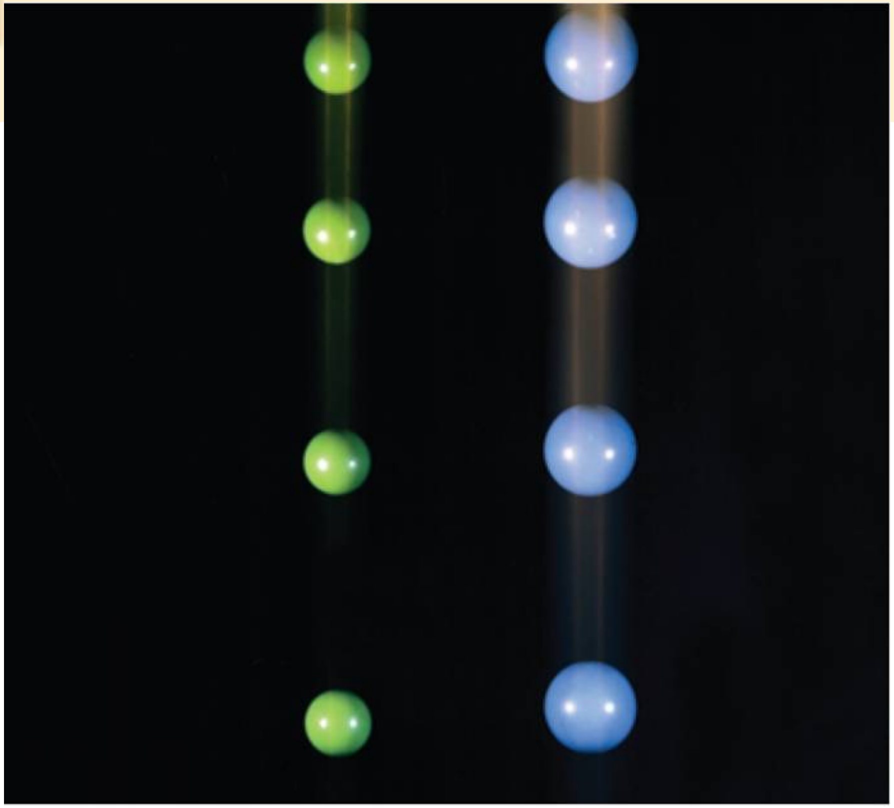
1 Functions and Limits



1.1 Four Ways to Represent a Function

Four Ways to Represent a Function

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets *D* and *E* are sets of real numbers. The set *D* is called the **domain** of the function.

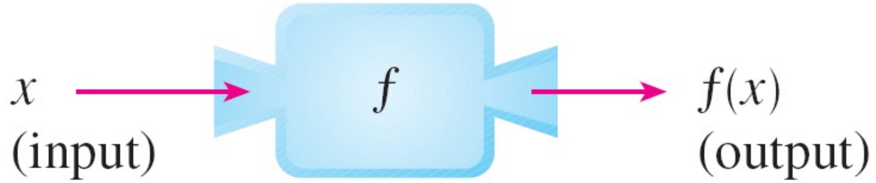
The number $f(x)$ is the **value of *f* at *x*** and is read "f of x." The **range** of *f* is the set of all possible values of $f(x)$ as *x* varies throughout the domain.

A symbol that represents an arbitrary number in the *domain* of a function *f* is called an **independent variable**.

Four Ways to Represent a Function

A symbol that represents a number in the *range* of *f* is called a **dependent variable**.

It's helpful to think of a function as a **machine** (see Figure 2).



Machine diagram for a function *f*

Figure 2

Four Ways to Represent a Function

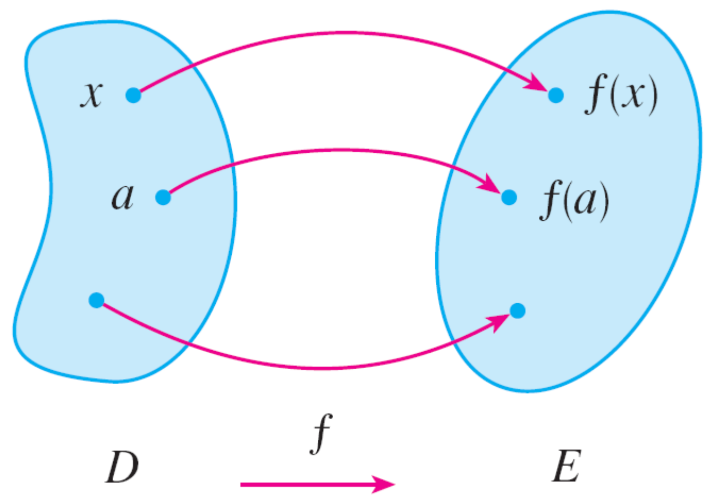
If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. In this scenario:

Domain: set of all possible inputs

Range: set of all possible outputs.

Four Ways to Represent a Function

Another way to picture a function is by an **arrow diagram** as in Figure 3.



Arrow diagram for *f*

Figure 3

Each arrow connects an element of *D* to an element of *E*. The arrow indicates that *f*(*x*) is associated with *x*, *f*(*a*) is associated with *a*, and so on.

Four Ways to Represent a Function

The most common method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

Geometrically, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

Four Ways to Represent a Function

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4).

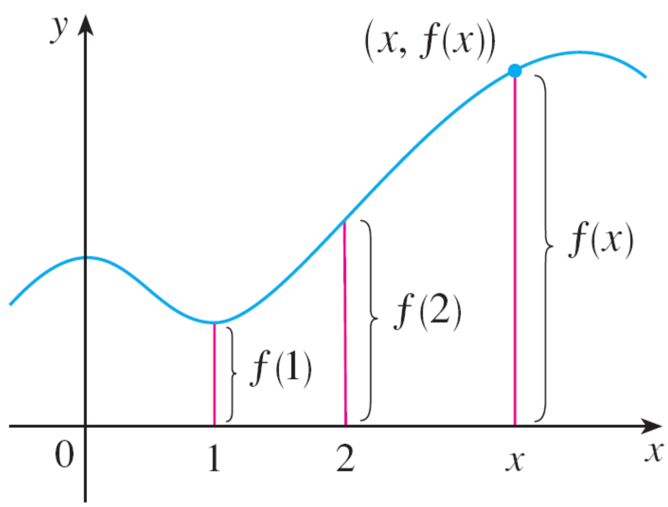


Figure 4

Four Ways to Represent a Function

The graph of *f* also allows us to picture the domain of *f* on the *x*-axis and its range on the *y*-axis as in Figure 5.

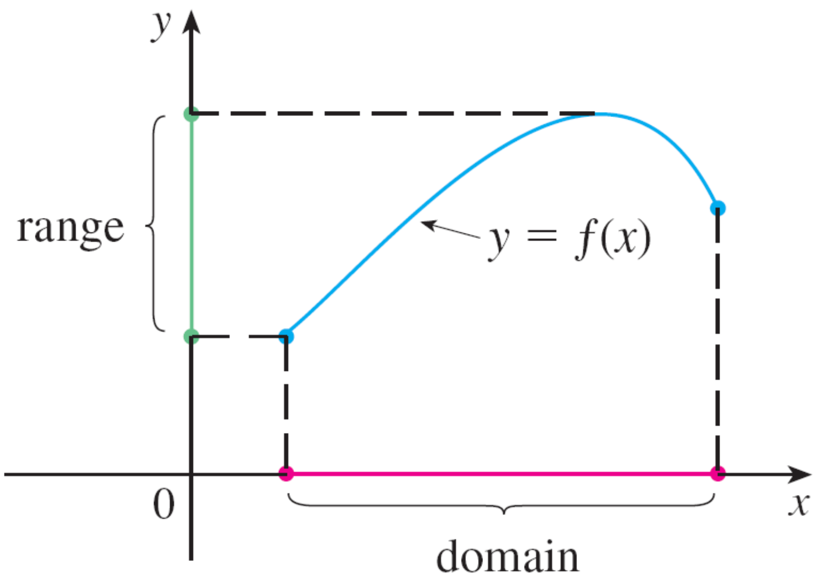
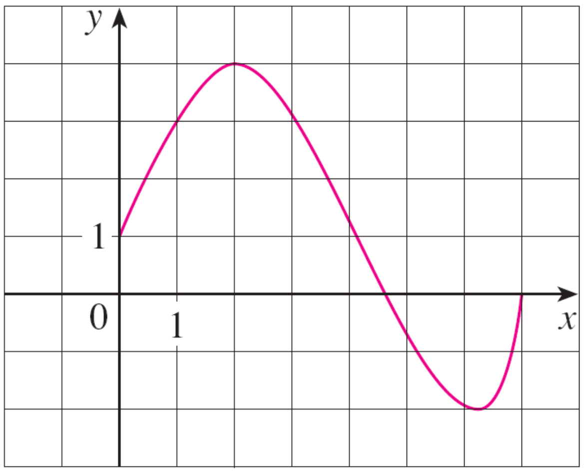


Figure 5

Example 1

The graph of a function *f* is shown in Figure 6.



The notation for intervals is given in Appendix A.

Figure 6

(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f?

Examples 2 and 3

Example 2. Sketch the graph and find the domain and range. (a) $f(x) = 2x-1$ (b) $g(x) = x^2$.

Example 3. If $f(x) = 2x^2-5x+1$ and $h \neq 0$, evaluate and simplify

$$\frac{f(a+h)-f(a)}{h}.$$

Representations of Functions

Representations of Functions

Here are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

Examples 4 - 6

Example 4. When you turn on a hot-water faucet, the temperature *T* of the water depends on how long the water has been running. Draw a rough graph of *T* as a function of the time *t* that has elapsed since the faucet was turned on.

Example 5. A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs $10 per m^2; material for the sides costs $6 per square meter. Express the cost of materials as a function of the width of the base.

Example 6. Find the domain of (a) $f(x) = \sqrt{x+2}$
(b) $f(x) = 1/(x^2 - x)$.

Representations of Functions

The graph of a function is a curve in the *xy*-plane. But the question arises: Which curves in the *xy*-plane are graphs of functions? This is answered by the following test.

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

Representations of Functions

The reason for the truth of the Vertical Line Test can be seen in Figure 13.

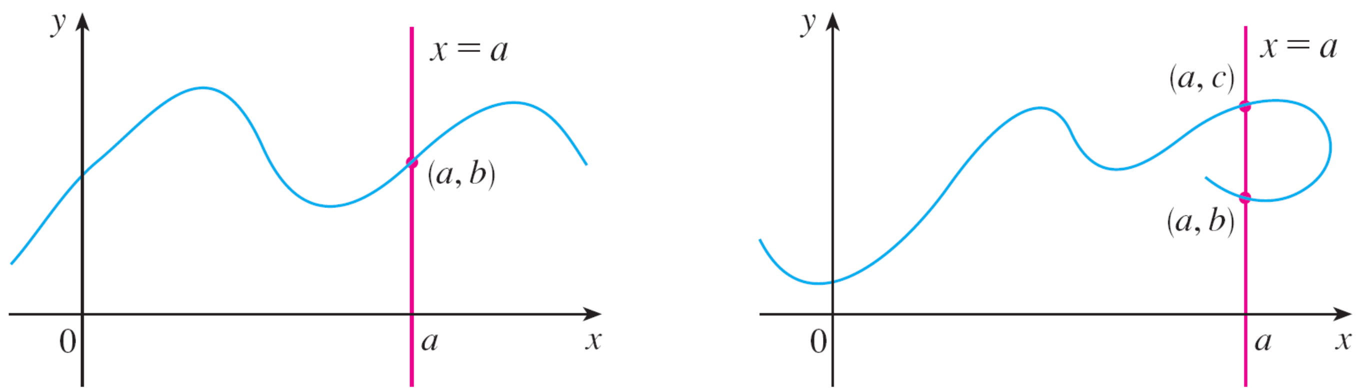
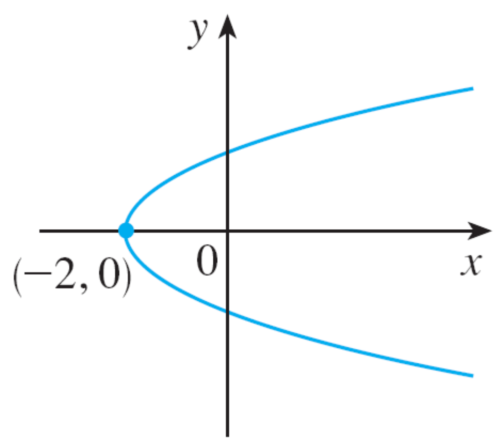


Figure 13

If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different values to a.

Representations of Functions

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x.



$x = y^2 - 2$

Figure 14(a)

Piecewise Defined Functions

Example 7

A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Piecewise Defined Functions

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number *a*, denoted by $|a|$, is the distance from *a* to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geq 0 \qquad \text{for every number } a$$

For example,

$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1$

$|3 - \pi| = \pi - 3$

Piecewise Defined Functions

In general, we have

$$\begin{aligned} |a| &= a \qquad \text{if } a \geqslant 0 \\ |a| &= -a \quad \text{if } a < 0 \end{aligned}$$

(Remember that if *a* is negative, then –*a* is positive.)

Example 8

Sketch the graph of the absolute value function $f(x) = |x|$.

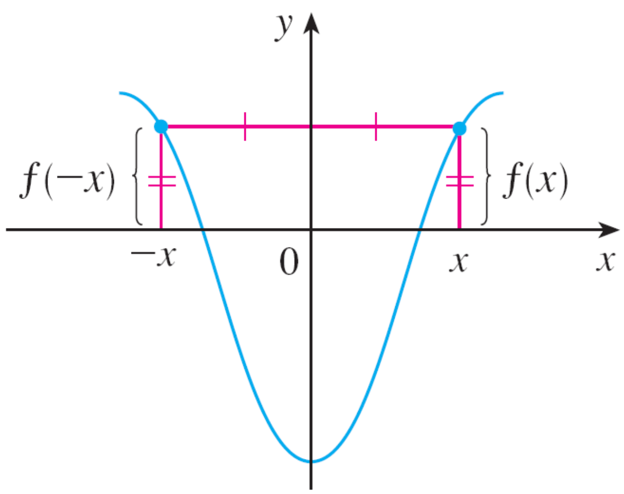
Symmetry

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 19).



An even function

Figure 19

Symmetry

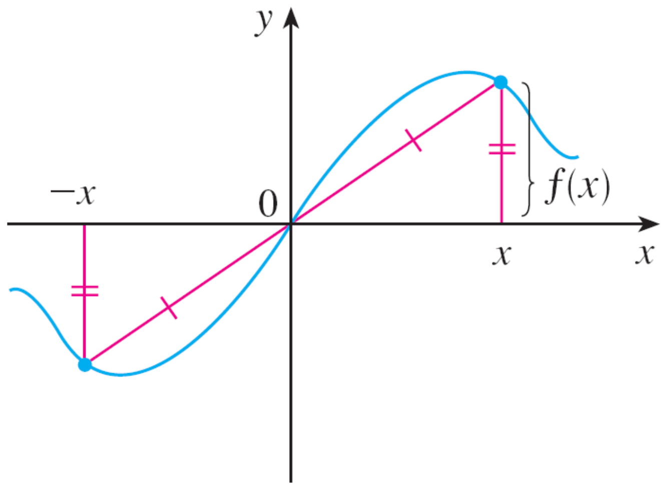
This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Symmetry

The graph of an odd function is symmetric about the origin (see Figure 20).



An odd function

Figure 20

If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

Example 11

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ **(b)** $g(x) = 1 - x^4$ **(c)** $h(x) = 2x - x^2$

Increasing and Decreasing Functions

Increasing and Decreasing Functions

The graph shown in Figure 22 rises from *A* to *B*, falls from *B* to *C*, and rises again from *C* to *D*. The function *f* is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$.

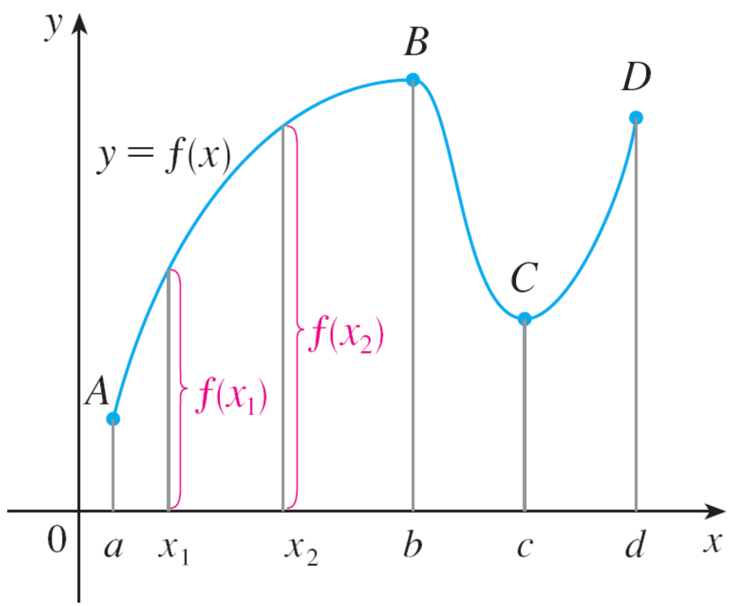


Figure 22

Increasing and Decreasing Functions

Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$.

We use this as the defining property of an increasing function.

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

Increasing and Decreasing Functions

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

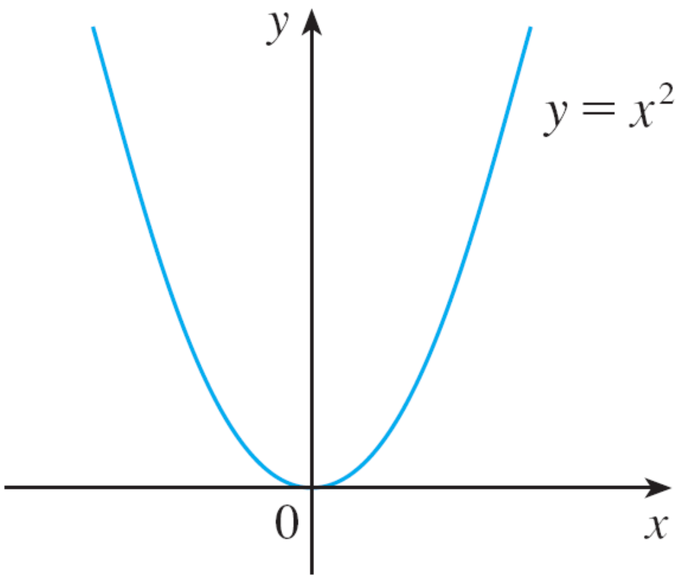


Figure 23

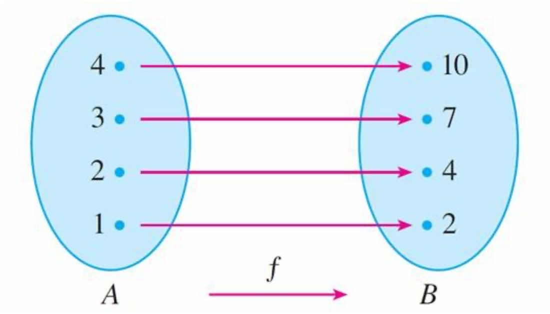
Inverse functions (Chapter 6.1)

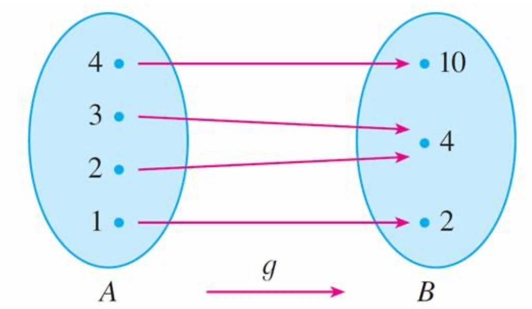
Inverse Functions

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Q: Which of the following functions is one-to one?

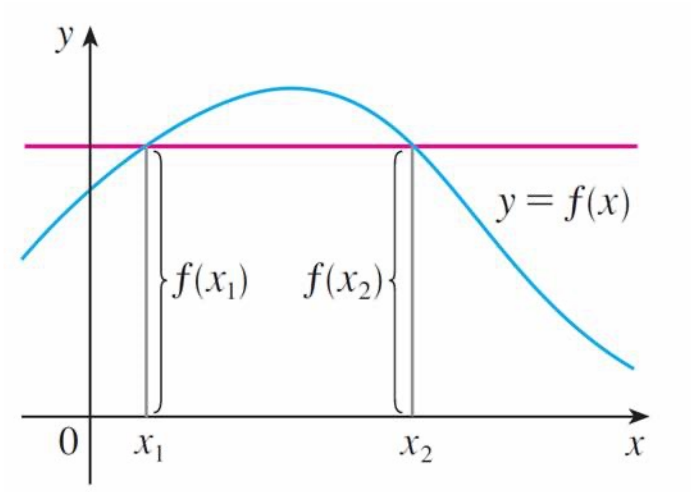




Note: increasing/decreasing functions are one-to-one.

Inverse Functions

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one.



This function is not one-to-one because $f(x_1) = f(x_2)$.

Figure 2

Inverse Functions

Therefore, we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1 and 2

Example 1. Is the function $f(x) = x^3$ one-to-one?

Example 2. Is the function $f(x) = x^2$ one-to-one?

Inverse Functions

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

$\boxed{2}$ **Definition** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if *f* maps *x* into *y*, then f^{-1} maps *y* back into *x*. (If were not one-to-one, then f^{-1} would not be uniquely defined.)

Inverse Functions

The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f.

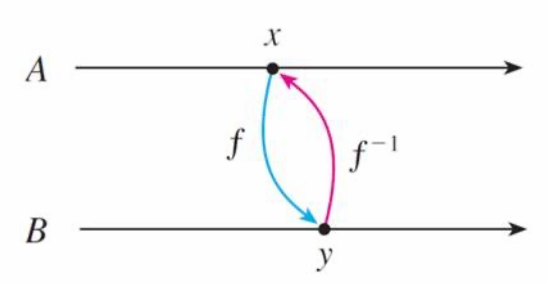


Figure 5

Note that

$$\boxed{\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned}}$$

Inverse Functions

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

Warning: This is different from the reciprocal $1/f(x)$ which could, however, be written as $[f(x)]^{-1}$.

Example 3

If $f(1) = 5$, $f(3) = 7$, and $f(8) = –10$, find $f^{–1}(7)$, $f^{–1}(5)$, and $f^{–1}(–10)$.

Inverse Functions

The letter *x* is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on *f*, we usually reverse the roles of *x* and *y* in Definition 2 and write

3
$$\boxed{f^{-1}(x) = y \iff f(y) = x}$$

By substituting for *y* in Definition 2 and substituting for *x* in $\boxed{3}$, we get the following **cancellation equations**:

4
$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in } A \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in } B \end{aligned}$$

Inverse Functions

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 7).

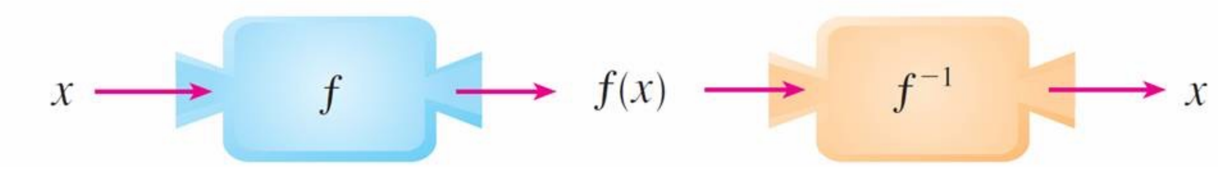


Figure 7

Thus f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

Inverse Functions

For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$

$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Inverse Functions

Now let's see how to compute inverse functions. If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$.

If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

5 How to Find the Inverse Function of a One-to-One Function f

Step 1 Write $y = f(x)$.

Step 2 Solve this equation for x in terms of y (if possible).

Step 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 4

Find the inverse function of $f(x) = x^3 + 2$.

Inverse Functions

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f.

Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.

But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

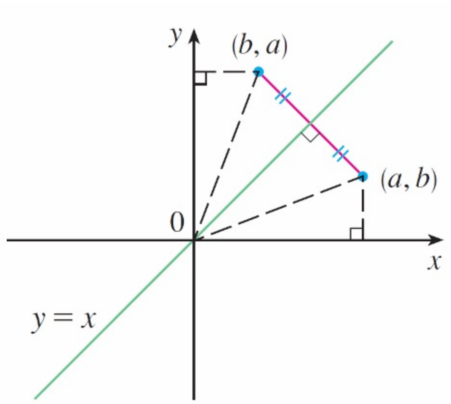


Figure 8

Inverse Functions

Therefore, as illustrated by Figure 9:

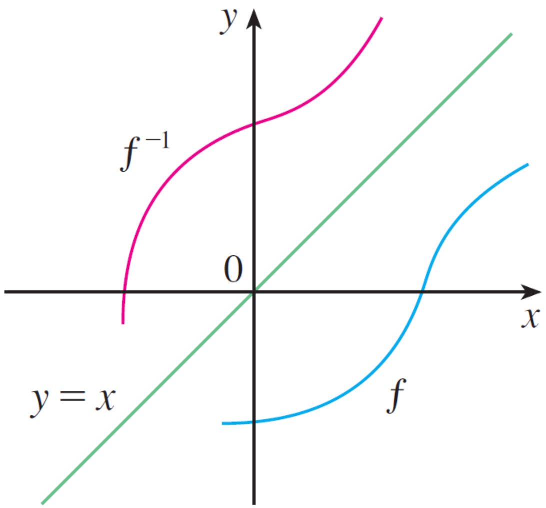
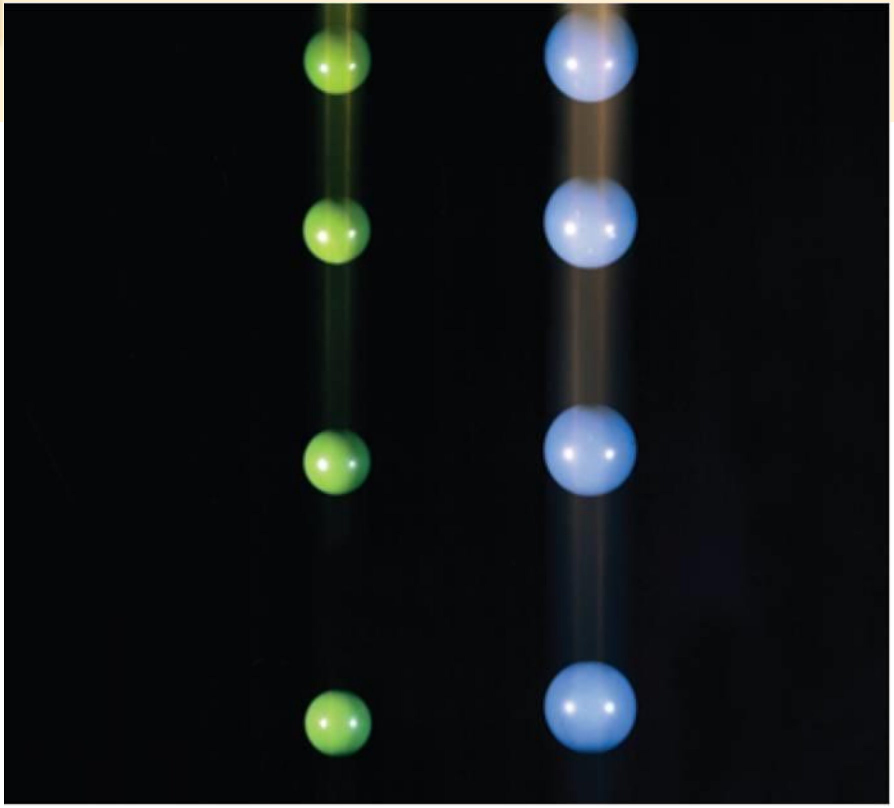


Figure 9

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

1

Functions and Limits



1.2 Mathematical Models: A Catalog of Essential Functions

Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions.

The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Figure 1 illustrates the process of mathematical modeling.

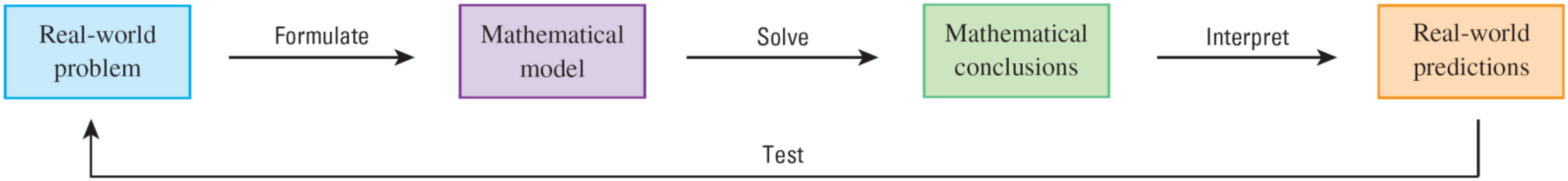


Figure 1

The modeling process

Mathematical Models: A Catalog of Essential Functions

- A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*.

- A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.

- There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

Linear Models

When we say that y is a **linear function** of x, we mean that the graph of the function is a line, so we can use the so-called *slope-intercept form* of the equation of a line to write a formula for the function as

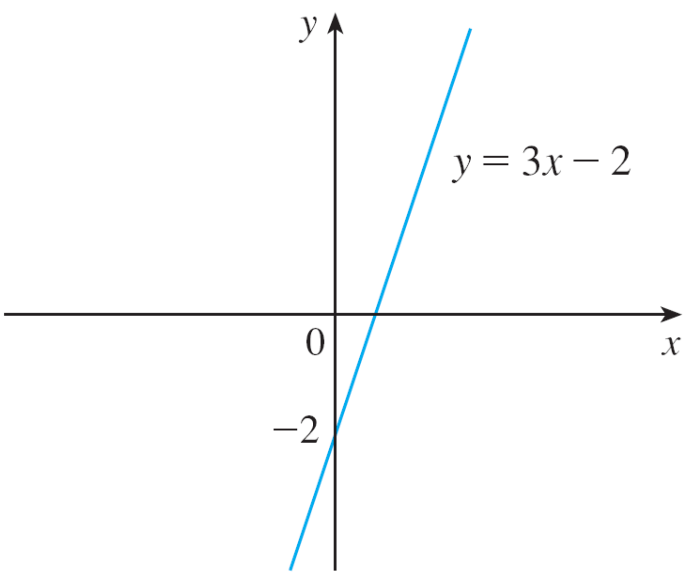
$$y = f(x) = mx + b$$

where m is the slope of the line and b is the y-intercept.

Linear Models

A characteristic feature of linear functions is that they grow at a constant rate.

For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values.



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

Figure 2

Linear Models

Notice that whenever *x* increases by 0.1, the value of *f*(*x*) increases by 0.3.

So *f*(*x*) increases three times as fast as *x*. Thus the slope of the graph $y = 3x - 2$, namely 3, can be interpreted as the rate of change of *y* with respect to *x*.

In general: if there $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two distinct points on the graph of a linear function, then the slope of the line is given by:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

and thus $\Delta y = m\Delta x$.

Linear Models

The so-called *point-slope form* of the equation of the line through $P(x_1,y_1)$ with slope m is

$$y - y_1 = m(x - x_1)$$

Example: Find both the point-slope and the slope-intercept form of the equation of the line through $P_1(1,2)$ and $P_2(2,4)$.

Example 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature *T* (in °C) as a function of the height *h* (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

Polynomials

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial.

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

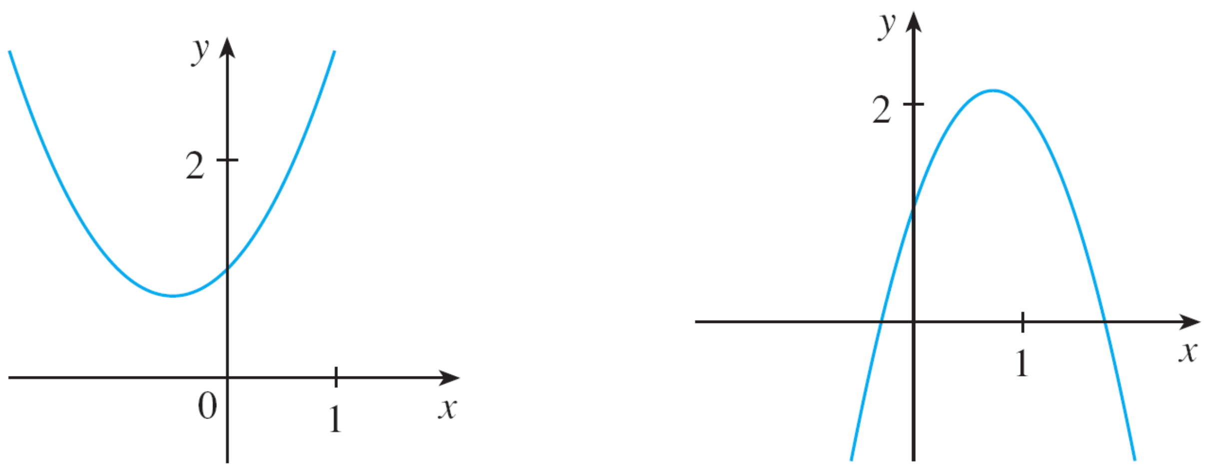
Polynomials

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function ($m \neq 0$).

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function** ($a \neq 0$).

Polynomials

Its graph is always a parabola obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)



(a) $y = x^2 + x + 1$

(b) $y = -2x^2 + 3x + 1$

The graphs of quadratic functions are parabolas.

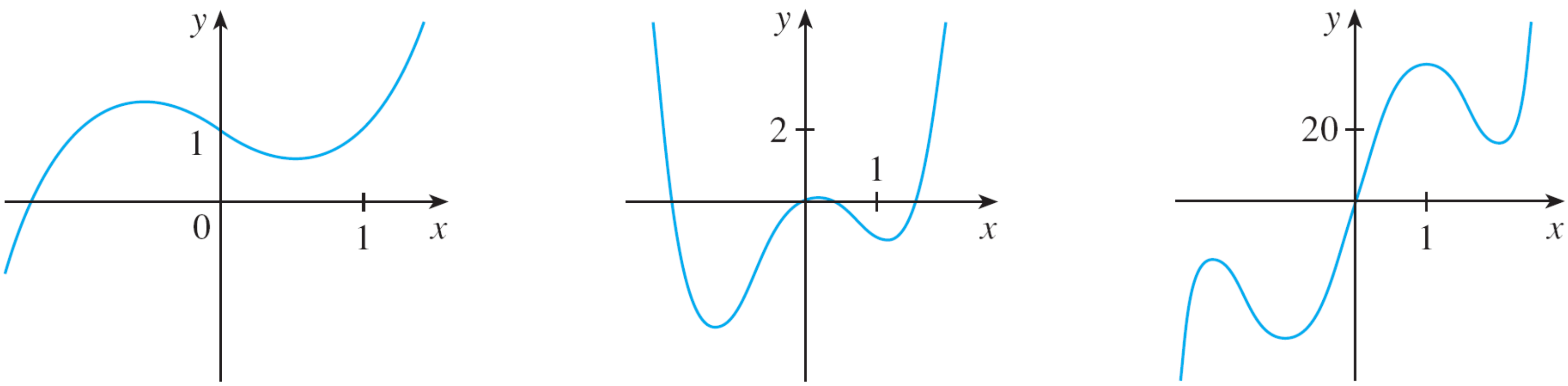
Figure 7

Polynomials

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c).



(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Figure 8

Power Functions

Power Functions

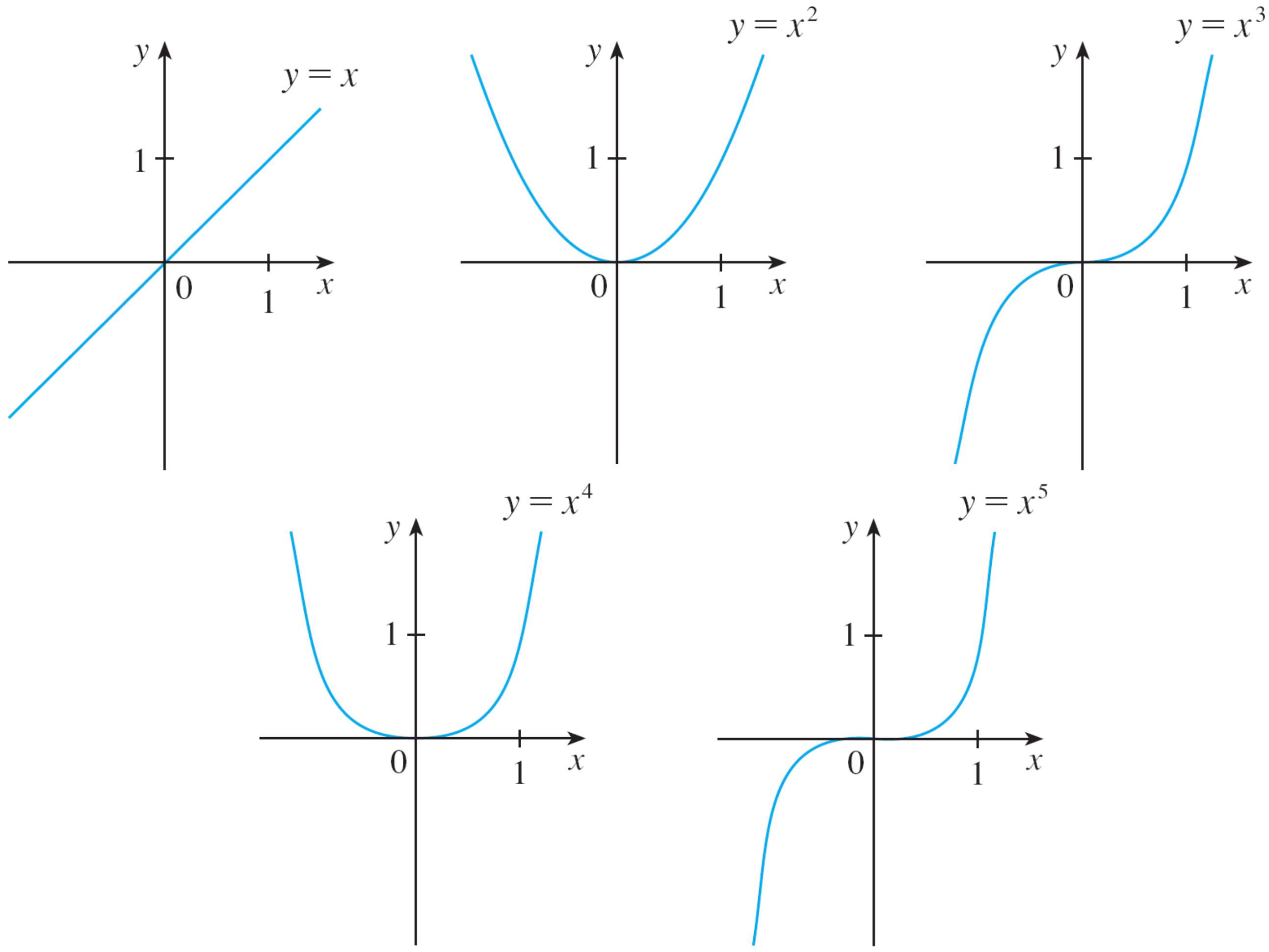
A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $\boldsymbol{a = n}$**, where** $\boldsymbol{n}$ **is a positive integer**

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.)

We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ (a parabola).

Power Functions



Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

Figure 11

Power Functions

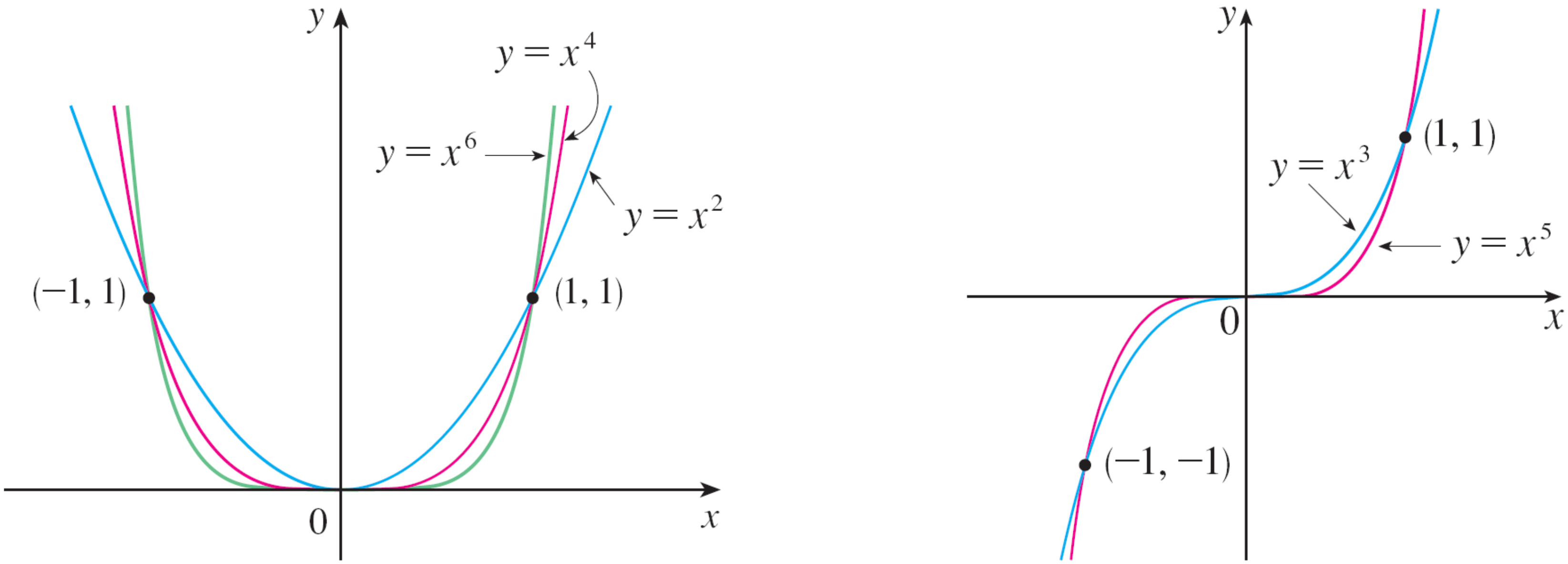
The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$.

If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$.

Power Functions

Notice from Figure 12, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)



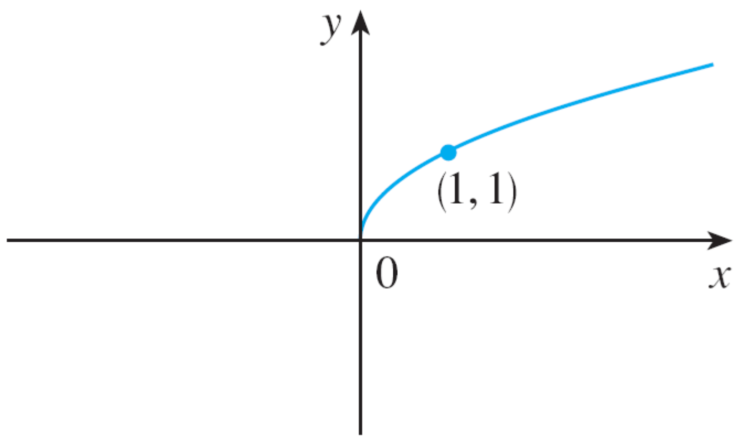
Families of power functions

Figure 12

Power Functions

(ii) *a* = 1/*n*, where *n* is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For *n* = 2 it is the square root function $f(x) = \sqrt{x}$, whose domain is [0, ∞) and whose graph is the upper half of the parabola $x = y^2$. [See Figure 13(a).]



$f(x) = \sqrt{x}$

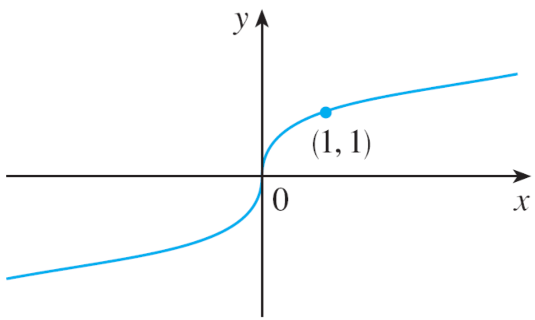
Graph of root function

Figure 13(a)

Power Functions

For other even values of *n*, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.

For *n* = 3 we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for *n* odd (*n* > 3) is similar to that of $y = \sqrt[3]{x}$.



$f(x) = \sqrt[3]{x}$

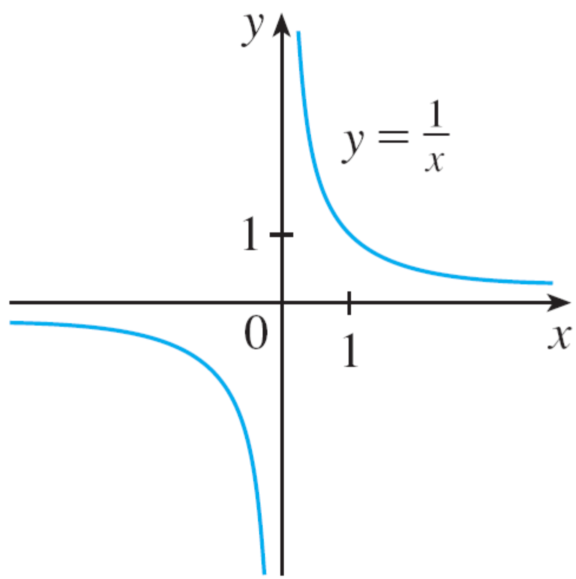
Graph of root function

Figure 13(b)

Power Functions

(iii) *a* = –1

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes.



The reciprocal function

Figure 14

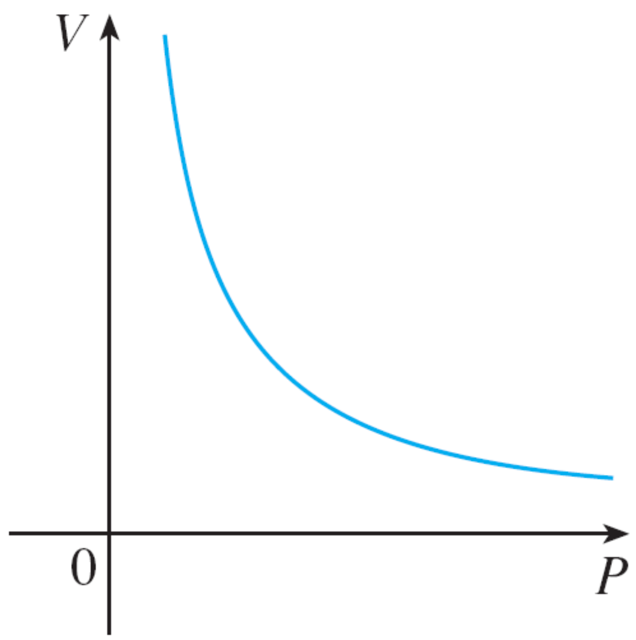
Power Functions

This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume V of a gas is inversely proportional to the pressure P:

$$V = \frac{C}{P}$$

where C is a constant.

Thus the graph of V as a function of P (see Figure 15) has the same general shape as the right half of Figure 14.



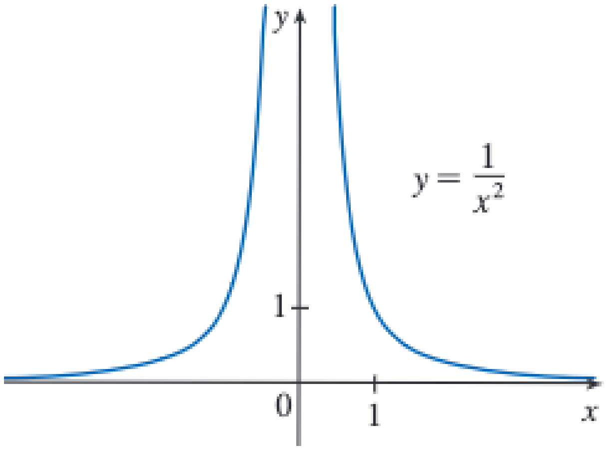
Volume as a function of pressure at constant temperature

Figure 15

Power Functions

(iv) n = – 2: $f(x) = \frac{1}{x^2}$

Many natural laws state that one quantity is inversely proportional to the square of another quantity. In other words, the first quantity is modeled by a function of the form $f(x) = C/x^2$ and we refer to this as an **inverse square law.**



Inverse square laws model gravitational force, loudness of sound, and electrostatic force between two charged particles

Rational Functions

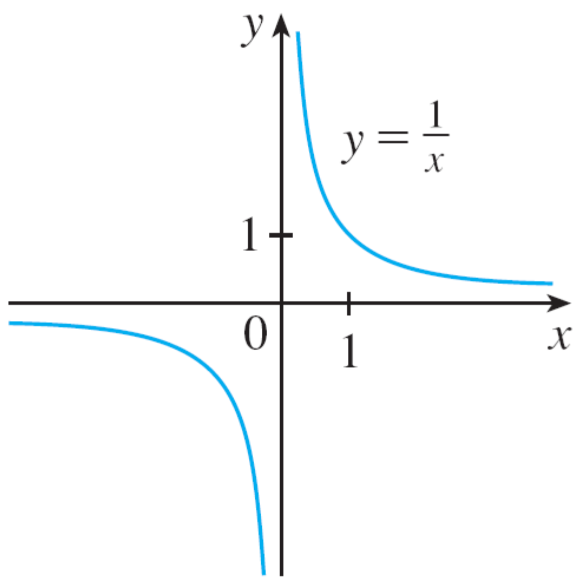
Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$.

