11.1 Sequences

Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the *first term,* a_2 is the *second term,* and in general a_n is the *nth term.* We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Note: for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers.

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But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Some sequences can be defined by giving a formula for the nth term.

Example 1

In the following examples give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula for the *n*th term, and a third by writing out the terms of the sequence. Notice that *n* doesn’t have to start at 1.

(a) $\left\{\frac{1}{2^n}\right\}$

(b) $\left\{\frac{n}{n+1}\right\}$

(c) $\left\{\sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots\right\}$

Examples 2 and 3

2. Find a_n for $\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \ldots\right\}$.

3. Fibonacci sequence: $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. This is an example of a *recursion*, or *recursive definition,* of a sequence. Write out the first few terms.

Sequences

A sequence such as the one in Example 1(b), $a_n = n/(n + 1)$, can be pictured either by plotting its terms on a number line, as in Figure 2, or by plotting its graph, as in Figure 3.

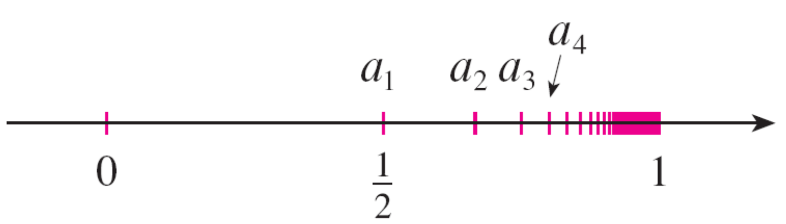


Figure 2

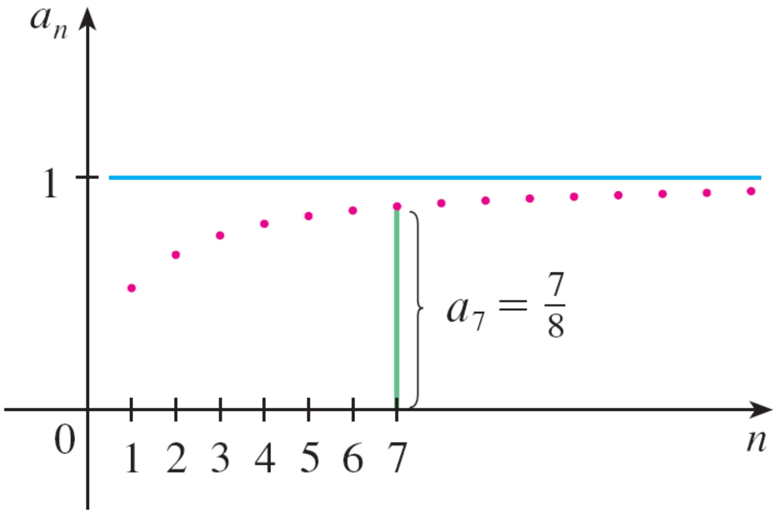


Figure 3

Sequences

Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \quad (2, a_2) \quad (3, a_3) \quad \ldots \quad (n, a_n) \quad \ldots$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n + 1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large.

Sequences

We indicate this by writing

$$\lim_{n \to \infty} \frac{n}{n + 1} = 1$$

In general, the notation

$$\lim_{n \to \infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large.

Sequences

Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity.

$\boxed{1}$ **Intuitive Definition of a Limit of a Sequence** A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Note: the sequence $\{a_n\}$ is divergent, if there is no L with the above property.

Sequences

Figure 4 illustrates Definition 1 by showing the graphs of two sequences that have the limit *L*.

