a)  $\iff$  (b).** Define a sequence  $(u_n)_{n \geq 0}$  by

$$u_n := s_2(n) \pmod{2}.$$

We show that  $(u_n)$  satisfies the recursion in (a) with  $u_0 = 0$ .

Write  $n$  in binary. Appending a 0 at the right corresponds to multiplying by 2, and appending a 1 at the right corresponds to multiplying by 2 and adding 1. Hence

$$s_2(2n) = s_2(n), \quad s_2(2n + 1) = s_2(n) + 1.$$

Reducing mod 2 gives

$$u_{2n} = u_n, \quad u_{2n+1} = 1 - u_n,$$

and clearly  $u_0 = s_2(0) = 0$ . Thus  $(u_n)$  satisfies the recursion in (a).

Conversely, any sequence  $(t_n)$  satisfying (a) is uniquely determined by  $t_0$ : once  $t_0$  is fixed, all  $t_n$  follow by repeatedly using the rules for  $t_{2n}$  and  $t_{2n+1}$ . Hence the sequence defined in (b) (which we have called  $(u_n)$ ) coincides with the sequence defined in (a), and (a) and (b) are equivalent.

**(b)  $\implies$  (d).** For  $k \geq 0$  let

$$T_k := t_0 t_1 \dots t_{2^k-1}$$

be the prefix of length  $2^k$  of the infinite sequence  $(t_n)$  defined by (b). We claim that

$$T_0 = 0, \quad T_{k+1} = T_k \overline{T_k}.$$

The first equality is immediate:  $T_0$  has length 1 and  $t_0 = s_2(0) \equiv 0$ . Now fix  $k \geq 0$  and  $0 \leq m < 2^k$ . Then  $2^k + m$  has binary expansion obtained from that of  $m$  by adding a leading 1, so

$$s_2(2^k + m) = 1 + s_2(m).$$

Thus

$$t_{2^k+m} \equiv s_2(2^k + m) \equiv 1 + s_2(m) \equiv 1 - t_m \pmod{2},$$

i.e.  $t_{2^k+m} = 1 - t_m$ . This shows precisely that the block  $(t_{2^k}, \dots, t_{2^{k+1}-1})$  is the bitwise complement of  $(t_0, \dots, t_{2^k-1})$ , so  $T_{k+1} = T_k \overline{T_k}$ . Hence the infinite sequence from (b) satisfies the block construction (d).

Conversely, the infinite word produced in (d) is uniquely determined by the rule  $u_{k+1} = u_k \overline{u_k}$  and the starting word  $u_0 = 0$ , so (b) and (d) define the same sequence.

**(c)  $\iff$  (d).** Let  $\mu$  be the substitution  $0 \mapsto 01$ ,  $1 \mapsto 10$  of (c), and let  $u_k$  be as in (d). We claim that

$$u_{k+1} = \mu(u_k) \quad \text{for all } k \geq 0.$$

Indeed, for a single letter  $a \in \{0, 1\}$  we have

$$\mu(a) = a(1-a),$$

so for a word  $u = u_0 u_1 \dots u_{m-1}$  we obtain

$$\mu(u) = u_0(1-u_0) u_1(1-u_1) \dots u_{m-1}(1-u_{m-1}) = u \overline{u}.$$

Therefore

$$u_{k+1} = u_k \overline{u_k} = \mu(u_k),$$

and by induction we get  $u_k = \mu^k(0)$ . Thus the sequence of words  $(u_k)$  in (d) coincides with the sequence  $(w_k)$  in (c), and the corresponding infinite words (and hence the sequences  $(t_n)$ ) are the same.

**(c)  $\implies$  (a).** Let  $(v_n)_{n \geq 0}$  be the sequence defined by (c), i.e. the  $n$ -th symbol of the unique infinite word  $w$  with  $\mu(w) = w$  and  $w$  starting with 0. Each letter  $a$  of

$w$  produces in  $\mu(w)$  the two-letter block  $a(1-a)$ . Therefore, in positions  $2n$  and  $2n+1$  of  $w$  we see exactly the two letters generated from the letter in position  $n$ :

$$v_{2n} = v_n, \quad v_{2n+1} = 1 - v_n.$$

Also  $v_0 = 0$ . Hence  $(v_n)$  satisfies the recursion in (a), and by the uniqueness argument above,  $(v_n)$  coincides with  $(t_n)$  from (a).

Combining (a)  $\iff$  (b), (b)  $\iff$  (d), and (c)  $\iff$  (d), we conclude that all four descriptions define one and the same sequence  $(t_n)_{n \geq 0}$ , the Thue–Morse sequence.  $\square$

### Non-periodicity (without proofs)

**Remark 7.** The Thue–Morse sequence  $(t_n)$  is not eventually periodic. That is, there do not exist integers  $N \geq 0$  and  $p \geq 1$  such that

$$t_{n+p} = t_n \quad \text{for all } n \geq N.$$

**Remark 8.** In fact, the Thue–Morse sequence is much stronger: it contains no *overlaps* and in particular no substring of the form  $xxx$  where  $x$  is any nonempty finite word (it is *cube-free*). This deep property goes back to work of A. Thue and is a cornerstone of combinatorics on words.

### Optional reading: Gauss's arithmetic–geometric mean iteration

**Definition 10** (Arithmetic and geometric means). For two positive real numbers  $a, b > 0$  define their *arithmetic mean* and *geometric mean* by

$$A(a, b) = \frac{a+b}{2}, \quad G(a, b) = \sqrt{ab}.$$

**Definition 11** (Gauss (AGM) sequences). Fix  $a_0, b_0 > 0$ . Define two sequences  $(a_n)$  and  $(b_n)$  by

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

These are called the *arithmetic* and *geometric* mean iterations.

The following facts are classical and fundamental.

**Proposition 6** (Monotonicity). For all  $n \geq 0$ ,

$$a_{n+1} \leq a_n, \quad b_{n+1} \geq b_n.$$

Thus  $(a_n)$  is decreasing and  $(b_n)$  is increasing.

*Proof.* Since  $a_n \geq b_n > 0$ , the arithmetic–geometric mean inequality gives

$$\sqrt{a_n b_n} \leq \frac{a_n + b_n}{2}.$$

Thus

$$b_{n+1} \leq a_{n+1}.$$

But clearly

$$a_{n+1} = \frac{a_n + b_n}{2} \leq \frac{a_n + a_n}{2} = a_n,$$

and

$$b_{n+1} = \sqrt{a_n b_n} \geq \sqrt{b_n b_n} = b_n.$$

□

**Proposition 7** (Squeezing). For all  $n$ ,

$$b_n \leq a_n.$$

Moreover, the difference  $a_n - b_n$  tends to 0.

*Proof.* The inequality  $b_n \leq a_n$  is preserved by the iteration:

$$a_{n+1} - b_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n} = \frac{(\sqrt{a_n} - \sqrt{b_n})^2}{2} \geq 0.$$

The right-hand side is 0 exactly when  $a_n = b_n$ .

Also,

$$a_{n+1} - b_{n+1} = \frac{(\sqrt{a_n} - \sqrt{b_n})^2}{2} \leq \frac{a_n - b_n}{2},$$

so the positive sequence  $(a_n - b_n)$  is decreasing and is bounded below by 0, hence convergent. The last inequality shows that its only possible limit is 0. □

**Theorem 7** (Convergence to a common limit: the AGM). The sequences  $(a_n)$  and  $(b_n)$  converge to the same limit:

$$a_n \longrightarrow M(a_0, b_0), \quad b_n \longrightarrow M(a_0, b_0),$$

where  $M(a_0, b_0)$  is called the *arithmetic–geometric mean* of  $a_0$  and  $b_0$ .

*Proof.* We already know:

- $(a_n)$  is decreasing and bounded below by 0, hence convergent;
- $(b_n)$  is increasing and bounded above by  $a_0$ , hence convergent;
- $a_n - b_n \rightarrow 0$ .

Let

$$\lim_{n \rightarrow \infty} a_n = L_a, \quad \lim_{n \rightarrow \infty} b_n = L_b.$$

Since  $a_n - b_n \rightarrow 0$  we have  $L_a = L_b$ . Denote this common limit by

$$M(a_0, b_0).$$

□

**Remark 9.** Gauss showed that the AGM has deep connections with elliptic integrals:

$$\pi = 2 M(1, \sqrt{1 - k^2}) \cdot \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta.$$

This relation lies at the heart of the fastest algorithms for computing  $\pi$ .

## 2.5.1 :: :: problems for tutorial: powers of $1 \pm \sqrt{2}$ and almost-integers

**Problem 1** (Uniqueness of representation). Let  $a, b, c, d \in \mathbb{Q}$  and suppose

$$a + b\sqrt{2} = c + d\sqrt{2}.$$

Prove that  $a = c$  and  $b = d$ .

**Problem 2.** For each  $n \geq 0$  write

$$(1 - \sqrt{2})^n = a_n - b_n\sqrt{2}$$

with real numbers  $a_n, b_n$ .

1. Compute explicitly  $a_0, b_0$ , then  $a_1, b_1$ , then  $a_2, b_2$ .
2. Using the binomial theorem or induction on  $n$ , show that in fact  $a_n, b_n \in \mathbb{Z}$  for all  $n \geq 0$ .
3. Show that the sequences  $(a_n)$  and  $(b_n)$  satisfy the same recurrence

$$x_{n+2} = 2x_{n+1} - x_n$$

with appropriate initial conditions.

4. Show that for all  $n \geq 0$  the same integers  $a_n, b_n$  from the previous problem also satisfy

$$(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}.$$

**Problem 3** (Integer part of the sum). Let

$$u_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n.$$

1. Express  $u_n$  in terms of  $a_n$  and deduce that  $u_n \in \mathbb{Z}$  for all  $n$ .
2. Find a recurrence relation for  $u_n$  (it should again be order 2).
3. Compute the first several values of  $u_n$  (for example, for  $n = 0, 1, 2, 3, 4$ ).
4. Let  $k$  be a positive integer. Show that if

$$n > k \log_2 10,$$

then  $|(\sqrt{2} - 1)^n| < 10^{-k}$ .

5. Show that for such even  $n$  the first  $k$  digits to the right of the decimal point in  $(1 + \sqrt{2})^n$  are nines. Solutions are in Section 5.4

## 2.5.2 :: :: problems for workshop: second-order linear recurrences

Consider a second-order linear recurrence  $u_{n+2} = c_1 u_{n+1} + c_2 u_n$ . Its characteristic polynomial is  $z^2 - c_1 z - c_2$ , and knowledge of its roots gives a closed form for  $(u_n)$ .

- If the characteristic polynomial has two distinct roots  $z_1, z_2$ , then

$$u_n = Az_1^n + Bz_2^n$$

for uniquely determined constants  $A, B$  (complex in general), obtained from  $u_0, u_1$ .

- If the characteristic roots are complex conjugates  $r e^{\pm i\varphi}$  and  $u_0, u_1$  are real, then

$$u_n = r^n (A \cos(n\varphi) + B \sin(n\varphi)) \quad (A, B \in \mathbb{R}).$$

- If the characteristic polynomial has a double root  $r$ , then

$$u_n = (A + Bn)r^n.$$

### 1. (Distinct complex roots; modulus–argument form)

Solve the recurrence

$$u_{n+2} - 6u_{n+1} + 10u_n = 0, \quad u_0 = 2, \quad u_1 = 1,$$

and express the answer in the form

$$u_n = R \cdot \rho^n \cos(n\varphi - \theta).$$

### 2. (Repeated root)

Consider the recurrence

$$u_{n+2} - 2\lambda u_{n+1} + \lambda^2 u_n = 0, \quad u_0 = \alpha, \quad u_1 = \beta, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

- Prove that  $u_n = (A + Bn)\lambda^n$ .
- Express  $A, B$  in terms of  $\alpha, \beta, \lambda$ .
- Solve

$$u_{n+2} - 6u_{n+1} + 9u_n = 0, \quad u_0 = 2, \quad u_1 = 9.$$

## Solution.

The characteristic equation is

$$z^2 - 6z + 10 = 0.$$

The roots are

$$z = 3 \pm i.$$

Write them in polar form:

$$3+i = \rho e^{i\varphi}, \quad \rho = \sqrt{3^2+1} = \sqrt{10}, \quad \varphi = \arctan(1/3).$$

Hence

$$u_n = A(3+i)^n + B(3-i)^n = \rho^n (C e^{in\varphi} + \bar{C} e^{-in\varphi}) = \rho^n (\alpha \cos(n\varphi) + \beta \sin(n\varphi))$$

with real  $\alpha, \beta$ .

Use initial conditions:

$$u_0 = 2 = \alpha, \quad u_1 = 1 = \rho(\alpha \cos \varphi + \beta \sin \varphi).$$

Thus

$$1 = \sqrt{10}(2 \cos \varphi + \beta \sin \varphi),$$

so

$$\beta = \frac{1}{\sqrt{10} \sin \varphi} - 2 \cot \varphi = \frac{1}{1} - 2 \cdot \frac{\cos \varphi}{\sin \varphi}$$

(using  $\sin \varphi = \frac{1}{\sqrt{10}}$  and  $\cos \varphi = \frac{3}{\sqrt{10}}$ ). Hence

$$\beta = 1 - 2 \cdot 3 = -5.$$

Thus

$$u_n = (\sqrt{10})^n (2 \cos(n\varphi) - 5 \sin(n\varphi)).$$

Now rewrite in amplitude-phase form:

$$a \cos(n\varphi) + b \sin(n\varphi) = \sqrt{a^2 + b^2} \cos(n\varphi - \theta), \quad \theta = \arctan \frac{b}{a}.$$

Here

$$R = \sqrt{2^2 + (-5)^2} = \sqrt{29}, \quad \theta = \arctan \left( \frac{-5}{2} \right).$$

$$u_n = \sqrt{29} (\sqrt{10})^n \cos(n\varphi - \theta), \quad \varphi = \arctan(1/3).$$

### 2.5.3 :: :: 8th homework: Collatz sequence

**Definition 12** (Collatz map). Define  $T : \mathbb{N}_{\geq 1} \rightarrow \mathbb{N}_{\geq 1}$  by

$$T(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ 3n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

For  $k \geq 0$ , let  $T^k$  denote the  $k$ -fold iterate of  $T$  and the *trajectory* of  $n$  be  $(n, T(n), T^2(n), \dots)$ .

**Very famous open Collatz (or  $3n+1$ ) conjecture:** For every  $n \in \mathbb{N}_{\geq 1}$  there exists  $k \geq 0$  such that  $T^k(n) = 1$ . Equivalently, every trajectory reaches the cycle  $1 \mapsto 4 \mapsto 2 \mapsto 1$ .

1. ~~Prove Collatz conjecture~~ Starting from 18, apply the Collatz map until you hit 1.

**(Complex conjugate roots with real data)** Let  $u_{n+2} - 2u_{n+1} + 5u_n = 0$ , with  $u_0 = 0$ ,  $u_1 = 1$ .

- (a) Find a closed form for  $u_n$  using only real functions.
- (b) Show that for all  $n \geq 0$ ,

$$u_{n+1}^2 + 5u_n^2 - 2u_{n+1}u_n = 5^n.$$

*Hint:* Either use the closed form, or consider  $z_n = (1 + 2i)^n$  and relate  $|z_n|^2 = 5^n$  to  $u_n$ .

2. **(Minimal polynomial from samples)** A sequence  $(u_n)$  satisfies a second-order linear homogeneous recurrence with constant coefficients and has values  $u_0 = 1$ ,  $u_1 = 2$ ,  $u_2 = 5$ ,  $u_3 = 12$ .

- (a) Find the recurrence.
- (b) Decide whether it has a double root or distinct complex roots.
- (c) Solve for  $u_n$ .

**Problem 1** (Complex conjugate roots with real data). Let  $(u_n)_{n \geq 0}$  satisfy

$$u_{n+2} - 2u_{n+1} + 5u_n = 0, \quad u_0 = 0, \quad u_1 = 1.$$

1. Find a closed form for  $u_n$  using only real functions.
2. Show that for all  $n \geq 0$ ,

$$u_{n+1}^2 + 5u_n^2 - 2u_{n+1}u_n = 5^n.$$

*Proof.* (1) **Closed form.** The characteristic polynomial is

$$r^2 - 2r + 5 = 0.$$

Its roots are

$$r_{1,2} = 1 \pm 2i.$$

Thus the general complex solution is

$$u_n = A(1+2i)^n + B(1-2i)^n,$$

for some constants  $A, B \in \mathbb{C}$ . Since the initial data  $u_0, u_1$  are real and the roots are complex conjugate, we will get a real sequence for  $B = \overline{A}$ , but we can instead solve directly from  $u_0, u_1$ .

From  $u_0 = 0$  we get

$$0 = u_0 = A + B \Rightarrow B = -A.$$

From  $u_1 = 1$  we get

$$1 = u_1 = A(1+2i) + B(1-2i) = A(1+2i) - A(1-2i) = A((1+2i) - (1-2i)) = A(4i).$$

Hence

$$A = \frac{1}{4i}.$$

Therefore

$$u_n = \frac{(1+2i)^n - (1-2i)^n}{4i}.$$

Now write  $1+2i$  in polar form. Its modulus is

$$\rho = |1+2i| = \sqrt{1^2 + 2^2} = \sqrt{5},$$

and we can choose an angle  $\varphi$  with

$$\cos \varphi = \frac{1}{\sqrt{5}}, \quad \sin \varphi = \frac{2}{\sqrt{5}}, \quad \varphi = \arctan 2.$$

Then

$$1 + 2i = \sqrt{5} e^{i\varphi}, \quad 1 - 2i = \sqrt{5} e^{-i\varphi}.$$

Hence

$$(1 + 2i)^n = (\sqrt{5})^n e^{in\varphi}, \quad (1 - 2i)^n = (\sqrt{5})^n e^{-in\varphi}.$$

Substituting into the formula for  $u_n$ ,

$$\begin{aligned} u_n &= \frac{(\sqrt{5})^n e^{in\varphi} - (\sqrt{5})^n e^{-in\varphi}}{4i} = \frac{(\sqrt{5})^n}{4i} (e^{in\varphi} - e^{-in\varphi}) \\ &= \frac{(\sqrt{5})^n}{4i} \cdot 2i \sin(n\varphi) = \frac{(\sqrt{5})^n}{2} \sin(n\varphi), \end{aligned}$$

where  $\varphi = \arctan 2$ .

Thus a closed real form is

$$u_n = \frac{(\sqrt{5})^n}{2} \sin(n \arctan 2).$$

**(2) Quadratic identity.** We will prove

$$u_{n+1}^2 + 5u_n^2 - 2u_{n+1}u_n = 5^n \quad \text{for all } n \geq 0,$$

using a complex-number interpretation.

Define

$$z_n = (1 + 2i)^n.$$

Write  $z_n$  in terms of its real and imaginary parts:

$$z_n = a_n + ib_n, \quad a_n, b_n \in \mathbb{R}.$$

Then

$$|z_n|^2 = a_n^2 + b_n^2 = |1 + 2i|^{2n} = 5^n.$$

On the other hand, from part (1) we have

$$u_n = \frac{(1 + 2i)^n - (1 - 2i)^n}{4i} = \frac{z_n - \overline{z_n}}{4i} = \frac{b_n}{2},$$

since  $z_n - \overline{z_n} = 2ib_n$ . Thus

$$b_n = 2u_n.$$

Next, compute  $z_{n+1}$ :

$$z_{n+1} = (1 + 2i)z_n = (1 + 2i)(a_n + ib_n) = (a_n - 2b_n) + i(2a_n + b_n).$$

Hence

$$b_{n+1} = 2a_n + b_n.$$

Using  $u_{n+1} = b_{n+1}/2$  and  $u_n = b_n/2$ , we get

$$u_{n+1} = \frac{b_{n+1}}{2} = \frac{2a_n + b_n}{2} = a_n + \frac{b_n}{2} = a_n + u_n.$$

Thus

$$a_n = u_{n+1} - u_n.$$

Now plug  $a_n$  and  $b_n$  into  $|z_n|^2$ :

$$5^n = a_n^2 + b_n^2 = (u_{n+1} - u_n)^2 + (2u_n)^2.$$

Expanding,

$$5^n = u_{n+1}^2 - 2u_{n+1}u_n + u_n^2 + 4u_n^2 = u_{n+1}^2 + 5u_n^2 - 2u_{n+1}u_n.$$

This is exactly the desired identity:

$$\boxed{u_{n+1}^2 + 5u_n^2 - 2u_{n+1}u_n = 5^n \quad \text{for all } n \geq 0.}$$

This completes the proof. □

## 2.6 :: countable and uncountable sets, Cantor–Bernstein theorem, restricted infinite product

**Simple shifts on countable sets.** Define

$$f : \mathbb{N} \rightarrow \{2, 3, 4, \dots\}, \quad f(n) = n + 1.$$

This is clearly a bijection: every integer  $k \geq 2$  has a unique preimage  $k - 1 \in \mathbb{N}$ .

This illustrates a basic idea:

$$\mathbb{N} \sim \mathbb{N} \setminus \{1\}.$$

Here the notation “ $A \sim B$ ” means: *there exists a bijection between A and B*.

**A bijection between  $(0, 1]$  and  $(0, 1)$ .** We now construct an explicit bijection

$$(0, 1] \longrightarrow (0, 1).$$

Consider the countable set

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N}, n \geq 1 \right\} \subset (0, 1].$$

We will “shift” this set one step to the right.

Define  $f : (0, 1] \rightarrow (0, 1)$  by

$$f(x) = \begin{cases} \frac{1}{n+1}, & \text{if } x = \frac{1}{n} \text{ for some } n \geq 1, \\ x, & \text{if } x \notin S. \end{cases}$$

- $f$  is injective: If  $x \notin S$ , then  $f(x) = x$ , and these values are never equal to any  $\frac{1}{n+1}$ . If  $x = \frac{1}{n}$  and  $y = \frac{1}{m}$  with  $n \neq m$ , then  $f(x) = \frac{1}{n+1} \neq \frac{1}{m+1} = f(y)$ .
- $f$  is surjective: Let  $y \in (0, 1)$ . If  $y \notin \{1/(n+1) : n \geq 1\}$ , then  $y \notin f(S)$ , so  $f(y) = y$ . If  $y = \frac{1}{n+1}$ , then  $y = f(\frac{1}{n})$ .

Hence  $f$  is a bijection and

$$(0, 1] \sim (0, 1).$$

## Homotheties and translations in the plane

**Definition 13** (Homothety). Let  $\lambda > 0$ . The *homothety* with center at the origin and ratio  $\lambda$  is the map

$$H_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad H_\lambda(x) = \lambda x.$$

It is a bijection with inverse  $H_{1/\lambda}$ .

For example, if

$$D_r = \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\| = \sqrt{x_1^2 + x_2^2} < r\}$$

is the open disc of radius  $r$  centered at the origin, then

$$H_\lambda : D_r \longrightarrow D_{\lambda r}$$

is a bijection between discs of different radii.

**Definition 14** (Parallel transport (translation)). For a fixed vector  $v \in \mathbb{R}^2$ , the *translation* by  $v$  is

$$T_v : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T_v(x) = x + v.$$

This is a bijection with inverse  $T_{-v}$ .

Thus if  $D$  and  $D'$  are two discs of the same radius with different centers, a suitable translation  $T_v$  gives a bijection  $D \rightarrow D'$ . Similarly one constructs a bijection between two discs with arbitrarily radii and centers.

## Disc with and without boundary

Let

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

be the open unit disc, and

$$\overline{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

the closed unit disc (disc with boundary). We want a bijection

$$\overline{D} \longrightarrow D.$$

Idea: mimic the “shifting a countable set” trick using a countable family of concentric circles with radii  $\{\frac{1}{n}\}$ ,  $n = 1, 2, 3, \dots$ .

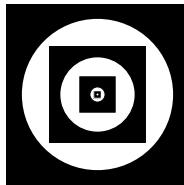
## Cantor–Schröder–Bernstein theorem

**Theorem 8** (Cantor–Bernstein (Schröder–Bernstein)). Let  $A, B$  be sets. Suppose there are injective maps

$$f : A \rightarrow B, \quad g : B \rightarrow A.$$

Then there exists a bijection  $h : A \rightarrow B$ .

We will not prove this theorem in this course. The idea of the proof can be indicated on the example with a square and a disc. Fix injections, and apply them many times (see picture below). Now the bijection from the square to the disc is given by a constant map on the white parts and homothety shift inside of the black parts:



(A full proof will appear in the Set Theory course.) Instead we use it as a powerful tool:

*To show that  $|A| = |B|$ , it suffices to construct injections both ways.*

### Example: $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$

We show that

$$\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$$

by constructing *two injections* and then applying the Cantor–Schröder–Bernstein theorem.

**Injection  $\mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ .** Define

$$f : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}, \quad f(n) = (n, 1).$$

If  $f(n) = f(m)$ , then  $(n, 1) = (m, 1)$ , hence  $n = m$ . So  $f$  is injective.

**Injection**  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . Use the Fundamental Theorem of Arithmetic. For each pair  $(m, n) \in \mathbb{N} \times \mathbb{N}$ , set

$$g(m, n) = 2^m 3^n \in \mathbb{N}.$$

If  $g(m, n) = g(m', n')$ , then

$$2^m 3^n = 2^{m'} 3^{n'}.$$

By uniqueness of prime factorization, we must have  $m = m'$  and  $n = n'$ . Thus  $g$  is injective.

