3.4 Limits at Infinity; Horizontal Asymptotes

Limits at Infinity; Horizontal Asymptotes

In this section we let *x* become arbitrarily large (positive or negative) and see what happens to *y*. We will find it very useful to consider this so-called *end behavior* when sketching graphs.

Let's begin by investigating the behavior of the function *f* defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as *x* becomes large.

Limits at Infinity; Horizontal Asymptotes

The following table contains approximate values of the function.

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

Limits at Infinity; Horizontal Asymptotes

The graph of *f* has been drawn by a computer in Figure 1.

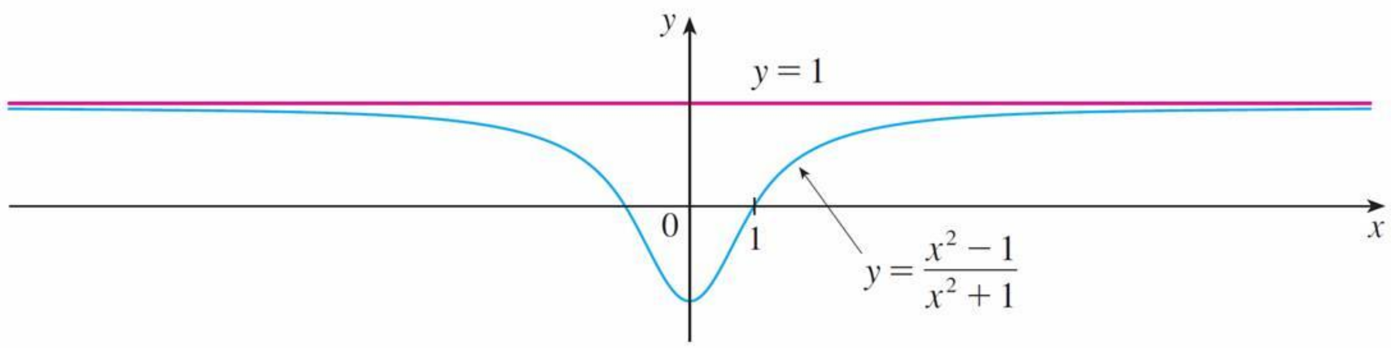


Figure 1

Limits at Infinity; Horizontal Asymptotes

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large.

This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

$\boxed{1}$ **Intuitive Definition of a Limit at Infinity** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

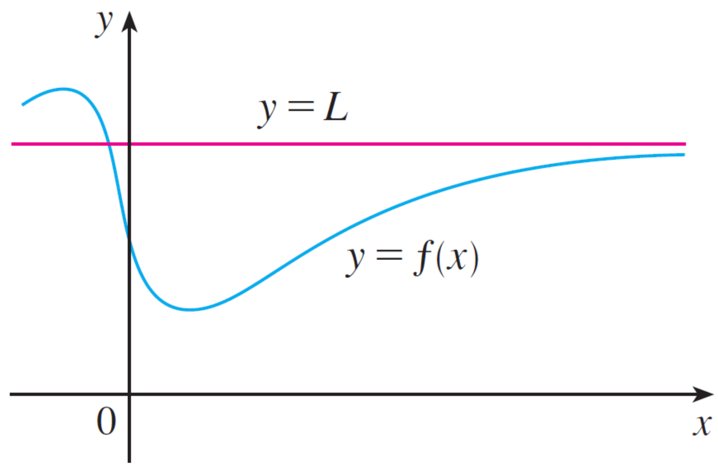
Limits at Infinity; Horizontal Asymptotes

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

Limits at Infinity; Horizontal Asymptotes

Geometric illustrations of Definition 1 are shown in Figure 2.

