# Examples

**Example 1.** Find $\int x^3 \cos(x^4 + 2)\, dx$.

**Example 2.** Evaluate $\int \sqrt{2x+1}\, dx$.

**Example 3.** Find

$$\int \frac{x}{\sqrt{1-4x^2}}\, dx.$$

**Example 4.** Evaluate $\int \cos 5x \; dx$.

**Example 5.** Evaluate $\int \sqrt{1+x^2}\, x^5\, dx$.

# Definite Integrals

# Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem.

For example,

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4 = \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4$$

$$= \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2} = \tfrac{1}{3}(27-1) = \tfrac{26}{3}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

# Definite Integrals

$\boxed{5}$ **The Substitution Rule for Definite Integrals** If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

# Examples

**Example 6.** Evaluate $\int_0^4 \sqrt{2x+1}\,dx$ .

**Example 7.** Evaluate

$$\int_1^2 \frac{dx}{(3-5x)^2}.$$

# 6.1 Calculus of Inverse Functions

# Inverse Functions

Recall the definition of inverse function.

**2 Definition** Let $f$ be a one-to-one function with domain $A$ and range $B$. Then its **inverse function** $f^{-1}$ has domain $B$ and range $A$ and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any $y$ in $B$.

This definition says that if $f$ maps $x$ into $y$, then $f^{-1}$ maps $y$ back into $x$. (If were not one-to-one, then $f^{-1}$ would not be uniquely defined.)

# Inverse Functions

The principle of interchanging $x$ and $y$ to find the inverse function also gives us the method for obtaining the graph of $f^{-1}$ from the graph of $f$.

Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point $(a, b)$ is on the graph of $f$ if and only if the point $(b, a)$ is on the graph of $f^{-1}$.

But we get the point $(b, a)$ from $(a, b)$ by reflecting about the line $y = x$. (See Figure 8.)

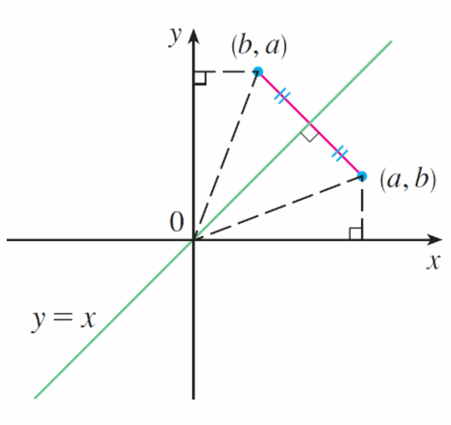



**Figure 8**

# Inverse Functions

Therefore, as illustrated by Figure 9:

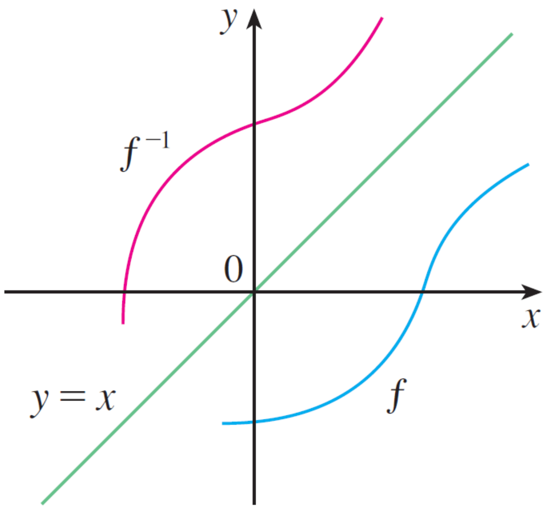



**Figure 9**

The graph of $f^{-1}$ is obtained by reflecting the graph of $f$ about the line $y = x$.

# The Calculus of Inverse Functions

# The Calculus of Inverse Functions

Now let's look at inverse functions from the point of view of calculus. Suppose that $f$ is both one-to-one and continuous. We think of a continuous function as one whose graph has no break in it. (It consists of just one piece.)

Since the graph of $f^{-1}$ is obtained from the graph of $f$ by reflecting about the line $y = x$, the graph of $f^{-1}$ has no break in it. Thus we might expect that $f^{-1}$ is also a continuous function.

# The Calculus of Inverse Functions

This geometrical argument does not prove the following theorem but at least it makes the theorem plausible.

$\boxed{6}$ **Theorem** If $f$ is a one-to-one continuous function defined on an interval, then its inverse function $f^{-1}$ is also continuous.

For a rigorous proof see Appendix F.

Now suppose that *f* is a one-to-one differentiable function. Geometrically we can think of a differentiable function as one whose graph has no corner or kink in it.

# The Calculus of Inverse Functions

We get the graph of $f^{-1}$ by reflecting the graph of $f$ about the line $y = x$, so the graph of $f^{-1}$ has no corner or kink in it either.

We therefore expect that $f^{-1}$ is also differentiable (except where its tangents are vertical). In fact, we can predict the value of the derivative of $f^{-1}$ at a given point by a geometric argument.

