Example 6

Find $F'(x)$ if $F(x) = (6x^3)(7x^4)$.

Example 7

If $h(x) = xg(x)$ and it is known that $g(3) = 5$ and $g'(3) = 2$, find $h'(3)$.

Example 8

Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Find *y'*.

General Power Functions

The Quotient Rule can be used to extend the Power Rule to the case where the exponent is a negative integer.

If n is a positive integer, then

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

Example 9

(a) If $y = \frac{1}{x}$, then find $\frac{dy}{dx}$.

(b) Find $\frac{d}{dt}\left(\frac{6}{t^3}\right)$

General Power Functions

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 10

(a) If $f(x) = x^{\pi}$, find $f'(x)$.

(b) Find $\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^2}}\right)$.

Example 11

Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

General Power Functions

The differentiation rules enable us to find tangent lines without having to resort to the definition of a derivative.

They also enable us to find *normal lines*.

The **normal line** to a curve C at point P is the line through P that is perpendicular to the tangent line at P.

Example 12

Find equations of the tangent line and normal line to the curve

$$y = \frac{\sqrt{x}}{1+x^2}$$

at the point $(1, \frac{1}{2})$.

Example 12

The curve and its tangent and normal lines are graphed in Figure 5.

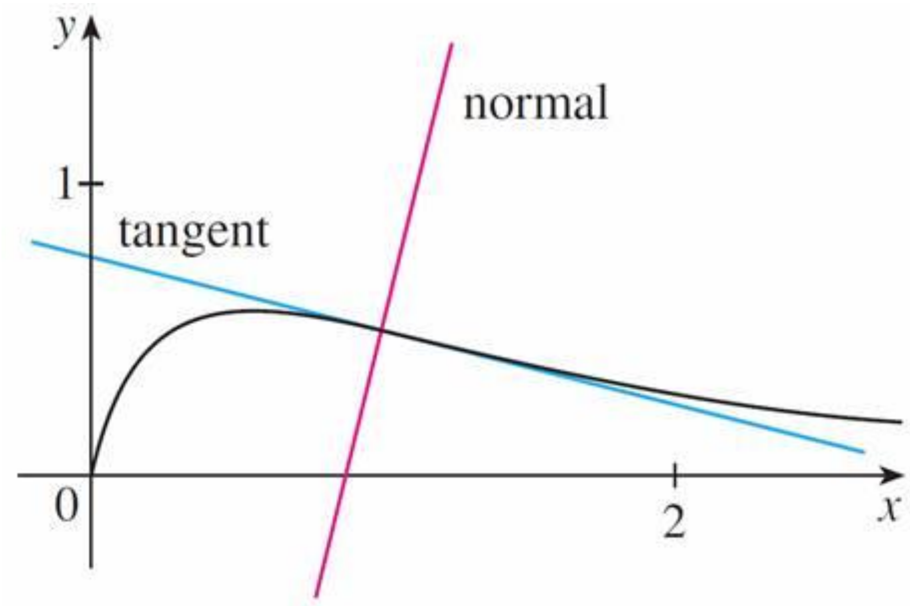


Figure 5

2.4 Derivatives of Trigonometric Functions

Derivatives of Trigonometric Functions

In particular, it is important to remember that when we talk about the function f defined for all real numbers x by

$$f(x) = \sin x$$

it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x. A similar convention holds for the other trigonometric functions cos, tan, csc, sec, and cot.

All of the trigonometric functions are continuous at every number in their domains.

Derivatives of Trigonometric Functions

If we sketch the graph of the function $f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f', then it looks as if the graph of f' may be the same as the cosine curve. (See Figure 1).