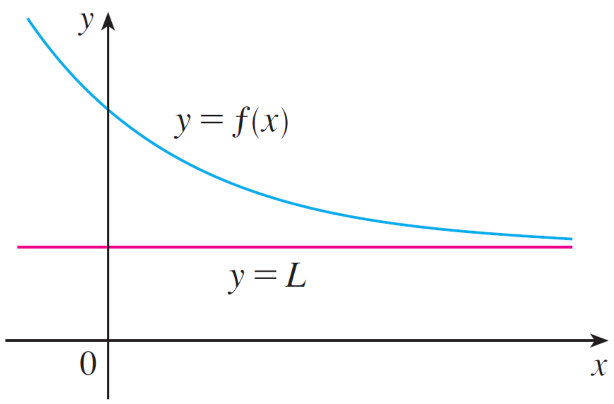


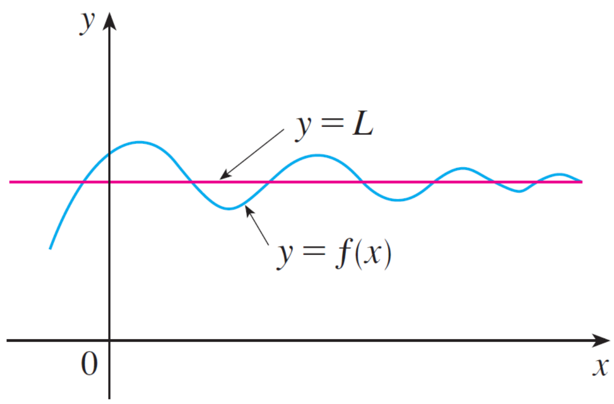
Examples illustrating $\lim_{x \to \infty} f(x) = L$

Figure 2

Limits at Infinity; Horizontal Asymptotes

Notice that there are many ways for the graph of *f* to approach the line *y* = *L* (which is called a *horizontal asymptote*) as we look to the far right of each graph.





Limits at Infinity; Horizontal Asymptotes

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1.

By letting x decrease through negative values without bound, we can make $f(x)$ as close to 1 as we like.

This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

The general definition is as follows.

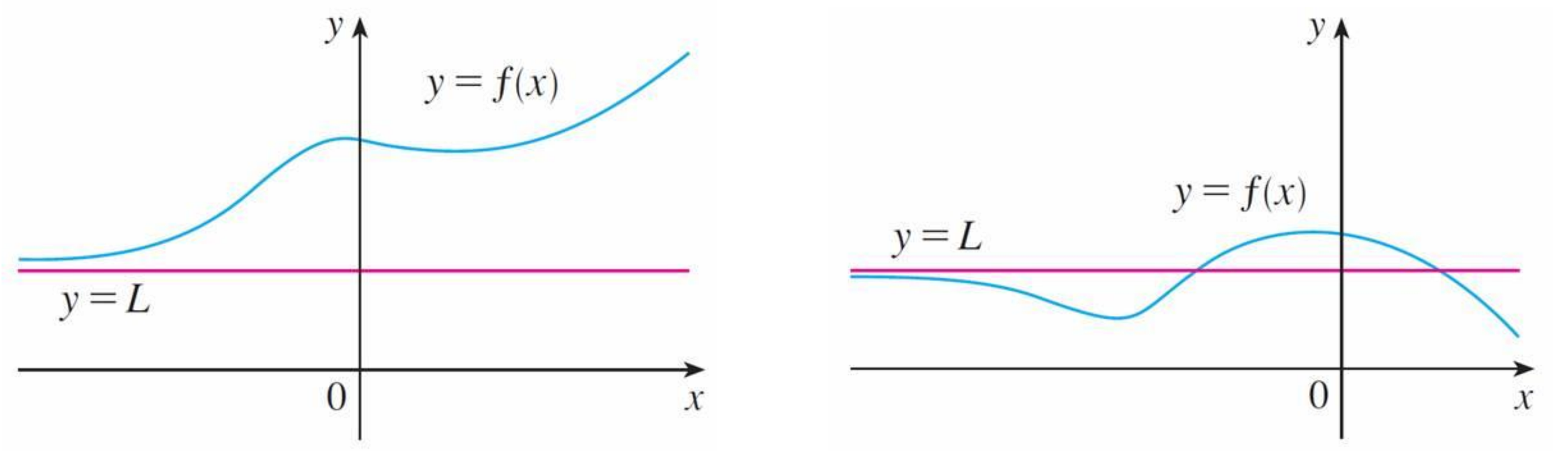
$\boxed{2}$ **Definition** Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Limits at Infinity; Horizontal Asymptotes

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.



