# Recap: the Definite Integral

**2 Definition of a Definite Integral** If $f$ is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so $x_i^*$ lies in the $i$th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of $f$ from $a$ to $b$** is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that $f$ is **integrable** on $[a, b]$.

**Note:** We saw that, for example, continuous functions are integrable.

# The Midpoint Rule

# The Midpoint Rule

We often choose the sample point $x_i^*$ to be the right endpoint of the *i*th subinterval because it is convenient for computing the limit.

But if the purpose is to find an *approximation* to an integral, it is usually better to choose $x_i^*$ to be the midpoint of the interval, which we denote by $\overline{x}_i$.

# The Midpoint Rule

Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

**Midpoint Rule**

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\,\Delta x = \Delta x\,[f(\overline{x}_1) + \cdots + f(\overline{x}_n)]$$

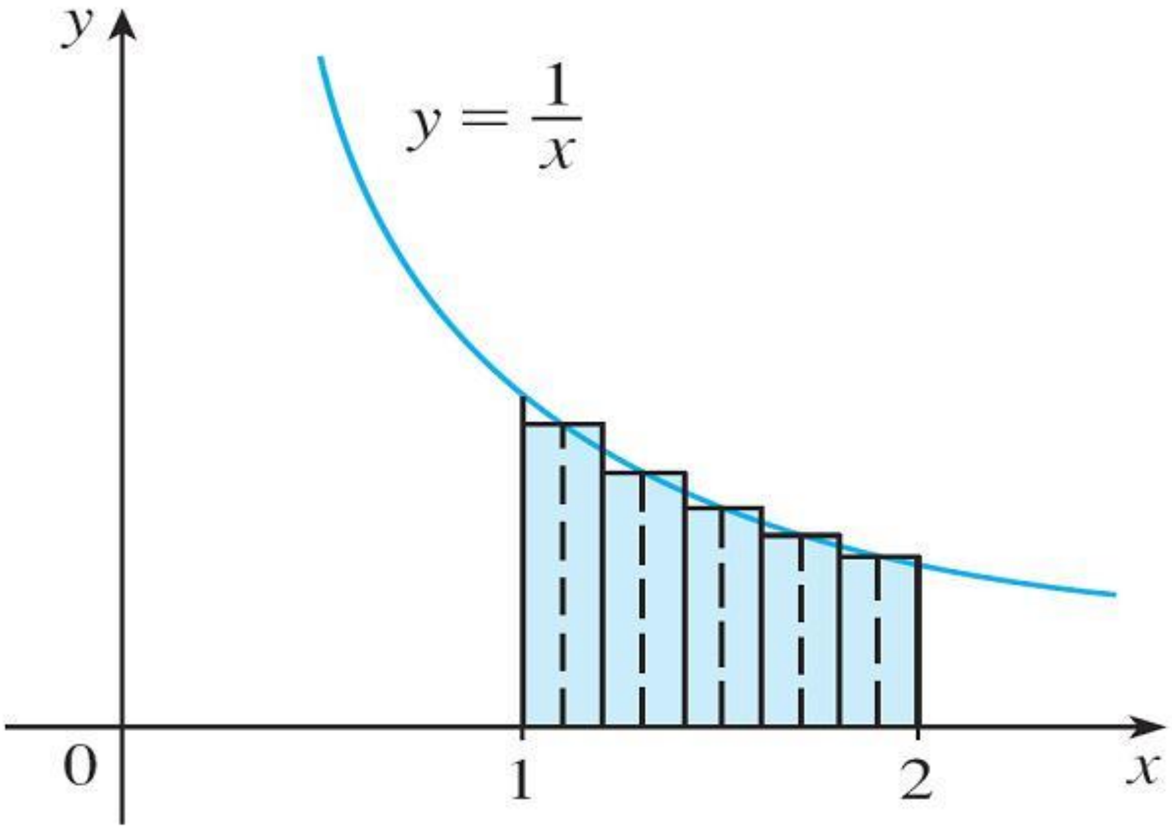
where $$\Delta x = \frac{b-a}{n}$$

and $$\overline{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

# Example 5

Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\,dx$.

# Example



# Properties of the Definite Integral

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When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$.

But the definition as a limit of Riemann sums makes sense even if $a > b$.

Notice that if we reverse $a$ and $b$, then $\Delta x$ changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

# Properties of the Definite Integral

If $a = b$, then $\Delta x = 0$ and so

$$\boxed{\int_a^a f(x)\,dx = 0}$$

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner. We assume that *f* and *g* are continuous functions.

