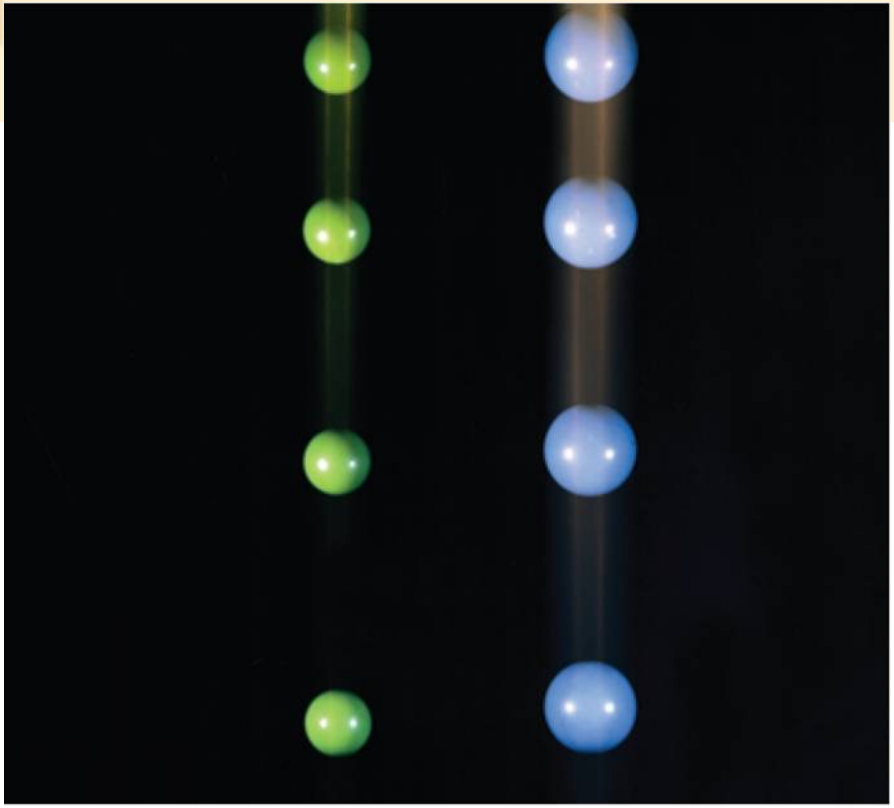
1 Functions and Limits



1.1 Four Ways to Represent a Function

Four Ways to Represent a Function

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

We usually consider functions for which the sets *D* and *E* are sets of real numbers. The set *D* is called the **domain** of the function.

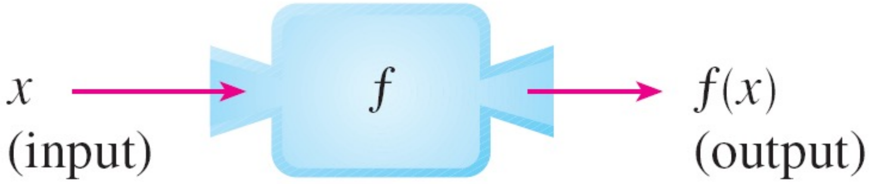
The number $f(x)$ is the **value of *f* at *x*** and is read "*f* of *x*." The **range** of *f* is the set of all possible values of $f(x)$ as *x* varies throughout the domain.

A symbol that represents an arbitrary number in the *domain* of a function *f* is called an **independent variable**.

Four Ways to Represent a Function

A symbol that represents a number in the *range* of f is called a **dependent variable**.

It's helpful to think of a function as a **machine** (see Figure 2).



Machine diagram for a function f

Figure 2

Four Ways to Represent a Function

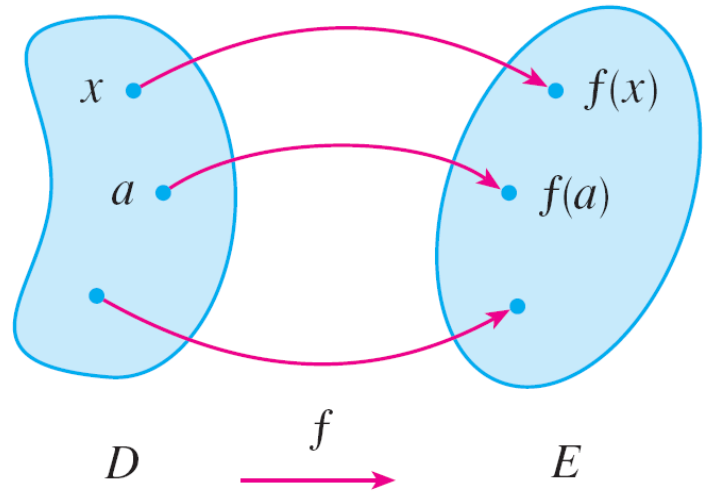
If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. In this scenario:

Domain: set of all possible inputs

Range: set of all possible outputs.