Examples illustrating $\lim_{x \to \infty} f(x) = L$

Figure 3

Limits at Infinity; Horizontal Asymptotes

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Limits at Infinity; Horizontal Asymptotes

The curve $y = f(x)$ sketched in Figure 4 has both $y = -1$ and $y = 2$ as horizontal asymptotes because

$$\lim_{x \to \infty} f(x) = -1 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 2$$

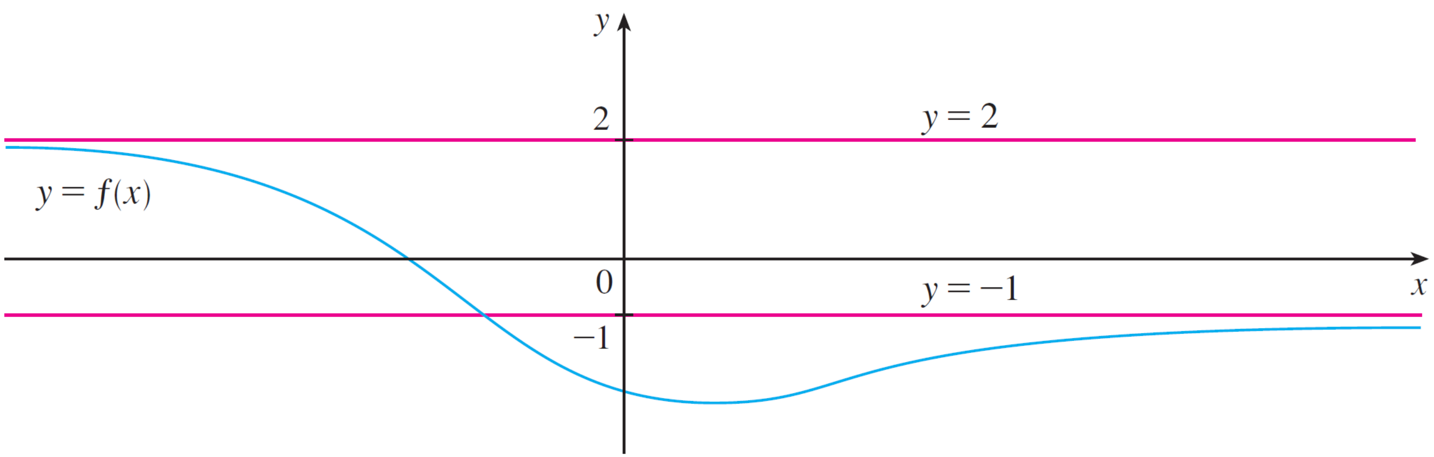
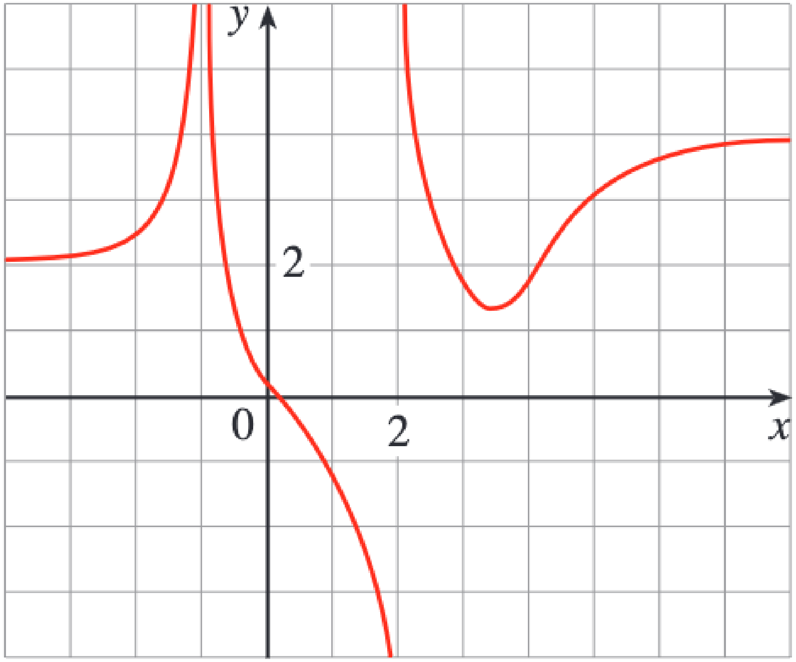


Figure 4

Example 1

Find the infinite limits, limits at infinity, and asymptotes for the function *f* whose graph is shown below.



