14.4 Tangent Planes and Linear Approximations

Tangent Planes

Tangent Planes

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_o)$ be a point on S.

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2.

Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

Tangent Planes

Then the **tangent plane** to the surface *S* at the point *P* is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)

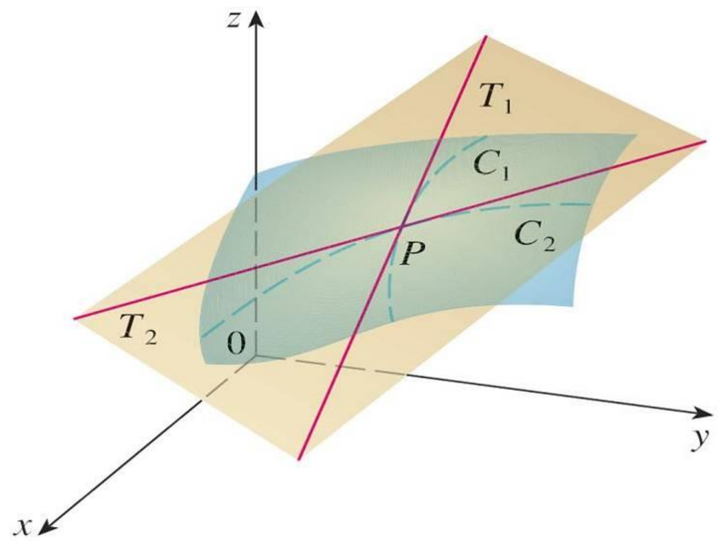


Figure 1

The tangent plane contains the tangent lines T_1 and T_2.

Tangent Planes

If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane.

Therefore, you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Tangent Planes

Here the vector <u>*n*</u>=(*A*,*B*,*C*) is a normal vector of the plane.

We may parameterize the curve C_1 using the vector function

$$\mathbf{r}_1(x) = \langle x, y_0, f(x, y_0)\rangle$$

and parameterize the curve C_2 using the vector function

$$\mathbf{r}_2(y) = \langle x_0, y, f(x_0, y)\rangle$$

Then the tangent vector of $\boldsymbol{r}_1$ at the point $(x_0, y_0, f(x_0, y_0)$ is

$$\mathbf{r}_1{}'(x_0) = \langle 1,0, f_x(x_0, y_0)\rangle$$

and the tangent vector of $\boldsymbol{r}_1$ at the point $(x_0, y_0, f(x_0, y_0)$ is

$$\mathbf{r}_2{}'(y_0) = \left\langle 0,1, f_y(x_0, y_0)\right\rangle$$

Tangent Planes

Hence a normal vector of the tangent plane is given by

$$\mathbf{n} = \mathbf{r}_1{}'(x_0) \times \mathbf{r}_2{}'(y_0) = \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \rangle$$

Hence we arrive at the following formula.

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

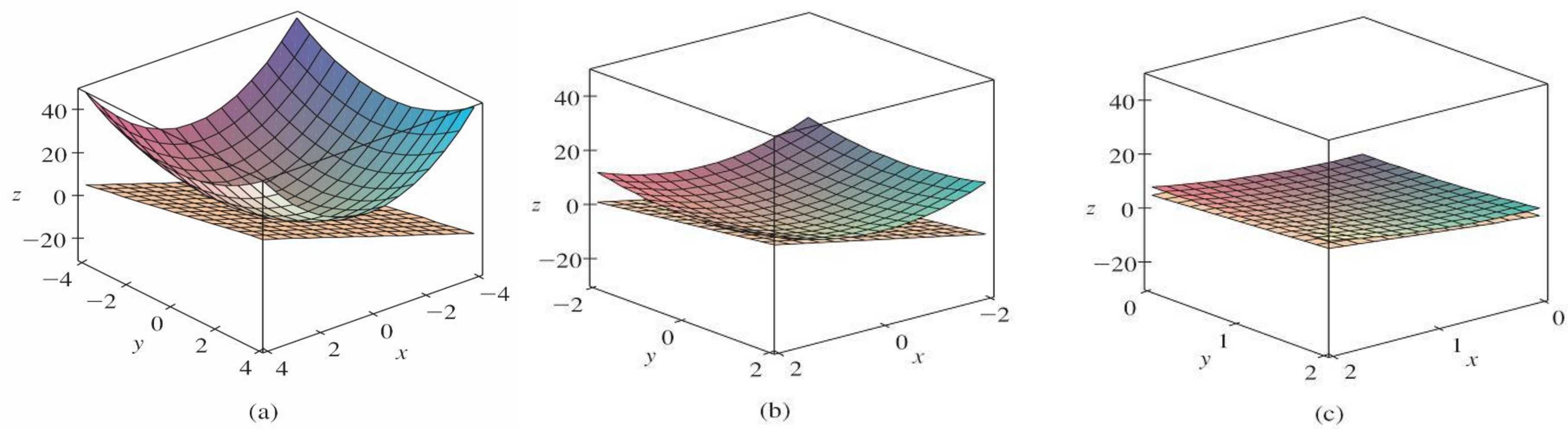
$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

