15.2 Double Integrals over General Regions

Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval.

But for double integrals, we want to be able to integrate a function *f* not just over rectangles but also over regions *D* of more general shape, such as the one illustrated in Figure 1.

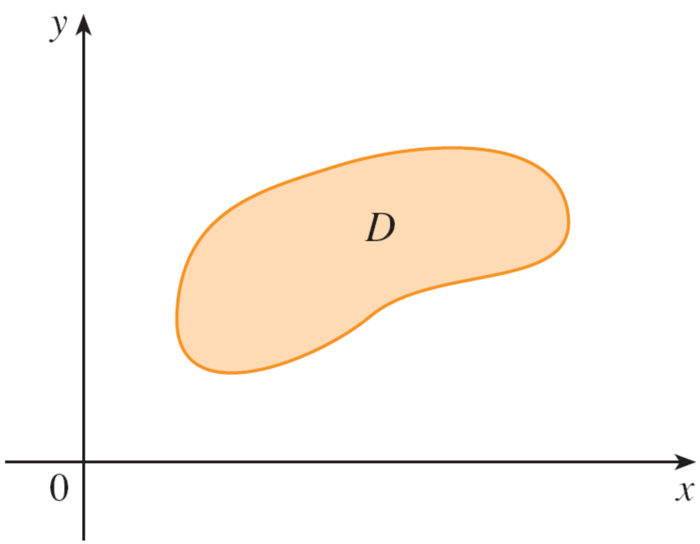


Figure 1

Double Integrals over General Regions

We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2.

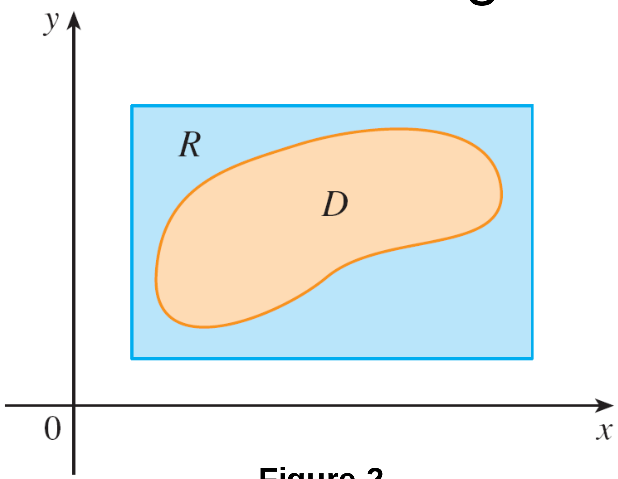


Figure 2

Then we define a new function F with domain R by

$$\boxed{1} \qquad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Double Integrals over General Regions

If *F* is integrable over *R*, then we define the **double integral of *f* over *D*** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because *R* is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined.

Double Integrals over General Regions

The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral.

This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geq 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f).

Double Integrals over General Regions

You can see that this is reasonable by comparing the graphs of f and F in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of F.

