2.5 The Chain Rule

The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = x^2 + 1$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$.

We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g.

The Chain Rule

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the *Chain Rule.*

This seems plausible as $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$ and $\Delta u \to 0$ as $\Delta x \to 0$ (by the continuity of u) and thus letting $\Delta x \to 0$ we expect $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

The flaw in this argument is that Δu might be 0 even if $\Delta x \neq 0$.

The Chain Rule

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

The Chain Rule

The Chain Rule can be written either in the prime notation

$$\boxed{2} \qquad (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

or, if $y = f(u)$ and $u = g(x)$, in Leibniz notation:

$$\boxed{3} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Equation 3 is easy to remember because if dy/du and du/dx were quotients, then we could cancel du.

Remember, however, that du has not been defined and du/dx should not be thought of as an actual quotient.

Examples

Example 1. Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

Example 2. Differentiate (a) $y = \sin(x^2)$ (b) $y = \sin^2 x$

Example 3. Differentiate $y = (x^3 - 1)^{100}$.

Example 4. Find f' if $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$.

Example 5. Find the derivative of the function

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

Examples

Example 6. Differentiate $y = (2x + 1)^5(x^3 - x + 1)^4$.

Chain rule for more than 2 functions

If we want to differentiate

$$F(x) = f\big(g(h(x))\big),$$

then we can use the chain rule twice to get

$$\begin{aligned} F'(x) &= f'\big(g(h(x))\big)\frac{d}{dx}[g(h(x))] \\ &= f'\big(g(h(x))\big)g'\big(h(x)\big)h'(x) \end{aligned}$$

Example 7. Differentiate $y = \sin(\cos(\tan x))$.

Example 8. Differentiate $y = \sqrt{\sec x^3}$.

2.6 Implicit Differentiation

Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$.

Some functions, however, are defined implicitly by a relation between x and y such as

$$\boxed{1} \qquad x^2 + y^2 = 25$$

or

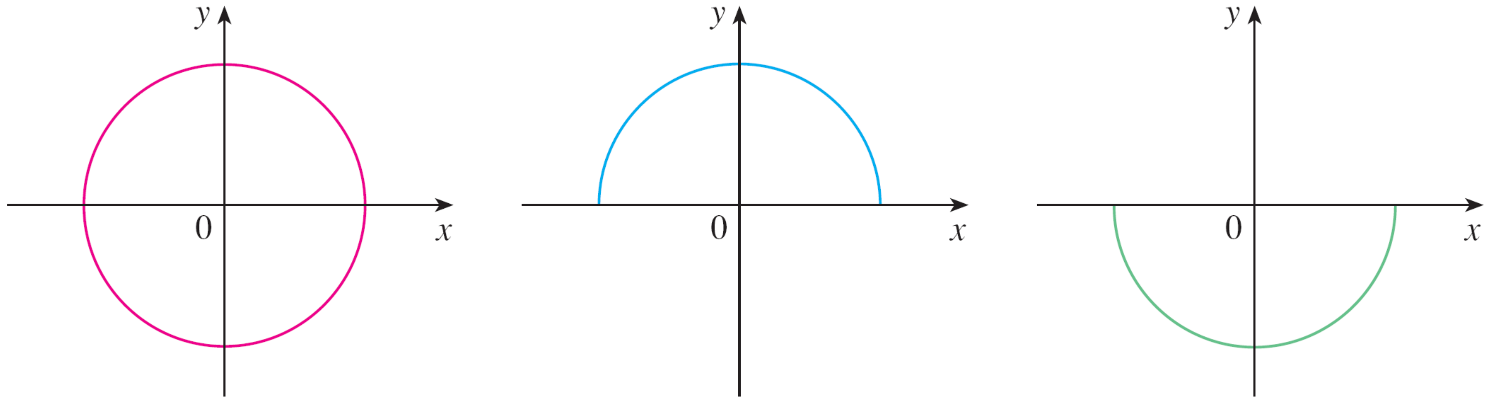
$$\boxed{2} \qquad x^3 + y^3 = 6xy$$

Implicit Differentiation

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x.