Tangent Planes

Figure 2(a) shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1. In parts (b) and (c) we zoom in toward the point (1, 1, 3) by restricting the domain of the function $f(x, y) = 2x^2 + y^2$.



The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

Figure 2

Linear Approximations

Linear Approximations

In Example 1 we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is $z = 4x + 2y - 3$. Therefore, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near (1, 1). The function L is called the *linearization* of f at (1, 1) and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at (1, 1).

Linear Approximations

For instance, at the point (1.1, 0.95) the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is quite close to the true value of

$$f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225.$$

But if we take a point farther away from (1, 1), such as (2, 3), we no longer get a good approximation.

In fact, $L(2, 3) = 11$ whereas $f(2, 3) = 17$.

Linear Approximations

In general, we know from $\boxed{2}$ that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$$\boxed{3} \qquad L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

$$\boxed{4} \qquad f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

Linear Approximations

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

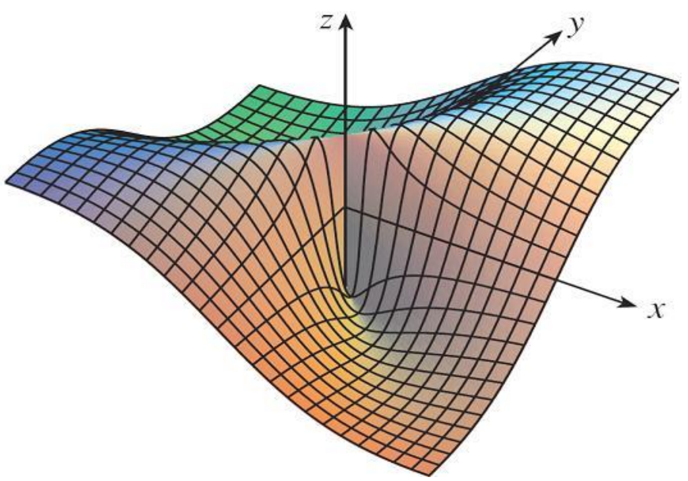


Figure 4

$f(x, y) = \dfrac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$,
$f(0, 0) = 0$

You can verify that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous (even f is not continuous!).

Linear Approximations

The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$.

So a function of two variables can behave badly even though both of its partial derivatives exist.

To rule out such behavior, we formulate the idea of a differentiable function of two variables.

Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximations

If f is differentiable at a, then

5 $$\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

6 $$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Thus, the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Linear Approximations

By analogy with $\boxed{5}$ we define the differentiability of a function of two variables as follows.

$\boxed{7}$ **Definition** If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation $\boxed{4}$ is a good approximation when (x, y) is near (a, b).

In other words, the tangent plane approximates the graph of f well near the point of tangency.

Linear Approximations

It’s sometimes hard to use Definition 7 directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 2

Show that $f(x, y) = xe^{xy}$ is differentiable at (1,0) and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Differentials

Differentials

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number.