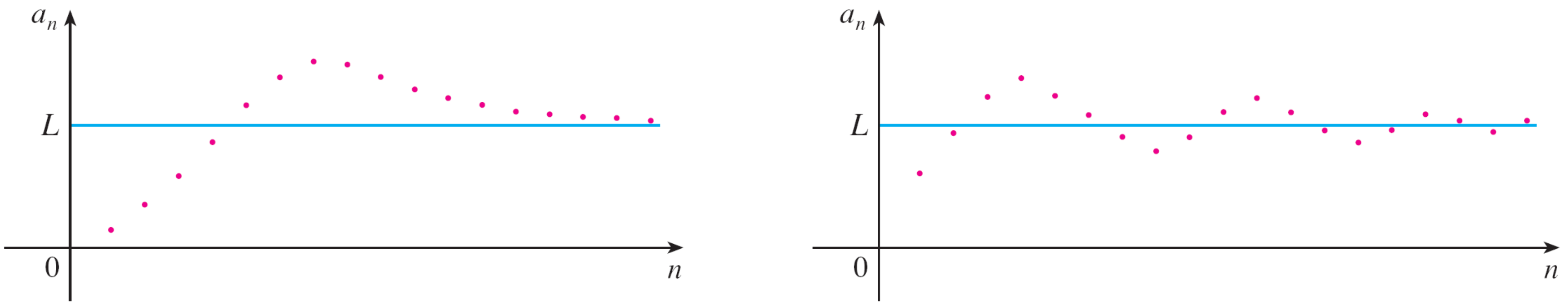


Figure 4

Graphs of two sequences with $\lim_{n \to \infty} a_n = L$

Sequences

A more precise version of Definition 1 is as follows.

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

Note: 1. If a sequence $\{a_n\}$ has the limit *L*, then we can also say that $\{a_n\}$ *converges* to *L* (as *n* approaches ∞).

2. The limit, if it exists, is unique.

3. $\{a_n\}$ is divergent, if there is no *L* with the above property.

Sequences

Definition 2 is illustrated by Figure 5, in which the terms a_1, a_2, a_3 , . . . are plotted on a number line.

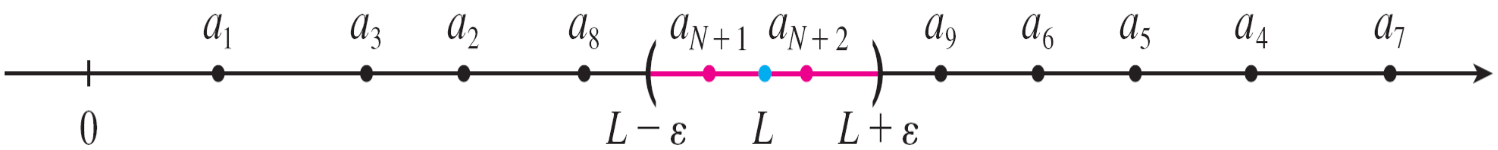


Figure 5

No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

Or, equivalently, no matter how small an interval

$(L - \varepsilon, L + \varepsilon)$ is chosen there are only finitely many terms of the sequence outside of that interval.

Sequences

Another illustration of Definition 2 is given in Figure 6. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.

