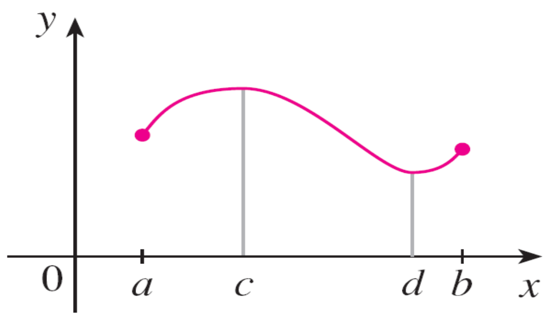
Maximum and Minimum Values

It appears that at local maximum and minimum points the tangent lines are horizontal and therefore each has slope 0.



We know that the derivative is the slope of the tangent line, so it appears that $f'(c) = 0$ and $f'(d) = 0$. The following theorem says that this is always true if the derivative exists.

4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 5

If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$.

But f has no maximum or minimum at 0, as you can see from its graph in Figure 11.

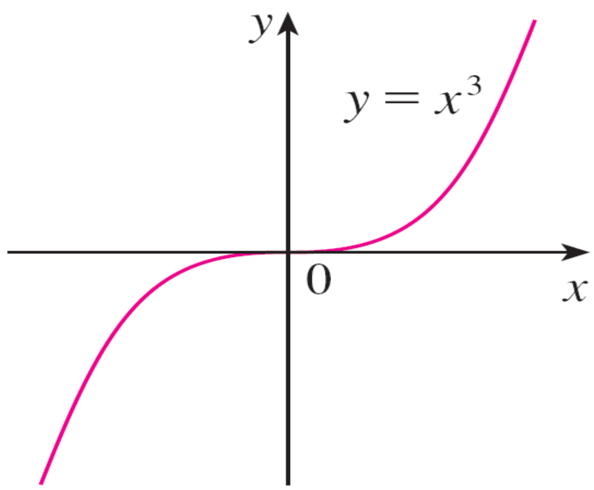


Figure 11

If $f(x) = x^3$, then $f'(0) = 0$ but f has no maximum or minimum.

Example 5

The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at (0, 0).

Instead of having a maximum or minimum at (0, 0), the curve crosses its horizontal tangent there.

Example 6

The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value can't be found by setting $f'(x) = 0$ because, $f'(0)$ does not exist. (see Figure 12)

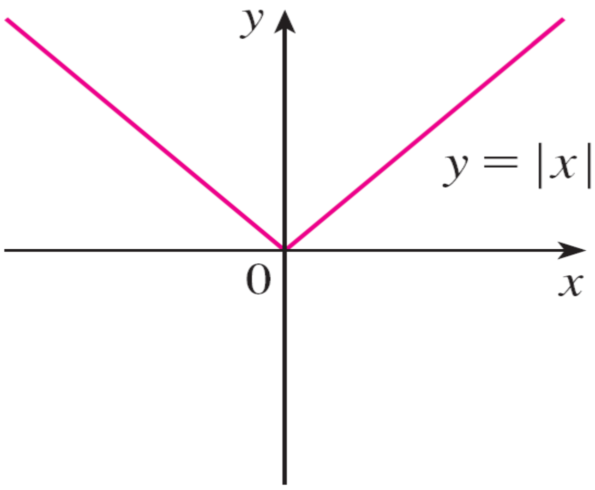


Figure 12

If $f(x) = |x|$, then $f(0) = 0$ is a minimum value, but $f'(0)$ does not exist.

Maximum and Minimum Values

Examples 5 and 6 show that we must be careful when using Fermat's Theorem. Example 5 demonstrates that even when $f'(c) = 0$, f doesn't necessarily have a maximum or minimum at c. (In other words, the converse of Fermat's Theorem is false in general.)

Furthermore, there may be an extreme value even when $f'(c)$ does not exist (as in Example 6).

Maximum and Minimum Values

Fermat’s Theorem does suggest that we should at least *start* looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

In terms of critical numbers, Fermat’s Theorem can be rephrased as follows.

7 If f has a local maximum or minimum at c, then c is a critical number of f.

Example 7

Find the critical numbers of (a) $f(x) = x^3 - 3x^2 + 1$

(b) $f(x) = x^{\frac{3}{5}}(4 - x)$

Maximum and Minimum Values

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval.

Thus the following three-step procedure always works.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Examples 8 and 9

Find the absolute maximum and minimum values of the function on the given interval.

Example 8. $f(x) = x^3 - 3x^2 + 1$ on $\left[-\frac{1}{2}, 4\right]$.

Example 9. $f(x) = x - 2\sin x$ on $[0, 2\pi]$.

3.2 The Mean Value Theorem

The Mean Value Theorem

We will see that many of the results depend on one central fact, which is called the Mean Value Theorem. But to arrive at the Mean Value Theorem we first need the following result.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem

Before giving the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses.