**Conclusion.** We have constructed injections

$$f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \quad \text{and} \quad g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}.$$

By the Cantor–Schröder–Bernstein theorem, there exists a bijection

$$h : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}.$$

Hence  $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$ .

**Important corollary:** So a countable union of countable sets is countable.

### A bijection between $[0, 1]^2$ and $[0, 1]$

We now apply Cantor–Bernstein to show that

$$[0, 1]^2 \sim [0, 1].$$

**Step 1: Injection**  $[0, 1] \rightarrow [0, 1]^2$ . This is easy:

$$f : [0, 1] \rightarrow [0, 1]^2, \quad f(x) = (x, 0).$$

Clearly injective.

**Step 2: Injection**  $[0, 1]^2 \rightarrow [0, 1]$ . We first build an injection  $(0, 1)^2 \rightarrow (0, 1)$  by interleaving binary digits.

Write

$$x = 0.x_1 x_2 x_3 \dots, \quad y = 0.y_1 y_2 y_3 \dots$$

in binary (avoiding the ambiguous expansions ending with repeating 1's). Define

$$\varphi(x, y) = 0.x_1 y_1 x_2 y_2 x_3 y_3 \dots$$

(again in binary). Then  $\varphi : (0, 1)^2 \rightarrow (0, 1)$  is injective.

To extend this to  $[0, 1]^2 \rightarrow [0, 1]$ , we can modify finitely or countably many special points (those with two binary expansions), or simply note:

- There is an injection  $[0, 1]^2 \rightarrow (0, 1)^2$  (for instance, squeeze the square slightly into its interior).
- There is an injection  $(0, 1)^2 \rightarrow (0, 1)$  (the map  $\varphi$  above).
- There is an injection  $(0, 1) \rightarrow [0, 1]$  (inclusion).

Composing these, we obtain an injection

$$g : [0, 1]^2 \rightarrow [0, 1].$$

**Step 3: Apply Cantor–Bernstein.** We now have:

an injection  $[0, 1] \rightarrow [0, 1]^2$  and an injection  $[0, 1]^2 \rightarrow [0, 1]$ .

By Cantor–Bernstein, there exists a bijection

$$h : [0, 1]^2 \longrightarrow [0, 1].$$

## Other examples

Using similar ideas and the Cantor–Bernstein theorem, one can show:

- $\mathbb{R} \sim (0, 1) \sim (0, 1)^2 \sim \mathbb{R}^2 \sim \mathbb{R}^n$  for any fixed  $n \geq 1$ .
- Any nonempty open interval  $I \subset \mathbb{R}$  has the same cardinality as  $\mathbb{R}$ .
- Any open disc in the plane has the same cardinality as  $\mathbb{R}$  and as the entire plane  $\mathbb{R}^2$ .

The key patterns to remember and practice in everyday life:

1. Removing or adding a finite or countable set (e.g. a boundary, a few special points) does not change cardinality.
2. Cantor–Bernstein allows us to turn *two* injections into a bijection.

**Example 3.** Let  $I = \mathbb{N}$  and let all  $A_i = \mathbb{Z}$ , where  $\mathbb{Z}$  is considered as a set with the distinguished point 0. Recall that the restricted direct product

$$\prod_i^* A_i$$

consists of all families  $(a_i)_{i \in \mathbb{N}}$  such that  $a_i = 0$  for all but finitely many  $i$ .

1. Verify that  $\prod_i^* A_i$  is countable.
2. Verify that the full product  $\prod_i A_i = \mathbb{Z}^{\mathbb{N}}$  has the cardinality of the continuum.

*Proof.* **(1) The restricted product is countable.**

By the definition of the restricted product with the distinguished point 0, we have

$$\prod_i^* A_i = \left\{ (a_i)_{i \in \mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} \mid a_i = 0 \text{ for all but finitely many } i \right\}.$$

In other words, each element is a sequence of integers which is zero from some point on.

For each  $k \geq 0$  define

$$S_k = \left\{ (a_i)_{i \in \mathbb{N}} \in \prod_i^* A_i \mid a_i = 0 \text{ for all } i > k \right\}.$$

Then every sequence with finite support belongs to some  $S_k$ , so

$$\prod_i^* A_i = \bigcup_{k \geq 0} S_k.$$

For fixed  $k$ , the set  $S_k$  is in bijection with  $\mathbb{Z}^k$  via

$$(a_1, \dots, a_k, 0, 0, \dots) \longleftrightarrow (a_1, \dots, a_k).$$

Since  $\mathbb{Z}$  is countable, so is every finite power  $\mathbb{Z}^k$ . Hence each  $S_k$  is countable.

Therefore  $\prod_i^* A_i$  is a countable union of countable sets, so it is countable.

**(2) The full product has the cardinality of the continuum.**

The full product

$$\prod_i A_i = \mathbb{Z}^{\mathbb{N}}$$

is the set of all sequences of integers  $(a_i)_{i \in \mathbb{N}}$ .

First, we show that  $\mathbb{Z}^{\mathbb{N}}$  is *uncountable*. Consider the subset

$$\{0, 1\}^{\mathbb{N}} = \{(a_i)_{i \in \mathbb{N}} : a_i \in \{0, 1\}\} \subset \mathbb{Z}^{\mathbb{N}}.$$

The set  $\{0, 1\}^{\mathbb{N}}$  is in bijection with the power set  $\mathcal{P}(\mathbb{N})$  via characteristic functions:

$$E \subset \mathbb{N} \iff (\chi_E(i))_{i \in \mathbb{N}},$$

where  $\chi_E(i) = 1$  if  $i \in E$  and 0 otherwise. Thus

$$|\{0, 1\}^{\mathbb{N}}| = |\mathcal{P}(\mathbb{N})|.$$

By Cantor's theorem,  $\mathcal{P}(\mathbb{N})$  is uncountable, so  $\{0, 1\}^{\mathbb{N}}$  is uncountable, and hence  $\mathbb{Z}^{\mathbb{N}}$  is uncountable as well.

Next, we show that  $\mathbb{Z}^{\mathbb{N}}$  has the cardinality of the continuum (the same as  $\mathbb{R}$ ). Since  $\mathbb{Z}$  is countable, there is a bijection  $\mathbb{Z} \cong \mathbb{N}$ , which induces a bijection

$$\mathbb{Z}^{\mathbb{N}} \cong \mathbb{N}^{\mathbb{N}}.$$

It is a standard result that

$$|\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|$$

(e.g. by encoding infinite sequences of natural numbers as real numbers in  $(0, 1)$  via base expansions, and conversely by sending  $(a_i)$  to  $\sum_{n=1}^{\infty} 10^{-\sum_i^n a_i}$ , then Cantor-Bernstein theorem).

Thus

$$|\mathbb{Z}^{\mathbb{N}}| = |\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|,$$

i.e. the full product  $\prod_i A_i$  has the cardinality of the continuum.  $\square$

**Theorem 9** (Cantor). The set of real numbers in the interval  $(0, 1)$  is uncountable.

*Proof.* Suppose, for contradiction, that the set  $(0, 1)$  is countable. Then all real numbers in  $(0, 1)$  can be listed as a sequence

$$r_1, r_2, r_3, \dots$$

We will derive a contradiction by constructing a real number in  $(0, 1)$  that is not equal to any  $r_n$ .

Write each  $r_n$  in its decimal expansion:

$$r_1 = 0.d_{11}d_{12}d_{13}\dots, \quad r_2 = 0.d_{21}d_{22}d_{23}\dots, \quad r_3 = 0.d_{31}d_{32}d_{33}\dots, \quad \dots$$

where each  $d_{nk} \in \{0, 1, \dots, 9\}$  is the  $k$ -th digit of  $r_n$  after the decimal point.

We may assume that none of the  $r_n$  has a decimal expansion ending in infinitely many 9's (for example, we write 0.5000... instead of 0.4999...). This is always possible, because every real number has at least one such expansion.

Now we construct a new real number  $x \in (0, 1)$  by specifying its decimal digits:

$$x = 0.c_1c_2c_3\dots,$$

where, for each  $n \geq 1$ , we define  $c_n$  by

$$c_n = \begin{cases} 1, & \text{if } d_{nn} \neq 1, \\ 2, & \text{if } d_{nn} = 1. \end{cases}$$

In particular,  $c_n \in \{1, 2\}$  and  $c_n \neq d_{nn}$  for every  $n$ .

We claim that  $x$  is not equal to any  $r_n$  in the list. Fix  $n \geq 1$ . Then  $r_n$  has decimal expansion  $0.d_{n1}d_{n2}\dots$ , while  $x$  has decimal expansion  $0.c_1c_2\dots$ . By construction we have  $c_n \neq d_{nn}$ , so  $x$  and  $r_n$  differ at least in the  $n$ -th digit after the decimal point. Hence  $x \neq r_n$ .

Since  $n$  was arbitrary,  $x$  is a real number in  $(0, 1)$  that does not appear in the list  $r_1, r_2, r_3, \dots$ . This contradicts our assumption that the list contains all real numbers in  $(0, 1)$ .

Therefore  $(0, 1)$  is not countable; that is, it is uncountable. In particular, the set of all real numbers  $\mathbb{R}$  is uncountable as well.  $\square$

## 2.6.1 :: :: problems for tutorial: explicit bijections

**Problem 1.** Construct an explicit bijection

$$\mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}.$$

*Idea:* Arrange pairs  $(m, n)$  in a table and follow diagonals.

**Problem 2.** Construct an explicit bijection between  $\mathbb{N}$  and the set of all finite sequences of natural numbers:

$$\mathcal{F} = \bigcup_{n \geq 0} \mathbb{N}^n.$$

*Hint:* Use prime factorizations: encode  $(a_1, \dots, a_k)$  as  $2^{a_1+1}3^{a_2+1}\cdots p_k^{a_k+1}$ , where  $p_k$  is the  $k$ -th prime.

**Problem 3.** Construct an explicit bijection between the open interval  $(0, 1)$  and the real line  $\mathbb{R}$ .

*Hint:* First find a bijection  $(0, 1) \rightarrow (0, \infty)$  (e.g. using a rational function), then  $(0, \infty) \rightarrow \mathbb{R}$ .

Solutions are in Section 5.4

## 2.6.2 :: :: problems for workshop: binary strings

**Problem 1** (Vertices of the cube and subsets). Show that the set of vertices of the  $n$ -dimensional unit cube  $\{0, 1\}^n$  is in bijection with the set of all subsets of  $\{1, 2, \dots, n\}$ . Give the bijection explicitly in both directions.

**Problem 2** (Binary strings and lattice paths in a strip). Show that the set of binary strings of length  $n$  with exactly  $k$  ones is in bijection with the set of lattice paths from  $(0, 0)$  to  $(n - k, k)$  using steps  $(1, 0)$  and  $(0, 1)$ .

**Problem 3** (Lattice paths and binomial coefficients). Consider lattice paths on the square grid from  $(0, 0)$  to  $(m, n)$  that only move one step right or one step up. Construct a bijection between such paths and  $m$ -element subsets of a set of size  $m + n$ . Deduce that the number of such paths is  $\binom{m+n}{m}$ .

**Problem 4.** Let  $n \in \mathbb{N}$  have binary expansion

$$n = \sum_{k=0}^m \varepsilon_k 2^k, \quad \varepsilon_k \in \{0, 1\}.$$

Thus  $\varepsilon_0$  is the least significant bit,  $\varepsilon_1$  is the next bit, etc.

Fix  $k \geq 0$  and consider the integer

$$q_k = \left\lfloor \frac{n}{2^k} \right\rfloor.$$

Prove that the  $(k + 1)$ -st binary digit  $\varepsilon_k$  of  $n$  is determined by the parity of  $q_k$ :

$$\varepsilon_k \equiv q_k \pmod{2}.$$

In other words, the  $(k + 1)$ -st bit of  $n$  equals 1 if and only if  $\lfloor n/2^k \rfloor$  is odd, and equals 0 if and only if  $\lfloor n/2^k \rfloor$  is even.

:: :: **homework**

1. Prove that the union of a finite set and a countable set is countable.
2. A sequence  $(s_n)_{n=0}^{\infty}$  of natural numbers is called *eventually constant* if there exist  $n_0 \in \mathbb{N}$  and  $s \in \mathbb{N}$  such that  $s_n = s$  for all  $n \geq n_0$ . Show that the set of eventually constant sequences of natural numbers is countable.
3. A sequence  $(s_n)_{n=0}^{\infty}$  of natural numbers is (eventually) *periodic* if there exist  $n_0, p \in \mathbb{N}$  with  $p \geq 1$  such that for all  $n \geq n_0$  we have  $s_{n+p} = s_n$ . Show that the set of all periodic sequences of natural numbers is countable.
4. A sequence  $(s_n)_{n=0}^{\infty}$  of natural numbers is called an *arithmetic progression* if there exists  $d \in \mathbb{N}$  such that  $s_{n+1} = s_n + d$  for all  $n \in \mathbb{N}$ . Prove that the set of all arithmetic progressions is countable.

### 3 Elementary number theory, equivalence relation, and real numbers

I am able to discuss the series “If the liver is a mirror image of the sky” with capable scholars. I can solve convoluted reciprocals and calculations that do not come out evenly. I have read cunningly written text in Sumerian, obscure Akkadian, the interpretation of which is difficult.

Aššurbanipal, Assirian king, 685–631 BC

#### 3.1 :: remainders, congruences

We begin with the division algorithm.

**Division with remainder.** For any integer  $m$  and any natural number  $n$ , there exist unique integers  $k$  and  $r$  such that

$$m = kn + r, \quad 0 \leq r < n.$$

The number  $r$  is called the *remainder* of  $m$  upon division by  $n$ .

**Lemma.** The remainder upon division by  $n$  of the sum and of the product of two integers  $m_1, m_2$  depends only on the remainders of  $m_1$  and  $m_2$  upon division by  $n$ .

**Proof.** Write

$$m_1 = k_1n + r_1, \quad m_2 = k_2n + r_2,$$

with  $0 \leq r_1, r_2 < n$ .

Then

$$m_1 + m_2 = (k_1 + k_2)n + (r_1 + r_2).$$

Thus the remainder of  $m_1 + m_2$  upon division by  $n$  is the same as the remainder of  $r_1 + r_2$ .

Similarly,

$$m_1 m_2 = (k_1 n + r_1)(k_2 n + r_2) = k_1 k_2 n^2 + (k_1 r_2 + k_2 r_1)n + r_1 r_2.$$

Hence the remainder of  $m_1 m_2$  upon division by  $n$  is the same as the remainder of  $r_1 r_2$ .

This proves the lemma. □

The lemma motivates the following fundamental notion.

**Definition (congruence).** Let  $n$  be a natural number. Two integers  $a$  and  $b$  are called *congruent modulo  $n$* , written

$$a \equiv b \pmod{n},$$

if  $a$  and  $b$  have the same remainder upon division by  $n$ , that is, if  $a - b$  is divisible by  $n$ .

### Basic properties of congruences.

For all integers  $a, b, c$  and any natural number  $n$ :

1.  $a \equiv a \pmod{n}$ .
2. If  $a \equiv b \pmod{n}$ , then  $b \equiv a \pmod{n}$ .
3. If  $a \equiv b \pmod{n}$  and  $b \equiv c \pmod{n}$ , then  $a \equiv c \pmod{n}$ .
4. If  $a \equiv b \pmod{n}$  and  $c \equiv d \pmod{n}$ , then

$$a + c \equiv b + d \pmod{n}.$$

5. If  $a \equiv b \pmod{n}$  and  $c \equiv d \pmod{n}$ , then

$$ac \equiv bd \pmod{n}.$$

**Proof.** The first three properties follow directly from the definition.

For addition, write  $a - b = kn$  and  $c - d = \ell n$ . Then

$$(a + c) - (b + d) = (k + \ell)n,$$

so  $a + c \equiv b + d \pmod{n}$ .

For multiplication,

$$ac - bd = a(c - d) + d(a - b),$$

and both terms on the right-hand side are divisible by  $n$ , hence so is  $ac - bd$ . □

**Theorem 10** (Existence of a modular inverse). Let  $n$  be a natural number and let  $d$  be an integer such that

$$\gcd(d, n) = 1.$$

Then there exists an integer  $u$  such that

$$du \equiv 1 \pmod{n}.$$

In other words,  $d$  has a multiplicative inverse modulo  $n$ .

*Proof.* Since  $\gcd(d, n) = 1$ , by Bézout's identity there exist integers  $u$  and  $v$  such that

$$du + nv = 1.$$

Rewriting this equality gives

$$du = 1 - nv.$$

The right-hand side differs from 1 by a multiple of  $n$ . Therefore,

$$du \equiv 1 \pmod{n}.$$

This shows that the residue class of  $u$  modulo  $n$  is a multiplicative inverse of  $d$  modulo  $n$ .  $\square$

**Remark 10.** The integer  $u$  is not unique, but its remainder modulo  $n$  is unique. Indeed, if  $u_1$  and  $u_2$  both satisfy

$$du_1 \equiv du_2 \equiv 1 \pmod{n},$$

then

$$d(u_1 - u_2) \equiv 0 \pmod{n}.$$

Since  $\gcd(d, n) = 1$ , this implies  $u_1 - u_2$  is divisible by  $n$ , so  $u_1 \equiv u_2 \pmod{n}$ .

### Decimal representation of integers.

Any nonnegative integer  $m$  can be written in base 10 as

$$m = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_k \cdot 10^k,$$

where  $0 \leq a_i \leq 9$  are the digits of  $m$ .

We now apply congruences to divisibility tests.

### Divisibility by 3 and 9.

Since

$$10 \equiv 1 \pmod{3} \quad \text{and} \quad 10 \equiv 1 \pmod{9},$$

we have

$$10^k \equiv 1 \pmod{3}, \quad 10^k \equiv 1 \pmod{9}$$

for all  $k$ .

Therefore,

$$m \equiv a_0 + a_1 + \cdots + a_k \pmod{3},$$

and similarly modulo 9.

Hence:

- $m$  is divisible by 3 if and only if the sum of its digits is divisible by 3;
- $m$  is divisible by 9 if and only if the sum of its digits is divisible by 9.

### **Divisibility by 11.**

Since

$$10 \equiv -1 \pmod{11},$$

we have

$$10^k \equiv (-1)^k \pmod{11}.$$

Thus

$$m \equiv a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^k a_k \pmod{11}.$$

Therefore, an integer is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.

### **Divisibility by 37.**

Since

$$1000 = 10^3 \equiv 1 \pmod{37},$$

we may group the decimal expansion of  $m$  into blocks of three digits:

$$m = b_0 + b_1 \cdot 1000 + b_2 \cdot 1000^2 + \cdots,$$

where each  $b_i$  is an integer between 0 and 999.

Then

$$m \equiv b_0 + b_1 + b_2 + \cdots \pmod{37}.$$

Hence, an integer is divisible by 37 if and only if the sum of its three-digit blocks is divisible by 37.

### **Divisibility rules for powers of 2 and 5**

We first formulate simple divisibility rules for powers of 2 and 5.

#### **Divisibility by powers of 2.**

Let  $k \geq 1$ . An integer written in decimal form

$$m = a_0 + a_1 \cdot 10 + \cdots + a_r \cdot 10^r$$

is divisible by  $2^k$  if and only if the integer formed by its last  $k$  decimal digits is divisible by  $2^k$ .

**Proof.** We write

$$m = A \cdot 10^k + B,$$

where  $B$  is the number formed by the last  $k$  digits of  $m$ .

Since  $10^k = 2^k \cdot 5^k$  is divisible by  $2^k$ , the term  $A \cdot 10^k$  is divisible by  $2^k$ . Therefore, the divisibility of  $m$  by  $2^k$  depends only on  $B$ .  $\square$

### Divisibility by powers of 5.

Let  $k \geq 1$ . An integer  $m$  is divisible by  $5^k$  if and only if the integer formed by its last  $k$  decimal digits is divisible by  $5^k$ .

**Proof.** The proof is identical to the previous one, since

$$10^k = 2^k \cdot 5^k$$

is divisible by  $5^k$ .  $\square$

**Remark.** These rules explain the familiar tests:

- divisibility by 2 depends on the last digit;
- divisibility by 4 depends on the last two digits;
- divisibility by 5 depends on the last digit;
- divisibility by 25 depends on the last two digits.

### Existence of a power of 10 congruent to 1

We now prove a fundamental fact that leads to divisibility rules for many other numbers.

**Theorem.** Let  $n$  be a natural number such that  $\gcd(n, 10) = 1$  (that is,  $n$  is not divisible by 2 or 5). Then there exists a natural number  $N$  such that

$$10^N \equiv 1 \pmod{n}.$$

**Proof.** Consider the sequence of remainders obtained by dividing

$$10^1, 10^2, 10^3, \dots, 10^n$$

by  $n$ .

There are only  $n - 1$  possible nonzero remainders modulo  $n$ . Since  $\gcd(10, n) = 1$ , none of the powers  $10^k$  is divisible by  $n$ , so all these remainders are nonzero.

By the pigeonhole principle, two of the numbers  $10^i$  and  $10^j$  with  $1 \leq i < j \leq n$  have the same remainder modulo  $n$ . Thus

$$10^i \equiv 10^j \pmod{n}.$$

Subtracting,

$$10^i(10^{j-i} - 1) \equiv 0 \pmod{n}.$$

Since  $\gcd(10^i, n) = 1$ , we may cancel  $10^i$ , obtaining

$$10^{j-i} \equiv 1 \pmod{n}.$$

Setting  $N = j - i$  completes the proof. □

### Divisibility rules derived from $10^N \equiv 1 \pmod{n}$

Assume now that  $n$  is not divisible by 2 or 5, and let  $N$  be a natural number such that

$$10^N \equiv 1 \pmod{n}.$$

Write a nonnegative integer  $m$  in decimal form and group its digits into blocks of length  $N$ :

$$m = b_0 + b_1 \cdot 10^N + b_2 \cdot 10^{2N} + \dots,$$

where each  $b_i$  is an integer between 0 and  $10^N - 1$ .

Using the congruence  $10^N \equiv 1 \pmod{n}$ , we obtain

$$m \equiv b_0 + b_1 + b_2 + \dots \pmod{n}.$$

**Divisibility rule.** An integer  $m$  is divisible by  $n$  if and only if the sum of its blocks of  $N$  digits is divisible by  $n$ .

### Examples.

- For  $n = 3$  and  $n = 9$ , we may take  $N = 1$ , recovering the rule for the sum of digits.
- For  $n = 11$ , we may take  $N = 2$ , leading to a rule based on two-digit blocks.
- For  $n = 37$ , we may take  $N = 3$ , leading to a rule based on three-digit blocks.

**Example 4.** Check the divisibility of the number

$$M = 987\,654\,321\,012\,345$$

by 1665, and compute the remainder of  $M$  upon division by 1665.

Note that

$$5 \cdot 9 \cdot 37 = 1665.$$

We compute the remainders of  $M$  modulo 5, 9, and 37, and then combine them.

**Step 1. Remainder modulo 5.**

The last digit of  $M$  is 5. Hence

$$M \equiv 0 \pmod{5}.$$

**Step 2. Remainder modulo 9.**

Write  $M$  in decimal form and compute the sum of its digits:

$$9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 + 1 + 2 + 3 + 4 + 5 = 60.$$

Since

$$60 = 9 \cdot 6 + 6,$$

we obtain

$$M \equiv 6 \pmod{9}.$$

**Step 3. Remainder modulo 37.**

Since

$$1000 = 10^3 \equiv 1 \pmod{37},$$

we group the number into blocks of three digits:

$$M = 987 \cdot 10^{12} + 654 \cdot 10^9 + 321 \cdot 10^6 + 012 \cdot 10^3 + 345.$$

Using  $10^3 \equiv 1 \pmod{37}$ , we obtain

$$M \equiv 987 + 654 + 321 + 12 + 345 \pmod{37}.$$

Compute the sum:

$$987 + 654 = 1641, \quad 1641 + 321 = 1962, \quad 1962 + 12 = 1974, \quad 1974 + 345 = 2319.$$

Now divide:

$$2319 = 37 \cdot 62 + 25,$$

hence

$$M \equiv 25 \pmod{37}.$$

#### Step 4. Summary of remainders.

We have shown that

$$M \equiv 0 \pmod{5},$$

$$M \equiv 6 \pmod{9},$$

$$M \equiv 25 \pmod{37}.$$

#### Step 5. Computing the remainder modulo 1665.

We now find the unique integer  $r$ ,  $0 \leq r < 1665$ , such that

$$r \equiv 0 \pmod{5}, \quad r \equiv 6 \pmod{9}, \quad r \equiv 25 \pmod{37}.$$

From  $r \equiv 0 \pmod{5}$ , write  $r = 5k$ . Then

$$5k \equiv 6 \pmod{9}.$$

Since  $5 \cdot 2 = 10 \equiv 1 \pmod{9}$ , the inverse of 5 modulo 9 is 2, and hence

$$k \equiv 6 \cdot 2 = 12 \equiv 3 \pmod{9}.$$

Thus

$$k = 9t + 3, \quad r = 45t + 15.$$

Now impose the condition modulo 37:

$$45t + 15 \equiv 25 \pmod{37}.$$

Since  $45 \equiv 8 \pmod{37}$ , this becomes

$$8t + 15 \equiv 25 \pmod{37},$$

$$8t \equiv 10 \pmod{37}.$$

Because  $8 \cdot 14 = 112 \equiv 1 \pmod{37}$ , the inverse of 8 modulo 37 is 14, and therefore

$$t \equiv 10 \cdot 14 = 140 \equiv 29 \pmod{37}.$$

Hence

$$t = 37s + 29.$$

Substituting back,

$$r = 45(37s + 29) + 15 = 1665s + 1320.$$

The smallest nonnegative solution is

$$r = 1320.$$

**Final result.**

$$987\,654\,321\,012\,345 \equiv 1320 \pmod{1665}.$$

Thus the number is not divisible by  $5 \cdot 9 \cdot 37$ , and its exact remainder upon division by 1665 is 1320.