Four Ways to Represent a Function

Another way to picture a function is by an **arrow diagram** as in Figure 3.



Arrow diagram for *f*

Figure 3

Each arrow connects an element of *D* to an element of *E*. The arrow indicates that *f*(*x*) is associated with *x*, *f*(*a*) is associated with *a*, and so on.

Four Ways to Represent a Function

The most common method for visualizing a function is its graph. If f is a function with domain D, then its **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}$$

Geometrically, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

Four Ways to Represent a Function

The graph of a function *f* gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4).

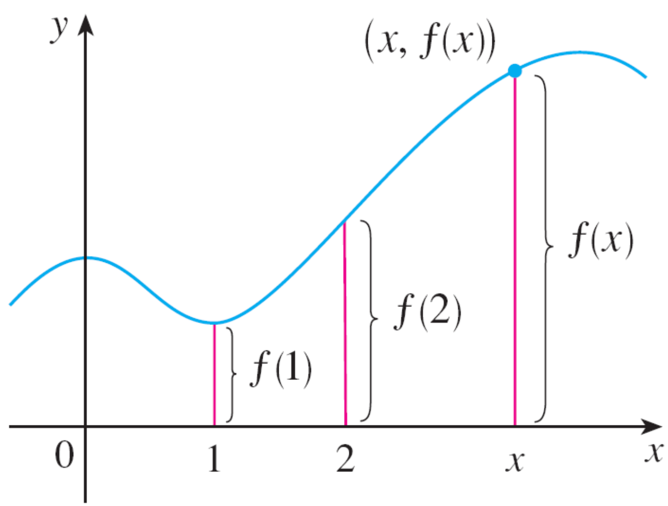


Figure 4

Four Ways to Represent a Function

The graph of *f* also allows us to picture the domain of *f* on the *x*-axis and its range on the *y*-axis as in Figure 5.

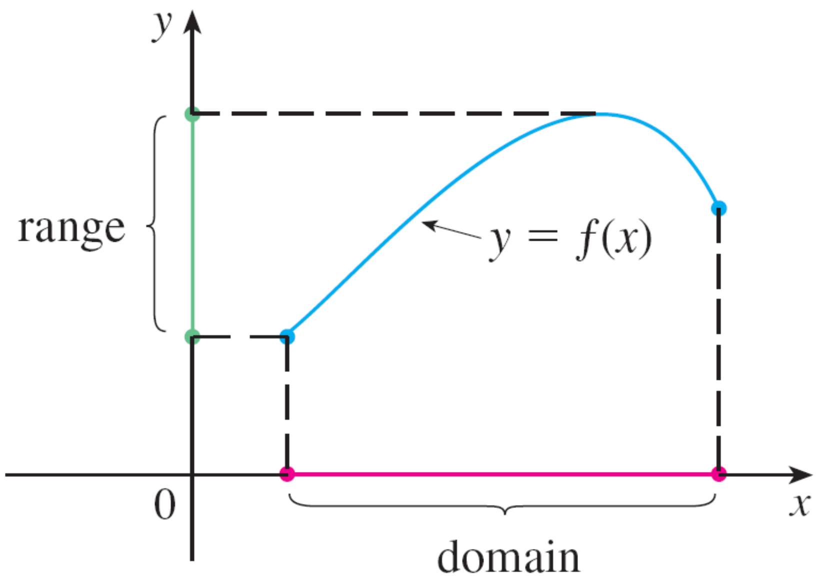
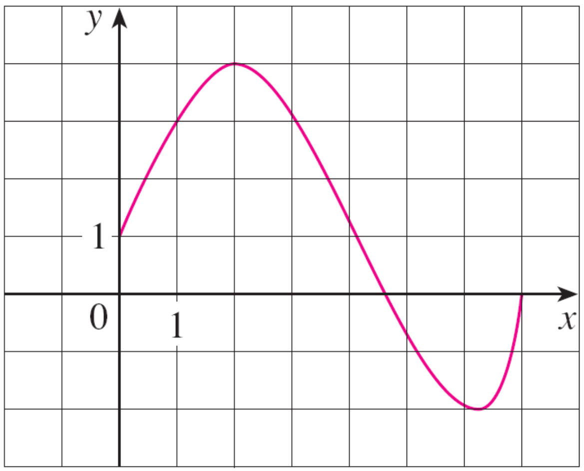