The Mean Value Theorem

Figure 1 shows the graphs of four such functions.

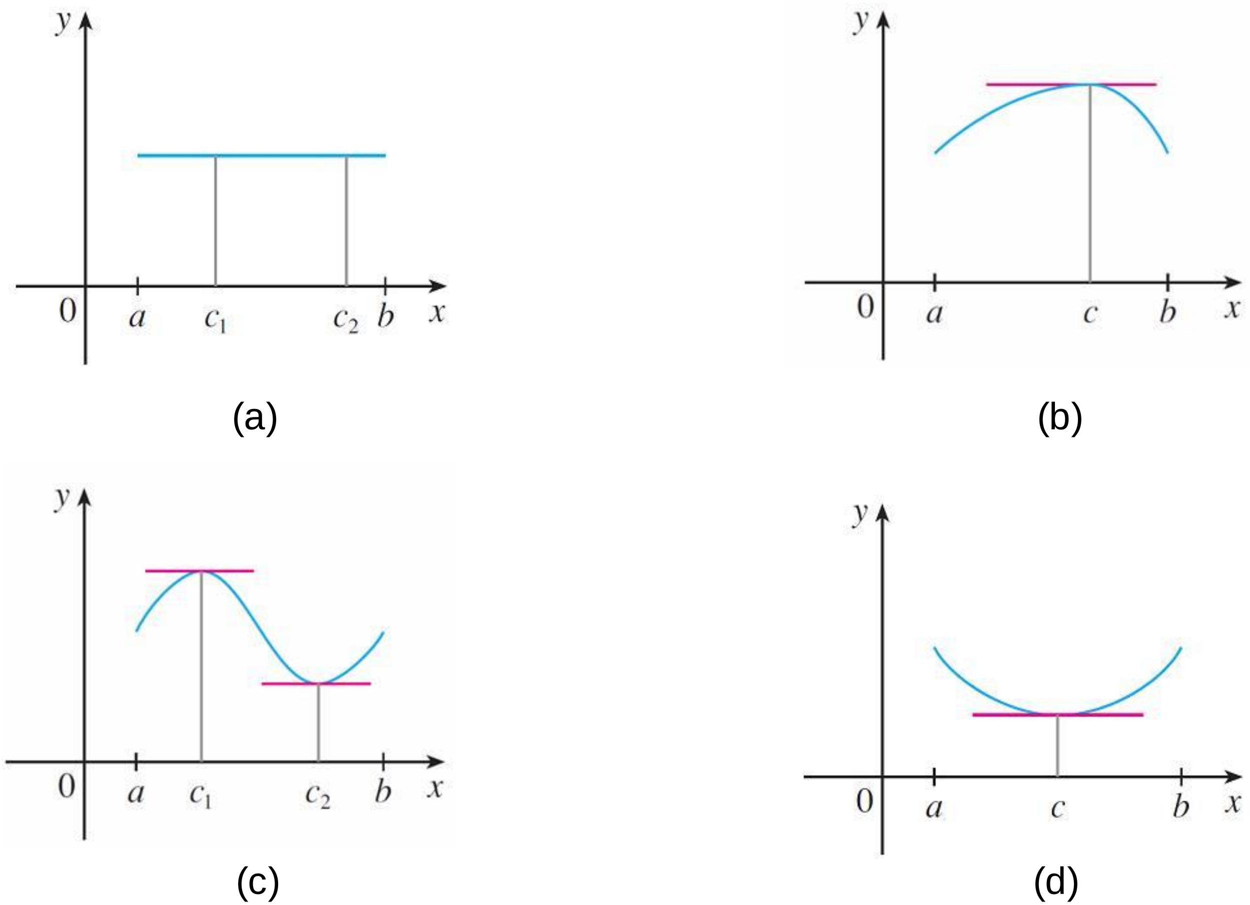


Figure 1

The Mean Value Theorem

In each case it appears that there is at least one point $(c, f(c))$ on the graph where the tangent is horizontal and therefore $f'(c) = 0$.

Thus Rolle's Theorem is plausible.

Example 1

Apply Rolle's Theorem to the position function $s = f(t)$ of a (rectilinearly) moving object.

Example 2

Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

The Mean Value Theorem

Our main use of Rolle's Theorem is in proving the following important theorem, which was first stated by another French mathematician, Joseph-Louis Lagrange.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

1 $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

2 $$f(b) - f(a) = f'(c)(b - a)$$

The Mean Value Theorem

Before proving this theorem, we can see that it is reasonable by interpreting it geometrically. Figures 3 and 4 show the points $A(a, f(a))$ and $B(b, f(b))$ on the graphs of two differentiable functions.

