Functions of Three or More Variables

Functions of Three or More Variables

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one for functions of two variables.

Differentiability

Definition. If $w = f(x, y, z)$ is defined at and near (a, b, c), then f is **differentiable** at (a, b, c) if the increment

$$\Delta w = f(a + \Delta x, b + \Delta y, c + \Delta z) - f(a, b, c)$$

can be expressed near (a, b, c) in the form

$$\Delta w = f_x(a, b, c)\Delta x + f_y(a, b, c)\Delta y + f_z(a, b, c)\Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are functions of $\Delta x, \Delta y, \Delta z$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0,0,0)$.

Note: Similarly to the two variable case if the partial derivatives f_x, f_y, f_z exist near (a, b, c) and are continuous at (a, b, c), then f is differentiable at (a, b, c).

Functions of Three or More Variables

For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

The **differential** dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$$

Example 6

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

14.5 The Chain Rule

The Chain Rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function:
If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\boxed{1} \qquad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

The Chain Rule

The first version (Theorem 1) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t. This means that z is indirectly a function of t, $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t.

Theorem 1 (Chain rule, case 1). Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The Chain Rule

We assume that f is differentiable. Recall that this is the case when f_x and f_y are continuous.

Since we often write $\partial z/\partial x$ in place of $\partial f/\partial x$, we can rewrite the Chain Rule in the form

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Example 1

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

A multivariable Taylor's formula

Suppose that f is a function of 2 variables and suppose that aall second order partial derivatives exist in an open disc D with center $\mathbf{a} = (a_1, a_2)$ and are continuous at **a**. If $\mathbf{x} = (x_1, x_2) \in D,$ then there is **b** on the line segment joining **a** and **x** such that

$$f(\mathbf{x}) = f(\mathbf{a}) + f_x(\mathbf{a})(x_1 - a_1) + f_y(\mathbf{a})(x_2 - a_2)$$
$$+\frac{1}{2}\left(f_{xx}(\mathbf{b})(x_1 - a_1)^2 + f_{yy}(\mathbf{b})(x_2 - a_2)^2 + 2f_{xy}(\mathbf{b})(x_1 - a_1)(x_2 - a_2)\right).$$

Remark. The above can be naturally extended to functions of n variables.

The Chain Rule

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t:

$$x = g(s, t),\ y = h(s, t).$$

Then z is indirectly a function of s and t and we wish to find $\partial z/\partial s$ and $\partial z/\partial t$.

Recall that in computing $\partial z/\partial t$ we hold s fixed and compute the ordinary derivative of z with respect to t.

Therefore, we can apply Theorem 1 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The Chain Rule

A similar argument holds for $\partial z/\partial s$ and so we have proved the following version of the Chain Rule.

Theorem 2 (Chain rule, case 2). Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Case 2 of the Chain Rule contains three types of variables: *s* and *t* are **independent** variables, *x* and *y* are called **intermediate** variables, and *z* is the **dependent** variable.

The Chain Rule

Notice that Theorem 2 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule.

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2.

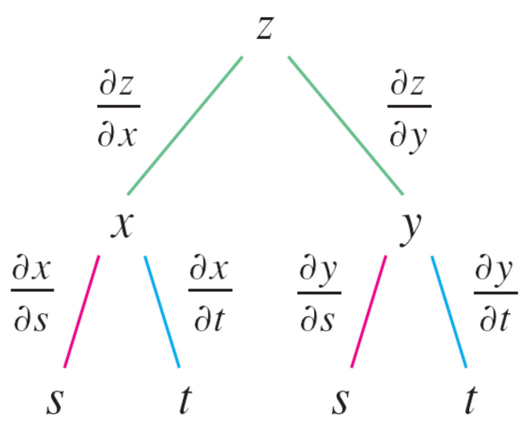

Figure 2

The Chain Rule

We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then we draw branches from x and y to the independent variables s and t.

On each branch we write the corresponding partial derivative. To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly, we find $\partial z/\partial t$ by using the paths from z to t.

Example 3

If $z = e^x \sin y$ where $x = st^2$and $y = s^2 t$ find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

The Chain Rule

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, \ldots, x_n$, each of which is, in turn, a function of m independent variables $t_1, \ldots, t_m$. Notice that there are n terms in the Theorem, one for each intermediate variable.

Theorem (Chain rule, general version). Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Examples 4 and 5

Example 4. Write out the chain rule for the case where and $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, and $t = t(u,v)$.

Example 5. If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$ and $t = 0$.

Implicit Differentiation

Implicit Differentiation

The Chain Rule can be used to give a more complete description of the process of implicit differentiation. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f.

If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x.

Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Implicit Differentiation

But $dx/dx = 1$, so if $\partial F/\partial y \neq 0$ we solve for dy/dx and obtain

$$\boxed{\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

To derive this equation, we assumed that $F(x, y) = 0$ defines y implicitly as a function of x.

Implicit Function Theorem

The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid:

It states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a differentiable function of x near the point (a, b) and the derivative of this function shown in the previous slide.

Example 8

Find y' if $x^3 + y^3 = 6xy$.

Implicit Differentiation

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Implicit Differentiation

But $\dfrac{\partial}{\partial x}(x) = 1$ and $\dfrac{\partial}{\partial x}(y) = 0$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} = -\frac{F_x}{F_z}.$$

Implicit Function Theorem

Similarly

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Again, a version of the **Implicit Function Theorem** gives conditions under which our assumption is valid:

If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given as above.

Example 9

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz + 4 = 0$.