For instance, if we solve Equation 1 for y, we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$.

The graphs of f and g are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)



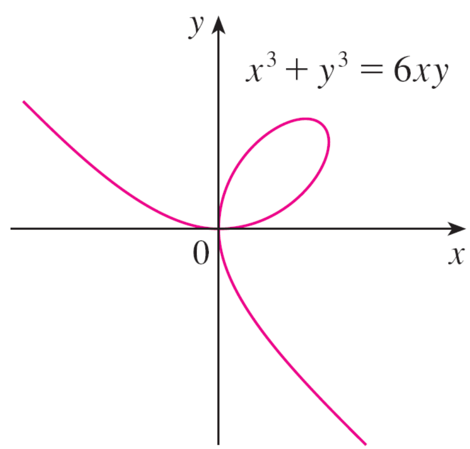
(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

Figure 1

Implicit Differentiation

It's not easy to solve Equation 2 for *y* explicitly as a function of *x* by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.)

Nonetheless, (2) is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines *y* as several functions of *x*.

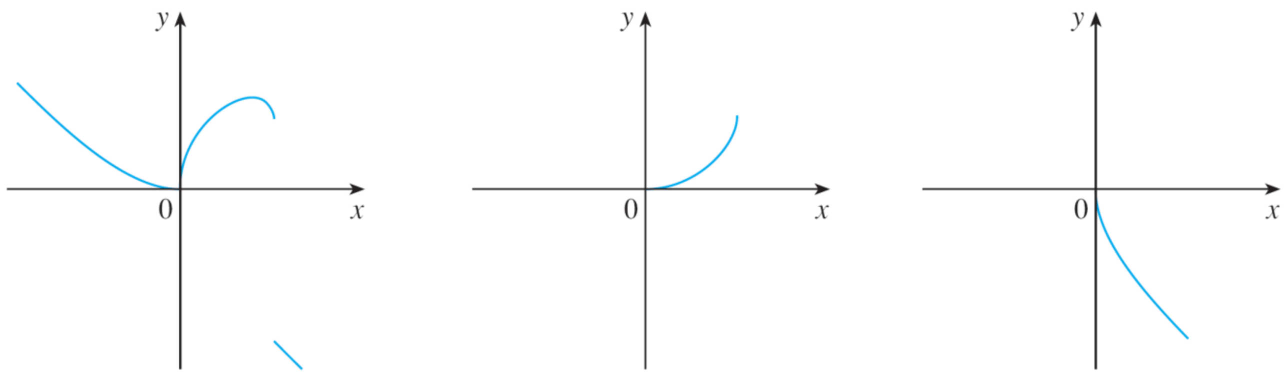


The folium of Descartes

Figure 2

Implicit Differentiation

The graphs of three such functions are shown in Figure 3.



Graphs of three functions defined by the folium of Descartes

Figure 3

When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6x f(x)$$

is true for all values of x in the domain of f.

Implicit Differentiation

Fortunately, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of **implicit differentiation**.

This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Example 1

(a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4).

Example 2

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent line to the folium of Descartes $x^3 + y^3 = 6xy$ at the point (3, 3).

(c) At what point in the (open) first quadrant is the tangent line horizontal?

Examples 3 and 4

Example 3. Find y' if $\sin(x+y) = y^2 \cos x$.

Example 4. Find y'' if $x^4 + y^4 = 16$.

How to Prove the Chain Rule

Proof of the chain rule

In order to properly prove the chain rule one needs the following observation:

Proposition. A function $y = f(x)$, is differentiable at a if and only if there is a number A such that for Δx small nonnegative:

$$\Delta y = A\,\Delta x + \varepsilon\,\Delta x \quad \text{where} \quad \varepsilon \to 0 \text{ as } \Delta x \to 0$$

and ε is a continuous function of Δx at 0. In this case $f'(a) = A$.

Here: $\Delta y = f(a + \Delta x) - f(a)$.