Limits at Infinity; Horizontal Asymptotes

4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

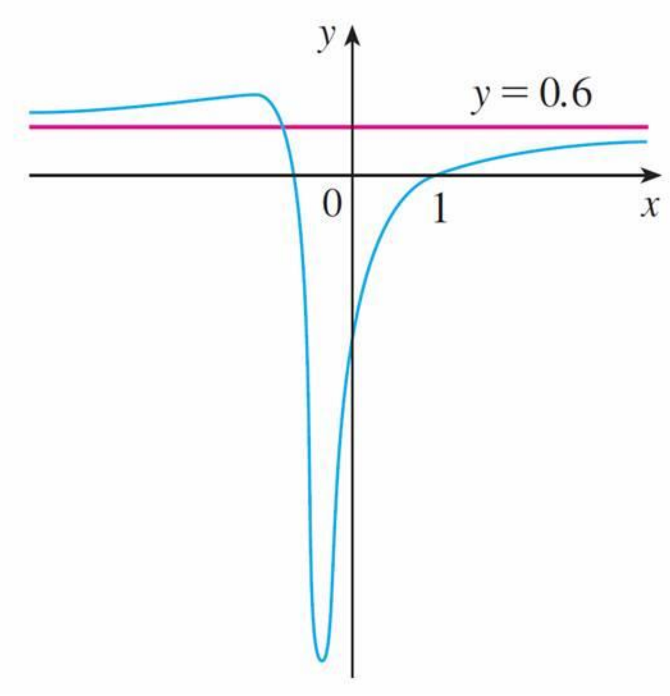
Example 3

Evaluate $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$ and indicate which properties of limits are used at each stage.

Example 3

∞

Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$.



$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

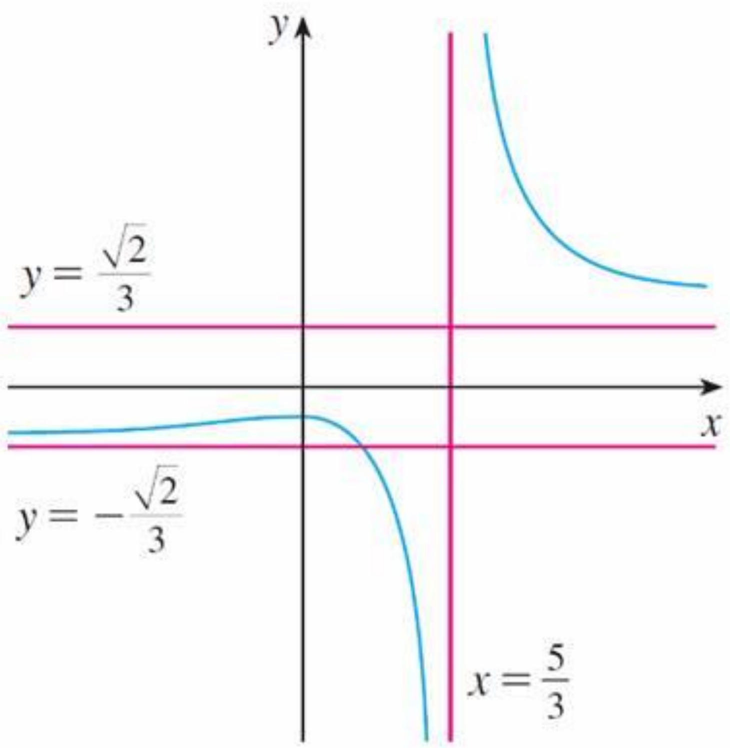
Figure 7

Example 4

Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Example 4



$$y = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Example 5

Evaluate

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 1} - x \right).$$

Examples 6 and 7

Example 6. Evaluate

$$\lim_{x\to\infty} \sin\frac{1}{x}.$$

Example 7. Evaluate

$$\lim_{x\to\infty} \sin x\,.$$

Infinite Limits at Infinity

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Example 8 – 10

Example 8. Find

$$\lim_{x\to\infty} x^3 \text{ and } \lim_{x\to-\infty} x^3 .$$

Example 9. Find

$$\lim_{x\to\infty} (x^2 - x).$$

Example 10. Find

$$\lim_{x\to\infty} \frac{x^2 + x}{3 - x}.$$

Precise Definitions

Precise Definitions

Definition 1 can be stated precisely as follows.