**Remark.** This method does not merely test divisibility: it allows one to compute the exact remainder of a very large number using only its decimal expansion and elementary congruence rules.

### 3.1.1 :: tutorial, method of the infinite descent

Every integer  $t$  can be written in exactly one of the three forms

$$t = 3k, \quad t = 3k + 1, \quad t = 3k + 2$$

for some integer  $k$ .

**Problem 1** (Squares in the three cases). Let  $k$  be an integer.

1. Expand  $(3k)^2$  and show it is divisible by 3.
2. Expand  $(3k + 1)^2$  and show it has the form  $3n + 1$  for some integer  $n$ .
3. Expand  $(3k + 2)^2$  and show it has the form  $3n + 1$  for some integer  $n$ .

Conclude: every square integer is either divisible by 3 or has the form  $3n + 1$ .

**Lemma 5.** If  $m^2$  is divisible by 3, then  $m$  is divisible by 3.

**Problem 2.** Prove that the Diophantine equation

$$x^2 + y^2 = 3z^2$$

has no integer solutions other than  $x = y = z = 0$ .

Solution: Assume there is a nonzero solution. Among all nonzero solutions choose one with  $|z|$  minimal.

1. Show that  $z \neq 0$ .
2. Show that  $x^2 + y^2$  is divisible by 3, hence at least one of  $x^2, y^2$  is not of the form  $3n + 1$ . Deduce that *both*  $x^2$  and  $y^2$  are divisible by 3.
3. Use the lemma to conclude 3 divides  $x$  and  $y$ .
4. Write  $x = 3x_1, y = 3y_1$  and substitute back to show 3 divides  $z$  as well.
5. Put  $z = 3z_1$  and divide the equation by 9 to obtain a new solution

$$x_1^2 + y_1^2 = 3z_1^2$$

with  $|z_1| < |z|$ . Contradiction. This is another instance of the Method of Extreme from the beginning of this course.

**Lemma 6.** 1. If  $u^2$  is even then  $u$  is even.

2. If  $u^2$  is divisible by 4 then  $u$  is even.

3. If  $u^4$  is even then  $u$  is even.

**Lemma 7** (Warm up before Primitive Pythagorean triples). Prove that if  $xy = z^2$ , with  $x, y, z$  positive integers, and  $\gcd(x, y) = 1$  then  $x$  and  $y$  are both squares

**Problem 3** (Primitive Pythagorean triples). Suppose  $a, b, c$  are integers with

$$a^2 + b^2 = c^2,$$

$\gcd(a, b) = 1$ , and  $a$  and  $b$  are not both odd. Prove that there exist integers  $m > n \geq 1$  such that

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

up to swapping  $a$  and  $b$ .

(Hint: Remark that if  $(a, b, c)$  are Primitive Pythagorean triples then  $a, b$  and  $c$  have no common factors. Try to understand why. If  $a$  is even, write  $a^2 = (c - b)(c + b)$ . You will also need to use the warm up lemma above. )

**Problem 4** (Fermat's infinite descent for fourth powers (no non-zero integer solutions of  $x^4 + y^4 = z^4$ )). We will prove even stronger result. Assume that there exists a nonzero integer solution of

$$x^4 + y^4 = w^2.$$

Among all such solutions, choose one with  $z > 0$  minimal.

1. **Primitive reduction.** Show that you may assume  $\gcd(x, y) = 1$ , and then deduce that

$$\gcd(x, y) = \gcd(x, w) = \gcd(y, w) = 1.$$

2. **Parity choice.** Show that  $x$  and  $y$  cannot both be odd. After possibly exchanging  $x$  and  $y$ , assume that  $x$  is odd and  $y$  is even. Conclude that  $x^2$  is odd and  $y^2$  is even.

3. **First primitive Pythagorean triple.** Rewrite the equation as

$$(x^2)^2 + (y^2)^2 = w^2$$

and prove that  $(x^2, y^2, w)$  is a primitive Pythagorean triple.

4. **First parametrization.** Use the classification of primitive Pythagorean triples to show that there exist coprime integers  $u > v \geq 1$  of opposite parity such that

$$x^2 = u^2 - v^2, \quad y^2 = 2uv, \quad w = u^2 + v^2.$$

5. **Mod 4 parity refinement.** Reduce  $x^2 = u^2 - v^2$  modulo 4 and deduce that  $u$  must be odd and  $v$  must be even.

6. **Square-factor forcing from  $y^2 = 2uv$ .** Using only  $\gcd(u, v) = 1$  and the fact that  $u$  is odd, show that  $\gcd(u, 2v) = 1$ . Deduce from  $y^2 = 2uv$  that both  $u$  and  $2v$  are perfect squares. Conclude that there exist coprime integers  $a, b \geq 1$  such that

$$u = a^2, \quad v = 2b^2.$$

7. **Second primitive Pythagorean triple.** Substitute  $u = a^2$  and  $v = 2b^2$  into  $x^2 = u^2 - v^2$  and show that

$$x^2 = a^4 - 4b^4, \quad \text{hence} \quad x^2 + (2b^2)^2 = (a^2)^2.$$

Prove that  $(x, 2b^2, a^2)$  is a primitive Pythagorean triple.

8. **Second parametrization.** Apply the primitive Pythagorean parametrization again to obtain coprime integers  $c > d \geq 1$  (of opposite parity) such that

$$x = c^2 - d^2, \quad 2b^2 = 2cd, \quad a^2 = c^2 + d^2.$$

Deduce that  $b^2 = cd$ .

9. **Squares again.** Using  $\gcd(c, d) = 1$  and  $b^2 = cd$ , prove that  $c$  and  $d$  are perfect squares:

$$c = r^2, \quad d = s^2$$

for some coprime integers  $r, s \geq 1$ .

10. **Descended solution.** Show that the last identity  $a^2 = c^2 + d^2$  becomes

$$a^2 = r^4 + s^4,$$

so  $(r, s, a)$  is a new nonzero solution of

$$X^4 + Y^4 = W^2.$$

11. **Strict descent.** Prove the inequality chain

$$0 < a \leq a^2 = u \leq u^2 < u^2 + v^2 = w,$$

and conclude that  $0 < a < w$ . Explain why this contradicts the choice of  $w = z^2$  minimal.

12. **Conclusion.** Deduce that there are no nonzero integer solutions to  $x^4 + y^4 = w^2$  with  $\gcd(x, y, w) = 1$ , and therefore the original equation  $x^4 + y^4 = z^4$  has no solutions with  $xyz \neq 0$ .

See <https://www.math.uci.edu/~ndonalds/math180b/5descent.pdf> for more examples of descent. The following text is taken from there:

In 1637 Fermat stated his famous Last Theorem in the margin of his copy of Diophantus' *Arithmetica*.

**Theorem.** *If  $n \geq 3$  is a natural number, then the equation  $x^n + y^n = z^n$  has no non-zero integer solutions.*

In perhaps the most famous claim in mathematical history, Fermat stated that the margin was too small to contain his argument. After his death in 1665, an edited version of the *Arithmetica* was published including Fermat's unproved notes and comments. As proofs to essentially everything else slowly appeared, the last theorem steadily grew in notoriety. In 1994 (after 350 years!) Andrew Wiles finally proved the last theorem via a related result regarding elliptic curves. Only the most romantic modern mathematicians now believe that Fermat had a valid argument.

## :: workshop

[we do not use congruences here! congruences will be introduced later!]

**Example 5.** Prove that the number  $n^3 - n$  is divisible by 6 for all integers  $n$ .

**Solution.** Factor the given expression:

$$n^3 - n = (n - 1)n(n + 1).$$

We obtain the product of three consecutive integers. One of them is divisible by 3, hence the product is divisible by 3. At least one of the three consecutive integers is even, so the product is even. A number divisible by both 2 and 3 is divisible by 6.

**Example 6.** Prove that there exist infinitely many integers that cannot be represented as the sum of three squares.

**Solution.** The square of an integer, when divided by 8, leaves a remainder of 0, 1, or 4. To verify this, it suffices to check the squares of all possible remainders modulo 8, namely the numbers from 0 to 7. Therefore, a sum of three squares cannot have remainder 7 when divided by 8.

**Example 7.** Prove that a number whose decimal representation consists of three ones and several zeros cannot be a square.

**Solution.** If such a number existed, it would be divisible by 3 but not divisible by 9 (by the divisibility tests for 3 and 9). However, if a number is divisible by 3 and is a perfect square, then it must be divisible by 9. This contradiction shows that such a number cannot be a square.

Prove:

1. If  $p$  is a prime number greater than 3, prove that  $p^2 - 1$  is divisible by 24.
2. For which integers  $n$  is the number  $2^n - 1$  divisible by 7?
3. It is known that the sum of several natural numbers is divisible by 6. Prove that the sum of the cubes of these numbers is also divisible by 6.
4. Prove that if an integer arithmetic progression contains a perfect square, then it contains infinitely many perfect squares.
5. A six-digit number is divisible by 7. Prove that if its last digit is moved to the front, the resulting number is also divisible by 7.
6. Find three pairwise coprime integers such that the sum of any two of them is divisible by the third.

### 3.1.2 :: :: homework: Egyptian fractions

## Egyptian Fractions

**Definition 15.** A *unit fraction* (or *Egyptian fraction*) is a fraction of the form  $\frac{1}{n}$ , where  $n$  is a positive integer. A representation of a rational number as a sum of distinct unit fractions is called an *Egyptian decomposition*.

**Example 8.** Consider dividing five loaves equally among eight people. Each person should receive  $\frac{5}{8}$  of a loaf. Instead of cutting every loaf into eighths, one can note that

$$\frac{5}{8} = \frac{1}{2} + \frac{1}{8},$$

so it suffices to divide four loaves in half and the fifth loaf into eight parts. This exemplifies the efficiency of Egyptian decomposition.

The ancient Egyptians expressed all other fractions as sums of distinct unit fractions. This problem is nontrivial because of the requirement that all denominators be distinct.

**Lemma 8.** For any positive integer  $n$ , the following identity holds:

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

*Proof.* Multiplying both sides by  $n(n+1)$  gives  $n+1 = n+1$ , verifying the equality.  $\square$

Repeated application of this lemma allows one to expand a unit fraction into a sum of distinct unit fractions with successively increasing denominators. For instance:

$$\frac{3}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{1}{n} + \frac{2}{n+1} + \frac{2}{n(n+1)} = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{(n+1)(n+2)} + \frac{1}{n(n+1)} + \frac{1}{n(n+1)+1} + \frac{1}{(n(n+1)+1)n(n+1)}.$$

However, the denominators in this procedure grow rapidly, making it impractical for general use.

## The Fibonacci Algorithm

**Definition 16.** Given a proper fraction  $\frac{k}{n}$  with  $k < n$ , the *Fibonacci algorithm* produces an Egyptian decomposition by setting

$$\frac{k}{n} = \frac{1}{\lceil \frac{n}{k} \rceil} + \frac{k \lceil \frac{n}{k} \rceil - n}{n \lceil \frac{n}{k} \rceil},$$

where  $\lceil x \rceil$  denotes the smallest integer greater than or equal to  $x$ . The procedure is then applied recursively to the remaining fraction until a complete decomposition is obtained.

**Theorem 11.** The Fibonacci algorithm terminates after at most  $k$  steps and yields a decomposition into distinct unit fractions.

*Proof.* At each step, the numerator decreases while denominators strictly increase. Since the numerator is a positive integer, the process must terminate after finitely many steps, producing distinct denominators.  $\square$

**Example 9.** The fraction  $\frac{5}{121}$  admits a Egyptian decomposition:

$$\frac{5}{121} = \frac{1}{33} + \frac{1}{121} + \frac{1}{363},$$

which is compact, whereas the Fibonacci algorithm yields a much longer expansion:

$$\frac{5}{121} = \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} + \frac{1}{873960180193} + \frac{1}{1527612795642093418846225}.$$

The inefficiency arises from the algorithm's *greedy* nature — it always selects the largest available unit fraction  $\frac{1}{\lceil n/k \rceil}$ . In this case, beginning with a bigger denominator (33 rather than 25) yields a shorter decomposition.

## Open Problems

In 1985, M. Vose constructed an algorithm that expresses  $\frac{k}{n}$  as a sum of at most  $C\sqrt{\log n}$  unit fractions (where  $C$  is a constant independent of  $k$  and  $n$ ). Lower estimates are of the order  $\log(\log n)$ . The exact bounds remain unknown. An even more elementary open question is the following:

**Erdős–Straus Conjecture.** For every integer  $n \geq 2$ , the fraction  $\frac{4}{n}$  can be written as the sum of three disctinct unit fractions

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Although this conjecture has been computationally verified for all  $n \leq 10^{14}$ , no general proof has yet been found.

# How to Read Mathematical Texts

A general approach may be outlined as follows (apply instructions to the above example):

## 1. Focus on the Definitions

The most important part of any mathematical text is its *definitions*. Locate them first. Try to understand each definition and visualize some examples in your mind: what objects satisfy it, and what do not?

If the text provides examples (as it does for the *first definition*), study them carefully. If not (as for the *second definition*), create your own examples to clarify the meaning.

**Exercise 1.** Examine the definition of the *Fibonacci algorithm*. Apply it to the fraction a)  $\frac{5}{13}$  b)  $\frac{7}{19}$  and write down the sequence of unit fractions that you obtain.

## 2. Examine the Theorems

The next essential elements are the *theorems*. Find the theorem in the text and try to understand its *proof*.

How can you do this effectively?

Read the *first sentence* of the proof aloud. Check that you fully understand its meaning. Then try to question it — imagine disagreeing with the statement, and see if you can find a counterexample. Afterward, check how this sentence relates to the example you have already studied.

Continue with the *second sentence* in the same way: read, question, and test it against your example.

Finally, return to the statement of the theorem. Ask yourself: have we really proven what the theorem claims?

In our case the answer is **NO**. This often happens: mathematical proofs in textbooks or articles may be written concisely, leaving some

steps implicit. Treat the written proof as a *hint* or a guide, and try to reconstruct the argument on your own. Doing so will help you understand the logical structure of the reasoning.

**Exercise 2.** Write a full proof of Theorem 11 (i.e. prove that the process will terminate after at most  $k$  steps).

### 3. Read It Again

Once you have carefully examined the definitions, examples, and theorems, reread the entire text. You will likely find that what initially seemed complicated now appears as a connected and coherent narrative.

Reading mathematics is a skill developed through patience and active engagement. Always ask yourself: what is being defined, what is being claimed, and why is it true?

### 4. Look at the open problems.

Explore the open problems—they might have simple solutions that no one has noticed yet. Trying to solve them can be both fascinating and a lot of fun.

**Exercise 3.** For  $n = 4, 5, 6, 7$  find the Egyptian decomposition

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

### 3.2 :: equivalence relations, partitions

**Example 10.** We call two integer numbers  $x, y$  equivalent if  $x - y$  is divisible by 3. We denote this  $x \sim y$ . Note that if  $x \sim y$  and  $y \sim z$ , then  $x \sim z$ . Also  $x \sim x$ . If  $x \sim y$ , then  $y \sim x$ . Then, all the integer numbers are divided in three groups: with remainder 0, remainder 1, and remainder 2, when we divide by 3. Similar things are called equivalence relations.

**Binary relations.** Let  $X$  be a set. A *binary relation*  $R$  on  $X$  is a subset

$$R \subseteq X \times X.$$

If  $(x, y) \in R$ , we write  $x R y$ .

**Basic properties of relations.** A relation  $R$  on  $X$  may have the following properties:

- *Reflexive*: for all  $x \in X$ ,  $x R x$  ( $\forall x \in X$ ,  $(x, x) \in R$ )
- *Symmetric*: for all  $x, y \in X$ , if  $x R y$  then  $y R x$  (if  $(x, y) \in R$ , then  $(y, x) \in R$ ).
- *Transitive*: for all  $x, y, z \in X$ , if  $x R y$  and  $y R z$ , then  $x R z$  (if  $(x, y) \in R$ ,  $(y, z) \in R$ , then  $(x, z) \in R$ ).
- *Antisymmetric*: for all  $x, y \in X$ , if  $x R y$  and  $y R x$ , then  $x = y$ .

**Definition.** An *equivalence relation* on a set  $X$  is a reflexive, symmetric, and transitive binary relation  $R$ .

We shall denote equivalence as follows:

$$x R y \iff x \sim y$$

(read “ $x$  is equivalent to  $y$ ”), but other symbols are often used as well:  $\cong$ ,  $\equiv$ , and so on (the names of equivalence relations that will appear later, and the meanings of these symbols, will be clear from the context).

By definition:

1.  $x \sim x$  (reflexivity);
2.  $x \sim y \implies y \sim x$  (symmetry);
3.  $x \sim y$  and  $y \sim z \implies x \sim z$  (transitivity).

For each element  $x \in X$ , let  $\bar{x}$  (or  $[x]$ ) denote the corresponding *equivalence class*, defined as the set of all  $y \in X$  equivalent to  $x$ :

$$\bar{x} = [x] = \{y \in X \mid y \sim x\}.$$

The element  $x$  itself is then called a *representative* of the class  $\bar{x}$ . Thus, by definition, if  $\bar{x} = \bar{y} \iff x \sim y$ , then  $\bar{x}$  is uniquely determined by any of its representatives.

The conditions (a)–(c) can now be reformulated as:

1.  $x \in \bar{x}$ ;
2.  $y \in \bar{x} \iff x \in \bar{y}$ ;
3.  $y \in \bar{x}$  and  $z \in \bar{y} \implies z \in \bar{x}$ .

On any set, there are two obvious equivalence relations:

- the *identity relation*, when  $x \sim y \iff x = y$ ;
- the *universal relation*, when  $x \sim y$  for all  $x, y$ .

**Lemma 9** (Equivalence classes do not intersect or coincide). Let  $R$  be an equivalence relation on a set  $X$ , and let  $x, y \in X$ . Then either

$$[x] = [y] \quad \text{or} \quad [x] \cap [y] = \emptyset.$$

*Proof.* Assume that  $[x] \cap [y] \neq \emptyset$ . Then there exists an element  $z \in X$  such that

$$z \in [x] \quad \text{and} \quad z \in [y].$$

By definition of equivalence classes, this means

$$x R z \quad \text{and} \quad y R z.$$

Since  $R$  is symmetric, from  $y R z$  we obtain  $z R y$ . By transitivity of  $R$ , from  $x R z$  and  $z R y$  we conclude

$$x R y.$$

Now let  $u \in [x]$ . Then  $x R u$ , and together with  $x R y$  and symmetry we obtain

$$y R u,$$

so  $u \in [y]$ . Hence  $[x] \subseteq [y]$ . By symmetry of the argument, also  $[y] \subseteq [x]$ , and therefore  $[x] = [y]$ .

Thus, if two equivalence classes intersect, they must coincide.  $\square$

**Example 11.** We call two integer numbers  $x, y$  equivalent if  $x - y$  is divisible by 3. We denote this by  $x \sim y$ .

Note that:

- $x \sim x$  for every integer  $x$ ;
- if  $x \sim y$ , then  $y \sim x$ ;
- if  $x \sim y$  and  $y \sim z$ , then  $x \sim z$ .

Thus  $\sim$  is an equivalence relation.

All integers are divided into three groups (equivalence classes):

$$\overline{0} = \{x | x \equiv 0 \pmod{3}\}, \overline{1} = \{x | x \equiv 1 \pmod{3}\}, \overline{2} = \{x | x \equiv 2 \pmod{3}\}$$

These are the integers with remainder 0, 1, and 2 upon division by 3.

### Arithmetic with equivalence classes.

We now observe that we can perform arithmetic operations directly with these equivalence classes.

For example:

- if  $x \in \bar{2}$  and  $y \in \bar{2}$ , then  $x + y \in \bar{1}$ , since  $2 + 2 = 4$  leaves remainder 1;
- if  $x \in \bar{2}$  and  $y \in \bar{1}$ , then  $x + y \in \bar{0}$ ;
- if  $x \in \bar{2}$  and  $y \in \bar{2}$ , then  $xy \in \bar{1}$ , since  $2 \cdot 2 = 4$  leaves remainder 1.

Thus we may write informally:

$$\bar{2} + \bar{2} = \bar{1}, \quad \bar{2} + \bar{1} = \bar{0}, \quad \bar{2} \cdot \bar{2} = \bar{1}.$$

### Addition table modulo 3.

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

### Multiplication table modulo 3.

$\cdot$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{1}$

### Why we need equivalence relations.

In order for such tables to make sense, we must ensure that the result does *not depend on the chosen representatives*. For example, if  $x \sim x'$  and  $y \sim y'$ , then we must have

$$x + y \sim x' + y', \quad xy \sim x'y'.$$

This requirement is precisely guaranteed by working with *equivalence relations* and their associated *partitions into equivalence classes*.

## Conclusion.

The set of equivalence classes

$$\mathbb{Z}/3\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}\}$$

together with the operations defined above forms a new mathematical object, obtained from the integers by identifying numbers that differ by a multiple of 3.

This construction will appear repeatedly in what follows.

Let us recall several other examples of equivalence relations that we shall encounter later:

- **Equivalence (isomorphism) of sets.** Consider some set  $X$  of sets, for example  $2^Z$  for a fixed set  $Z$ . Then we set  $A \sim B \iff |A| = |B|$ , i.e., there exists a bijection between  $A$  and  $B$ .
- **Equality up to sign.** Let  $X = \mathbb{Z}$ , and define  $m \sim n \iff m \mid n$  and  $n \mid m$ .
- **Congruence modulo  $m$ .** Let  $X = \mathbb{Z}$  and  $m$  a fixed positive integer. We set  $x \sim y \iff x \equiv y \pmod{m}$  (recall, this means that  $x - y$  is divisible by  $m$ ).

Many examples of equivalence relations appeared in elementary geometry. In addition to the already mentioned congruence, similarity, parallelism, etc., we can also give the following examples.

- **Equality of lengths.** Let  $X$  be the set of vectors in the plane or in three-dimensional space. We say that  $x \sim y$  if the lengths of the vectors  $x$  and  $y$  coincide.
- **Equality of areas.** Let  $X$  be the “set of figures” on the plane. We shall say that two figures  $A$  and  $B$  are equivalent if they have the same area (or if the area is undefined for both).

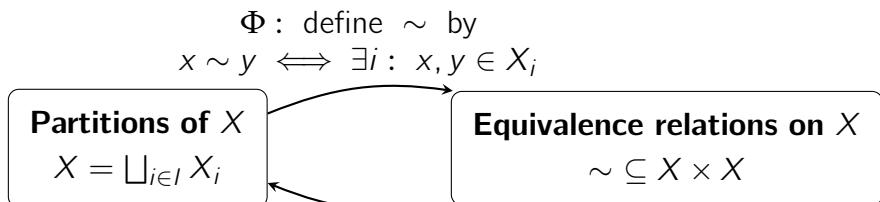
## Partitions

On the ancient pages of the Chinese encyclopedia “Heavenly Empire of Benevolent Knowledge” it is written that animals are divided into: a) belonging to the emperor, b) embalmed, c) tame, d) suckling pigs, e) sirens, f) fabulous, g) stray dogs, h) included in this classification, i) frenzied, j) innumerable, k) drawn with the finest brush of camel hair, l) others, m) that have just broken a flower vase, n) that resemble flies from afar. (*Jorge Luis Borges*, “The Analytical Language of John Wilkins”.)

A *partition* of a set  $X$  is a representation of  $X$  as a disjoint union of some family of non-empty subsets:

$$X = \bigsqcup_{i \in I} X_i \quad (\text{i.e. } X_i \cap X_j = \emptyset \text{ for } i \neq j, X_i \neq \emptyset).$$

The notion of partition coincides essentially with that of an equivalence relation. Namely, to each partition one can associate an equivalence relation, for which  $x \sim y$  if and only if  $x, y \in X_i$  for some  $i$ . Conversely, every equivalence relation on  $X$  defines a partition of  $X$  into disjoint equivalence classes.



$$\begin{aligned} \Psi : \text{classes } [x] &= \{y : y \sim x\} \\ \Rightarrow X &= \bigsqcup_{[x] \in X/\sim} [x] \end{aligned}$$

$\Psi \circ \Phi = \text{id} \quad \text{and} \quad \Phi \circ \Psi = \text{id}$   
(so this correspondence is a bijection)

To choose one representative from each class  $X_i$ , let  $\{x_i \mid i \in I\}$  be a *system of representatives* of the equivalence relation  $\sim$  (or a *transversal* to it).