A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14.



The reciprocal function

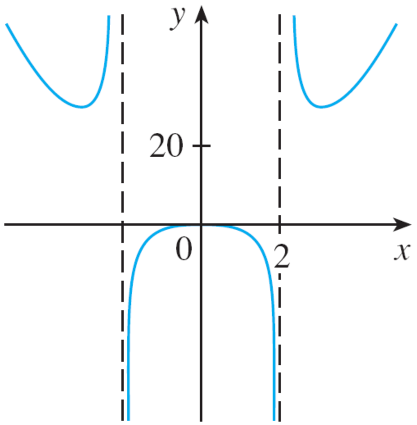
Figure 14

Rational Functions

The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 16.



$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

Figure 16

Algebraic Functions

Algebraic Functions

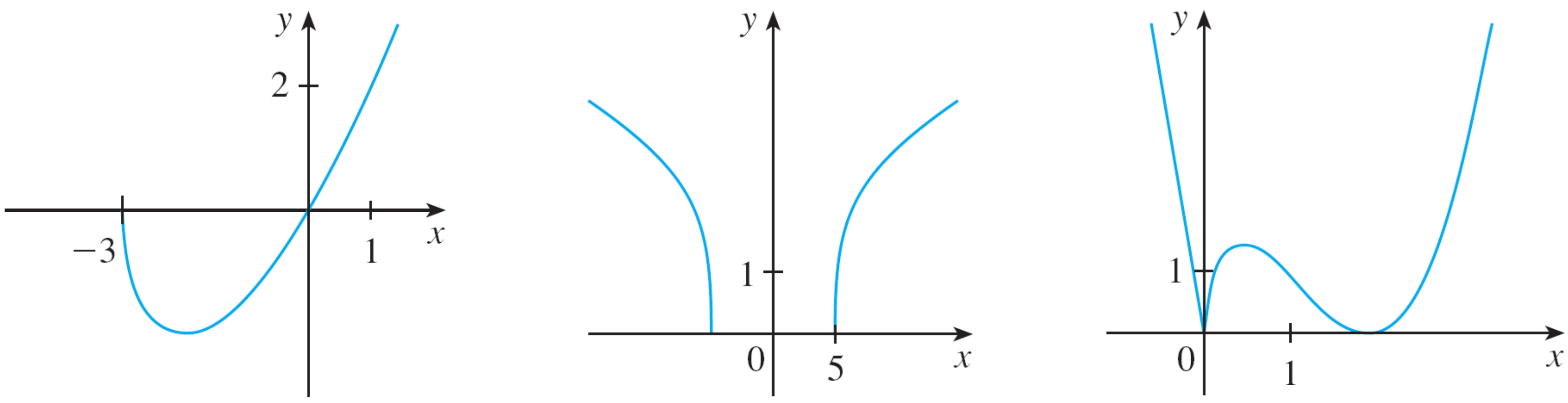
A function *f* is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function.

Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

Algebraic Functions

The graphs of algebraic functions can assume a variety of shapes. Figure 17 illustrates some of the possibilities.



(a) $f(x) = x\sqrt{x+3}$ (b) $g(x) = \sqrt[4]{x^2 - 25}$ (c) $h(x) = x^{2/3}(x-2)^2$

Figure 17

Algebraic Functions

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity *v* is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

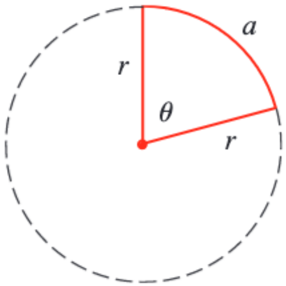
where m_0 is the rest mass of the particle and
$c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

Trigonometric Functions

Radian measure

The radian measure of an angle θ is given by

$$\theta = \frac{a}{r}$$



Note: when *r = 1* then the radian measure of θ equals to the arc length of the corresponding arc.

Example: (a) Find the radian measure of 60° (b) Express $\frac{5\pi}{4}$ rad in degrees.

Radian measure

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	$270°$	$360°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Standard position

- The standard position of an angle occurs when we place its vertex at the origin of a rectangular coordinate system and its initial side on the positive x-axis as in Figure 3.
- A positive angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side.
- Likewise, negative angles are obtained by clockwise rotation as in Figure 4

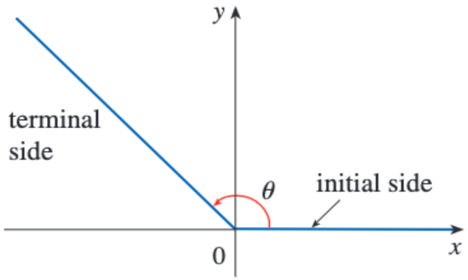


FIGURE 3 $\theta \geqslant 0$

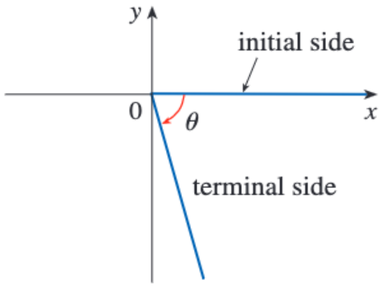
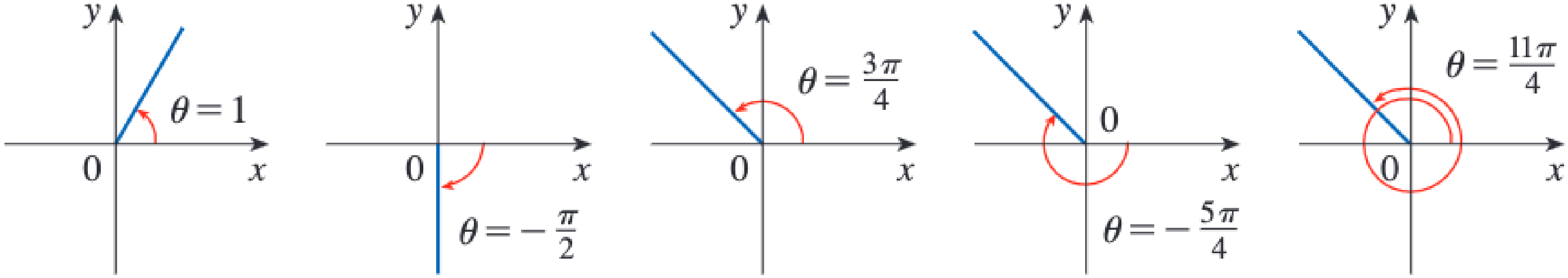


FIGURE 4 $\theta < 0$

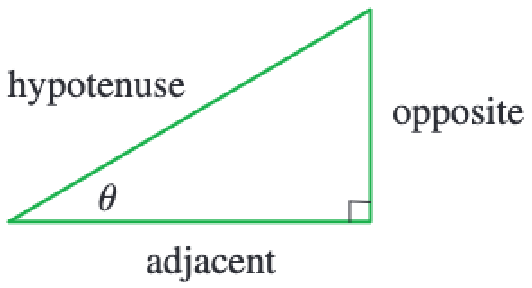
Standard position

For example:



Trigonometric functions

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows:



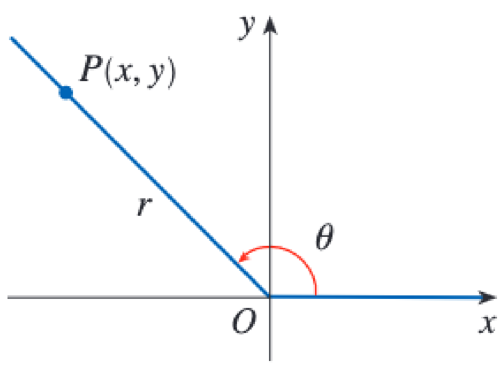
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Trigonometric functions

This definition doesn’t apply to obtuse or negative angles, so for a general angle in standard position we let *P(x,y)* be any point on the terminal side of θ and we let *r* be the distance |*OP*| as in the figure below. Then we define:



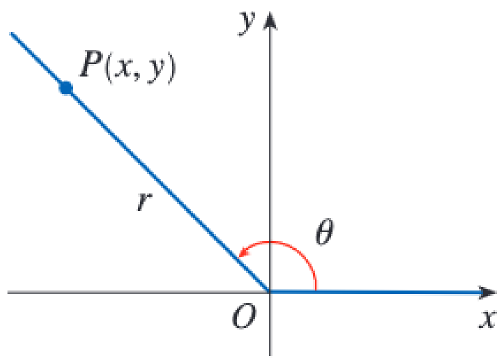
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Trigonometric functions

Since division by 0 is not defined, $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$ and $\cot\theta$ and $\csc\theta$ are undefined when $y = 0$. Notice that these previous definitions are consistent when θ is an acute angle.



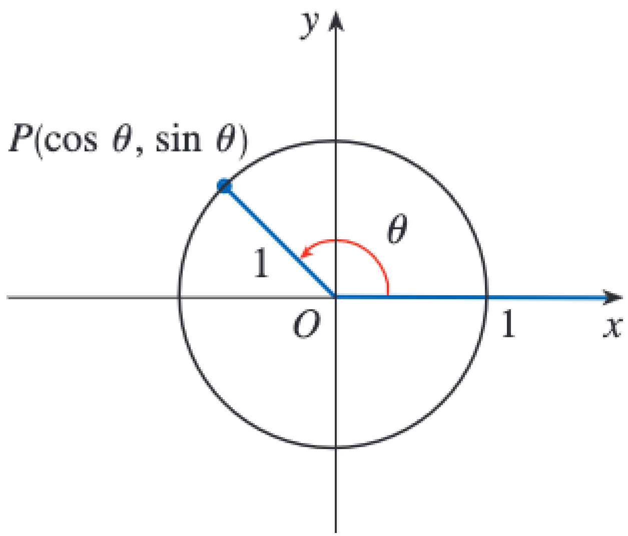
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

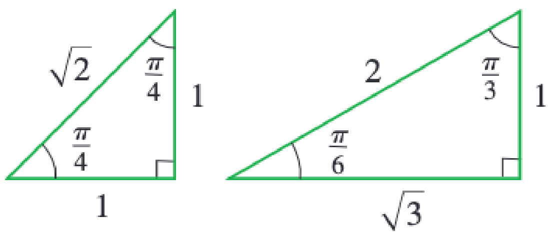
Trigonometric functions

If we put r = 1 in the previous definition and draw a unit circle with center the origin and label θ as in the figure below, then the coordinates of *P* are $(\cos\theta, \sin\theta)$.



Trigonometric functions

The exact trigonometric ratios for certain angles can be read from the triangles in the following figure:



For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

Trigonometric functions

Example: Find the exact trigonometric ratios for $\theta = \frac{2\pi}{3}$.

Example: If $\cos\theta = \frac{2}{5}$, find the other five trigonometric functions of θ.

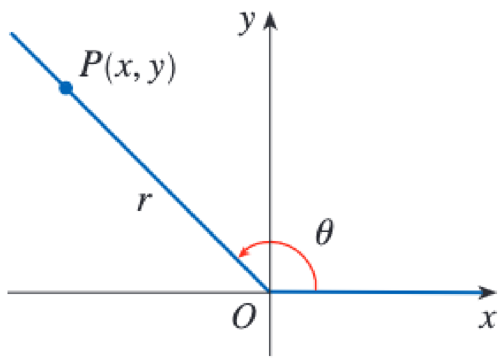
Trigonometric functions

The following table gives some values of found by the same method as before:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

Trigonometric functions

Recall: for a general angle in standard position, we let *P(x,y)* be any point on the terminal side of θ and we let *r* be the distance |*OP*| as in the figure below. Then we define:



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Trigonometric Functions

Thus, the graphs of the sine and cosine functions are as shown in Figure 18.

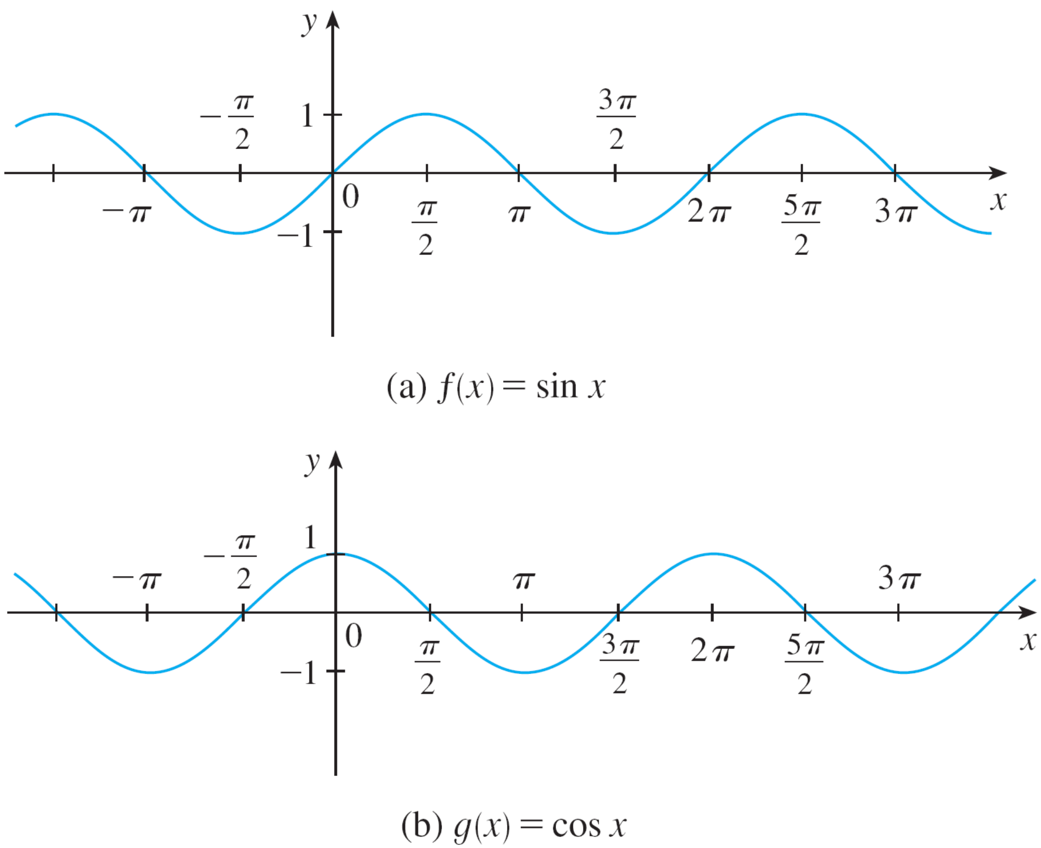


Figure 18

Trigonometric Functions

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval [–1, 1].

Thus, for all values of *x*, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \qquad |\cos x| \leq 1$$

Trigonometric Functions

Also, the zeros of the sine and cosine functions:

$\sin x = 0$	when	$x = n\pi$	n an integer
$\cos x = 0$	when	$x = \pi/2 + n\pi$	n an integer

An important property of the sine and cosine functions is that they are periodic functions and have period 2π.

This means that, for all values of x,

$$\boxed{\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x}$$

Important trigonometric identities

Pythagoraen theorem:

$$\sin^2(x) + \cos^2(x) = 1$$

The sine function is odd, the cosine function is even:

$$\sin(-x) = -\sin x; \quad \cos(-x) = \cos x$$

Addition formulae:

$$\boxed{\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \end{aligned}}$$

Trigonometric formulae

Example: Derive the substraction formulae and double angel formulae for sine and cosine.

Example: Derive the half-angle (or linearization) formulae:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Example: Derive the product identities:

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

Example 5

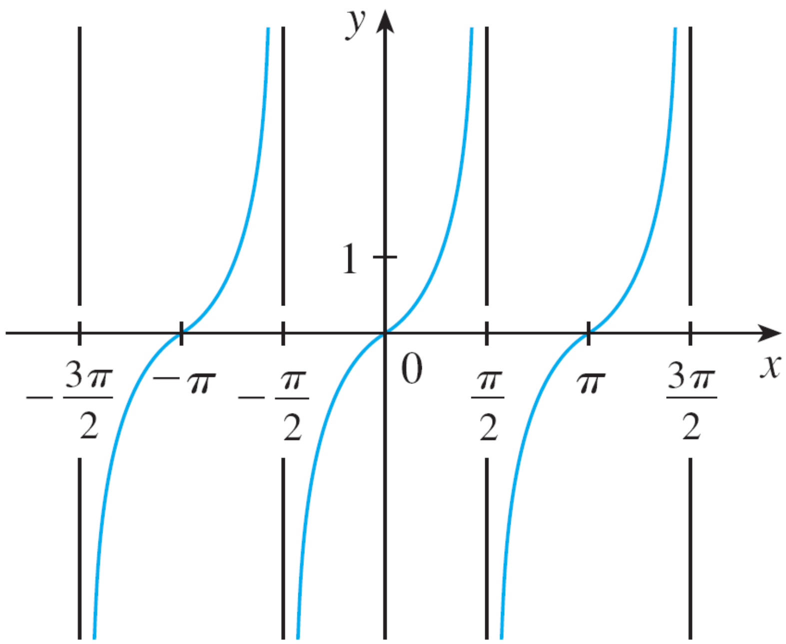
Find the domain of $f(x) = \frac{1}{1-2\cos x}$.

Trigonometric Functions

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 19. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$.



$y = \tan x$

Figure 19

Its range is $(-\infty, \infty)$.

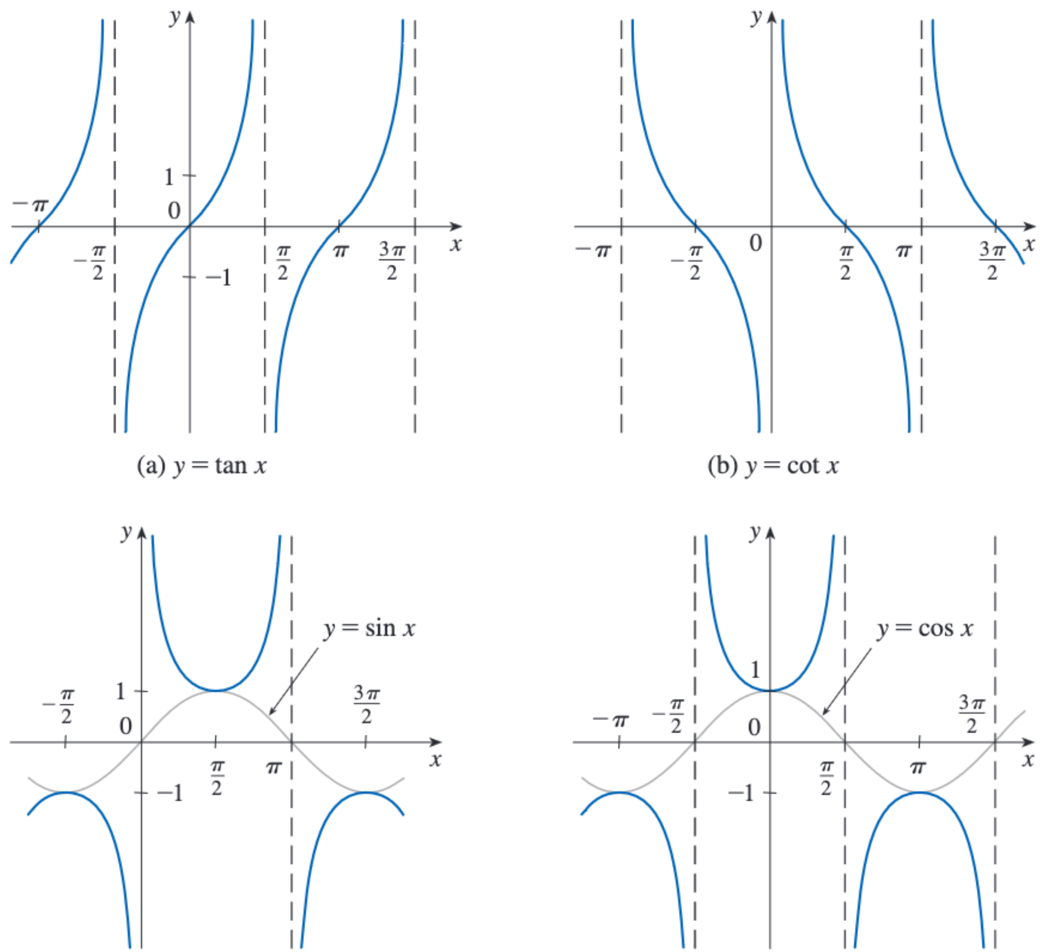
Trigonometric Functions

Notice that the tangent function has period π :

$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions; their graphs are shown on the next slide.

Trigonometric functions



(a) $y = \tan x$

(b) $y = \cot x$

(c) $y = \csc x$

(d) $y = \sec x$

Exponential Functions

Exponential Functions

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant.

The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 20. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

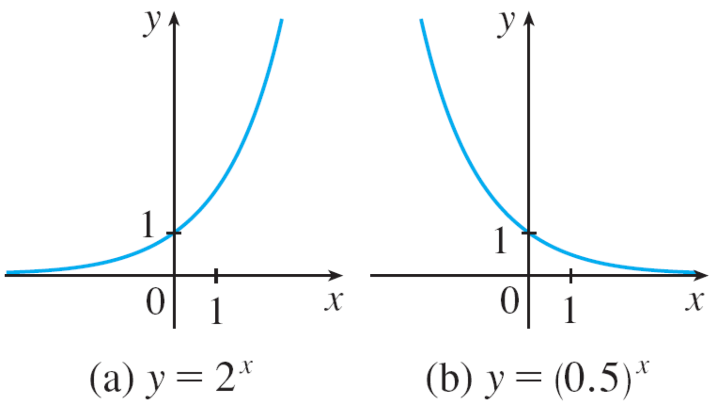


Figure 20

Exponential Functions

Exponential functions are useful for modeling many natural phenomena, such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

Logarithmic Functions

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions. Figure 21 shows the graphs of four logarithmic functions with various bases.

In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

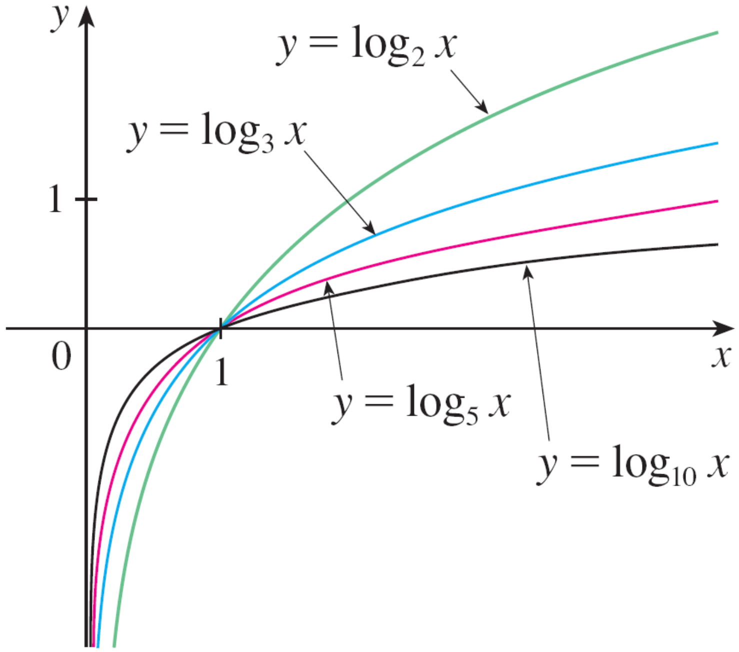


Figure 21

Example 6

Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

(d) $u(t) = 1 - t + 5t^4$

1.3 New Functions from Old Functions

Transformations of Functions

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions.

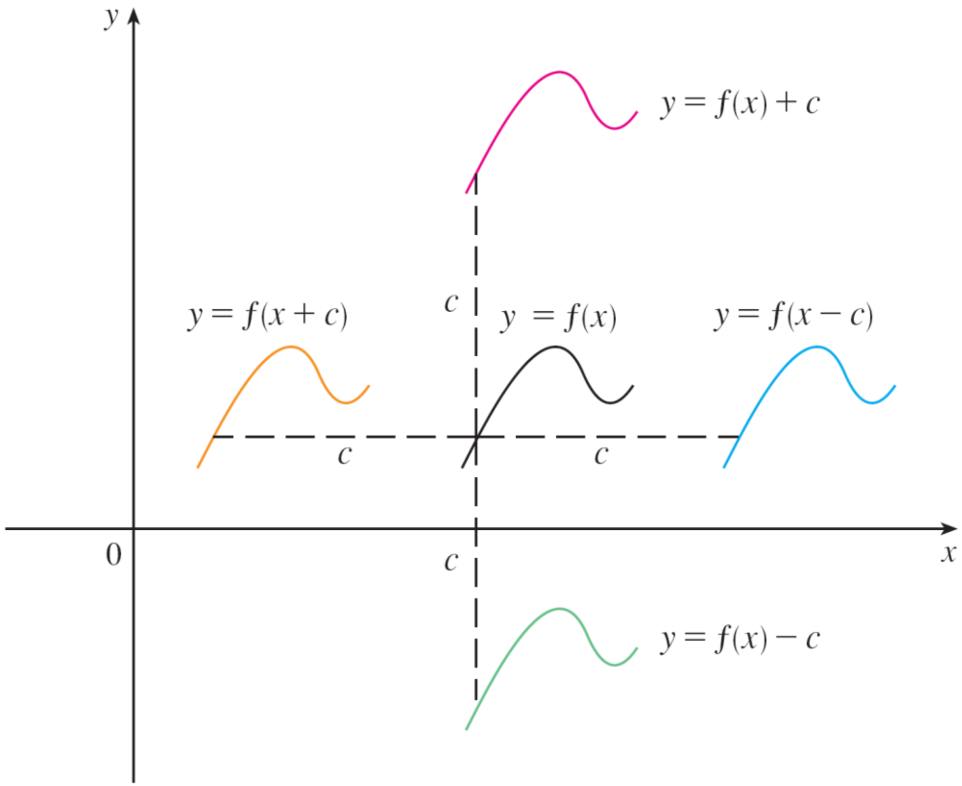
This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c).

Transformations of Functions

Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x).

Therefore, the graph of $y = f(x - c)$, is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).



Translating the graph of f

Figure 1

Transformations of Functions

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Example: Show that the graph of the cosine function is the same as the graph of the sine function shifted to the left by $\pi/2$; that is, $\cos x = \sin(x + \frac{\pi}{2})$.

Transformations of Functions

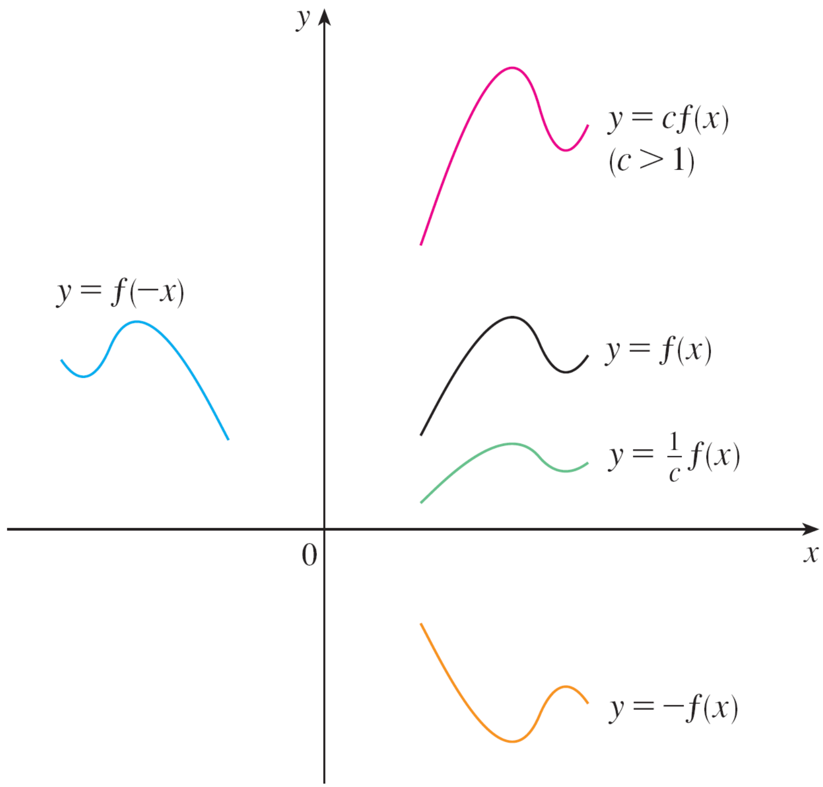
Now let's consider the **stretching** and **reflecting** transformations.

If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c).

Transformations of Functions

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis because the point (x, y) is replaced by the point $(x, -y)$.

(See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)



Stretching and reflecting the graph of *f*

Figure 2

Transformations of Functions

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Transformations of Functions

Figure 3 illustrates these stretching transformations when applied to the cosine function with *c* = 2.

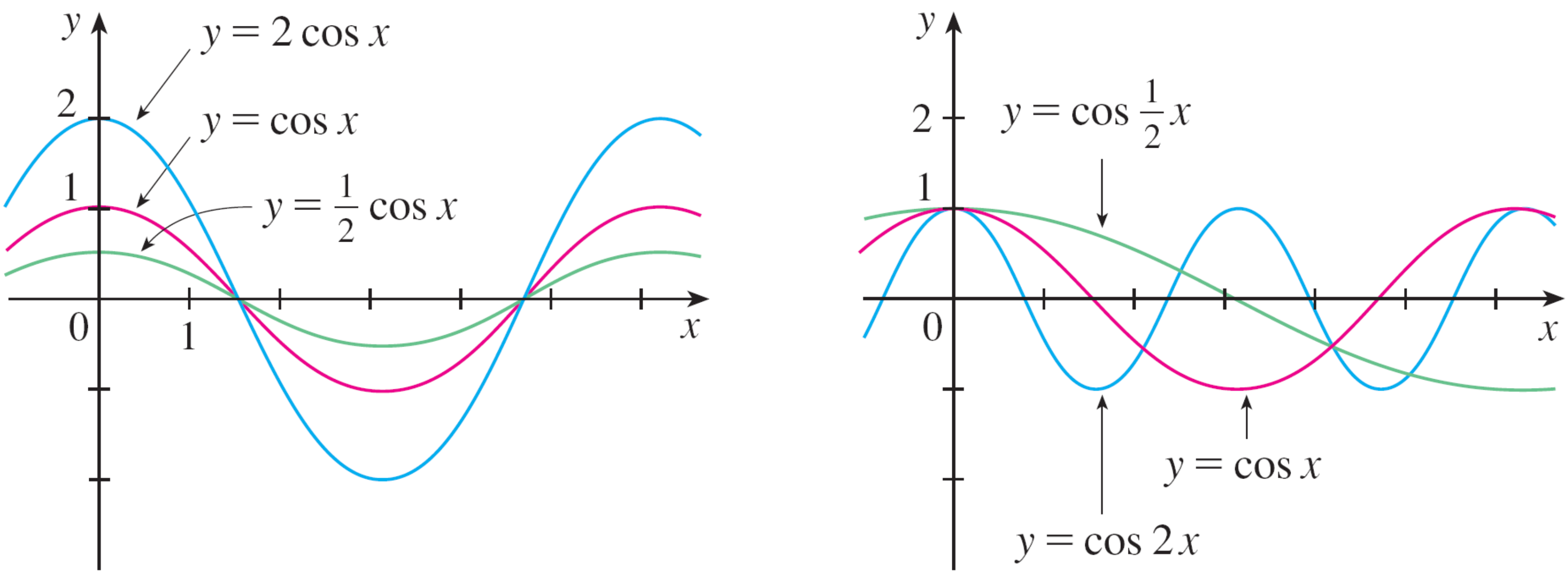


Figure 3

Transformations of Functions

For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y-coordinate of each point on the graph of $y = \cos x$ by 2.

This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

Examples

1. Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

2. Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

3. Sketch the graph of (a) $y = \sin(2x)$ and (b) $y = 1 - \sin x$

Transformations of Functions

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$.

This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: The part of the graph that lies above the x-axis remains the same; the part that lies below the x-axis is reflected about the x-axis.

Example 5. Sketch the graph of the function $y = |x^2 - 1|$.

Combinations of Functions

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad (f - g)(x) = f(x) - g(x)$$

If the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined.

For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Combinations of Functions

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$.

For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

Combinations of Functions

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$.

Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

Combinations of Functions

In general, given any two functions *f* and *g*, we start with a number *x* in the domain of *g* and find its image *g*(*x*). If this number *g*(*x*) is in the domain of *f*, then we can calculate the value of *f*(*g*(*x*)).

The result is a new function *h*(*x*) = *f*(*g*(*x*)) obtained by substituting *g* into *f*. It is called the *composition* (or *composite*) of *f* and *g and* is denoted by $f \circ g$ ("*f* circle *g*").

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

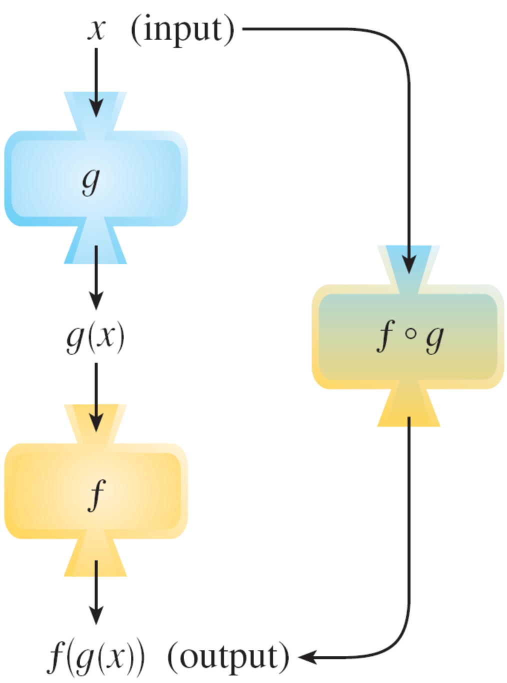
$$(f \circ g)(x) = f(g(x))$$

Combinations of Functions

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

Figure 11 shows how to picture $f \circ g$ in terms of machines.



The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Figure 11

Examples

6. If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

7. If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each function and its domain. (a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

Combinations of Functions

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Examples: composition

8. Find $f \circ g \circ h$ if $f(x) = \frac{x}{x+1}$, $g(x) = x^{10}$ and $h(x) = x + 3$.

9. Given $F(x) = \cos^2(x + 9)$, find functions f, g, h such that

$$F = f \circ g \circ h.$$

1.4 The Tangent and Velocity Problems

Tangent and velocity problems

Before introducing the notion of the limit of a function we consider two motivating problems: finding equations of *tangent lines* to curves and calculating the *instantaneous velocity* of a moving object knowing its position.

The Tangent Problem

The Tangent Problems

The word *tangent* is derived from the Latin word *tangens*, which means “touching.”

Thus, a tangent to a curve is a line that touches the curve.

In other words, a tangent line should have the same direction as the curve at the point of contact.

The Tangent Problems

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once, as in Figure 1(a).

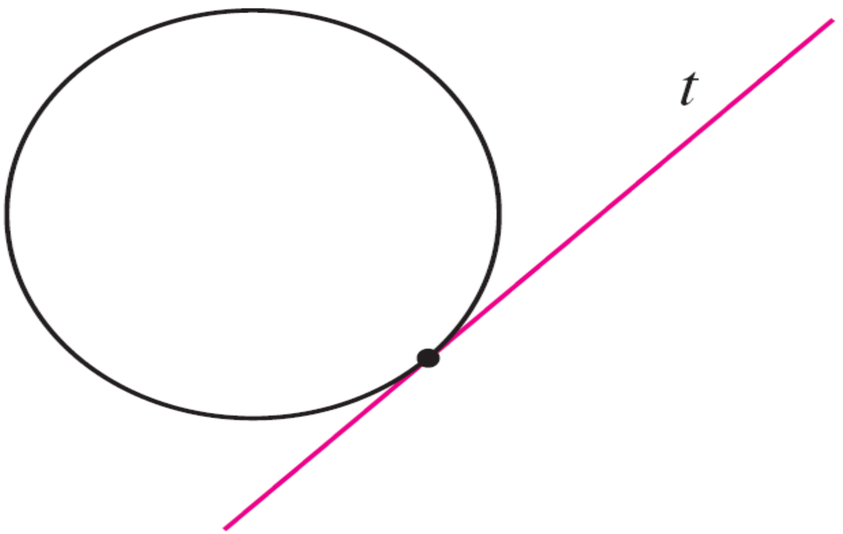


Figure 1(a)

The Tangent Problems

For more complicated curves this definition is inadequate. Figure 1(b) shows two lines *l* and *t* passing through a point *P* on a curve *C*.

The line *l* intersects *C* only once, but it certainly, does not look like what we think of as a tangent.

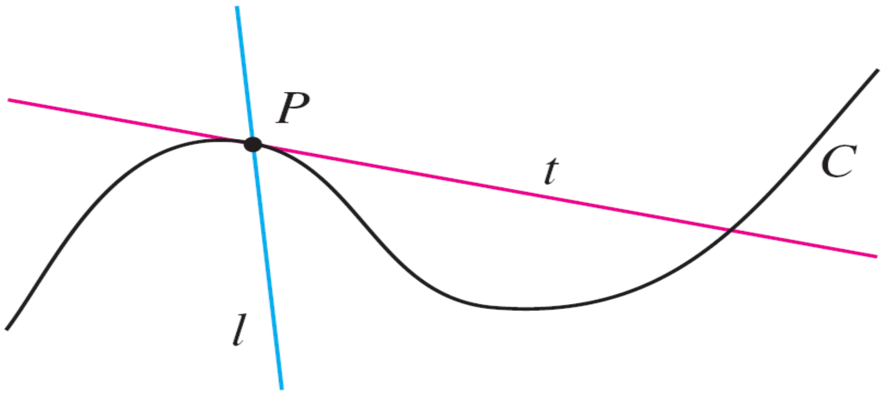


Figure 1(b)

The line *t*, on the other hand, looks like a tangent but it intersects *C* twice.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution:

We will be able to find an equation of the tangent line t as soon as we know its slope m.

The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope.

Example 1 – *Solution*

cont'd

But observe that we can compute an approximation to *m* by choosing a near by point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line *PQ*. [A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

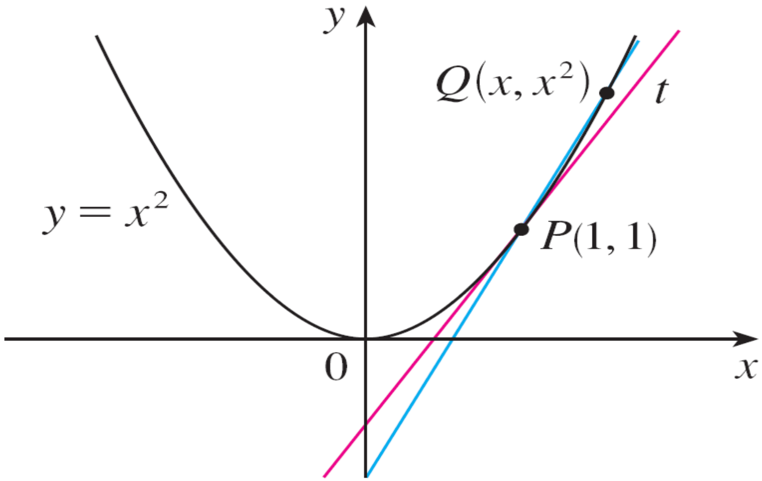


Figure 2

Example 1 – *Solution*

cont'd

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1}$$

$$= \frac{1.25}{0.5}$$

$$= 2.5$$

Example 1 – *Solution*

cont'd

The tables in the margin show the values of m_{PQ} for several values of x close to 1.

The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2.

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

Example 1 – *Solution*

cont'd

This suggests that the slope of the tangent line *t* should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

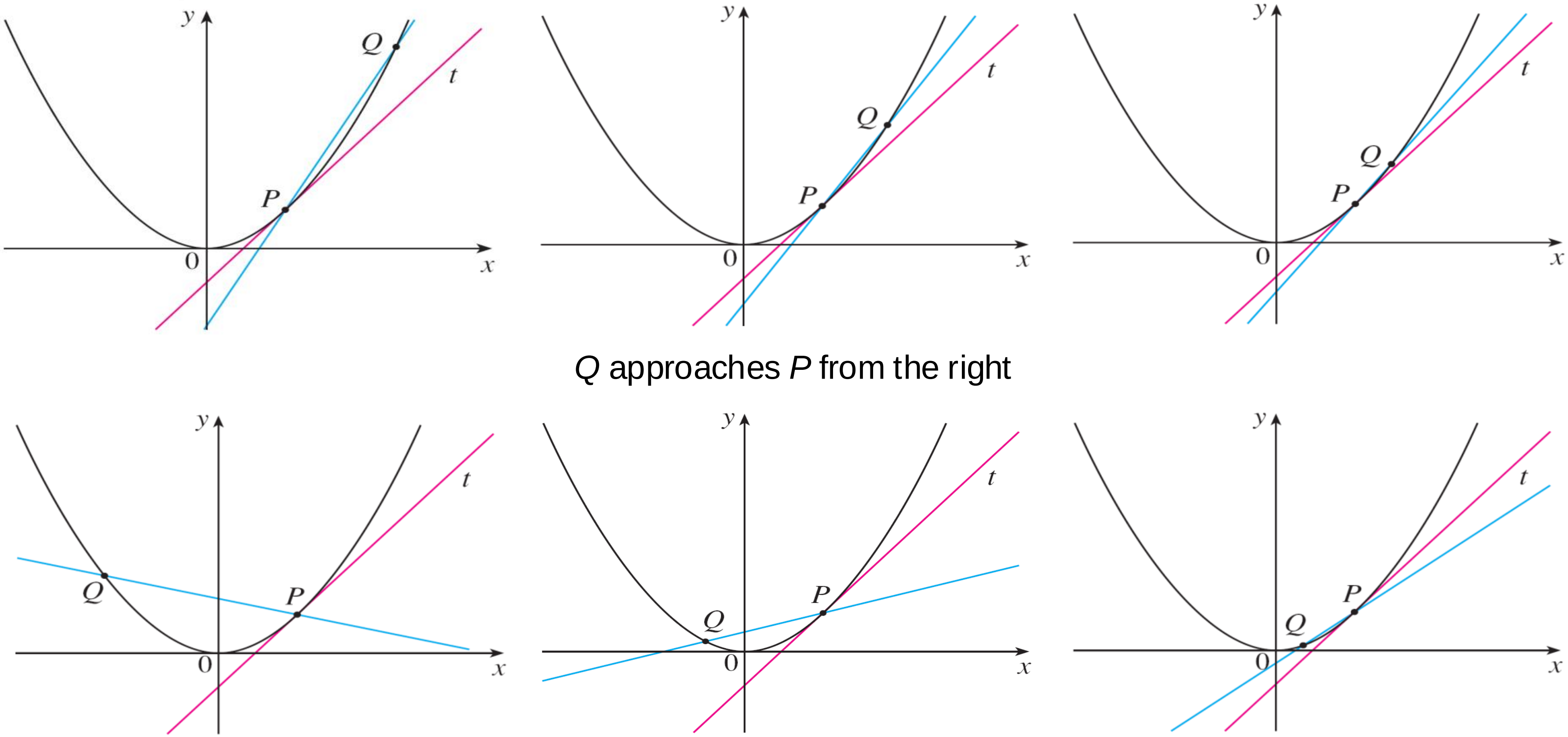
$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line to write the equation of the tangent line through (1, 1) as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Example 1 – *Solution*

Figure 3 illustrates the limiting process that occurs in this example.



Q approaches *P* from the right

Q approaches *P* from the left

Figure 3

Example 1 – *Solution*

cont'd

As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line t.

The Velocity Problem

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Solution:

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.)

If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation $s(t) = 4.9t^2$

Example 3 – *Solution*

cont'd

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved.

However, we can approximate the desired quantity by computing the average velocity over the brief time interval of length Δt from $t = 5$ to $t = 5+\Delta t$ and calculate:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

Example 3 – *Solution*

cont'd

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

Example 3 – *Solution*

cont'd

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s.

The **instantaneous velocity** when t = 5 is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at t = 5.

Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

1.5 The Limit of a Function

The Limit of a Function

In this section we will introduce an intuitive definition of the limit of a function and discuss means and pitfalls of finding a limit graphically/numerically.

To find, for example, the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function *f* defined by $f(x) = \frac{x-1}{x^2-1}$ for values of *x* near 1.

The Limit of a Function

$$f(x) = \frac{x-1}{x^2-1}$$

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

The Limit of a Function

In fact, it appears that we can make the values of $f(x)$ as close as we like to 1/2 by taking x sufficiently close to 1.

We express this by saying "the limit of the function $f(x) = \frac{x-1}{x^2-1}$ as x approaches 1 is equal to 1/2."

The notation for this is

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

The Limit of a Function

We can illustrate what is happening by considering the graph of *f* in Figure 3.

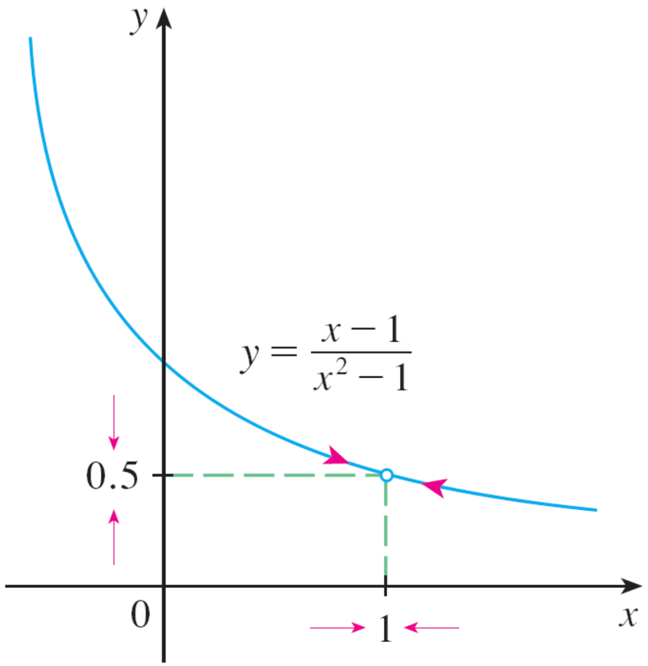


Figure 3

The Limit of a Function

In general, we use the following notation.

$\boxed{1}$ **Intuitive Definition of a Limit** Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

This says that the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

The Limit of a Function

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is $f(x) \to L$ as $x \to a$

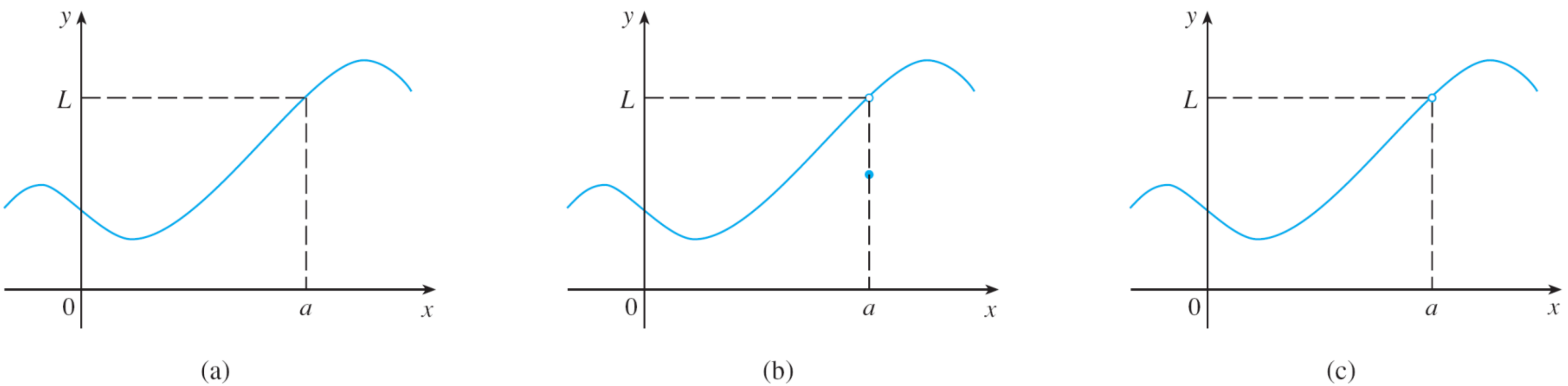
which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near a*.

