# Oriented Surfaces

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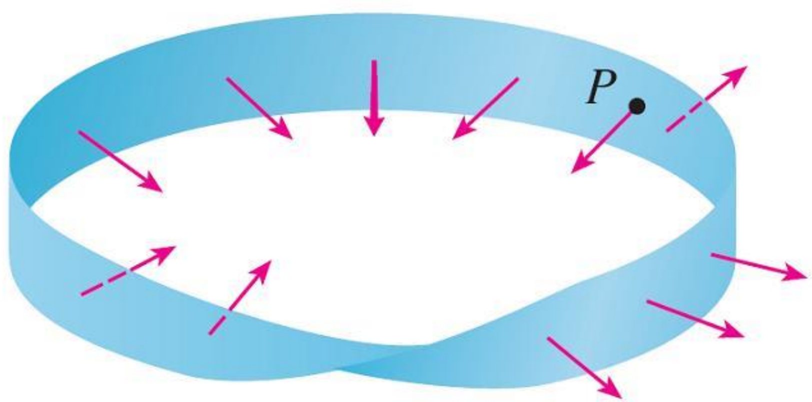
To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).]

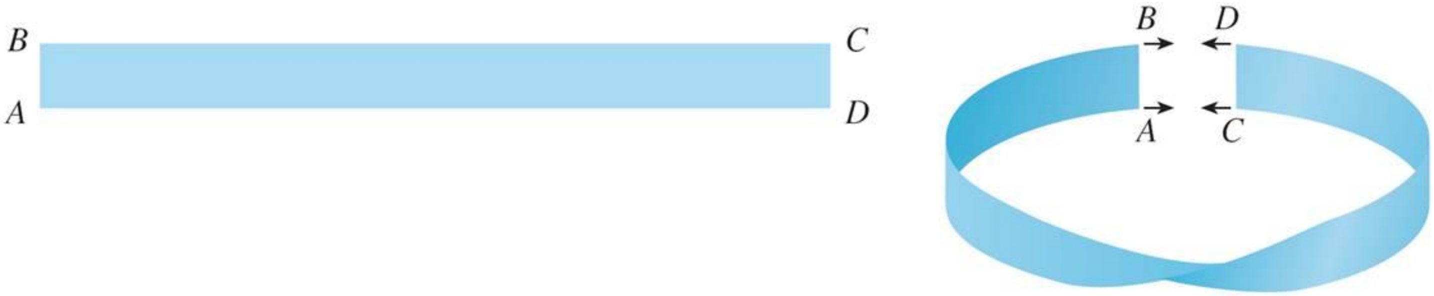



A Möbius strip

**Figure 4**

# Oriented Surfaces

You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5.




Constructing a Möbius strip

**Figure 5**

# Oriented Surfaces

If an ant were to crawl along the Möbius strip starting at a point $P$, it would end up on the "other side" of the strip (that is, with its upper side pointing in the opposite direction).

Then, if the ant continued to crawl in the same direction, it would end up back at the same point $P$ without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.)

Therefore, a Möbius strip really has only one side.

From now on we consider only orientable (two-sided) surfaces.

# Oriented Surfaces

We start with a surface $S$ that has a tangent plane at every point $(x, y, z)$ on $S$ (except at any boundary point).

There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at $(x, y, z)$. (See Figure 6.)