**Proposition.** The set of equivalence classes forms a partition of  $X$ . In other words,  $X$  can be represented as a disjoint union

$$X = \bigsqcup_{i \in I} X_i, \quad i \in I,$$

of distinct classes.

**Proof.** First note that  $X = \bigcup_i X_i$ . Indeed, since we chose one  $x_i$  in each class, for any  $x \in X$  there exists  $i$  such that  $x \sim x_i$ , and therefore  $x \in X_i$ . It remains to show that different classes do not intersect. Indeed, if  $x \in X_i$  and  $x \in X_j$  for  $i \neq j$ , then  $x \sim x_i$  and  $x \sim x_j$ , whence  $x_i \sim x_j$ , a contradiction. Thus the classes are disjoint, and the claim follows.

**Definition.** Let  $X$  be a set and let  $\sim$  be an equivalence relation on  $X$ . The set

$$X/\sim = \{\bar{x} \mid x \in X\}$$

of all equivalence classes is called the *quotient set* of  $X$  by the equivalence relation  $\sim$ .

The map

$$\pi: X \longrightarrow X/\sim, \quad x \longmapsto \bar{x},$$

is called the *canonical projection* (or *quotient map*).

By definition, the map  $\pi$  sends each element of  $X$  to the equivalence class to which it belongs.

**Example 12** (Congruence modulo  $n$ ). Let  $n \geq 1$  be a fixed integer. On the set  $\mathbb{Z}$  we define an equivalence relation by

$$x \sim y \iff x - y \text{ is divisible by } n.$$

This relation is called *congruence modulo  $n$* .

The equivalence class of an integer  $k$  consists of all integers having the same remainder as  $k$  upon division by  $n$ :

$$\bar{k} = \{ m \in \mathbb{Z} : m \equiv k \pmod{n} \}.$$

Every integer  $k$  can be written uniquely in the form

$$k = qn + r, \quad 0 \leq r < n,$$

and therefore

$$\bar{k} = \bar{r}.$$

Thus there are exactly  $n$  distinct equivalence classes, namely

$$\bar{0}, \bar{1}, \dots, \overline{n-1}.$$

The quotient set

$$\mathbb{Z}/n\mathbb{Z}$$

(or simply  $\mathbb{Z}_n$ ) is the set of these equivalence classes.

The canonical projection

$$\pi: \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}, \quad k \longmapsto \bar{k},$$

sends each integer to its remainder class modulo  $n$ .

**Fibers.** Let  $f: X \rightarrow Y$  be a map. For  $y \in Y$  the *fiber* of  $f$  over  $y$  is the set

$$f^{-1}(y) = \{x \in X : f(x) = y\}.$$

Fibers describe how the map  $f$  partitions the set  $X$ : every element  $x \in X$  lies in exactly one fiber, namely the fiber over  $f(x)$ , and two fibers are either disjoint or equal.

In particular, for an equivalence relation  $\sim$  on  $X$  the canonical projection

$$\pi: X \rightarrow X/\sim, \quad x \mapsto \bar{x},$$

has the property that its fibers are exactly the equivalence classes:

$$\pi^{-1}(\bar{x}) = \bar{x}.$$

Thus the quotient set  $X/\sim$  can be viewed as the set of all fibers of  $\pi$ .

:: tutorial: equivalence relations

**Problem 1: Equivalence relation generated by a function.** Let  $A$  be a set and let  $f: A \rightarrow A$  be a function. Consider its graph

$$\Gamma_f = \{(x, f(x)) : x \in A\} \subseteq A \times A.$$

Let  $\sim$  denote the smallest equivalence relation on  $A$  that contains  $\Gamma_f$ , that is, the intersection of all equivalence relations on  $A$  containing  $\Gamma_f$ .

1. Show that for every  $x \in A$ , one has  $x \sim f(x)$ .
2. Show that if  $x \sim y$ , then there exist nonnegative integers  $k, \ell$  such that

$$f^k(x) = f^\ell(y).$$

3. Conversely, show that if there exist nonnegative integers  $k, \ell$  such that

$$f^k(x) = f^\ell(y),$$

then  $x \sim y$ .

4. Deduce that

$$x \sim y \iff \exists k, \ell \in \mathbb{N} \text{ such that } f^k(x) = f^\ell(y).$$

5. Describe the equivalence class  $[x]$  of an element  $x \in A$  in terms of the forward orbit

$$x, f(x), f^2(x), \dots$$

of  $x$  under  $f$ .

**Problem 2: Remainders modulo  $n$** 

Let  $n \geq 1$  be a fixed integer, and define a function

$$f_n: \mathbb{Z} \rightarrow \mathbb{Z}, \quad f_n(x) = \text{the remainder of } x \text{ upon division by } n.$$

1. Show that for every integer  $x$ , the sequence

$$x, f_n(x), f_n(f_n(x)), \dots$$

stabilizes after the first step, and that its eventual value lies in  $\{0, 1, \dots, n-1\}$ .

2. Show that for integers  $x, y \in \mathbb{Z}$ , the eventual values of the sequences

$$x, f_n(x), f_n^2(x), \dots \quad \text{and} \quad y, f_n(y), f_n^2(y), \dots$$

are equal if and only if  $x \equiv y \pmod{n}$ .

3. Let  $\Gamma_{f_n} \subset \mathbb{Z} \times \mathbb{Z}$  be the graph of  $f_n$ , and let  $\sim$  be the smallest equivalence relation on  $\mathbb{Z}$  that contains  $\Gamma_{f_n}$ . Show that

$$x \sim y \iff x \equiv y \pmod{n}.$$

4. Deduce that the equivalence classes of  $\sim$  are precisely the residue classes

$$\bar{r} = \{k \in \mathbb{Z} : k \equiv r \pmod{n}\}, \quad r = 0, 1, \dots, n-1.$$

**Problem 3:** Let  $A$  be a set, and let  $\{\sim_i\}_{i \in I}$  be a family of equivalence relations on  $A$ .

1. Show that the intersection  $\sim = \bigcap_{i \in I} \sim_i$  is an equivalence relation on  $A$ .
2. Prove or disprove:  $\approx = \bigcup_{i \in I} \sim_i$  is an equivalence relation.

## **Solution.**

Let  $\{R_i\}_{i \in I}$  be a family of equivalence relations on a set  $X$ .

*Intersection.* Define

$$R = \bigcap_{i \in I} R_i.$$

Then:

- $R$  is reflexive, since each  $R_i$  is reflexive;
- $R$  is symmetric, since each  $R_i$  is symmetric;
- $R$  is transitive, since each  $R_i$  is transitive.

Hence the intersection of equivalence relations is again an equivalence relation.

*Union.* Define

$$R = \bigcup_{i \in I} R_i.$$

In general,  $R$  is *not* transitive. Indeed, it may happen that

$$x R_1 y \quad \text{and} \quad y R_2 z,$$

but there is no index  $i$  such that both relations belong to the same  $R_i$ , so  $x R z$  need not hold.

Therefore, the union of equivalence relations is *not* necessarily an equivalence relation.  $\square$

### 3.2.1 :: workshop: Fundamental Theorem of Arithmetic as a problem sequence

Throughout this workshop,  $\mathbb{N} = \{1, 2, 3, \dots\}$ . We say that an integer  $p > 1$  is *prime* (or *irreducible*) if its only positive divisors are 1 and  $p$ .

We will use the *minimal counterexample* principle several times: assume a statement  $(S)$  is false and choose a counterexample with the smallest possible value of some natural number parameter; then derive a strictly smaller counterexample. It contradicts to the choice of the initial counterexample, and this contradiction implies that the statement  $(S)$  is true.

**Problem 1** (Division algorithm). Let  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}$ . Prove that there exist unique integers  $q, r$  such that

$$a = bq + r, \quad 0 \leq r < b.$$

You need to prove a) existence, b) uniqueness.

**Problem 2.** Prove that every integer  $n \geq 2$  can be written as a product of primes.

*Hint (minimal counterexample in n).* Assume not, and let  $n \geq 2$  be the smallest integer that is not a product of primes. Show that  $n$  cannot be prime, so  $n = ab$  with  $1 < a < n$  and  $1 < b < n$ . Apply minimality to  $a$  and  $b$  and multiply their factorizations.

We want to prove the following theorem.

**Theorem 12** (Fundamental Theorem of Arithmetic). Every integer  $n \geq 2$  can be written as a product of primes, and this factorization is unique up to the order of the factors.

We have already proven the existence of a factorization. Show that it is enough to prove the following:

**Problem 3** (A prime dividing a product of primes). Let  $p$  be a prime and suppose

$$p \mid q_1 q_2 \cdots q_k$$

where each  $q_i$  is prime. Prove that  $p = q_i$  for some  $i$ .

(Hint: choose minimal  $n$  with two different factorizations

$$n = q_1 q_2 \cdots q_k = p_1 p_2 \cdots p_l$$

(the minimal counterexample), then apply the above problem to  $p = p_1$  to obtain a smaller counterexample).

Now reduce the above problem to the following one.

**Problem 4** (Cancellation from prime divisibility). Let  $p$  be a prime and suppose  $p \mid AB$  and  $p \nmid A$ . Prove that  $p \mid B$ .

(Hint: we can use this problem repeatedly: first apply it to  $p \mid q_1(q_2 \cdots q_k)$ , then continue...)

Thus we reduced the fundamental theorem of arithmetic to the following problem: (Euclid's lemma) Let  $p$  be a prime. We want to prove that for all integers  $A, B$ ,

$$p \mid AB \implies p \mid A \text{ or } p \mid B.$$

We will present here so called descent form proof of this statement.

Assume the statement is false for some prime  $p$ . Choose a counterexample with *minimal* prime  $p$ , and among those choose one with *minimal* product  $AB > 0$  such that

$$p \mid AB, \quad p \nmid A, \quad p \nmid B.$$

Write  $AB = pq$ .

**Problem 5.** Show that one may assume  $0 < A, B < p$ . (Hint: if  $A > p$ , write  $A = pk + r$  with  $0 \leq r < p$  and use that  $p(q - kB) = rB$  which contradicts minimality.)

**Problem 6.** Assume now that  $0 < A \leq B < p$  and  $AB = pq$ . Prove that

$$1 \leq q < p.$$

*Hint.* From  $AB = pq$  and  $A, B < p$ , compare  $pq$  with  $p^2$ .

**Problem 7.** Assume  $0 < A \leq B < p$  and  $AB = pq$ . Show that  $q \neq 1$ .

**Problem 8** (A prime divisor of  $q$  yields a smaller counterexample). Assume  $q > 1$  and let  $q'$  be a prime divisor of  $q$ . Prove that  $q'$  does *not* divide  $A$  and does *not* divide  $B$ .

*Hint.* If  $q' \mid A$  (or  $q' \mid B$ ), divide the equality  $AB = pq$  by  $q'$  and explain why this produces a counterexample with a smaller value of  $AB$ , contradicting the minimal choice of  $AB$ .

**Problem 9** (Constructing a smaller prime counterexample). Let  $q'$  be a prime divisor of  $q$ . Show that

$$q' \mid AB, \quad q' \nmid A, \quad q' \nmid B,$$

so  $(q', A, B)$  is a counterexample to Euclid's Lemma. *Hint.* Since  $q' \mid q$  and  $AB = pq$ , we have  $q' \mid AB$  and  $q' < p$ . This contradicts the minimality of  $p$ .

**Remark (prime factorization may fail in other rings).** The Fundamental Theorem of Arithmetic is special to the ring of integers  $\mathbb{Z}$ . In other rings, one can define *irreducible* elements (elements that cannot be written as a product of two non-units), but factorization into irreducibles may fail to be unique.

For example, in the ring

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\},$$

one has the identity

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}),$$

and each of the four factors  $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$  is irreducible in  $\mathbb{Z}[\sqrt{-5}]$ .

Moreover, they do not differ by multiplication by a unit (here the only units are  $\pm 1$ ), so these give genuinely different decompositions of the same element. Thus, outside  $\mathbb{Z}$  one must be careful: factorization into irreducibles may fail to be unique, and one should not expect the same proof strategy as in  $\mathbb{Z}$ .

One reason is that in  $\mathbb{Z}[\sqrt{-5}]$  there is *no* division algorithm of the Euclidean type (no analogue of  $a = bq + r$  with a remainder that is guaranteed to be “smaller” than  $b$  with respect to a Euclidean norm  $|r| < |b|$ ). For instance, consider the usual norm

$$N(a + b\sqrt{-5}) = (a + b\sqrt{-5})(a - b\sqrt{-5}) = a^2 + 5b^2.$$

Try to “divide”  $1 + \sqrt{-5}$  by 2:

$$1 + \sqrt{-5} = 2 \cdot 0 + (1 + \sqrt{-5}) \quad \text{and} \quad 1 + \sqrt{-5} = 2 \cdot 1 + (-1 + \sqrt{-5}).$$

Both remainders have the same norm

$$N(1 + \sqrt{-5}) = 1^2 + 5 \cdot 1^2 = 6, \quad N(-1 + \sqrt{-5}) = (-1)^2 + 5 \cdot 1^2 = 6,$$

while

$$N(2) = 4.$$

So neither remainder is “better” (smaller) than the other, and in fact *no* choice of  $q \in \mathbb{Z}[\sqrt{-5}]$  makes the remainder  $r = (1 + \sqrt{-5}) - 2q$  satisfy  $N(r) < N(2)$ . This illustrates that the familiar Euclidean-division mechanism behind many arguments in  $\mathbb{Z}$  is not available in  $\mathbb{Z}[\sqrt{-5}]$ .

:: homework

1. A bus ticket with a six digit number is called *lucky* if the sum of its digits is divisible by 7. Can two consecutive tickets both be lucky?
2. Write the addition and multiplication tables for  $\mathbb{Z}/n\mathbb{Z}$  for  $n = 4, 5, 6, 7$ .

3. **Congruence modulo  $\mathbb{Z}$ .** Let now  $X = \mathbb{R}$ . We shall say that two real numbers are congruent modulo  $\mathbb{Z}$ , and write  $x \equiv y \pmod{\mathbb{Z}}$ , if their fractional parts coincide. Recall that the fractional part  $\{x\} \in [0, 1)$  of a real number  $x \in \mathbb{R}$  is the difference  $x - \text{Ent}(x)$ , where  $[x]$  the largest integer  $n$  such that  $n \leq x$ . By definition,  $x = [x] + \{x\}$ . Thus,

$$x \equiv y \pmod{\mathbb{Z}} \iff \{x\} = \{y\}.$$

Show that this is an equivalence relation.

4. Consider two partitions  $A = \bigsqcup_{i \in I} X_i = \bigsqcup_{j \in J} Y_j$  of the same set  $A$ . Show that the family of all *nonempty* pairwise intersections also forms a partition of  $A$ , namely

$$A = \bigsqcup_{k \in K} Z_k, \quad K := \{(i, j) \in I \times J : X_i \cap Y_j \neq \emptyset\}, \quad Z_{(i,j)} := X_i \cap Y_j.$$

This partition  $\{Z_k\}_{k \in K}$  is called the *common refinement* of the partitions  $A = \bigsqcup_{i \in I} X_i$  and  $A = \bigsqcup_{j \in J} Y_j$ .

5. Consider the partition  $\mathbb{Z} = \bigsqcup_{i \in I} X_i$  induced by the equivalence relation  $\pmod{2}$ . What are the sets  $X_i$ ?

Consider the partition  $\mathbb{Z} = \bigsqcup_{j \in J} Y_j$  induced by the equivalence relation  $\pmod{3}$ . What are the sets  $Y_j$ ?

Find the common refinement of these two partitions. To which equivalence relation does it correspond?

### 3.3 :: construction of natural, rational, real numbers

We assume here that the rational numbers  $\mathbb{Q}$  have been already constructed (see the tutorial) based on the construction of the natural numbers  $\mathbb{N}$  (see the workshop).

The rational numbers  $\mathbb{Q}$  are already rich, but they are not complete: there exist Cauchy sequences of rational numbers that do not converge to a rational limit (for example, rational approximations to  $\sqrt{2}$ ). To obtain a complete number system, we construct the real numbers  $\mathbb{R}$  from  $\mathbb{Q}$ .

#### Cauchy sequences in $\mathbb{Q}$

**Definition 17** (Cauchy sequence). A sequence  $(a_n)_{n \geq 1}$  of rational numbers is called *Cauchy* if for every  $\varepsilon \in \mathbb{Q}$  with  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ ,

$$|a_n - a_m| < \varepsilon.$$

**Definition 18** (Null sequence). A sequence  $(a_n)_{n \geq 1}$  in  $\mathbb{Q}$  is called a *null sequence* if for every  $\varepsilon \in \mathbb{Q}$  with  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$|a_n| < \varepsilon.$$

**Example 13** (Constant sequences). Let  $c \in \mathbb{Q}$  and define  $a_n := c$  for all  $n \geq 1$ . Then  $(a_n)$  is Cauchy, since for all  $m, n$ ,

$$|a_n - a_m| = |c - c| = 0 < \varepsilon.$$

It is a null sequence if and only if  $c = 0$ . Indeed, if  $c \neq 0$ , take  $\varepsilon := |c|/2 > 0$ ; then  $|a_n| = |c| \not< \varepsilon$  for any  $n$ .

**Example 14** (A sequence jumping over zero). Define  $a_n := (-1)^{\frac{n+1}{n}} \in \mathbb{Q}$ . This is a null sequence: given  $\varepsilon \in \mathbb{Q}$  with  $\varepsilon > 0$ , choose  $N \in \mathbb{N}$  such

that  $1/N < \varepsilon$  (for instance, any  $N > \frac{1}{\varepsilon}$  works). Then for all  $n \geq N$ ,

$$|a_n| = \left| (-1)^n \frac{1}{n} \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

In particular,  $(a_n)$  is Cauchy. Indeed, for all  $m, n \geq N$ ,

$$|a_n - a_m| \leq |a_n| + |a_m| = \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N}.$$

So if we choose  $N$  large enough that  $2/N < \varepsilon$ , then  $|a_n - a_m| < \varepsilon$  for all  $m, n \geq N$ .

**Exercise 12.** Prove that the a) sum b) product of two null-sequences is a null sequence.

**Exercise 13.** Prove that the a) sum b) product of two Cauchy sequences is a Cauchy sequence.

### An equivalence relation on Cauchy sequences

Let  $\mathcal{C}$  denote the set of all Cauchy sequences of rational numbers. This is a subset of  $\mathbb{Q}^{\mathbb{N}}$ .

**Definition 19** (Equivalence of Cauchy sequences). For two Cauchy sequences  $(a_n)$  and  $(b_n)$  in  $\mathcal{C}$  we write

$$(a_n) \sim (b_n)$$

if the sequence of differences  $(a_n - b_n)$  is a null sequence, i.e. if for every  $\varepsilon \in \mathbb{Q}$  with  $\varepsilon > 0$  there exists  $N$  such that for all  $n \geq N$ ,

$$|a_n - b_n| < \varepsilon.$$

**Exercise 14.** Prove that  $\sim$  is an equivalence relation on  $\mathcal{C}$ : it is reflexive, symmetric, and transitive.

## Definition of $\mathbb{R}$

**Definition 20** (Real numbers). The set of *real numbers* is the quotient set

$$\mathbb{R} := \mathcal{C} / \sim .$$

An element of  $\mathbb{R}$  is an equivalence class of Cauchy sequences. If  $(a_n)$  is a Cauchy sequence, we denote its equivalence class by

$$[(a_n)_{n=1}^{\infty}] = [(a_n)] \in \mathbb{R}.$$

**Remark 11.** Two Cauchy sequences represent the same real number if and only if their difference tends to 0. Thus, each real number can be thought of as a “limit” of rational numbers, without assuming limits exist inside  $\mathbb{Q}$ .

**Remark 12.** If we already believe that there exists a set of real numbers and know limits, then the equivalence class corresponding to a real number  $x$  is the set of all sequences converging to  $x$ . In our construction situation is reversed: we do not know real numbers and limits, but we define exactly the same sets of sequences as equivalence classes, and then define real numbers as equivalence classes.

## Arithmetic operations on $\mathbb{R}$

We now define addition and multiplication by applying the operations termwise.

**Definition 21** (Addition and multiplication). Let  $x = [(a_n)]$  and  $y = [(b_n)]$  be real numbers. Define

$$x + y := [(a_n + b_n)], \quad x \cdot y := [(a_n b_n)].$$

Define also

$$-x := [(-a_n)].$$

**Problem 1** (Well-definedness). Prove that these definitions do not depend on the choice of representatives: if  $(a_n) \sim (a'_n)$  and  $(b_n) \sim (b'_n)$ , then

$$(a_n + b_n) \sim (a'_n + b'_n) \quad \text{and} \quad (a_n b_n) \sim (a'_n b'_n).$$

Hint: use Exercise 12.

### Order and absolute value

To compare real numbers we define an order relation in terms of sequences.

**Remark 13.** The absolute value  $|x|$  for  $x \in \mathbb{Q}$  is defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

All inequalities above are inequalities of rational numbers.

**Definition 22** (“Eventually”). A property  $P(n)$  is said to hold *eventually* (or *for all sufficiently large  $n$* ) if there exists  $N \in \mathbb{N}$  such that  $P(n)$  holds for all  $n \geq N$ . For example,  $(a_n)$  is *eventually non-negative* if  $\exists N$  such that  $a_n \geq 0$  for all  $n \geq N$ .

**Lemma 10** (Trichotomy lemma). Every Cauchy sequence  $(a_n)$  of rational numbers falls in one of three cases:

1.  $(a_n)$  has infinitely many positive and infinitely many negative terms; then  $(a_n)$  is a null sequence.
2.  $(a_n)$  is eventually non-negative; then either  $(a_n)$  is a null sequence, or there exists  $\varepsilon \in \mathbb{Q}$  with  $\varepsilon > 0$  such that  $a_n \geq \varepsilon$  eventually.
3.  $(a_n)$  is eventually non-positive; then either  $(a_n)$  is a null sequence, or there exists  $\varepsilon \in \mathbb{Q}$  with  $\varepsilon < 0$  such that  $a_n \leq \varepsilon$  eventually.

*Proof.* **Case analysis.**

**Case 1: infinitely many positive and infinitely many negative terms.**

Let  $\delta \in \mathbb{Q}$  with  $\delta > 0$ . Since  $(a_n)$  is Cauchy, there exists  $N$  such that

$$|a_n - a_m| < \delta \quad \text{for all } m, n \geq N.$$

Because there are infinitely many positive and negative terms, we can choose indices  $p, q \geq N$  with  $a_p > 0$  and  $a_q < 0$ . Then for  $m > N$  we have  $|a_m - a_q| < \delta$  so  $a_m < \delta$  and  $|a_m - a_p| < \delta$  so  $a_m > -\delta$ , so  $|a_m| < \delta$  so  $(a_n)$  is a null-sequence.

**Case 2: not Case 1.** Then  $(a_n)$  does *not* have infinitely many positive and negative terms simultaneously, i.e. one of the following holds:

$(a_n)$  is eventually non-negative      or       $(a_n)$  is eventually non-positive.

(Indeed, if  $(a_n)$  is not eventually non-negative, then there are infinitely many  $n$  with  $a_n < 0$ ; similarly, if it is not eventually non-positive, there are infinitely many  $n$  with  $a_n > 0$ . If both failures happened, we would be in Case 1.)

We treat the “eventually non-negative” case; the other is symmetric.

Assume  $(a_n)$  is eventually non-negative:  $\exists N_0$  such that  $a_n \geq 0$  for all  $n \geq N_0$ . If  $(a_n)$  is a null sequence, we are done. Otherwise,  $(a_n)$  is not null, so by definition there exists  $\varepsilon_0 \in \mathbb{Q}$  with  $\varepsilon_0 > 0$  such that for every  $N$  there exists  $n \geq N$  with

$$|a_n| \geq \varepsilon_0.$$

For  $n \geq N_0$  we have  $a_n \geq 0$ , hence  $|a_n| = a_n$ , and the condition becomes: for every  $N$  there exists  $n \geq N$  with  $a_n \geq \varepsilon_0$ .

Now apply the Cauchy property with  $\varepsilon_0/2$ : there exists  $N_1$  such that

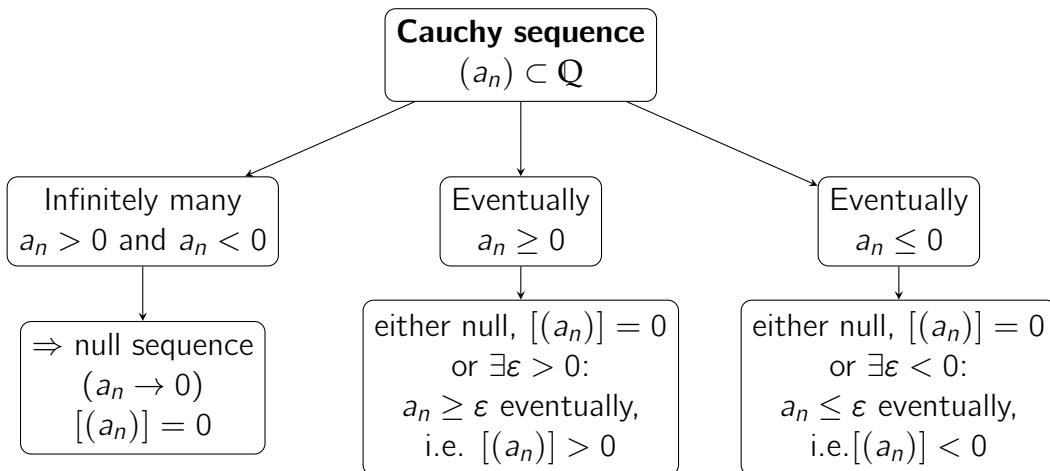
$$|a_n - a_m| < \varepsilon_0/2 \quad \text{for all } m, n \geq N_1.$$

Choose an index  $n_* \geq \max(N_0, N_1)$  with  $a_{n_*} \geq \varepsilon_0$  (it exists by the previous paragraph). Then for every  $m \geq \max(N_0, N_1)$ ,

$$a_m \geq a_{n_*} - |a_m - a_{n_*}| > \varepsilon_0 - \varepsilon_0/2 = \varepsilon_0/2.$$

Thus, with  $\varepsilon := \varepsilon_0/2 > 0$ , we have  $a_m \geq \varepsilon$  eventually, as required.