The Limit of a Function

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$.

But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.



$\lim_{x \to a} f(x) = L$ in all three cases

Figure 2

The Limit of a Function

For example, if we change *f* in the first example slightly by giving it the value 2 when $x = 1$ and calling the resulting function *g*, then *g* still has the same limit as *f* as *x* approaches 1 :

$$g(x) = \begin{cases} \dfrac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

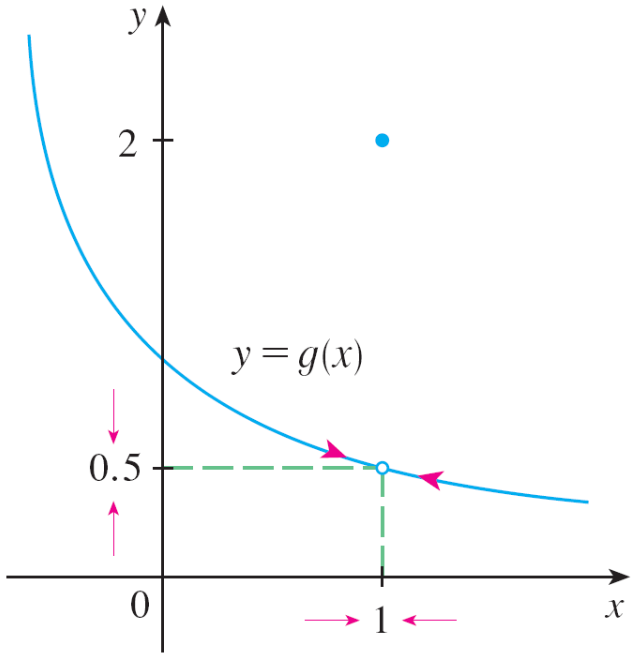


Figure 4

Example 1

Guess the value of

$$\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}.$$

Example 2

Guess the value

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

One-Sided Limits

One-Sided Limits

The function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

$H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right.

We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

One-Sided Limits

The symbol "$t \to 0^-$" indicates that we consider only values of t that are less than 0.

Likewise, "$t \to 0^+$" indicates that we consider only values of t that are greater than 0.

One-Sided Limits

$\boxed{2}$ **Intuitive Definition of One-Sided Limits** We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the left*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *less than* a.

We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the right*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *greater than* a.

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a.

One-Sided Limits

These definitions are illustrated in Figure 6.

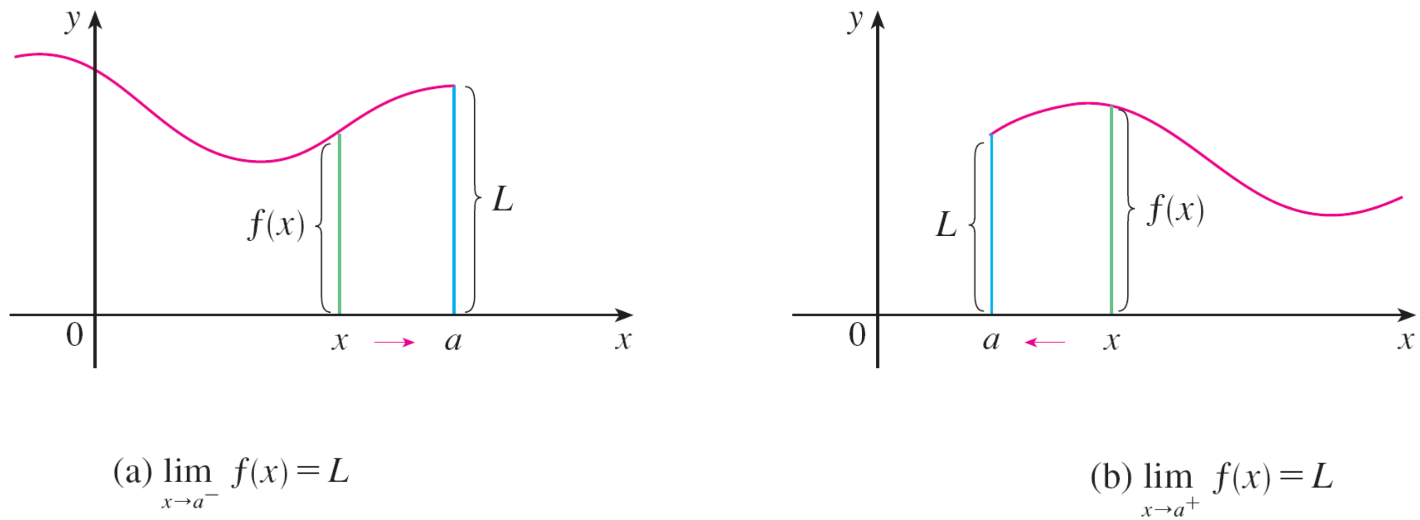


Figure 6

One-Sided Limits

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

$$\boxed{3} \qquad \lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Example 4

The graph of a function g is shown in Figure 7. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

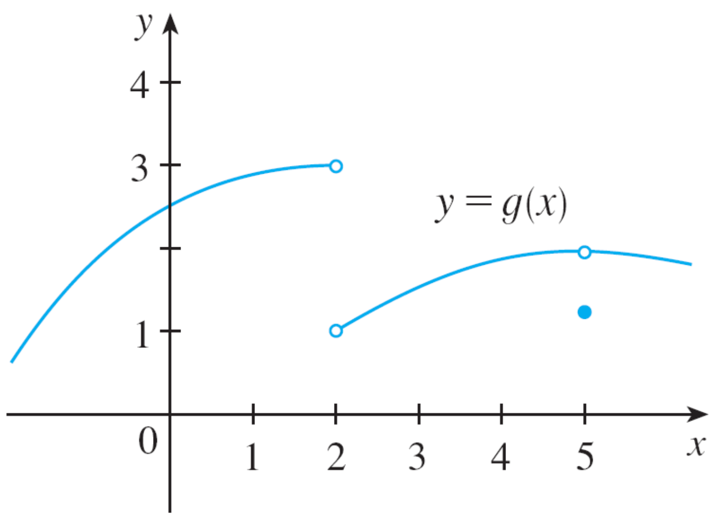


Figure 7

How can a limit fail to exist?

We have seen that a limit fails to exist at a number a if the left- and right-hand limits are not equal (as in Example 4). The next two examples illustrate additional ways that a limit can fail to exist.

Example 5. Investigate

$$\lim_{x \to 0} \sin \frac{\pi}{x}.$$

Example 6. Find

$$\lim_{x \to 0} \frac{1}{x^2}$$

if it exists.

Infinite Limits

Infinite Limits

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Another notation for

$$\lim_{x \to a} f(x) = \infty$$

is

$$f(x) \to \infty \text{ as } x \to a.$$

Infinite Limits

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

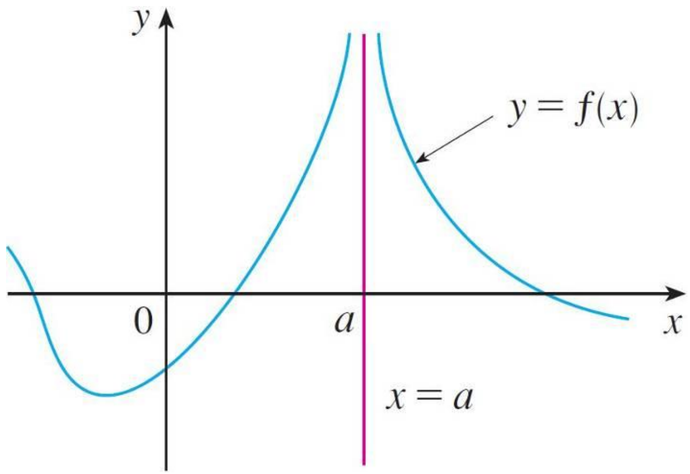
"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

Infinite Limits

This definition is illustrated graphically in Figure 12.

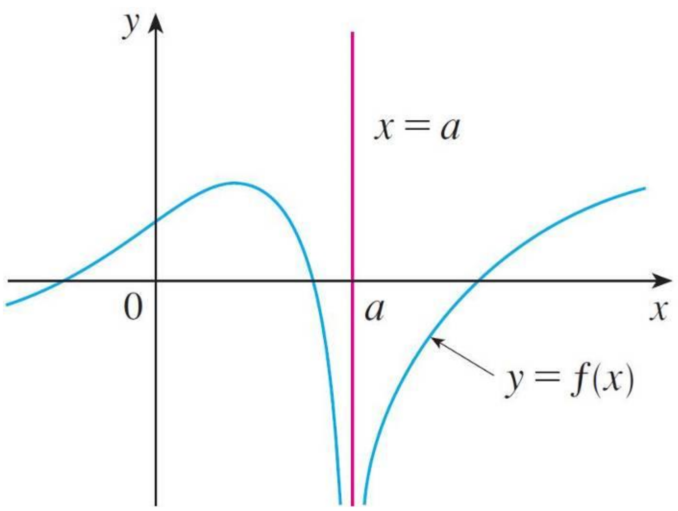


$$\lim_{x \to a} f(x) = \infty$$

Figure 12

Infinite Limits

A similar sort of limit, for functions that become large negative as *x* gets close to *a*, is defined in Definition 4 and is illustrated in Figure 11.



$$\lim_{x \to a} f(x) = -\infty$$

Figure 11

Infinite Limits

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example, we have

$$\lim_{x \to 0} \left(-\frac{1}{x^2}\right) = -\infty$$

Infinite Limits

Similar definitions can be given for the one-sided infinite limits

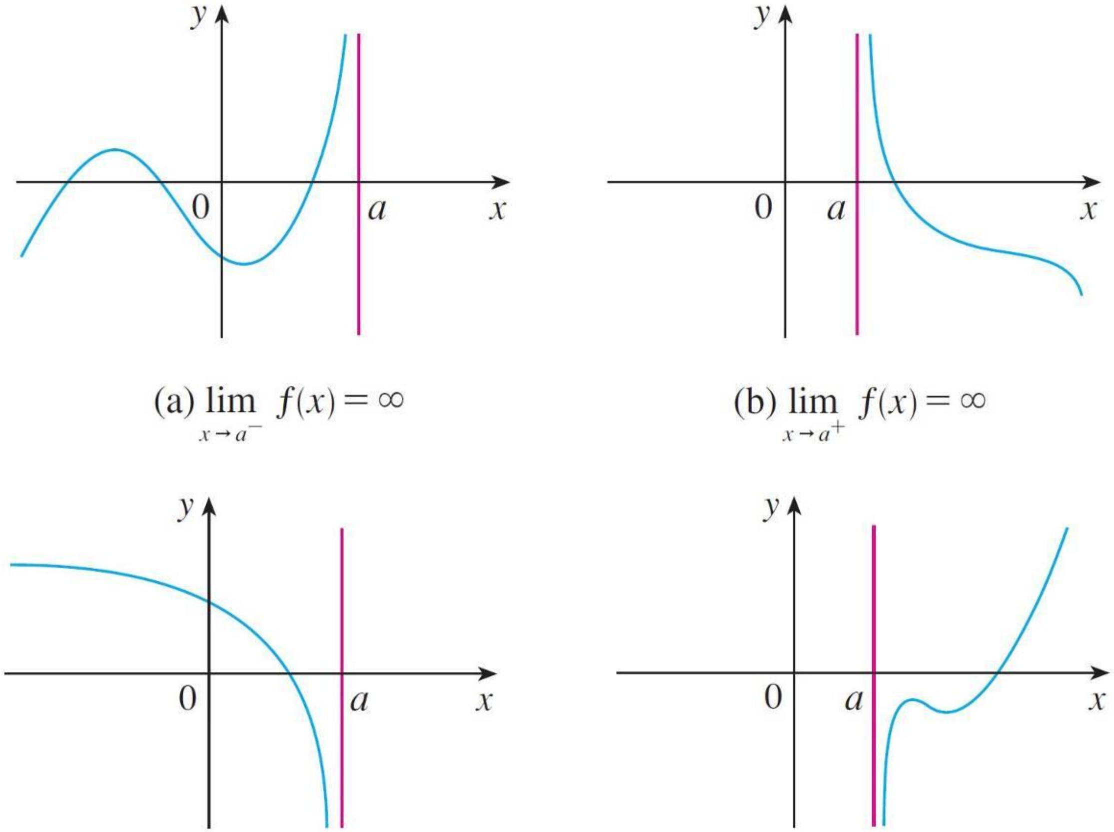
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that "$x \to a^-$" means that we consider only values of that are less than a, and similarly "$x \to a^+$" means that we consider only $x > a$.

Infinite Limits

Illustrations of these four cases are given in Figure 14.



(a) $\lim_{x \to a^-} f(x) = \infty$

(b) $\lim_{x \to a^+} f(x) = \infty$

(c) $\lim_{x \to a^-} f(x) = -\infty$

(d) $\lim_{x \to a^+} f(x) = -\infty$

Figure 14

Infinite Limits

$\boxed{6}$ **Definition** The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Examples 7 and 8

7. Does the curve $y = \frac{2x}{x-3}$ have a vertical asymptote?

8. Find the vertical asymptotes of $f(x) = \tan x$.

1.6

Calculating Limits Using the Limit Laws

Calculating Limits Using the Limit Laws

In this section we use the following properties of limits, called the *Limit Laws,* to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\displaystyle \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\displaystyle \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\displaystyle \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\displaystyle \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

Calculating Limits Using the Limit Laws

These five laws can be stated verbally as follows:

Sum Law

1. The limit of a sum is the sum of the limits.

Difference Law

2. The limit of a difference is the difference of the limits.

Constant Multiple Law

3. The limit of a constant times a function is the constant times the limit of the function.

Calculating Limits Using the Limit Laws

Product Law

4. The limit of a product is the product of the limits.

Quotient Law

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$.

Remark: These laws hold for one-sided limits, too.

Example 1

Use the Limit Laws and the graphs of *f* and *g* in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

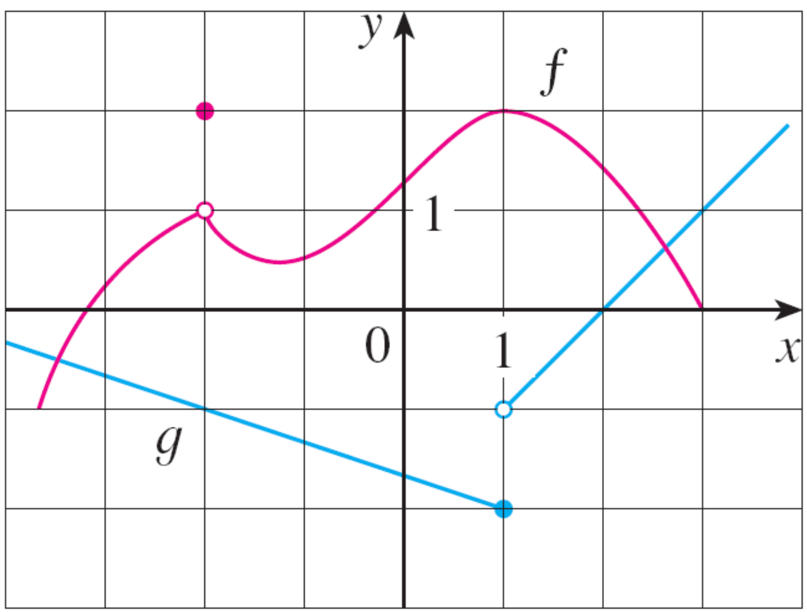


Figure 1

Calculating Limits Using the Limit Laws

If we use the Product Law repeatedly with $g(x) = f(x)$ (or, using induction), we obtain the following law.

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

In applying these six limit laws, we need to use two special limits:

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$).

Calculating Limits Using the Limit Laws

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

9. $\lim\limits_{x \to a} x^n = a^n$ where n is a positive integer

A similar limit holds for roots as follows.

10. $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

More generally, we have the following law.

Root Law

11. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$ where n is a positive integer

$\left[\text{If } n \text{ is even, we assume that } \lim\limits_{x \to a} f(x) > 0.\right]$

Example 2

Evaluate the following limit and justify each step.

(a)

$$\lim_{x \to 5} (2x^2 - 3x + 4)$$

(b)

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

Calculating Limits Using the Limit Laws

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a.*

In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

Example 3

Find

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Example 4

Find

$$\lim_{x \to 1} g(x)$$

where

$$g(x) = \begin{cases} x + 1 \text{ if } x \neq 1 \\ \pi \qquad \text{ if } x = 1. \end{cases}$$

Example 5

Evaluate

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}.$$

Calculating Limits Using the Limit Laws

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

Example 7 and 8

Example 7. Show that

$$\lim_{x \to 0} |x| = 0.$$

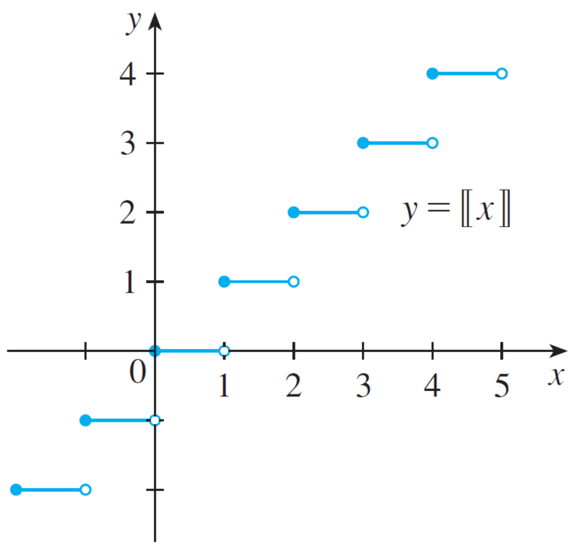
Example 8. Prove that

$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist.

Example 10

The **greatest integer function** is defined $[\![x]\!]$ = the largest integer that is less than or equal to *x*. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}\,]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.



Greatest integer function

Calculating Limits Using the Limit Laws

The next two theorems give two additional properties of limits.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Calculating Limits Using the Limit Laws

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7.

It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

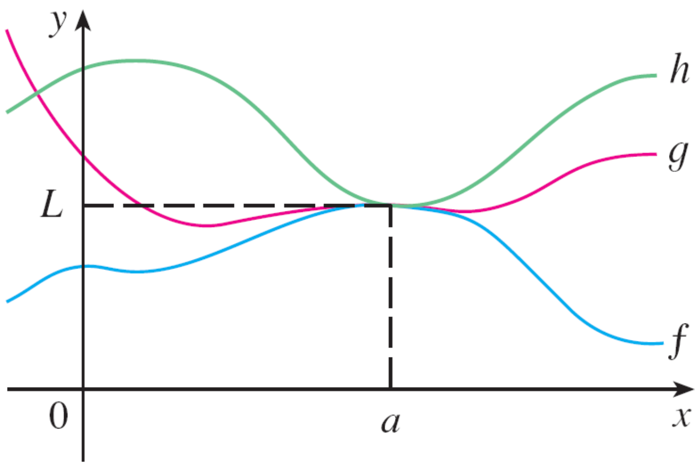


Figure 7

Example 11

Show that

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$$

1.7 The Precise Definition of a Limit

The Precise Definition of a Limit

The intuitive definition of a limit is inadequate for some purposes because such phrases as “*x* is close to 2” and “*f*(*x*) gets closer and closer to *L*” are vague.

In order to be able to prove conclusively that

$$\lim_{x \to 0} \left(x^3 + \frac{\cos 5x}{10{,}000} \right) = 0.0001 \qquad \text{or} \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

The Precise Definition of a Limit

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \to 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:
How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The Precise Definition of a Limit

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad |x - 3| < \delta \qquad \text{but } x \neq 3$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < \delta$$

The Precise Definition of a Limit

Notice that if $0 < |x - 3| < (0.1)/2 = 0.05$ then

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6|$$

$$= 2|x - 3| < 2(0.05) = 0.1$$

that is,

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < 0.05$$

Thus an answer to the problem is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5.

The Precise Definition of a Limit

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 provided that x differs from 3 by less than (0.01)/2 = 0.005:

$$|f(x) - 5| < 0.01 \qquad \text{if} \qquad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \qquad \text{if} \qquad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01 and 0.001 that we have considered are *error tolerances* that we might allow.

The Precise Definition of a Limit

For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below *any* positive number.

And, by the same reasoning, we can! If we write ε (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

$$\boxed{1} \qquad |f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

The Precise Definition of a Limit

This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3 because $\boxed{1}$ says that we can make the values of $f(x)$ within an arbitrary distance ε from 5 by taking the values of x within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

Note that $\boxed{1}$ can be rewritten as follows: if

$$3 - \delta < x < 3 + \delta \qquad (x \neq 3)$$

then

$$5 - \varepsilon < f(x) < 5 + \varepsilon$$

and this is illustrated in Figure 1.

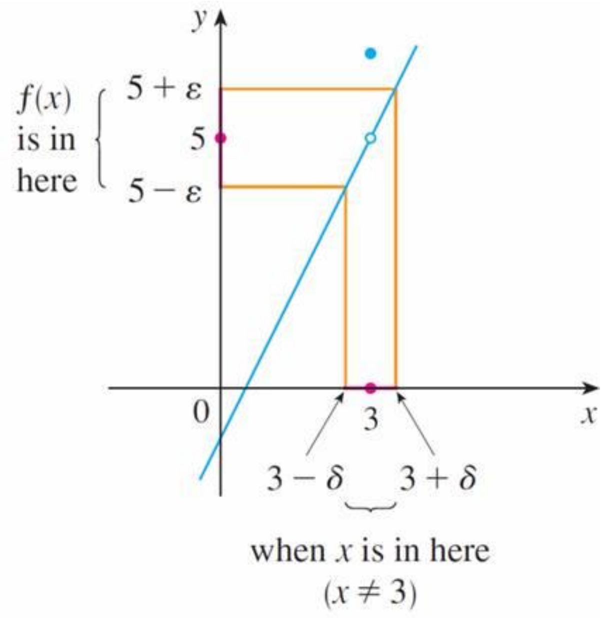


Figure 1

The Precise Definition of a Limit

By taking the values of x ($\neq 3$) to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the values of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$.

Using $\boxed{1}$ as a model, we give a precise definition of a limit.

$\boxed{2}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

The Precise Definition of a Limit

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from $f(x)$ to L, and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

$$\lim_{x \to a} f(x) = L$$

means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0).

The Precise Definition of a Limit

Alternatively,

$$\lim_{x \to a} f(x) = L$$

the values of $f(x)$ can be made as close as we please to L by taking x close enough to a (but not equal to a).

The Precise Definition of a Limit

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as $a - \delta < x < a + \delta$.

Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$.

The Precise Definition of a Limit

Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

$$\lim_{x \to a} f(x) = L$$

means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

The Precise Definition of a Limit

We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where *f* maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.

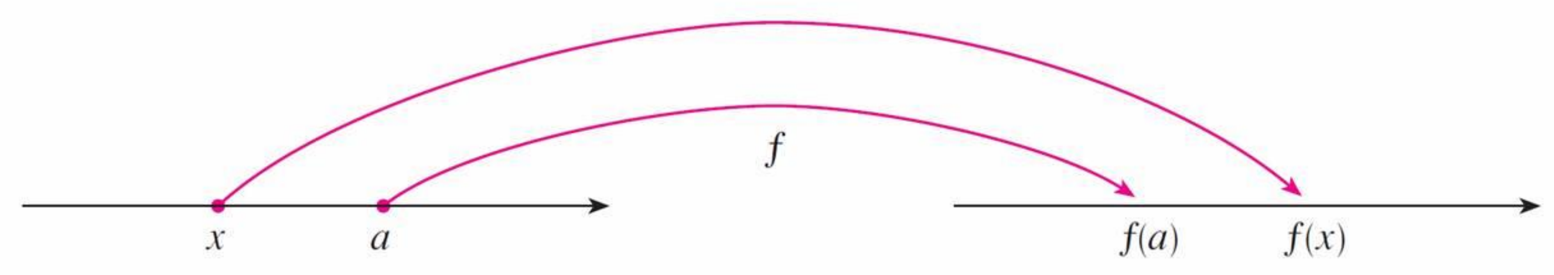


Figure 2

The Precise Definition of a Limit

The definition of limit says that if any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find an interval $(a - \delta, a + \delta)$ around a such that f maps all the points in $(a - \delta, a + \delta)$ (except possibly a) into the interval $(L - \varepsilon, L + \varepsilon)$. (See Figure 3.)

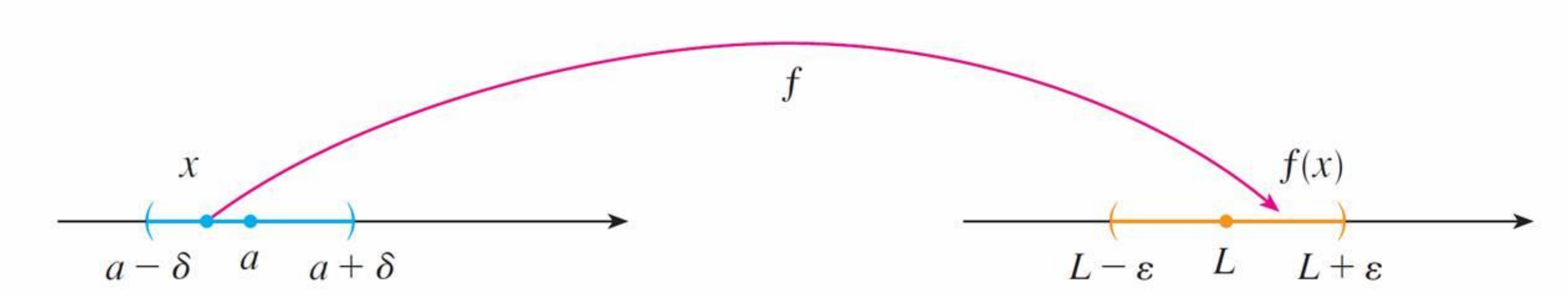


Figure 3

The Precise Definition of a Limit

Another geometric interpretation of limits can be given in terms of the graph of a function. If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f. (See Figure 4.)

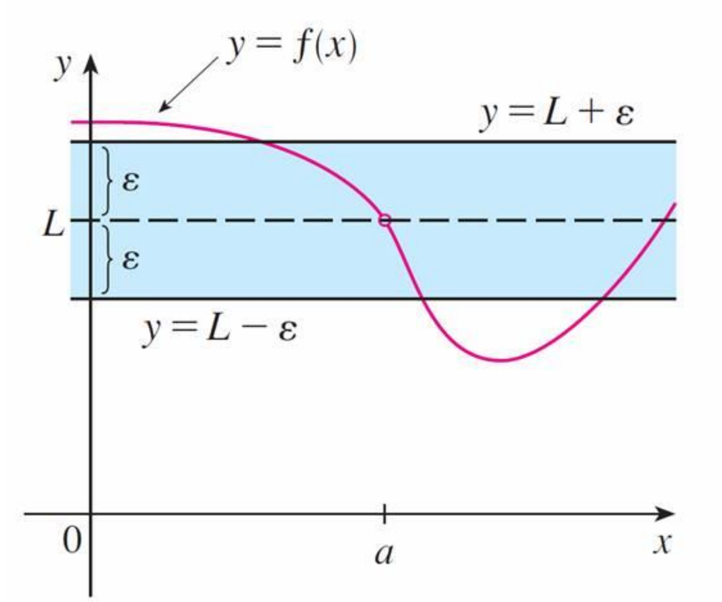


Figure 4

The Precise Definition of a Limit

If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$ (See Figure 5.) You can see that if such a δ has been found, then any smaller δ will also work.

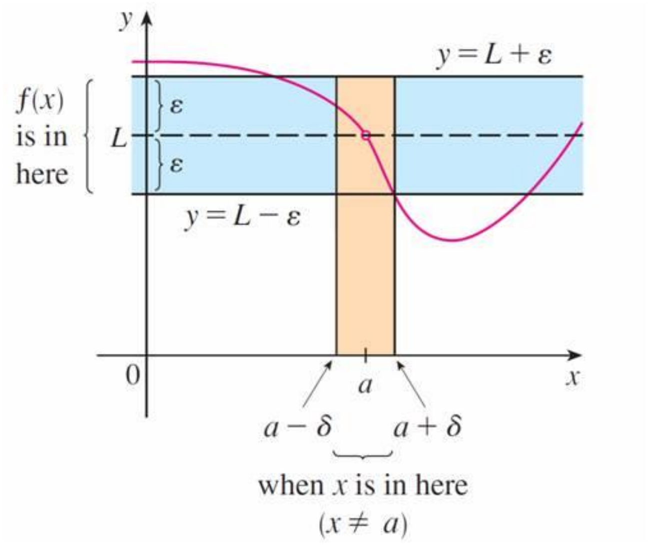


Figure 5

The Precise Definition of a Limit

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number ε, no matter how small it is chosen. Figure 6 shows that if a smaller ε is chosen, then a smaller δ may be required.

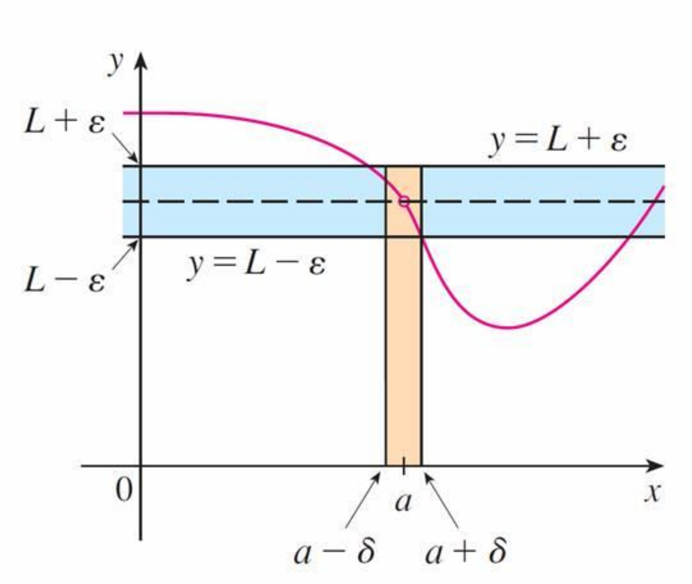


Figure 6

Examples 2 and 3

Example 2. Prove that $\lim_{x \to 3} (4x - 5) = 7$.

Example 3. Prove that

$$\lim_{x \to 3} x^2 = 9.$$

The Precise Definition of a Limit

The intuitive definitions of one-sided limits can be precisely reformulated as follows.

3 Definition of Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a - \delta < x < a \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

The Precise Definition of a Limit

4 Definition of Right-Hand Limit

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 3

Use Definition 4 to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Uniqeness of the limit

Prove that the limit, if it exists, is unique.

Limit laws. Proof of the sum law.

Prove that if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist, then

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Infinite Limits

Infinite Limits

Infinite limits can also be defined in a precise way.

$\boxed{6}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M$$

Infinite Limits

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given number M) by taking x close enough to a (within a distance δ, where δ depends on M, but with $x \neq a$). A geometric illustration is shown in Figure 10.

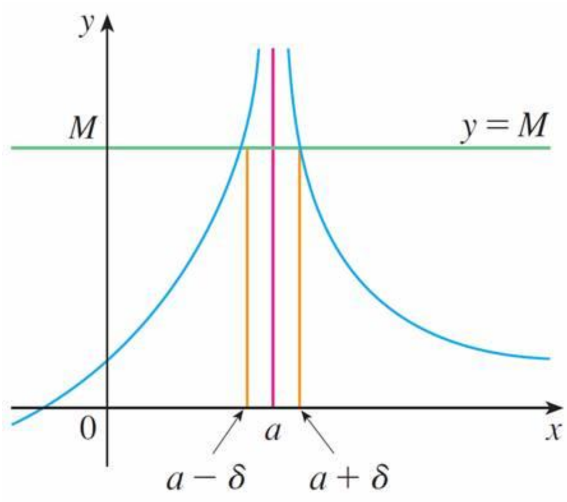


Figure 10

Infinite Limits

Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that if we restrict to x lie in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$.

You can see that if a larger M is chosen, then a smaller δ may be required.

Example 5

Use Definition 6 to prove that $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

Infinite Limits

$\boxed{7}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$

1.8 Continuity

Continuity

The limit of a function as *x* approaches *a* can often be found simply by calculating the value of the function at *a*. Functions with this property are called *continuous at a*.

We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

Continuity

Notice that Definition 1 implicitly requires three things if *f* is continuous at *a*:

1. *f*(*a*) is defined (that is, *a* is in the domain of *f*)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

The definition says that *f* is continuous at *a* if *f*(*x*) approaches *f*(*a*) as *x* approaches *a*. Thus a continuous function *f* has the property that a small change in *x* produces only a small change in *f*(*x*).

Continuity

In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents.

Continuity

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

Example 1

Figure 2 shows the graph of a function *f*. At which numbers is *f* discontinuous? Why?

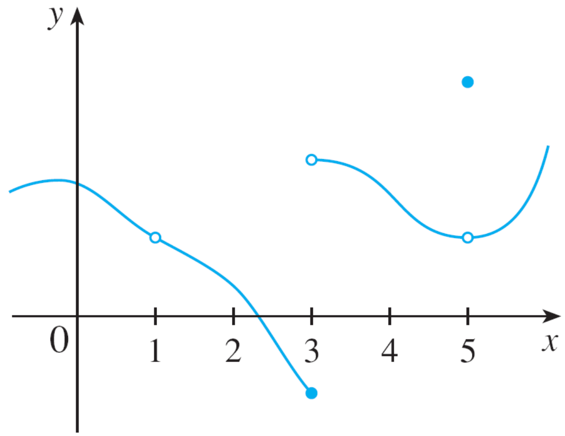


Figure 2

Example 2

Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

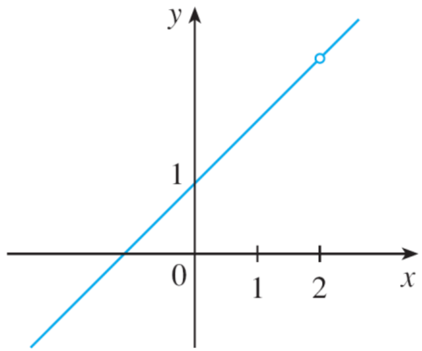
(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

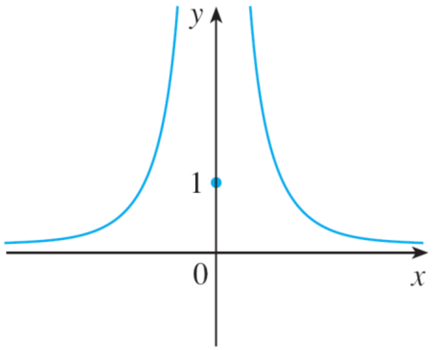
(d) $f(x) = [\![x]\!]$

Continuity

Figure 3 shows the graphs of the functions in Example 2.



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

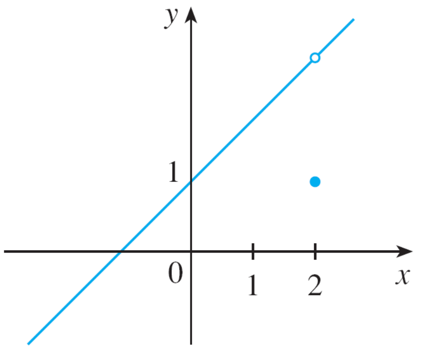


(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

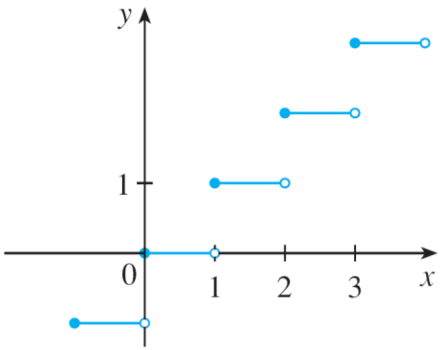
Graphs of the functions in Example 2

Figure 3

Continuity



(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



(d) $f(x) = [\![x]\!]$

Graphs of the functions in Example 2

Figure 3

Continuity

In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph.

The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.]

The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.

Continuity

2 Definition A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

Thus a function *f* is continuous at *a* if and only if it is continuous both from the left and right at *a*.

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Examples 3 and 4

Example 3. Discuss the function $f(x) = [\![x]\!]$ for one sided continuity.

Example 4. Show, using limit laws, that the function

$$f(x) = 1 - \sqrt{1 - x^2}$$

is continuous on [-1, 1].

Continuity

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

$\boxed{4}$ **Theorem** If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

Continuity

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) f/g.

The following theorem was stated as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Continuity

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r.

Likewise, if a ball is thrown vertically into the air with a velocity of 15 m/s, then the height of the ball in meters t seconds later is given by the formula $h = 15t - 4.9t^2$.

Again, this is a polynomial function, so the height is a continuous function of the elapsed time.

Example 5

Find

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}.$$

Continuity

It turns out that most of the familiar functions are continuous at every number in their domains.

From the appearance of the graphs of the sine and cosine functions, we would certainly guess that they are continuous.

We know from the definitions of $\sin\theta$ and $\cos\theta$ that the coordinates of the point P in Figure 5 are $(\cos\theta, \sin\theta)$. As $\theta \to 0$, we see that P approaches the point $(1, 0)$ and so $\cos\theta \to 1$ and $\sin\theta \to 0$.

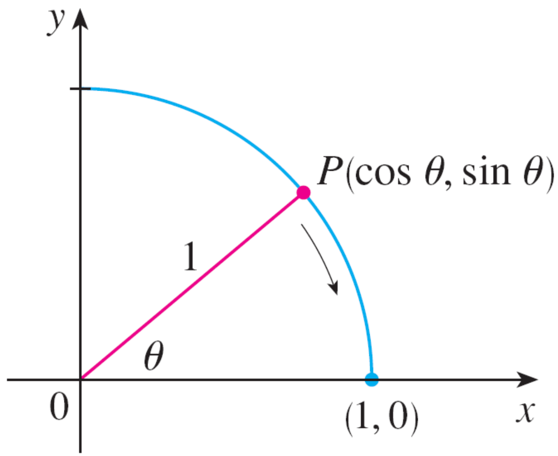


Figure 5

Continuity

Thus

$$\boxed{6} \qquad \lim_{\theta \to 0} \cos \theta = 1 \qquad \lim_{\theta \to 0} \sin \theta = 0$$

Since cos 0 = 1 and sin 0 = 0, the equations in $\boxed{6}$ assert that the cosine and sine functions are continuous at 0.

The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere.

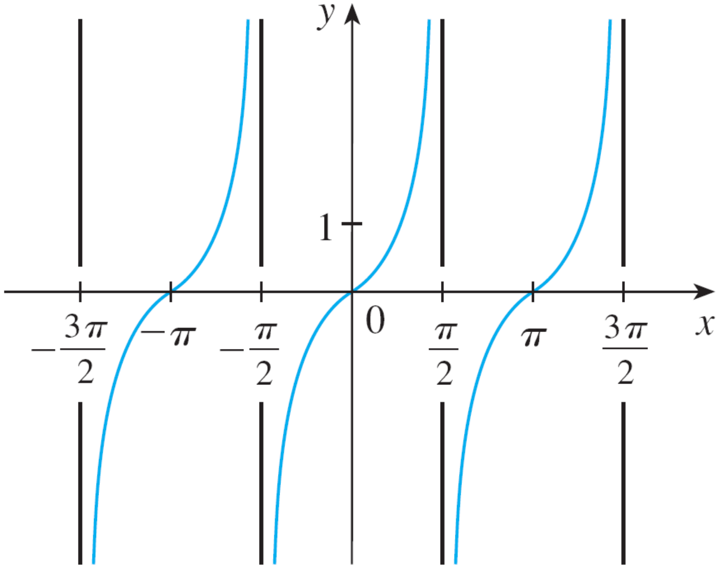
It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where cos x = 0.

Continuity

This happens when x is an odd integer multiple of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, and so on (see Figure 6).



$y = \tan x$

Figure 6

Continuity

$\boxed{7}$ **Theorem** The following types of functions are continuous at every number in their domains:

polynomials	rational functions
root functions	trigonometric functions

Example 6 and 7

Example 6. On what intervals is each function continuous?

(a) $f(x) = x^{100} - 2x^{37} + 75$

(b) $g(x) = \dfrac{x^2 + 2x + 17}{x^2 - 1}$

(c) $h(x) = \sqrt{x} + \dfrac{x + 1}{x - 1} - \dfrac{x + 1}{x^2 + 1}$

Example 7. Evaluate

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$$

Continuity

Another way of combining continuous functions *f* and *g* to get a new continuous function is to form the composite function $f \circ g$. This is a consequence of the following theorem.

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Intuitively, Theorem 8 is reasonable because if *x* is close to *a*, then *g*(*x*) is close to *b*, and since *f* is continuous at *b*, if *g*(*x*) is close to *b*, then *f*(*g*(*x*)) is close to *f*(*b*).

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example 8

Where are the following functions continuous?

(a) $f(x) = \sin x^2$

(b) $\frac{1}{\sqrt{x^2+7}-4}$

Intermediate Value Theorem

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Continuity

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values *f*(*a*) and *f*(*b*). It is illustrated by Figure 7.

Note that the value *N* can be taken on once [as in part (a)] or more than once [as in part (b)].

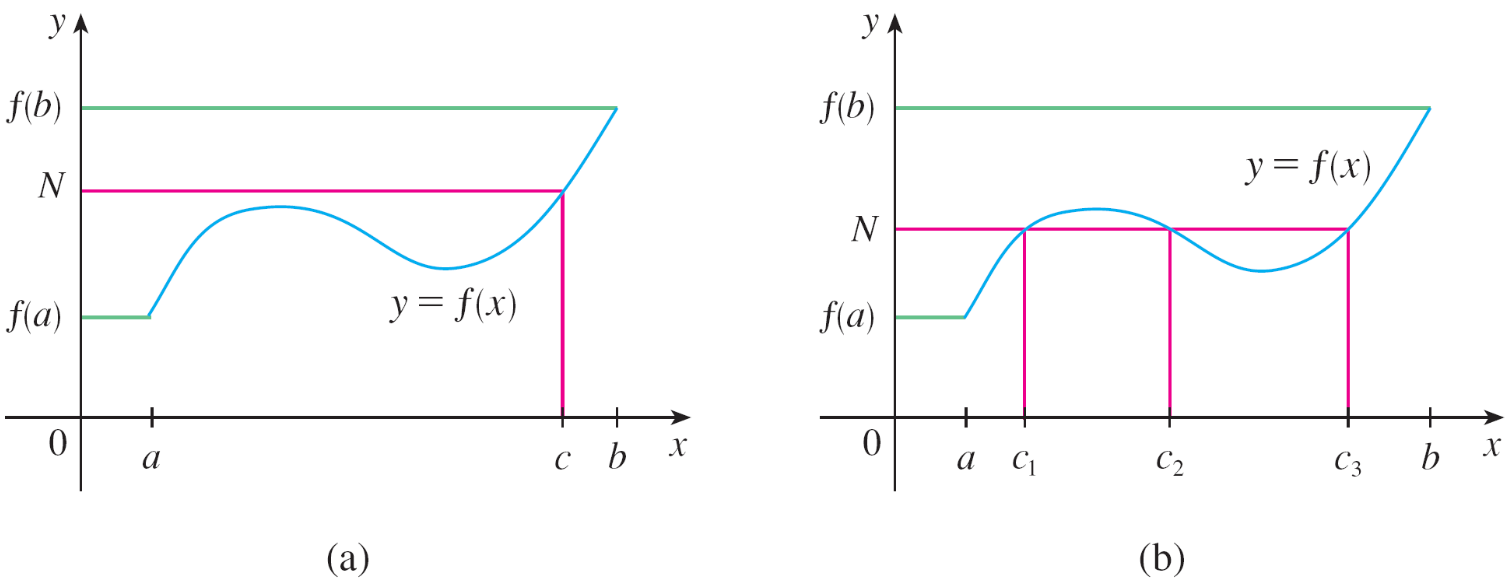


Figure 7

Continuity

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true.

In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 8, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

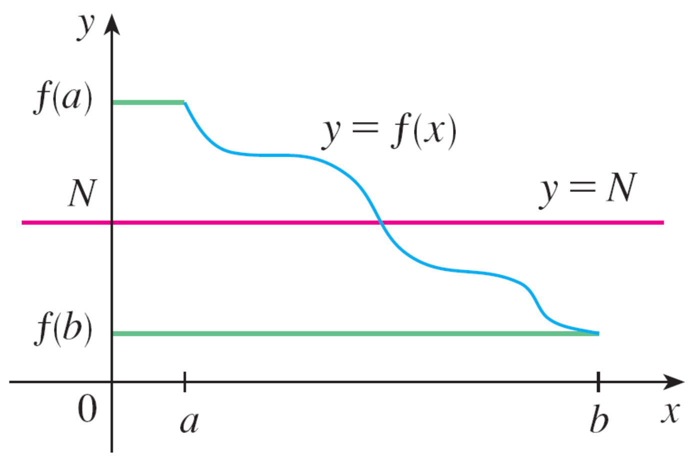


Figure 8

Example 8

It is important that the function *f* in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions.

One use of the important Intermediate Value Theorem is in locating solutions of equations.

Example 9. Show that there is a solution of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

Proof of the Intermediate Value Theorem

The following theorem is the key for proving the Intermediate Value Theorem.

Theorem (Cantor). Let

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$$

Then the set

$$\bigcap_{n=1}^{\infty} [a_n, b_n]$$

is nonempty.

2.1 Derivatives and Rates of Change

Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit.

This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

Tangents

Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a.

Tangents

If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)

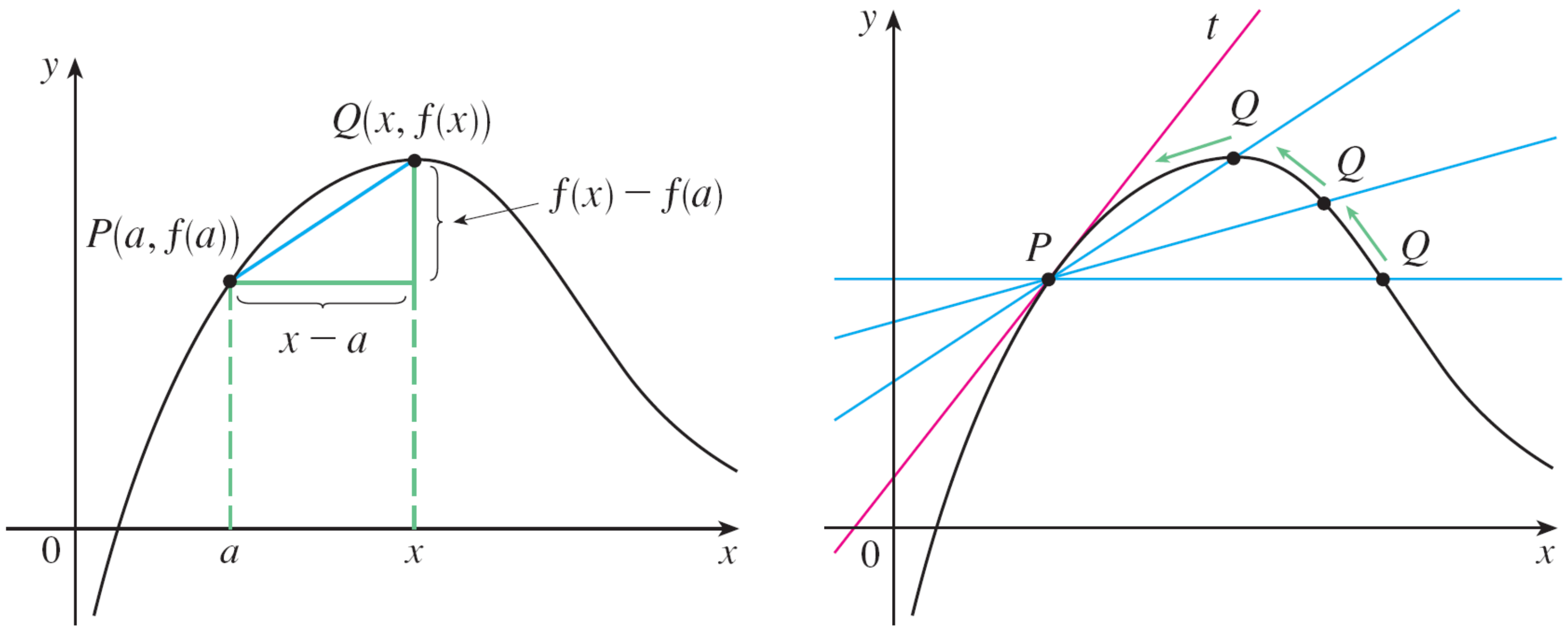


Figure 1

Tangents

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Tangents

If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.)