The differential of y is then defined as

$$dy = f'(x)\,dx \tag{9}$$

Differentials

Figure 6 shows the relationship between the increment Δy and the differential dy: Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

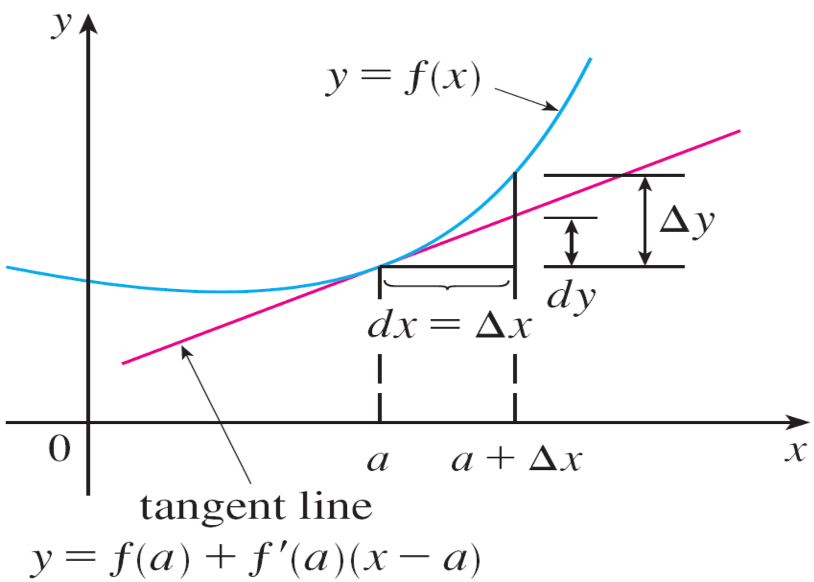


Figure 6

Differentials

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

$$\boxed{10} \qquad dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

Sometimes the notation df is used in place of dz.

Differentials

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation $\boxed{4}$ can be written as

$$f(x, y) \approx f(a, b) + dz$$

Differentials

Figure 7 is the three-dimensional counterpart of Figure 6 and shows the geometric interpretation of the differential dz and the increment Δz: dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