# The Calculus of Inverse Functions

In Figure 11 the graphs of $f$ and its inverse $f^{-1}$ are shown. If $f(b) = a$, then $f^{-1}(a) = b$ and $(f^{-1})'(a)$ is the slope of the tangent line $L$ to the graph of $f^{-1}$ at $(a, b)$, which is $\Delta y/\Delta x$.

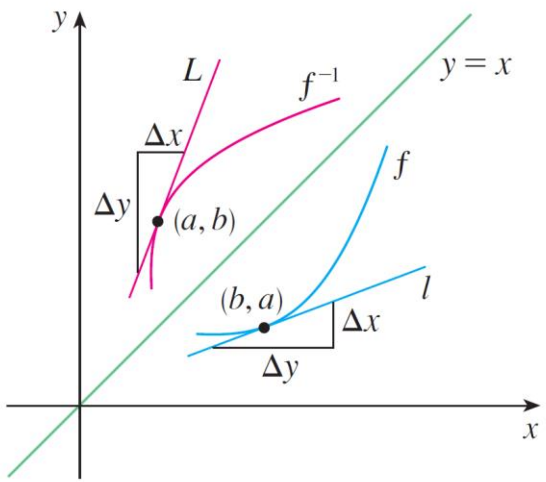



**Figure 11**

Reflecting in the line $y = x$ has the effect of interchanging the $x$- and $y$-coordinates.

# The Calculus of Inverse Functions

So the slope of the reflected line [the tangent to the graph of *f* at (*b*, *a*) is $\Delta x/\Delta y$. Thus the slope of *L* is the reciprocal of the slope of *l*, that is,

$$(f^{-1})'(a) = \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x/\Delta y} = \frac{1}{f'(b)}$$

**7 Theorem** If $f$ is a one-to-one differentiable function with inverse function $f^{-1}$ and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at $a$ and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

# The Calculus of Inverse Functions

**Note 1:**

Replacing *a* by the general number *x* in the formula of Theorem 7, we get

$$\boxed{8} \qquad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

If we write $y = f^{-1}(x)$, then $f(y) = x$, so Equation 8, when expressed in Leibniz notation, becomes

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

# The Calculus of Inverse Functions

**Note 2:**

If it is known in advance that $f^{-1}$ is differentiable, then its derivative can be computed more easily than in the proof of Theorem 7 by using implicit differentiation. If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating the equation $f(y) = x$ implicitly with respect to $x$, remembering that $y$ is a function of $x$, and using the Chain Rule, we get

$$f'(y)\frac{dy}{dx} = 1$$

Therefore
$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{\dfrac{dx}{dy}}$$

# Example 7

If $f(x) = 2x + \cos x$, find $(f^{-1})'(1)$.

# 6.2* The natural logarithmic function

# The natural logarithmic function

We start with the definition.

**1 Definition** The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

- If $x$ >1, then ln $x$ can be interpreted geometrically as the area under the hyperbola $y$ = 1/$t$ from $t$ = 1 to $t$ = $x$.
- For 0 < x < 1, $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$. Thus ln $x$ can be interpreted geometrically as the negative of the area under the hyperbola $y$ = 1/$t$ from $t$ = x to $t$ = *1*.
- $\ln 1 = 0$.

# The natural logartithmic function

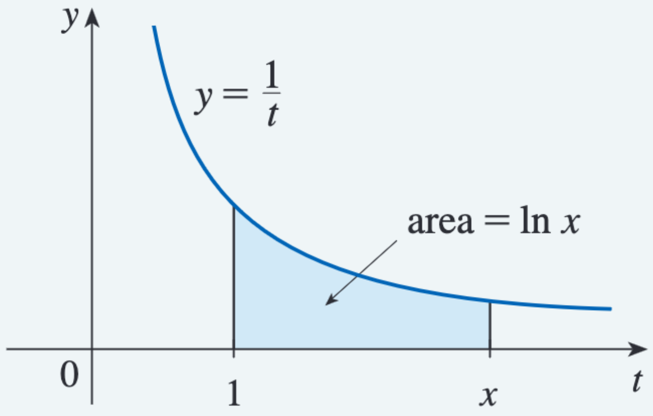



**FIGURE 1**

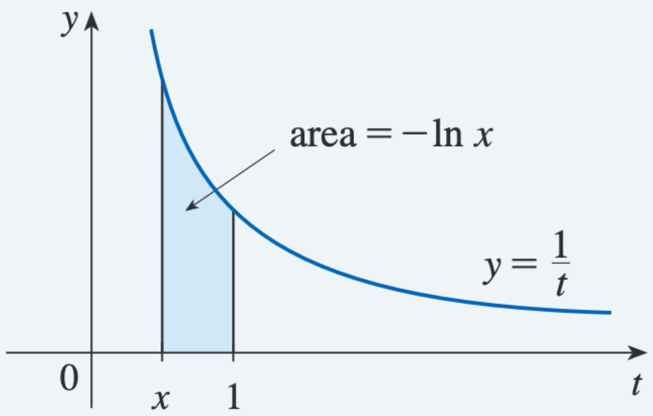



**FIGURE 2**

# Properties

**Theorem.**

- $\frac{d}{dx} \ln x = \frac{1}{x}, x > 0$
- $\ln(xy) = \ln x + \ln y\,, x, y > 0$
- $\ln \frac{x}{y} = \ln x - \ln y\,, x, y > 0$
- $\ln x^r = r \ln x\,, x > 0, r$ rational

