1.8 Continuity

Continuity

The limit of a function as *x* approaches *a* can often be found simply by calculating the value of the function at *a*. Functions with this property are called *continuous at a*.

We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

1 Definition A function f is **continuous at a number** $\boldsymbol{a}$ if

$$\lim_{x \to a} f(x) = f(a)$$

Continuity

Notice that Definition 1 implicitly requires three things if *f* is continuous at *a*:

1. *f*(*a*) is defined (that is, *a* is in the domain of *f*)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

The definition says that *f* is continuous at *a* if *f*(*x*) approaches *f*(*a*) as *x* approaches *a*. Thus a continuous function *f* has the property that a small change in *x* produces only a small change in *f*(*x*).

Continuity

In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents.

Continuity

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

Example 1

Figure 2 shows the graph of a function *f*. At which numbers is *f* discontinuous? Why?

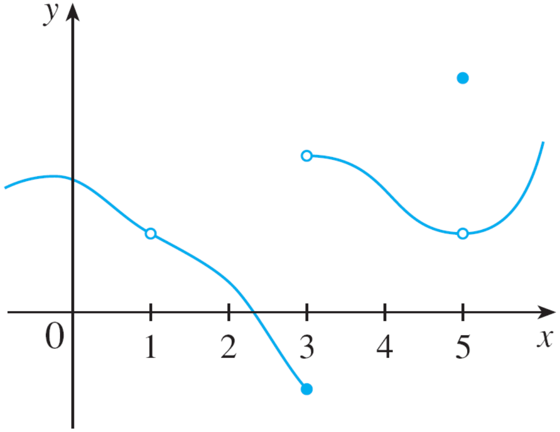


Figure 2

Example 2

Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

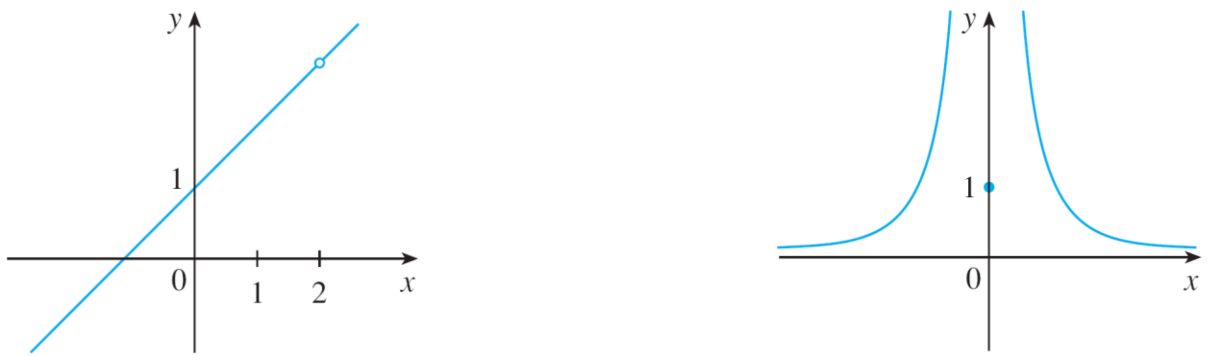
(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Continuity

Figure 3 shows the graphs of the functions in Example 2.



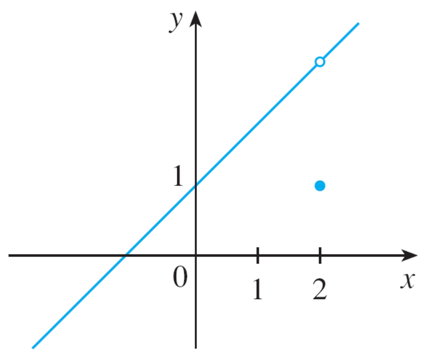
(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

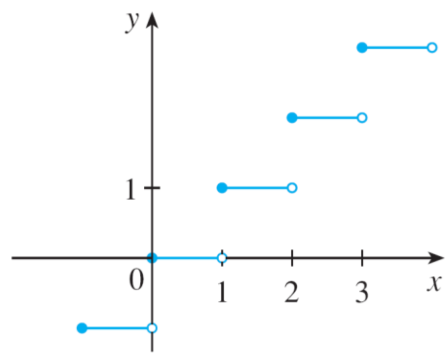
Graphs of the functions in Example 2

Figure 3

Continuity



(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



(d) $f(x) = [\![x]\!]$

Graphs of the functions in Example 2

Figure 3

Continuity

In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph.

The kind of discontinuity illustrated in parts (a) and (c) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.]

The discontinuity in part (b) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.

Continuity

2 Definition A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

Thus a function *f* is continuous at *a* if and only if it is continuous both from the left and right at *a*.

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Examples 3 and 4

Example 3. Discuss the function $f(x) = [\![x]\!]$ for one sided continuity.