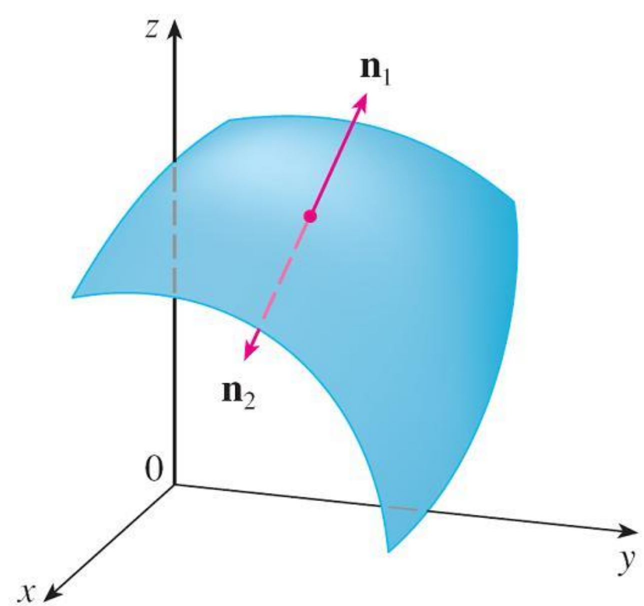
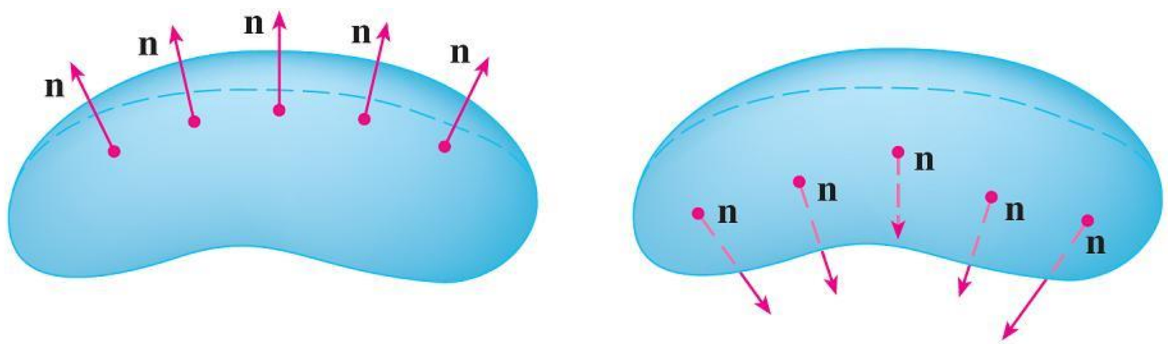



**Figure 6**

# Oriented Surfaces

If it is possible to choose a unit normal vector **n** at every such point $(x, y, z)$ so that **n** varies continuously over $S$, then $S$ is called an **oriented surface** and the given choice of **n** provides $S$ with an **orientation**.

There are two possible orientations for any orientable surface (see Figure 7).




The two orientations of an orientable surface

**Figure 7**

# Recap: Graphs

We saw: a surface $S$ with equation $z = g(x, y)$ can be seen as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

Thus

$$\boxed{3} \qquad \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

# Oriented Surfaces

Thus, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\boxed{5} \qquad \mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\,\mathbf{i} - \dfrac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$$

Since the **k**-component is positive, this gives the *upward* orientation of the surface.

# Oriented Surfaces

If $S$ is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\boxed{6} \qquad \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and the opposite orientation is given by $-\mathbf{n}$.

For instance, the parametric representation is

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \, \mathbf{i} + a \sin \phi \sin \theta \, \mathbf{j} + a \cos \phi \, \mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$.

# Oriented Surfaces

We saw last time that

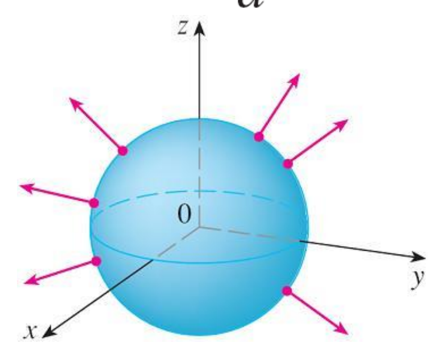
$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2 \phi \cos \theta\, \mathbf{i} + a^2 \sin^2 \phi \sin \theta\, \mathbf{j} + a^2 \sin \phi \cos \phi\, \mathbf{k}$$

and $$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin \phi$$

So the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin \phi \cos \theta\, \mathbf{i} + \sin \phi \sin \theta\, \mathbf{j} + \cos \phi\, \mathbf{k} = \frac{1}{a}\mathbf{r}(\phi, \theta)$$

Observe that $\mathbf{n}$ points in the same direction as the position vector, that is, outward from the sphere (see Figure 8).