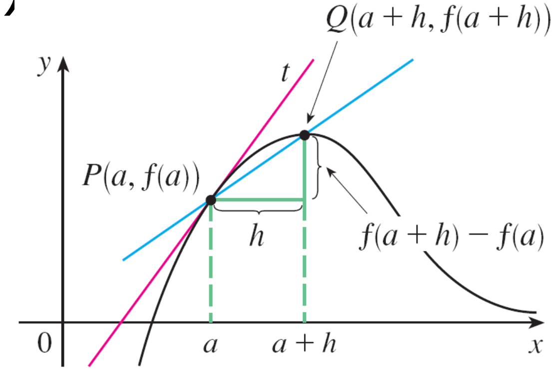


Figure 3

Tangents

Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

$$\boxed{2} \qquad \boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Example 2

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3, 1)

Velocities

Velocities

In general, suppose an object moves along a straight line (a.k.a rectilinear motion) according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t.

The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) – f(a)$. (See Figure 5.)

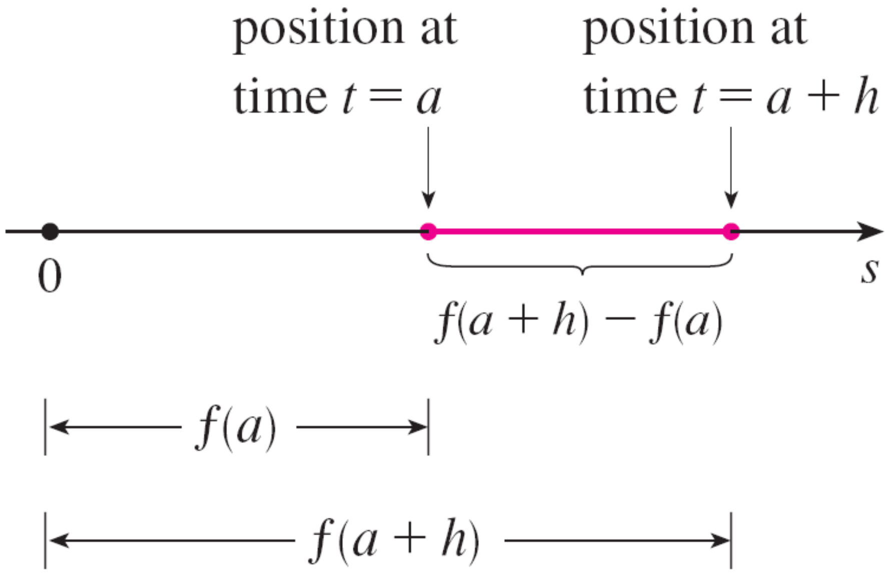


Figure 5

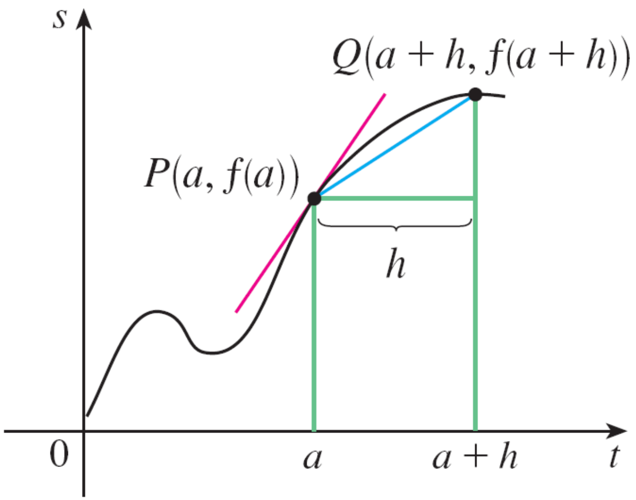
Velocities

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a + h) - f(a)}{h}$$

Velocities

which is the same as the slope of the secant line *PQ* in Figure 6.



$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$
$$= \text{average velocity}$$

Figure 6

Velocities

Now suppose we compute the average velocities over shorter and shorter time intervals [*a*, *a* + *h*].

In other words, we let *h* approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) *v*(*a*) at time *t* = *a* to be the limit of these average velocities:

3

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

This means that the velocity at time *t* = *a* is equal to the slope of the tangent line at *P*.

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Derivatives

Derivatives

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Equation 3).

In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics.

Since this type of limit occurs so widely, it is given a special name and notation.

Derivatives

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore, an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

5

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 4

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

Derivatives

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1 or 2.

Since, by Definition 4, this is the same as the derivative $f'(a)$, we can now say the following.

> The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Derivatives

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

Rates of Change

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus, y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

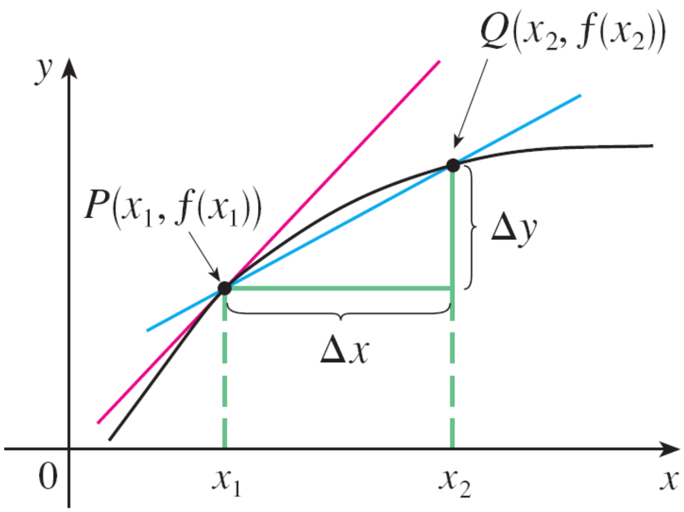
$$\Delta y = f(x_2) - f(x_1)$$

Rates of Change

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line *PQ* in Figure 8.



average rate of change $= m_{PQ}$

instantaneous rate of change $=$ slope of tangent at P

Figure 8

Rates of Change

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0.

The limit of these average rates of change is called the (**instantaneous**) **rate of change of *y* with respect to *x*** at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

$$\boxed{6} \qquad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

Rates of Change

We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$.

Rates of Change

This means that when the derivative is large (and therefore the curve is steep, as at the point *P* in Figure 9), the *y*-values change rapidly.

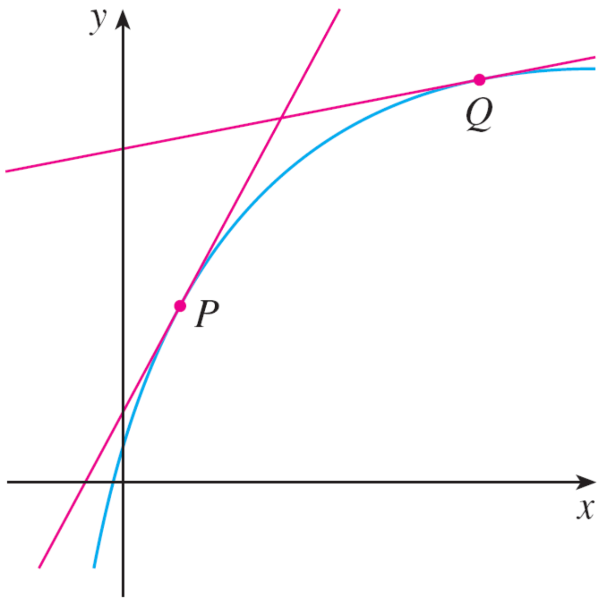


Figure 9

The *y*-values are changing rapidly at *P* and slowly at *Q*.

Rates of Change

When the derivative is small, the curve is relatively flat (as at point Q) and the y-values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t.

In other words, $f'(a)$ *is the velocity of the particle at time* $t = a$.

The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

Example 6

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x meters of this fabric is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

The Derivative as a Function

We have considered the derivative of a function *f* at a fixed number *a*:

1 $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here we change our point of view and let the number *a* vary. If we replace *a* in Equation 1 by a variable *x*, we obtain

2 $$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

The Derivative as a Function

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of *f*** and defined by Equation 2.

We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been “derived” from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

Example 1 – Derivative of a Function given by a Graph

The graph of a function *f* is given in Figure 1. Use it to sketch the graph of the derivative *f*′.

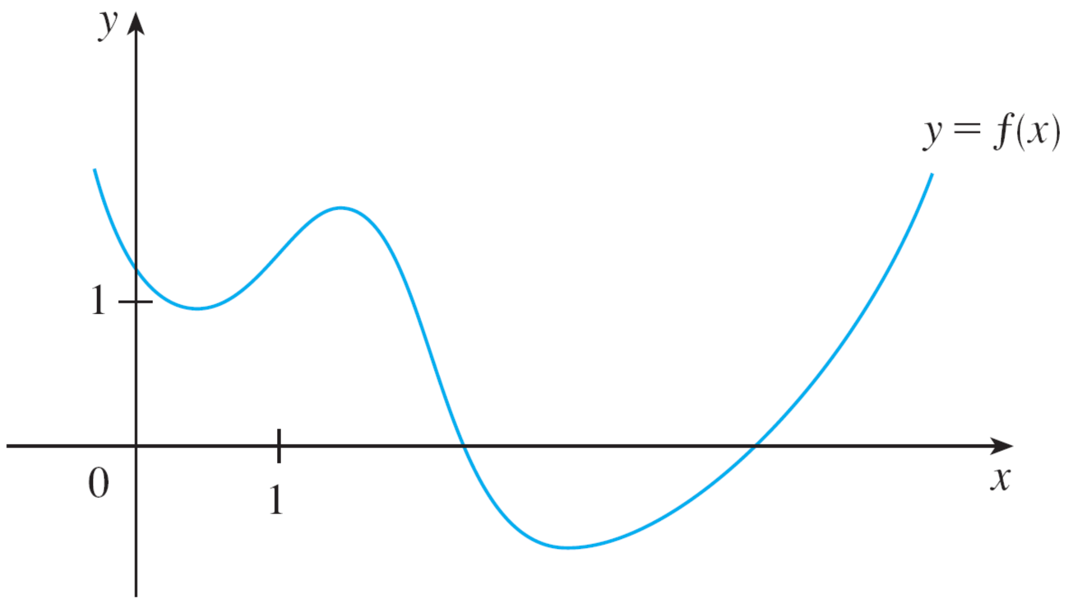


Figure 1

Example 3

If $f(x) = \sqrt{x}$, find the derivative of *f*. State the domain of *f'*.

Homework: read Example 4 in the book

Find *f'* if $f(x) = \frac{1-x}{2+x}$.

Other Notations

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

Other Notations

The symbol *dy*/*dx*, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation.

We can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Other Notations

If we want to indicate the value of a derivative *dy*/*dx* in Leibniz notation at a specific number *a*, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$.

$\boxed{3}$ **Definition** A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 5

Where is the function $f(x) = |x|$ differentiable?

Continuity vs differentiability

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a, then f is continuous at a.

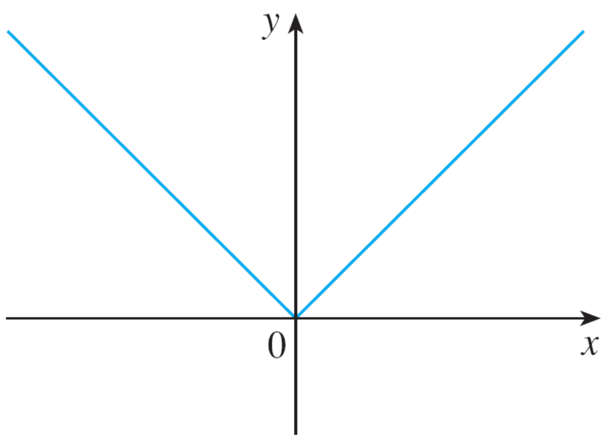
The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable.

How Can a Function Fail to Be Differentiable?

How Can a Function Fail to Be Differentiable?

We saw that the function $y = |x|$ is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$.

In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]



$y = f(x) = |x|$

Figure 5(a)

How Can a Function Fail to Be Differentiable?

Theorem 4 gives another way for a function not to have a derivative. It says that if *f* is not continuous at *a*, then *f* is not differentiable at *a*. So, at any discontinuity (for instance, a jump discontinuity) *f* fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, *f* is continuous at *a* and

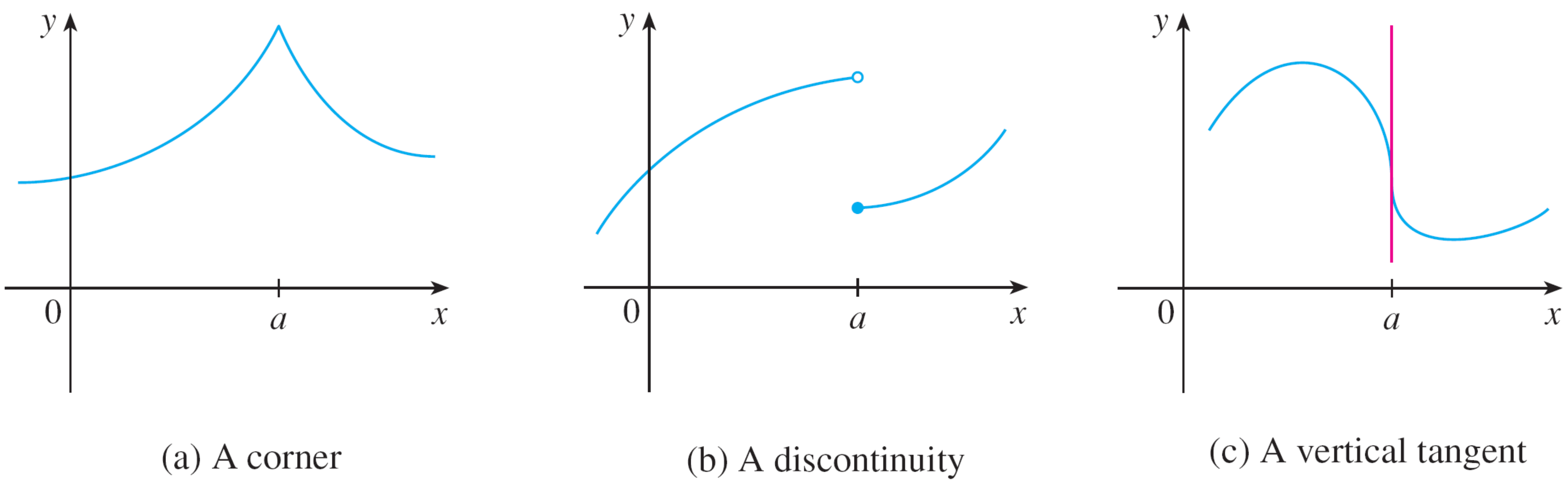
$$\lim_{h \to 0+} \frac{f(a+h) - f(a)}{h} = \infty \text{ or } -\infty$$

and

$$\lim_{h \to 0-} \frac{f(a+h) - f(a)}{h} = \infty \text{ or } -\infty$$

How Can a Function Fail to Be Differentiable?

Figure 7 illustrates examples of the three possibilities that we have discussed.



Three ways for *f* not to be differentiable at *a*

Figure 7

Vertical tangent

1. Show that the function $f(x) = \sqrt[3]{x}$ is not differentiable at *a = 0* (vertical tangent).

2. Show that the function

$$f(x) = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$$

is not differentiable at *a = 0* (vertical tangent).

Higher Derivatives

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f.

Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 6

If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Higher Derivatives

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration,* which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Higher Derivatives

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Higher Derivatives

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$.

If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Higher Derivatives

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$.

In general, the *n*th derivative of *f* is denoted by $f^{(n)}$ and is obtained from *f* by differentiating *n* times.

If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Higher Derivatives

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line.

Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Higher Derivatives

Thus, the jerk j is the rate of change of acceleration.

It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

Example 7

If $f(x) = x^3 - x$, find $f''(x)$ and $f^{(4)}(x)$.

2.3 Differentiation Formulas

Differentiation Formulas

Let's start with the two simplest functions.

Example. Let $f(x) = c$, for all x, where c is a constant. Then $f'(x) = 0$ for all x.

Example. Let $f(x)=x$. Then $f'(x) = 1$ for all x.

Differentiation Formulas

Theorem. Suppose that *f* and *g* are differentiable at *x* and let *c* be a constant. Then *f* + *g*, *f* - *g*, *cf*, *f*·*g*, *f*/*g* (if g(x) ≠ 0) are all differentiable at x. Furthermore

(a) $(f + g)'(x) = f'(x) + g'(x)$

(b) $(cf)'(x) = cf'(x)$

(c) $(f - g)'(x) = f'(x) - g'(x)$

(d) $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

(e)

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Sum of multiple functions

Remark: The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f + g + h)'(x) = [(f + g) + h]'(x) = (f + g)'(x) + h'(x)$$

$$= f'(x) + g'(x) + h'(x).$$

Power Rule

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 1

(a) If $f(x) = x^6$, find $f'(x)$.

(b) If $y = x^{1000}$, find y'.

(c) If $y = t^4$, find $\dfrac{dy}{dt}$.

(d) Find $\dfrac{d}{dr}(r^3)$.

Example 2

(a) Find $\dfrac{d}{dx}(3x^4)$

(b) Find $\dfrac{d}{dx}(-x)$

Example 3

Find $\dfrac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$.

Example 4

Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Example 5

The equation of motion of a particle is

$$s = 2t^3 - 5t^2 + 3t - 4,$$

where *s* is measured in centimeters and *t* in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?

Example 6

Find $F'(x)$ if $F(x) = (6x^3)(7x^4)$.

Example 7

If $h(x) = xg(x)$ and it is known that $g(3) = 5$ and $g'(3) = 2$, find $h'(3)$.

Example 8

Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Find *y'*.

General Power Functions

The Quotient Rule can be used to extend the Power Rule to the case where the exponent is a negative integer.

If n is a positive integer, then

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

Example 9

(a) If $y = \frac{1}{x}$, then find $\frac{dy}{dx}$.

(b) Find $\frac{d}{dt}\left(\frac{6}{t^3}\right)$

General Power Functions

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 10

(a) If $f(x) = x^{\pi}$, find $f'(x)$.

(b) Find $\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^2}}\right)$.

Example 11

Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

General Power Functions

The differentiation rules enable us to find tangent lines without having to resort to the definition of a derivative.

They also enable us to find *normal lines*.

The **normal line** to a curve C at point P is the line through P that is perpendicular to the tangent line at P.

Example 12

Find equations of the tangent line and normal line to the curve

$$y = \frac{\sqrt{x}}{1+x^2}$$

at the point $(1, \frac{1}{2})$.

Example 12

The curve and its tangent and normal lines are graphed in Figure 5.

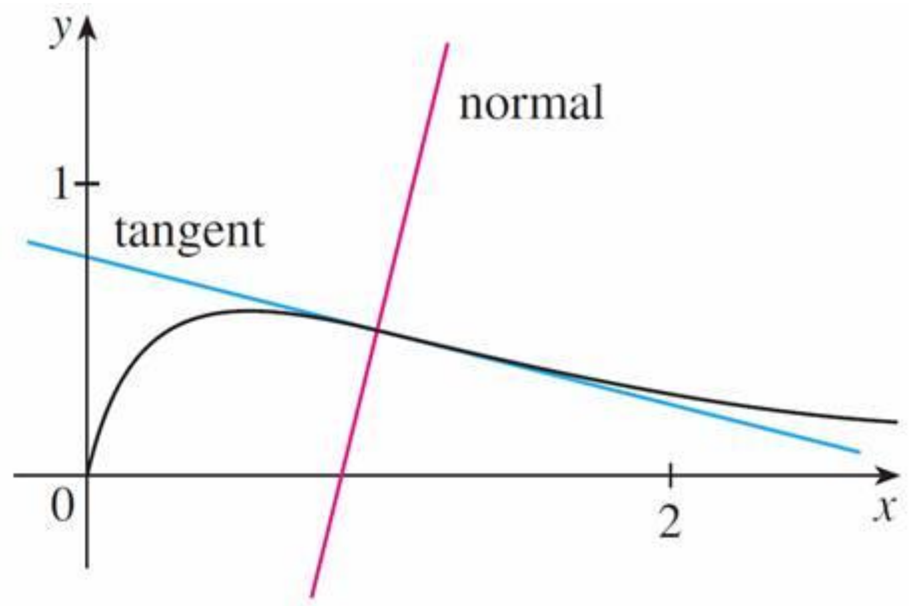


Figure 5

2.4 Derivatives of Trigonometric Functions

Derivatives of Trigonometric Functions

In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x. A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot.

All of the trigonometric functions are continuous at every number in their domains.

Derivatives of Trigonometric Functions

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f', then it looks as if the graph of f' may be the same as the cosine curve. (See Figure 1).

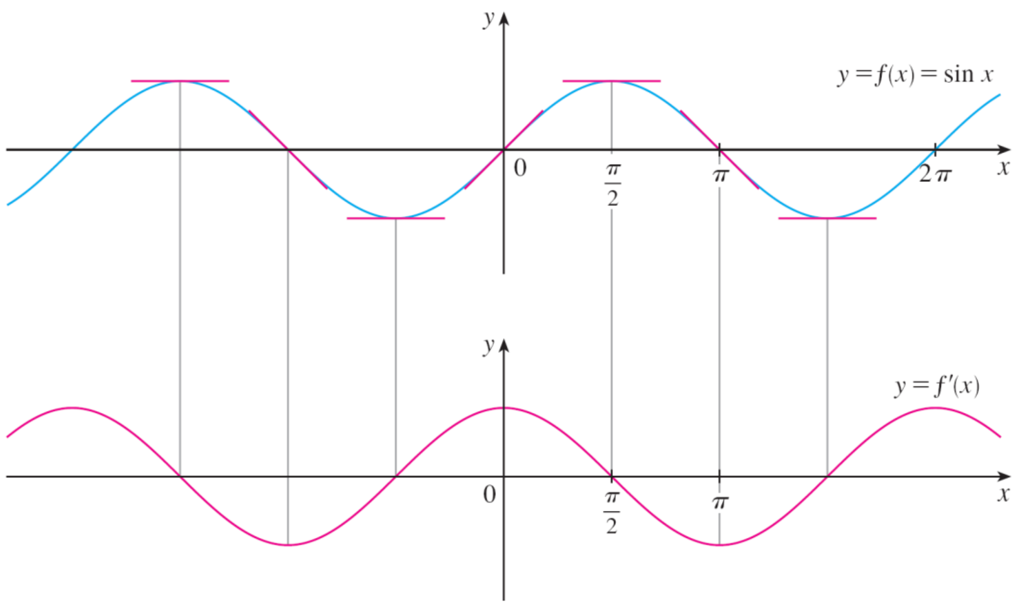


Figure 1

Derivatives of Trigonometric Functions

In order to calculate the derivative of $f(x) = \sin x$ we first show that

2

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

and as consequence we will show that:

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0.$$

Derivatives of Trigonometric Functions

Now we can prove the formula for the derivative of the sine function:

4

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Example 1

Differentiate $y = x^2 \sin x$.

Derivatives of Trigonometric Functions

Using the same methods as in the proof of Formula 4, one can prove that

5

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 4 and 5:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

Derivatives of Trigonometric Functions

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 4 and 5 to get:

6

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

The derivatives of the remaining trigonometric functions, csc, sec, and cot, can also be found easily using the Quotient Rule.

Derivatives of Trigonometric Functions

We collect all the differentiation formulas for trigonometric functions in the following table. Remember that they are valid only when *x* is measured in radians.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Trigonometric Functions

Trigonometric functions are often used in modeling real-world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. In the following example we discuss an instance of simple harmonic motion.

Example 2

Differentiate

$$f(x) = \frac{\sec x}{1+\tan x}.$$

For what values of *x* does the graph of *f* have a horizontal tangent?

Example 3

An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. (See Figure 5 and note that the downward direction is positive.)

Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

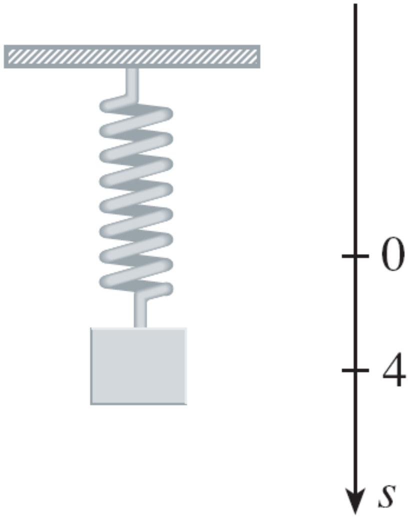
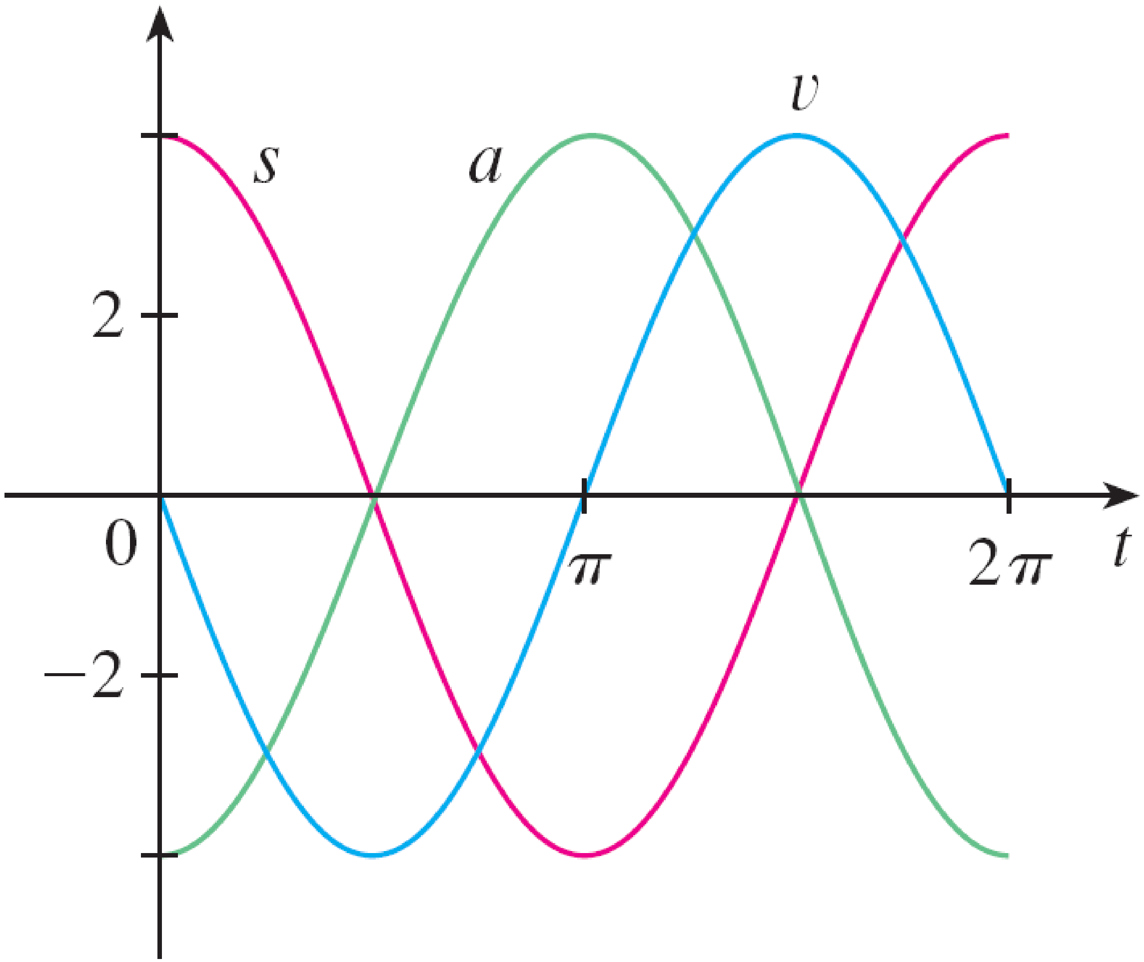


Figure 5

Example 3



Example 4

Find the 27th derivative of cos *x*.

Examples 5 - 7

Example 5. Find

$$\lim_{x \to 0} \frac{\sin 7x}{5x}.$$

Example 6. Find

$$\lim_{x \to 0} x \cot x.$$

Example 7. Find

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta}.$$

2.5 The Chain Rule

The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$.

We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g.

The Chain Rule

It turns out that the derivative of the composite function *f* ∘ *g* is the product of the derivatives of *f* and *g*. This fact is one of the most important of the differentiation rules and is called the *Chain Rule.*

This seems plausible as $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$ and $\Delta u \to 0$ as $\Delta x \to 0$ (by the continuity of *u*) and thus letting $\Delta x \to 0$ we expect $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

The flaw in this argument is that Δu might be 0 even if $\Delta x \neq 0$.

The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule

The Chain Rule can be written either in the prime notation

$$\boxed{2} \qquad (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

$$\boxed{3} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Equation 3 is easy to remember because if dy/du and du/dx were quotients, then we could cancel du.

Remember, however, that du has not been defined and du/dx should not be thought of as an actual quotient.

Examples

Example 1. Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

Example 2. Differentiate (a) $y = \sin(x^2)$ (b) $y = \sin^2 x$

Example 3. Differentiate $y = (x^3 - 1)^{100}$.

Example 4. Find f' if $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$.

Example 5. Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

Examples

Example 6. Differentiate $y = (2x + 1)^5(x^3 - x + 1)^4$.

Chain rule for more than 2 functions

If we want to differentiate

$$F(x) = f\big(g(h(x))\big),$$

then we can use the chain rule twice to get

$$F'(x) = f'\big(g(h(x))\big)\frac{d}{dx}[g(h(x))]$$
$$= f'\big(g(h(x))\big)g'\big(h(x)\big)h'(x)$$

Example 7. Differentiate $y = \sin(\cos(\tan x))$.

Example 8. Differentiate $y = \sqrt{\sec x^3}$.

2.6 Implicit Differentiation

Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$.

Some functions, however, are defined implicitly by a relation between x and y such as

$$x^2 + y^2 = 25 \tag{1}$$

or

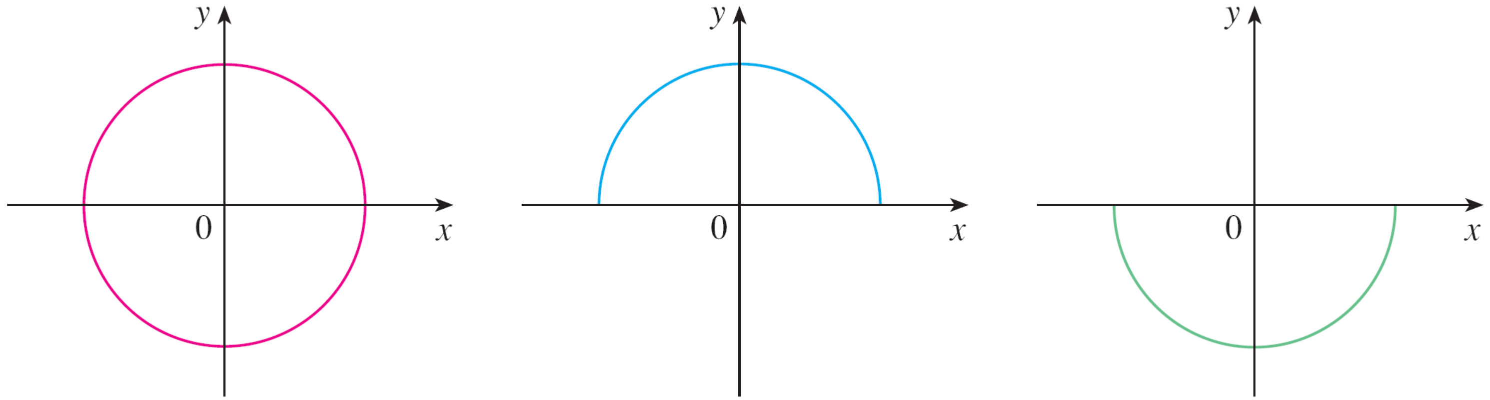
$$x^3 + y^3 = 6xy \tag{2}$$

Implicit Differentiation

In some cases it is possible to solve such an equation for *y* as an explicit function (or several functions) of *x*.

For instance, if we solve Equation 1 for *y*, we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$.

The graphs of *f* and *g* are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)



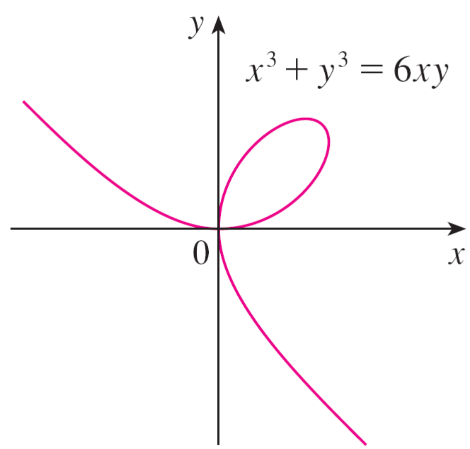
(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

Figure 1

Implicit Differentiation

It’s not easy to solve Equation 2 for *y* explicitly as a function of *x* by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.)

Nonetheless, (2) is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines *y* as several functions of *x*.

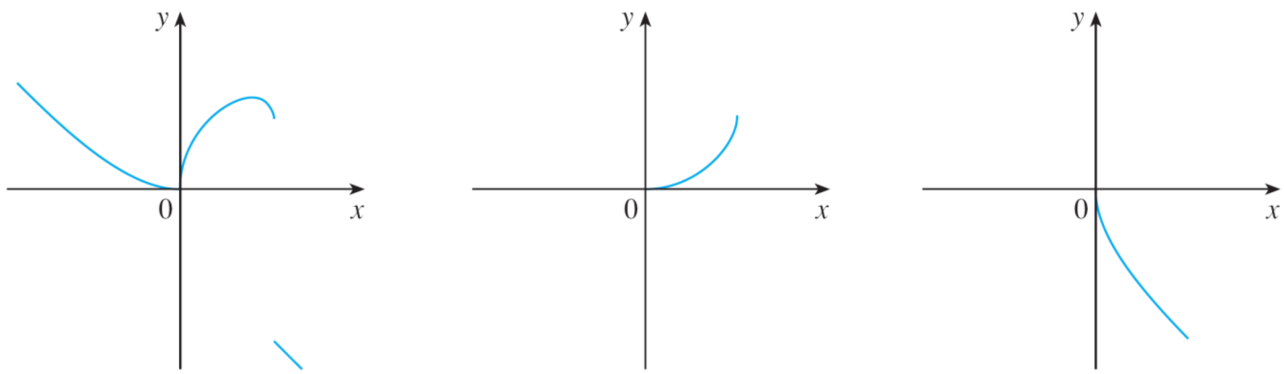


The folium of Descartes

Figure 2

Implicit Differentiation

The graphs of three such functions are shown in Figure 3.



Graphs of three functions defined by the folium of Descartes

Figure 3

When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6x f(x)$$

is true for all values of x in the domain of f.

Implicit Differentiation

Fortunately, we don’t need to solve an equation for *y* in terms of *x* in order to find the derivative of *y*. Instead we can use the method of **implicit differentiation**.

This consists of differentiating both sides of the equation with respect to *x* and then solving the resulting equation for *y*'.

In the examples and exercises of this section it is always assumed that the given equation determines *y* implicitly as a differentiable function of *x* so that the method of implicit differentiation can be applied.

Example 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

Example 2

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent line to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).

(c) At what point in the (open) first quadrant is the tangent line horizontal?

Examples 3 and 4

Example 3. Find y' if $\sin(x+y) = y^2 \cos x$.

Example 4. Find y'' if $x^4 + y^4 = 16$.

How to Prove the Chain Rule

Proof of the chain rule

In order to properly prove the chain rule one needs the following observation:

Proposition. A function $y = f(x)$, is differentiable at a if and only if there is a number A such that for Δx small nonnegative*:*

$$\Delta y = A\,\Delta x + \varepsilon\,\Delta x \quad \text{where} \quad \varepsilon \to 0 \text{ as } \Delta x \to 0$$

and ε is a continuous function of Δx at 0. In this case $f'(a) = A$.

Here: $\Delta y = f(a + \Delta x) - f(a)$.

2.9 Linear Approximations and Differentials

Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line.

This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f.

Linear Approximations and Differentials

So we settle for the easily computed values of the linear function *L* whose graph is the tangent line of *f* at (*a*, *f*(*a*)). (See Figure 1.)

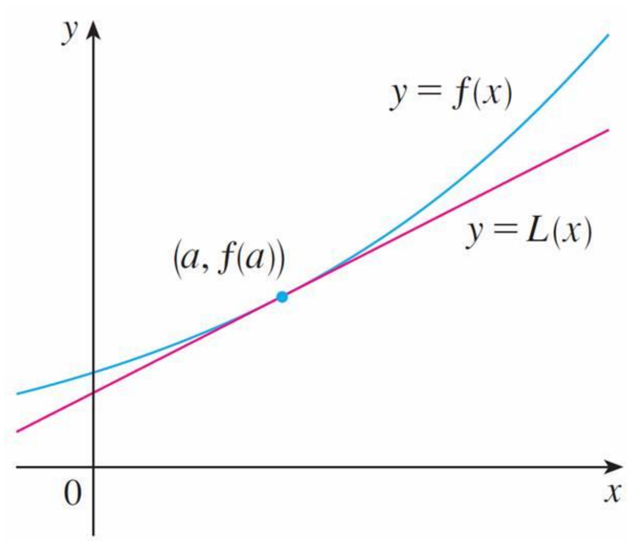


Figure 1

Linear Approximations and Differentials

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$\boxed{1} \qquad f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a.

Linear Approximations and Differentials

The linear function whose graph is this tangent line, that is,

$$\boxed{2} \qquad L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Example 1

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Example 1 – *Solution*

cont'd

The linear approximation is illustrated in Figure 2.

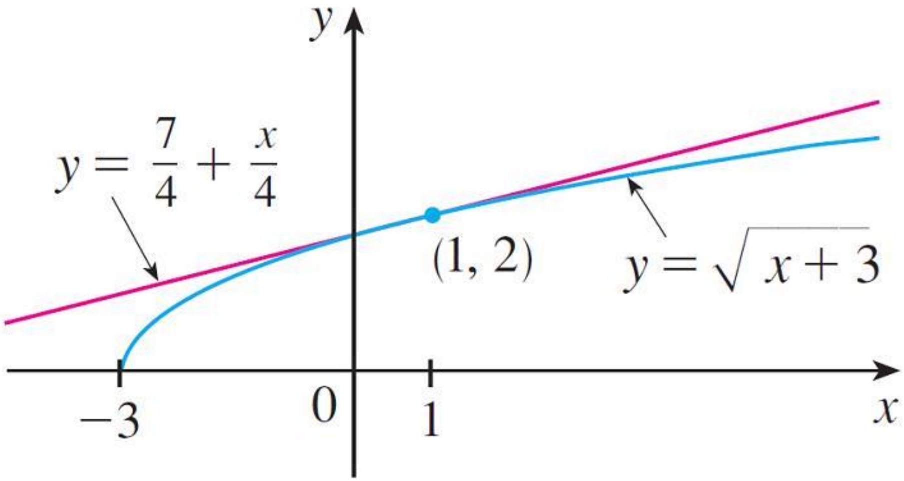


Figure 2

Linear Approximations and Differentials

In the following table we compare the estimates from the linear approximation in Example 1 with the true values.

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 . . .
$\sqrt{3.98}$	0.98	1.995	1.99499373 . . .
$\sqrt{4}$	1	2	2.00000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117 . . .
$\sqrt{4.1}$	1.1	2.025	2.02484567 . . .
$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

Applications to Physics

Applications to Physics

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation.

For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain the expression $a_T = -g \sin \theta$ for tangential acceleration and then replace $\sin \theta$ by θ with the remark that $\sin \theta$ is very close to θ if θ is not too large.

Applications to Physics

You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$ and so the linear approximation at 0 is

$$\sin x \approx x$$

So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Applications to Physics

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays.*

In paraxial (or Gaussian) optics, both $\sin\theta$ and $\cos\theta$ are replaced by their linearizations. In other words, the linear approximations

$$\sin\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1$$

are used because θ is close to 0.

Differentials

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials.*

If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number.

The **differential** dy is then defined in terms of dx by the equation

$$\boxed{3} \qquad dy = f'(x)\,dx$$

Differentials

So dy is a dependent variable; it depends on the values of x and dx.

If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.

Differentials

The geometric meaning of differentials is shown in Figure 5.

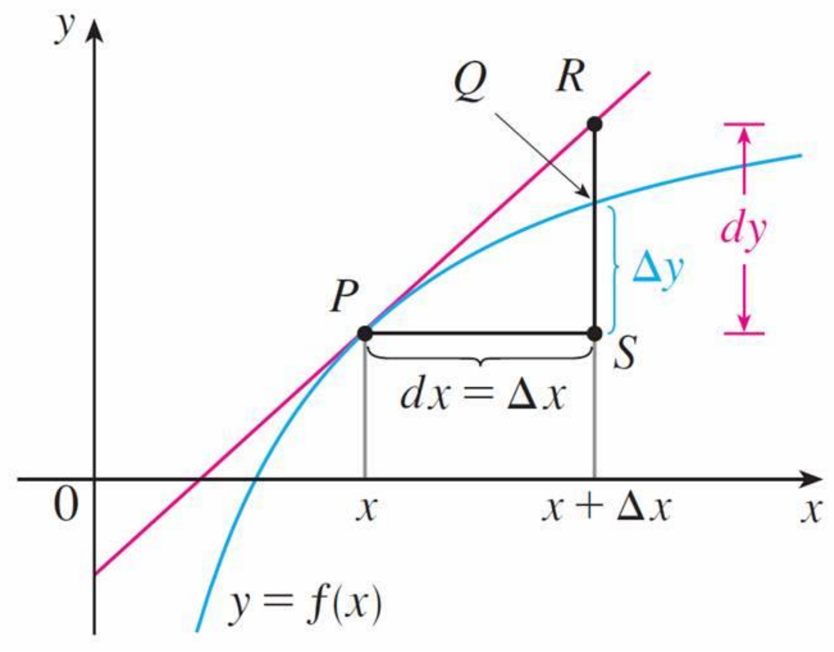


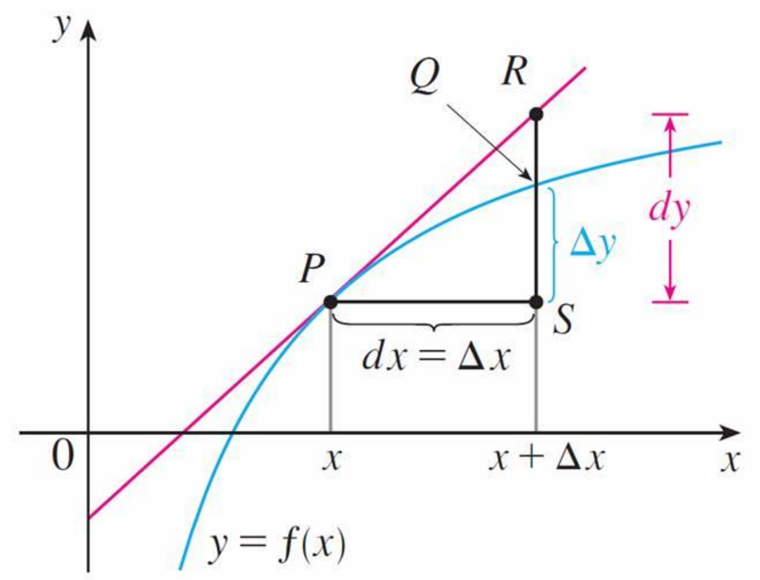
Figure 5

Differentials

Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)\,dx = dy$.



Example 3

Therefore, dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Example 3. Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

Example 4

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

3.1 Maximum and Minimum Values

Maximum and Minimum Values

Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something. These can be done by finding the maximum or minimum values of a function.

Let's first explain exactly what we mean by maximum and minimum values.

We see that the highest point on the graph of the function *f* shown in Figure 1 is the point (3, 5).

In other words, the largest value of *f* is *f*(3) = 5. Likewise, the smallest value is *f*(6) = 2.

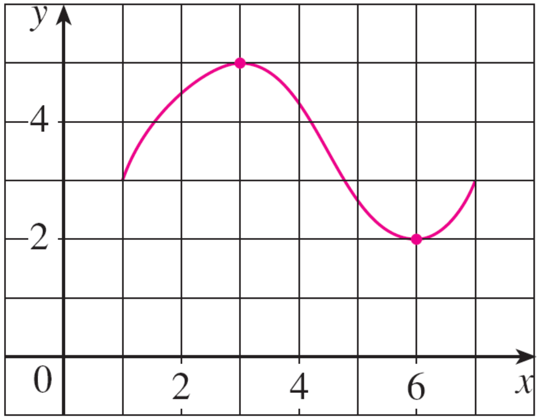


Figure 1

Maximum and Minimum Values

We say that $f(3) = 5$ is the *absolute maximum* of f and $f(6) = 2$ is the *absolute minimum*. In general, we use the following definition.

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

An absolute maximum or minimum is sometimes called a **global** maximum or minimum.

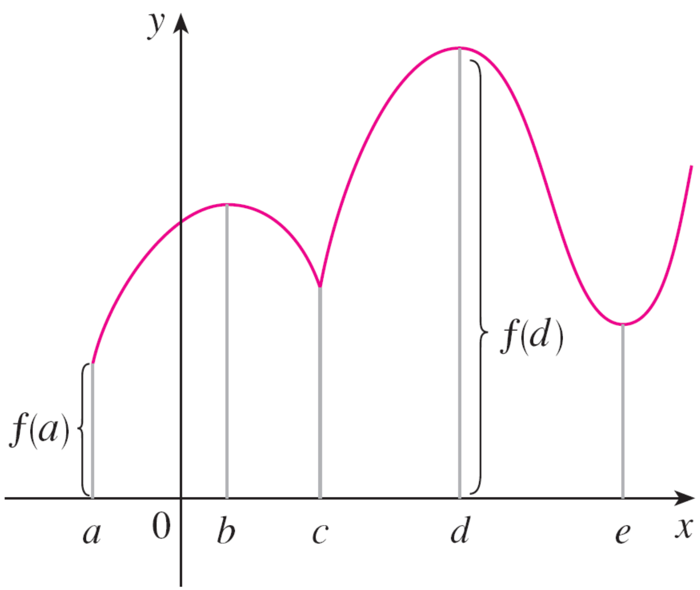
The maximum and minimum values of f are called **extreme values** of f.

Maximum and Minimum Values

Figure 2 shows the graph of a function *f* with absolute maximum at *d* and absolute minimum at *a*.

Note that (d, $f(d)$) is the highest point on the graph and (a, $f(a)$) is the lowest point.

In Figure 2, if we consider only values of x near b [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a *local maximum value* of *f*.



Abs min $f(a)$, abs max $f(d)$
loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

Figure 2

Maximum and Minimum Values

Likewise, $f(c)$ is called a *local minimum value* of f because $f(c) \le f(x)$ for x near c [in the interval (b, d), for instance].

The function f also has a local minimum at e. In general, we have the following definition.

$\boxed{2}$ **Definition** The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

In Definition 2 (and elsewhere), if we say that something is true **near** c, we mean that it is true on some open interval containing c.

Maximum and Minimum Values

For instance, in Figure 3 we see that $f(4) = 5$ is a local minimum because it's the smallest value of f on the interval I.