**Case 3: eventually non-positive.** The proof is analogous: if  $(a_n)$  is not null, then there exists  $\varepsilon_0 > 0$  such that for all  $N$  one can find  $n \geq N$  with  $|a_n| \geq \varepsilon_0$ ; for  $n$  large enough we have  $a_n \leq 0$ , hence  $|a_n| = -a_n$  and so  $a_n \leq -\varepsilon_0$  infinitely often. Cauchy with  $\varepsilon_0/2$  then implies  $a_n \leq -\varepsilon_0/2$  eventually; take  $\varepsilon := -\varepsilon_0/2 < 0$ .  $\square$



**Definition 23** (Nonnegative real numbers). A real number  $x = [(a_n)]$  is called *nonnegative*, written  $x \geq 0$ , if there exists a Cauchy sequence  $(a_n)$  representing  $x$  such that  $a_n \geq 0$  for all sufficiently large  $n$  (i.e. for all  $n \geq N$  for some  $N$ ).

**Definition 24** (Order on  $\mathbb{R}$ ). For real numbers  $x, y \in \mathbb{R}$ , define

$$x \leq y \iff y - x \geq 0.$$

The Trichotomy lemma states that any real number is 0, bigger than 0, or less than 0. Consequently, for any two real numbers  $x, y$  one has  $x > y$  or  $x < y$  or  $x = y$ .

**Definition 25** (Absolute value). For  $x \in \mathbb{R}$  define

$$|x| := \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

### Embedding of $\mathbb{Q}$ into $\mathbb{R}$

**Definition 26** (Constant sequences). For  $q \in \mathbb{Q}$ , consider the constant sequence  $(q, q, q, \dots)$ , which is Cauchy. Define

$$\iota_{\mathbb{Q}} : \mathbb{Q} \rightarrow \mathbb{R}, \quad q \mapsto [(q, q, q, \dots)].$$

This is why we used the same symbol 0 for a real number representing the null sequences. It is customary to denote the equivalence class  $[(q, q, q, \dots)]$  by  $q$  itself.

**Exercise 15.** Prove that  $\iota_{\mathbb{Q}}$  is injective and preserves addition and multiplication:

$$\iota_{\mathbb{Q}}(q_1 + q_2) = \iota_{\mathbb{Q}}(q_1) + \iota_{\mathbb{Q}}(q_2), \quad \iota_{\mathbb{Q}}(q_1 q_2) = \iota_{\mathbb{Q}}(q_1) \iota_{\mathbb{Q}}(q_2).$$

### Completeness

The main advantage of  $\mathbb{R}$  is that it is complete.

**Theorem 13** (Completeness (Cauchy completeness)). Every Cauchy sequence of real numbers converges to a real number.

**Remark 14.** In this construction, a real number *is* an equivalence class of Cauchy sequences of rationals, so completeness is built into the definition: limits of Cauchy sequences exist by construction. A full proof requires checking that a Cauchy sequence of classes admits a representative Cauchy sequence in  $\mathbb{Q}$  and that its class is the limit.

## From Cauchy-sequence reals to decimal expansions

Recall that in our construction a real number is an equivalence class

$$x = [(a_n)]$$

of Cauchy sequences of rationals, where two Cauchy sequences represent the same real number if their difference is a null sequence.

Our goal is to define a map

$$\text{Dec} : \mathbb{R} \longrightarrow \{\text{decimal expansions without an eventual tail of 9}\}$$

sending each real number to its (chosen) decimal representation.

### A stability lemma

**Lemma 11** (A Cauchy sequence cannot cross a point forever without converging to it). Let  $(a_n)$  be a Cauchy sequence of rational numbers,  $k$  is an integer number. Assume that there are infinitely many indices  $n$  with  $a_n > k$  and infinitely many indices  $m$  with  $a_m < k$ . Then  $(a_n)$  represents the real number 1, i.e. the sequence  $(a_n - k)$  is a null sequence.

*Proof.* Consider the sequence  $(a_n - k)$  and follow the same proof as Trichotomy lemma.  $\square$

### Step 1: the integer part

**Lemma 12** (Integer part). For each real number  $x \in \mathbb{R}$  there exists a unique integer  $n \in \mathbb{Z}$  such that

$$n \leq x < n + 1.$$

This integer is called the *integer part* (or *floor*) of  $x$  and is denoted by  $\lfloor x \rfloor = n$ .

*Proof.* Let  $x$  be represented by a Cauchy sequence  $(a_m)$  of rational numbers, that is,  $x = [(a_m)]$ .

Since  $(a_m)$  is Cauchy, there exists  $N$  such that for all  $m \geq N$ ,

$$|a_m - a_N| < \frac{1}{2}.$$

Hence all terms  $a_m$  with  $m \geq N$  lie in the interval

$$(a_N - \frac{1}{2}, a_N + \frac{1}{2}).$$

This interval has length 1, so it contains at most one integer. Denote this integer (if it exists) by  $k$ . If there is no integers in this interval then  $a_N = k + \frac{1}{2}$  and the integer part of  $x$  will be equal to  $k$ .

If  $(a_n)$  contains infinitely many elements bigger and less than  $k$  then  $[(a_n)] = k$ . If  $(a_n)$  is eventually bigger than  $k$  then  $x \geq k$  and clearly,  $x < k + 1$ . If  $(a_n)$  is eventually smaller than  $k$  then either it is smaller than  $k - \varepsilon$  for a rational  $\varepsilon > 0$ , and then  $n = k - 1$ ,  $k \leq x < k + 1 = n$  or  $[(a_n)] = k$  and then  $n = k$  again.  $\square$

### Step 2: reduction to the unit interval

Every real number  $x$  can be written uniquely in the form

$$x = \lfloor x \rfloor + \{x\},$$

where the *fractional part*  $\{x\}$  satisfies

$$0 \leq \{x\} < 1.$$

Therefore, in order to describe decimal representations of real numbers, it suffices to understand real numbers in the interval  $[0, 1)$ .

### Step 3: decimal digits

Let  $x \in [0, 1)$ . For each  $k \geq 1$ , define the  $k$ -th decimal digit of  $x$  by

$$c_k := \left\lfloor 10^k x \right\rfloor - 10 \left\lfloor 10^{k-1} x \right\rfloor,$$

so that  $c_k \in \{0, 1, \dots, 9\}$ .

Equivalently,  $c_k$  is the last digit of the integer  $\lfloor 10^k x \rfloor$ .

#### Step 4: reconstruction from decimal digits

Let  $x \in [0, 1)$ , and let  $(c_k)_{k \geq 1}$  be its decimal digits defined by

$$c_k := \lfloor 10^k x \rfloor - 10 \lfloor 10^{k-1} x \rfloor.$$

Define a sequence of rational numbers  $(a_n)$  by

$$a_n := \sum_{i=1}^n c_i 10^{-i}.$$

**Lemma 13.** For every  $n \geq 1$ , the following inequality holds:

$$a_n \leq x < a_n + 10^{-n}.$$

*Proof.* By construction,

$$\lfloor 10^n x \rfloor = \sum_{i=1}^n c_i 10^{n-i}.$$

Dividing by  $10^n$ , we obtain

$$\sum_{i=1}^n c_i 10^{-i} \leq x < \sum_{i=1}^n c_i 10^{-i} + 10^{-n},$$

which is exactly the desired inequality. □

Passing to equivalence classes of Cauchy sequences, this yields

$$[(a_n)] \leq x < [(a_n + 10^{-n})].$$

Since the sequence  $(10^{-n})$  converges to 0, it is a null sequence. Therefore,

$$[(a_n)] \leq x \leq [(a_n + 10^{-n})] = [(a_n)].$$

Consequently,

$$x = [(a_n)].$$

Thus the real number  $x$  is represented by the Cauchy sequence

$$a_n = \sum_{i=1}^n c_i 10^{-i},$$

which recovers  $x$  from its decimal digits.

**Remark 15.** Note that we could consider, for example,  $c_1 = 4, c_2 = c_3 = \dots = 9$ . But the corresponding sequence of  $a_n$  is equivalent to 0.5. So, if our algorithm starts to produce only 9 after some step  $N$ , it means that we performed the step  $N$  incorrectly and  $x = a_N + 10^{-N}$ .

### 3.3.1 :: tutorial: from natural numbers to integers and rationals

We assume here that natural numbers have been already constructed.

The natural numbers do not allow subtraction in general (e.g.  $2 - 5$  is not in  $\mathbb{N}$ ). To obtain a number system where subtraction is always possible, we construct the integers.

**Definition 27** (Integers as equivalence classes). Consider the set  $\mathbb{N} \times \mathbb{N}$  of pairs  $(a, b)$ . Define a relation  $\sim$  on  $\mathbb{N} \times \mathbb{N}$  by

$$(a, b) \sim (c, d) \iff a + d = b + c.$$

The set of integers is by definition the quotient (i.e. the set of equivalence classes)

$$\mathbb{Z} := (\mathbb{N} \times \mathbb{N}) / \sim.$$

We denote the class of  $(a, b)$  by  $[(a, b)]$ .

**Remark 16.** The class  $[(a, b)]$  is intended to represent the formal difference  $a - b$ . For example,  $[(5, 2)]$  corresponds to  $+3$ , and  $[(2, 5)]$  corresponds to  $-3$ .

**Definition 28** (Addition and multiplication in  $\mathbb{Z}$ ). Define

$$[(a, b)] + [(c, d)] := [(a + c, b + d)],$$

$$[(a, b)] \cdot [(c, d)] := [(ac + bd, ad + bc)].$$

**Problem 1.** Check that these operations are *well-defined*: if  $(a, b) \sim (a', b')$  and  $(c, d) \sim (c', d')$ , then the results of the above formulas define the same equivalence class.

**Remark 17.** The natural numbers embed into the integers via

$$\iota_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{Z}, \quad n \mapsto [(n, 0)].$$

Under this embedding, the class  $[(a, b)]$  behaves as  $a - b$ .

## Construction of the rational numbers

The integers do not allow division in general (e.g.  $1/2$  is not in  $\mathbb{Z}$ ). To obtain a number system where division by a nonzero number is always possible, we construct the rational numbers.

**Definition 29** (Rationals as equivalence classes of fractions). Consider the set

$$\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$$

of pairs  $(a, b)$  with  $b \neq 0$ . Define a relation  $\approx$  by

$$(a, b) \approx (c, d) \iff ad = bc.$$

The set of rational numbers is the quotient

$$\mathbb{Q} := (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})) / \approx.$$

We denote the class of  $(a, b)$  by  $\frac{a}{b}$ .

**Remark 18.** The condition  $ad = bc$  expresses the familiar equality of fractions:

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc.$$

For example,  $(1, 2) \approx (2, 4)$  because  $1 \cdot 4 = 2 \cdot 2$ .

**Definition 30** (Addition and multiplication in  $\mathbb{Q}$ ). Define

$$\frac{a}{b} + \frac{c}{d} := \frac{ad + bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd},$$

and

$$-\frac{a}{b} := \frac{-a}{b}.$$

**Problem 2.** Check that these operations are *well-defined*: if  $(a, b) \approx (a', b')$  and  $(c, d) \approx (c', d')$ , then

$$\frac{ad + bc}{bd} \approx \frac{a'd' + b'c'}{b'd'} \quad \text{and} \quad \frac{ac}{bd} \approx \frac{a'c'}{b'd'}.$$

**Definition 31** (Division in  $\mathbb{Q}$ ). If  $\frac{c}{d} \neq 0$  (equivalently  $c \neq 0$ ), define

$$\frac{a}{b} / \frac{c}{d} := \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

**Remark 19.** The integers embed into the rationals via

$$\iota_{\mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{Q}, \quad a \mapsto \frac{a}{1}.$$

Under this embedding,  $\mathbb{Z}$  becomes a subset of  $\mathbb{Q}$ .

### What we gain from the construction

- $\mathbb{N}$  is the basic structure governed by the Peano axioms.
- $\mathbb{Z}$  is obtained from  $\mathbb{N}$  by formally allowing subtraction.
- $\mathbb{Q}$  is obtained from  $\mathbb{Z}$  by formally allowing division by nonzero numbers.

These constructions are canonical: although one may build  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$  in different ways, the resulting structures are the same up to a unique isomorphism.

### 3.3.2 :: :: workshop: Axioms for the natural numbers

In mathematics, the natural numbers are not introduced as “obvious objects”: they are characterized by axioms. The standard axiomatization is due to Peano.

**Definition 32** (Peano axioms). A *system of natural numbers* consists of a set  $\mathbb{N}$ , a distinguished element  $0 \in \mathbb{N}$ , and a map

$$S : \mathbb{N} \rightarrow \mathbb{N} \quad (\text{the successor map}),$$

satisfying the following axioms:

(P1)  $0$  is not the successor of any natural number: for all  $n \in \mathbb{N}$ ,  $S(n) \neq 0$ .

(P2)  $S$  is injective: if  $S(m) = S(n)$ , then  $m = n$ .

(P3) (*Induction axiom*) If  $A \subseteq \mathbb{N}$  satisfies

$$0 \in A \quad \text{and} \quad (n \in A \Rightarrow S(n) \in A),$$

then  $A = \mathbb{N}$ .

**Remark 20.** Many texts use  $1$  instead of  $0$ ; this is only a change of notation. We will use  $0$  because it simplifies the definition of addition and multiplication.

**Theorem 14** (Principle of induction). Let  $P(n)$  be a statement depending on  $n \in \mathbb{N}$ . Assume:

- $P(0)$  is true;
- for every  $n \in \mathbb{N}$ ,  $P(n) \Rightarrow P(S(n))$ .

Then  $P(n)$  is true for all  $n \in \mathbb{N}$ .

*Proof.* Let  $A = \{n \in \mathbb{N} : P(n) \text{ is true}\}$ . By assumption,  $0 \in A$  and  $n \in A \Rightarrow S(n) \in A$ . By the induction axiom (P3),  $A = \mathbb{N}$ .  $\square$

## Defining addition and multiplication on $\mathbb{N}$

Using induction, we can define operations recursively.

**Definition 33** (Addition on  $\mathbb{N}$ ). Define addition  $+$ :  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by recursion on the second variable:

$$m + 0 := m, \quad m + S(n) := S(m + n).$$

**Definition 34** (Multiplication on  $\mathbb{N}$ ). Define multiplication  $\cdot$ :  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by recursion on the second variable:

$$m \cdot 0 := 0, \quad m \cdot S(n) := m \cdot n + m.$$

**Problem 1** (Left identity for addition). Prove that for every  $n \in \mathbb{N}$ ,

$$0 + n = n.$$

**Hint 4.** Induct on  $n$ . Base:  $0 + 0 = 0$ . Step: assume  $0 + n = n$ ; then  $0 + S(n) = S(0 + n) = S(n)$ .

**Problem 2** (Successor moves across addition). Fix  $m \in \mathbb{N}$ . Prove that for every  $n \in \mathbb{N}$ ,

$$S(m) + n = S(m + n).$$

**Hint 5.** Induct on  $n$ . Base:  $S(m) + 0 = S(m) = S(m + 0)$ . Step: use the recursion  $x + S(n) = S(x + n)$  twice.

**Problem 3** (Associativity of addition). Prove that for all  $l, m, n \in \mathbb{N}$ ,

$$(l + m) + n = l + (m + n).$$

**Hint 6.** Fix  $l, m$  and induct on  $n$ . Base is immediate. For the step, expand  $(l + m) + S(n)$  using the recursion and apply the induction hypothesis inside  $S(\cdot)$ .

**Problem 4** (Commutativity of addition). Prove that for all  $m, n \in \mathbb{N}$ ,

$$m + n = n + m.$$

**Hint 7.** Induct on  $n$ . Base:  $m + 0 = m = 0 + m$  (use the left identity problem). Step: assuming  $m + n = n + m$ , compute

$$m + S(n) = S(m + n) = S(n + m).$$

Now use the identity  $S(n) + m = S(n + m)$  from the previous problem (with roles swapped) to conclude  $m + S(n) = S(n) + m$ .

**Problem 5** (Left annihilator for multiplication). Prove that for every  $n \in \mathbb{N}$ ,

$$0 \cdot n = 0.$$

**Hint 8.** Induct on  $n$ . Base:  $0 \cdot 0 = 0$ . Step:  $0 \cdot S(n) = 0 \cdot n + 0$ , and use the induction hypothesis plus  $x + 0 = x$ .

**Problem 6** (Distributivity of multiplication over addition). Fix  $m \in \mathbb{N}$ . Prove that for all  $n, k \in \mathbb{N}$ ,

$$m \cdot (n + k) = m \cdot n + m \cdot k.$$

**Hint 9.** Induct on  $k$ . Base:  $m \cdot (n + 0) = m \cdot n$  and  $m \cdot 0 = 0$ . Step: write  $n + S(k) = S(n + k)$ , then expand  $m \cdot S(n + k)$  using the recursion for  $\cdot$ , and use the induction hypothesis plus associativity of  $+$ .

**Problem 7** (Right identity for multiplication (optional)). Let  $1 := S(0)$ . Prove that for every  $n \in \mathbb{N}$ ,

$$n \cdot 1 = n.$$

**Hint 10.** Use the recursion:  $n \cdot 1 = n \cdot S(0) = n \cdot 0 + n = 0 + n = n$ .

**Problem 8** (Associativity of multiplication). Prove that for all  $l, m, n \in \mathbb{N}$ ,

$$(l \cdot m) \cdot n = l \cdot (m \cdot n).$$

**Hint 11.** Fix  $l, m$  and induct on  $n$ . Base is immediate. Step: expand both sides at  $S(n)$  using the recursion for  $\cdot$ , then use distributivity and associativity of addition.

**Problem 9** (Commutativity of multiplication (challenge)). Prove that for all  $m, n \in \mathbb{N}$ ,

$$m \cdot n = n \cdot m.$$

**Hint 12.** A standard route is to first prove, by induction on  $m$ , the auxiliary identity

$$S(n) \cdot m = n \cdot m + m$$

(for fixed  $n$ ). Then induct on  $n$  to compare  $m \cdot S(n) = m \cdot n + m$  with  $S(n) \cdot m = n \cdot m + m$ , and use commutativity and associativity of addition.

:: homework puzzle: “All triangles are equilateral” — find the mistake

We will now present a (false) “proof” that every triangle is equilateral. Since an equilateral triangle is, in particular, isosceles, it is “enough” to show that an arbitrary triangle  $ABC$  is isosceles with  $|AB| = |AC|$ . (If we could do this for an arbitrary triangle, then repeating the same argument for the other vertices would force  $|AB| = |BC| = |CA|$ .)

So, find the mistake in the following argument.

Let a triangle  $ABC$  be given. Clearly, it is enough to prove that it is isosceles, i.e.

$$|AB| = |AC|.$$

Draw the bisector of angle  $A$  until it meets the perpendicular bisector of the side  $BC$  at a point  $O$  (Fig. 1). Let  $H$  be the midpoint of  $BC$ , so that  $BH = CH$  and  $OH \perp BC$ . Logically, three cases are possible:

1. The point  $O$  lies inside the triangle  $ABC$ .
2. The point  $O$  lies outside the triangle  $ABC$ .
3. The point  $O$  lies on the side  $BC$ .

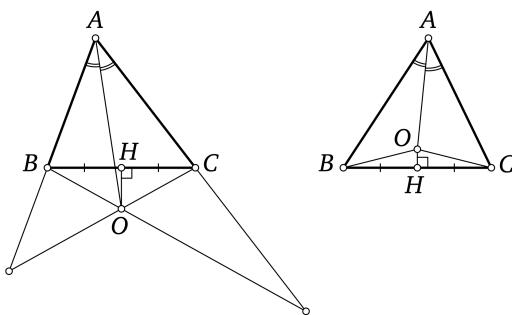


Figure 1: Equilateral triangles?

**(1)  $O$  lies inside  $ABC$ .** First we prove that the triangles  $HOB$  and  $HOC$  are congruent. Indeed, they have the common side  $OH$ ; the angles  $\angle CHO$  and  $\angle BHO$  are equal (both are right angles); and the sides  $BH$  and  $CH$  are equal by construction. Therefore, the segments  $BO$  and  $CO$  are equal.

Now we prove that the triangles  $AOB$  and  $AOC$  are congruent. They have the common side  $AO$ ; the angles  $\angle OAB$  and  $\angle OAC$  are equal by construction (since  $AO$  is the angle bisector); and the sides  $BO$  and  $CO$  are equal (as we have just proved). Hence  $|AB| = |AC|$ , and the triangle  $ABC$  is isosceles.

**(2)  $O$  lies outside  $ABC$ .** First we prove that the triangles  $HOB$  and  $HOC$  are congruent. Indeed, they have the common side  $OH$ ; the angles  $\angle CHO$  and  $\angle BHO$  are equal (both are right angles); and the sides  $BH$  and  $CH$  are equal by construction. Therefore, the segments  $BO$  and  $CO$  are equal.

Now we prove that the triangles  $AOB$  and  $AOC$  are congruent. They have the common side  $AO$ ; the angles  $\angle OAB$  and  $\angle OAC$  are equal by construction; and the sides  $BO$  and  $CO$  are equal (as we have just proved). Hence  $|AB| = |AC|$ , and the triangle  $ABC$  is isosceles.

**(3)  $O$  lies on  $BC$ .** In this case the points  $H$  and  $O$  coincide, and the lines  $AH$  and  $AO$  coincide as well. Therefore, the triangle  $ABC$  is isosceles.

This concludes the “proof”.

**Question.** Where is the mistake?

## 4 Order as a binary relation. Examples and lexicographic order

### 4.1 :: orders, lexicographic order, lexicographic induction

#### Order relations as binary relations

Recall that a *binary relation* on a set  $X$  is a subset  $R \subseteq X \times X$ . If  $(x, y) \in R$ , we write  $xRy$ .

**Definition 35** (Partial order). A binary relation  $\leq$  on a set  $X$  is called a *partial order* if it satisfies:

1. **Reflexive:** for all  $x \in X$ ,  $x \leq x$ .
2. **Antisymmetric:** for all  $x, y \in X$ , if  $x \leq y$  and  $y \leq x$ , then  $x = y$ .
3. **Transitive:** for all  $x, y, z \in X$ , if  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ .

A set  $X$  equipped with a partial order is called a *partially ordered set* (or *poset*).

**Definition 36** (Total (linear) order). A partial order  $\leq$  on  $X$  is called a *total order* (or *linear order*) if in addition it satisfies:

$$\text{for all } x, y \in X, \text{ either } x \leq y \text{ or } y \leq x.$$

This property is called *comparability*.

**Example 15** (Standard orders).

- The usual order on  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$  is a total order.

- The subset relation  $\subseteq$  on the power set  $\mathcal{P}(X)$  is a partial order (usually not total).
- Divisibility on  $\mathbb{N}$ :

$$a \preceq b \iff a | b$$

is a partial order on  $\mathbb{N}$  (again, not total).

**Example 16** (Why some orders are not total). On  $\mathcal{P}(\{1, 2, 3\})$  with  $\subseteq$ , the sets  $\{1\}$  and  $\{2\}$  are not comparable:

$$\{1\} \not\subseteq \{2\} \quad \text{and} \quad \{2\} \not\subseteq \{1\}.$$

Thus  $\subseteq$  is not a total order.

**Example 17.** Picture  $R \subset A^2$  for the total order on  $A = \{1, 2, 3, 4, 5\}$ .

### Lexicographic order

Lexicographic order is the mathematical version of the order used in dictionaries (see the cover page of these notes).

**Definition 37** (Lexicographic order on pairs). Let  $(A, \leq_A)$  and  $(B, \leq_B)$  be totally ordered sets. Define a relation  $\leq_{\text{lex}}$  on  $A \times B$  by:

$$(a, b) \leq_{\text{lex}} (a', b') \iff (a <_A a') \text{ or } (a = a' \text{ and } b \leq_B b').$$

**Problem 1.** Prove that if  $\leq_A$  and  $\leq_B$  are total orders, then  $\leq_{\text{lex}}$  is a total order on  $A \times B$ .