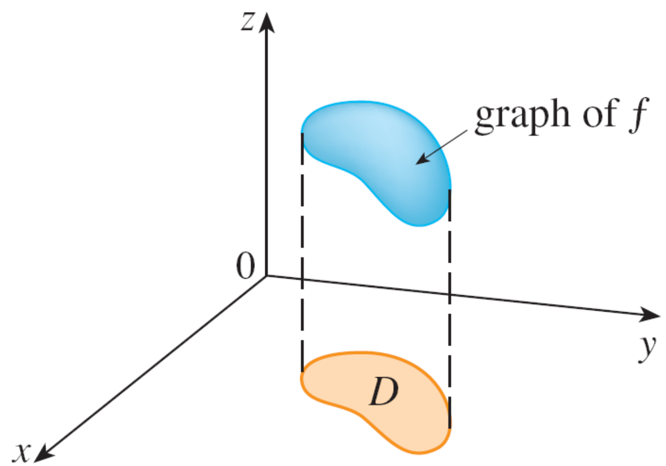


Figure 3

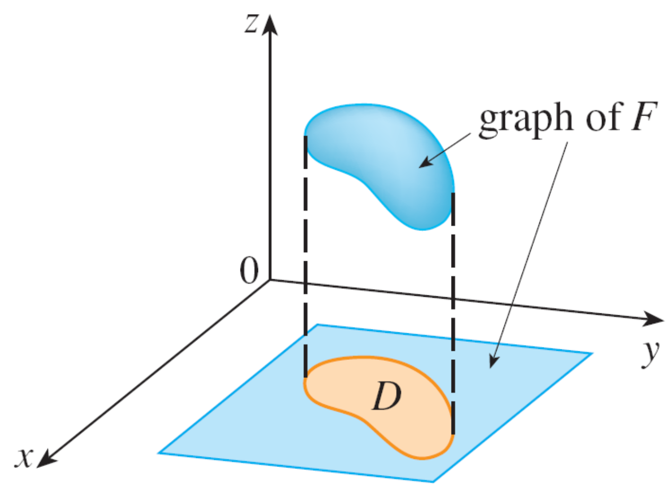


Figure 4

Double Integrals over General Regions

Figure 4 also shows that F is likely to have discontinuities at the boundary points of D.

Nonetheless, if f is continuous on D and the boundary curve of D is "well behaved", then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists.

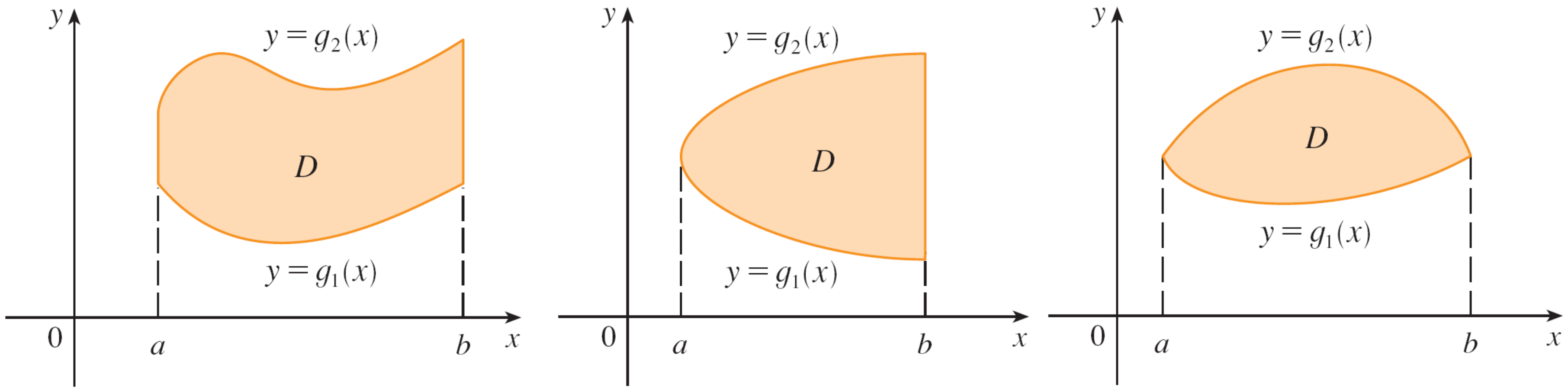
In particular, this is the case for **type I** and **type II** regions.

Double Integrals over General Regions

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.



Some type I regions

Figure 5

Double Integrals over General Regions

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D.

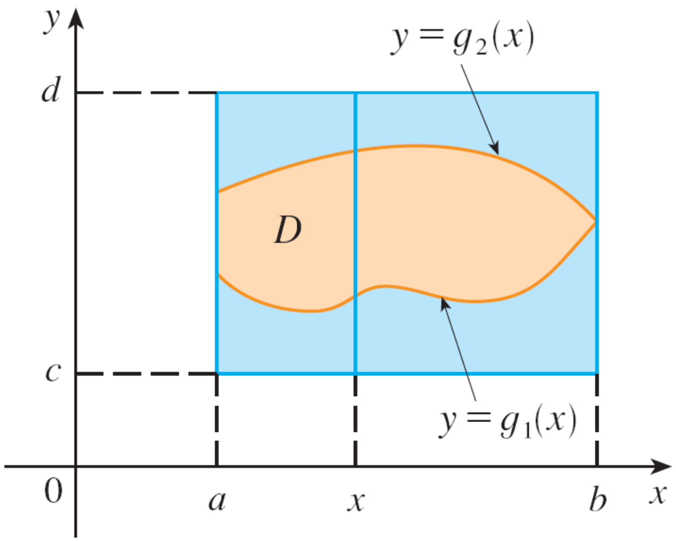


Figure 6

Double Integrals over General Regions

Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \le y \le g_2(x)$.