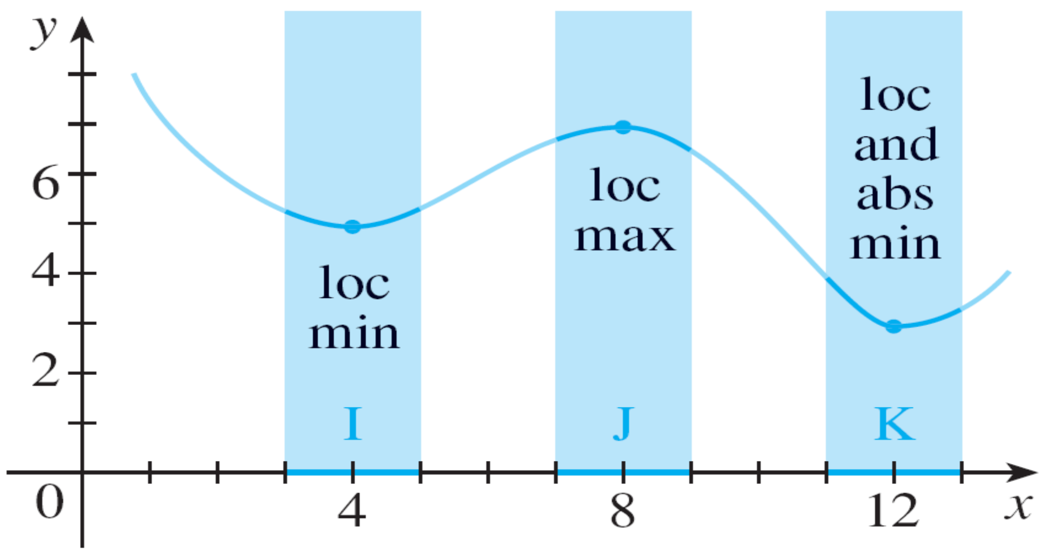


Figure 3

Maximum and Minimum Values

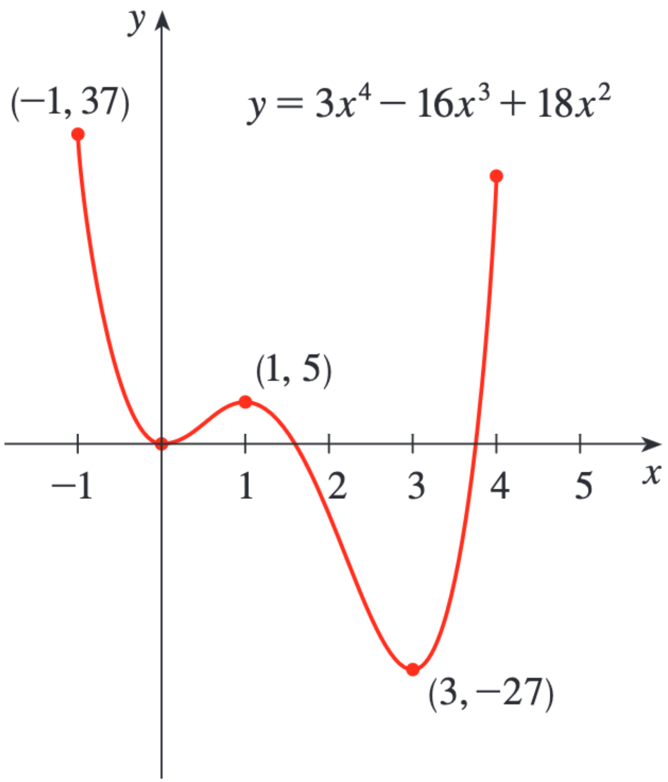
It's not the absolute minimum because $f(x)$ takes smaller values when x is near 12 (in the interval K, for instance).

In fact $f(12) = 3$ is both a local minimum and the absolute minimum.

Similarly, $f(8) = 7$ is a local maximum, but not the absolute maximum because f takes larger values near 1.

Example 1

Consider the function $f(x) = 3x^4 - 16x^3 + 12x^2$.



Examples 2 - 4

Discuss $f(x) = \cos x, f(x) = x^2, f(x) = x^3$.

Maximum and Minimum Values

The following theorem gives conditions under which a function is guaranteed to possess extreme values.

$\boxed{3}$ **The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Maximum and Minimum Values

The Extreme Value Theorem is illustrated in Figure 7.

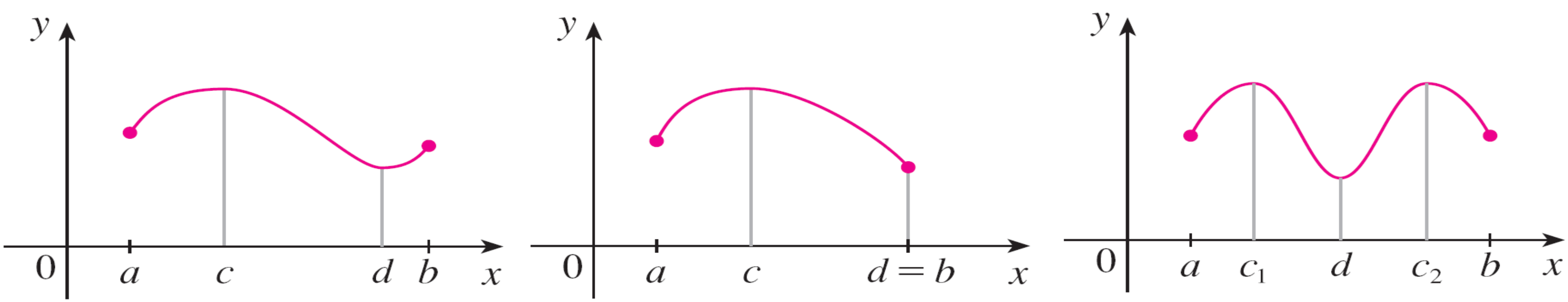


Figure 7

Note that an extreme value can be taken on more than once.

Maximum and Minimum Values

Figures 8 and 9 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

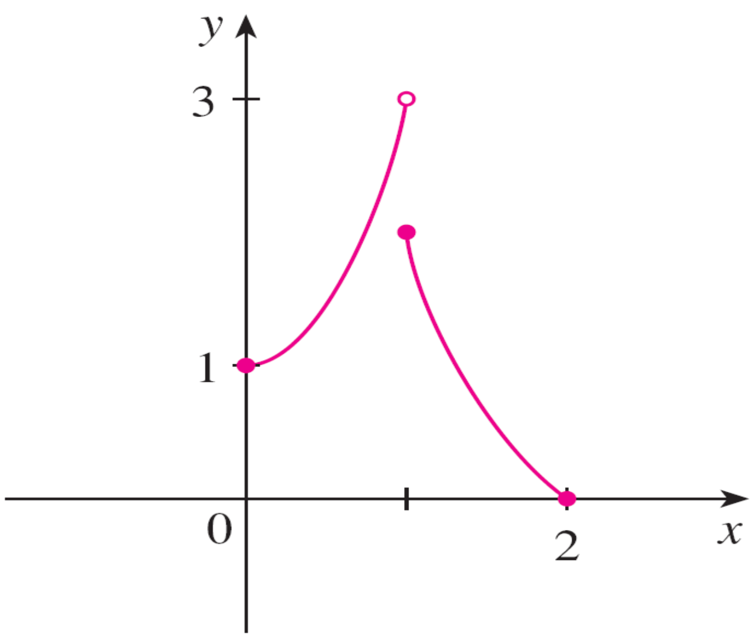


Figure 8 This function has minimum value $f(2) = 0$, but no maximum value.

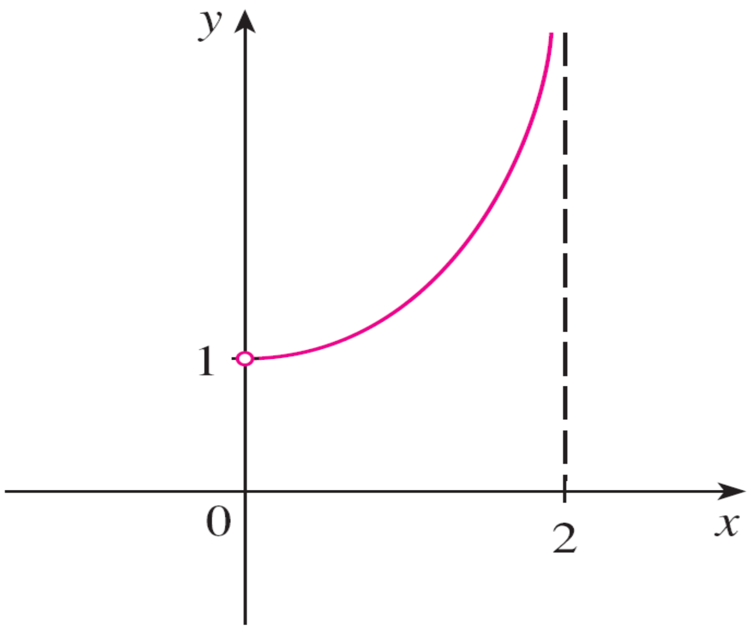


Figure 9 This continuous function g has no maximum or minimum.

Proof of the EVT

Definition. Given $A \subset \mathbb{R}$ we say that A is **bounded above** if there is a real number m such that $a \leq m$ for all $a \in A$. Such an m is called an **upper bound** of A.

Similar definition can be given for a set A to be **bounded below** and a **lower bound** of A.

Definition. A set $A \subset \mathbb{R}$ is **bounded** if it is both bounded above and below. A function f defined on [*a*, *b*] is bounded, bounded above/below on [*a*, *b*], if the set

$$A = \{f(x) | \ x \in [a, b]\}$$

is bounded, bounded above/below.

Proof of the EVT

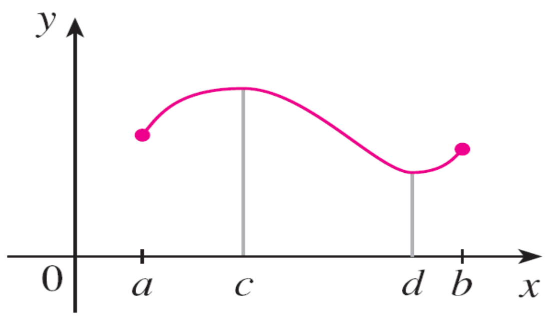
The **completeness axiom** of the real numbers assert that if $A \subset \mathbb{R}$ is bounded above, then A has a **least upper bound** M, called the **supremum** of A, denoted by $M = \sup A$. Thus, (i) $\sup A$ is an upper bound of A (ii) if m is an upper bound of A, then $\sup A \leq m$.

Similarly, $A \subset \mathbb{R}$ is bounded below, then A has a **greatest lower bound** K, called the **infimum** of A, denoted by $K = \inf A$. Thus, (i) $\inf A$ is a lower bound of A (ii) if k is a lower bound of A, then $\inf A \geq k$.

Examples: (0, 1), (0,1], [0,1).

Maximum and Minimum Values

It appears that at local maximum and minimum points the tangent lines are horizontal and therefore each has slope 0.



We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. The following theorem says that this is always true if the derivative exists.

4 **Fermat's Theorem** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 5

If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$.

But f has no maximum or minimum at 0, as you can see from its graph in Figure 11.

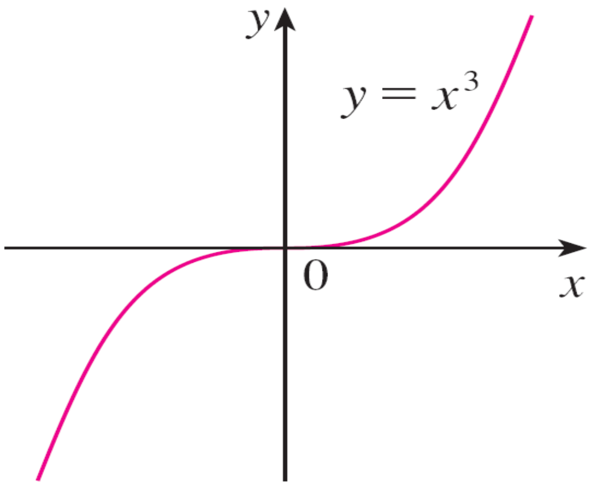


Figure 11

If $f(x) = x^3$, then $f'(0) = 0$ but f has no maximum or minimum.

Example 5

The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at (0, 0).

Instead of having a maximum or minimum at (0, 0), the curve crosses its horizontal tangent there.

Example 6

The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value can't be found by setting $f'(x) = 0$ because, $f'(0)$ does not exist. (see Figure 12)

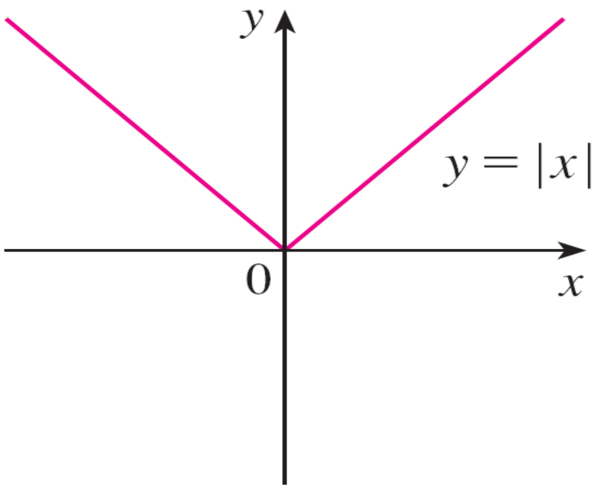


Figure 12

If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

Maximum and Minimum Values

Examples 5 and 6 show that we must be careful when using Fermat's Theorem. Example 5 demonstrates that even when $f'(c) = 0$, f doesn't necessarily have a maximum or minimum at c. (In other words, the converse of Fermat's Theorem is false in general.)

Furthermore, there may be an extreme value even when $f'(c)$ does not exist (as in Example 6).

Maximum and Minimum Values

Fermat's Theorem does suggest that we should at least *start* looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

In terms of critical numbers, Fermat's Theorem can be rephrased as follows.

7 If f has a local maximum or minimum at c, then c is a critical number of f.

Example 7

Find the critical numbers of (a) $f(x) = x^3 - 3x^2 + 1$

(b) $f(x) = x^{\frac{3}{5}}(4 - x)$

Maximum and Minimum Values

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval.

Thus the following three-step procedure always works.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Examples 8 and 9

Find the absolute maximum and minimum values of the function on the given interval.

Example 8. $f(x) = x^3 - 3x^2 + 1$ on $\left[-\frac{1}{2}, 4\right]$.

Example 9. $f(x) = x - 2\sin x$ on $[0, 2\pi]$.

3.2 The Mean Value Theorem

The Mean Value Theorem

We will see that many of the results depend on one central fact, which is called the Mean Value Theorem. But to arrive at the Mean Value Theorem we first need the following result.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

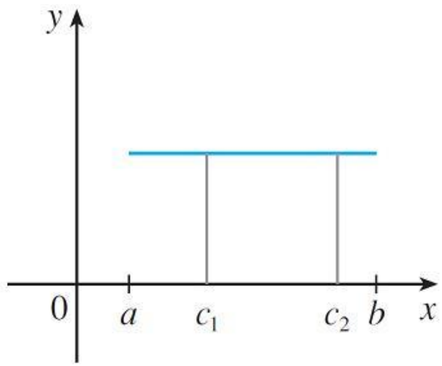
Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem

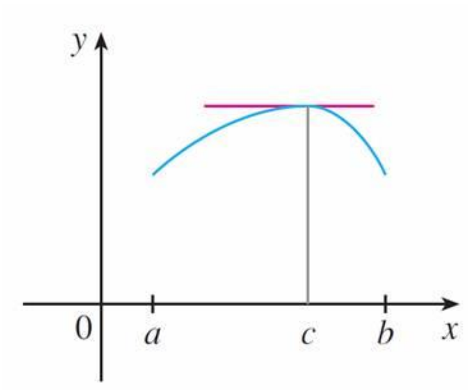
Before giving the proof let’s take a look at the graphs of some typical functions that satisfy the three hypotheses.

The Mean Value Theorem

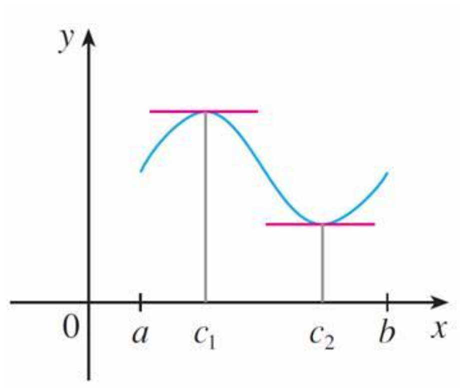
Figure 1 shows the graphs of four such functions.



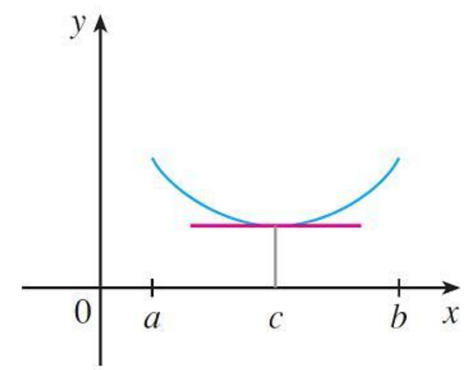
(a)



(b)



(c)



(d)

Figure 1

The Mean Value Theorem

In each case it appears that there is at least one point (c, $f(c)$) on the graph where the tangent is horizontal and therefore $f'(c) = 0$.

Thus Rolle's Theorem is plausible.

Example 1

Apply Rolle's Theorem to the position function $s = f(t)$ of a (rectilinearly) moving object.

Example 2

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

The Mean Value Theorem

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

The Mean Value Theorem

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions.

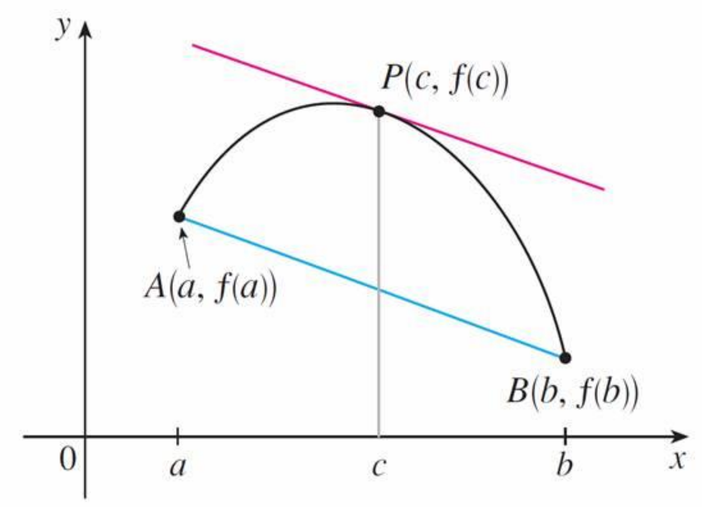


Figure 3

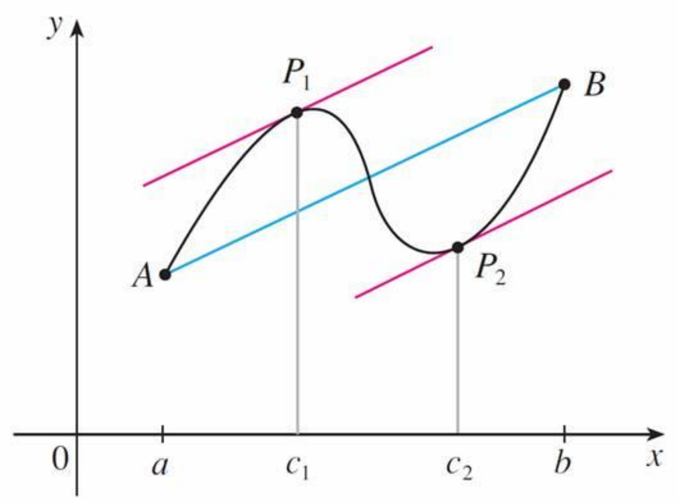


Figure 4

The Mean Value Theorem

The slope of the secant line *AB* is

$$\boxed{3} \qquad m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1.

The Mean Value Theorem

Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB.

In other words, there is a point P where the tangent line is parallel to the secant line AB.

Example 3

To illustrate the Mean Value Theorem with a specific function, let's consider

$$f(x) = x^3 - x,\ a = 0,\ b = 2.$$

Example 3

Figure 6 illustrates the calculation:

The tangent line at this value of *c* is parallel to the secant line *OB*.

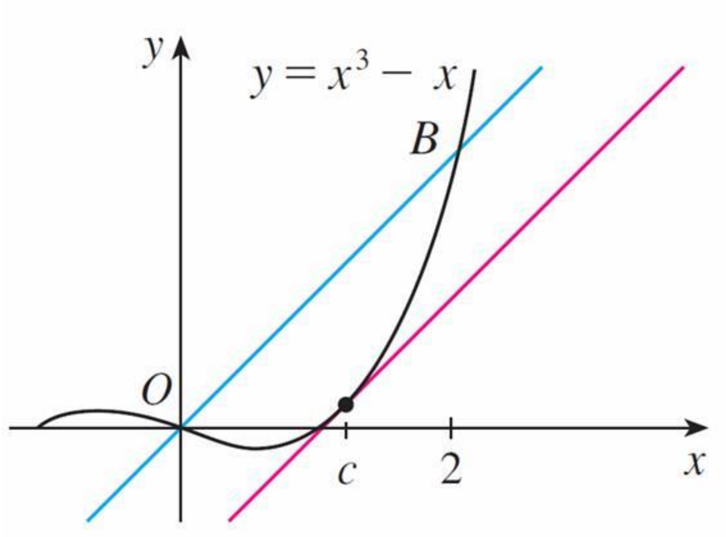


Figure 6

Example 4

Consider an object moves in a straight line with position function $s = f(t)$, between $t = a$ and $t = b$.

Example 5

Suppose that $f(0) = –3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

The Mean Value Theorem

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus.

One of these basic facts is the following theorem.

$\boxed{5}$ **Theorem** If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

$\boxed{7}$ **Corollary** If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

The Mean Value Theorem

Note:

Care must be taken in applying Theorem 5. Let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function.

This does not contradict Theorem 5 because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.

3.3 How Derivatives Affect the Shape of a Graph

What Does f' Say About f ?

What Does f' Say About f ?

To see how the derivative of *f* can tell us where a function is increasing or decreasing, look at Figure 1.

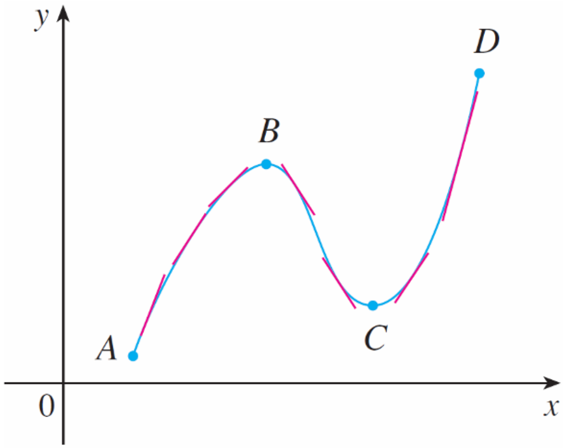


Figure 1

What Does f' Say About f ?

Between *A* and *B* and between *C* and *D*, the tangent lines have positive slope and so $f'(x) > 0$. Between *B* and *C* the tangent lines have negative slope and so $f'(x) < 0$. Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative.

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Note: Here the interval is open. However, it is clear from the proof, that the statement is also true on a closed interval $[a, b]$, provided that f is continuous at the endpoints a and b and $f'(x)>0$, resp., $f'(x)<0$ for all x in (a,b).

Example 1

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

What Does f' Say About f ?

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

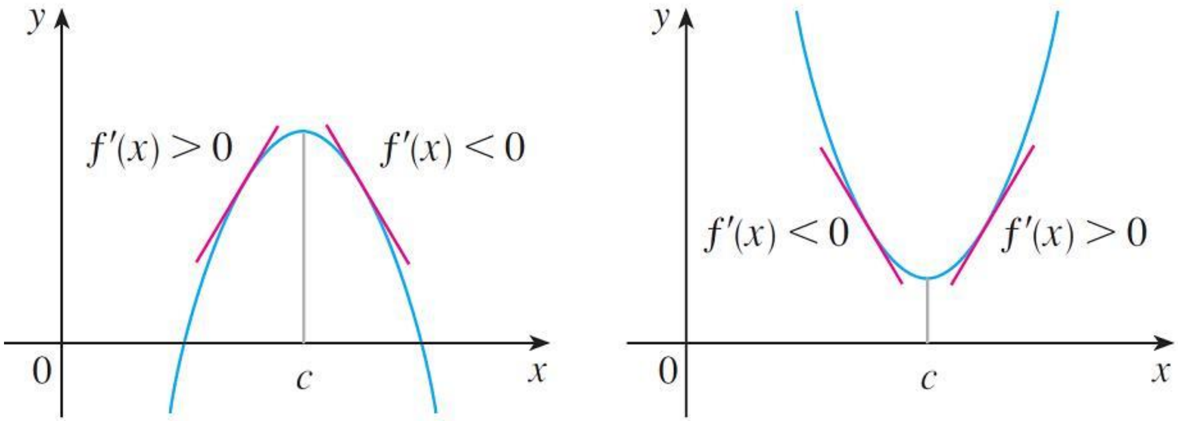
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Note: The continuity of f at c is important!

The First Derivative Test is a consequence of the I/D Test.

What Does f' Say About f ?

It is easy to remember the First Derivative Test by visualizing diagrams such as those in Figure 3.

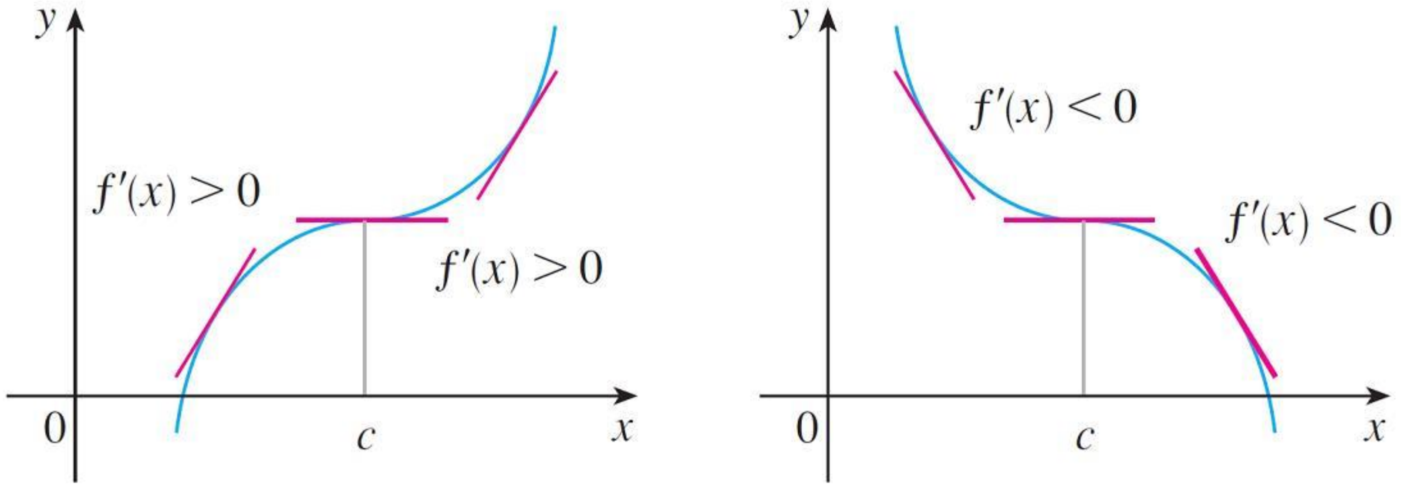


(a) Local maximum

(b) Local minimum

Figure 3

What Does f' Say About f ?



(c) No maximum or minimum

(d) No maximum or minimum

Figure 3

Example 2 and 3

Example 2. Find the local maximum and minimum values of the function from Example 1.

Example 3. Find the local maximum and minimum values of the function

$$g(x) = x + 2 \sin x \qquad 0 \leq x \leq 2\pi$$

What Does f'' Say About f ?

What Does f'' Say About f ?

Figure 5 shows the graphs of two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions.

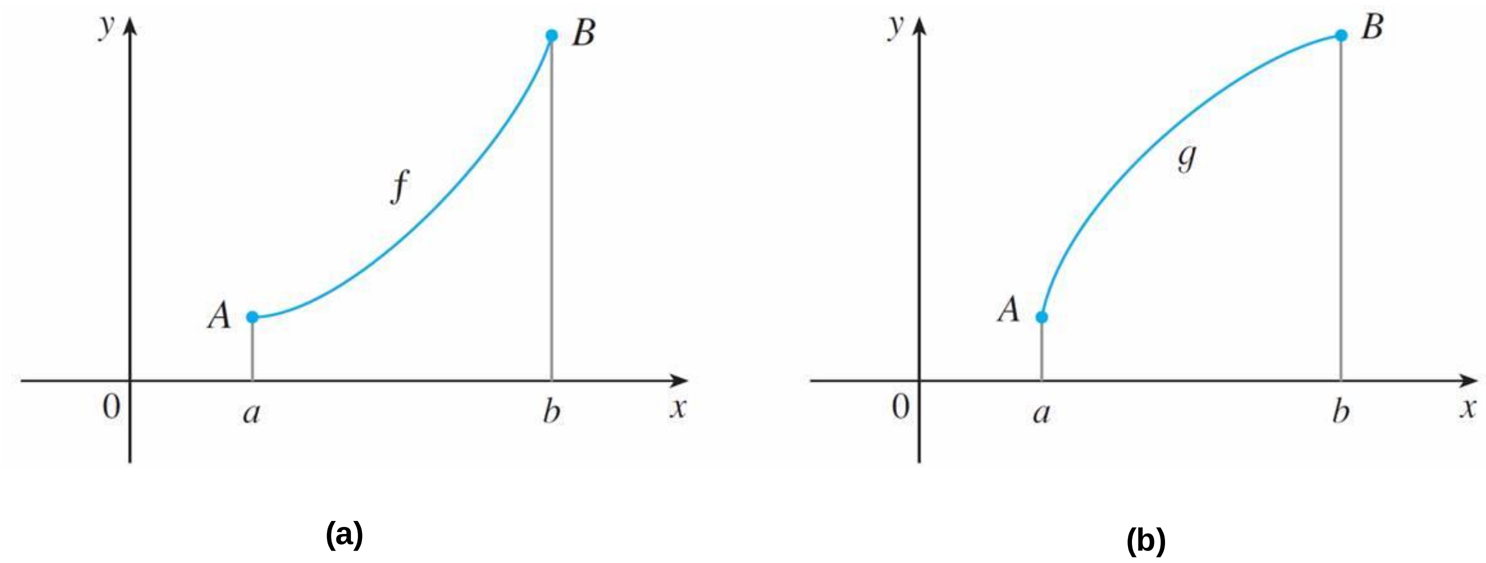
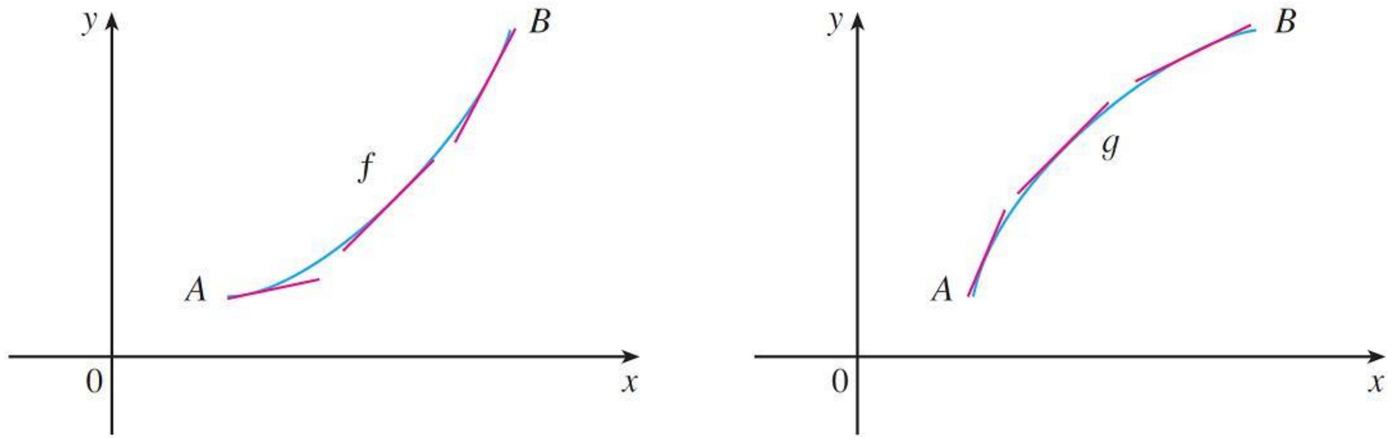


Figure 5

What Does f'' Say About f ?

In Figure 6 tangents to these curves have been drawn at several points. In (a) the curve lies above the tangents and f is called *concave upward* on (a, b). In (b) the curve lies below the tangents and g is called *concave downward* on (a, b).



(a) Concave upward

(b) Concave downward

Figure 6

What Does f'' Say About f ?

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

What Does f'' Say About f ?

Figure 7 shows the graph of a function that is concave upward (abbreviated CU) on the intervals (b, c), (d, e), and (e, p) and concave downward (CD) on the intervals (a, b), (c, d), and (p, q).

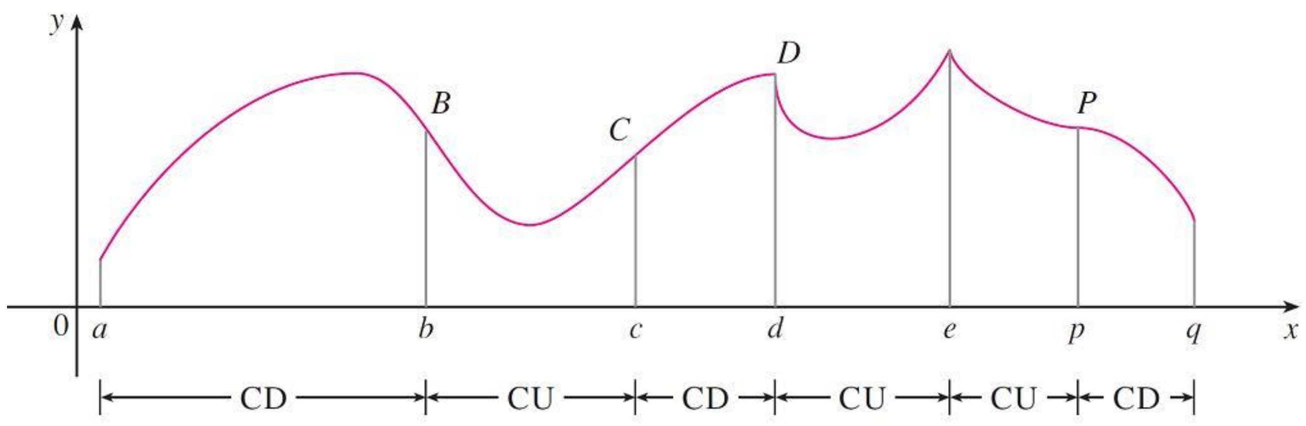


Figure 7

What Does f'' Say About f ?

Let's see how the second derivative helps determine the intervals of concavity. Looking at Figure 6(a), you can see that, going from left to right in (b, c), the slope of the tangent increases.

This means that the derivative f' is an increasing function and therefore its derivative f'' is positive.

Likewise, in Figure 6(b) the slope of the tangent decreases from left to right on (c, d), so f' decreases and therefore f'' is negative.

What Does f'' Say About f ?

This reasoning can be reversed and suggests that the following theorem is true.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 4

Figure 8 shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is *P* concave upward or concave downward?

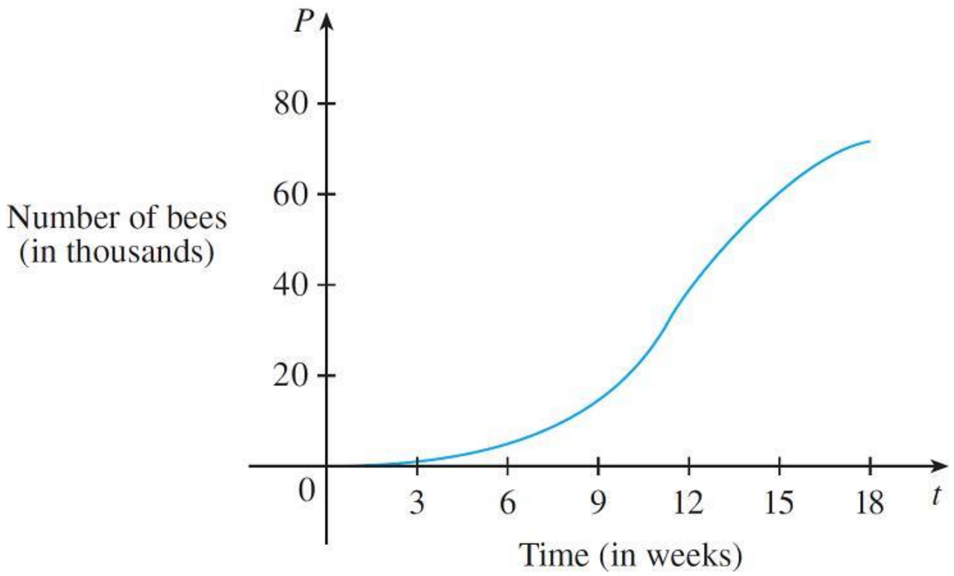


Figure 8

What Does f'' Say About f ?

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

What Does f'' Say About f ?

Note:

The Second Derivative Test is inconclusive when $f''(c) = 0$. In other words, at such a point there might be a maximum, there might be a minimum, or there might be neither.

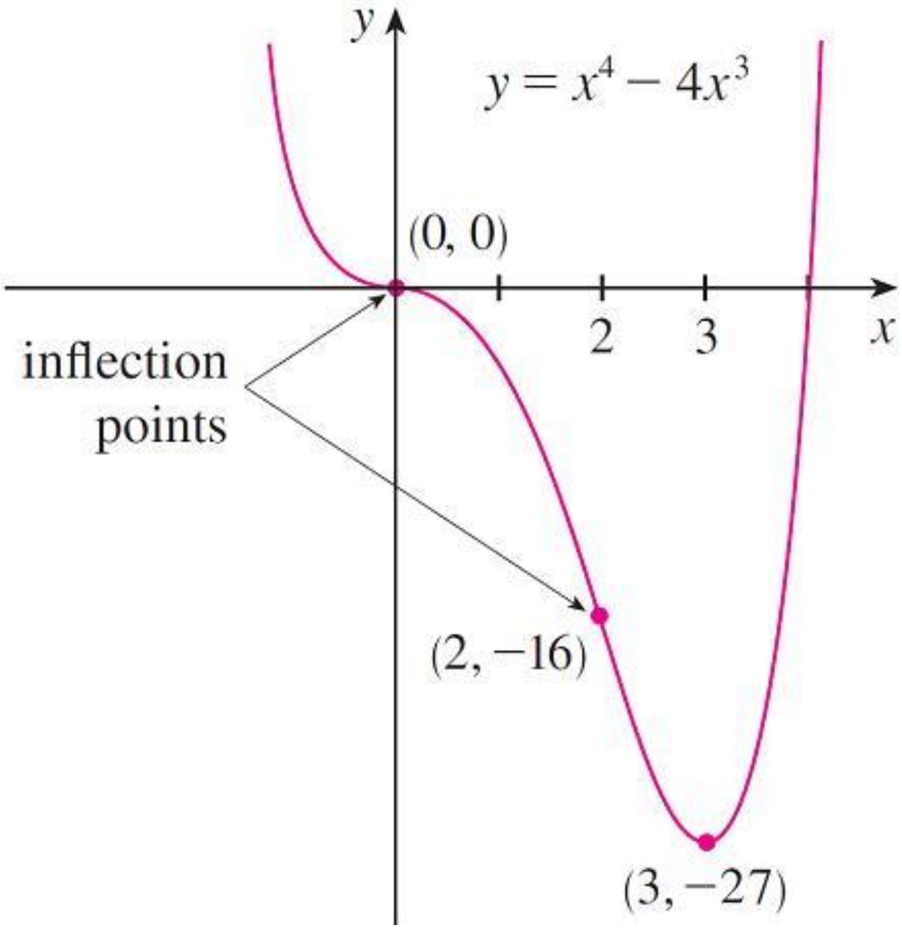
This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

Example 6

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Example 6

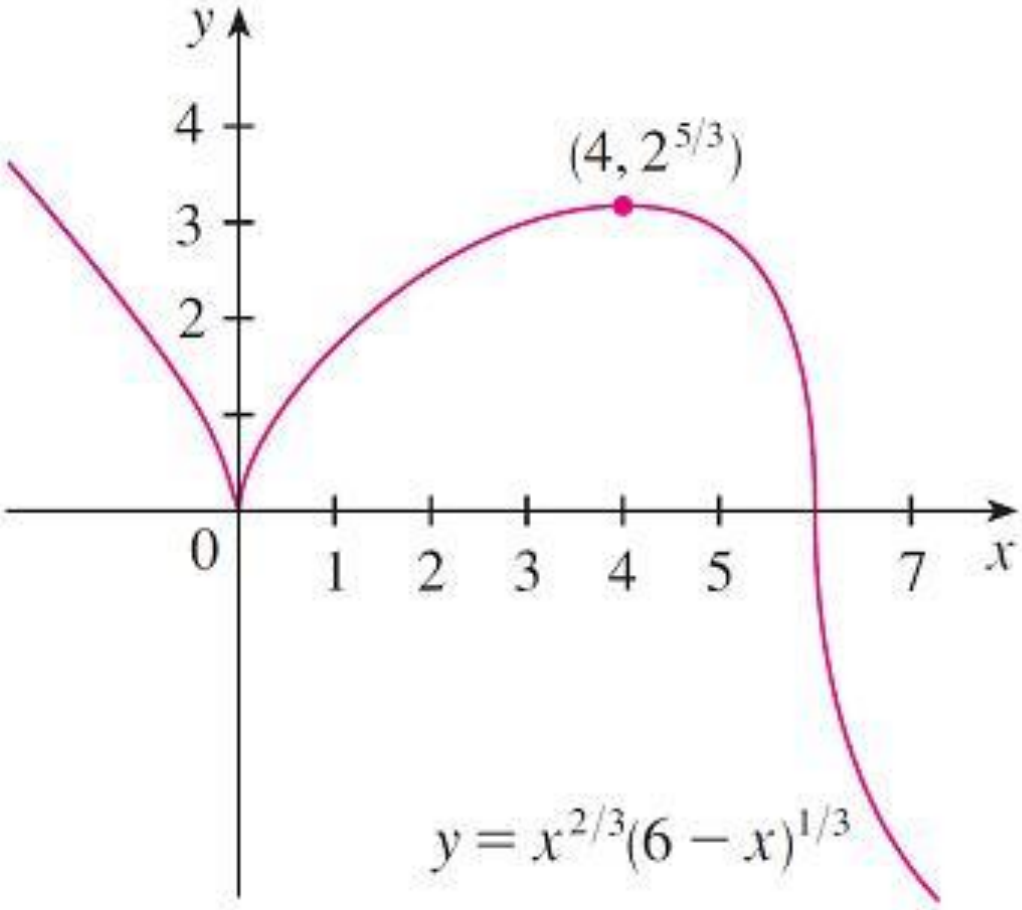
cont'd



Example 7

Sketch the graph of the function $f(x) = x^{2/3}(6 - x)^{1/3}$

Example 7



3.4 Limits at Infinity; Horizontal Asymptotes

Limits at Infinity; Horizontal Asymptotes

In this section we let *x* become arbitrarily large (positive or negative) and see what happens to *y*. We will find it very useful to consider this so-called *end behavior* when sketching graphs.

Let's begin by investigating the behavior of the function *f* defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as *x* becomes large.

Limits at Infinity; Horizontal Asymptotes

The following table contains approximate values of the function.

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

Limits at Infinity; Horizontal Asymptotes

The graph of *f* has been drawn by a computer in Figure 1.

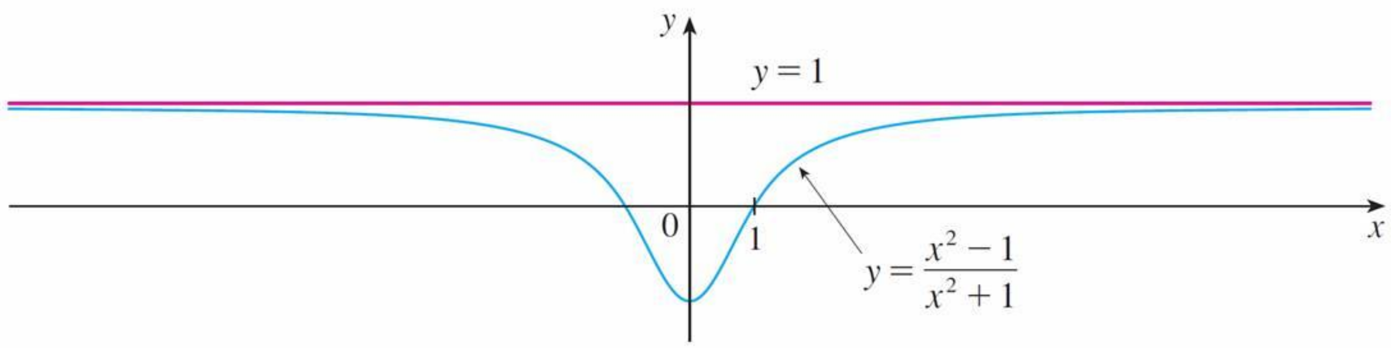


Figure 1

Limits at Infinity; Horizontal Asymptotes

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large.

This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

$\boxed{1}$ **Intuitive Definition of a Limit at Infinity** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

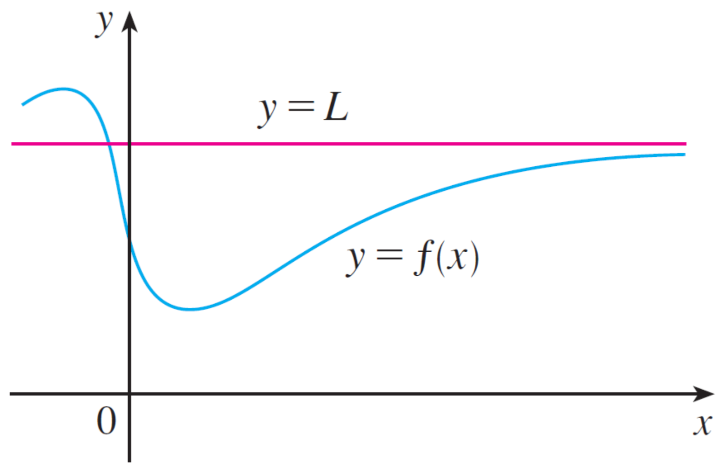
Limits at Infinity; Horizontal Asymptotes

Another notation for $\lim_{x\to\infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

Limits at Infinity; Horizontal Asymptotes

Geometric illustrations of Definition 1 are shown in Figure 2.

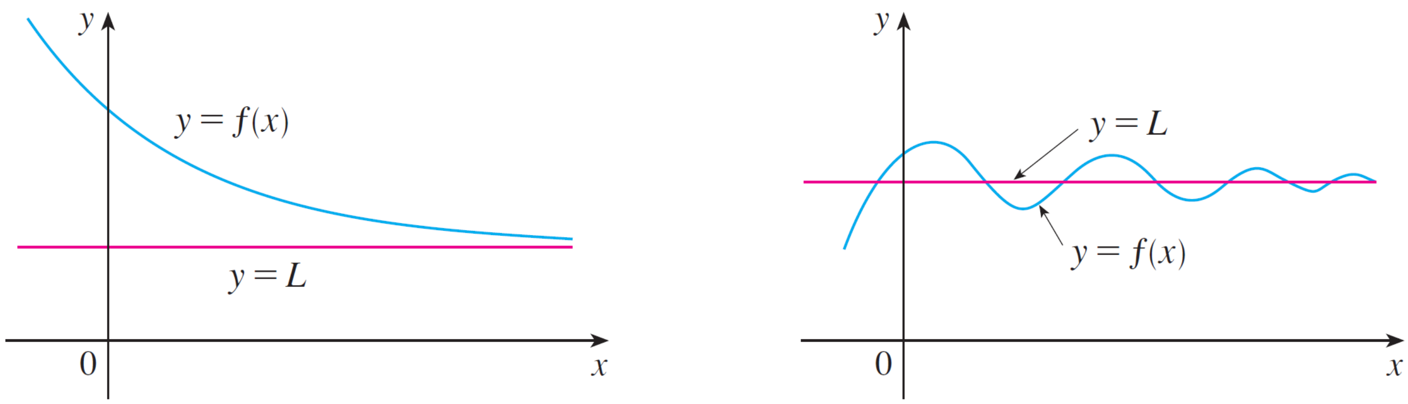


Examples illustrating $\lim_{x \to \infty} f(x) = L$

Figure 2

Limits at Infinity; Horizontal Asymptotes

Notice that there are many ways for the graph of *f* to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.



Limits at Infinity; Horizontal Asymptotes

Referring back to Figure 1, we see that for numerically large negative values of *x*, the values of *f*(*x*) are close to 1.

By letting *x* decrease through negative values without bound, we can make *f*(*x*) as close to 1 as we like.

This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

The general definition is as follows.