5 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

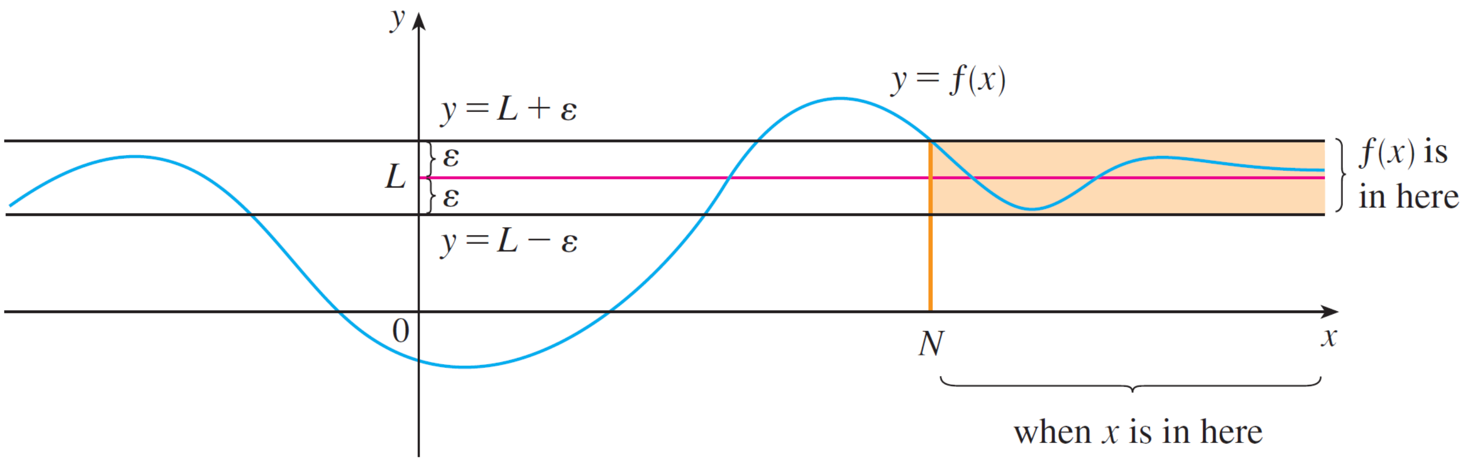
means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \qquad x > N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by taking x sufficiently large (larger than N, where depends on ε).

Precise Definitions

Graphically it says that by choosing *x* large enough (larger than some number *N*) we can make the graph of *f* lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 12.

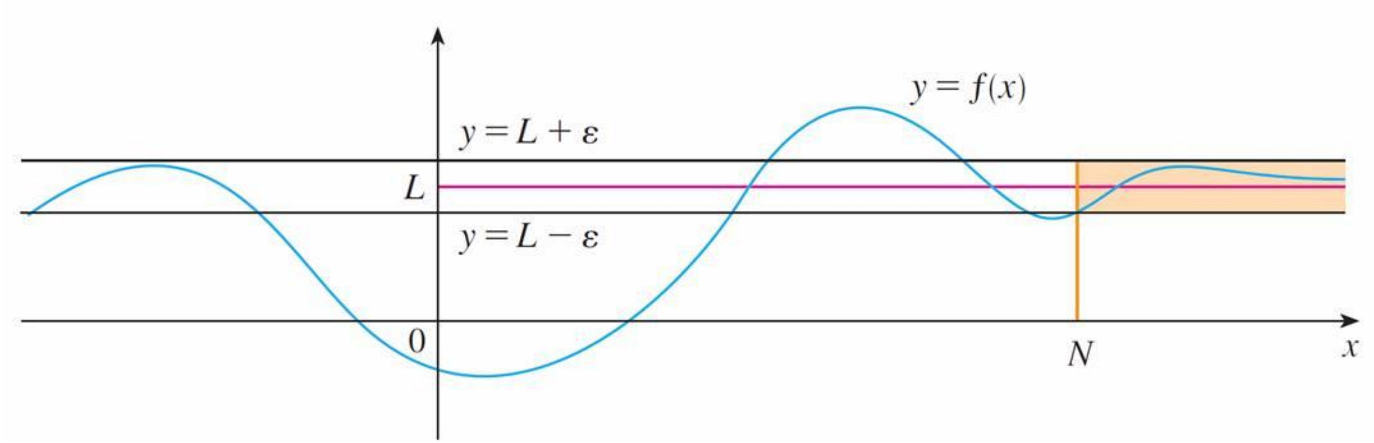


$$\lim_{x \to \infty} f(x) = L$$

Figure 12

Precise Definitions

This must be true no matter how small we choose *ε*. Figure 13 shows that if a smaller value of *ε* is chosen, then a larger value of *N* may be required.



