6.3* The natural exponential function

The natural exponential function

Since ln is an increasing function, it is one-to-one and therefore has an inverse function, which we denote by exp. Thus, according to the definition of an inverse function.

1 $$\boxed{\exp(x) = y \iff \ln y = x}$$

Cancellation equations:

- $\exp(\ln x) = x$ for all $x > 0$
- $\ln(\exp x) = x$ for all $x \in \mathbb{R}$

Elementary Properties

Theorem.

- $\exp 0 = 1$
- $\exp 1 = e$
- If r is a rational number, then $\exp r = e^r$

Because of the last property we will define, for irrational values of *x*, the number e^x by

$$e^x = \exp x$$

Elementary properties

With this notation we have

$$e^x = y \Leftrightarrow \ln y = x$$

and the cancellation equations:

- $e^{\ln x} = x$ for all $x > 0$
- $\ln e^x = x$ for all $x \in \mathbb{R}$

Examples

Example 1. Find x if $\ln x = 5$.

Example 2. Solve the equation $e^{5-3x} = 10$

Further properties

Theorem.

- Domain of $f(x) = e^x$ is $(-\infty, \infty)$, the range of exp in $(0, \infty)$
- The function $f(x) = e^x$ continuous and increasing
- $\lim\limits_{x \to \infty} e^x = \infty$
- $\lim\limits_{x \to -\infty} e^x = 0$
- Laws of exponents:

  a) $e^{x+y} = e^x e^y$, for all $x, y \in (-\infty, \infty)$

  b) $e^{x-y} = \frac{e^x}{e^y}$, for all $x, y \in (-\infty, \infty)$

  c) $(e^x)^r = e^{rx}$, for all $x \in (-\infty, \infty)$ and r rational

Graph of the natural exp function

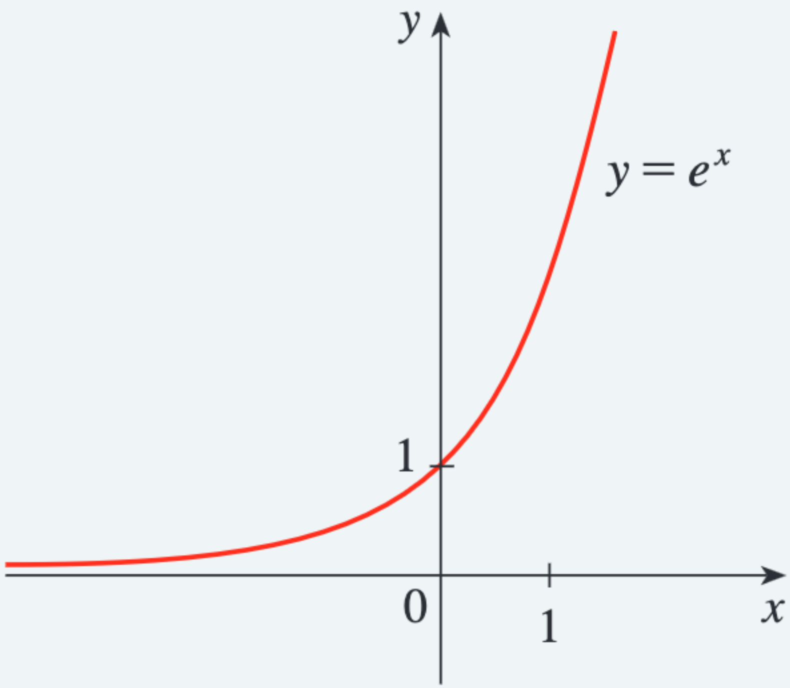


FIGURE 2
The natural exponential function

Example 3

Example 3. Find

$$\lim_{x\to\infty} \frac{e^{2x}}{e^{2x}+1}$$

Differentiation and integration

Theorem.

- $\frac{d}{dx} e^x = e^x$
- $\int e^x \, dx = e^x + C$

Examples

Example 4. Differentiate $y = e^{\tan x}$.

Example 6. Find the absolute maximum value of the function $f(x) = xe^{-x}$.

Example 8. Evaluate $\int x^2 e^{x^3}\, dx$.

Example 9. Find the area under the curve $y = e^{-3x}$ from 0 to 1.