# Propertises of the integral

**Properties of the Integral**

1. $\int_a^b c\,dx = c(b-a)$, where $c$ is any constant
2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where $c$ is any constant
4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

# Example 6

Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

# Properties of the Definite Integral

The next property tells us how to combine integrals of the same function over adjacent intervals:

**5.** $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

# Properties of the Definite Integral

This is not easy to prove in general, but for the case where $f(x) \geq 0$ and $a < c < b$ Property 5 can be seen from the geometric interpretation in Figure 15: The area under $y = f(x)$ from $a$ to $c$ plus the area from $c$ to $b$ is equal to the total area from $a$ to $b$.

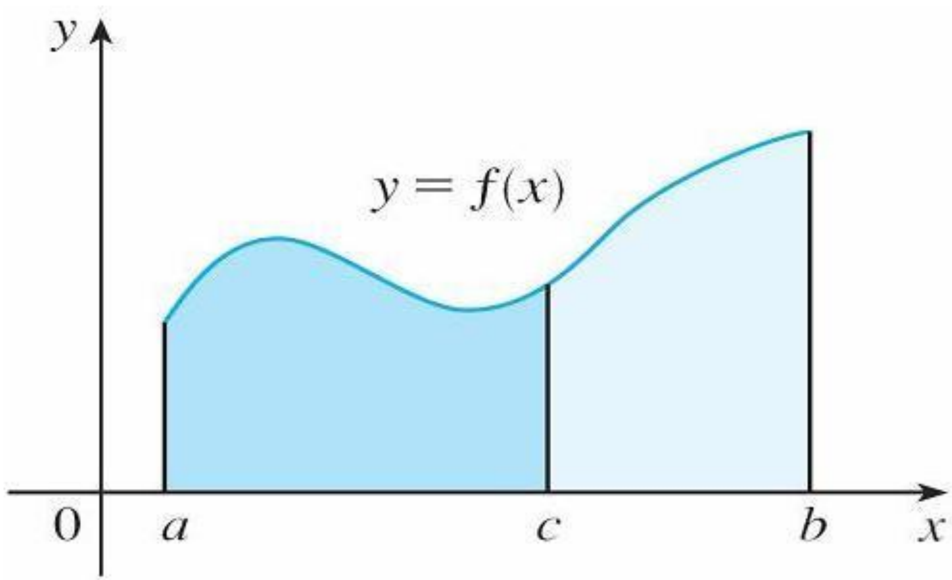


**Figure 15**

# Properties of the Definite Integral

Properties 1–5 are true whether *a* < *b*, *a* = *b*, or *a* > *b*. The following properties, in which we compare sizes of functions and sizes of integrals, are true only if *a* ≤ *b*.

**Comparison Properties of the Integral**

**6.** If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

**7.** If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

**8.** If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$$

# Example 9

Use Property 8 to estimate $\int_1^4 \sqrt{x}\, dx$.

# 4.3 The Fundamental Theorem of Calculus

# The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus.

It gives the precise inverse relationship between the derivative and the integral.

# The Fundamental Theorem of Calculus

The first part of the Fundamental Theorem deals with functions defined by an equation of the form

$$\boxed{1} \qquad g(x) = \int_a^x f(t)\,dt$$

where *f* is a continuous function on [*a*, *b*] and *x* varies between *a* and *b*. Observe that *g* depends only on *x*, which appears as the variable upper limit in the integral. If *x* is a fixed number, then the integral $\int_a^x f(t)\,dt$ is a definite number.

If we then let *x* vary, the number $\int_a^x f(t)\,dt$ also varies and defines a function of *x* denoted by *g*(*x*).

# The Fundamental Theorem of Calculus

If *f* happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of *f* from *a* to *x*, where *x* can vary from *a* to *b*. (Think of *g* as the "area so far" function; see Figure 1.)

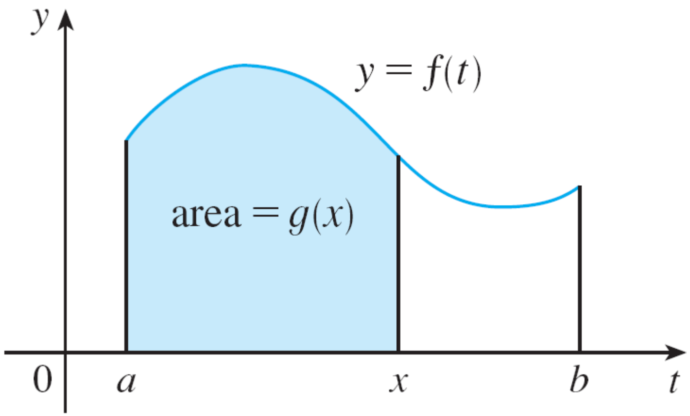


**Figure 1**

# The Fundamental Theorem of Calculus

We already saw that if we take $f(t) = t^2$ and $a = 0$, then we have

$$g(x) = \int_0^x f(t)\,dt = \int_0^x t^2\,dt = \frac{x^3}{3}$$

Notice that $g'(x) = x^2$, that is, $g' = f$. In other words, if $g$ is defined as the integral of $f$ by Equation 1, then $g$ turns out to be an antiderivative of $f$, at least in this case.