Example 4. Show, using limit laws, that the function

$$f(x) = 1 - \sqrt{1 - x^2}$$

is continuous on [-1, 1].

Continuity

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

$\boxed{4}$ **Theorem** If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

Continuity

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) f/g.

The following theorem was stated as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Continuity

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r.

Likewise, if a ball is thrown vertically into the air with a velocity of 15 m/s, then the height of the ball in meters t seconds later is given by the formula $h = 15t - 4.9t^2$.

Again, this is a polynomial function, so the height is a continuous function of the elapsed time.

Example 5

Find

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}.$$

Continuity

It turns out that most of the familiar functions are continuous at every number in their domains.

From the appearance of the graphs of the sine and cosine functions, we would certainly guess that they are continuous.

We know from the definitions of $\sin \theta$ and $\cos \theta$ that the coordinates of the point P in Figure 5 are $(\cos \theta, \sin \theta)$. As $\theta \to 0$, we see that P approaches the point $(1, 0)$ and so $\cos \theta \to 1$ and $\sin \theta \to 0$.

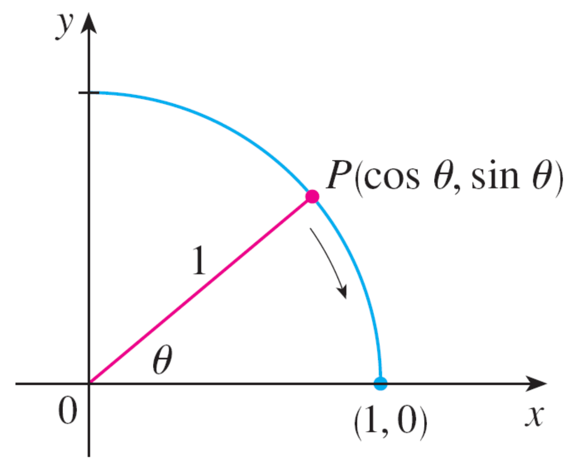


Figure 5

Continuity

Thus

$$\boxed{6} \qquad \lim_{\theta \to 0} \cos \theta = 1 \qquad \lim_{\theta \to 0} \sin \theta = 0$$

Since cos 0 = 1 and sin 0 = 0, the equations in $\boxed{6}$ assert that the cosine and sine functions are continuous at 0.

The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere.

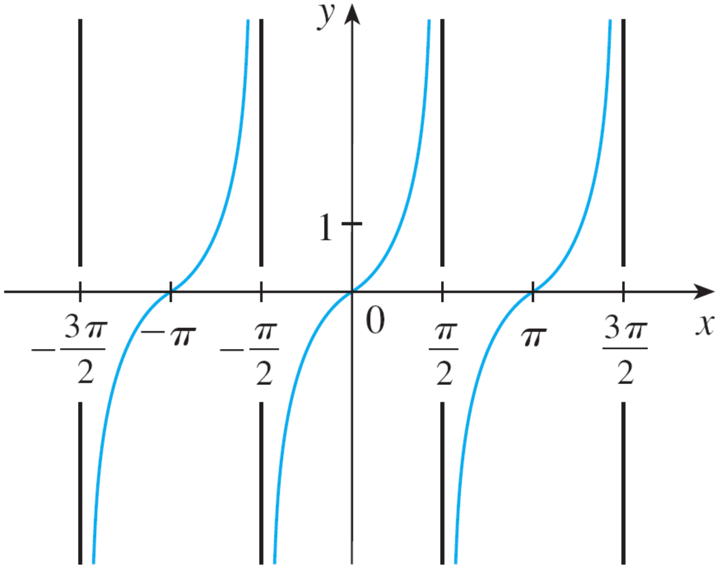
It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where cos x = 0.

Continuity

This happens when x is an odd integer multiple of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm3\pi/2, \pm5\pi/2$, and so on (see Figure 6).



$y = \tan x$

Figure 6

Continuity

$\boxed{7}$ **Theorem** The following types of functions are continuous at every number in their domains:

polynomials	rational functions
root functions	trigonometric functions

Example 6 and 7

Example 6. On what intervals is each function continuous?

(a) $f(x) = x^{100} - 2x^{37} + 75$

(b) $g(x) = \dfrac{x^2 + 2x + 17}{x^2 - 1}$

(c) $h(x) = \sqrt{x} + \dfrac{x+1}{x-1} - \dfrac{x+1}{x^2+1}$

Example 7. Evaluate

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$$

Continuity

Another way of combining continuous functions *f* and *g* to get a new continuous function is to form the composite function $f \circ g$. This is a consequence of the following theorem.

$\boxed{8}$ **Theorem** If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$.
In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Intuitively, Theorem 8 is reasonable because if *x* is close to *a*, then *g*(*x*) is close to *b*, and since *f* is continuous at *b*, if *g*(*x*) is close to *b*, then *f*(*g*(*x*)) is close to *f*(*b*).

