3.3 How Derivatives Affect the Shape of a Graph

What Does f' Say About f ?

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To see how the derivative of f can tell us where a function is increasing or decreasing, look at Figure 1.

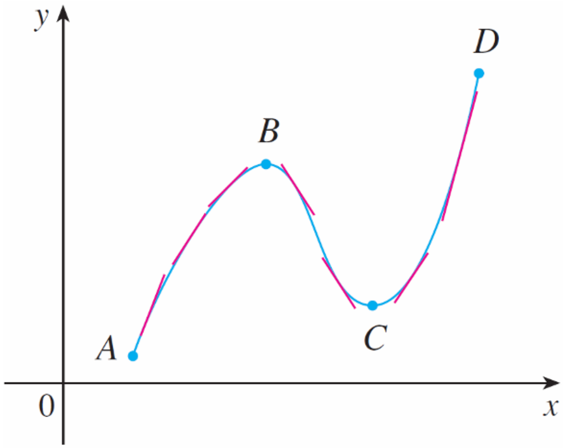


Figure 1

What Does f' Say About f ?

Between A and B and between C and D, the tangent lines have positive slope and so $f'(x) > 0$. Between B and C the tangent lines have negative slope and so $f'(x) < 0$. Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Note: Here the interval is open. However, it is clear from the proof, that the statement is also true on a closed interval $[a, b]$, provided that f is continuous at the endpoints a and b and $f'(x)>0$, resp., $f'(x)<0$ for all x in (a,b).

Example 1

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

What Does f' Say About f ?

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

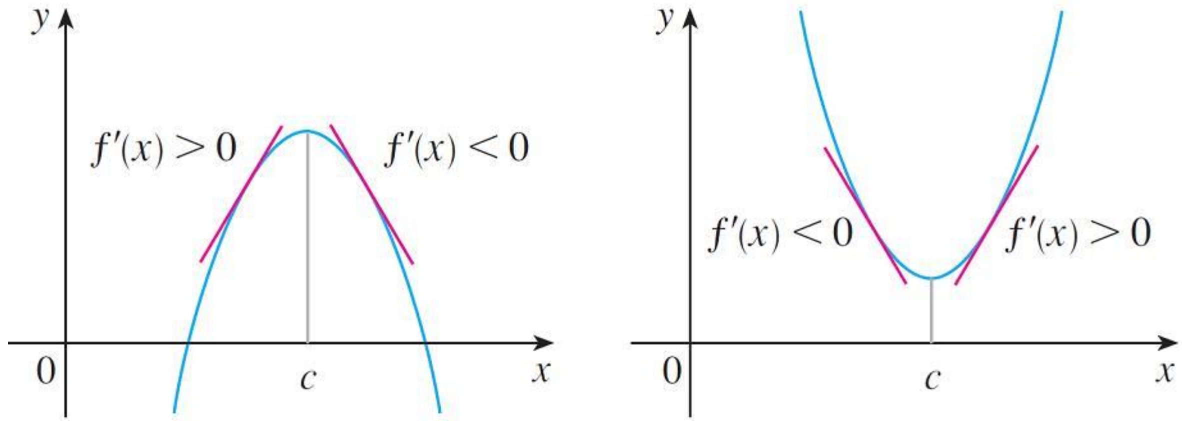
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Note: The continuity of *f* at *c* is important!

The First Derivative Test is a consequence of the I/D Test.

What Does f' Say About f ?

It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 3.