Double Integrals over General Regions

Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

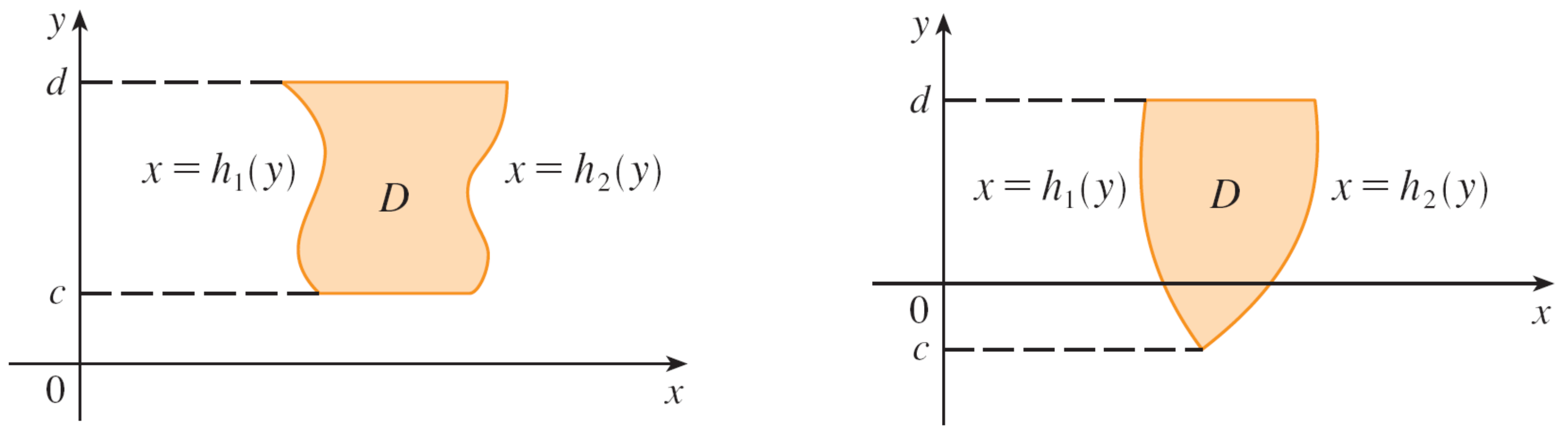
The integral on the right side of $\boxed{3}$ is an iterated integral, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

Double Integrals over General Regions

We also consider plane regions of **type II**, which can be expressed as

$$\boxed{4} \qquad D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.



Some type II regions

Figure 7

Double Integrals over General Regions

Using the same methods that were used in establishing $\boxed{3}$, we can show that

$$\boxed{5} \qquad \iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

Examples 1 - 4

Example 1. Evaluate $\iint_D (x+2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Example 2. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example 3. (different from the one in the book). Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = 2x$ and the parabola $y^2 = 2x + 6$.

Example 4. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 0$, $x = 2y$, and $z = 0$.

Changing the order of integration

Example 5. Evaluate the interated integral
$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

Properties of Double Integrals

Properties of Double Integrals

We assume that all of the following integrals exist. The first three properties of double integrals over a region *D* follow immediately from Definition 2.

6 $$\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

7 $$\iint_D c f(x, y)\, dA = c \iint_D f(x, y)\, dA$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then

8 $$\iint_D f(x, y)\, dA \geqslant \iint_D g(x, y)\, dA$$

Properties of Double Integrals

The next property of double integrals is similar to the property of single integrals given by the equation