6.4* General logarithmic and exponential functions

General exponential functions

Let $b > 0$. We saw in the previous section that if r is a rational number, then

$$b^r = (e^{\ln b})^r = e^{r \ln b}.$$

Therfore, we define for every irrational number *x*

$$b^x = e^{x \ln b}.$$

For example: $2^{\sqrt{3}} = e^{\sqrt{3} \ln 2} \approx 3.32$

Properties

Theorem. Let $b > 0$. Then

- $\ln b^r = r \ln b$ for all $r \in (-\infty, \infty)$ (so not just for rational r)
- *Laws of exponents:* let $a, b > 0$ and $x, y \in (-\infty, \infty)$. Then

  a) $b^{x+y} = b^x b^y$

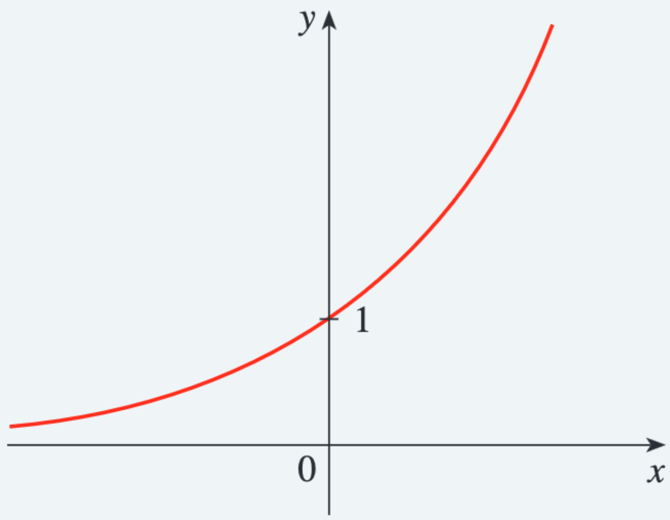
  b) $b^{x-y} = \frac{b^x}{b^y}$

  c) $(b^x)^y = b^{xy}$

  d) $a^x b^x = (ab)^x$

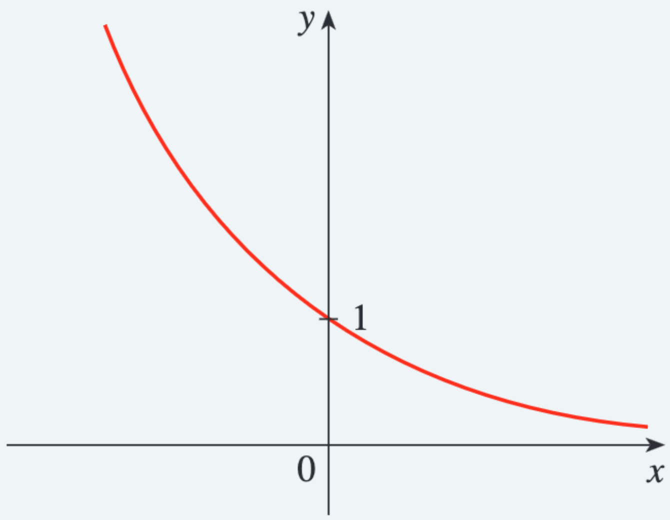
- $\frac{d}{dx} b^x = b^x \ln b$
- $\int b^x dx = \frac{b^x}{\ln b} + C,\ b \neq 1.$