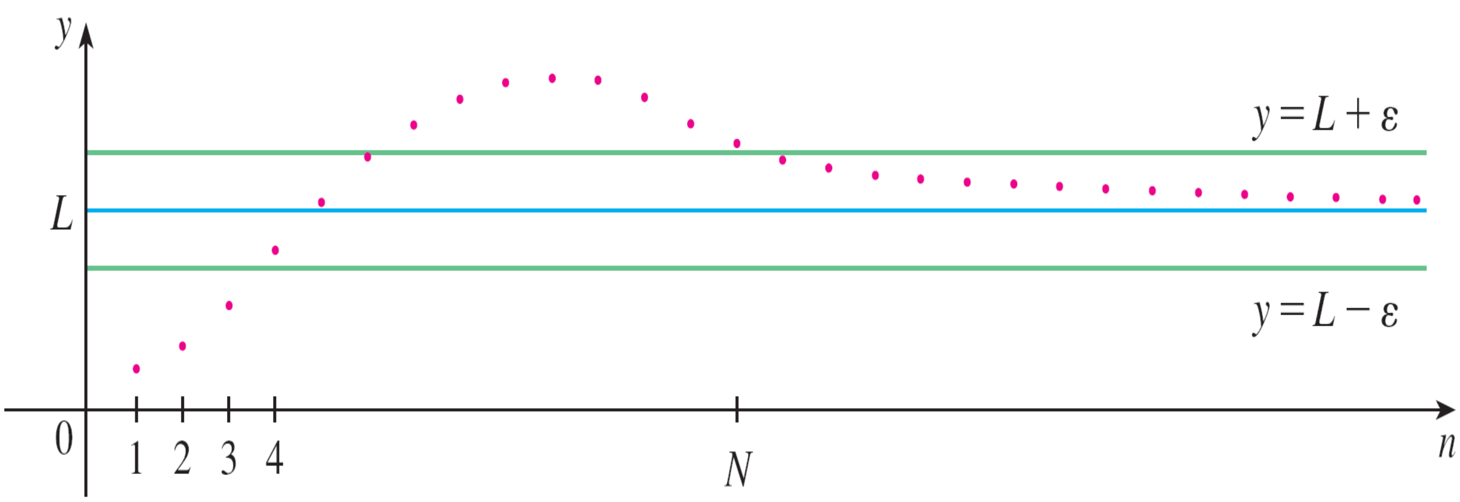
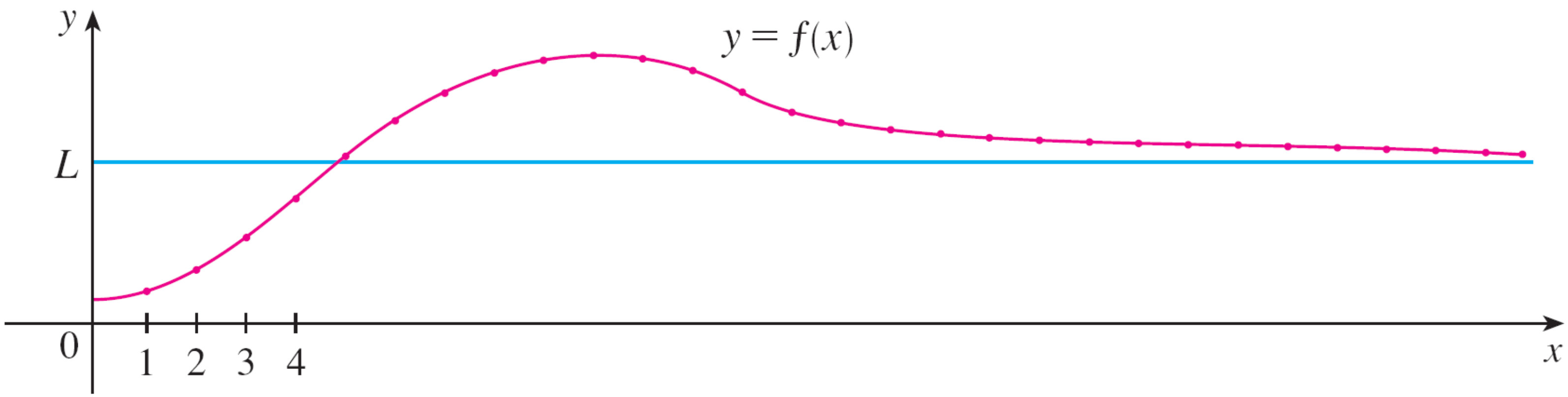


Figure 6

Sequences

The only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem.

Theorem 3. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim a_n = L$**.**



Sequences

Example. Since we know that $\lim_{x \to \infty}(1/x^r) = 0$ when $r > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0.$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$. Consider the definition:

Definition 3. We write $\lim_{n\to\infty} a_n = \infty$ if for every $M > 0$, there is an integer N such that $n > N \implies a_n > M$.