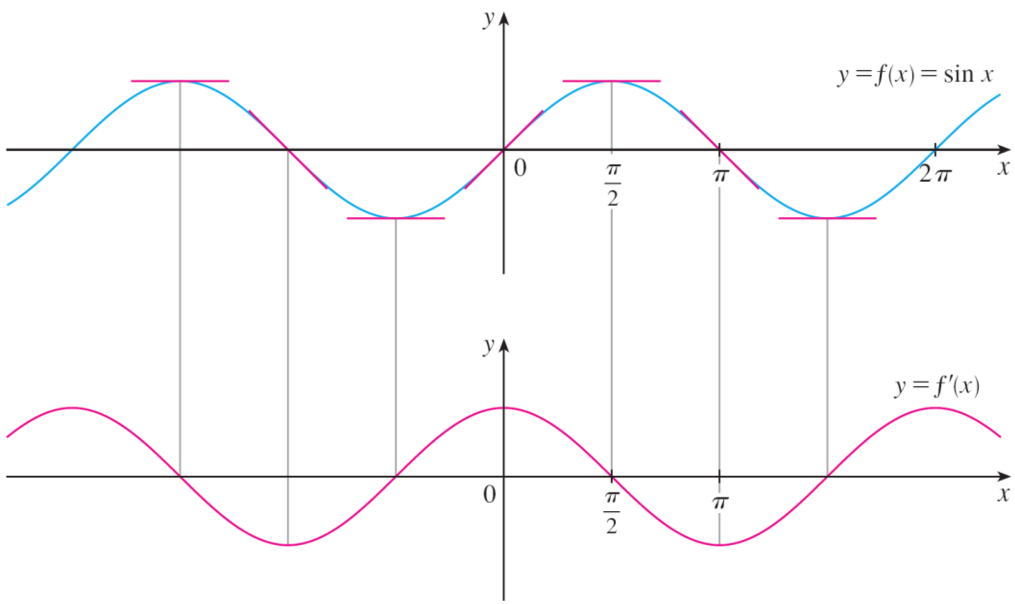


Figure 1

Derivatives of Trigonometric Functions

In order to calculate the derivative of $f(x) = \sin x$ we first show that

$$\boxed{2} \qquad \boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$

and as consequence we will show that:

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0.$$

Derivatives of Trigonometric Functions

Now we can prove the formula for the derivative of the sine function:

4 $$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Example 1

Differentiate $y = x^2 \sin x$.

Derivatives of Trigonometric Functions

Using the same methods as in the proof of Formula 4, one can prove that

5 $$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 4 and 5:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

Derivatives of Trigonometric Functions

The tangent function can also be differentiated by using the definition of a derivative, but it is easier to use the Quotient Rule together with Formulas 4 and 5 to get:

6
$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

The derivatives of the remaining trigonometric functions, csc, sec, and cot, can also be found easily using the Quotient Rule.

Derivatives of Trigonometric Functions

We collect all the differentiation formulas for trigonometric functions in the following table. Remember that they are valid only when *x* is measured in radians.

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Trigonometric Functions

Trigonometric functions are often used in modeling real-world phenomena. In particular, vibrations, waves, elastic motions, and other quantities that vary in a periodic manner can be described using trigonometric functions. In the following example we discuss an instance of simple harmonic motion.

Example 2

Differentiate

$$f(x) = \frac{\sec x}{1+\tan x}.$$

For what values of *x* does the graph of *f* have a horizontal tangent?

Example 3

An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time t = 0. (See Figure 5 and note that the downward direction is positive.)

Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion of the object.