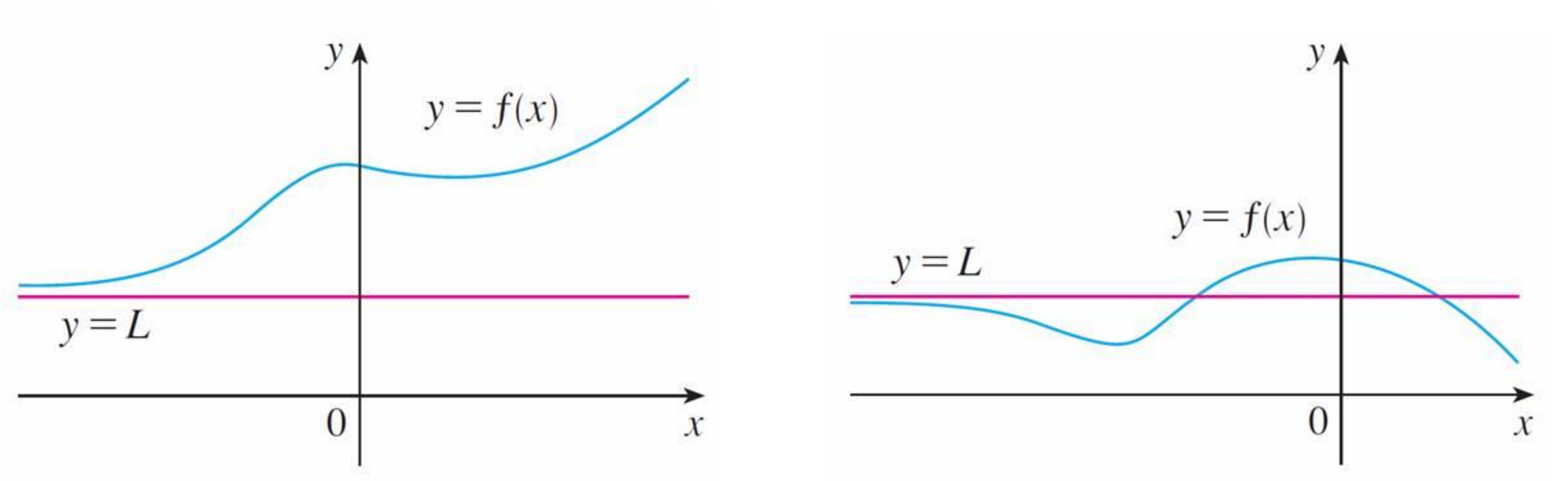
2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Limits at Infinity; Horizontal Asymptotes

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line *y* = *L* as we look to the far left of each graph.



Examples illustrating $\lim_{x \to \infty} f(x) = L$

Figure 3

Limits at Infinity; Horizontal Asymptotes

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Limits at Infinity; Horizontal Asymptotes

The curve $y = f(x)$ sketched in Figure 4 has both $y = –1$ and $y = 2$ as horizontal asymptotes because

$$\lim_{x \to \infty} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

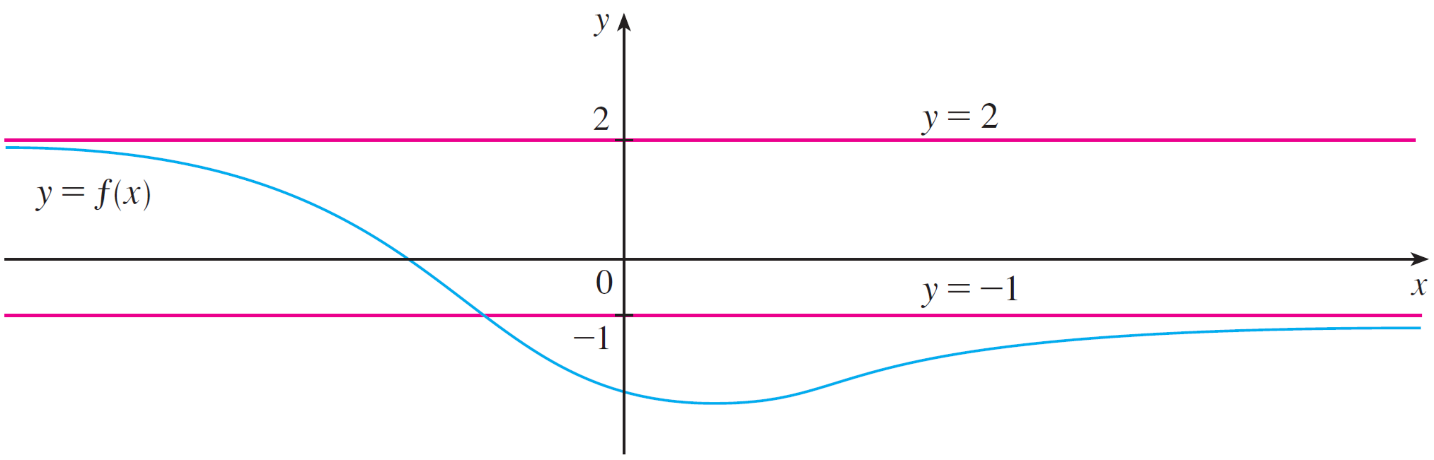
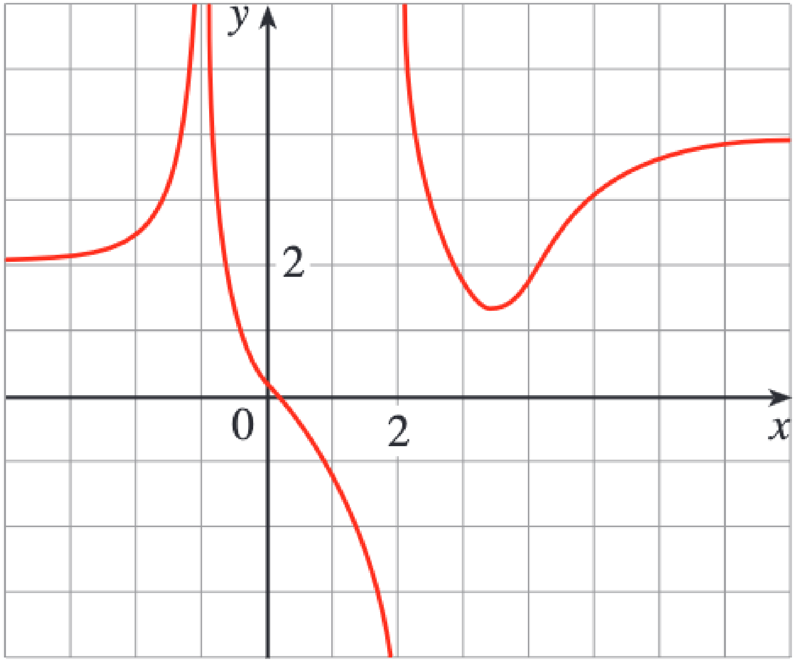


Figure 4

Example 1

Find the infinite limits, limits at infinity, and asymptotes for the function *f* whose graph is shown below.



Limits at Infinity; Horizontal Asymptotes

$\boxed{4}$ **Theorem** If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

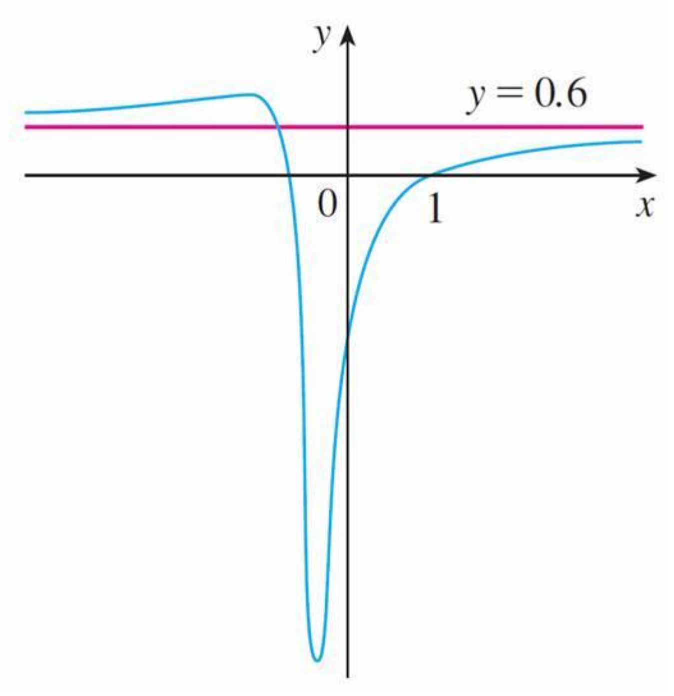
Example 3

Evaluate $\lim_{x \to \infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$ and indicate which properties of limits are used at each stage.

Example 3

∞

Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.



$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

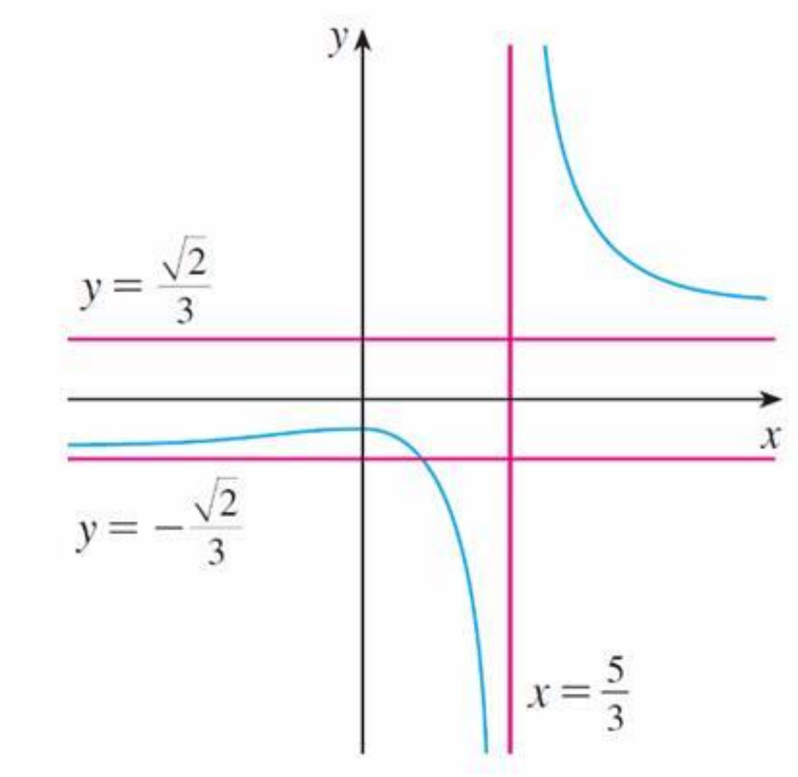
Figure 7

Example 4

Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Example 4



$$y = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Example 5

Evaluate

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x \right).$$

Examples 6 and 7

Example 6. Evaluate

$$\lim_{x \to \infty} \sin \frac{1}{x}.$$

Example 7. Evaluate

$$\lim_{x \to \infty} \sin x .$$

Infinite Limits at Infinity

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Example 8 – 10

Example 8. Find

$$\lim_{x\to\infty} x^3 \text{ and } \lim_{x\to-\infty} x^3 .$$

Example 9. Find

$$\lim_{x\to\infty}(x^2 - x).$$

Example 10. Find

$$\lim_{x\to\infty}\frac{x^2 + x}{3 - x}.$$

Precise Definitions

Precise Definitions

Definition 1 can be stated precisely as follows.

5 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

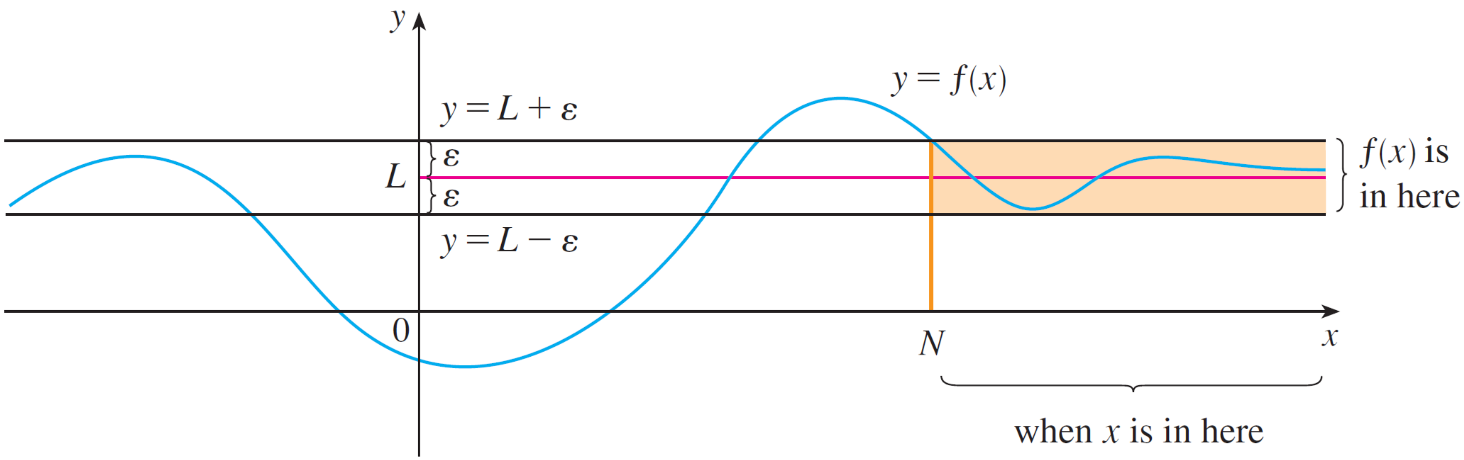
means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by taking x sufficiently large (larger than N, where depends on ε).

Precise Definitions

Graphically it says that by choosing *x* large enough (larger than some number *N*) we can make the graph of *f* lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 12.

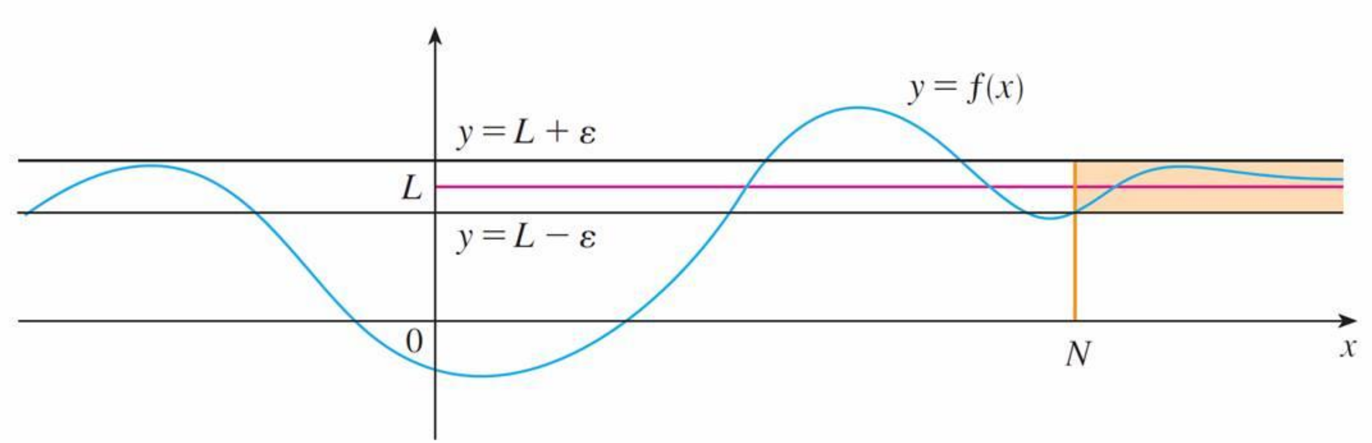


$$\lim_{x \to \infty} f(x) = L$$

Figure 12

Precise Definitions

This must be true no matter how small we choose ε. Figure 13 shows that if a smaller value of ε is chosen, then a larger value of *N* may be required.



$$\lim_{x \to \infty} f(x) = L$$

Figure 13

Precise Definitions

6 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

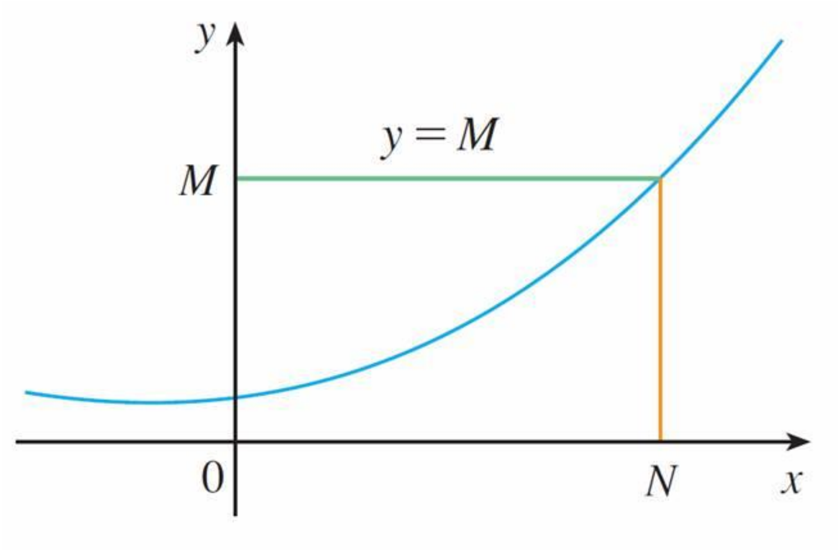
$$\text{if} \quad x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 13

Use Definition 5 to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Precise Definitions

Finally, we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 17.



$$\lim_{x \to \infty} f(x) = \infty$$

Figure 17

Precise Definitions

7 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$.

Guidelines for Sketching a Curve

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.)

But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of for which $f(x)$ is defined.

Guidelines for Sketching a Curve

B. Intercepts The *y*-intercept is *f*(0) and this tells us where the curve intersects the *y*-axis. To find the *x*-intercepts, we set $y = 0$ and solve for *x*. (You can omit this step if the equation is difficult to solve.)

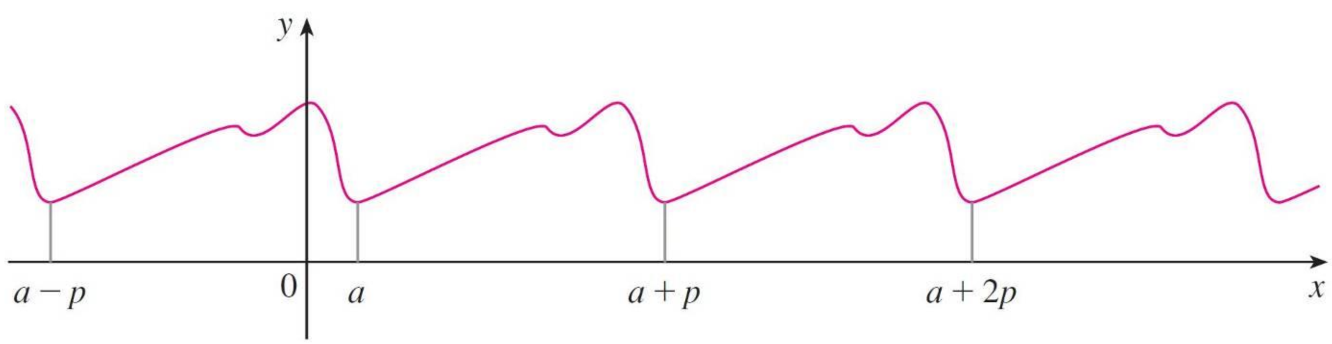
C. Symmetry

(i) If *f* is an **even function** or an **odd function** then you only have to sketch the curve for $x > 0$, then reflect about the y axis (even) or the origin (odd).

Guidelines for Sketching a Curve

(ii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.**

For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 4).



Periodic function: translational symmetry

Figure 4

Guidelines for Sketching a Curve

D. Asymptotes

(i) *Horizontal Asymptotes.* If either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$.

If it turns out that $\lim_{x \to \infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

Guidelines for Sketching a Curve

(ii) *Vertical Asymptotes.* The line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$\boxed{1}$

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.)

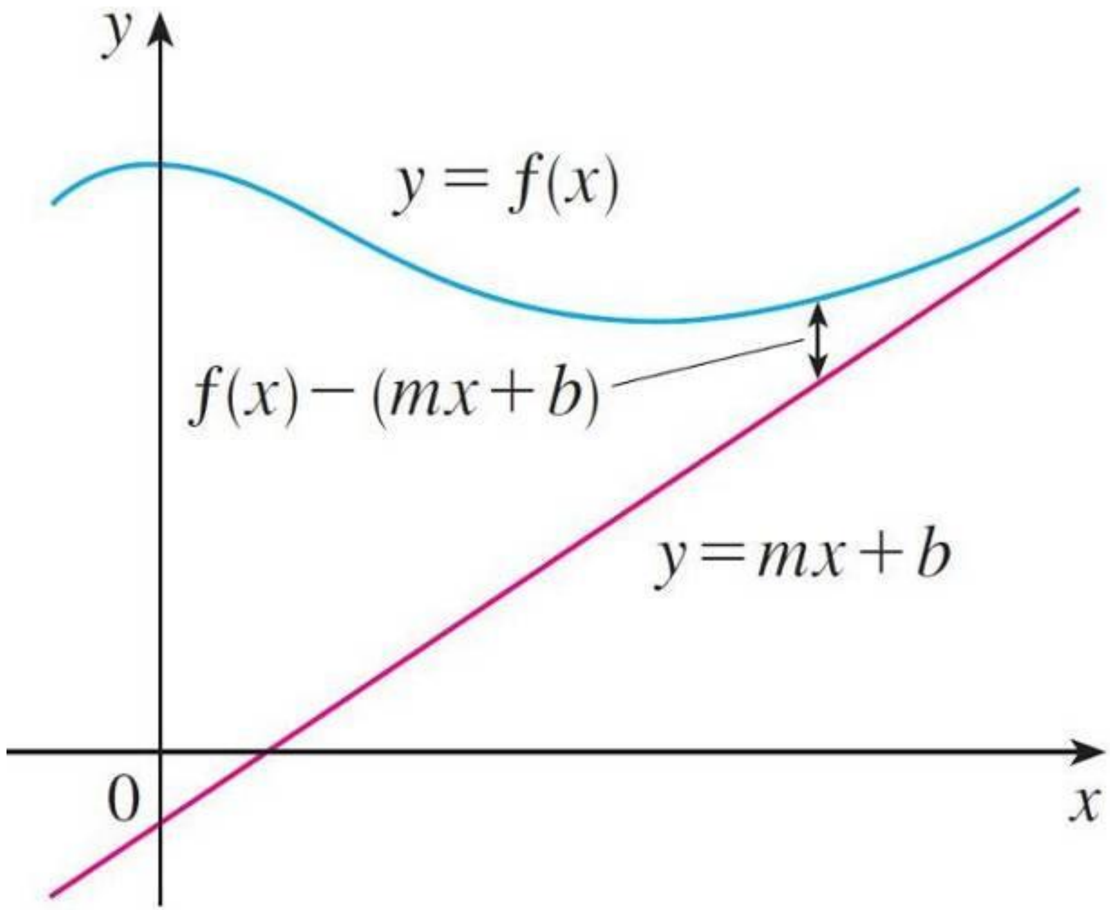
Guidelines for Sketching a Curve

Furthermore, in sketching the curve it is very useful to know exactly which of the statements in $\boxed{1}$ is true.

If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.* If $\lim_{x\to\infty} [f(x) - (mx + b)] = 0$,
then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0. (A similar situation may exist if we let $x \to \infty$.)

Slant Asymptotes



Slant Asymptotes

For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator.

In such a case the equation of the slant asymptote can be found by long division as in Example 4.

Guidelines for Sketching a Curve

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test if f is continuous at c. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$ and f'' is continuous at c.

Guidelines for Sketching a Curve

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes and f is continuous there.

Guidelines for Sketching a Curve

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points.

Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.

If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

Example 1

Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

3.7 Optimization Problems

Example 1

A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Example 2

A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Optimization Problems

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Example 3

Find the point on the parabola $y^2 = 2x$ that is closest to the point (1, 4).

Example 4

A woman launches her boat from point *A* on a bank of a straight river, 3 km wide, and wants to reach point *B*, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). She could row her boat directly across the river to point *C* and then run to *B*, or she could row directly to B, or she could row to some point *D* between *C* and *B* and then run to *B*. If she can row 6 km/h and run 8 km/h, where should she land to reach *B* as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the woman rows.)

Example 4

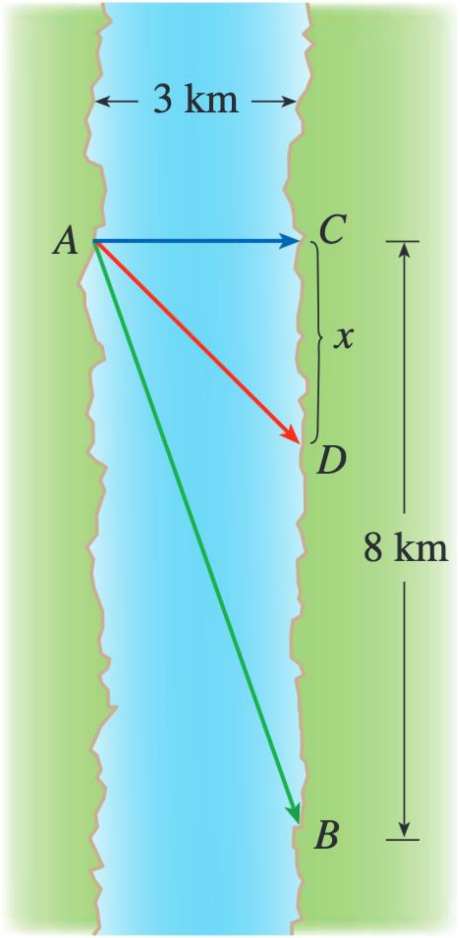


FIGURE 7

3.8 Newton's Method

Newton's Method

We aim to find a root of a function f; that is find a solution of

$$f(x) = 0.$$

There are a variety of methods, but probably the most of famous one is called **Newton's method**, also called the **Newton-Raphson method**. Numerical rootfinders usually use this or a variant of this method.

We will explain how this method works, partly to show what happens inside a calculator or computer, and partly as an application of the idea of linear approximation.

Newton's Method

The geometry behind Newton's method is shown in Figure 2, where the root that we are trying to find is labeled *r*.

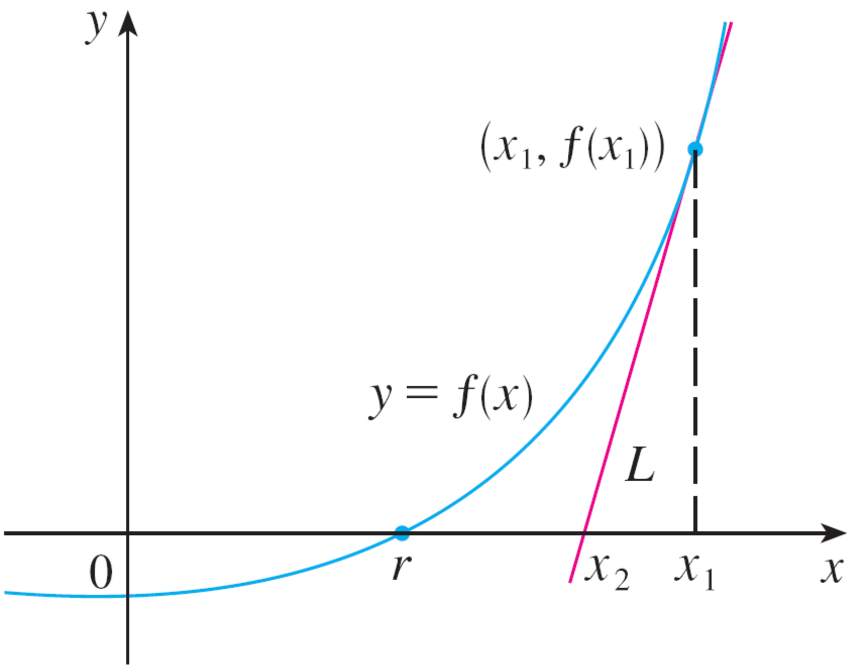


Figure 2

Newton's Method

In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

2

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence *converges* to r and we write

$$\lim_{n \to \infty} x_n = r$$

Warning

Although the sequence of successive approximations converges to the desired solution for functions of the type illustrated in Figure 3, in certain circumstances the sequence may not converge.

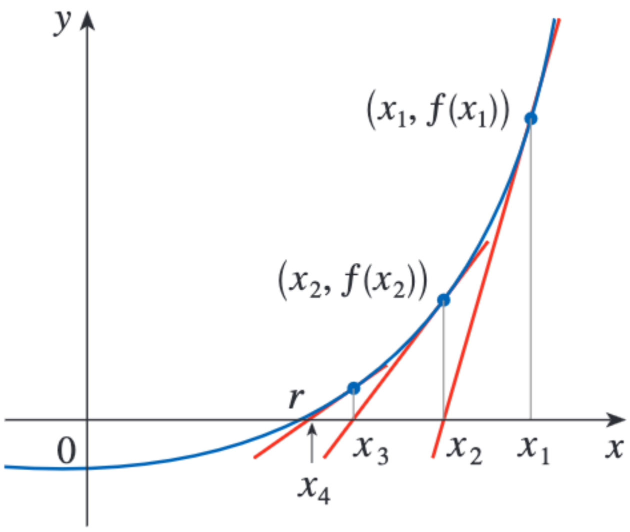


FIGURE 3

Warning

For example, consider the situation shown in Figure 4. You can see that x_2 is a worse approximation than x_1. This is likely to be the case when $f'(x_1)$ is close to 0. It might even happen that an approximation (such as x_3 in Figure 4) falls outside the domain of f. Then Newton's method fails and a better initial approximation x_1 should be chosen.

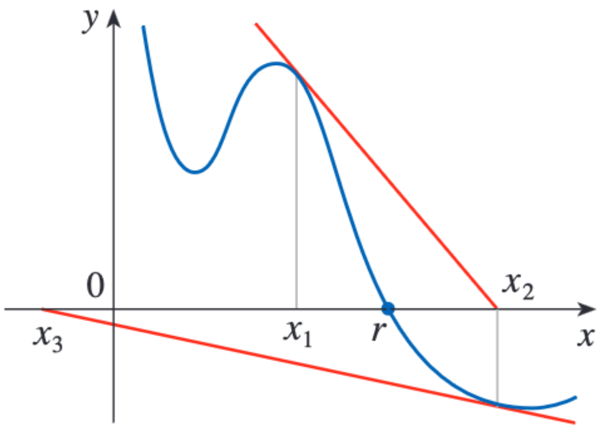


FIGURE 4

Example 1

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$. (Newton himself used this equation to illustrate his method)

Example 2

Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

3.9 Antiderivatives

Antiderivatives

A physicist who knows the velocity of a particle might wish to know its position at a given time.

An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period.

A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time.

Antiderivatives

In each case, the problem is to find a function *F* whose derivative is a known function *f*. If such a function *F* exists, it is called an *antiderivative* of *f*.

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Antiderivatives

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$.

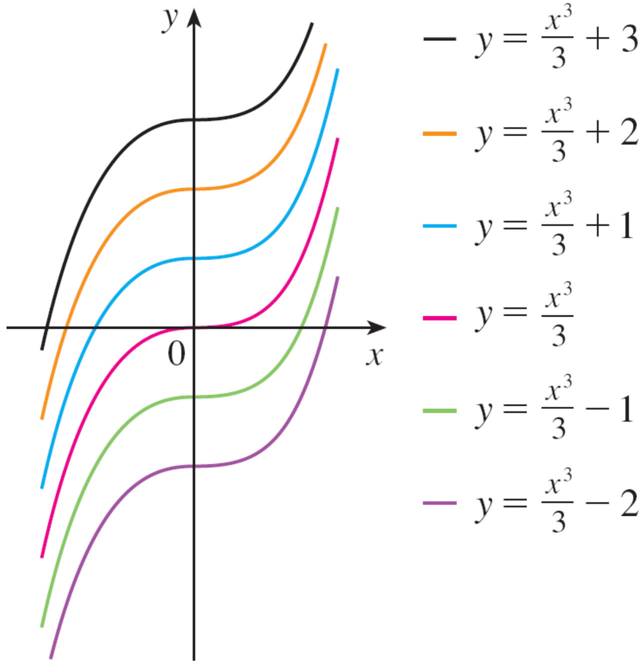
But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore, both F and G are antiderivatives of f.

Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f.

Antiderivatives

By assigning specific values to the constant *C*, we obtain a family of functions whose graphs are vertical translates of one another (see Figure 1).

This makes sense because each curve must have the same slope at any given value of *x*.



Members of the family of antiderivatives of $f(x) = x^2$

Figure 1

Antiderivatives

1 Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 1

Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ **(b)** $f(x) = x^n, \ n \geq 0$ **(c)** $f(x) = x^{-3}$

Antiderivatives

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives.

2 Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \; (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
		$\sec x \tan x$	$\sec x$

To obtain the most general antiderivative from the particular ones in Table 2, we have to add a constant (or constants), as in Example 1.

Antiderivatives

Each formula in the table is true because the derivative of the function in the right column appears in the left column.

In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function.

The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation $F' = f$, $G' = g$.)

Example 2

Find all functions *g* such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

Antiderivatives

An equation that involves the derivatives of a function is called a **differential equation**.

The general solution of a differential equation involves an arbitrary constant (or constants).

However, there may be some extra conditions given that will determine the constants and therefore uniquely specify the solution.

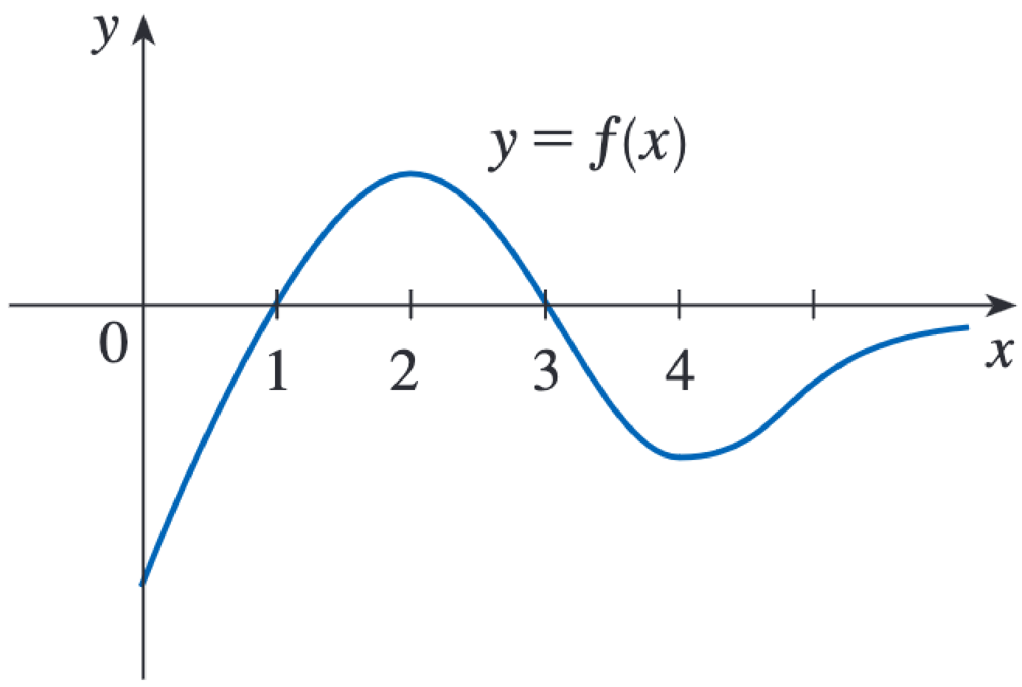
Example 3 and 4

Example 3. Find f if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

Example 4. Find f if $f''(x) = 12x^2 + 6x - 4, f(0) = 4$ and $f(1) = 1$.

Example 5

The graph of a function f is given below. Make a rough sketch of an antiderivative F, given that $F(0) = 2$.



Rectilinear Motion

Rectilinear Motion

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$.

This means that the position function is an antiderivative of the velocity function.

Likewise, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration.

If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

Examples 6 and 7

Example 6. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Example 7. A ball is thrown upward with a speed of 15 m/s from the edge of a cliff, 130 m above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

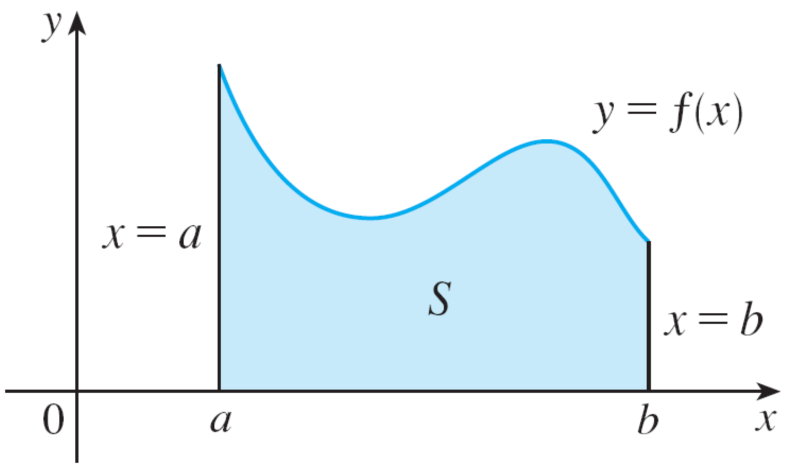
4.1 Areas and Distances

The Area Problem

The Area Problems

We begin by attempting to solve the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.



$S = \{(x, y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}$

Figure 1

The Area Problems

For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height.

The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

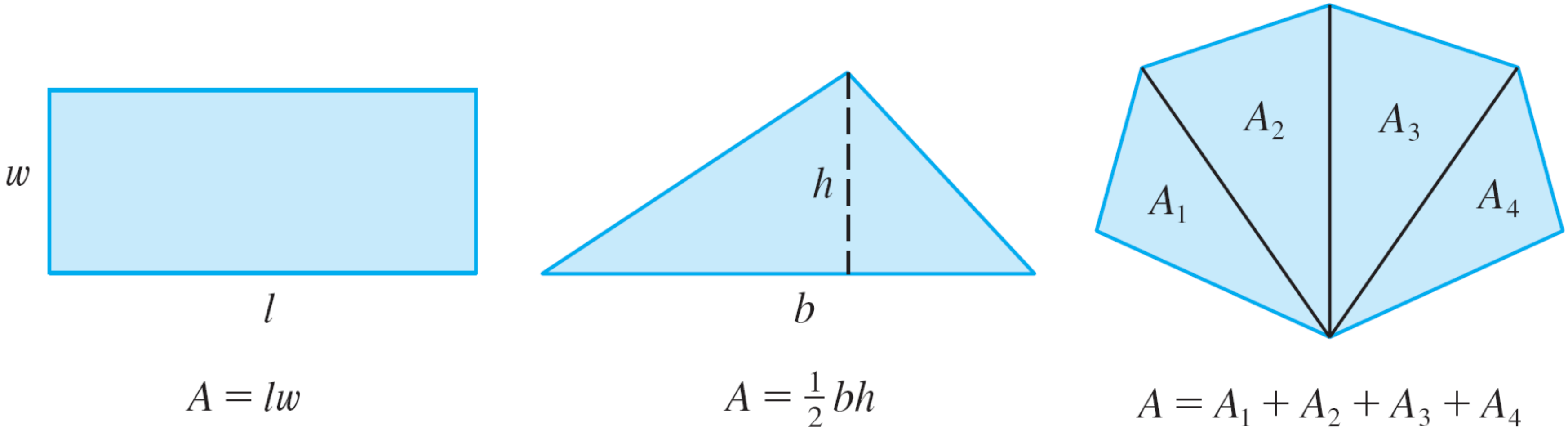


Figure 2

The Area Problems

However, it isn't so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations.

We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

Examples 1 and 2

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to t, where t>0 fixed (the parabolic region S illustrated in Figure 3 for t=1).

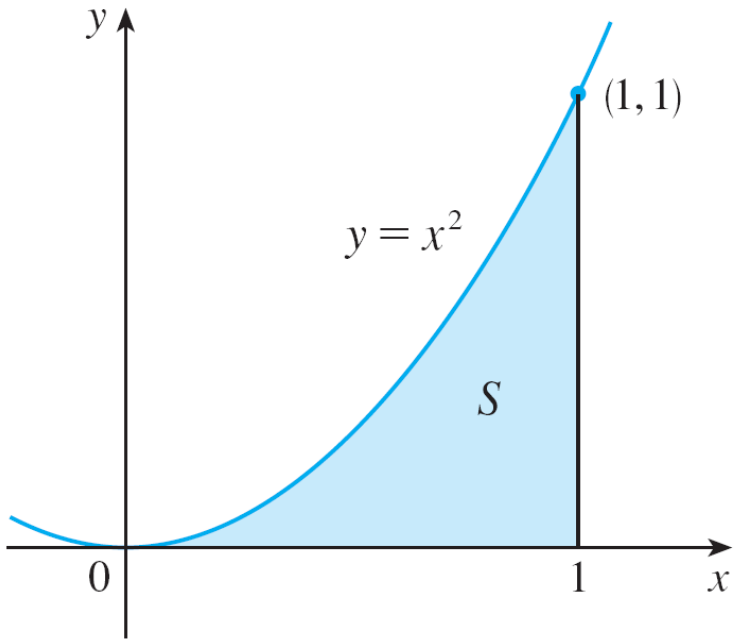
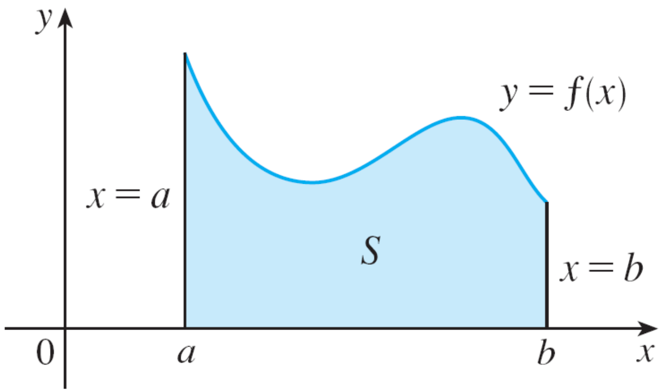


Figure 3

The Area Problems

Let's apply the idea of Example 1 to the more general region *S* of Figure 1. We start by subdividing *S* into *n* strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.



$S = \{(x, y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}$

Figure 1

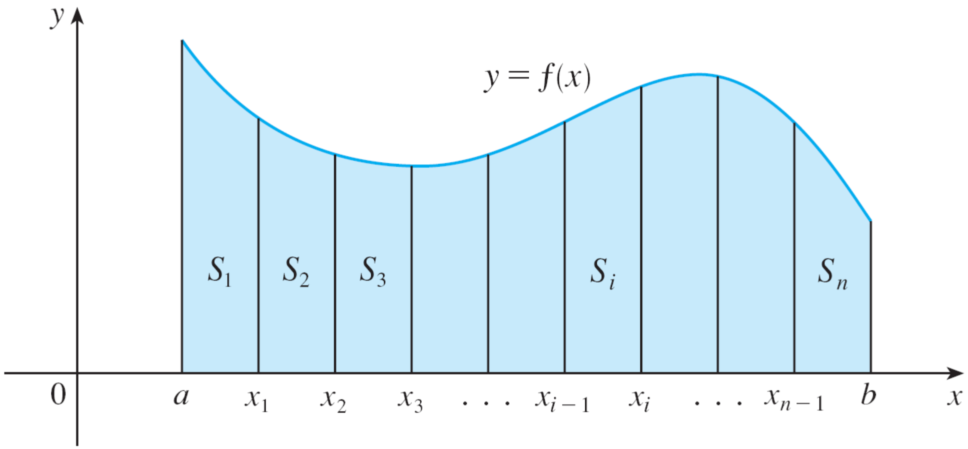


Figure 10

The width of the interval [*a*, *b*] is *b* – *a*, so the width of each of the *n* strips is

$$\Delta x = \frac{b - a}{n}$$

The Area Problems

These strips divide the interval [*a*, *b*] into *n* subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x,$$

$$x_2 = a + 2\,\Delta x,$$

$$x_3 = a + 3\,\Delta x,$$

$$\vdots$$

The Area Problems

Let's approximate the *i*th strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of *f* at the right endpoint (see Figure 11).

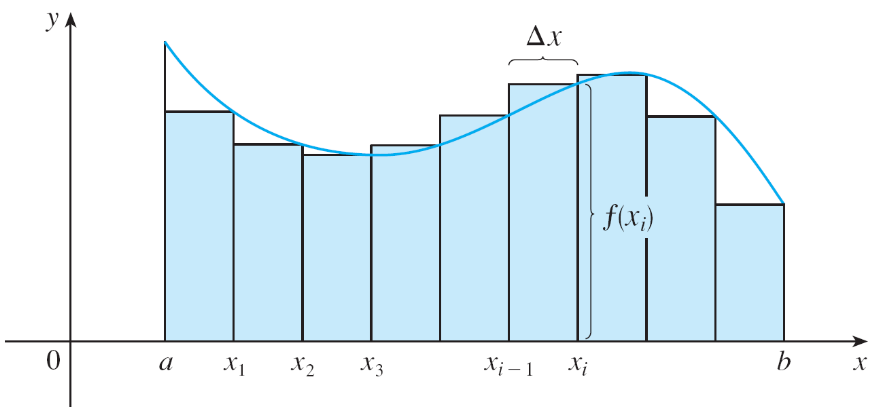


Figure 11

Then the area of the *i*th rectangle is $f(x_i)\,\Delta x$. What we think of intuitively as the area of *S* is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_n)\,\Delta x$$

The Area Problems

Figure 12 shows this approximation for $n = 2, 4, 8,$ and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

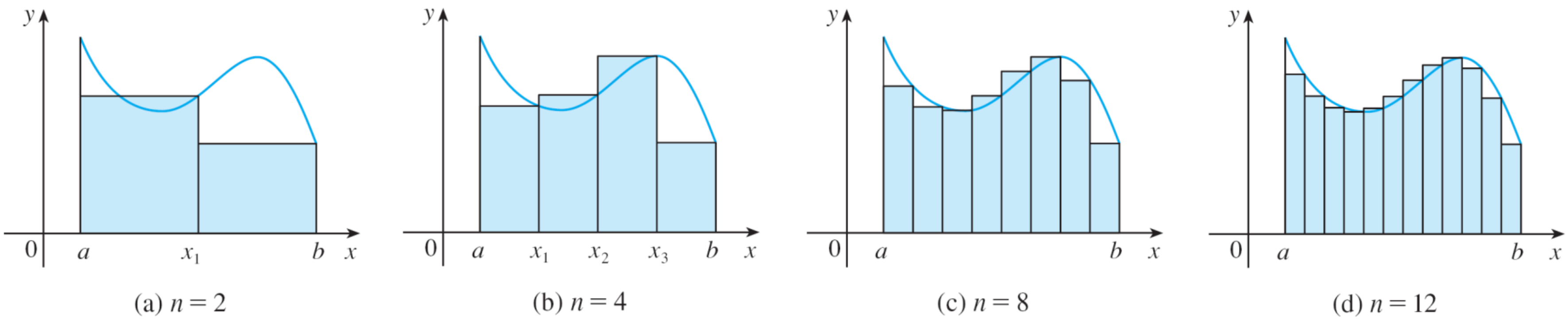


Figure 12

The Area Problems

Therefore, we define the area *A* of the region *S* in the following way.

$\boxed{2}$ **Definition** The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that *f* is continuous. It can also be shown that we get the same value if we use left endpoints:

$\boxed{3}$ $$A = \lim_{n \to \infty} L_n = \lim_{n \to \infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

The Area Problems

In fact, instead of using left endpoints or right endpoints, we could take the height of the *i*th rectangle to be the value of *f* at *any* number x_i* in the *i*th subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1*, x_2*, \ldots, x_n*$ the **sample points**.

Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints.