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

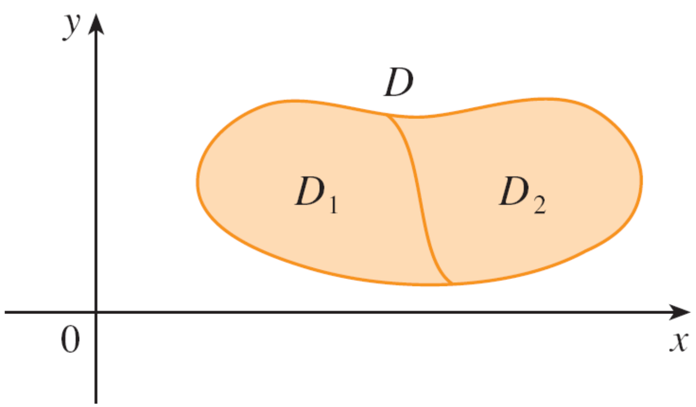
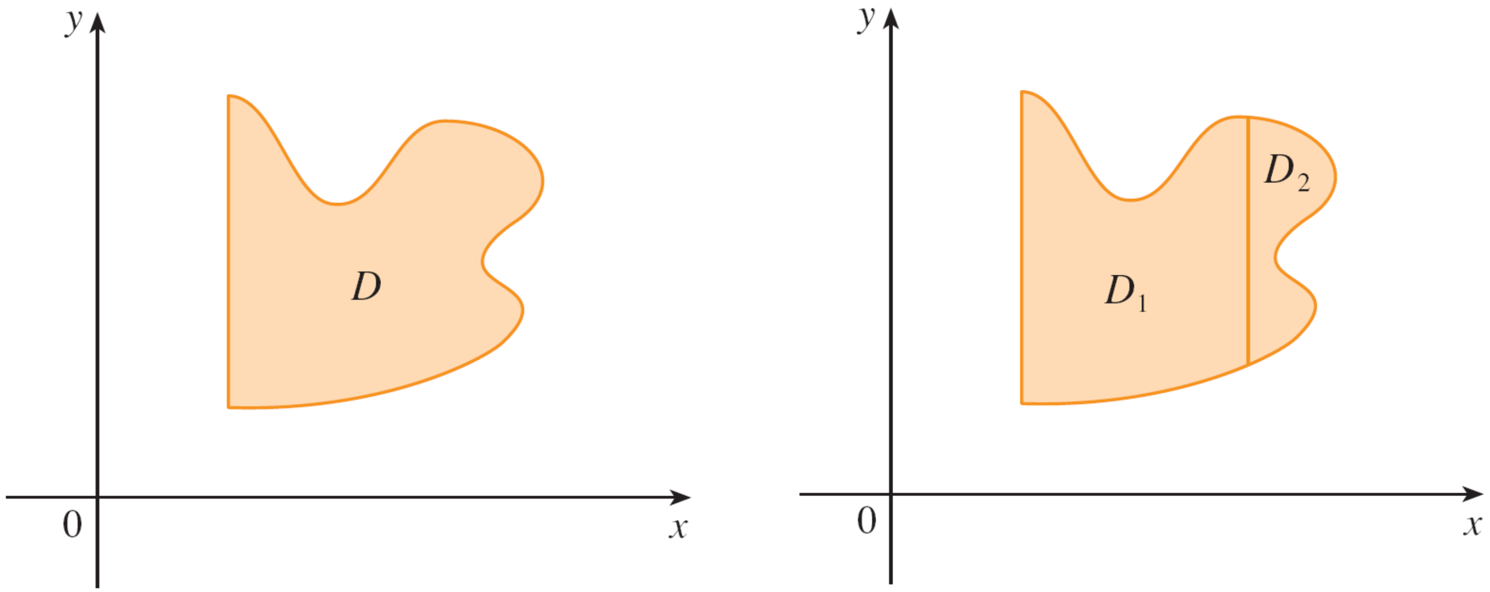


Figure 17

9 $$\boxed{\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA}$$

Properties of Double Integrals

Property 9 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure.



(a) D is neither type I nor type II. (b) $D = D_1 \cup D_2$; D_1 is type I, D_2 is type II.

Figure 18

Properties of Double Integrals

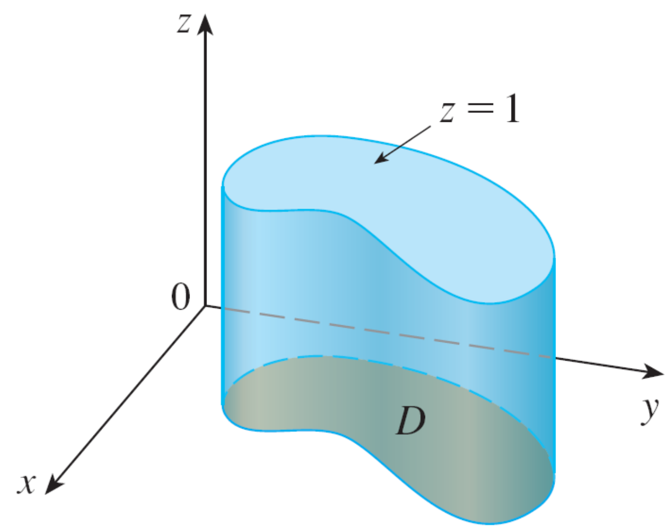
The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

10

$$\boxed{\iint_D 1\, dA = A(D)}$$

Properties of Double Integrals

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.



Cylinder with base *D* and height 1

Figure 19

Properties of Double Integrals

Finally, we can combine Properties 7, 8, and 10 to prove the following property.

11 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$