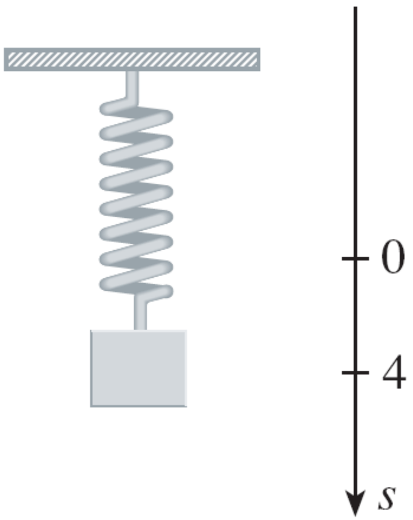
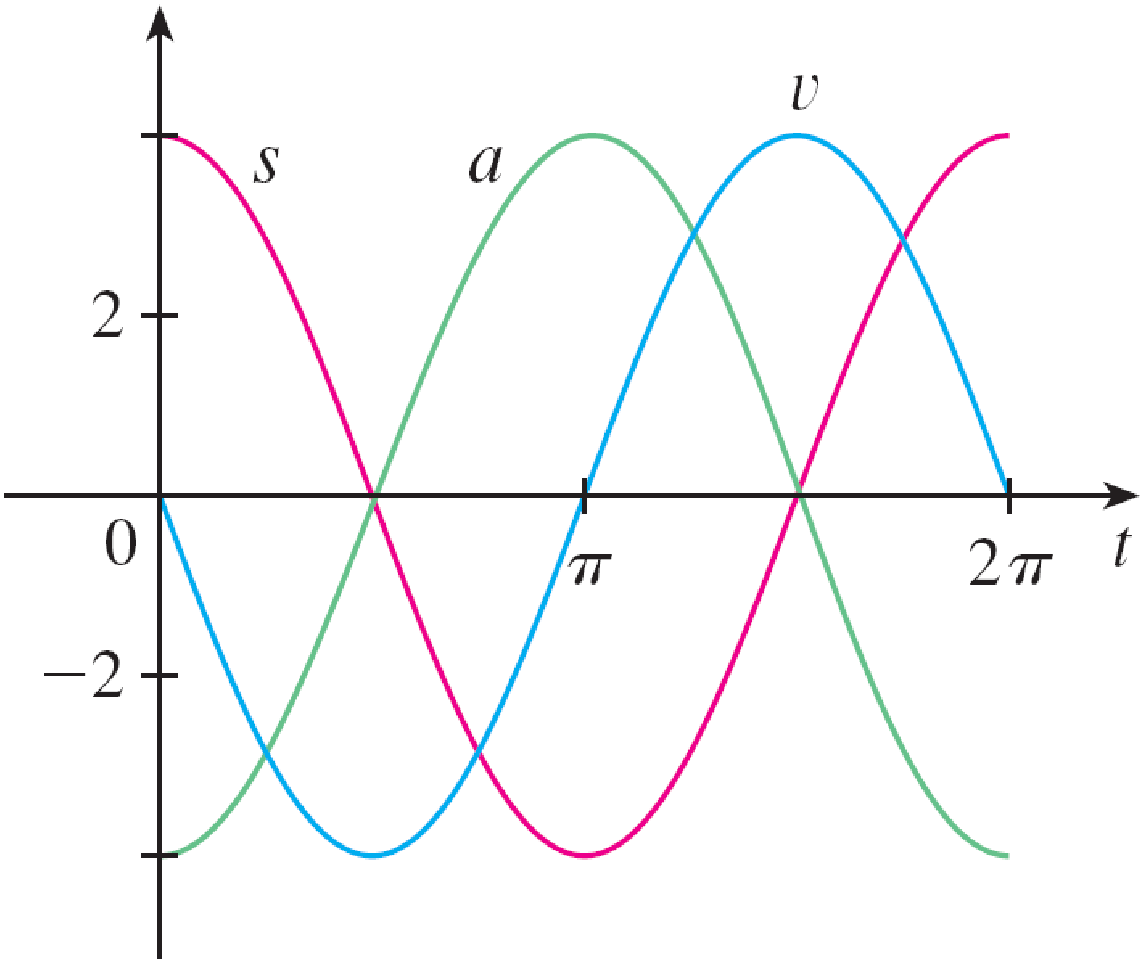


Figure 5

Example 3



Example 4

Find the 27th derivative of cos *x*.

Examples 5 - 7

Example 5. Find

$$\lim_{x \to 0} \frac{\sin 7x}{5x}.$$

Example 6. Find

$$\lim_{x \to 0} x \cot x.$$

Example 7. Find

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta}.$$