$$\lim_{x \to \infty} f(x) = L$$

Figure 13

Precise Definitions

6 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

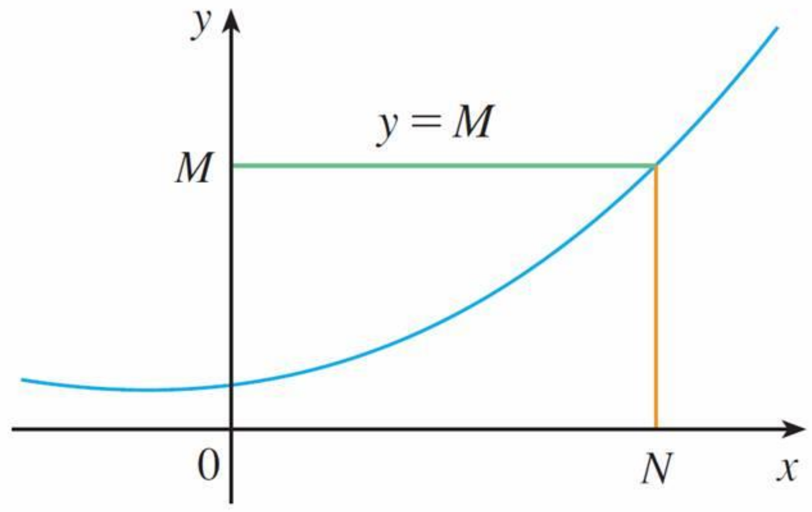
$$\text{if} \quad x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 13

Use Definition 5 to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Precise Definitions

Finally, we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 17.



$$\lim_{x \to \infty} f(x) = \infty$$

Figure 17

Precise Definitions

7 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$.

Guidelines for Sketching a Curve

Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.)

But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of for which $f(x)$ is defined.

Guidelines for Sketching a Curve

B. Intercepts The *y*-intercept is *f*(0) and this tells us where the curve intersects the *y*-axis. To find the *x*-intercepts, we set $y = 0$ and solve for *x*. (You can omit this step if the equation is difficult to solve.)

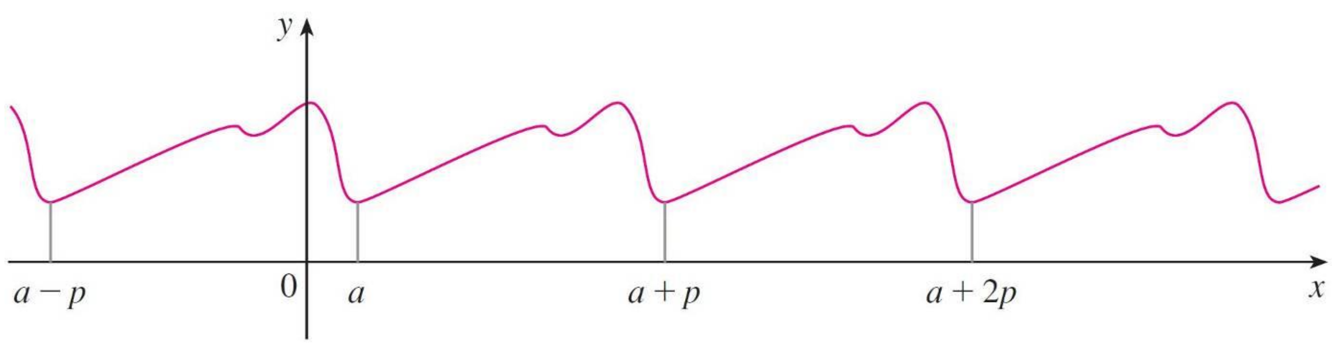
C. Symmetry

(i) If *f* is an **even function** or an **odd function** then you only have to sketch the curve for $x > 0$, then reflect about the y axis (even) or the origin (odd).

Guidelines for Sketching a Curve

(ii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.**

For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 4).