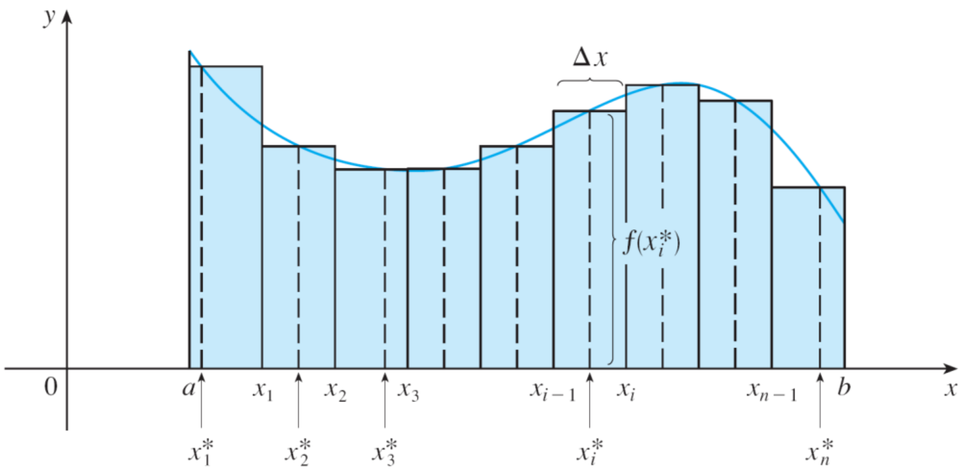


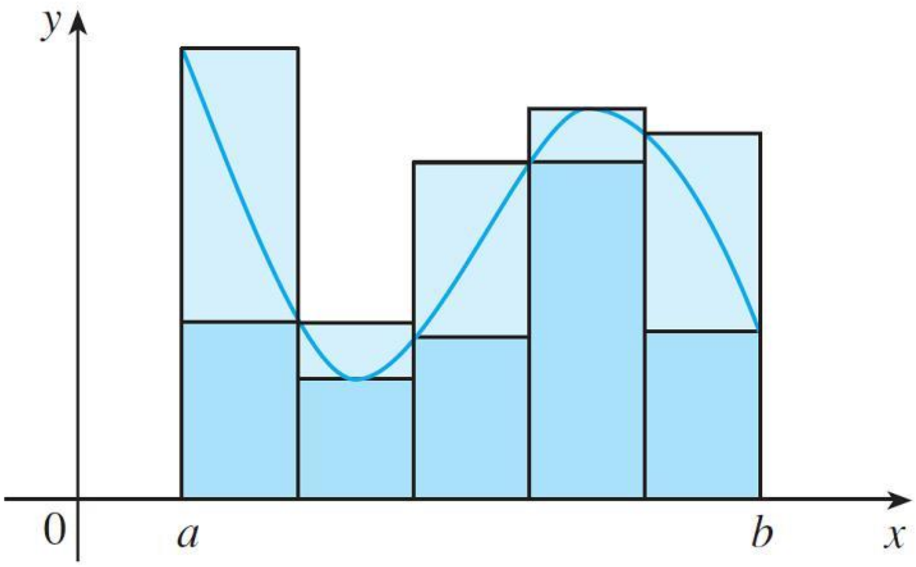
Figure 13

So a more general expression for the area of *S* is

$$\boxed{4} \qquad A = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

The Area Problems

Note: In general, we can form the so-called **lower** (and **upper**) **sums** by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of *f* on the *i*th subinterval. (See Figure 14)



Lower sums (short rectangles) and upper sums (tall rectangles)

Figure 14

The Area Problems

It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums for every* n.

The Area Problems

We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

The Distance Problem

The Distance Problems

Suppose an object moves with velocity $v = f(t)$, where $a \leq t \leq b$ and $f(t) \geq 0$ (so the object always moves in the positive direction).

We take velocity readings at times t_0 (= a), t_1, t_2,. . ., t_n (= b) so that the velocity is approximately constant on each subinterval.

If these times are equally spaced, then the time between consecutive readings is $\Delta t = (b - a)/n$. During the first time interval the velocity is approximately $f(t_0)$ and so the distance traveled is approximately $f(t_0)\,\Delta t$.

The Distance Problems

Similarly, the distance traveled during the second time interval is about $f(t_1)\,\Delta t$ and the total distance traveled during the time interval $[a, b]$ is approximately

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \cdots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

If we use the velocity at right endpoints instead of left endpoints, our estimate for the total distance becomes

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The Distance Problems

The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the *exact* distance *d* traveled is the *limit* of such expressions:

$$\boxed{5} \qquad d = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

4.2 The Definite Integral

The Definite Integral

We have seen that a limit of the form

$$\boxed{1} \quad \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

arises when we compute an area.

We also have seen that it arises when we try to find the distance traveled by an object.

It turns out that this same type of limit occurs in a wide variety of situations even when *f* is not necessarily a positive function.

The Definite Integral

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Note 1: The symbol $\int$ was introduced by Leibniz and is called an **integral sign**.

It is an elongated *S* and was chosen because an integral is a limit of sums.

Definite integral: precise definition

A function *f* defined on $[a, b]$ is integrable on $[a, b]$ if there is a number, denoted by

$$\int_a^b f(x)\, dx,$$

such that for every $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x) dx - \sum_{i=1}^{n} f(x_i^*)\, \Delta x \right| < \varepsilon$$

for all $n > N$ and for every choice of $x_i^* \in [x_{i-1,}\, x_i]$. In this case $\int_a^b f(x)\, dx$ is called the definite integral of *f* from *a* to *b*.

The Definite Integral

In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**.

For now, the symbol dx has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol.

The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

The Definite Integral

Note 2: The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on *x*. In fact, we could use any letter in place of *x* without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

Note 3: The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866).

The Definite Integral

So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

We know that if *f* happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1).

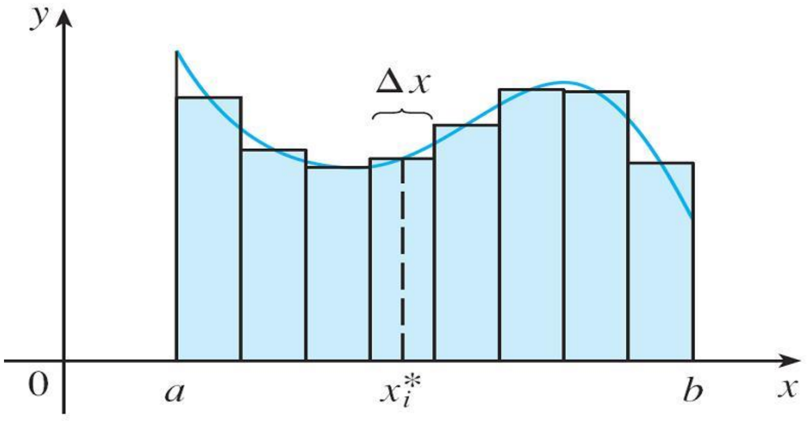


Figure 1

If $f(x) \geq 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

The Definite Integral

We see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b. (See Figure 2.)

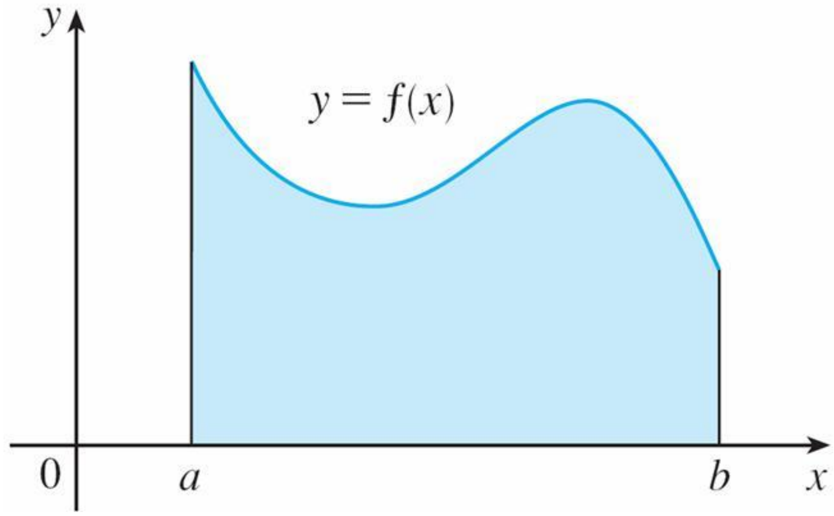


Figure 2

If $f(x) \geq 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

The Definite Integral

If *f* takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the *x*-axis and the *negatives* of the areas of the rectangles that lie below the *x*-axis (the areas of the blue rectangles *minus* the areas of the gold rectangles).

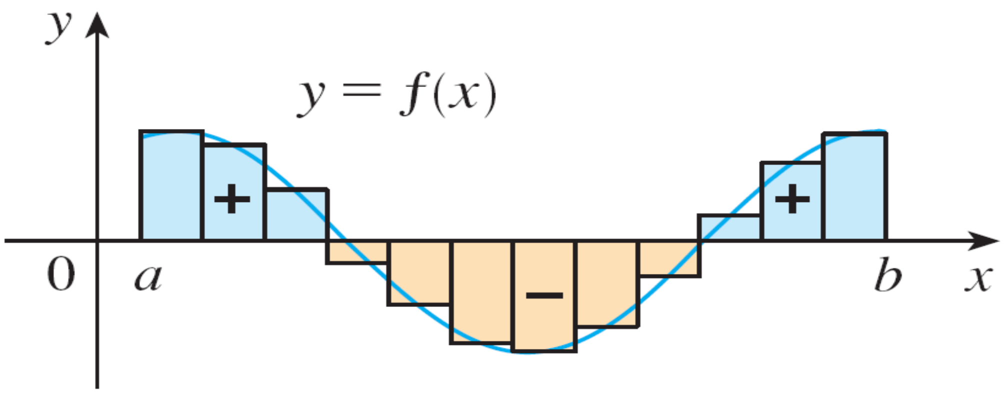


Figure 3

$\Sigma f(x_i^*)\ \Delta x$ is an approximation to the net area.

The Definite Integral

When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

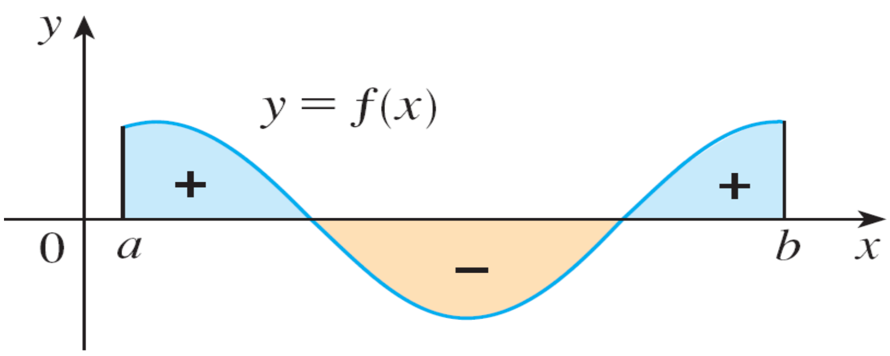


Figure 4

$\int_a^b f(x)\,dx$ is the net area.

The Definite Integral

Note 4: Although we have defined $\int_a^b f(x)\,dx$ by dividing [*a*, *b*] into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, max Δx_i, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The Definite Integral

Note 5: Not all functions are integrable. The following theorem shows that the most commonly occurring functions are in fact integrable. It is proved in more advanced courses.

> **3 Theorem** If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is integrable on $[a, b]$, then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points x_i^*.

The Definite Integral

To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

$\boxed{4}$ **Theorem** If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\quad \Delta x = \dfrac{b - a}{n} \quad$ and $\quad x_i = a + i\,\Delta x$

Note: The converse is not true. It can happen that the particular limit above exists, but *f* is not integrable on [a,b].

Recap: the Definite Integral

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \,(= a), x_1, x_2, \ldots, x_n \,(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Note: We saw that, for example, continuous functions are integrable.

The Midpoint Rule

The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the *i*th subinterval because it is convenient for computing the limit.

But if the purpose is to find an *approximation* to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\overline{x}_i$.

The Midpoint Rule

Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\,\Delta x = \Delta x\,[f(\overline{x}_1) + \cdots + f(\overline{x}_n)]$$

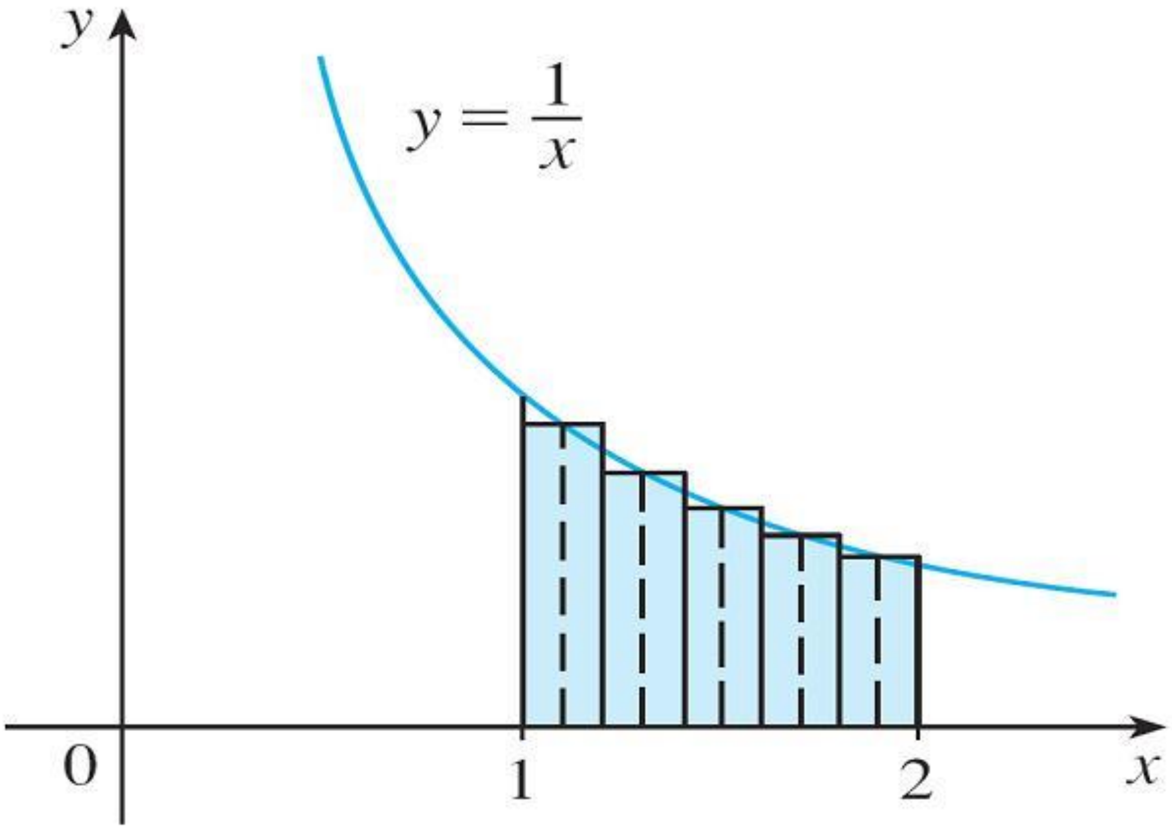
where $$\Delta x = \frac{b-a}{n}$$

and $$\overline{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

Example 5

Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\, dx$.

Example



Properties of the Definite Integral

Properties of the Definite Integral

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$.

But the definition as a limit of Riemann sums makes sense even if $a > b$.

Notice that if we reverse a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$$

Properties of the Definite Integral

If $a = b$, then $\Delta x = 0$ and so

$$\int_a^a f(x)\,dx = 0$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that f and g are continuous functions.

Propertises of the integral

Properties of the Integral

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Example 6

Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

Properties of the Definite Integral

The next property tells us how to combine integrals of the same function over adjacent intervals:

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Properties of the Definite Integral

This is not easy to prove in general, but for the case where $f(x) \geq 0$ and $a < c < b$ Property 5 can be seen from the geometric interpretation in Figure 15: The area under $y = f(x)$ from a to c plus the area from c to b is equal to the total area from a to b.

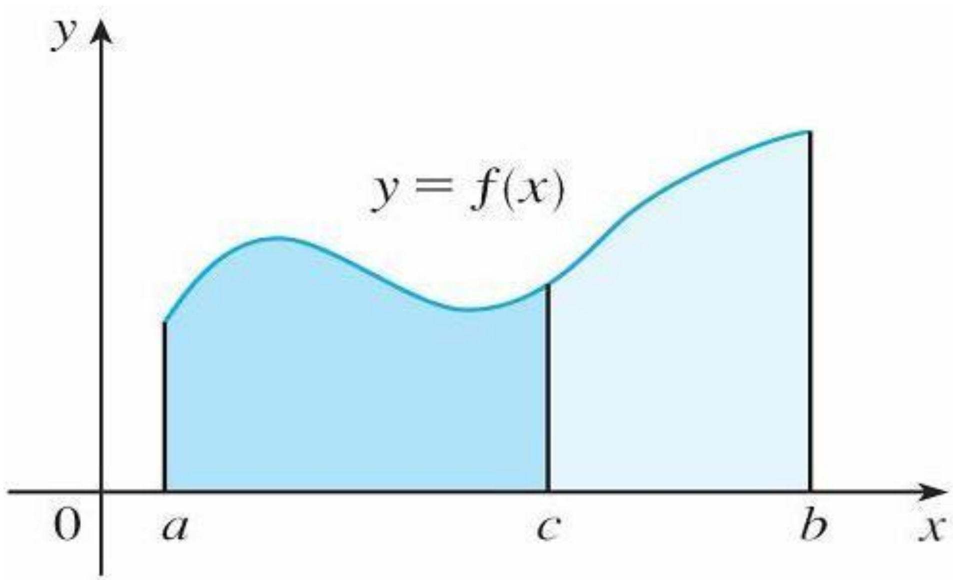


Figure 15

Properties of the Definite Integral

Properties 1–5 are true whether $a < b$, $a = b$, or $a > b$. The following properties, in which we compare sizes of functions and sizes of integrals, are true only if $a \leq b$.

Comparison Properties of the Integral

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$$

Example 9

Use Property 8 to estimate $\int_1^4 \sqrt{x}\, dx$.

4.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus.

It gives the precise inverse relationship between the derivative and the integral.

The Fundamental Theorem of Calculus

The first part of the Fundamental Theorem deals with functions defined by an equation of the form

$$\boxed{1} \qquad g(x) = \int_a^x f(t)\,dt$$

where *f* is a continuous function on [*a*, *b*] and *x* varies between *a* and *b*. Observe that *g* depends only on *x*, which appears as the variable upper limit in the integral. If *x* is a fixed number, then the integral $\int_a^x f(t)\,dt$ is a definite number.

If we then let *x* vary, the number $\int_a^x f(t)\,dt$ also varies and defines a function of *x* denoted by *g*(*x*).

The Fundamental Theorem of Calculus

If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. (Think of g as the "area so far" function; see Figure 1.)

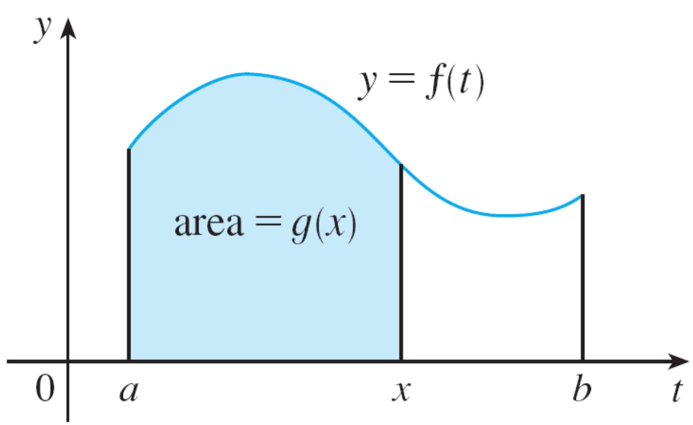


Figure 1

The Fundamental Theorem of Calculus

We already saw that if we take $f(t) = t^2$ and $a = 0$, then we have

$$g(x) = \int_0^x f(t)\,dt = \int_0^x t^2\,dt = \frac{x^3}{3}$$

Notice that $g'(x) = x^2$, that is, $g' = f$. In other words, if g is defined as the integral of f by Equation 1, then g turns out to be an antiderivative of f, at least in this case.

The Fundamental Theorem of Calculus

To see why this might be generally true we consider any continuous function f with $f(x) \geq 0$. Then $g(x) = \int_a^x f(t)\,dt$ can be interpreted as the area under the graph of f from a to x, as in Figure 1.

In order to compute $g'(x)$ from the definition of a derivative we first observe that, for $h > 0$, $g(x + h) - g(x)$ is obtained by subtracting areas, so it is the area under the graph of f from x to $x + h$ (the blue area in Figure 5).

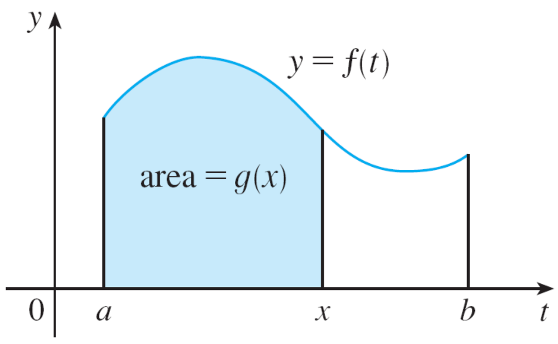


Figure 1

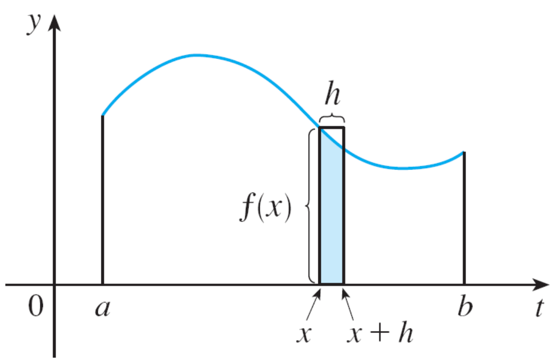


Figure 5

The Fundamental Theorem of Calculus

For small *h* you can see from the figure that this area is approximately equal to the area of the rectangle with height $f(x)$ and width *h*:

$$g(x + h) - g(x) \approx hf(x)$$

so

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

The Fundamental Theorem of Calculus

The fact that this is true, even when *f* is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Using Leibniz notation for derivatives, we can write this theorem as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

when *f* is continuous.

The Fundamental Theorem of Calculus

Roughly speaking, this equation says that if we first integrate *f* and then differentiate the result, we get back to the original function *f*.

Example 2

Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

Example 3

Although a formula of the form $g(x) = \int_a^x f(t)\,dt$ may seem like a strange way of defining a function, books on physics, chemistry, and statistics are full of such functions. For instance, the **Fresnel function**

$$S(x) = \int_0^x \sin(\pi t^2/2)\,dt$$

is named after the French physicist Augustin Fresnel (1788–1827), who is famous for his works in optics.

This function first appeared in Fresnel's theory of the diffraction of light waves, but more recently it has been applied to the design of highways.

The Fundamental Theorem of Calculus

The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

More precisely, *F* should be continuous on $[a, b]$, differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$.

Differentiation and Integration as Inverse Processes

Differentiation and Integration as Inverse Processes

We end this section by bringing together the two parts of the Fundamental Theorem.

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We noted that Part 1 can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

which says that if *f* is integrated and then the result is differentiated, we arrive back at the original function *f*.

Differentiation and Integration as Inverse Processes

Since $F'(x) = f(x)$, Part 2 can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This version says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function F, but in the form $F(b) – F(a)$.

Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are inverse processes. Each undoes what the other does.

Recap: The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\, dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We often use the notation

$$F(b) - F(a) = F(x)]_a^b$$

Other common notations are:

$$[F(x)]_a^b$$

or

$$F(x)|_a^b.$$

Examples

Example 4. Find

$$\frac{d}{dx}\int_{1}^{x^4} \sec t\, dt\,.$$

Example 5. Evaluate

$$\int_{-2}^{1} x^3 dx.$$

Example 6. Find the area under the parabola $y = x^2$ for $0 \leq x \leq 1$.

Examples

Example 7. Find the area under the cosine curve from $x = 0$ to $x = b$, where $0 \leq b \leq \frac{\pi}{2}$.

Example 8. What is wrong with the following calculation:

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\bigg]_{-1}^{3} = -\left(\frac{1}{3} - (-1)\right) = -\frac{4}{3}.$$

4.4 Indefinite Integrals and the Net Change Theorem

Indefinite Integrals and the Net Change Theorem

In this section we introduce a notation for antiderivatives, review the formulas for antiderivatives, and use them to evaluate definite integrals.

We also reformulate FTC2 in a way that makes it easier to apply to science and engineering problems.

Indefinite Integrals

Indefinite Integrals

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if *f* is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of *f*. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating *F*(*b*) – *F*(*a*), where *F* is an antiderivative of *f*.

We need a convenient notation for antiderivatives that makes them easy to work with.

Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of *f* and is called an **indefinite integral.**

Indefinite Integrals

Thus

$$\boxed{\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)}$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire *family* of functions (one antiderivative for each value of the constant *C*).

Indefinite Integrals

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*, whereas an indefinite integral $\int f(x)\,dx$ is a *function* (or family of functions).

The connection between them is given by Part 2 of the Fundamental Theorem:

If *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions.

Indefinite Integrals

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x\,\tan x\,dx = \sec x + C \qquad \int \csc x\,\cot x\,dx = -\csc x + C$$

The most general antiderivative *on a given interval* is obtained by adding a constant to a particular antiderivative.

Indefinite Integrals

Any formula can be verified by differentiating the function on the right side and obtaining the integrand.

For instance,

$$\int \sec^2 x \, dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

Indefinite Integrals

We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.

Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$.

Indefinite Integrals

This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\[2ex] -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

Examples

Example 1. Find the most general indefinite integral

$$\int 10x^4 - 2\sec^2 x \, dx.$$

Example 2. Evaluate $\int \frac{\cos\theta}{\sin^2\theta} d\theta$.

Example 3. Evaluate

$$\int_0^3 (x^3 - 6x) dx.$$

Examples

Example 4. Find

$$\int_{0}^{12} (x - 12\sin x)dx\,.$$

Example 5. Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt.$

Applications

Applications

Part 2 of the Fundamental Theorem says that if *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where *F* is any antiderivative of *f*. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Applications

We know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) - F(a)$ is the change in y when x changes from a to b.

[Note that y could, for instance, increase, then decrease, then increase again.

Although y might change in both directions, $F(b) - F(a)$ represents the *net* change in y.]

Applications

So we can reformulate FTC2 in words as follows.

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in the natural and social sciences. Here are a few instances of this idea:

• If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into (or out of) the reservoir at time t.

Applications

So

$$\int_{t_1}^{t_2} V'(t)\, dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If [C](t) is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[C]/dt$.

Applications

So

$$\int_{t_1}^{t_2} \frac{d[\mathrm{C}]}{dt}\, dt = [\mathrm{C}](t_2) - [\mathrm{C}](t_1)$$

is the change in the concentration of C from time t_1 to time t_2.

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\, dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

Applications

- If the rate of growth of a population is *dn*/*dt*, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\,dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2.

(The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

Applications

- If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$.

  So

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

  is the increase in cost when production is increased from x_1 units to x_2 units.

Applications

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\boxed{2} \qquad \int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2.

This was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

Applications

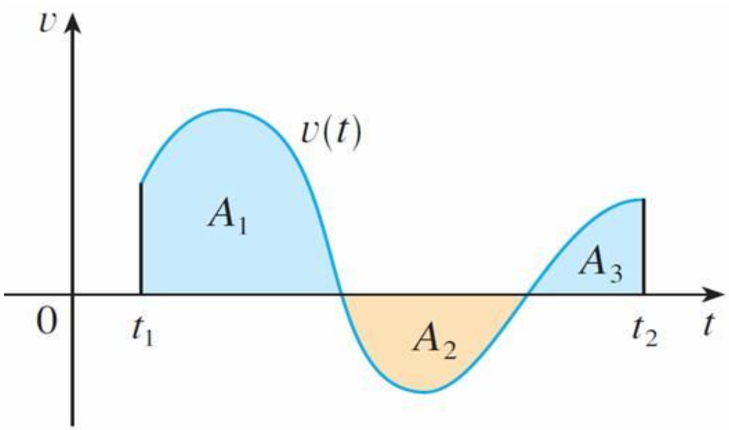
- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left).

  In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

$$\boxed{3} \qquad \int_{t_1}^{t_2} |v(t)|\, dt = \text{total distance traveled}$$

Applications

Figure 3 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

Figure 3

Applications

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\, dt = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2.

Example 6

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.

4.5 The Substitution Rule

The Substitution Rule

Because of the Fundamental Theorem, it’s important to be able to find antiderivatives.

But our antidifferentiation formulas don’t tell us how to evaluate integrals such as

$$\boxed{1} \qquad \int 2x\sqrt{1 + x^2}\, dx$$

To find this integral we use the problem-solving strategy of *introducing a new variable*; we change from the variable *x* to a new variable *u*.

The Substitution Rule

Suppose that we let u be the quantity under the root sign in $\boxed{1}$, $u = 1 + x^2$. Then the differential of u is $du = 2x\,dx$.

Notice that if the dx in the notation for an integral were to be interpreted as a differential, then the differential $2x\,dx$ would occur in $\boxed{1}$ and so, formally, without justifying our calculation, we could write

$$\boxed{2} \qquad \int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du$$

$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(x^2+1)^{3/2} + C$$

The Substitution Rule

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(x^2 + 1)^{3/2} + C\right] = \tfrac{2}{3} \cdot \tfrac{3}{2}(x^2 + 1)^{1/2} \cdot 2x = 2x\sqrt{x^2 + 1}$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))g'(x)\,dx$.

The Substitution Rule

$\boxed{4}$ **The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation.

Notice also that if $u = g(x)$, then $du = g'(x)\,dx$, so a way to remember the Substitution Rule is to think of dx and du in $\boxed{4}$ as differentials.

The Substitution Rule

Thus the Substitution Rule says: **It is permissible to operate with *dx* and *du* after integral signs as if they were differentials.**

Examples

Example 1. Find $\int x^3 \cos(x^4 + 2)\, dx$.

Example 2. Evaluate $\int \sqrt{2x+1}\, dx$.

Example 3. Find

$$\int \frac{x}{\sqrt{1-4x^2}}\, dx.$$

Example 4. Evaluate $\int \cos 5x\ dx$.

Example 5. Evaluate $\int \sqrt{1+x^2}\, x^5\, dx$.

Definite Integrals

Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem.

For example,

$$\int_0^4 \sqrt{2x + 1}\, dx = \int \sqrt{2x + 1}\, dx\Big]_0^4 = \tfrac{1}{3}(2x + 1)^{3/2}\Big]_0^4$$

$$= \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2} = \tfrac{1}{3}(27 - 1) = \tfrac{26}{3}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

Definite Integrals

5 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Examples

Example 6. Evaluate $\int_0^4 \sqrt{2x+1}\,dx$.

Example 7. Evaluate

$$\int_1^2 \frac{dx}{(3-5x)^2}.$$

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Examples

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Example 7. Evaluate

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6.1 Calculus of Inverse Functions

Inverse Functions

Recall the definition of inverse function.

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if f maps x into y, then f^{-1} maps y back into x. (If were not one-to-one, then f^{-1} would not be uniquely defined.)

Inverse Functions

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f.

Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.

But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

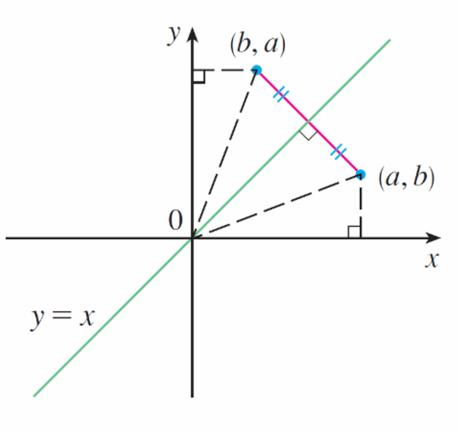


Figure 8

Inverse Functions

Therefore, as illustrated by Figure 9:

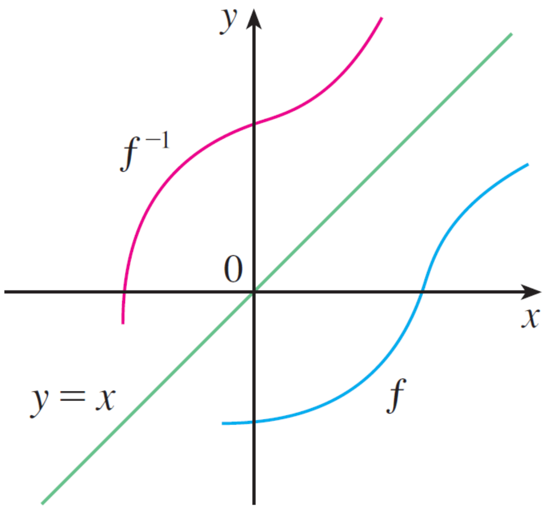


Figure 9

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

The Calculus of Inverse Functions

The Calculus of Inverse Functions

Now let's look at inverse functions from the point of view of calculus. Suppose that f is both one-to-one and continuous. We think of a continuous function as one whose graph has no break in it. (It consists of just one piece.)

Since the graph of f^{-1} is obtained from the graph of f by reflecting about the line $y = x$, the graph of f^{-1} has no break in it. Thus we might expect that f^{-1} is also a continuous function.

The Calculus of Inverse Functions

This geometrical argument does not prove the following theorem but at least it makes the theorem plausible.

> **6 Theorem** If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

For a rigorous proof see Appendix F.

Now suppose that *f* is a one-to-one differentiable function. Geometrically we can think of a differentiable function as one whose graph has no corner or kink in it.

The Calculus of Inverse Functions

We get the graph of f^{-1} by reflecting the graph of f about the line $y = x$, so the graph of f^{-1} has no corner or kink in it either.

We therefore expect that f^{-1} is also differentiable (except where its tangents are vertical). In fact, we can predict the value of the derivative of f^{-1} at a given point by a geometric argument.

The Calculus of Inverse Functions

In Figure 11 the graphs of f and its inverse f^{-1} are shown. If $f(b) = a$, then $f^{-1}(a) = b$ and $(f^{-1})'(a)$ is the slope of the tangent line L to the graph of f^{-1} at (a, b), which is $\Delta y/\Delta x$.

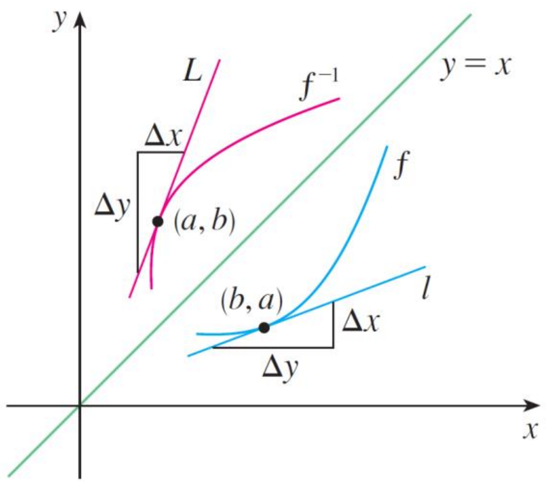


Figure 11

Reflecting in the line $y = x$ has the effect of interchanging the x- and y-coordinates.

The Calculus of Inverse Functions

So the slope of the reflected line [the tangent to the graph of f at (b, a) is $\Delta x/\Delta y$. Thus the slope of L is the reciprocal of the slope of l, that is,

$$(f^{-1})'(a) = \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x/\Delta y} = \frac{1}{f'(b)}$$

7 Theorem If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

The Calculus of Inverse Functions

Note 1:

Replacing *a* by the general number *x* in the formula of Theorem 7, we get

$$\boxed{8} \qquad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

If we write $y = f^{-1}(x)$, then $f(y) = x$, so Equation 8, when expressed in Leibniz notation, becomes

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

The Calculus of Inverse Functions

Note 2:
If it is known in advance that f^{-1} is differentiable, then its derivative can be computed more easily than in the proof of Theorem 7 by using implicit differentiation. If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating the equation $f(y) = x$ implicitly with respect to x, remembering that y is a function of x, and using the Chain Rule, we get

$$f'(y)\,\frac{dy}{dx} = 1$$

Therefore $$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{\dfrac{dx}{dy}}$$

Example 7

If $f(x) = 2x + \cos x$, find $(f^{-1})'(1)$.

6.2* The natural logarithmic function

The natural logarithmic function

We start with the definition.

$\boxed{1}$ **Definition** The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

- If x >1, then ln x can be interpreted geometrically as the area under the hyperbola y = 1/t from t = 1 to t = x.
- For 0 < x < 1, $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$. Thus ln x can be interpreted geometrically as the negative of the area under the hyperbola y = 1/t from t = x to t = *1*.
- $\ln 1 = 0$.

The natural logartithmic function

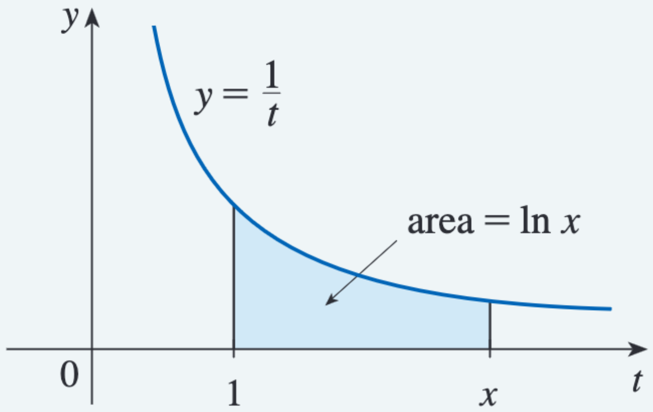


FIGURE 1

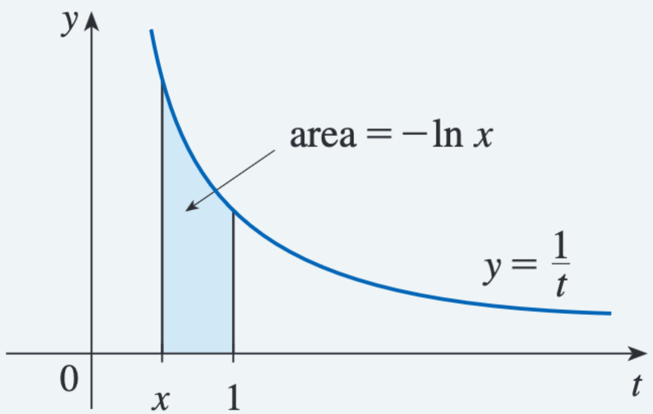


FIGURE 2

Properties

Theorem.

- $\frac{d}{dx}\ln x = \frac{1}{x}, x > 0$
- $\ln(xy) = \ln x + \ln y, x, y > 0$
- $\ln\frac{x}{y} = \ln x - \ln y, x, y > 0$
- $\ln x^r = r\ln x, x > 0, r$ rational

Example 2

Use the laws of logarithms to expand the expression

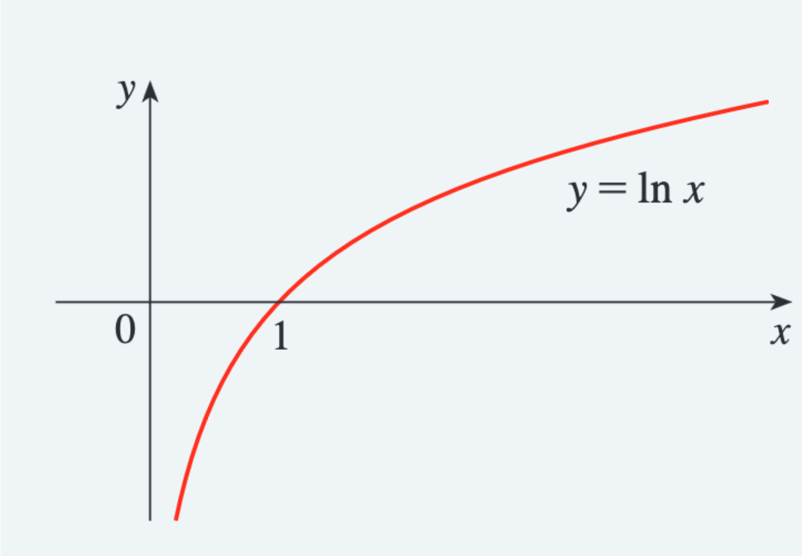
$$\ln \frac{(x^2 + 5)^4 \sin x}{x^3 + 1}$$

Limits and graph

Theorem.

- $\ln x$ is increasing a concave down
- $\lim_{x \to \infty} \ln x = \infty$
- $\lim_{x \to 0+} \ln x = -\infty$

Hence its graph looks like:

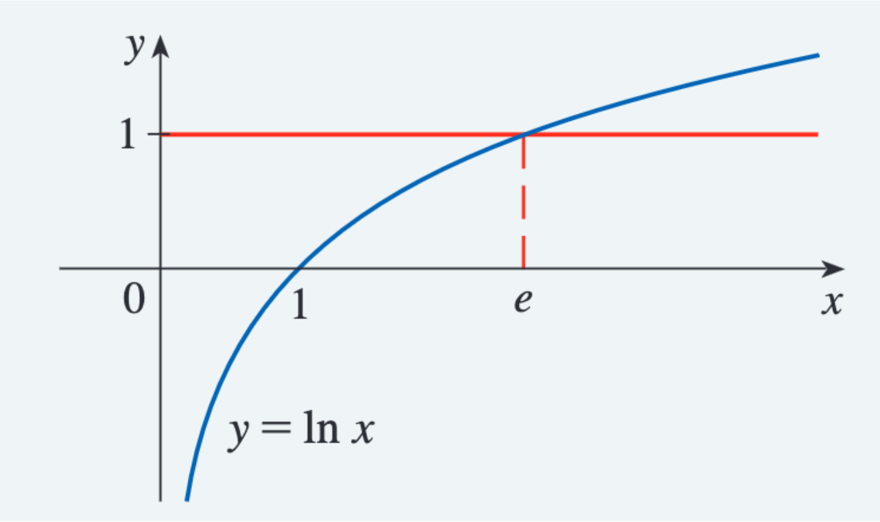


The number *e*

Since ln 1 = 0 and ln *x* is an increasing continuous function that takes on arbitrarily large values, the Intermediate Value Theorem shows that there is a number where ln *x* takes on the value 1. This important number is denoted by *e*.

5 Definition e is the number such that $\ln e = 1$.

Note: $e \approx 2.7182818253$



Examples

Example 5. Differentiate $y = \ln(x^3 + 1)$.

Example 7. Differentiate $y = \sqrt{\ln x}$

Example 8. Differentiate $y = \ln \frac{x+1}{\sqrt{x-2}}$.

Example 10. Differentiate $f(x) = \ln|x|$

Integrals

Example 10 shows that

8 $$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

Example 11. Evaluate $\int \frac{x}{x^2+1}\,dx$.

Example 12. Evaluate $\int_1^e \frac{\ln x}{x}\,dx$.

Example 13. Evaluate $\int \tan x\,dx$.

6.3* The natural exponential function

The natural exponential function

Since ln is an increasing function, it is one-to-one and therefore has an inverse function, which we denote by exp. Thus, according to the definition of an inverse function.

1 $$\boxed{\exp(x) = y \iff \ln y = x}$$

Cancellation equations:

- $\exp(\ln x) = x$ for all $x > 0$
- $\ln(\exp x) = x$ for all $x \in \mathbb{R}$

Elementary Properties

Theorem.

- $\exp 0 = 1$
- $\exp 1 = e$
- If r is a rational number, then $\exp r = e^r$

Because of the last property we will define, for irrational values of *x*, the number e^x by

$$e^x = \exp x$$

Elementary properties

With this notation we have

$$e^x = y \Leftrightarrow \ln y = x$$

and the cancellation equations:

- $e^{\ln x} = x$ for all $x > 0$
- $\ln e^x = x$ for all $x \in \mathbb{R}$

Examples

Example 1. Find x if $\ln x = 5$.

Example 2. Solve the equation $e^{5-3x} = 10$

Further properties

Theorem.

- Domain of $f(x) = e^x$ is $(-\infty, \infty)$, the range of exp in $(0, \infty)$
- The function $f(x) = e^x$ continuous and increasing
- $\lim_{x \to \infty} e^x = \infty$
- $\lim_{x \to -\infty} e^x = 0$
- Laws of exponents:

  a) $e^{x+y} = e^x e^y$, for all $x, y \in (-\infty, \infty)$

  b) $e^{x-y} = \frac{e^x}{e^y}$, for all $x, y \in (-\infty, \infty)$

  c) $(e^x)^r = e^{rx}$, for all $x \in (-\infty, \infty)$ and r rational

Graph of the natural exp function

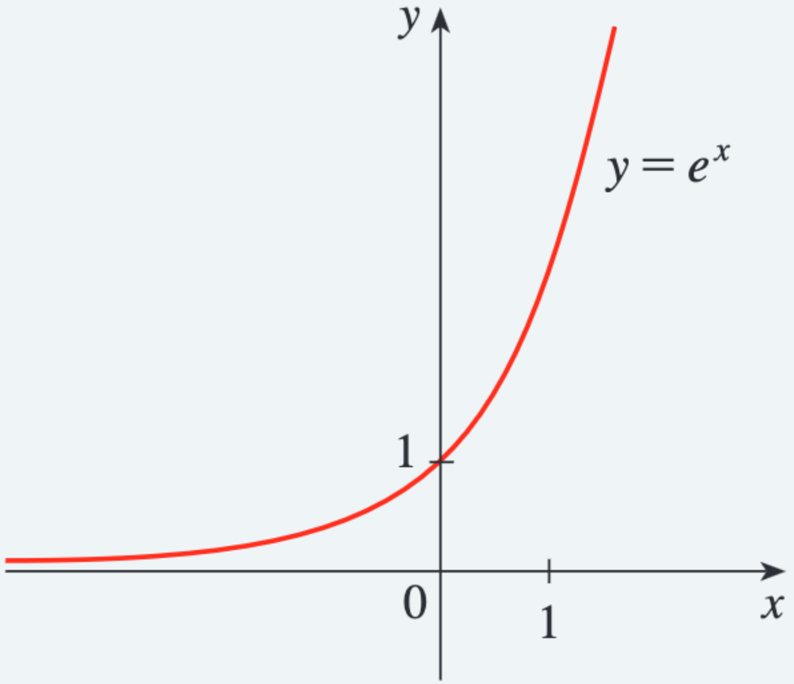


FIGURE 2
The natural exponential function

Example 3

Example 3. Find

$$\lim_{x\to\infty} \frac{e^{2x}}{e^{2x}+1}$$

Differentiation and integration

Theorem.

- $\frac{d}{dx} e^x = e^x$

- $\int e^x \, dx = e^x + C$

Examples

Example 4. Differentiate $y = e^{\tan x}$.

Example 6. Find the absolute maximum value of the function $f(x) = xe^{-x}$.

Example 8. Evaluate $\int x^2 e^{x^3}\, dx$.

Example 9. Find the area under the curve $y = e^{-3x}$ from 0 to 1.