Exponential graphs



$\lim_{x \to -\infty} b^x = 0, \ \lim_{x \to \infty} b^x = \infty$

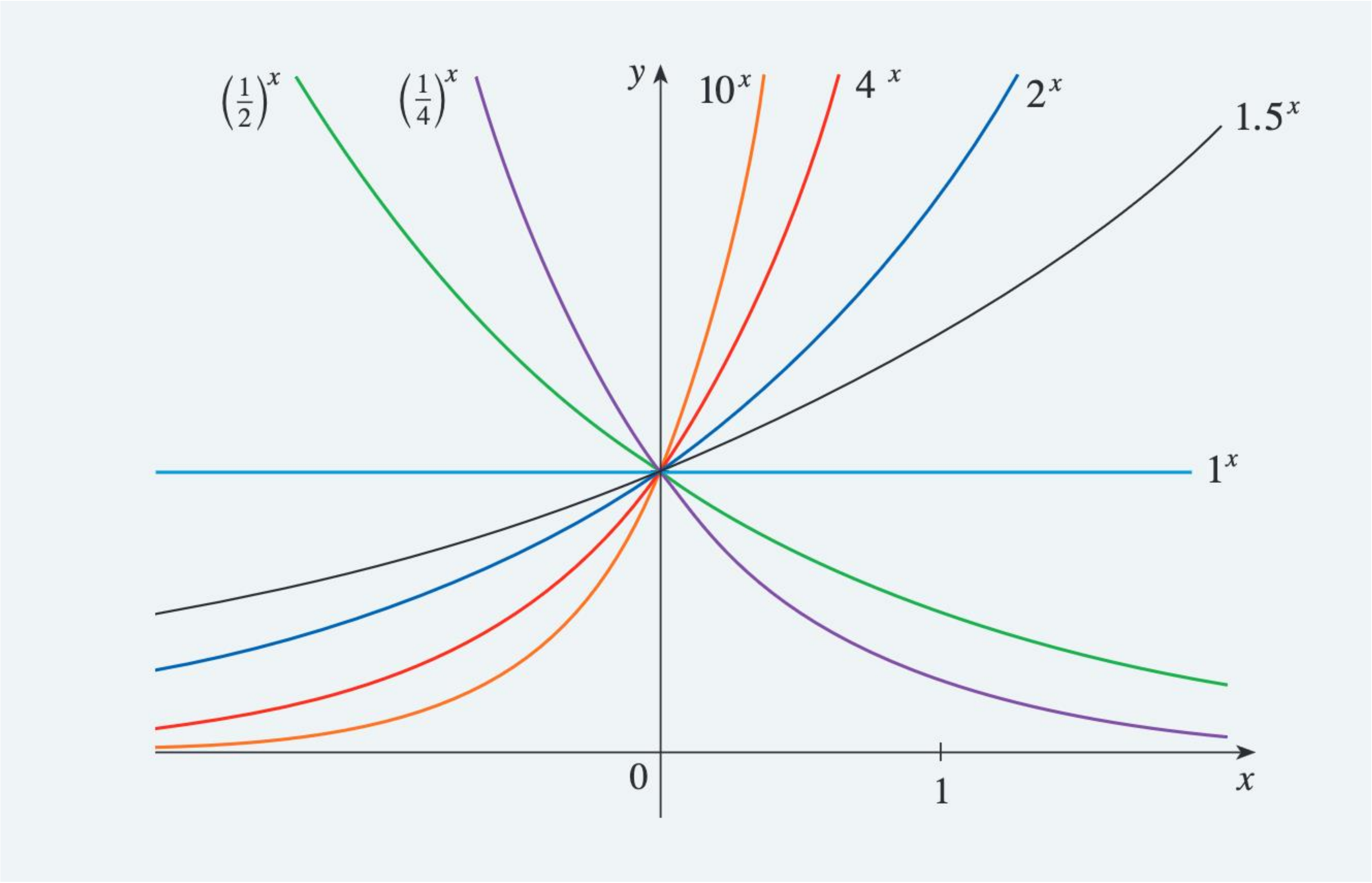
FIGURE 1 $y = b^x, \ b > 1$



$\lim_{x \to -\infty} b^x = \infty, \ \lim_{x \to \infty} b^x = 0$

FIGURE 2 $y = b^x, \ 0 < b < 1$

Exponential graphs



Example 6

Example 6. Evaluate $\int_1^5 2^x\,dx$.

Proof of the general power rule

Theorem. Let $f(x) = x^n, x > 0, n \in (-\infty, \infty)$. Then

$$\frac{d}{dx} f(x) = nx^{n-1}$$

General logarithmic function

For any $b > 0$, the function $f(x) = b^x$ is monotone increasing ($b > 1$)/decreasing ($b < 1$). Therefore it is one-to-one and hence has an inverse function, **called logharithmic function** with base b, denoted by $g(x) = \log_b x$. Hence:

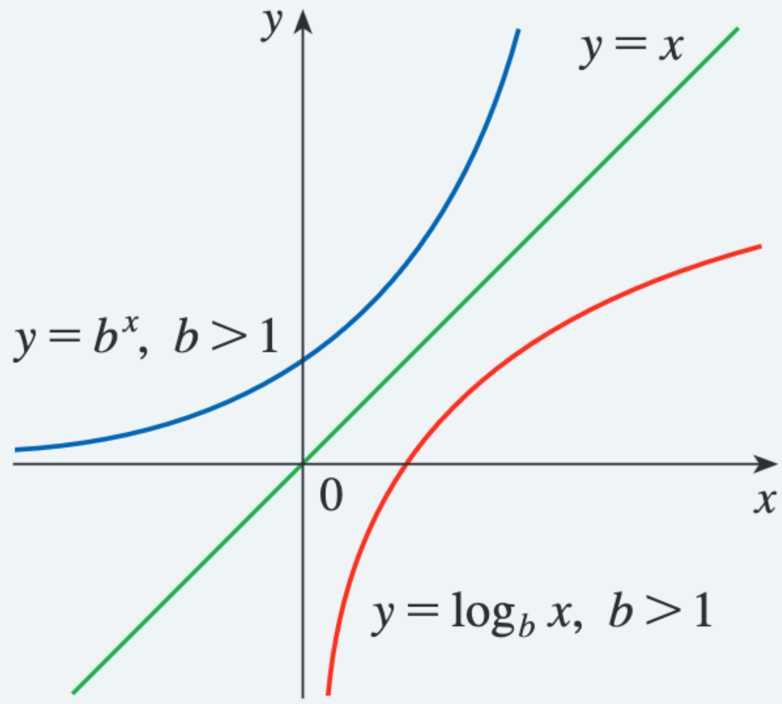
5 $$\boxed{\log_b x = y \iff b^y = x}$$

In particular: $\ln x = \log_e x$

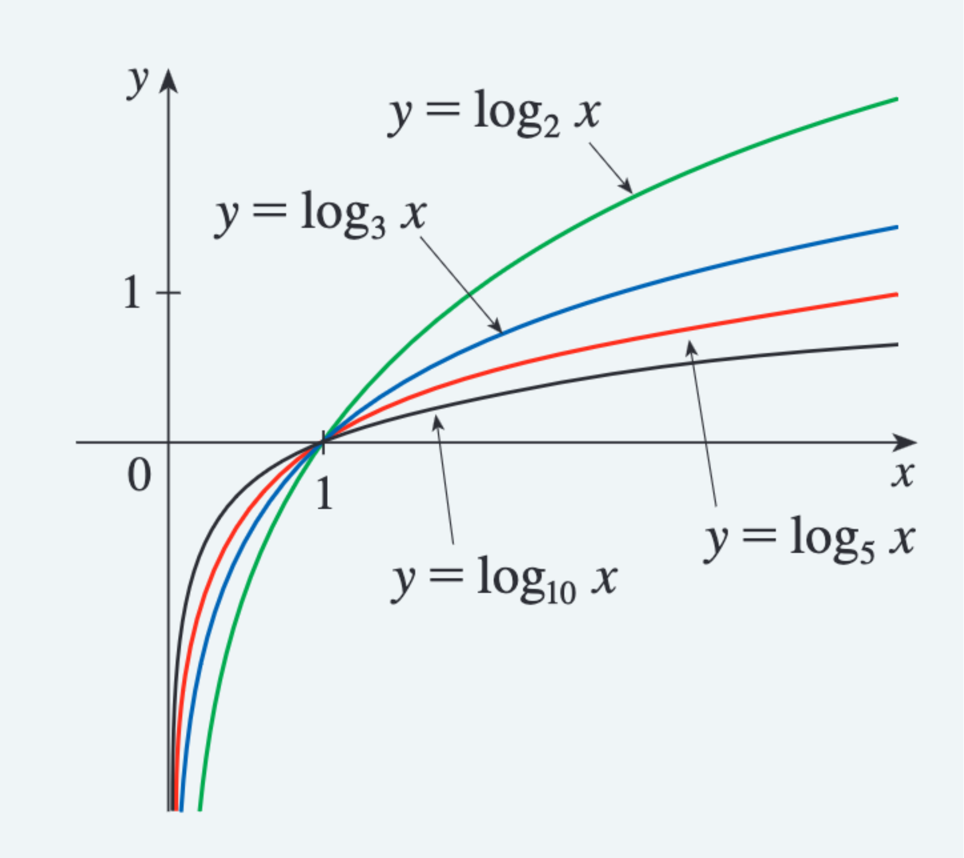
Properties of $\log_b x$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- Cancellation:
  a) $\log_b b^x = x$, $x \in (-\infty, \infty)$
  b) $b^{\log_b x} = x, x > 0$.
- Increasing for $b > 1,$ decreasing for $0 < b < 1$
- Change of base: $\log_b x = \dfrac{\ln x}{\ln b}$
- Laws of logarithm ($x, y > 0$, $r \in (-\infty, \infty)$):
  a) $\log_b(xy) = \log_b x + \log_b y$
  b) $\log_b \dfrac{x}{y} = \log_b x - \log_b y$
  c) $\log_b x^r = r \log_b x$

Graph of $\log_b x$



Graph of $\log_b x$



Derivative of $\log_b x$

Using the cange of base formula we get:

$$\frac{d}{dx}\log_b x = \frac{1}{x\ln b}$$

Example 9. Differentiate $\log_{10}(2 + \sin x)$.

The number *e* as a limit

Theorem. We have that

$$e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}},$$

or

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n}.$$