Periodic function: translational symmetry

Figure 4

Guidelines for Sketching a Curve

D. Asymptotes

(i) *Horizontal Asymptotes.* If either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$.

If it turns out that $\lim_{x \to \infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

Guidelines for Sketching a Curve

(ii) *Vertical Asymptotes.* The line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\boxed{1} \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.)

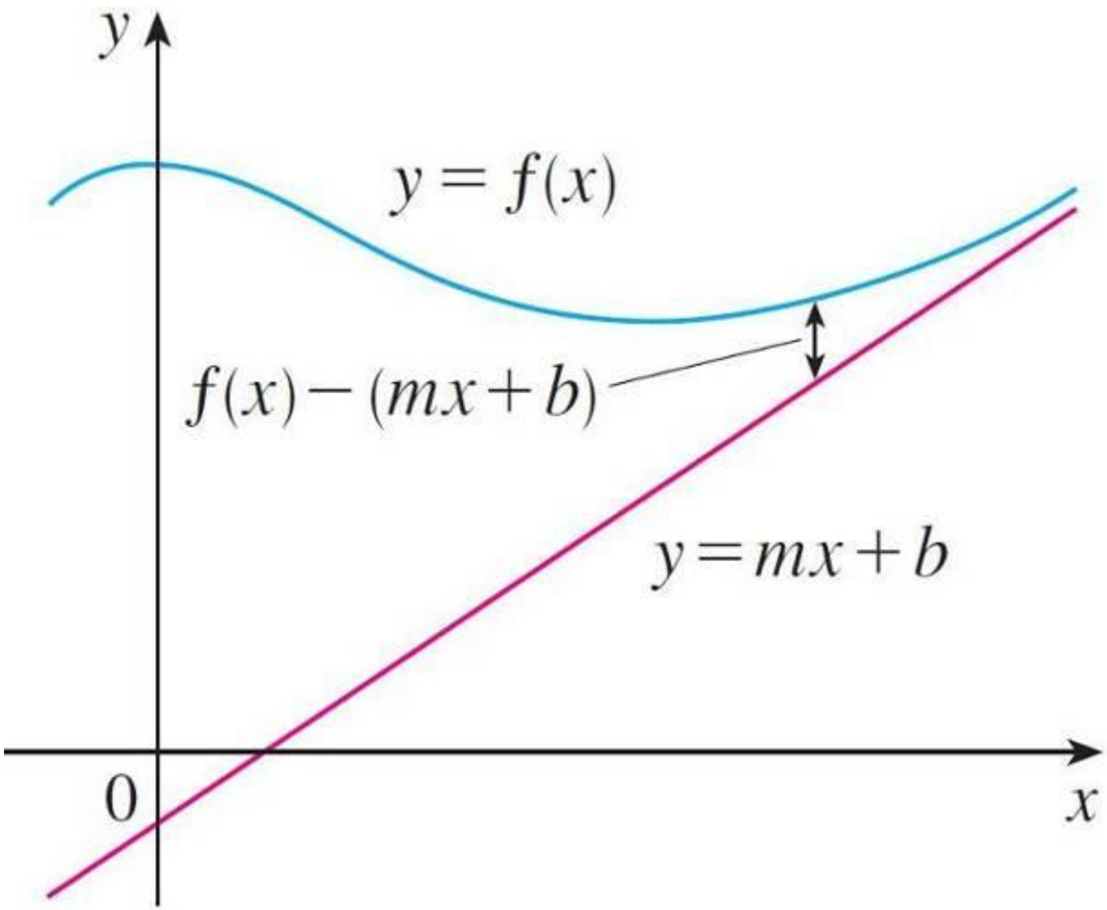
Guidelines for Sketching a Curve

Furthermore, in sketching the curve it is very useful to know exactly which of the statements in $\boxed{1}$ is true.

If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.* If $\lim_{x \to \infty} [f(x) - (mx + b)] = 0$,
then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0. (A similar situation may exist if we let $x \to \infty$.)

Slant Asymptotes



Slant Asymptotes

For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator.

In such a case the equation of the slant asymptote can be found by long division as in Example 4.

Guidelines for Sketching a Curve

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test if f is continuous at c. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$ and f'' is continuous at c.

Guidelines for Sketching a Curve

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes and f is continuous there.

Guidelines for Sketching a Curve

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points.

Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.

If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

Example 1

Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.