**Definition 38** (Lexicographic order on  $n$ -tuples). Let  $A_1, \dots, A_n$  be totally ordered sets. For  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  in  $A_1 \times \dots \times A_n$ , define

$$x \leq_{\text{lex}} y$$

if at the first position where  $x$  and  $y$  differ, the corresponding coordinate of  $x$  is smaller. Formally,  $x <_{\text{lex}} y$  if there exists  $k$  such that

$$x_1 = y_1, x_2 = y_2, \dots, x_{k-1} = y_{k-1}, \quad \text{and} \quad x_k < y_k.$$

### Applications of lexicographic order

**Example 18** (Sorting words). Let  $A$  be the set of letters with the usual alphabetic order. A word is a finite sequence of letters. If we order words lexicographically, then we recover dictionary order. For example:

$$\text{cat} < \text{cater} < \text{dog}.$$

**Example 19** (Sorting by two keys). Suppose each student has a pair (grade, name), where grades are integers and names are ordered alphabetically. Lexicographic order on

$$(\text{grade}, \text{name})$$

sorts students primarily by grade, and among equal grades, by name.

**Example 20** (Calendar order of dates). A date can be represented by a triple

$$(\text{year}, \text{month}, \text{day}) \in \mathbb{N}^3.$$

Lexicographic order on these triples gives the usual chronological order: first by year, then by month, then by day.

**Example 21** (Decimal expansions and order). A nonnegative real number can be represented (after choosing a unique decimal expansion without an eventual tail of 9) by

$$(b_0, b_1, b_2, \dots), \quad b_0 \in \mathbb{N}, \quad b_i \in \{0, 1, \dots, 9\}.$$

Comparing two such expansions digit by digit is a lexicographic comparison: the first digit where they differ determines which number is larger. This is exactly how we compare decimals in elementary school.

**Example 22** (Minimal counterexample arguments). Lexicographic order is often used to formalize “minimal counterexample” proofs when there are several parameters. For example, among counterexamples (page 132) described by a pair  $(p, AB)$ , one can choose a counterexample with minimal  $p$ , and among those with minimal  $AB$ . This is precisely choosing a minimal element with respect to lexicographic order on  $\mathbb{N}^2$ .

## Exercises

**Problem 2.** On  $\mathbb{N}^2$ , compare the following two orders:

- lexicographic order;
- order by sum:  $(a, b) \preceq (c, d)$  if  $a + b \leq c + d$ .

Which one is a total order? Which one is not?

### **:: :: tutorial and workshop: lexicographic induction**

Let two sequences of natural numbers be given

$$A = (a_1, \dots, a_n), \quad B = (b_1, \dots, b_n).$$

We say that  $A > B$  (that is,  $A$  is *lexicographically larger* than  $B$ ) if either  $a_1 > b_1$ , or  $a_1 = b_1$  and  $a_2 > b_2$ , or  $a_1 = b_1$ ,  $a_2 = b_2$  and  $a_3 > b_3$ , and so on.

1. Prove that:
  - (a) if  $A > B$  and  $B > C$ , then  $A > C$ ;
  - (b) any strictly decreasing sequence of strings (sequences) of length  $n$  is finite;
  - (c) every nonempty set of strings (sequences) of length  $n$  has a smallest element.
2. Let  $x > 1$  be a natural number. Every second Diego replaces it by the number
 
$$y = \frac{x}{p} (p - 1)^k,$$
 where  $p$  is some prime divisor of  $x$ , and the integer  $k$  is arbitrary (and may change from step to step). Prove that sooner or later Diego will obtain 1.
3. A businessman made a deal with the devil: every day the businessman gives the devil one banknote, and in return receives any number of banknotes he wants, but of strictly smaller denomination.

The businessman has no other source of banknotes, and initially he has only finitely many. Prove that no matter how the businessman exchanges banknotes, at some moment he will go bankrupt.

4. A computer screen shows a certain finite sequence of zeros and ones. One may perform the following operation: replace the block of digits “01” by either
  - (a) “1000”,
  - (b) “100...0”, where the number of zeros is arbitrary and may change from step to step.

Prove that no matter how we act, we cannot perform this operation infinitely many times.

5. Participants of some exam are seated one per desk arranged as a  $10 \times 10 \times 10$  cube. The exam consists of one problem. After the first hour, some students solved and wrote down the solution. From this moment on, each minute one student either copies the solution from a neighbor, or discovers a flaw in an already written solution and crosses it out. The proctor tries to prevent copying, so copying from a neighbor on the left, above, or behind is impossible. It is known that every written solution will eventually be crossed out. Prove that sooner or later all students will cross out all solutions in their notebooks, and thus all fail the exam.
6. Let  $a_1 < a_2 < \dots$  be an increasing sequence of natural numbers such that among the prime divisors of the numbers  $a_1, \dots, a_{100}$  there occur
  - (a) only 2 and 5,
  - (b) only 2, 3, and 5,

and moreover, for every natural number  $n > 100$ , the number  $a_n$  is defined as the smallest natural number greater than  $a_{n-1}$  that

is not divisible by any of the numbers  $a_1, a_2, \dots, a_{n-1}$ . Prove that this sequence contains only finitely many composite numbers.

## 4.2 :: afterword for a course

We hope you enjoyed the course. Solve the following problem (hint: by induction! since 100 looks like a random number)

**Exercise 16.** There is an island upon which a tribe resides. The tribe consists of 100 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness. All the tribespeople are highly logical and devout, and they all know that each other is also highly logical and devout (and they all know that they all know that each other is highly logical and devout, and so forth).

Of the 100 islanders, it turns out that  $n$  of them have blue eyes and  $100 - n$  of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 99 of the 100 tribespeople).

One day, a blue-eyed foreigner visits to the island and wins the complete trust of the tribe.

One evening, he addresses the entire tribe to thank them for their hospitality.

However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

What effect, if anything, does this *faux pas* have on the tribe? a) Consider the case  $n = 1$ . b) Consider the case  $n = 2$ . c) Consider the general case.

## 5 :: Materials

### 5.1 :: credits and what can be done differently

**Credits.** The first chapter consists of edited, AI-translated excerpts from the excellent Russian book by Kanel-Belov and Kovaldgi, *How to Solve Non-Standard Problems*.<sup>43</sup>

The chapters on naive set theory benefited substantially from N. Vavilov's *Not Quite Naive Set Theory* (in Russian). Most of the content is not intended to be original and was adapted from various sources.

A large portion of the text was produced using the following workflow: provide a proof strategy to the paid version of ChatGPT (for example: "here is the proof strategy [...]; write the details and correct the grammar"), and then edit the output carefully.

The advice on writing on pages 198 was taken from an excellent "Everything You Always Wanted To Know About Mathematics\*" (\*But didn't even know to ask)" by Brendan W. Sullivan, [https://www.math.cmu.edu/~jmackey/151\\_128/bws\\_book.pdf](https://www.math.cmu.edu/~jmackey/151_128/bws_book.pdf)

**What can be done differently.** It is better to **reduce the amount of material.** In particular, spend more time on foundational techniques (induction, the pigeonhole principle, etc.) in a tutorial-style format rather than a lecture format: solve problems slowly with students and discuss the reasoning in detail.

- Allocate more time to permutations (for example, dedicate an entire lecture to permutations).
- Allocate more time to the construction of the real numbers (two weeks rather than one).

---

<sup>43</sup><https://old.mccme.ru/free-books/olymp/KanKov.pdf>

- Ideally, the constructions of  $\mathbb{N}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$  should be done fully in lectures; tutorials and workshops should then be used to check complete proofs.

In our run of the course we mostly discussed proof strategies and proved only the key statements. In that sense, the construction part was only half-finished. This may be acceptable (many technical proofs are routine), but it may also be a drawback: proofs are the working substance of mathematics (the blood), while ideas are its organizing core (the heart). The heart without blood is not viable.

**How to teach writing.** Writing should likely be taught in a separate lecture (ideally in a preparatory course). One possible structure is:

1. Assign a first homework with full written solutions.
2. Grade it, collect the most common issues, and discuss them in a lecture.
3. In the same lecture, demonstrate how to rewrite a proof step-by-step (for example, on slides), showing several successive improvements.
4. Assign a second homework asking students to rewrite one of their own proofs according to the guidelines.
5. Repeat once more, and then add peer review: students read and critique each other's proofs.

This approach is time-consuming, but it teaches a real skill. See also the course *Mathematical Writing* by Donald E. Knuth, Tracy Larrabee, and Paul M. Roberts.

**About reading.** Provide a short text and ask a small set of comprehension questions (for example, via Moodle).

**Further references.** A useful point of comparison is the MIT course *Mathematics for Computer Science*.<sup>44</sup> It has a different syllabus, but it is helpful for seeing what is covered and how it is taught.

A very good book to read selectively is Courant and Robbins, *What Is Mathematics?*

As supplementary reading, I also recommend Timothy Gowers, *Mathematics: A Very Short Introduction*. It is particularly suitable for students who want to improve their English, and can be used as part of language practice.

## Comments to the arithmetic/geometric mean inequality. Notes from tutorial 1

In the tutorial, we proved that if  $n = 2^k$  for some positive integer  $k$ , and  $x_1, \dots, x_{2^k}$  are positive numbers, then the inequality

$$(x_1 \cdots x_{2^k})^{\frac{1}{2^k}} \leq \frac{x_1 + \cdots + x_{2^k}}{2^k} \quad (9)$$

holds. In particular, equality occurs if and only if

$$x_1 = x_2 = \cdots = x_{2^k}.$$

We are left to show the case when  $n$  is not a power of 2, that is, we need to fill the gaps between powers of 2.

Let  $x_1, \dots, x_n$  be arbitrary positive numbers, and choose  $k$  such that  $n \leq 2^k$ . Define

$$\alpha_i := \begin{cases} x_i, & i \leq n, \\ A, & n+1 \leq i \leq 2^k, \end{cases} \quad \text{where } A := \frac{x_1 + \cdots + x_n}{n}.$$

---

<sup>44</sup><https://ocw.mit.edu/courses/6-042j-mathematics-for-computer-science-fall-2010/pages/assignments/>

We apply inequality (9) to

$$(\alpha_1 \alpha_2 \cdots \alpha_{2^k})^{\frac{1}{2^k}},$$

which yields (remark that  $nA = x_1 + \cdots + x_n$ )

$$(x_1 \cdots x_n A^{2^k-n})^{\frac{1}{2^k}} \leq \frac{x_1 + \cdots + x_n + (2^k - n)A}{2^k} = A.$$

We obtain

$$(x_1 \cdots x_n)^{\frac{1}{2^k}} \leq A^{1 - \frac{2^k - n}{2^k}} = A^{\frac{n}{2^k}}.$$

Finally, raising both sides to the power  $\frac{2^k}{n}$  gives

$$(x_1 \cdots x_n)^{\frac{1}{n}} \leq A = \frac{x_1 + \cdots + x_n}{n}.$$

This completes the proof.

## Sufficiency of operations $\cup$ and $\Delta$

*Exercise: Express  $\cap$  and  $\setminus$  using only the operations  $\cup$  and  $\Delta$ .*

*Answer:*

1. We prove that

$$A \cap B = (A \cup B) \Delta (A \Delta B).$$

First, we show that  $(A \cap B) \subseteq (A \cup B) \Delta (A \Delta B)$ .

Let  $x \in (A \cap B)$ . Then  $x \in (A \cup B)$  and  $x \notin (A \Delta B)$ , so  $x \in (A \cup B) \setminus (A \Delta B) \subseteq (A \cup B) \Delta (A \Delta B)$ .

Next, we show that  $(A \cup B) \Delta (A \Delta B) \subseteq (A \cap B)$ .

Let  $x \in (A \cup B) \Delta (A \Delta B)$ . Then

$$x \in ((A \cup B) \setminus (A \Delta B)) \cup ((A \Delta B) \setminus (A \cup B)),$$

by the definition of the symmetric difference.

If  $x \in (A \cup B) \setminus (A \Delta B)$ , then  $x \notin (A \Delta B)$ , which implies  $x \in (A \cap B)$ . Moreover, it is clear that  $(A \Delta B) \setminus (A \cup B) = \emptyset$ . Thus,  $(A \cup B) \Delta (A \Delta B) \subseteq (A \cap B)$ , completing the proof.

2. We now prove that

$$A \setminus B = (A \cup B) \Delta B.$$

First, we show that  $A \setminus B \subseteq (A \cup B) \Delta B$ .

If  $x \in A \setminus B$ , then  $x \in (A \cup B)$  and  $x \notin B$ , so  $x \in (A \cup B) \setminus B \subseteq (A \cup B) \Delta B$ .

Now, we show that  $(A \cup B) \Delta B \subseteq A \setminus B$ .

Let  $x \in (A \cup B) \Delta B$ . Then either  $x \in (A \cup B) \setminus B$  or  $x \in B \setminus (A \cup B)$ , by definition of the symmetric difference.

If  $x \in (A \cup B) \setminus B$ , then  $x \in A \setminus B$ . On the other hand,  $B \setminus (A \cup B) = \emptyset$ . Therefore,  $(A \cup B) \Delta B \subseteq A \setminus B$ , completing the proof.

## 5.2 Basic Graph Theory: Trees, Paths, and Leaves

**Graph.** A graph  $G = (V, E)$  consists of a set of vertices  $V$  and a set of edges  $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$ .

**Path.** A path is a sequence of distinct vertices  $v_1, \dots, v_k$  such that each consecutive pair  $(v_i, v_{i+1})$  is an edge; its length is  $k - 1$ .

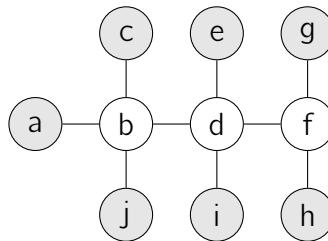
**Tree.** A tree is a connected graph with no cycles (acyclic). We will prove in tutorial that a tree on  $n$  vertices has exactly  $n - 1$  edges.

### A question before the tutorial

A leaf is a vertex of degree 1.

Why every tree has a leaf ?

### Example tree with leaves highlighted



In the picture above, the light vertices  $a, c, e, g, h, i, j$  are leaves (each has degree 1). Vertices  $b, d, f$  have degree  $\geq 2$ .

## 5.3 Corrections of students' homework

### Student 1

1. **Solution:** Obviously, we cannot cut a  $2 \times 2$  square without opposite corners into dominoes. Suppose we cannot cut a  $2^n \times 2^n$  square with opposite corners removed into dominoes. A  $2^{n+1} \times 2^{n+1}$  square can be cut into four  $2^n \times 2^n$  squares. Even though it is obvious, you should show a picture showing what the cut is We place two dominoes at the center of this  $2^{n+1} \times 2^{n+1}$  square so that they remove one corner from each of the four  $2^n \times 2^n$  squares. In the case of two  $2^n \times 2^n$  squares, their diagonals are effectively absent, meeting the above condition. The other two  $2^n \times 2^n$  squares have one corner missing. here you should also show a picture Since the total grid count is even and dominoes cover an even number of cells, the coverable cells in the remaining squares are odd Here you are assuming that a domino should be contained in one of the four squares, which is not in the assumption. Hence the argument is wrong. Hence, we prove that a  $2^{n+1} \times 2^{n+1}$  square with opposite corners removed cannot be tiled with dominoes.
2. **Solution:** According to Problem 1, we have proven by induction that a  $2^n \times 2^n$  square without opposite corners cannot be cut into dominoes. Therefore, we cannot cut a  $2^n \times 2^n$  square without opposite corners into dominoes.

### 3. **Solution:**

- When  $n = 7$ , a large square is directly divided into four equal small squares, then one of these small squares is divided into four smaller squares, making a total of seven squares. You should show a picture
- When  $n = 8$ , first divide the square into two rectangles, and then divide it into eight squares. Show a picture. I am afraid

you cannot obtain 8 squares in this form

- When  $n = 9$ , we can directly divide a square into nine squares.  
*show a picture*

We will prove this proposition by induction.

Suppose we can cut a square into  $n$  squares for each  $n$ . For a square that has been cut into  $n$  squares, cut one small square into four smaller squares. Then we prove we can cut a square into  $n + 3$  squares for each  $n$ . If you are going to apply induction, you could say: We already know that a square can be cut into 7,8 or 9 squares. Let  $n \geq 9$  and suppose that a given square can be cut into  $k$  squares for any  $k$  between 7 and  $n$ . We are doing to cut the original square into  $n + 1$  small squares. Let  $k = n + 1 - 3$ . Then  $k$  is between 7 and  $n$  so the original square can be cut into  $k$  squares by the inductive hypothesis. Choose one small square and cut it into four smaller squares. Then the original square has been cut into  $k - 1 + 4 = k + 3 = n + 1$  squares. By induction, we can say that the original square can be cut into  $n$  squares for any  $n \geq 7$ .

Alternatively, you could also say: Note that if the given square can be cut into  $n$  squares then it can be also cut into  $n + 3$  squares. Indeed, for a square that has been cut into  $n$  squares, cut one small square into four smaller squares. Since any number  $n \geq 10$  can be written as  $7 + 3k$ ,  $8 + 3k$  or  $9 + 3k$  for  $k \in \mathbb{N}$  and we know how to cut a square into 7, 8 and 9 squares, then we can cut a square into  $n$  square for any  $n \geq 10$  too.

Hence, we prove it.

4. **Solution:** Totally, we have 51 odds, 50 evens, the sum of all numbers is odd (why?). Suggestion: At the beginning we have 51 odd numbers, 50 even numbers, and the sum of all the numbers

is  $101 \times 102/2 = 101 \times 51$ . We can do three operations: We analyze the 3 possible operations

- Odd minus even results in an odd, and the parity of the sum remains unchanged. **you should give an explanation for this.**
- Odd minus odd results in an even, and the parity of the sum remains unchanged. **you should give an explanation for this.**
- Even minus even results in an even, and the parity of the sum remains unchanged. **you should give an explanation for this.**

Hence, the parity of the sum is odd. (**Suggestion: Hence the parity of the sum doesn't change after any sequence of these operations. Since at the beginning the sum is odd, it will be always odd.**) When the last number is 0, the parity of the sum becomes even, leading to a contradiction. Thus, we prove it cannot be zero.

## Student 2

### Problem 1: Tiling a Square Without Opposite Corners

Suppose we don't have the corners  $(1, 1)$  and  $(4, 4)$ .

- For row 1: We still have  $(1, 1)$ ,  $(1, 3)$ ,  $(1, 4)$
- Let  $(1, 2)$  and  $(2, 2)$  be a domino (**Explain why it is enough to consider this case**). Then  $(1, 3)$  and  $(2, 3)$  should be a domino (**why?, why can't  $(1, 3)$  and  $(1, 4)$  be covered by a domino instead?**),  $(4, 4)$  and  $(5, 4)$  should be a domino.
- For row 2: We just have  $(2, 1)$ , then let  $(2, 1)$  and  $(3, 1)$  be a domino.
- For row 3: We have  $(3, 2)$ ,  $(3, 3)$ ,  $(3, 4)$ , let **(. Let)**  $(3, 4)$  and  $(4, 4)$ ,  $(3, 5)$  and  $(4, 5)$  be dominoes. **Why do you consider these cases only?** Then we still have  $(3, 4)$ .

- For row 4: We still have (4, 1)

But (4, 1) and (3, 4) can't be a domino. Thus, we cannot cut a square without opposite corners into dominoes.

### Problem 2: Chessboard Coloring Argument

We can color the  $8 \times 8$  square like a chessboard. Then a domino should cover one black square and one white square.

If we remove two black corners, then the removed cells are of the same color, and we can assume that they are black. Hence we have 30 black squares and 32 white squares. Thus, we can have 30 dominoes but still have 2 white squares left. Thus we can use at most 30 dominoes, leaving at least 2 white cells uncovered.

Therefore, we cannot cut an  $8 \times 8$  square without opposite corners into dominoes.

### Problem 3: Dividing a Square into Smaller Squares

- If  $n = 7$ , then we can cut a square into a square  $2 \times 2$  and 6 squares  $1 \times 1$ . how? there are no pictures.
- If  $n = 8$ , then we can cut a square into a square  $3 \times 3$  and a square  $1 \times 1$ . how? there are no pictures.
- If  $n = 9$ , then we can cut a square into 9 squares  $1 \times 1$ . how? there are no pictures.

For  $n > 9$ , if we can cut a square into  $k$  squares, then cut one of the squares into 4 smaller squares, thus we will get  $k + 3$  squares.

- For  $n = 7$ , then we can get  $n = 10 = 7 + 3$ ,  $n = 13 = 10 + 3$ , ...
- For  $n = 8$ , then we can get  $n = 11 = 8 + 3$ ,  $n = 14 = 11 + 3$ , ...

- For  $n = 9$ , then we can get  $n = 12 = 9 + 3$ ,  $n = 15 = 12 + 3$ , ...

In all, we can cut a square into 10 squares for each  $n > 6$ .

#### **Problem 4: Parity Argument with Numbers**

From 1 to 101, there are 51 odd numbers and 50 even numbers.

Since Note that:

- odd - odd = even
- even - odd = odd
- even - even = even
- odd - even = odd

Thus:

- If we remove two odd numbers, the count of odd numbers will decrease by 2
- If we remove an odd number and an even number, the counts will remain the same
- If we remove two even numbers, the counts will remain the same

Therefore, no matter what two numbers we remove, the count of odd numbers is always an odd number (at least 1).

Hence, this number cannot be zero.

#### **Student 3**

##### **problem 1**

You could begin by saying: Assume that it is possible and let us consider a covering. We write  $(a, b) \rightarrow (c, d)$  to indicate that the cells  $(a, b)$  and  $(c, d)$  are covered by a domino.

The first line:

$(1, 2) \rightarrow (1, 3) \Rightarrow$  must  $(1, 4)$  y  $(2, 4)$  You could say. If  $(1, 2) \rightarrow (1, 3)$  then necessarily  $(1, 4) \rightarrow (2, 4)$

The second line:

$(2, 3) \rightarrow (2, 2) \Rightarrow$  must implies  $(2, 1) \rightarrow (3, 1)$ . Why do you assume  $(2, 3) \rightarrow (2, 2)$ ?

The third line:

$(3, 3) \rightarrow (3, 4) \Rightarrow$  must  $(3, 2) \rightarrow (4, 2)$  Same comment

So  $(4, 1)$  is solo is false However, it is now impossible to cover  $(4, 1)$

if  $(3, 2) \rightarrow (3, 3)$  that  $(3, 4)$  must  $\rightarrow (4, 4)$

but  $(4, 4)$  is not exist so is false. is not exist doesn't exist in English.  
You may say: but then  $(4, 4)$  can not be covered.

It seems that you are considering two cases. Be more explicit.

## Problem 2

Color the  $8 \times 8$  grid in black and white like a chessboard  
 $8 \times 8 = 64$ : it has 32 white cells and 32 black cells

reduce opposite corners  $64 - 2 = 62$  After removing two opposite corners, there are 62 cells left

because the opposite corners have the same color

therefore it has 30 black (white) and 32 white (black)

Since two opposite corners have the same color, we may assume that

there are now 30 black cells and 32 white cells

because every domino have one black one-white color

but now the number (white and black) is not equal But every domino covers one black cell and one white cell and the number of white and black cells in the table are unequal; hence such a covering is impossible

so it is false . But every domino covers one black cell and one white cell and the number of white and black cells are unequal; hence such a covering is impossible

### Problem 3

Let the large square cut into a  $4 \times 2 \times 2$  squares You might say: Let the large square be cut into four  $2 \times 2$ -squares. Include a figure

Let let one of the small squares be cut to into a  $2 \times 2$  smaller squares like (figure)

it It totally has 7 squares and repeat . Repeat this process

We we can know what cut a square in  $7 + 3n$  is correct. We therefore know how to cut a square into  $7 + 3n$  squares

$n = 8$  like (figure) (The cut in the figure has only 7 squares

$n = 9$  like (figure)

so  $9 + 3n$  is correct so  $n > 6$  is correct. You probably want to say: Therefore, we can cut a square into  $7 + 3n$ ,  $8 + 3n$  and  $9 + 3n$  for any non-negative integer  $n$

## Student 4

### Problem 1

(figure) Establish a plane right angle coordinate system and each block can match a coordinate point.

Then we notice that  $(1, 2)$  can only connect with  $(1, 3)$  or  $(2, 2)$  to create a domino.

**(Case 1):**  $P(1, 2)$  connect with  $(1, 3)$ ,

$\Rightarrow (1, 4)$  can only connect with  $(2, 4)$

$\Rightarrow (3, 4)$  can only connect with  $(3, 3)$

$\Rightarrow (2, 3)$  can only connect with  $(2, 2)$

and  $(4, 3)$  can only connect with  $(4, 2)$

$\Rightarrow (3, 2)$  can only connect with  $(3, 1)$

Then we find that  $(2, 1)$  and  $(4, 1)$  cannot create into a domino.

Therefore this case is invalid.

**(Case 2):**  $P(1, 2)$  connect with  $(2, 2)$

$\Rightarrow (2, 1)$  can only connect with  $(3, 1)$

$\Rightarrow (4, 1)$  can only connect with  $(4, 2)$

$\Rightarrow (3, 2)$  can only connect with  $(3, 3)$

then we find that  $(4, 3)$  cannot connect with anything. Therefore, this case is also invalid.

So we cannot cut a square without opposite corners into domino.

### Problem 2

(figure: chess board)

We can cover some blocks with shadow to make sure that each block was surrounded by four blocks with shadow like the picture. Then we can find that a domino must be created by a white block and a shadowed block.

