Estimates of sums

2 **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx$$

Corollary:

3

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant s \leqslant s_n + \int_{n}^{\infty} f(x)\,dx$$

Examples 5 and 6

Examples 5 and 6.

(a) Approximate the sum of the series $\Sigma\ 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

(c) Use equation (3) with $n = 10$ to estimate the sum of the series.

11.4 The Comparison Tests

The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

$$\boxed{1} \qquad \sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series $\sum_{n=1}^{\infty} 1/2^n$, which is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Because the series $\boxed{1}$ is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is.

The Comparison Tests

The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series $\boxed{1}$ has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series).

This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

The Comparison Tests

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

The Comparison Tests

In using the Comparison Test we must, of course, have some known series $\Sigma\, b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p-series [$\Sigma\, 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$]

- A geometric series [$\Sigma\, ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$]

Note: Although the condition $a_n \leq b_n$ or $a_n \geq b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

Examples 1 and 2

Example 1. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

converges or diverges.

Example 2. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$

converges or diverges.

The Comparison Tests

Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because $\Sigma\, b_n = \Sigma \left(\frac{1}{2}\right)^n$ is convergent and $a_n > b_n$.

The Comparison Tests

Nonetheless, we have the feeling that $\Sigma\ 1/(2^n - 1)$ ought to be convergent because it is very similar to the convergent geometric series $\sum \left(\frac{1}{2}\right)^n$.

In such cases the following test can be used.

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Examples 3 and 4

Example 3. Test the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

for convergence or divergence.

Example 4. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$$

converges or diverges.

11.5 Alternating Series

Alternating Series

In this section we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series,* whose terms alternate in sign.

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

Alternating Series

We see from these examples that the *n*th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

Remark. As before, one only needs that $b_{n+1} \leq b_n$ for all $n \geq N$.

Examples 1-3

Test the following series for for convergence or divergence.

Example 1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Example 2.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$$

Example 3.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$$