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞. Note that the analogue of Theorem 3 holds in this case too ($L = \infty$).

Sequences

The Limit Laws also hold for the limits of sequences and their proofs are similar.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

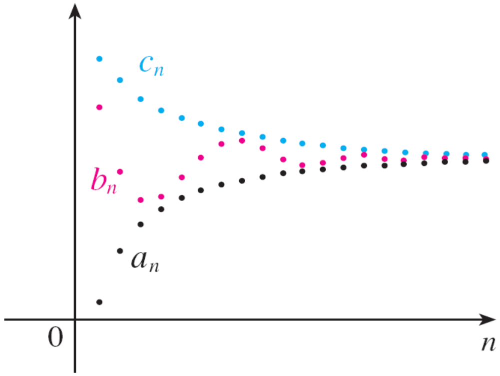
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \text{ if } p > 0 \text{ and } a_n > 0$$

Sequences

The Squeeze Theorem can also be adapted for sequences as follows.

Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Note: Similarly, if $a_n \leq b_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} b_n = \infty$.

Sequences

Another useful fact about limits of sequences is given by the following theorem.

Theorem 6. $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$.

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent.

7 Theorem If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Examples 4 -11

4. Find $\lim\limits_{n\to\infty} \frac{n}{n+1}$ if it exists.

5. Find $\lim\limits_{n\to\infty} \frac{n}{\sqrt{10+n}}$ if it exists.

6. Find $\lim\limits_{n\to\infty} \frac{\ln(n)}{n}$ if it exists.

7. Find $\lim\limits_{n\to\infty} (-1)^n$ if it exists.

8. Find $\lim\limits_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

9. Find $\lim\limits_{n\to\infty} \sin\frac{\pi}{n}$ if it exists.

10. Find $\lim\limits_{n\to\infty} \frac{n!}{n^n}$ if it exists.

11. For what values of r is the sequence $\{r^n\}$ convergent ?

Sequences

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

$\boxed{11}$ **Definition** A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Example 13: Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

Sequences

For instance, the sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = n/(n + 1)$ is bounded because $0 < a_n < 1$ for all n.

We know that not every bounded sequence is convergent (for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leq a_n \leq 1$ but is divergent). One may show though that every convergent sequence is bounded.

While not every monotonic sequence is convergent ($a_n = n$) but if a sequence is both bounded *and* monotonic, then it must be convergent.

Sequences

This fact is stated in Theorem 12, and intuitively you can understand why it is true by looking at Figure 14.

If $\{a_n\}$ is increasing and $a_n \leq M$ for all n, then the terms are forced to crowd together and approach some number L.

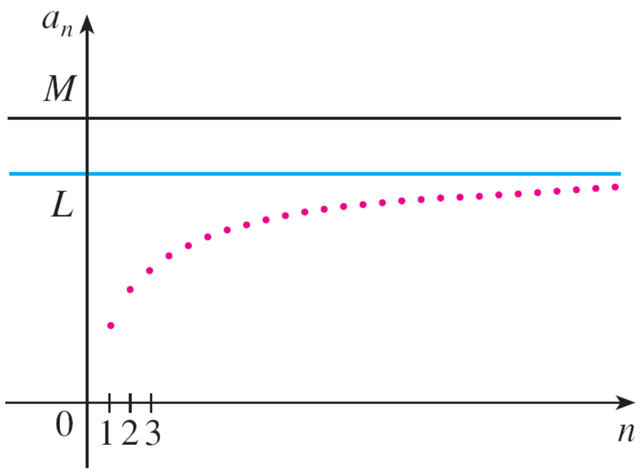


Figure 14

Monotonic Sequence Theorem

Theorem. If a sequence is nondecreasing and bounded above, then it is convergent. Similarly, if If a sequence is nondecreasing and bounded below, then it is convergent.

Note: Stewart only states the theorem for increasing and decreasing sequences, but the statement is true in this slightly greater generality too.