# Example 2

Use the laws of logarithms to expand the expression

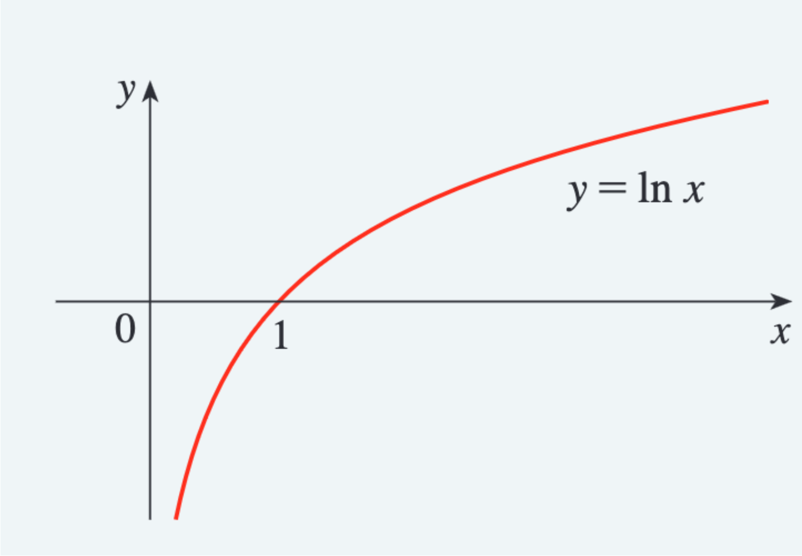
$$\ln \frac{(x^2 + 5)^4 \sin x}{x^3 + 1}$$

# Limits and graph

**Theorem.**

- $\ln x$ is increasing a concave down
- $\lim_{x\to\infty} \ln x = \infty$
- $\lim_{x\to 0+} \ln x = -\infty$

Hence its graph looks like:

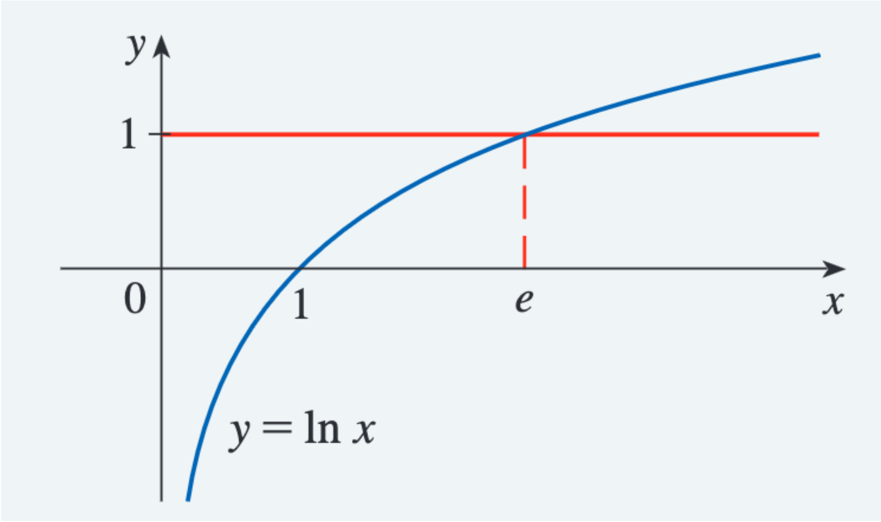


# The number *e*

Since ln 1 = 0 and ln *x* is an increasing continuous function that takes on arbitrarily large values, the Intermediate Value Theorem shows that there is a number where ln *x* takes on the value 1. This important number is denoted by *e*.

**5 Definition** $e$ is the number such that $\ln e = 1$.

Note: $e \approx 2.7182818253$



# Examples

**Example 5.** Differentiate $y = \ln(x^3 + 1)$.

**Example 7.** Differentiate $y = \sqrt{\ln x}$

**Example 8.** Differentiate $y = \ln \frac{x+1}{\sqrt{x-2}}$.

**Example 10.** Differentiate $f(x) = \ln|x|$

# Integrals

Example 10 shows that

**8** $$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

**Example 11.** Evaluate $\int \frac{x}{x^2+1}\,dx$.

**Example 12.** Evaluate $\int_1^e \frac{\ln x}{x}\,dx$.

**Example 13.** Evaluate $\int \tan x\,dx$.