11.2 Series

Series

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

π = 3.14159 26535 89793 23846 26433 83279 50288 . . .

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

where the three dots ($\cdots$) indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π.

Series

In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$\boxed{1} \qquad a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

Series

It would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms we get the cumulative sums 1, 3, 6, 10, 15, 21, . . . and, after the nth term, we get $n(n + 1)/2$, which becomes very large as n increases.

However, if we start to add the terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots$$

we get $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots, 1 - 1/2^n, \ldots .$

Series

The table shows that as we add more and more terms, these *partial sums* become closer and closer to 1.

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

Series

In fact, by adding sufficiently many terms of the series we can make the partial sums as close as we like to 1.

So it seems reasonable to say that the sum of this infinite series is 1 and to write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series (1) has a sum.

Series

We consider the **partial sums**

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit.

Series

If $\lim_{n \to \infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\Sigma\, a_n$.

$\boxed{2}$ **Definition** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\Sigma\, a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Series

Thus, the sum of a series is the limit of the sequence of partial sums.

So, when we write $\sum_{n=1}^{\infty} a_n = s$ we mean that by adding sufficiently many terms of the series we can get as close as we like to the number *s*.

Notice that $$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

Example 2

Example 2. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges and find its sum.

Geometric series

An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** *r*.

$\boxed{4}$ The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

Examples 3 and 4

Example 3. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Example 4. Is the series

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$$

convergent or divergent?

Examples 6 and 7

Example 6. Write the number

$$2.3\overline{17} = 2.3171717\ldots$$

as a ratio for integers.

Example 7. Find the sum of the series

$$\sum_{n=0}^{\infty} x^n,$$

where $|x| < 1$.

A necessary condition for conv.

6 **Theorem** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

Warning: The converse of Theorem 6 is not true in general.

If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\Sigma\, a_n$ is convergent as the following example shows:

Example 8. Show that the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Corollary: test for divergence

$\boxed{7}$ **The Test for Divergence** If $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 9. Show that the series

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$$

is divergent.

Series: limit laws

8 Theorem If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 10. Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right).$$

11.3 The Integral Test and Estimates of Sums

The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series $\Sigma\ 1/[n(n + 1)]$ because in each of those cases we could find a simple formula for the nth partial sum s_n.

But usually, it isn't easy to discover such a formula. However, in many cases one can determine whether a series converges or diverges without having an explicit formula for the nth partial sum s_n. Moreover, in many cases, one may obtain good estimates for the sum of a convergent series.

The Integral Test

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:

(a) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Note: It is enough if the function f is ultimately decreasing, that is, it is decreasing for $x > N$ for some positive integer *N*. Similarly, the sum and the integral does not have to start at 1.

Warning: In general, $\int_1^{\infty} f(x) dx \neq \sum_{n=1}^{\infty} a_n$.

The p-series

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the ***p*-series**.

$\boxed{1}$ The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Example 4

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.