11.8 Power Series

Power Series

A **power series** is a series of the form

$$\boxed{1} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence.

A power series may converge for some values of x and diverge for other values of x.

Power Series

The sum of the series is a function

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Power Series

More generally, a series of the form

$$\boxed{2} \qquad \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series in** $(x - a)$ or a **power series centered at *a*** or a **power series about *a***.

Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 2 we have adopted the convention that $(x - a)^0 = 1$ even when $x = a$.

Notice also that when $x = a$ all of the terms are 0 for $n \geq 1$ and so the power series (2) always converges when $x = a$.

Examples 1 - 3

For what values of x is the series convergent?

Example 1.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

Example 2.

$$\sum_{n=0}^{\infty} n!\, x^n$$

Example 3.

$$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$$

Power Series

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number *R* in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

Power Series

The **interval of convergence** of a power series is the interval that consists of all values of *x* for which the series converges.

In case (i) the interval consists of just a single point *a*.

In case (ii) the interval is $(-\infty, \infty)$.

In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$.

Power Series

When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints.

Thus in case (iii) there are four possibilities for the interval of convergence:

$(a - R, a + R) \quad (a - R, a + R] \quad [a - R, a + R) \quad [a - R, a + R]$

The situation is illustrated in Figure 3.

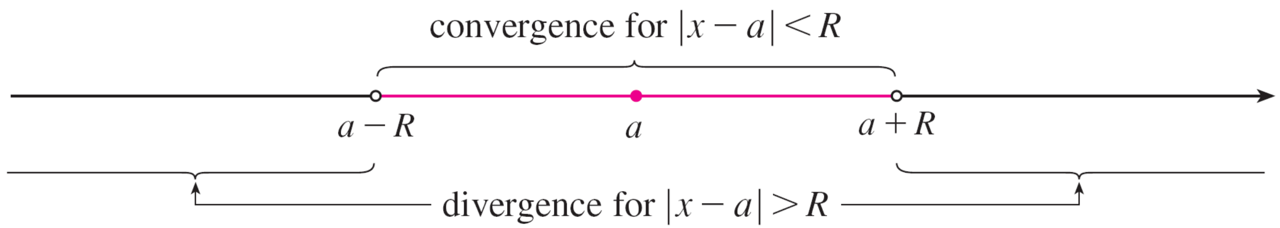


Figure 3

Examples 4 - 5

Find the radius of convergence and interval of convergence of the series.

Example 4.

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Example 5.

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

11.9 Representations of Functions as Power Series

Representations of Functions as Power Series

We start with an equation that we have seen before:

$$\boxed{1} \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

We have obtained this equation by observing that the series is a geometric series with *a* = 1 and *r* = *x*.

But here our point of view is different. We now regard Equation 1 as expressing the function $f(x) = 1/(1 - x)$ as a sum of a power series.

Examples 1 - 3

Example 1. Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

Example 2. Find a power series representation for $1/(x + 2)$.

Example 3. Find a power series representation of $x^3 /(x + 2)$.

Differentiation and Integration of Power Series

Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series.

We would like to be able to differentiate and integrate such functions, and the following theorem says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial.

This is called **term-by-term differentiation and integration**.

Differentiation and Integration of Power Series

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

(ii) $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Examples 4 - 6

Example 4. Express $1/(1 - x)^2$ as a power series. What is the radius of convergence?

Example 5. Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

Example 6. Find a power series representation for $\tan^{-1} x$ and its radius of convergence.

Example 8

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry.

In particular, the sum of the power series,

$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, is called a **Bessel function of order 0.**

Example 8.

(a) Find the domain of J_0.

(b) Find the derivative of J_0.

Bessel function

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

as the sum of a series we mean that, for every real number *x*, $J_0(x) = \lim_{n \to \infty} s_n(x)$ where $s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$.

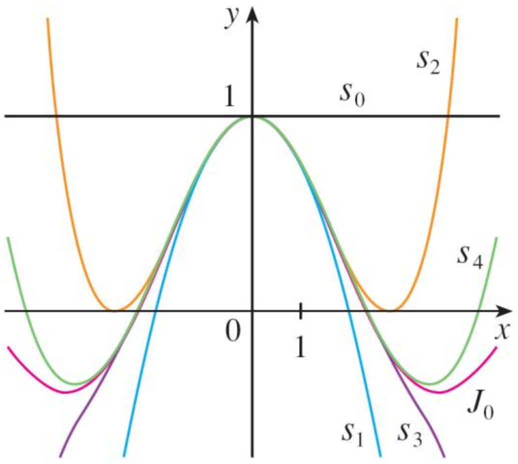
Power Series

The first few partial sums are

$$s_0(x) = 1 \qquad s_1(x) = 1 - \frac{x^2}{4} \qquad s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} \qquad s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$

Figure 2 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function J_0, but notice that the approximations become better when more terms are included.



Partial sums of the Bessel function J_0

Figure 2

Power Series

Figure 3 shows a more complete graph of the Bessel function.

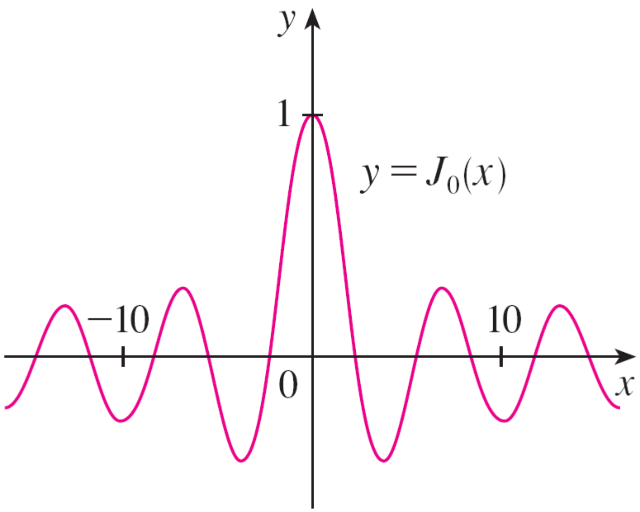


Figure 3