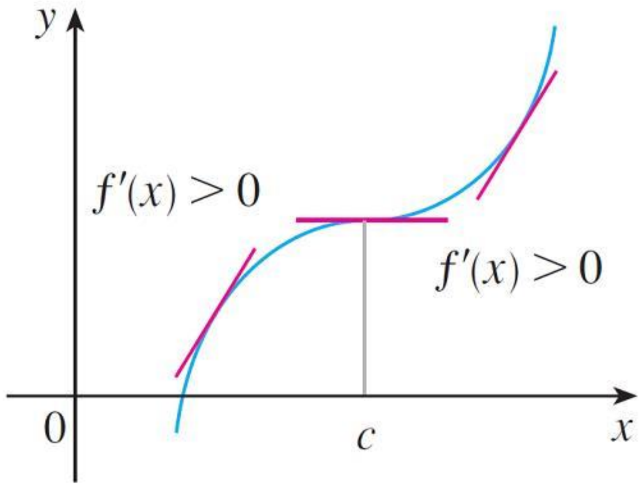


(a) Local maximum

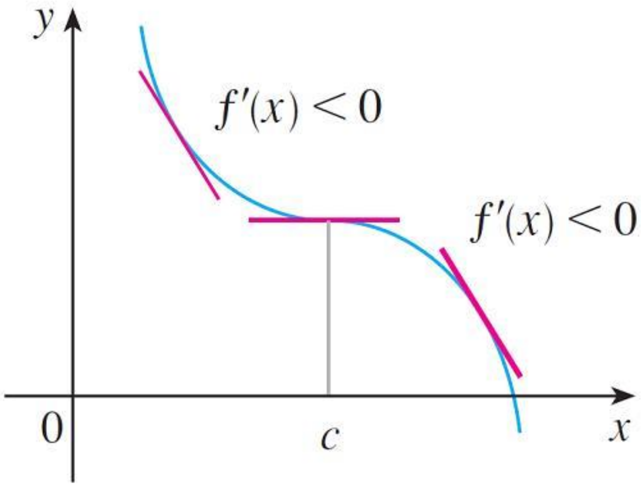
(b) Local minimum

Figure 3

What Does f' Say About f ?



(c) No maximum or minimum



(d) No maximum or minimum

Figure 3

Example 2 and 3

Example 2. Find the local maximum and minimum values of the function from Example 1.

Example 3. Find the local maximum and minimum values of the function

$$g(x) = x + 2 \sin x \qquad 0 \le x \le 2\pi$$

What Does f'' Say About f ?

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Figure 5 shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions.

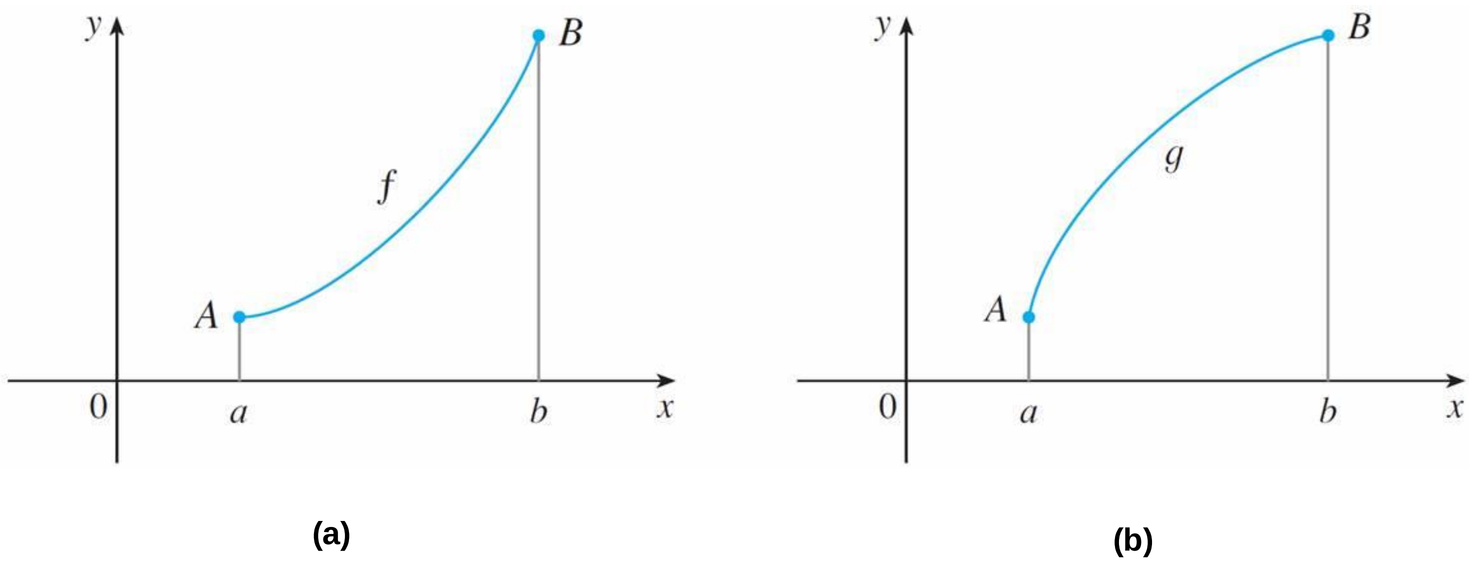
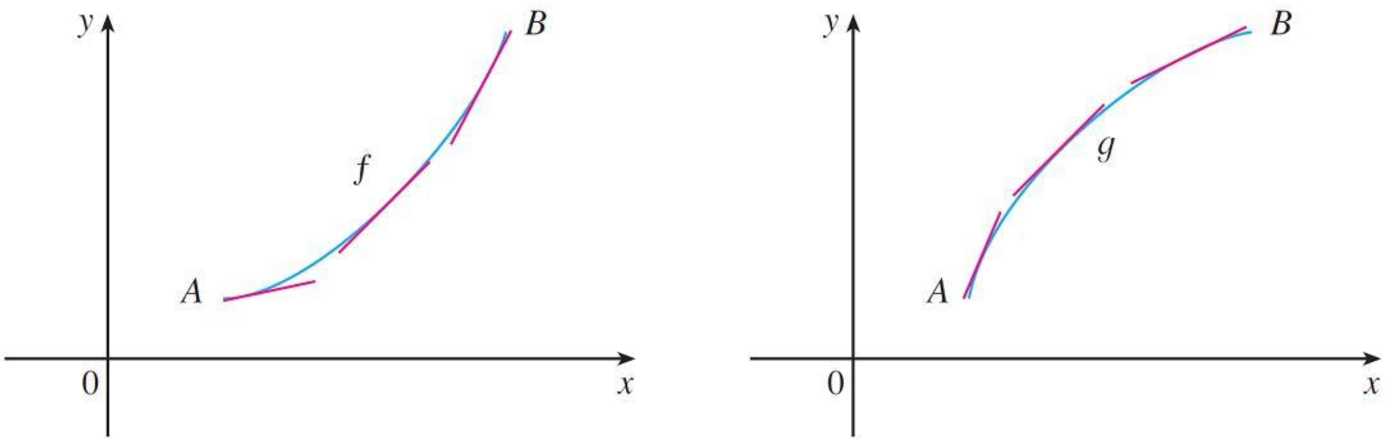