$\boxed{9}$ **Theorem** If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example 8

Where are the following functions continuous?

(a) $f(x) = \sin x^2$

(b) $\frac{1}{\sqrt{x^2+7}-4}$

Intermediate Value Theorem

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

$\boxed{10}$ **The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Continuity

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 7.

Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

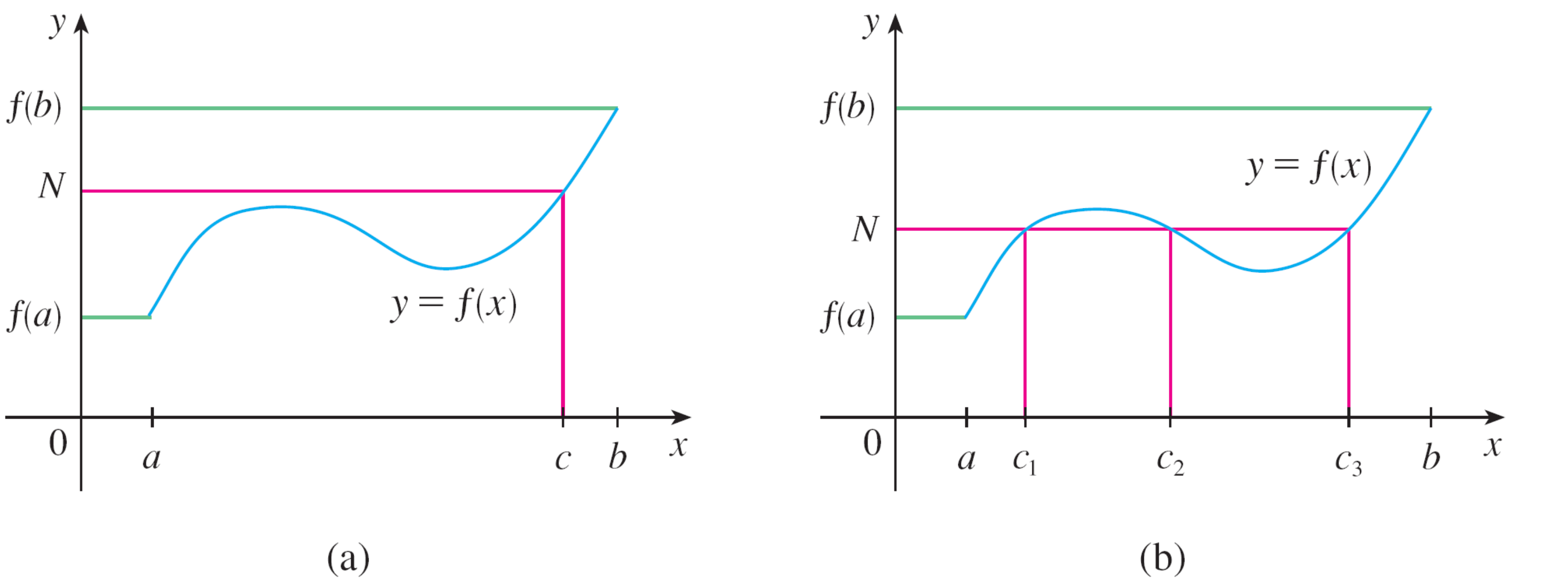


Figure 7

Continuity

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true.

In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 8, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

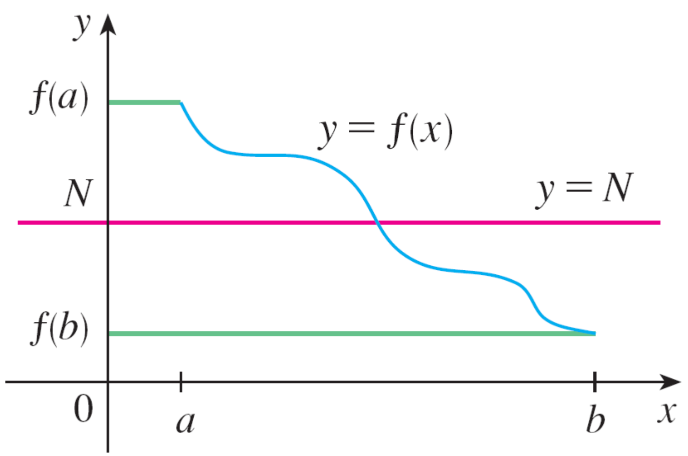


Figure 8

Example 8

It is important that the function *f* in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions.

One use of the important Intermediate Value Theorem is in locating solutions of equations.

Example 9. Show that there is a solution of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

Proof of the Intermediate Value Theorem

The following theorem is the key for proving the Intermediate Value Theorem.

Theorem (Cantor). Let

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$$

Then the set

$$\bigcap_{n=1}^{\infty} [a_n, b_n]$$

is nonempty.