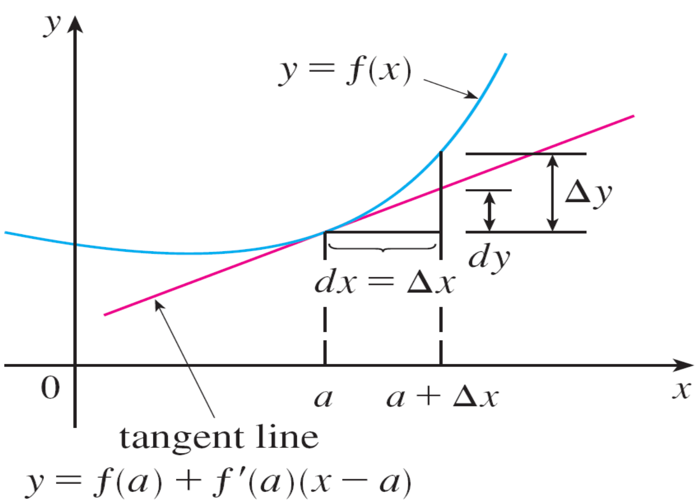


Figure 6

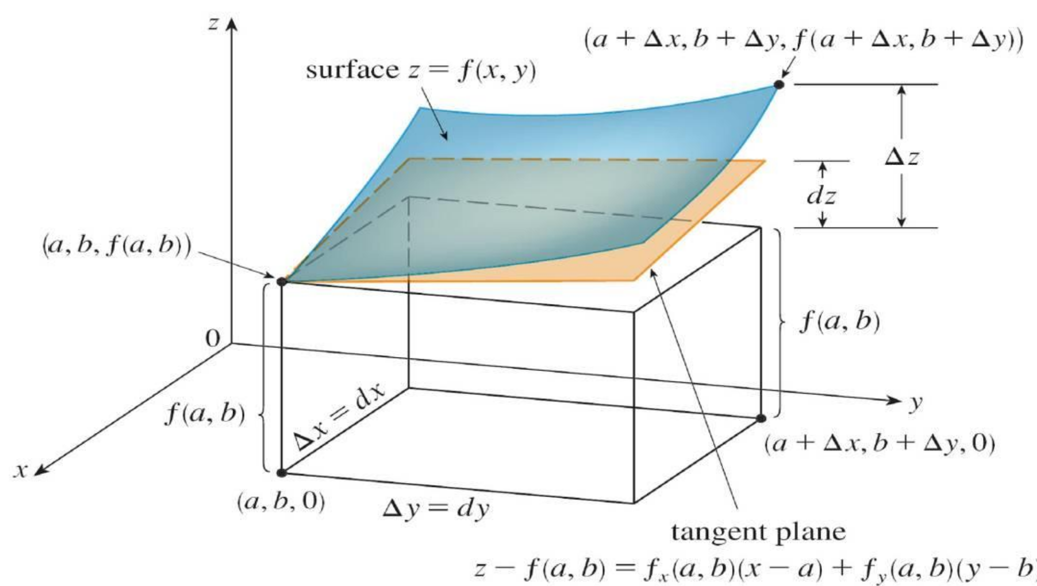
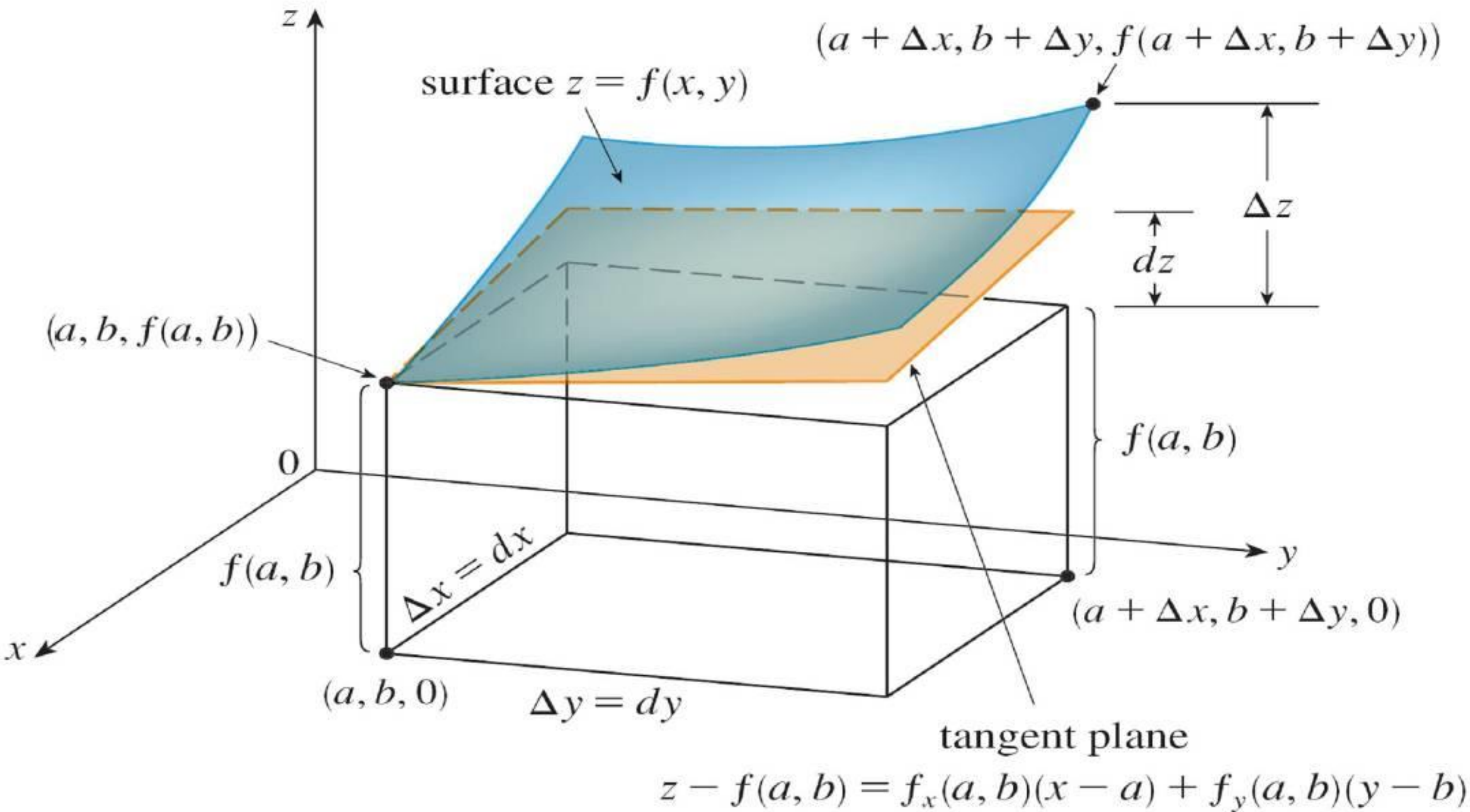


Figure 7

Differentials



Example 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.