Figure 5

What Does f'' Say About f ?

In Figure 6 tangents to these curves have been drawn at several points. In (a) the curve lies above the tangents and f is called *concave upward* on (a, b). In (b) the curve lies below the tangents and g is called *concave downward* on (a, b).



(a) Concave upward

(b) Concave downward

Figure 6

What Does f'' Say About f ?

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

What Does f'' Say About f ?

Figure 7 shows the graph of a function that is concave upward (abbreviated CU) on the intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), (c, d), and (p, q).

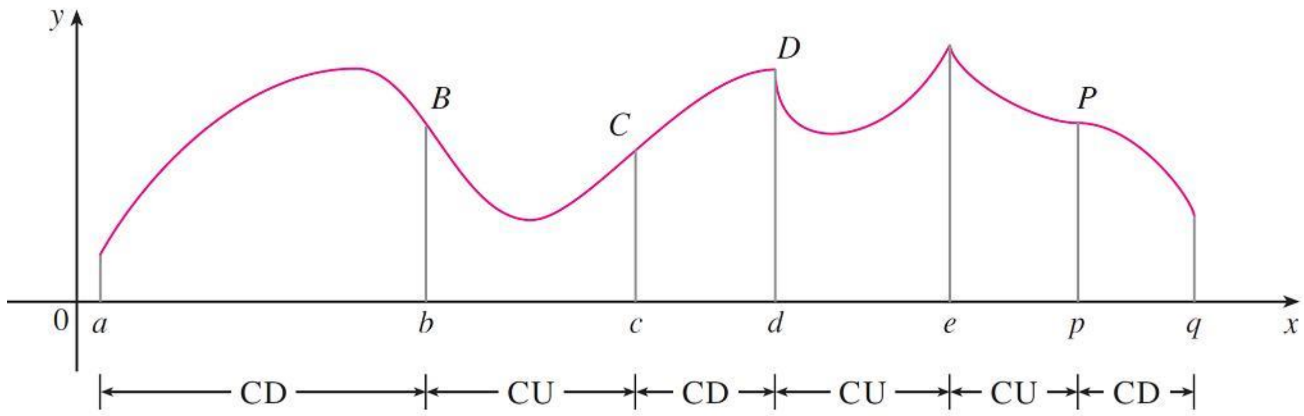


Figure 7

What Does f'' Say About f?

Let's see how the second derivative helps determine the intervals of concavity. Looking at Figure 6(a), you can see that, going from left to right in (b, c), the slope of the tangent increases.

This means that the derivative f' is an increasing function and therefore its derivative f'' is positive.

Likewise, in Figure 6(b) the slope of the tangent decreases from left to right on (c, d), so f' decreases and therefore f'' is negative.

What Does f'' Say About f ?

This reasoning can be reversed and suggests that the following theorem is true.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 4

Figure 8 shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is *P* concave upward or concave downward?