However, there exist  $8 \times 8 / 2 = 32$  white blocks but only exist  $8 \times 8 / 2 - 2 = 30$  shadowed blocks. Then there must exist two white blocks

that cannot be connected into a domino. Therefore, we cannot cut a square  $8 \times 8$  without opposite corners into domino.

### Problem 3

Assume that we can cut a square in

$$k \text{ squares}, k \in \mathbb{N}$$

Then we can choose one square from  $k$  squares and cut it into four  $2 \times 2$  squares. Then we can get 3 more squares. Then we can cut a square in  $k + 3$  squares

$$k \in \mathbb{N}$$

Then we only need to find that a square can be cut into  $3n$  squares,  $3n + 1$  squares and  $3n + 2$  squares.  $\exists n \in \mathbb{N}$ . This phrase is badly written. You may want to say. Every  $n \geq 7$  is of the form  $9 + 3n$ ,  $8 + 3n$  or  $7 + 3n$ .

1) cut a square in  $3n$  squares Cut of a square into  $9 + 3n$  squares if  $n = 3$ , then  $3n = 9$  By the above argument, it is enough to cut a square into 9 squares.

then we can cut it into a square  $3 \times 3$  valid no figure is shown

2) cut a square in  $3n + 1$  squares Cut of a square into  $7 + 3n$  squares if  $n = 1$ , then  $3n + 1 = 4$  then we can cut it into a square  $2 \times 2$  valid no figure is shown. Same comment as above

3) cut a square in  $3n + 2$  squares if  $n = 2$ , then  $3n + 2 = 8$  then we can cut it like the picture (figure) to get 8 squares, valid Same comment  
Therefore, we can cut a square in  $n$  squares,  $n > 6$  and  $n \in \mathbb{N}$ .

### Problem 4

Assume that the remaining number is zero. There exist At the beginning there are 51 odd numbers and 50 odd (even)numbers and odd number will

not effect the parity (**I don't understand this phrase**), and the difference of odd numbers must be an even number.

So after repeating this operation to 50 odd numbers and 50 even numbers, the outcome must be an even number, then the difference between this even number and the last odd number must be an odd number. **This part of the argument is unclear**

However, zero is an even number Then contradiction **This is a contradiction** Then the number remained cannot be zero. **The remaining number cannot be zero**

## Student 5

### Problem 1

We assume first domino horizontal in row.

Row 1 has squares (1,1) removed, so possible dominoes are (1,2)–(1,3) or (1,3)–(1,4).

If (1,2)–(1,3)) Suppose that (1,2)–(1,3) is a domino. At this time Row 1 has (1,4) left - need to (2,4) so (1,4)–(2,4) should be a domino. Now Row 2 has (2,1)–(2,3) left. Placing force (2,1)–(2,2) force (2,3) left **why do you assume that (2,1)–(2,2) is a domino?**

(2,3) pair with (3,3) but (3,3)'s adjacent squares. You might say that (3,4) cannot be part of a domino.

Later create (3,3) unpaired square. **this part is unclear**

Others situation is similar to above. **why?**

### Problem 2

(1)  $8 \times 8$  has 32 black and 32 white squares. Removing 2 opposite corners. These corners are **of** the same color.

Thus the remain 30 black and 32 white. **Thus we may assume that the remaining cells are 30 black cells and 32 white cells** A domino always

covers 1 black **cell** and 1 white **cell**, but the number of white **cells** is not equal to the number of black **cells**.

Therefore It's impossible to tile an  $8 \times 8$  square.

### Problem 3

$n = 6$ : Divide the original square into 1 large square ( $3/4$  area) and 5 smaller equal square ( $1/16$  area) in the remaining area. (figure) In the figure there are 8 squares. Besides  $3/4+5/16$  is not 1

$n = 7$ : Divide the original square into 1 large square ( $2/3$  area) and 6 smaller equal square ( $1/9$ ) in the remaining area. No figure is shown and same problem as above

$n \geq 8$ : start with  $n = 6$  or  $n = 11$  split one small square into 4 equal smaller squares. This argument is unclear

### Problem 4

The sum of 1 to 101 is odd:  $S = n(n + 1)/2 = 101 \times 102/2 = 5151$  because  $a + b$  and  $|a - b|$  have same parity. Don't begin a sentence with because. You could say: Note that  $a + b$  and  $|a - b|$  have the same parity. By the way: Why this is true? after 100 operations the final number equals the total sum's parity (odd). Before saying this, you should say, since  $a + b$  and  $|a - b|$  have the same parity, the parity of the total sum is unchanged under any operation. Since it was odd at the beginning, it must be always odd; in particular, it can never be 0.

Since 0 is even so the final number cannot be 0.

### Student 6

#### Problem 1

Solution: Assume that we can cut a square  $4 \times 4$  without opposite corners into domino. Analyze the following programm (program: Why do you call it a program?):

(figure)

Number each grid cells, we can get that there is just one case, since the programmer is symmetry **symmetric**, then . Then 1-2, and could only be 3-4, 11-10, 12-13, 14-8, 7-6. Finally we can see that 5 and 9 are not covered which is contradiction. Therefore, we cannot cut a square  $4 \times 4$  without opposite corners into domino.

## Problem 2

Solution: Color the table  $8 \times 8$  into black and white, like chess **end a sentence with a period**

then **Then (the first word in a sentence should begin with a capital letter)**each domino contains 1 white and 1 black cell

(figure) here 32 white cells 30 black cells

but a white cell needs to be paired with a black cell, thus the number of white cells and the number of black cells need to be equal. Therefore we cannot cut a square  $8 \times 8$  without opposite corners into dominos.

## Problem 3

Solution: Base step:

If  $n = 7$ , the result is true. proof: Start with a  $2 \times 2$  square and divide one of the square into 4 smaller squares. This gives  $\frac{1}{4} + 4 = 7$  squares.

If  $n = 8$ , the result is true. proof: Divide a  $3 \times 3$  square into 9 smaller squares and merge 2 adjacent squares to form a  $2 \times 2$  square. This gives  $9 - 2 - 1 = 8$  squares. **no figure. It seems that this is incorrect**

If  $n = 9$ , the result is true. proof: Simply divide the square into 9 equal smaller squares. **You should include a figure**

Inductive step: Suppose the result is true for  $n$  ( $n > 6$ )

n.t.p. the result is true for  $n + 3$ . Since we already know that the result is true for 7,8,9 we only need to prove that the result is true for  $n + 3$

Proof: To get  $n + 3$  squares from  $n$  squares, take a configuration of  $n$  squares and divide one of the squares into 4 smaller squares. This increase the total count by 3. Therefore we can cut a square in  $n$  squares for each  $n > 6$

### Problem 4

Proof: Consider the sum modulo 2 of all numbers on the board. Initially, the numbers are 1, 2, ..., 101. The sum is  $S = 101 \times 102/2 = 5151$

Since 5151 is odd, we have  $S \equiv 1 \pmod{2}$ . Now, pick two numbers  $a$  and  $b$ , erase them, and write  $|a - b|$ .

If  $a > b$ , then  $|a - b| = a - b$ . The new sum  $S = S - a - b + (b - a) = S - 2b$

If  $b \geq a$ , then  $|a - b| = b - a$ . The new sum  $S = S - a - b + (b - a) = S - 2a$

Thus  $S = S - 2 \cdot \min(a, b)$

That is, the sum changes by an even number. Therefore  $S \equiv S \pmod{2}$  **What do you really want to say here? Maybe you want to say that if  $S'$  is obtained after any number of operations then  $S' \equiv S \equiv 1 \pmod{2}$ .** Suppose  $X$  remains. Thus the sum is  $X$ . That is, **The sum is  $X$  and  $X \equiv S \equiv 1 \pmod{2}$**

Hence  $X$  is odd, and in particular  $X \neq 0$ .

### Student 7

#### Problem 1

Proof: Here is a square  $4 \times 4$  with one corner. (figure)

Assume the top-left cell has the coordinate  $(1, 1)$  and the bottom-right cell has the coordinate  $(4, 4)$ .

First case, we start to cover the  $(1, 2)$  cell.

A.  $(1, 2)$  cell covers the cell  $(1, 3)$  (**You want to say  $(1, 2)$  and  $(1, 3)$  are covered by the same domino. The same applies everywhere** ), then

the cell  $(1, 4)$  must cover  $(2, 4)$ , then the cell  $(3, 4)$  covers the cell  $(3, 3)$ , then the cell  $(2, 3)$  covers the cell  $(2, 2)$ , then the cell  $(2, 1)$  covers the cell  $(3, 1)$ , then the cell  $(3, 2)$  covers the cell  $(4, 2)$ . **Don't use too many commas in a row. Use perios instead** Finally, we find that the cell  $(4, 1)$  and the cell  $(4, 3)$  can't be covered.

B.  $(1, 2)$  cell covers the cell  $(2, 2)$ , then the cell  $(2, 1)$  covers the cell  $(3, 1)$ , then the cell  $(4, 1)$  covers the cell  $(4, 2)$ , then the cell  $(4, 3)$  covers the cell  $(3, 3)$ , then the cell  $(3, 4)$  covers the cell  $(4, 4)$ . Then we find that the cell  $(3, 3)$  and the cell  $(4, 3)$  can't be covered, so it is false. **This last conclusion should be written in a separate paragraph.**

**You could finish the proof here by symmetry**

Second case A, Cell  $(2,1)$  covers the cell  $(2,2)$ , the cell  $(1,2)$  covers the cell  $(1,3)$ , the cell  $(1,4)$  covers the cell  $(2,4)$ , the cell  $(2,3)$  covers the cell  $(3,3)$ . Then we find that the cell  $(3,4)$  can't be covered.

B cell  $(2,1)$  covers the cell  $(3,1)$ : the cell  $(4,1)$  covers the cell  $(4,3)$ , the cell  $(4,3)$  covers the cell  $(3,3)$ , the cell  $(3,2)$  covers the cell  $(2,2)$ , the cell  $(1,2)$  covers the cell  $(1,3)$ , the cell  $(2,3)$  covers the cell  $(2,4)$ . Then we find that the cell  $(1,4)$  and the cell  $(3,4)$  can't be covered.

In the two cases we find that this problem has no solution.

## Problem 2

Proof: Color the  $8 \times 8$  chessboard in a checkerboard pattern, where any two adjacent squares have different colors. Since squares at opposite corners are always **of** the same color, **we may** assume the squares at opposite corner are white, **Use a period here. You could say: After removing them, we have ....** then we remove them, then we have 32 black squares and 30 white squares.

When we place a  $1 \times 2$  or  $2 \times 1$  domino, we find that it must cover a black square and a white square. But the black squares is not equal to the white squares **But the number of black squares ... after we removed the two white squares.**

Thus, a complete tiling of the  $8 \times 8$  chessboard with two opposite corners removed is impossible.

### Problem 3

Proof:  $n = 7$ : we can cut the square then we have four squares, then we choose one square to cut, then we get  $4 - 1 + 4 = 7$  squares. Use periods and figures: We first cut the square into 4 squares as shown in the figure. Then we pick one of the small squares and cut it in a similar fashion. We get  $4 - 1 + 4 = 7$  squares

$n = 8$ : we can cut the square like this picture (This is not a valid cut). Then we get 8 squares.

$n = 9$ : we can cut the square like this picture. Then we get 9 squares.

Assume any element  $K \geq 9$ , the squares can be cut then we get  $K$  squares. Let  $K \geq 9$  and assume that it is possible to cut a square into  $K$  squares. We can choose one squares to cut to get 4 squares Use a period then we get  $K - 1 + 4 = K + 3$  squares.

Since we have proved it exist when  $n = 7, 8, 9$ . Then for all  $n > 6$ ,  $\Rightarrow n + 3$  when  $n = 7, n = 10, n = 13, n = 16$ , when  $n = 8, n = 11, n = 14, n = 17$ , when  $n = 9, n = 12, n = 15, n = 16$ . Thus for all  $n > 6$  is exist. You may want to say: Since we have proved that it is possible to cut any square into 7,8 or 9 squares, the argument in the previous paragraphh says that it is also possible to cut a square into  $7 + 3 = 10, 10 + 3 = 13, \dots$ , squares, or  $8 + 3 = 11, 11 + 3 = 14, \dots$ , squares, or  $9 + 3 = 12, 12 + 3 = 15, \dots$ . In other words, it is possible to cut a square into  $n$  squares for any  $n \geq 7$

### Problem 4

Proof: Firstly, we add all the numbers  $1 + 2 + 3 + \dots + 101 = \frac{101 \times 102}{2} = 5151$ . When we choose two numbers such as  $a, b$ ,

if  $a + b$  is odd then  $|a - b|$  is odd too because  $|a - b| = |a + b - 2b|$   
odd-even=odd,

if  $a + b$  is even, then  $|a - b|$  is even too because  $|a - b| = |a + b - 2b|$   
even-even=even. **Don't use too many commas. Use periods instead.**

Then we find that the parity of the total number remains the same after the change as it was before.

Assume the number is 0, then we find that the sum is even, but the initial total number  $101 \times 55$  is odd because odd  $\times$  odd = odd.

Thus the number is 0 that is false **what do you want to express here**, so the number isn't 0.

## Student 8

### Problem 1

Mark the horizontal and vertical coordinates from 1 to 4.

**Case 1.** Domino covers (1,3), (1,4)  
(1,2) (2,2) ✓ since (1,1) is removed  
consider (4,1)

1. 1-1 Domino covers (4,1) (4,2)  
sub (i) (4,3) - (3,3) ✓  
sub (ii) (2,4) ✓  
 $(2, 4) \rightarrow (2, 3) \rightarrow (2, 1)$  and  $(3, 1)$  covered  $(3, 2)$  and  $(3, 4)$  isolated

$(2, 4) \rightarrow (3, 4) \rightarrow (2, 3)$  isolated

Above, it's contradiction in both subcases.

2. 1.2 covers (4,1)  $\rightarrow (3,1)$  ✓  
 $\rightarrow (4,2)$  ✓  
cover (4,3)  $\Rightarrow (3,2)$  (3,3) ✓ leaving (2,1) ✗  
cover (3,2)  $\Rightarrow (4,3)(3,3)$  ✓ leaving (2,1) ✗

Above, it's contradiction

**Case 2** covers (1,4) (2,4) ✓

Then (1,3) ✓

2.1 covers (1,2) (1,3) ✓

Then (1,2) is covered. There are 3 cases (2,1) (3,1) (4,1)

(a) (4,1) (4,2) ✓

Then (4,3) covered with (3,3) ✓ and (2,1) must be covered

-(2,1) covered with (2,2) then (2,3) becomes isolated

-(2,1) If covered with (3,1) then (2,2), (2,3) (3,2) (3,4) remain but no domino can cover (2,3) and (3,4).

### **Contradiction**

(b) covers (4,1) - (3,1)

(4,2) ✓

(4,2) → (4,3) ✓ → (3,2) covered with (3,3) leaving (2,1) ×

(4,2) → (3,2) ⇒ (4,3) (3,3) ✓ leaving (2,1) ×

### **Contradiction**

2.2 covers (1,3) - (2,3)

Then (1,2) - (2,2) ✓

(a) (4,1) - (4,2) covered

⇒ (4,3) covered with (3,3)

(2,1) must covered with (3,1) ✓

leaving (3,2) and (3,4) ×

Contradiction

(b)  $(3,1) - (4,1)$  ✓

Then  $(4,2)$  ✓

$(4,2) \rightarrow (4,3) \Rightarrow (3,2)(3,3)$  ✓

$(2,1)$  is isolated

$(4,2) \rightarrow (3,2) \Rightarrow (4,3) (3,3)$

$(2,1)$  is isolated.

Since all cases lead to contradiction, it's impossible. The proof us unnecessary long. It could be reduced to Case 1 by symmetry. Moreover, while it is possible to follow it, the language is imprecise

## Problem 2

At the beginning there are... There are 64 squares. Black is 32 and White is 32. 32 black and 32 white

Since we remove the corners, hence the color of opposite corners are the same. Suppose the corners are both black, then black is 30, white is 32. This could be phrased as: Since we remove opposite corners, and opposite corners have the same colors, we may assume that there are now 30 black squares and 32 white squares.

Since Remove 'since' each domino should cover 1 black and 1 white square, but the number of black squares < the number of white squares.

Hence it can't be covered.

Conclusion: We can't cover a  $8 \times 8$  square without opposite corners by dominoes.

## Problem 3

Proof:

- For  $n = 7$ : This gives  $4 - 1 + 4 = 7$  squares no figure is shown
- For  $n = 8$ : This gives  $9 - 4 + 1 = 6$  squares no figure is shown

Induction:

Assume we can partition a square into  $k$  squares for some  $k \geq 6$ . Then we can partition into  $k + 3$  by:

1. Select any one square
2. Divide it into 4 smaller squares
3. This increases the total count by 3

Hence, it's true. *incomplete. You should also consider  $n = 9$  as a base case*

#### Problem 4

Proof:

Odd numbers: 51

Even numbers: 50

- Case 1: Erase 2 odd numbers  $\Leftrightarrow$   
Their difference is even  $\Leftrightarrow$  The count of odd numbers decreases by 2.
- Case 2: Erase 2 even numbers  $\rightarrow$   
Their difference is ~~odd~~ even  $\rightarrow$  The count of odd numbers remains unchanged.
- Case 3: Erase one odd and one even  $\rightarrow$   
The difference is odd  $\rightarrow$  The count of odd numbers remains unchanged.

We find no matter how you operate, the parity of odd count remains the same.

Therefore, the count of odd numbers is always odd.

If the last remaining number were 0, then the count of odd numbers would be 0.

This is a contradiction, hence it cannot be 0.

## Student 9

### Problem 1

Label the rows and columns of the  $4 \times 4$  square from 1 to 4, suppose that the removed opposite corners are  $(1, 1)$ ,  $(4, 4)$ .

For  $(1, 2)$ , it has two situations: covered with  $(1, 3)$ , or with  $(2, 2)$ .

1. if covered with  $(1, 3)$  (**Begin the sentence with capital I**), so  $(1, 4)$  must be covered with  $(2, 4)$ ;  $(3, 4)$  must be covered with  $(3, 3)$ ;  $(4, 3)$  must be covered with  $(4, 2)$ ;  $(4, 1)$  must be covered with  $(3, 1)$ ; and  $(2, 1)$  must be covered with  $(2, 2)$ ;

Eventually, there have  $(2, 3)$  and  $(3, 2)$  **Eventually  $(2, 3)$  and  $(3, 2)$  are the only remaining cells**, and they cannot be covered with a domino.

So in this case, we cannot cut a square  $4 \times 4$  without opposite corners into domino.

2. if covered with  $(2, 2)$  **begin a sentence with capital I**, so  $(2, 1)$  must be covered with  $(3, 1)$ ;  $(4, 1)$  must be covered with  $(4, 2)$ ;  $(4, 3)$  must be covered with  $(3, 3)$ ;  $(3, 4)$  must be covered with  $(2, 4)$ ;  $(1, 4)$  must be covered with  $(1, 3)$ .

Eventually, there have  $(2, 3)$  and  $(3, 2)$ , and they cannot be covered with a domino.

So in this case, we cannot cut a square  $4 \times 4$  without opposite corners into domino.

Therefore, this proposition is true.

### Problem 2

Color the  $8 \times 8$  square in a checkerboard pattern (black and white alternatively). Each domino must cover one black and one white square. There are 64 squares in this square. After removing two opposite corners

(the same color) there are 30 black/white squares and 32 white/black squares left. Since each domino must cover one black square and one white square, so the number of black and white squares must be equal. So it is impossible to tile the space.

### Problem 3

Proof: For  $n = 7$ , divide the square into three  $2 \times 2$  squares and four  $1 \times 1$  squares, totaling  $3 + 4 = 7$  squares.

For  $n = 8$ , divide the square into one  $3 \times 3$  square and seven  $1 \times 1$  squares, totaling  $1 + 7 = 8$  squares.

For  $n = 9$ , divide the square into nine  $1 \times 1$  squares.

For  $n \geq 7$ , we can split one square into four smaller squares based on  $n = 7$  (each operation increases the number of squares by 3). Thus, all integers  $n > 6$  satisfy the situation. **This argument is incomplete. The base cases are 7,8 and 9.**

### Problem 4

Proof: The initial sum  $1 + 2 + \dots + 101 = \frac{101 \times 102}{2} = 101 \times 51$ , which is odd. Each operation "erase two numbers  $a, b$  and write  $|a - b|$ " doesn't change the parity of the total sum **why?**. The remaining number must have the same parity as the initial sum, while 0 is even. Therefore, the remaining number cannot be 0.

1. A snail crawls in the plane at a constant speed and every 15 minutes turns 90 degrees (sometimes left, sometimes right). Prove that it can return to its starting point only after an integer number of hours.

Solution: We introduce a system of orthogonal coordinates so that the starting point is  $(0, 0)$  and the snail moves either horizontally or vertically. In addition, we can assume that the snail turns precisely when it reaches a point with integer coordinates. For example, the initial path could be  $(0, 0) \rightarrow (0, 1) \rightarrow (1, 1) \rightarrow (1, 2) \rightarrow (0, 2) \rightarrow (1, 1) \rightarrow (-1, 1)$ .

Note that in each movement  $(x, y) \rightarrow (x', y')$  the parity of the quantity  $x + y$  will change. Hence the number of movements to return to the starting point must be even. In other words, the snail can return to its starting point only after an integer multiple of 30 minutes. However, any 30 minutes the parity of  $x$  changes. Hence, the snail can return to the starting point only after an even multiple of 30 minutes, in other words, only after an integer multiple of 60 minutes.

2. Prove that among any 52 integers there exist two whose sum or difference is divisible by 100.

Solution: We partition the numbers  $0, 1, 2, \dots, 99$  into 51 groups:  $\{0\}$ ,  $\{1, 99\}$ ,  $\{2, 98\}$ , ...,  $\{49, 51\}$  and  $\{50\}$ . There are 51 groups. Let  $S$  be a set of 52 integers. The remainder in the division by 100 of each of these integers is a number in one of these groups. Since we have 52 integers and 51 groups, there must be two integers  $s_1, s_2 \in S$  in the same group. If  $s_1$  and  $s_2$  have the same remainder, say  $s_1 = 100k_1 + r$  and  $s_2 = 100k_2 + r$  then  $s_1 - s_2 = 100(k_1 - k_2)$  is divisible by 100. If  $s_1$  and  $s_2$  have different remainders then the group containing these remainders is of the form  $\{r, 100 - r\}$  and we may assume that  $s_1$  has remainder  $r$  and  $s_2$  remainder  $100 - r$ . Hence  $s_1 = 100k_1 + r$  and  $s_2 = 100k_2 + 100 - r$ . Hence

$s_1 + s_2 = 100(k_1 + k_2 + 1)$  is a multiple of 100.

3. One hundred numbers are arranged on a circle. Each number is equal to the arithmetic mean of its neighbours. Prove that all the numbers are equal.

Solution: Let  $L$  be the maximum of the numbers on the circle. We say that a number among the 100 numbers is maximal if it is equal to  $L$ . We want to show that every number is maximal. It is enough to show that if a number on the circle is maximal, then so are its two neighbours. Let  $a$  and  $b$  be its neighbours. Then  $a \leq L$  and  $b \leq L$ . We may assume  $a \leq b$ . Then  $L = \frac{a+b}{2} \leq \frac{b+b}{2} = b \leq L$ . Hence  $b = L$ . Now  $L = \frac{a+L}{2}$ , and from this we get  $2L = a + L$  and so  $a = L$ .

4. Each student in the class took part in at least one of two hikes. In each hike, the proportion of boys did not exceed  $2/5$ . Prove that in the whole class, the proportion of boys does not exceed  $4/7$ .

Solution:

*Proof.* Assume the snail moves at constant speed  $v$ , so each 15-minute segment covers distance  $s = v \times 15$  minutes. Without loss of generality, scale units so  $s = 1$ .

The snail's path consists of unit-length moves, each perpendicular to the previous one due to the 90-degree turns. Thus, the moves alternate between horizontal ( $x$ -direction) and vertical ( $y$ -direction), assuming the initial move is horizontal; the case of initial vertical is symmetric.

After  $n$  moves: - If  $n$  is even, there are  $n/2$  horizontal and  $n/2$  vertical moves. - If  $n$  is odd, there are  $(n+1)/2$  horizontal and  $(n-1)/2$  vertical moves.

Each horizontal move is  $\pm 1$  in the  $x$ -direction, and each vertical move is  $\pm 1$  in the  $y$ -direction. For the snail to return to the origin, the net displacement in both  $x$  and  $y$  must be zero.

The net  $x$ -displacement is the number of  $+1$  minus the number of  $-1$  in horizontal moves. Let  $r$  be the number of  $+1$  and  $\ell$  the number of  $-1$ , so  $r + \ell =$  number of horizontal moves, and net  $x = r - \ell$ . For net  $x = 0$ ,  $r = \ell$ , so the number of horizontal moves must be even.

Similarly, the number of vertical moves must be even.

- If  $n = 4k$ , both are  $2k$ , even. - If  $n = 4k + 1$ , horizontal:  $2k + 1$  (odd), vertical:  $2k$  (even) — impossible. - If  $n = 4k + 2$ , both  $2k + 1$  (odd) — impossible. - If  $n = 4k + 3$ , horizontal:  $2k + 2$  (even), vertical:  $2k + 1$  (odd) — impossible.

Thus, return is possible only if  $n$  is a multiple of 4.