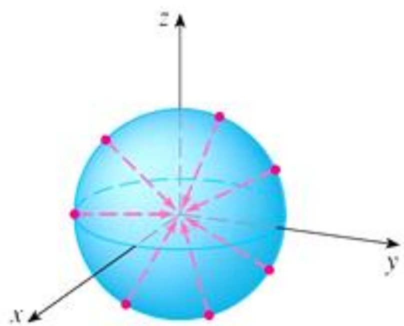



Positive orientation

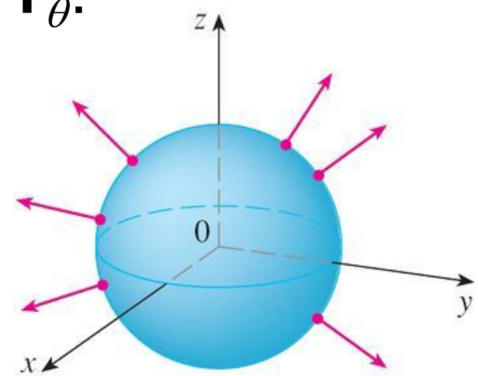
**Figure 8**

# Oriented Surfaces

The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.




Negative orientation
**Figure 9**




Positive orientation
**Figure 8**

For a **closed surface**, that is, a surface that is the boundary of a solid region *E*, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from *E*, and inward-pointing normals give the negative orientation (see Figures 8 and 9).

# Surface Integrals of Vector Fields

# Surface Integrals of Vector Fields

Suppose that $S$ is an oriented surface with unit normal vector $\mathbf{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing *through* $S$. (Think of $S$ as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.)

Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$.

# Surface Integrals of Vector Fields

We divide $S$ into small patches $S_{ij}$, as in Figure 10 (compare with Figure 1).

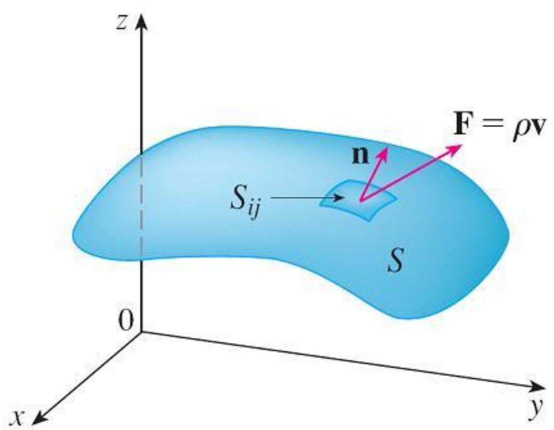



**Figure 10**

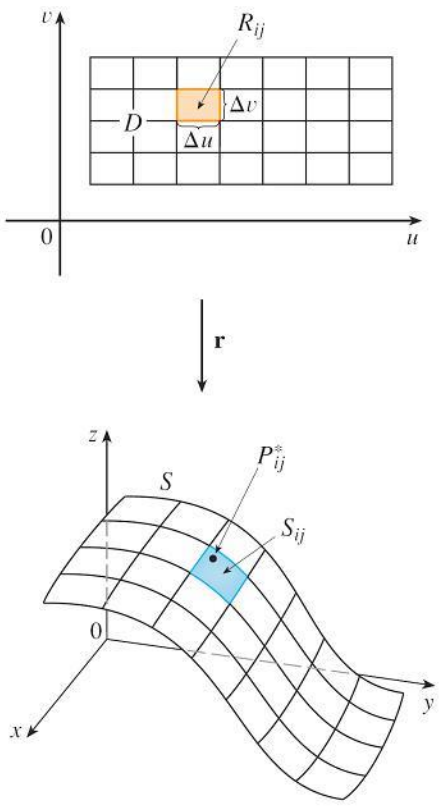



**Figure 1**

# Surface Integrals of Vector Fields

Then $S_{ij}$ is nearly planar and so we can approximate the mass of fluid per unit time crossing $S_{ij}$ in the direction of the normal **n** by the quantity

$$(\rho \mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

where $\rho$, **v**, and **n** are evaluated at some point on $S_{ij}$. (Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector **n** is $\rho\mathbf{v} \cdot \mathbf{n}$.)