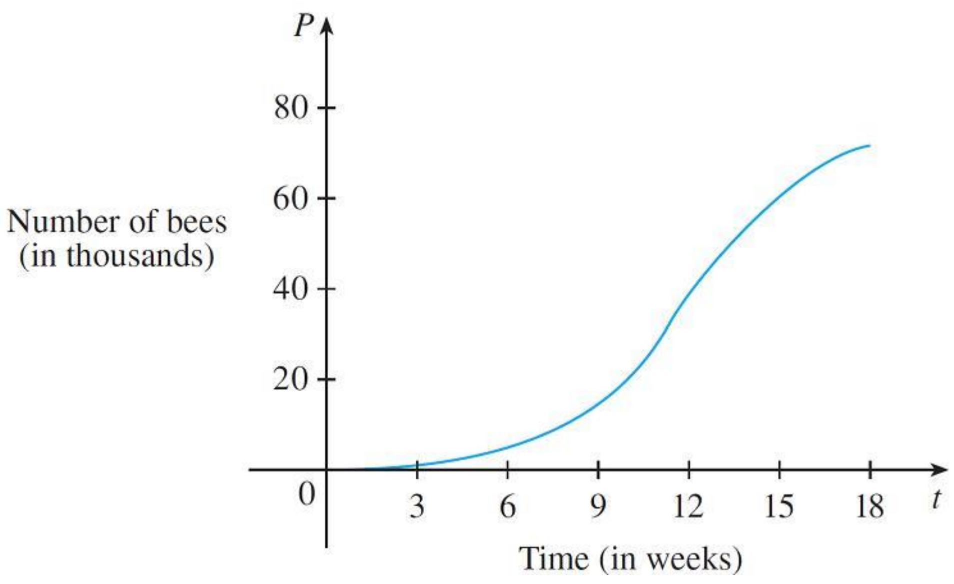


Figure 8

What Does f'' Say About f ?

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

What Does f'' Say About f ?

Note:
The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither.

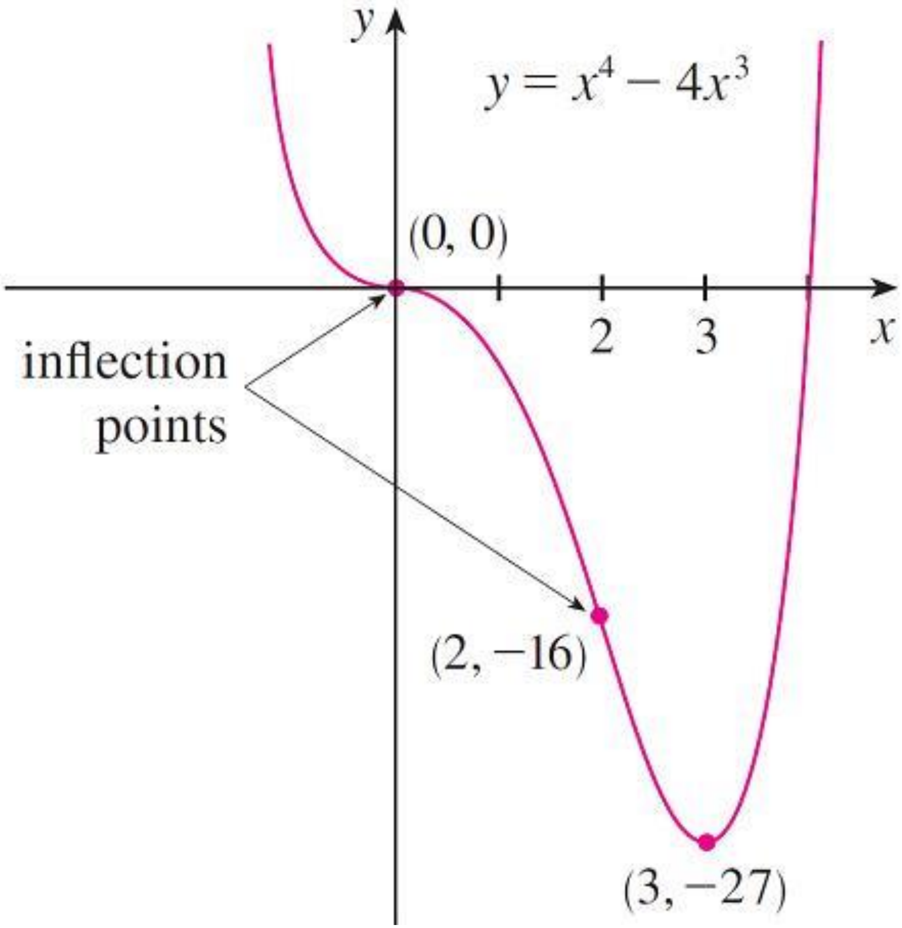
This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

Example 6

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Example 6

cont'd



Example 7

Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$

Example 7