# The Fundamental Theorem of Calculus

To see why this might be generally true we consider any continuous function $f$ with $f(x) \geq 0$. Then $g(x) = \int_a^x f(t)\, dt$ can be interpreted as the area under the graph of $f$ from $a$ to $x$, as in Figure 1.

In order to compute $g'(x)$ from the definition of a derivative we first observe that, for $h > 0$, $g(x + h) - g(x)$ is obtained by subtracting areas, so it is the area under the graph of $f$ from $x$ to $x + h$ (the blue area in Figure 5).

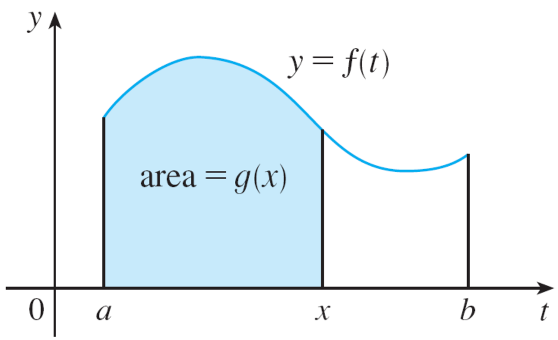


**Figure 1**

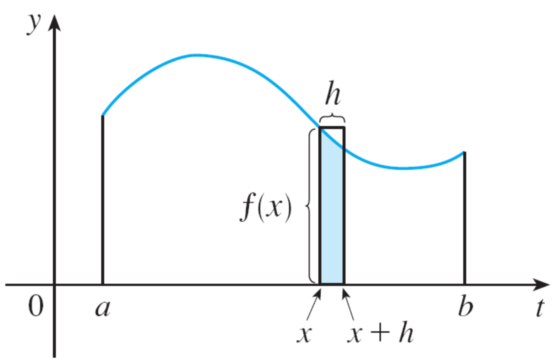


**Figure 5**

# The Fundamental Theorem of Calculus

For small $h$ you can see from the figure that this area is approximately equal to the area of the rectangle with height $f(x)$ and width $h$:

$$g(x + h) - g(x) \approx hf(x)$$

so

$$\frac{g(x + h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x)$$

# The Fundamental Theorem of Calculus

The fact that this is true, even when *f* is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

**The Fundamental Theorem of Calculus, Part 1** If $f$ is continuous on $[a, b]$, then the function $g$ defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on $(a, b)$, and $g'(x) = f(x)$.

Using Leibniz notation for derivatives, we can write this theorem as

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

when *f* is continuous.

# The Fundamental Theorem of Calculus

Roughly speaking, this equation says that if we first integrate $f$ and then differentiate the result, we get back to the original function $f$.

# Example 2

Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

# Example 3

Although a formula of the form $g(x) = \int_a^x f(t)\,dt$ may seem like a strange way of defining a function, books on physics, chemistry, and statistics are full of such functions. For instance, the **Fresnel function**

$$S(x) = \int_0^x \sin(\pi t^2/2)\,dt$$

is named after the French physicist Augustin Fresnel (1788–1827), who is famous for his works in optics.

This function first appeared in Fresnel's theory of the diffraction of light waves, but more recently it has been applied to the design of highways.

# The Fundamental Theorem of Calculus

The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

**The Fundamental Theorem of Calculus, Part 2** If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$, that is, a function such that $F' = f$.

More precisely, *F* should be continuous on $[a, b]$, differentiable on $(a, b)$ and $F'(x) = f(x)$ for all $x \in (a, b)$.

## Differentiation and Integration as Inverse Processes

# Differentiation and Integration as Inverse Processes

We end this section by bringing together the two parts of the Fundamental Theorem.

**The Fundamental Theorem of Calculus** Suppose $f$ is continuous on $[a, b]$.

**1.** If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$.

**2.** $\int_a^b f(x)\, dx = F(b) - F(a)$, where $F$ is any antiderivative of $f$, that is, $F' = f$.

We noted that Part 1 can be rewritten as

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

which says that if $f$ is integrated and then the result is differentiated, we arrive back at the original function $f$.

# Differentiation and Integration as Inverse Processes

Since $F'(x) = f(x)$, Part 2 can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This version says that if we take a function $F$, first differentiate it, and then integrate the result, we arrive back at the original function $F$, but in the form $F(b) – F(a)$.

Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are inverse processes. Each undoes what the other does.