6.4* General logarithmic and exponential functions

General exponential functions

Let $b > 0$. We saw in the previous section that if r is a rational number, then

$$b^r = (e^{\ln b})^r = e^{r \ln b}.$$

Therfore, we define for every irrational number *x*

$$b^x = e^{x \ln b}.$$

For example: $2^{\sqrt{3}} = e^{\sqrt{3} \ln 2} \approx 3.32$

Properties

Theorem. Let $b > 0$. Then

- $\ln b^r = r \ln b$ for all $r \in (-\infty, \infty)$ (so not just for rational r)
- *Laws of exponents:* let $a, b > 0$ and $x, y \in (-\infty, \infty)$. Then

  a) $b^{x+y} = b^x b^y$

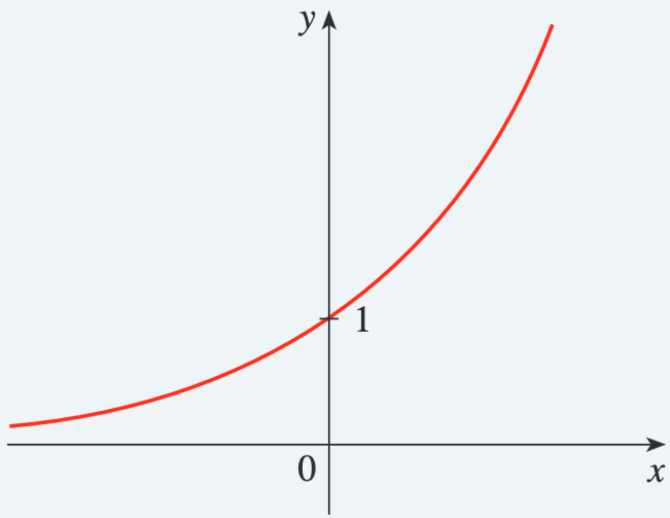
  b) $b^{x-y} = \frac{b^x}{b^y}$

  c) $(b^x)^y = b^{xy}$

  d) $a^x b^x = (ab)^x$

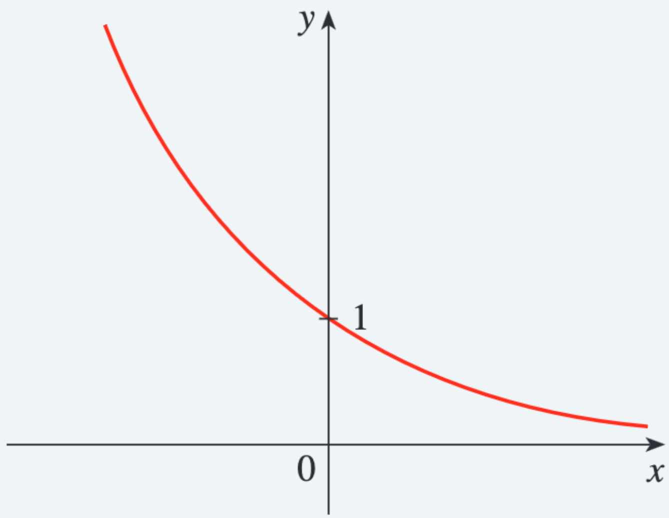
- $\frac{d}{dx} b^x = b^x \ln b$
- $\int b^x dx = \frac{b^x}{\ln b} + C,\ b \neq 1.$

Exponential graphs



$$\lim_{x \to -\infty} b^x = 0, \quad \lim_{x \to \infty} b^x = \infty$$

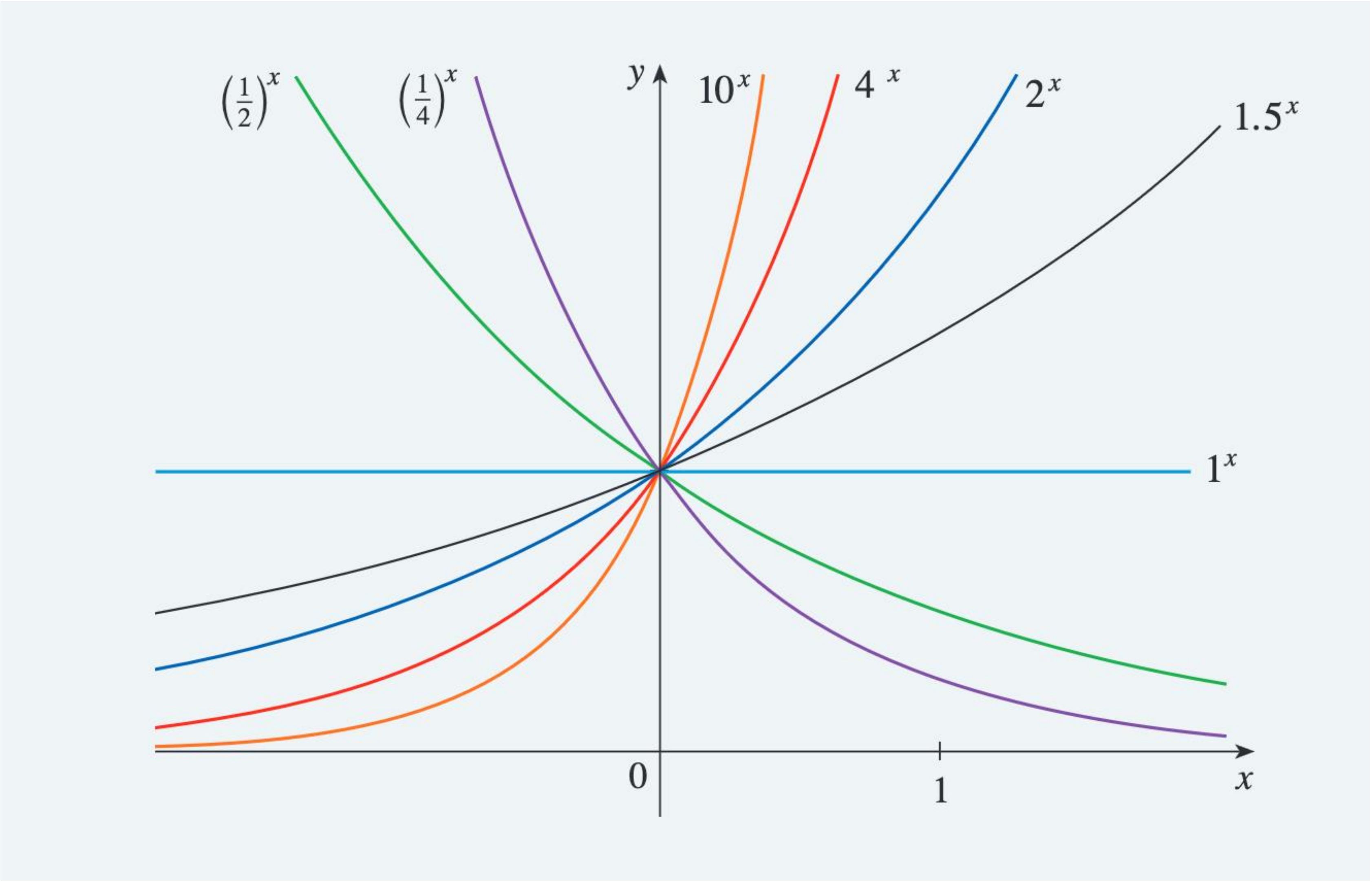
FIGURE 1 $y = b^x, \; b > 1$



$$\lim_{x \to -\infty} b^x = \infty, \quad \lim_{x \to \infty} b^x = 0$$

FIGURE 2 $y = b^x, \; 0 < b < 1$

Exponential graphs



Example 6

Example 6. Evaluate $\int_1^5 2^x\,dx$.

Proof of the general power rule

Theorem. Let $f(x) = x^n, x > 0, n \in (-\infty, \infty)$. Then

$$\frac{d}{dx} f(x) = nx^{n-1}$$

General logarithmic function

For any $b > 0$, the function $f(x) = b^x$ is monotone increasing ($b > 1$)/decreasing ($b < 1$). Therefore it is one-to-one and hence has an inverse function, **called logharithmic function** with base b, denoted by $g(x) = \log_b x$. Hence:

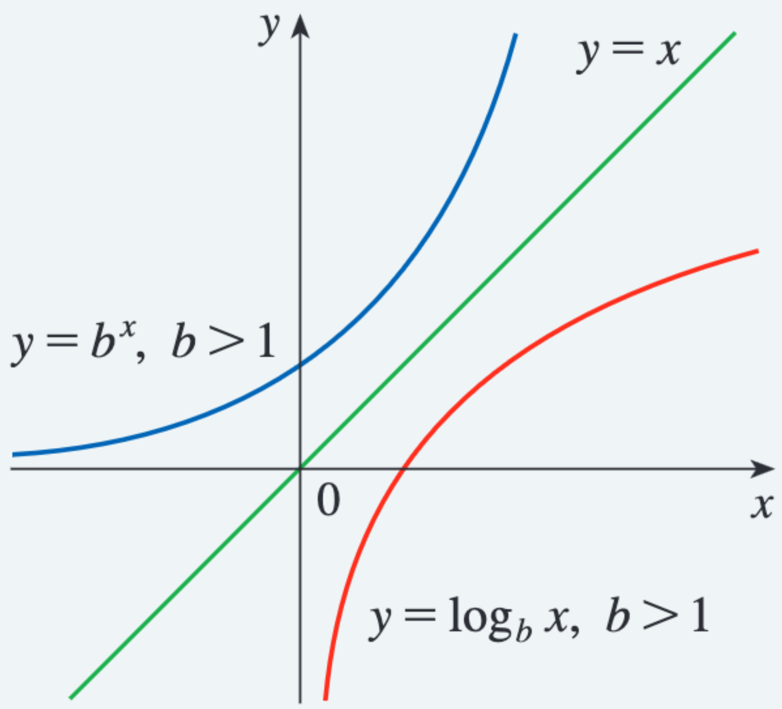
5 $$\boxed{\log_b x = y \iff b^y = x}$$

In particular: $\ln x = \log_e x$

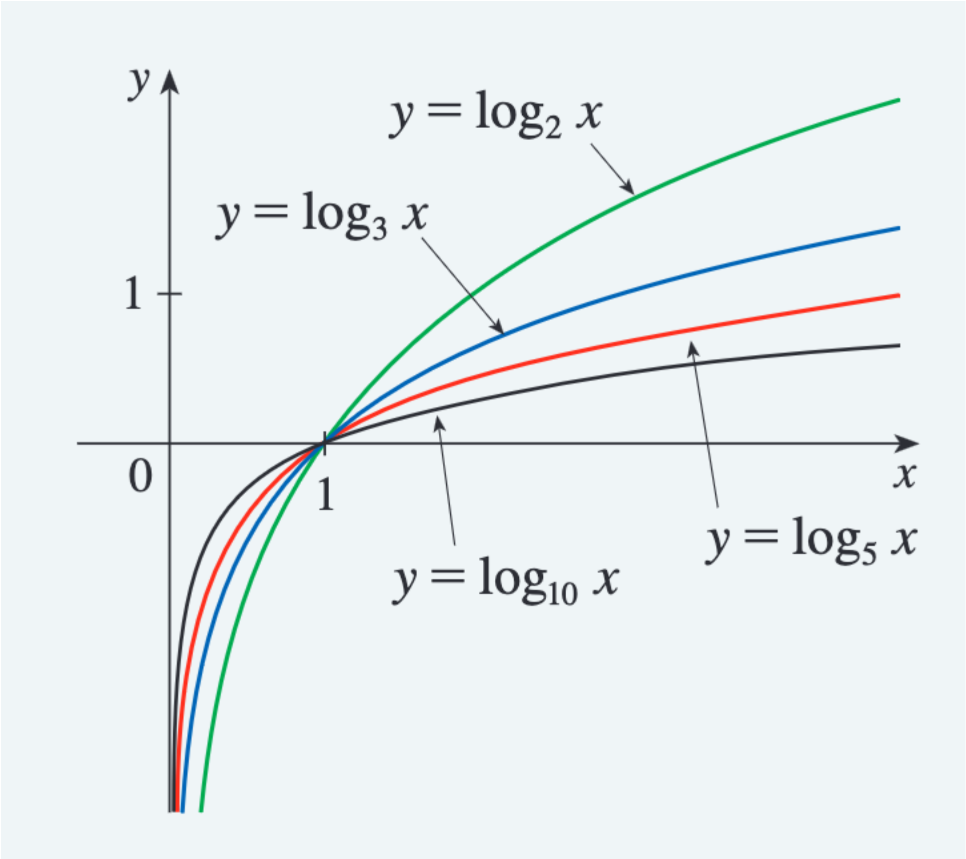
Properties of $\log_b x$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Cancellation:
  a) $\log_b b^x = x$, $x \in (-\infty, \infty)$
  b) $b^{\log_b x} = x, x > 0$.
- Increasing for $b > 1,$ decreasing for $0 < b < 1$
- Change of base: $\log_b x = \frac{\ln x}{\ln b}$
- Laws of logarithm ($x, y > 0$, r ∈ $(-\infty, \infty)$):
  a) $\log_b(xy) = \log_b x + \log_b y$
  b) $\log_b \frac{x}{y} = \log_b x - \log_b y$
  c) $\log_b x^r = r \log_b x$

Graph of $\log_b x$



Graph of $\log_b x$



Derivative of $\log_b x$

Using the cange of base formula we get:

$$\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$$

Example 9. Differentiate $\log_{10}(2+\sin x)$.

The number *e* as a limit

Theorem. We have that

$$e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}},$$

or

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n}.$$

6.6 Inverse Trigonometric Functions

Inverse Trigonometric Functions

You can see from Figure 1 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test).

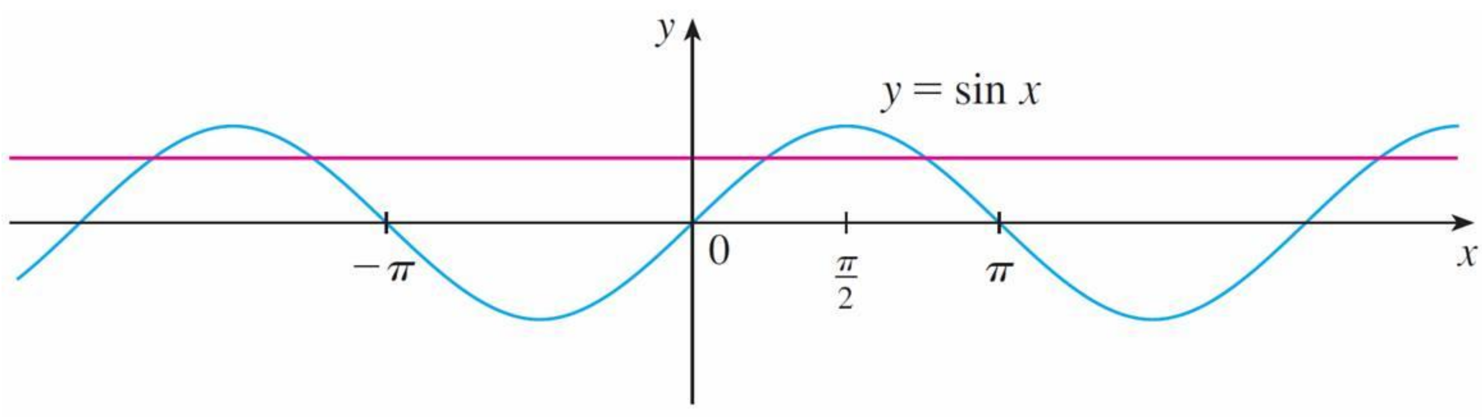
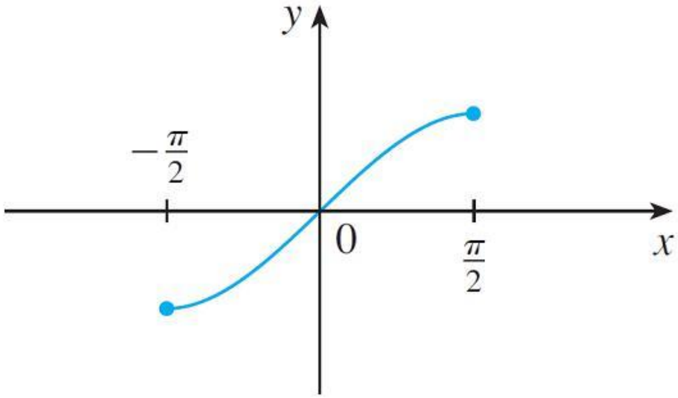


Figure 1

Inverse Trigonometric Functions

But the function $f(x) = \sin x$, $-\pi/2 \le x \le \pi/2$ *is* one-to-one (see Figure 2).The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.



$y = \sin x, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Figure 2

Inverse Trigonometric Functions

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

1

$$\boxed{\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}}$$

Thus, if $-1 \leq x \leq 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

Example 1

Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan(\arcsin \frac{1}{3})$.

$$\frac{1}{2}$$

Inverse Trigonometric Functions

The cancellation equations for inverse functions become, in this case,

$$\boxed{2} \qquad \begin{aligned} &\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ &\sin(\sin^{-1} x) = x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}$$

Example

Evaluate

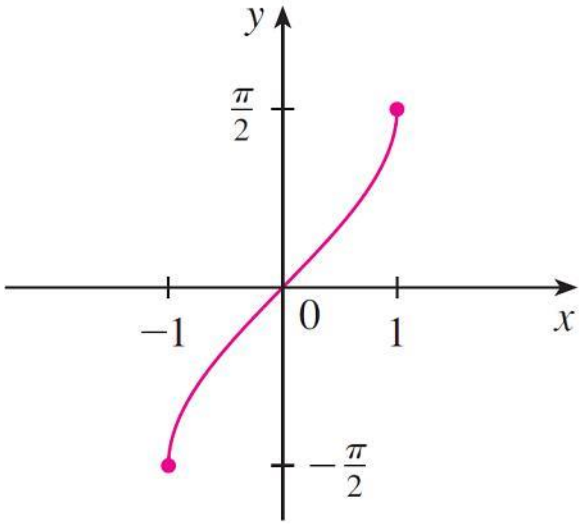
a) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

b) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$

c) $\sin(\sin^{-1} 2)$

Inverse Trigonometric Functions

The inverse sine function, $\sin^{-1}$, has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$, and its graph, shown in Figure 4, is obtained from that of the restricted sine function (Figure 2) by reflection about the line $y = x$.



$y = \sin^{-1} x = \arcsin x$

Figure 4

Derivative of $\sin^{-1} x$

Theorem. The function $f(x) = \sin^{-1} x$ is differentiable on $(-1, 1)$ and its derivative is given by

3

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1}$$

Example 2

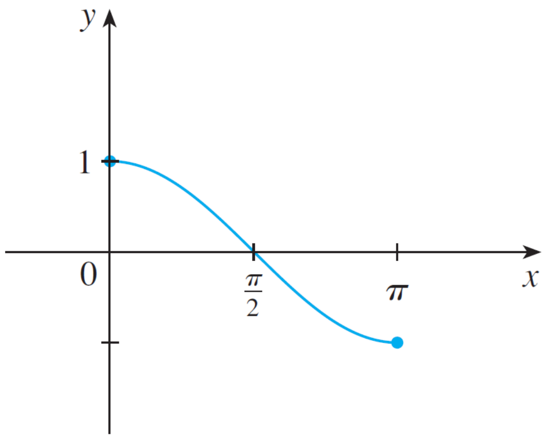
If $f(x) = \sin^{-1}(x^2 - 1)$, find (a) the domain of f, (b) $f'(x)$, and (c) the domain of f'.

Inverse Trigonometric Functions

The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x$, $0 \le x \le \pi$ is one-to-one (see Figure 6) and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

4

$$\boxed{\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi}$$



$y = \cos x$, $0 \le x \le \pi$

Figure 6

Inverse Trigonometric Functions

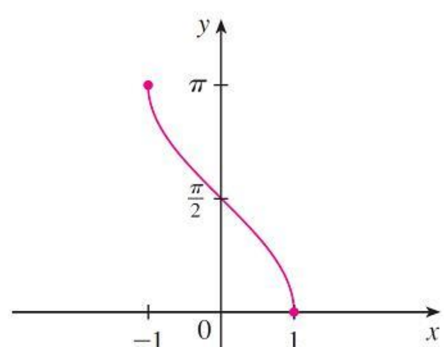
The cancellation equations are

5

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leqslant x \leqslant \pi$$

$$\cos(\cos^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

The inverse cosine function, $\cos^{-1}$, has domain [–1, 1] and range [0, π] and is a continuous function whose graph is shown in Figure 7.



$y = \cos^{-1}x = \arccos x$

Figure 7

Inverse Trigonometric Functions

Its derivative is given by

$$\boxed{6} \qquad \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

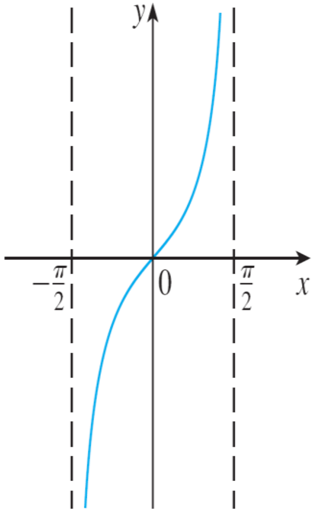
The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2\ ,\ \pi/2)$.

Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$.

Inverse Trigonometric Functions

It is denoted by $\tan^{-1}$ or arctan. (See Figure 8.)

7 $$\boxed{\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}}$$



$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

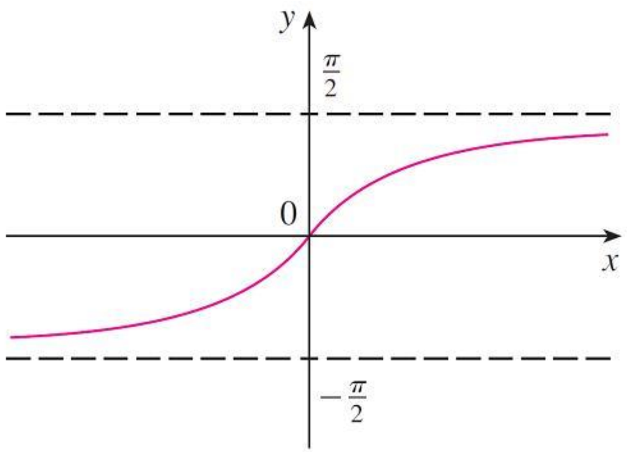
Figure 8

Example 3

Simplify the expression $\cos(\tan^{-1} x)$.

Inverse Trigonometric Functions

The inverse tangent function, $\tan^{-1}x$ = arctan, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 10.



$y = \tan^{-1}x = \arctan x$

Figure 10

Inverse Trigonometric Functions

We know that

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \quad \text{and} \quad \lim_{x \to -(\pi/2)^+} \tan x = -\infty$$

and so the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan.

Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

Inverse Trigonometric Functions

This fact is expressed by the following limits:

8

$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2} \qquad \lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Example 4

Evaluate $\lim_{x \to 2^+} \arctan\left(\dfrac{1}{x - 2}\right)$.

The derivative of $\tan^{-1} x$

Theorem. The function $f(x) = \tan^{-1} x$ is differentaiable on $(-\infty, \infty)$ and

9

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

Inverse Trigonometric Functions

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

$$\boxed{10} \quad y = \csc^{-1}x \ \ (|x| \geqslant 1) \quad \Longleftrightarrow \quad \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1}x \ \ (|x| \geqslant 1) \quad \Longleftrightarrow \quad \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$y = \cot^{-1}x \ \ (x \in \mathbb{R}) \quad \Longleftrightarrow \quad \cot y = x \quad \text{and} \quad y \in (0, \pi)$$

Inverse trigonometric functions

For example, in case of the secant function:

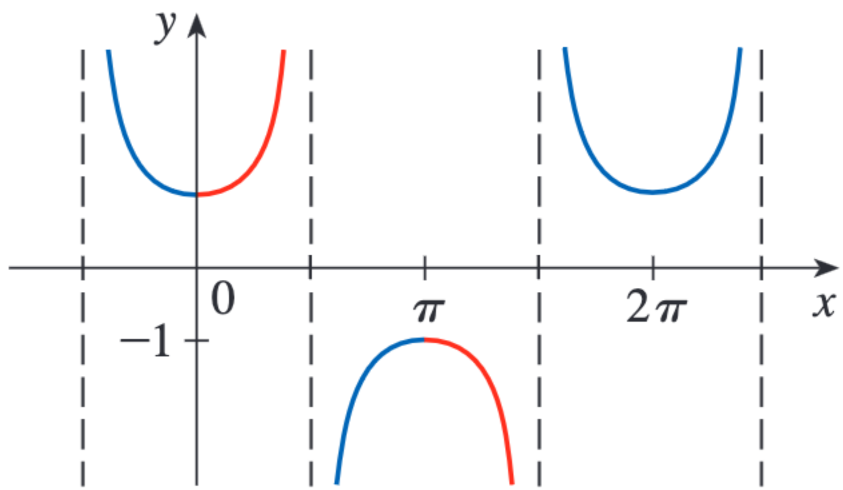


FIGURE 11
$y = \sec x$

Inverse Trigonometric Functions

We collect in Table 11 the differentiation formulas for all of the inverse trigonometric functions.

11 Table of Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

Example 5

Differentiate (a) $y = \dfrac{1}{\sin^{-1} x}$ and (b) $f(x) = x \arctan \sqrt{x}$.

Integrals od Inverse Trigonometric Functions

12 $$\boxed{\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C}$$

13 $$\boxed{\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C}$$

Example 7

Find

$$\int_0^{1/4} \frac{1}{\sqrt{1 - 4x^2}}\, dx.$$

Integrals of Inverse Trigonometric Functions

Example 8. Show that

$$\boxed{14} \qquad \boxed{\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}$$

Example 9

Find $\int \frac{x}{x^4 + 9}\, dx.$

6.7 Hyperbolic Functions

Hyperbolic Functions

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names.

In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle.

Hyperbolic Functions

For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Graphs of hyperbolic functions

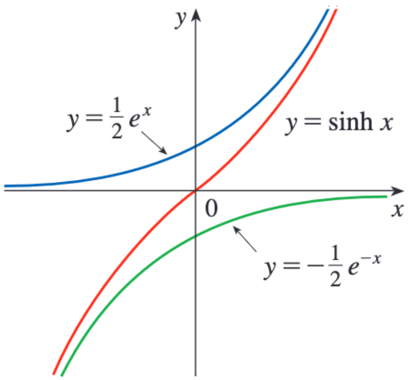


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

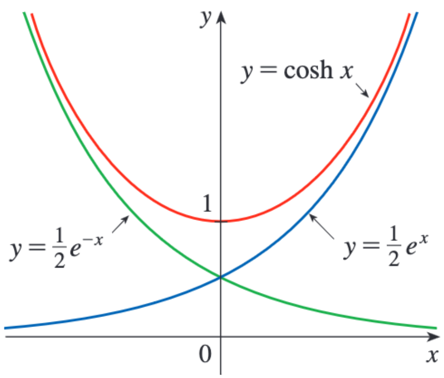


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

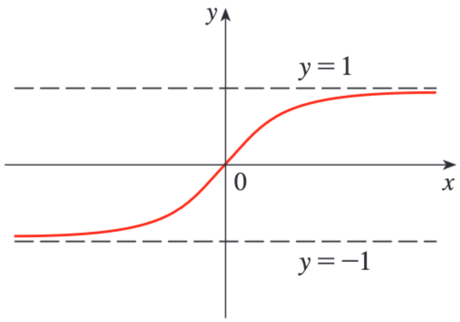
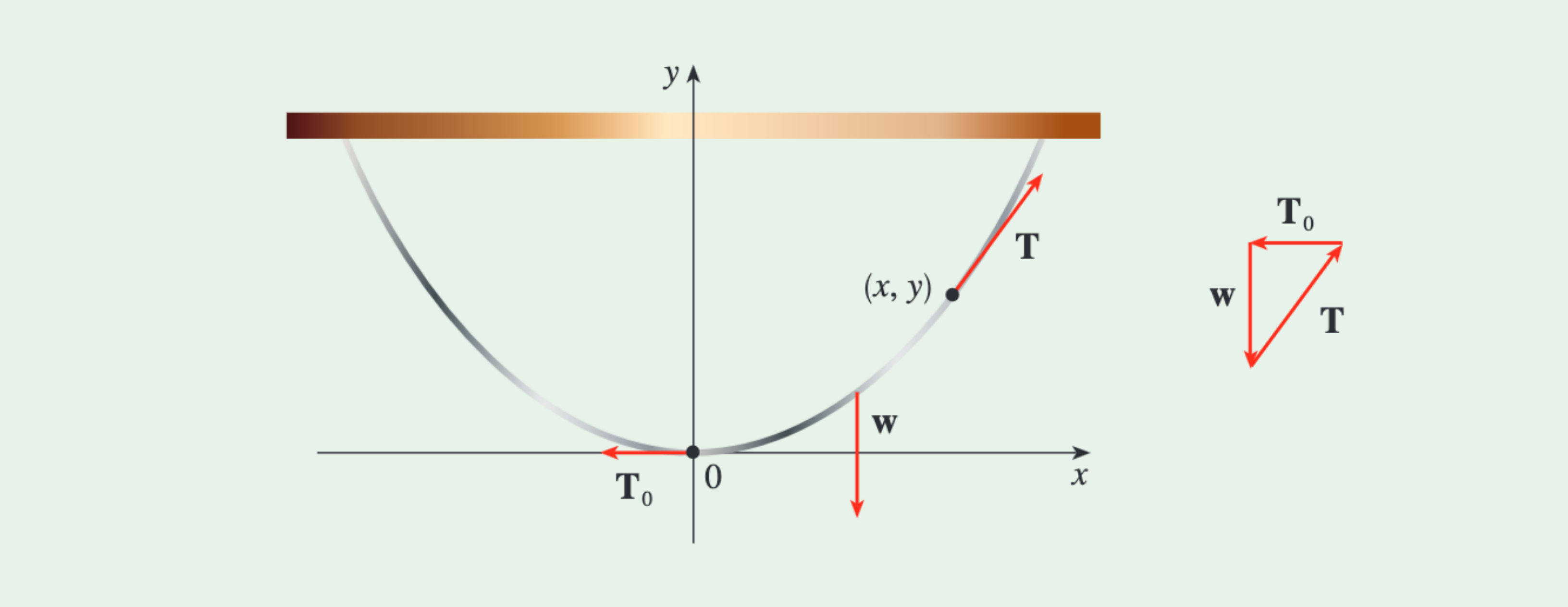


FIGURE 3
$y = \tanh x$

Famous application: catenary curve

Suppose that a chain (or cable) of uniform linear mass density ρ is hanging between two points, as shown in the figure. We place the origin at the vertex of the catenary, and let (x, y) be any point on the curve.



Famous application: catenary curve

The chain is in equlipbrium so one has

$$\boldsymbol{T_0} + \boldsymbol{T} + \boldsymbol{w} = \boldsymbol{0}.$$

One can then show that the shape of the chain is given by:

$$y = a\cosh\frac{x}{a} - a,$$

where $a = \frac{|\boldsymbol{T_0}|}{g\rho}$.

For more details: see Stewart Section 12.3 (p. 884), Discovery Project

Hyperbolic Functions

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

We list some of them here.

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Example 1

Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and

(b) $1 - \tanh^2 x = \text{sech}^2 x$.

Hyperbolic Functions

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

Hyperbolic Functions

We list the differentiation formulas for the hyperbolic functions as Table 1.

$\boxed{1}$ Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Example 2

Find $\frac{d}{dx} \cosh \sqrt{x}$.

Inverse Hyperbolic Functions

Inverse Hyperbolic Functions

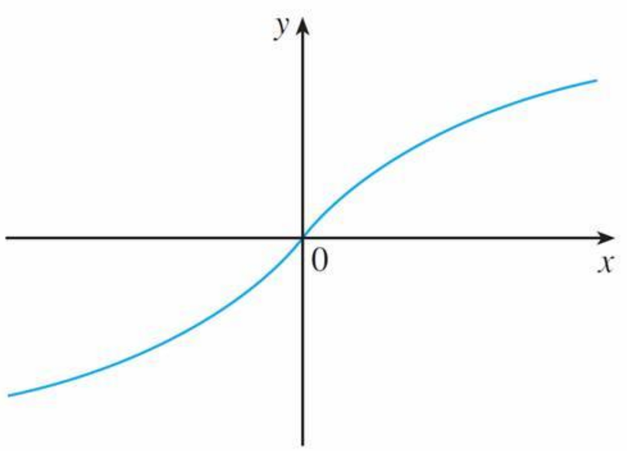
The sinh and tanh are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. The cosh is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one.

The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

$$\boxed{2} \qquad \boxed{\begin{array}{lcll} y = \sinh^{-1}x & \iff & \sinh y = x & \\ y = \cosh^{-1}x & \iff & \cosh y = x & \text{and} \quad y \geqslant 0 \\ y = \tanh^{-1}x & \iff & \tanh y = x & \end{array}}$$

Inverse Hyperbolic Functions

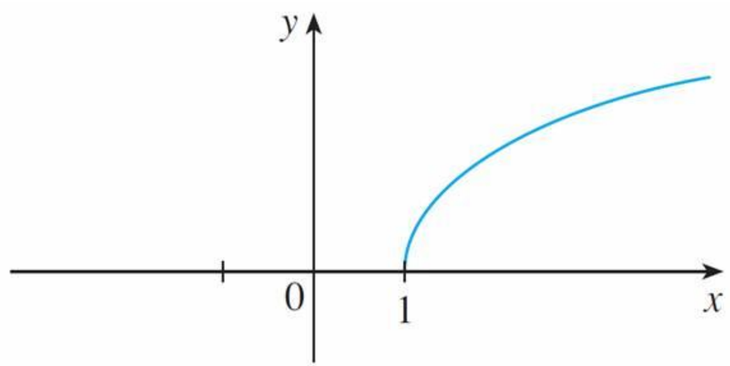
We can sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ in Figures 8, 9, and 10.



$y = \sinh^{-1} x$

domain = $\mathbb{R}$ range = $\mathbb{R}$

Figure 8



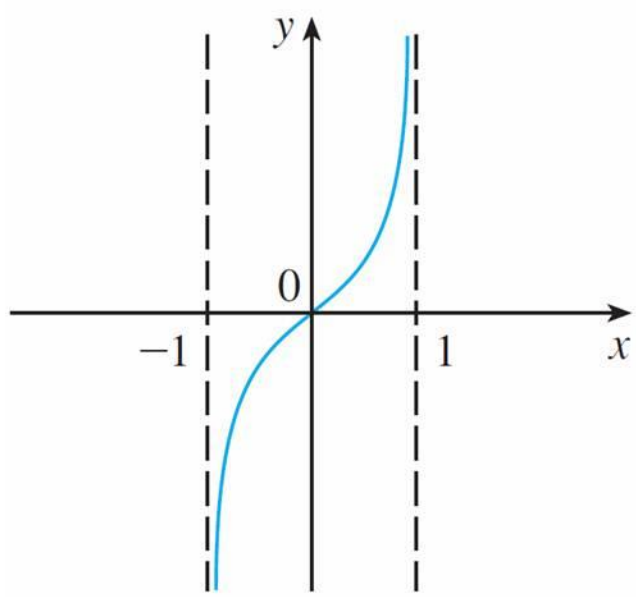
$y = \cosh^{-1} x$

domain = $[1, \infty)$ range = $[0, \infty)$

Figure 9

Inverse Hyperbolic Functions

cont'd



$y = \tanh^{-1} x$

domain = (–1, 1)   range = $\mathbb{R}$

Figure 10

Inverse Hyperbolic Functions

Since the hyperbolic functions are defined in terms of exponential functions, it’s not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms.

In particular, we have:

3	$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$	$x \in \mathbb{R}$
4	$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right)$	$x \geqslant 1$
5	$\tanh^{-1}x = \frac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$	$-1 < x < 1$

Example 3

Show that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Inverse Hyperbolic Functions

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$$

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable.

Example 4

Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$.

Examples 5 and 6

Example 5. Find $\frac{d}{dx}[\tanh^{-1}(\sin x)]$.

Example 6. Evaluate

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}}.$$

6.8 Indeterminate Forms and l'Hospital's Rule

Indeterminate Forms and l'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although *F* is not defined when *x* = 1, we need to know how *F* behaves *near* 1. In particular, we would like to know the value of the limit

$$\boxed{1} \qquad \lim_{x \to 1} \frac{\ln x}{x - 1}$$

Indeterminate Forms and l'Hospital's Rule

In computing this limit we can't apply Law 5 of limits because the limit of the denominator is 0. In fact, although the limit in $\boxed{1}$ exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type $\mathbf{\frac{0}{0}}$**.

Indeterminate Forms and l'Hospital's Rule

For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)}$$

$$= \lim_{x \to 1} \frac{x}{x + 1}$$

$$= \frac{1}{2}$$

We used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Indeterminate Forms and l'Hospital's Rule

But these methods do not work for limits such as $\boxed{1}$, so in this section we introduce a systematic method, known as *l'Hospital's Rule,* for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of *F* and need to evaluate its limit at infinity:

$\boxed{2}$ $$\lim_{x \to \infty} \frac{\ln x}{x - 1}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$.

Indeterminate Forms and l'Hospital's Rule

There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞; if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer will be some finite positive number.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type ∞/∞**.

Indeterminate Forms and l'Hospital's Rule

L'Hospital's Rule applies to this type of indeterminate form.

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Indeterminate Forms and l'Hospital's Rule

Note 1:

L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of *f* and *g* before using l'Hospital's Rule.

Note 2:

L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

Indeterminate Forms and l'Hospital's Rule

Proof of a special case: in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$.

Examples

Example 1. Find $\lim_{x \to 1} \dfrac{\ln x}{x - 1}$.

Example 2. Find $\lim_{x \to \infty} \dfrac{e^x}{x^2}$.

Example 3. Find $\lim_{x \to 0} \dfrac{\tan x - x}{x^3}$

Example 4. Find $\lim_{x \to \pi -} \dfrac{\sin x}{1 - \cos x}$

Indeterminate Products

Indeterminate Products

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x\to a} [f(x)\, g(x)]$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$).

Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**.

Indeterminate Products

We can deal with it by writing the product *fg* as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use l'Hospital's Rule.

Example 6

Evaluate $\lim_{x \to 0^+} x \ln x$.

Indeterminate Differences

Indeterminate Differences

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is called an **indeterminate form of type** $\infty - \infty$.

Example 8

Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$.

Indeterminate Powers

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0

2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0

3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Indeterminate Powers

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

Example 11

Find $\lim_{x \to 0+} x^x$.

7 Techniques of Integration

7.1 Integration by Parts

Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts.*

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Integration by Parts

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\, dx = f(x)g(x)$$

or $$\int f(x)g'(x)\, dx + \int g(x)f'(x)\, dx = f(x)g(x)$$

We can rearrange this equation as

$$\boxed{1} \qquad \boxed{\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx}$$

Formula 1 is called the **formula for integration by parts**.

Integration by Parts

It is perhaps easier to remember in the following notation.

Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

$$\boxed{2} \qquad \boxed{\int u\,dv = uv - \int v\,du}$$

Examples

1. Find $\int x \sin x \, dx$.

2. Evaluate $\int \ln x \; dx$.

3. Evaluate $\int t^2 e^t \, dt$.

4. Evaluate $\int e^x \sin x \, dx$.

Integration by Parts

If we combine the formula for integration by parts with Part 2 of Fundamental Theorem of Calculus, we can evaluate definite integrals by parts.

Evaluating both sides of Formula 1 between *a* and *b*, assuming f' and g' are continuous, and using the Fundamental Theorem, we obtain

6

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx$$

Definite integrals

Example 5. Calculate $\int_0^1 \tan^{-1} x \, dx$.

Reduction formulas

Example 6. Prove the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

7.2 Trigonometric Integrals

Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

We start with powers of sine and cosine.

Examples 1- 2

Example 1. Find $\int \sin^5 x \cos^2 x \, dx$.

Example 2. Evaluate $\int \cos^3 x \; dx$.

Examples 3 and 4

Example 3. Evaluate $\int_0^{\pi} \sin^2 x \, dx$.

Example 4. Find $\int \sin^4 x \;\; dx$.

Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$
$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Trigonometric Integrals

We can use a similar strategy to evaluate integrals of the form $\int \tan^m x \sec^n x \, dx$.

Since $(d/dx) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

Examples 5 - 6

Example 5. Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Example 6. Evaluate $\int \tan^5 \theta \sec^7 \theta \, d\theta$.

Trigonometric Integrals

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \sec^n x \, dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \; \sec x \; \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \; \sec x \; \tan x \, dx$$

Then substitute $u = \sec x$.

Trigonometric Integrals

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity.

We will sometimes need to be able to integrate tan *x* by using the formula given below:

$$\boxed{\int \tan x \, dx = \ln |\sec x| + C}$$

Trigonometric Integrals

We will also need the indefinite integral of secant:

1 $$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

We could verify Formula 1 by, for example, differentiating the right side.

Examples 7 - 8

Example 7. Find $\int \tan^3 x \, dx$.

Example 8. Find $\int \sec^3 x \; dx$.

Trigonometric Integrals

Finally, we can make use of another set of trigonometric identities:

2 To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

$$\text{(a)} \quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)} \quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)} \quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Example 9

Evaluate $\int \sin 4x \cos 5x \, dx$.

7.3 Trigonometric Substitution

Trigonometric Substitution

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$.

If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult.

Trigonometric Substitution

If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta}$$

$$= \sqrt{a^2(1 - \sin^2\theta)}$$

$$= \sqrt{a^2 \cos^2\theta}$$

$$= a\left|\cos \theta\right|$$

Trigonometric Substitution

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin\theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse.

To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one.

Trigonometric Substitution

In this case, if we replace *u* by *x* and *x* by *t* in the Substitution Rule, we obtain

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt$$

This kind of substitution is called *inverse substitution*.

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function.

Trigonometric Substitution

This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Trigonometric Substitution

In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one.

Example 1

Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\,dx.$

Example 2

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Trigonometric Substitution

Note:

Since the integral in Example 2 was a definite integral, we changed the limits of integration and did not have to convert back to the original variable x.

Example 3

Find $\int \frac{1}{x^2\sqrt{x^2 + 4}}\, dx.$

Example 5

Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

Trigonometric Substitution

Note:
As Example 5 illustrates, hyperbolic substitutions can be used in place of trigonometric substitutions and sometimes they lead to simpler answers.

But we usually use trigonometric substitutions because trigonometric identities are more familiar than hyperbolic identities.

Examples

Example 6. Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx.$

Example 7. Evaluate $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx.$

7.8 Improper Integrals

Improper Integrals

In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a, b]$. In either case the integral is called an *improper* integral.

Type 1: Infinite Intervals

Type 1: Infinite Intervals

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$.

You might think that, since S is infinite in extent, its area must be infinite, but let's take a closer look.

The area of the part of S that lies to the left of the line $x = t$ (shaded in Figure 1) is

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t$$

$$= 1 - \frac{1}{t}$$

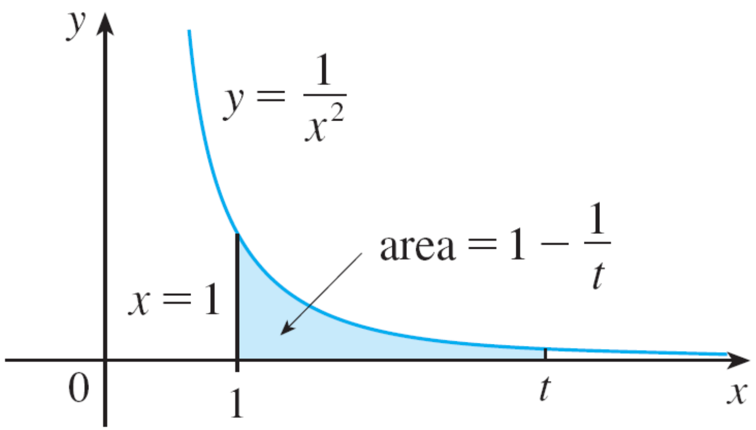


Figure 1

Type 1: Infinite Intervals

Notice that $A(t) < 1$ no matter how large t is chosen. We also observe that

$$\lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left(1 - \frac{1}{t} \right) = 1$$

The area of the shaded region approaches 1 as $t \to \infty$ (see Figure 2), so we say that the area of the infinite region S is equal to 1 and we write

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^2}\,dx = 1$$

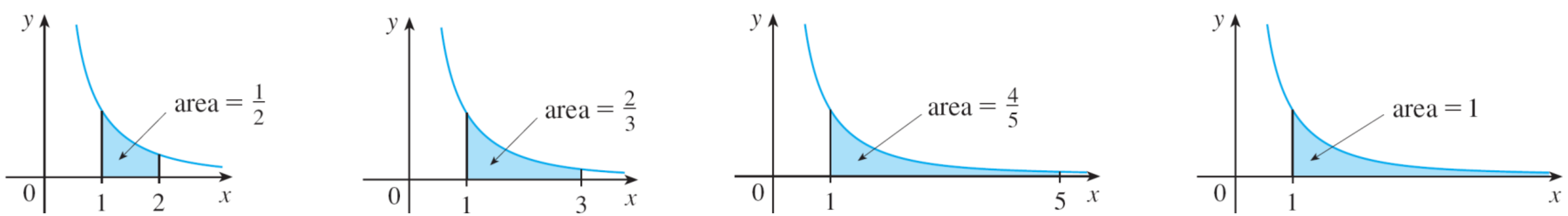


Figure 2

Type 1: Infinite Intervals

Using this example as a guide, we define the integral of *f* (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

Type 1: Infinite Intervals

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used.

Type 1: Infinite Intervals

Any of the improper integrals in Definition 1 can be interpreted as an area provided that *f* is a positive function.

For instance, in case (a) if $f(x) \geq 0$ and the integral $\int_a^\infty f(x)\,dx$ is convergent, then we define the area of the region $S = \{(x, y) \mid x \geq a,\ 0 \leq y \leq f(x)\}$ in Figure 3 to be

$$A(S) = \int_a^\infty f(x)\,dx$$

This is appropriate because $\int_a^\infty f(x)\,dx$ is the limit as $t \to \infty$ of the area under the graph of *f* from *a* to *t*.

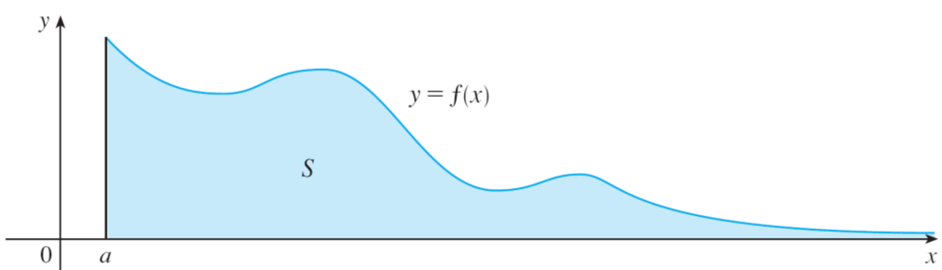


Figure 3

Examples

Example 1. Determine whether the integral $\int_1^\infty (1/x)\,dx$ is convergent or divergent.

Example 2. Evaluate $\int_{-\infty}^{0} xe^x\,dx$.

Example 3. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Example 4. For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

Type 1: Infinite Intervals

We summarize the findings of Example 4 as follows:

2 $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Type 2: Discontinuous Integrands

Type 2: Discontinuous Integrands

Suppose that *f* is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at *b*.

Let *S* be the unbounded region under the graph of *f* and above the *x*-axis between *a* and *b.* (For Type 1 integrals, the regions extended indefinitely in a horizontal direction. Here the region is infinite in a vertical direction.)

The area of the part of *S* between *a* and *t* (the shaded region in Figure 7) is

$$A(t) = \int_a^t f(x)\,dx$$

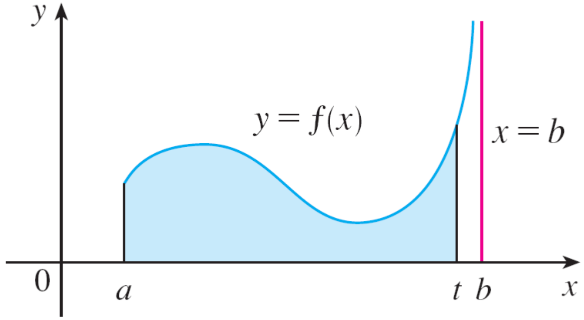


Figure 7

Type 2: Discontinuous Integrands

If it happens that $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A and we write

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

We use this equation to define an improper integral of Type 2 even when f is not a positive function, no matter what type of discontinuity f has at b.

Type 2: Discontinuous Integrands

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Examples

Example 5. Find $\int_{2}^{5} \frac{1}{\sqrt{x-2}}\, dx$.

Example 6. Determine whether $\int_{0}^{\frac{\pi}{2}} \sec x\, dx$ is convergent.

Example 7. Evaluate $\int_{0}^{3} \frac{1}{x-1}\, dx$ if possible.

Example 8. Evaluate $\int_{0}^{1} \ln x\, dx$.

Example. For what values of *p* does the integral $\int_{0}^{1} \frac{1}{x^p}\, dx$ converge?

A Comparison Test for Improper Integrals

A Comparison Test for Improper Integrals

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent.

In such cases the following theorem is useful. Although we state it for Type 1 integrals, a similar theorem is true for Type 2 integrals.

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

A Comparison Test for Improper Integrals

We omit the proof of the Comparison Theorem, but Figure 12 makes it seem plausible.

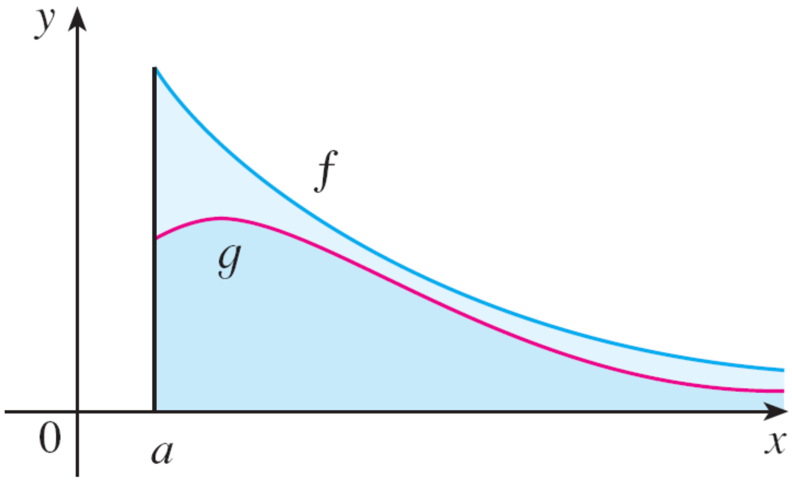


Figure 12

If the area under the top curve $y = f(x)$ is finite, then so is the area under the bottom curve $y = g(x)$.

A Comparison Test for Improper Integrals

And if the area under $y = g(x)$ is infinite, then so is the area under $y = f(x)$. [Note that the reverse is not necessarily true: If $\int_a^\infty g(x)\, dx$ is convergent, $\int_a^\infty f(x)\, dx$ may or may not be convergent, and if $\int_a^\infty f(x)\, dx$ is divergent, $\int_a^\infty g(x)\, dx$ may or may not be divergent.]

Examples

Example 9. Show that $\int_0^\infty e^{-x^2} dx$ is convergent.

Example 10. Show that $\int_1^\infty \frac{1+e^{-x}}{x} \, dx$ is divergent.