The time elapsed is  $n \times 15$  minutes =  $n/4$  hours, which is an integer if and only if  $n$  is a multiple of 4.  $\square$

## 5.4 Notes from tutorial 8

**Question 4.** Show that if  $n > k \log_2 10$ , then

$$|(\sqrt{2} - 1)^n| < 10^{-k}.$$

**Solution.** Since

$$\sqrt{2} - 1 = \frac{1}{\sqrt{2} + 1} < \frac{1}{2},$$

we have

$$0 < (\sqrt{2} - 1)^n < \left(\frac{1}{2}\right)^n = 2^{-n}.$$

Now we explain why  $n > k \log_2 10$  implies  $2^{-n} < 10^{-k}$ . Recall that  $\log_2 10$  is *the number  $x$  such that  $2^x = 10$* . Because the exponential function  $2^x$  is strictly increasing, the inequality  $n > k \log_2 10$  implies

$$2^n > 2^{k \log_2 10} = (2^{\log_2 10})^k = 10^k.$$

Taking reciprocals reverses the inequality (since all numbers are positive), giving

$$2^{-n} < 10^{-k}.$$

Combining with  $(\sqrt{2} - 1)^n < 2^{-n}$ , we obtain

$$|(\sqrt{2} - 1)^n| < 10^{-k}.$$

**Question 5.** Let  $n$  be even. Show that the first  $k$  digits after the decimal point of  $(1 + \sqrt{2})^n$  are equal to 9.

**Solution.** Define

$$u_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n.$$

Since  $n$  is even,  $(1 - \sqrt{2})^n = (\sqrt{2} - 1)^n > 0$ , and  $u_n \in \mathbb{Z}$ . Set

$$\varepsilon = |(\sqrt{2} - 1)^n| = (\sqrt{2} - 1)^n.$$

By Question 4, if  $n > k \log_2 10$  then  $0 < \varepsilon < 10^{-k}$ .

We write

$$(1 + \sqrt{2})^n = u_n - (1 - \sqrt{2})^n = u_n - \varepsilon.$$

Thus  $(1 + \sqrt{2})^n$  is an integer minus a positive number  $\varepsilon$  satisfying  $0 < \varepsilon < 10^{-k}$ . Hence the fractional part of  $(1 + \sqrt{2})^n$  is

$$\{(1 + \sqrt{2})^n\} = 1 - \varepsilon.$$

Since  $\varepsilon < 10^{-k}$ , its decimal expansion begins with at least  $k$  zeros, so  $1 - \varepsilon$  begins with at least  $k$  nines. Therefore the first  $k$  digits after the decimal point of  $(1 + \sqrt{2})^n$  are all equal to 9.

For even  $n > k \log_2 10$ , the first  $k$  decimal digits of  $(1 + \sqrt{2})^n$  are 9.

## Explaining the Diagonal Bijection $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

**Convention.**  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

### 1. The Diagonal Idea

Arrange all ordered pairs  $(a, b) \in \mathbb{N} \times \mathbb{N}$  in a grid:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$\dots$
$m = 0$	(0, 0)	(1, 0)	(2, 0)	(3, 0)	$\dots$
$m = 1$	(0, 1)	(1, 1)	(2, 1)	(3, 1)	$\dots$
$m = 2$	(0, 2)	(1, 2)	(2, 2)	(3, 2)	$\dots$
$m = 3$	(0, 3)	(1, 3)	(2, 3)	(3, 3)	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Instead of reading rows or columns, we read **\*\*diagonals\*\*** (top to bottom) of constant sum

$$a + b = 0, 1, 2, 3, \dots$$

For example:

$$\begin{aligned} a + b = 0 & : (0, 0) \\ a + b = 1 & : (1, 0), (0, 1) \\ a + b = 2 & : (2, 0), (1, 1), (0, 2) \\ a + b = 3 & : (3, 0), (2, 1), (1, 2), (0, 3) \\ & \vdots \end{aligned}$$

The idea is:

- First list all pairs on diagonal  $a + b = 0$  - Then all pairs on diagonal  $a + b = 1$  - Then all on diagonal  $a + b = 2$ , and so on.

This reading order goes through **\*every\*** pair exactly once, so it defines a bijection.

## 2. Counting How Many Pairs Come Before a Given Diagonal

Diagonal  $s$  (i.e. all pairs with  $a + b = s$ ) contains exactly  $s + 1$  pairs.

The total number of pairs listed before diagonal  $s$  is therefore:

$$0 + 1 + 2 + \cdots + s = \frac{s(s+1)}{2}.$$

This number is called the  **$s$ -th triangular number**, denoted

$$T_s = \frac{s(s+1)}{2}.$$

## 3. Locating $(a, b)$ Along Its Diagonal

If a pair has sum

$$s = a + b,$$

then:

- there are  $T_s$  pairs on earlier diagonals, and - within the diagonal  $a + b = s$ , we move  **$b$  steps** from the start (because the diagonal is listed as  $(s, 0), (s-1, 1), (s-2, 2), \dots, (0, s)$ ).

Therefore the position (the natural number assigned to  $(a, b)$ ) is

$$\boxed{\pi(a, b) = T_{a+b} + b = \frac{(a+b)(a+b+1)}{2} + b.}$$

## 4. Why This is a Bijection

First the following claim:

*If  $m+n < m'+n'$ , then  $\pi(m, n) < \pi(m', n')$ .*

*Proof:* Subject to the constraint  $m + n = k$ , the maximum value of  $\pi(m, n)$  is  $\pi(0, k) = \frac{k(k+1)}{2} + k$  and the minimum value of  $\pi(m, n)$

is  $\pi(k, 0) = \frac{k(k+1)}{2}$ . Now if  $k' \geq k + 1$ , then  $\pi(k', 0) = \frac{k'(k'+1)}{2} \geq \frac{(k+1)(k+2)}{2} = \frac{k(k+1)}{2} + k + 1 > \frac{k(k+1)}{2} + k = \pi(0, k)$ .

Check that you understand why this is enough to prove the claim above.

**Surjective.** Given  $n \in \mathbb{N}$ , let  $T_k = \frac{k(k+1)}{2}$  be the largest triangular number not exceeding  $n$ . Let  $b = n - T_k$  and  $a = k - b$ , then  $\pi(a, b) = n$ .

**Injective.** It follows that the equality  $\pi(a, b) = \pi(a', b')$  implies that  $a + b = a' + b'$ . But then  $a = a'$  and  $b = b'$ , which gives the injectivity of  $\pi(a, b)$ .

## 5. Small Table

$(a, b)$	$\pi(a, b)$
$(0, 0)$	0
$(1, 0)$	1
$(0, 1)$	2
$(2, 0)$	3
$(1, 1)$	4
$(0, 2)$	5
$(3, 0)$	6
$(2, 1)$	7

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$\dots$
$m = 0$	0	1	3	6	$\dots$
$m = 1$	2	4	7		$\dots$
$m = 2$	5	8			$\dots$
$m = 3$	9				$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

## 6. Summary

The formula

$$\pi(a, b) = \frac{(a+b)(a+b+1)}{2} + b$$

is nothing more than:

1. Count all diagonals before the one containing  $(a, b)$ : this gives  $T_{a+b}$ .
2. Then walk  $b$  steps along that diagonal.

This converts planar coordinates  $(a, b)$  into a single natural number by following the diagonal ordering.

## 5.5 How to write

Taken from "Everything You Always Wanted To Know About Mathematics\* (\*But didn't even know to ask)", Brendan W. Sullivan, [https://www.math.cmu.edu/~jmackey/151\\_128/bws\\_book.pdf](https://www.math.cmu.edu/~jmackey/151_128/bws_book.pdf)

### 1.2.2 Write Right

Another interesting aspect of mathematics is that, as much as it is a language unto itself, we rely on an external language to convey the mathematical thoughts and insights we have. Try rewriting any of the definitions and proofs we've looked at before without using any words. It's tough, isn't it? Accordingly, we want the written language we use to convey mathematical ideas to follow the same types of standards we apply to the mathematical "sentences" we write: we want them to be *precise*, *logical*, and *clear*.

Now, deciding on a precise, logical, and clear definition for each of these three words is a difficult task, in itself. However, we can all agree that it would be ideal for a proof to be:

- **precise:** no individual statement should be untrue or interpretable in multiple ways that would make the truth debatable;
- **logical:** each step should follow from previous steps with proper motivation and explanation; and,
- **clear:** steps should be connected and described with proper English grammar and usage, helping the reader to see what's going on.

Let's examine a few "proofs" that disregard these standards and somehow fail to fit the definition of *proof* that we have so far.

## Unclear Arguments

The last two bad “proofs” failed because of issues with precision and logical correctness. The third condition we require of a good proof is that it be *clear*: we want the writing to explain what the proof-writer accomplishes in each step and why that accomplishment is relevant. In other words, we don’t want the reader to stop at any point and ask, “What does that sentence mean?” or “Where did that come from?” or similar questions born from confusion. If it helps, think about writing a proof in terms of explaining it to your friend in your class, or the grader who will be reading your homework assignment, or a family member of similar intelligence. Reread your own writing and try to anticipate any questions that might arise or any clarifications that might be asked of you, and then address those issues by rewriting.

There are several ways that a proof could fail this condition and come across as unclear. For one, the words and sentences might fail to properly explain the steps and motivations of the proof, and this could actually be because there are too many words (obscuring the proof by overburdening the reader) or because there are too few words (not giving the reader enough information to work with) or because the words chosen are confusing (not properly explaining the proof). These are issues with the *language* of the proof.

Mathematically, any number of issues could arise, in terms of clarity. Perhaps the proof-writer suddenly introduces a variable without stating what type of number it is (an integer, a real number, etc.) or skips a few steps of arithmetic/algebra or uses new notation without defining what it means first or… None of these acts is technically wrong or illogical, but they can certainly cause confusion for a reader. Can you think of any other ways that a proof can be unclear? Try to think of a language-based one and a mathematical one.

### Bad “Proof” #3

Let’s state a simple fact about a polynomial function and then examine a “proof” about that fact. Read the argument carefully and try to pinpoint some sentences or mathematical steps that are *unclear*.

**Fact:** Consider the polynomial function  $f(x) = x^4 - 8x^2 + 16$ . This function satisfies  $f(x) \geq 0$  for any value of  $x$ .

*“Proof.”* No matter what the value of  $x$  is that we plug into the function  $f$  of  $x$  we can write the value that the function puts out by factoring the polynomial,

choosing when to have some of the verification of details to the reader can be tricky. In this particular instance, showing all of the steps is important.

That being said, though, the second proof we showed here is much clearer. Moreover, it completely avoids the case analysis that appears in the first “proof!” There was an issue with clarity of one of the cases in the first “proof”, but rather than simply expose the details in the amended version, we opted to scrap that technique altogether and use a shorter, more direct proof. Now, this is not to say that the technique of the first proof is incorrect. Were we to fill in the gaps of the argument of the first “proof”, we would obtain a completely correct proof. However, given the fact that techniques such as this are often used in the cases where  $-2 < x < 2$  and where  $x = 2$  are actually identical, in a sense: the factors satisfy  $(x-2)^2 \geq 0$  and  $(x+2)^2 \geq 0$  in both cases. In fact, this is true of the first case, where  $x < -2$ , as well! So why separate this argument into three separate cases when the same ultimate observation is applied to all three of them? In this case, it is best to combine them into one (also using the knowledge that when  $x = 2$  or  $x = -2$ , one of the factors is 0). Again, using that expanded technique is certainly not incorrect; rather, it just adds some unnecessary length to the proof.

We mentioned the term “case” and the phrase “case analysis” in the above paragraphs without properly defining or explaining what we mean. By now, we’ve got to postpone our discussion of these terms until we thoroughly discuss logic in Chapter 4. If you’re itching for immediate gratification regarding this issue, though, you can skip ahead to Section 1.4.4 and check out the “Hungarian friends” problem, which contains some intricate case analyses.

### 1.2.3 Pick Logic

We have used the word “logical”, and its associated forms, quite frequently, already, without fully explaining what we mean by it. We realize this seems to go against the precision and clarity that we have been so strongly advocating thus far, but we also have to admit that, unfortunately, it is extremely difficult to provide a thorough definition of *logic*.

#### Games

If you’re looking for a decent heuristic understanding of logic, try thinking about it in terms of “logic puzzles” like Sudoku or Kakuro. These puzzles/games are built around very specific rules that are established and agreed upon from the very beginning, and then the solver is presented with a starting board and expected to apply those rules in a rigorous manner until the puzzle is solved. For instance, in a standard 9x9 Sudoku grid, each digit from 1 through 9 must appear exactly once in every row, column, and 3x3 box. This allows the solver to systematically place more and more numbers in the grid, continually narrowing down the large number of potential “solutions” to find the unique answer that the starting grid of numbers yields. An important aspect of this solving process is that at no time is it necessary (or wise, at that) to *guess*;

like this:

$$f(x) = x^4 - 8x^2 + 16 = (x-2)^2(x+2)^2$$

Now, any number  $z$  must be less than  $-2$ , or greater than  $2$ , or strictly between  $-2$  and  $2$ , or equal to one of them. When  $z > 2$  then  $z-2$  and  $z+2$  are both greater than  $0$  so  $f(z) > 0$ . When  $z < -2$  then both terms are negative and a negative squared is positive so  $f(z) > 0$ , too. When  $-2 < z < 2$ , a similar thing happens, and when  $z = 2$  or  $z = -2$  one of the terms is  $0$  so  $f = 0$ . Therefore, what we were trying to prove has to be true.  $\square$

What is there to criticize in this proof? First of all, is it correct? Is it precise? Logical? Clear? Where is it unclear? Try to identify the statements, both linguistic and mathematical, that are even slightly unclear, and try to amend them appropriately. Without pointing out any of the individual errors, we offer below a much better, *clearer* proof of the fact above.

*Proof.* We begin by factoring the function  $f(x)$  by considering it as a quadratic function in the variable  $x^2$

$$f(x) = (x^2)^2 - 8x^2 + 16 = (x^2 - 4)^2$$

Next, we can factor  $x^2 - 4 = (x+2)(x-2)$  and rewrite the original function as

$$f(x) = ((x+2)(x-2))^2 = (x+2)^2(x-2)^2$$

Now, for any real number  $x$ ,  $(x+2)^2 \geq 0$  and  $(x-2)^2 \geq 0$ , since a squared quantity is always nonnegative. A product of two nonnegative terms is also nonnegative, so  $f(x) = (x+2)^2(x-2)^2 \geq 0$ , for any value of  $x$ .  $\square$

What are the differences between the first “proof” and this second proof? Does your rewritten proof look like this second one, as well?

One of the critiques of the first “proof” is that it does not fully explain the situation where  $-2 < x < 2$ ; rather, it merely says that something “similar” happens and does not actually carry out any of the details. This is a common situation in proofs (where some steps of a proof are “left to the reader”) and it is a convention that means that sometimes proofs in mathematics and algebra can make reading a proof easier, faster, and more enjoyable. However, it should be used sparingly and with caution. It is important, as a proof-writer, to make sure that those steps do work, even if you are not going to present them in your proof; you should consider providing the reader with a short summary or hint as to how those steps would actually work. Also, a proof-writer should try not to use this technique on steps that are crucial to the ultimate result of the proof.

In this particular case, the actual steps of factoring are skipped completely and the analysis of the case where  $-2 < x < 2$  is only mentioned in passing, yet these are essential components of the proof! It is such a short proof, anyway, that showing these steps does not represent a great sacrifice in brevity or clarity. Again, this brings up the point of proof-writing as an art, as much as a science:

every step should be guided by a rational choice given the current situation and the established rules of the puzzle, and within that framework, the puzzle is guaranteed to be solvable (given enough time, of course).

Mathematical logic is a little different, perhaps, but the essence is the same: there are established rules of how to play the game and every move should be guided by those rules and current knowledge, and nothing else. This is what mean when we say that writing mathematical proofs should be governed by *logic*: every step, from the truth to another, should follow the agreed-upon rules of logic. However, there are also “unwinnable games” or “puzzles” that we’re playing in a proof (and in mathematics, in general) is not as clear-cut as a Sudoku puzzle. Even more confusing, though, is the idea that sometimes we start playing an unwinnable game and don’t realize it!

This idea of an “unwinnable game” is an astounding, surprising, and downright powerful conclusion of the work of mathematician Kurt Gödel, a 20th century Austrian logician. His *Incompleteness Theorems* address an inherent problem with strong logical systems: there can be True statements that aren’t *provable* within that system. We are unable to provide a thoroughly detailed explanation of some terms here (namely, *logical system* and *provable*), but hopefully you see that there is something weird going on here. How could this be possible? If anything is *True* in mathematics, can’t we somehow show that it is *True*? How else would we know that it is *True*?

### Some Mathematical History

To begin to address these natural questions, let’s step a little further back in time and discuss the beginnings of logic as a full-fledged branch of mathematics. One thing to keep in mind throughout this discussion is that we can’t completely address every topic up, and that there’s a lot more to learn, and we’ll do our best to cover it. But, one of the benefits of learning mathematics is that one topic brings up so many other questions and concepts to think about, and these can be addressed, as well, with more mathematics. Context is important, though, and for the context of this book, we just don’t have the time and space to address all of these tangentially-related topics. We are not trying to hide anything from you or sweep some issues under the rug; rather, we’re just dealing with the reality of making sure we’re not forcing you to read 10,000 pages the entire history of mathematics just to get our point across!

You will probably study many of the people that we mention here (and the work they did) further along in your mathematical careers. At that point, you’ll have a deeper understanding and appreciation for the history of mathematics and its connection with the material, and you’ll be better-equipped to tackle the issues therein. For now, we are merely introducing these people out of interest. Mathematics has a rich and interesting history, and it helps to be aware of it! Here, we will try to present a concise yet meaningful interpretation of logic—its history, motivations, and meaning—that fits with the current context.

The mathematicians and philosophers in the mid- to late-19th century who first studied the ideas that would evolve into modern logic were interested in

many of the same issues we are trying to investigate here: How do we know something is True? How can we express that truth? What types of "somethings" can we even declare to be True or not? Breaking down mathematical language to its very roots, these mathematicians studied ways to combine a fixed set of symbols in very specific ways to create more complicated statements, but in the grand scheme of things, this was actually rather simple. This is not meant to be a knock against their efforts; one must start somewhere, after all, and these people were working from the ground up.

One of the first major efforts was to investigate the foundations of arithmetic, or the study of the **natural numbers** (1, 2, 3, 4, ...). Much like Euclid sought to study geometry by establishing a short list of accepted truths, or **axioms**, and then derive truths from these given assumptions, Italian mathematician Giuseppe Peano established a set of axioms for the natural numbers, while others approached the topic from a slightly different viewpoint. Meanwhile, this newfound appreciation for being rigorous and decisive about truths and proving them led to a desire to bring up some issues with Euclid's axioms specifically: the parallel postulate.

This work on geometry and arithmetic naturally led into further, intricate study of other areas of mathematics and fervent attempts to axiomatize fields like analysis of the real numbers. Karl Weierstrass, in studying this topic, produced some mind-blowing examples of functions with strange properties. For instance, try to define a continuous function that is not differentiable anywhere. (If you're unfamiliar with these terms from calculus, don't worry about it; suffice it to say, it's difficult.) Finally, Richard Dedekind was able to establish a rigorous, logical definition of the real numbers, derived entirely from the natural numbers and based on some vague, physical notion that a continuum of numbers must exist.

Later on, this study branched off slightly into the study of **sets**, or collections of objects. The groundwork for much of this area was laid by Georg Cantor towards the end of the 19th century. He was the first to truly study the theory of infinite sets, establishing the controversial idea that there are different "sizes" of infinity. That is, he showed that some infinite sets are strictly bigger than other infinite sets. This idea was so controversial at the time, that he was hated by many other mathematicians! Nowadays, we realize Cantor was right. (This also gives you a flavor of what we'll discuss later in Section 7.6.) Take this as an intriguing example: the set of integers and the set of even integers are the same size (they are both the same size as the set of *all* integers). However, the set of all real numbers is *strictly bigger*!

Indeed, some mathematicians were quite shocked by Cantor's discoveries, and even the great Bernhard Riemann thought the development of set theory would be the scourge of mathematics (at first, anyway). This was not the case, though, and it has flourished since then, with many mathematicians working on ways to represent all of mathematics in just the right way and understand the "foundations" of mathematics. In a way, you can think of set theory as the study of the basic objects that all mathematicians are working with, ultimately, in a way similar to the fact that all of chemistry is done by appropriately combining

elements of the periodic table in more and more complicated ways.

A further development from these topics was the study of symbolic logic, which is a bit more concrete than the abstract ideas we've mentioned so far, and whose basic ideas we will be studying frequently in the beginning chapters of this book. This area covers how we can combine mathematical equations and statements in ways that produce new, valid mathematical statements. These are mathematical statements that are able to be confirmed as true via a proof. This is an incredibly important component of mathematics, in general, and this book, in particular. Individual viewpoints are certainly more nuanced and specific than this, but, in general, most mathematicians are of the mindset that there are many mathematical truths out there waiting to be discovered and we spend our time learning about the truths we have already uncovered with the hopes of exposing even more of those truths. It's like a giant archaeological dig, whereby studying the bones and artifacts we've already unearthed will help us predict what kinds of other treasures we will find and where, which tells us where to look and what to do once we get there. In a way, logic is the process that is abstracted from the digging by one step: logic is the study of the digging process. It tells us how we can actually take our mathematical knowledge and learn from it and combine it with other knowledge to prove further truths from that.

Now, this is not a precise analogy, mind you, and the study of abstract logic is far more complicated and intricate. For our purposes, in this book, though, it is a sufficiently reasonable way to think of logic. We will learn about some of the first principles and basic operations of symbolic logic and apply this knowledge to our study of writing proofs. It will help us to actually understand what a proof even is, it will help guide the construction of proofs that we want to write, it will allow us to critique proofs that may be incorrect, and it will ultimately help us understand just how mathematics works, as a whole.

### Applications of Logic: Theoretical Computer Science

One very important application of the ideas and results of logic is in the development and study of computer science, particularly theoretical computer science and computability theory. The part of the branch of mathematics was initially motivated by one of David Hilbert's Twenty-Three Problems: this was a list of famously unsolved conjectures in the world of mathematics at the time of their publishing, in 1900. Problem number ten dealt with solving **Diofantine Equations**, which are equations of the form

$$a_1x_1^{p_1} + a_2x_2^{p_2} + a_3x_3^{p_3} + \cdots + a_nx_n^{p_n} = c$$

where  $a_1, a_2, \dots, a_n$  and  $c$  are fixed, given constants,  $p_1, \dots, p_n$  are fixed natural numbers, and  $x_1, x_2, \dots, x_n$  are variables that are left to be determined so that they make the equation true.

Given an equation like this, one might wonder whether there are any solutions at all and, if so, just how many there are. Furthermore, if we're given that the fixed constants  $a_i$  and  $c$  are all rational numbers, we might wonder

whether we can ensure that there is a solution where all of the variables  $x_i$  are also rational numbers. Some theoretical results have been established regarding this particular problem, but Hilbert's tenth problem, as stated in 1900, asked whether there was "a process according to which it can be determined in a finite number of operations" whether there is a solution to a given equation where all of the variables  $x_i$  are rational numbers. They didn't have a proper notion or definition of this term at the time, but what Hilbert was asking for was an **algorithm** that would take in the values of the constants  $a_i$  and  $c$  and output True or False depending on whether there exists a solution with the desired property. An important part of his question was that this "process" takes a finite number of steps before outputting an answer.

A student at Cambridge in the United Kingdom by the name of Alan Turing began working on this problem years later by thinking of a physical machine that would be executing the steps required to output an answer to the posed problem. Some subsequent publications of his described his invention, what we now call a *Turing Machine*, which is an interesting theoretical device that could be used to answer some problems in formal logic, but also represents many of the ideas that go into building modern computers. We say it's a *theoretical device* because the nature of its definition ensures that it is not physically feasible to build and operate, but it handles some theoretical problems quite well, including the aforementioned tenth problem of Hilbert. More specifically, this machine gave rise to a proper definition for what we mean when we say that something is computable, or able to be determined in a finite number of steps, and this helped to establish a proper notion of an **algorithm**. It would be unfair of us to discuss these topics without also mentioning Alonzo Church, who was working on similar problems at the same time as Turing. Their names, together, are placed on the *Church-Turing thesis* which relates the work of the Turing machine to the more theoretical, formal logic-based notion of computability.

### What Will We Do with Logic?

While all of these topics in set theory and logic are inherently interesting and immensely important to mathematics, in general, we simply don't have enough time and space to discuss them in detail. Instead, let's focus a bit more on the notions of logic that we'll be using in writing and critiquing mathematical proofs.

We will consider: (1) what kinds of "things" we can actually state and prove, (2) how we can combine "things" that we know to be true to produce more complex truths, and (3) how we can explain how we arrived at the conclusion that those "things" are, indeed, True. For lack of a better term, we say "things" since we don't yet have a formal definition of **mathematical statement**, which is really the type of "thing" that we will be proving. In essence, a mathematical statement is a combination of symbols and sentences from the languages of mathematics and English (in this book, at least) that can be verified as either True or False, but not both or neither. A proof, then, amounts to arranging a sequence of steps and explanations that use true mathematical statements