# Surface Integrals of Vector Fields

By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over $S$:

$$\boxed{7} \qquad \iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS$$

and this is interpreted physically as the rate of flow through $S$.

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$ and the integral in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

# Surface Integrals of Vector Fields

A surface integral of this form occurs frequently in physics, even when **F** is not $\rho\mathbf{v}$, and is called the *surface integral* (or *flux integral*) of **F** over *S*.

> $\boxed{8}$ **Definition** If **F** is a continuous vector field defined on an oriented surface $S$ with unit normal vector **n**, then the **surface integral of F over *S*** is
>
> $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$
>
> This integral is also called the **flux** of **F** across $S$.

In words, Definition 8 says that the surface integral of a vector field over *S* is equal to the surface integral of its normal component over *S*.

# Surface Integrals of Vector Fields

If $S$ is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS$$

$$= \iint_D \left[ \mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where $D$ is the parameter domain. Thus we have

**9**

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

# Example 4

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

# Surface Integrals of Vector Fields

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface $S$ given by a graph $z = g(x, y)$, we can think of $x$ and $y$ as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k}\right)$$

# Surface Integrals of Vector Fields

Thus Formula 9 becomes

**10**

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left( -P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

This formula assumes the upward orientation of *S*; for a downward orientation we multiply by –1.

Similar formulas can be worked out if *S* is given by $y = h(x, z)$ or $x = k(y, z)$.

# Example 5

**Example 5.** Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and $S$ is the boundary of the solid region $E$ enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations.

# Surface Integrals of Vector Fields

For instance, if **E** is an electric field, then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of **E** through the surface *S*. One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface *S* is

$$Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} \tag{11}$$

where $\varepsilon_0$ is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ C^2/N \cdot m^2$.)

# Surface Integrals of Vector Fields

Therefore, if the vector field **F** in Example 4 represents an electric field, we can conclude that the charge enclosed by $S$ is $Q = \frac{4}{3}\pi\varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point $(x, y, z)$ in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\,\nabla u$$

where $K$ is an experimentally determined constant called the **conductivity** of the substance. This is called Fourier's law of heat conduction.

# Surface Integrals of Vector Fields

The rate of heat flow across the surface *S* in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

# 16.8 Stokes' Theorem

# Stokes' Theorem

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem.

Whereas Green's Theorem relates a double integral over a plane region *D* to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface *S* to a line integral around the boundary curve of *S* (which is a space curve).

Figure 1 shows an oriented surface with unit normal vector **n**.

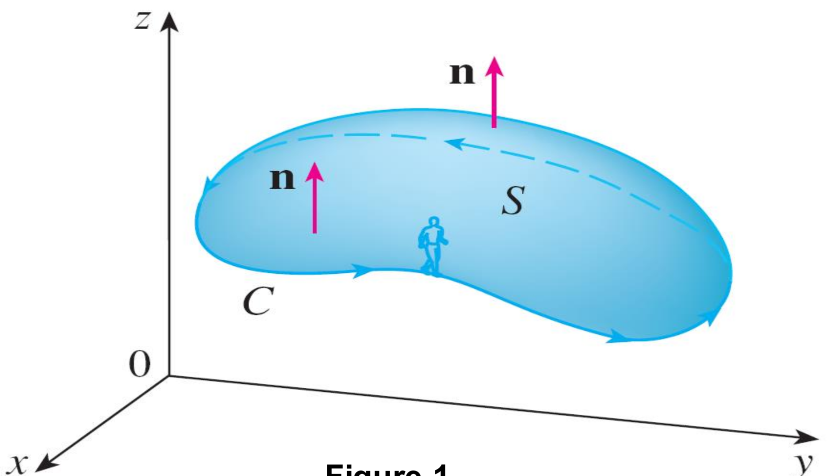



**Figure 1**

# Stokes' Theorem

The orientation of *S* induces the **positive orientation of the boundary curve *C*** shown in the figure.

This means that if you walk in the positive direction around *C* with your head pointing in the direction of **n**, then the surface will always be on your left.

**Stokes' Theorem** Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $C$ with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains $S$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$