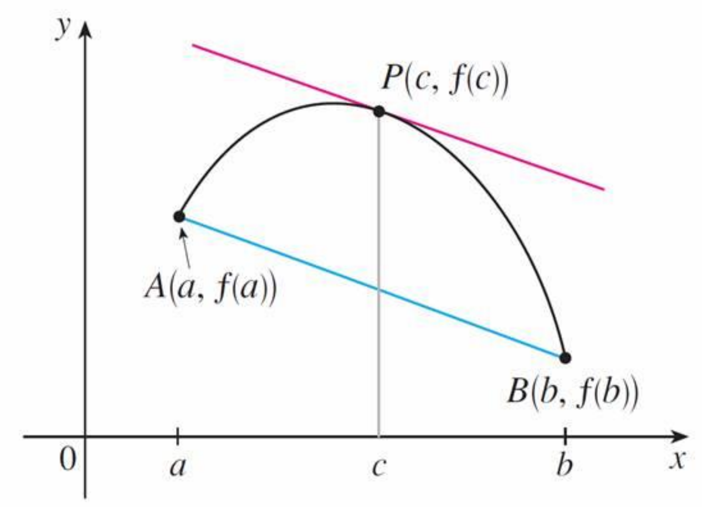


Figure 3

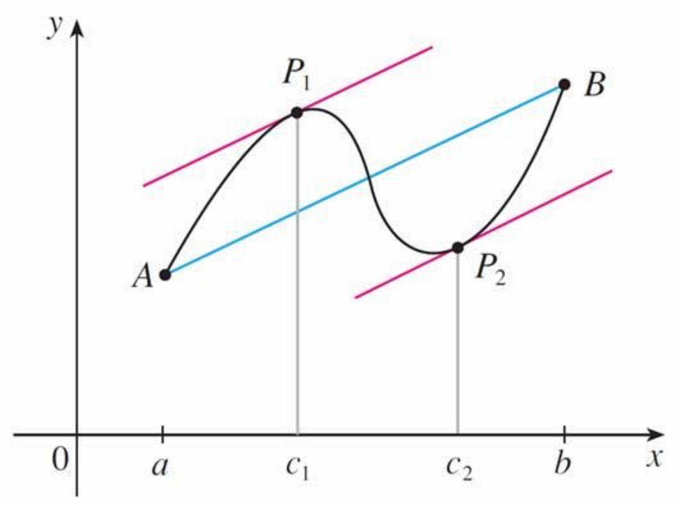


Figure 4

The Mean Value Theorem

The slope of the secant line *AB* is

$$\boxed{3} \qquad m_{AB} = \frac{f(b) - f(a)}{b - a}$$

which is the same expression as on the right side of Equation 1.

The Mean Value Theorem

Since $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$, the Mean Value Theorem, in the form given by Equation 1, says that there is at least one point $P(c, f(c))$ on the graph where the slope of the tangent line is the same as the slope of the secant line AB.

In other words, there is a point P where the tangent line is parallel to the secant line AB.

Example 3

To illustrate the Mean Value Theorem with a specific function, let's consider

$$f(x) = x^3 - x,\ a = 0,\ b = 2.$$

Example 3

Figure 6 illustrates the calculation:

The tangent line at this value of *c* is parallel to the secant line *OB*.

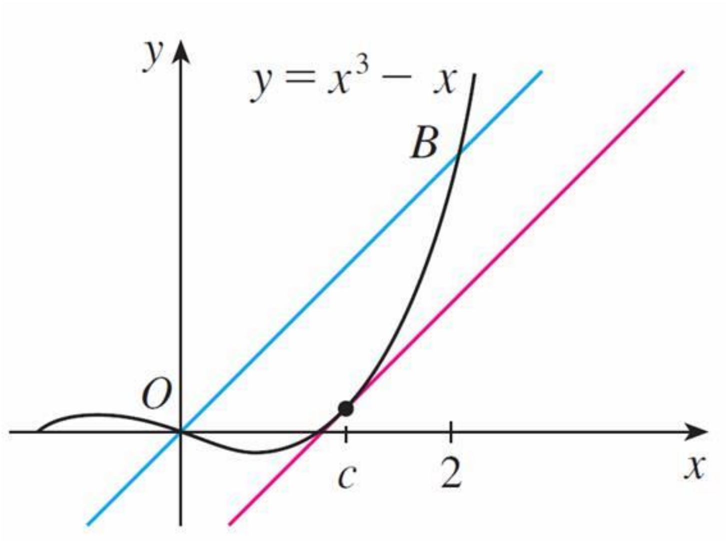


Figure 6

Example 4

Consider an object moves in a straight line with position function $s = f(t)$, between $t = a$ and $t = b$.

Example 5

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

The Mean Value Theorem

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus.

One of these basic facts is the following theorem.

$\boxed{5}$ **Theorem** If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

$\boxed{7}$ **Corollary** If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

The Mean Value Theorem

Note:

Care must be taken in applying Theorem 5. Let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function.

This does not contradict Theorem 5 because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.