Figure 5

Example 1

The graph of a function f is shown in Figure 6.



The notation for intervals is given in Appendix A.

Figure 6

(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f?

Examples 2 and 3

Example 2. Sketch the graph and find the domain and range. (a) $f(x) = 2x-1$ (b) $g(x) = x^2$.

Example 3. If $f(x) = 2x^2-5x+1$ and $h \neq 0$, evaluate and simplify

$$\frac{f(a+h)-f(a)}{h}.$$

Representations of Functions

Representations of Functions

Here are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

Examples 4 - 6

Example 4. When you turn on a hot-water faucet, the temperature *T* of the water depends on how long the water has been running. Draw a rough graph of *T* as a function of the time *t* that has elapsed since the faucet was turned on.

Example 5. A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per m^2; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

Example 6. Find the domain of (a) $f(x) = \sqrt{x+2}$
(b) $f(x) = 1/(x^2 - x)$.

Representations of Functions

The graph of a function is a curve in the *xy*-plane. But the question arises: Which curves in the *xy*-plane are graphs of functions? This is answered by the following test.

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

Representations of Functions

The reason for the truth of the Vertical Line Test can be seen in Figure 13.

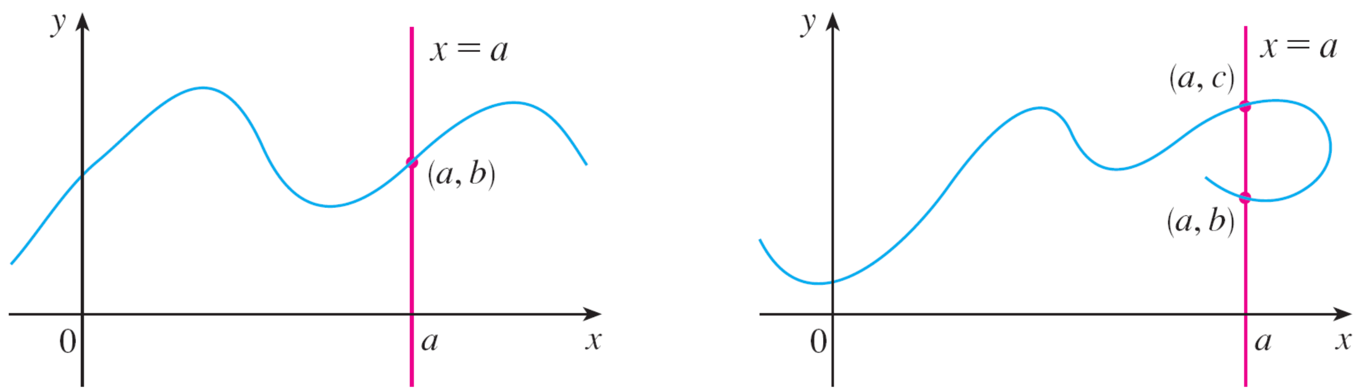
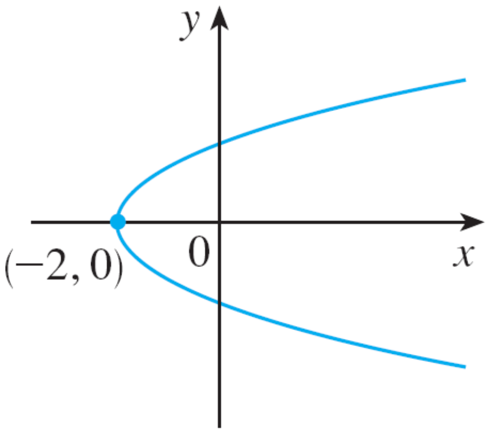


Figure 13

If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different values to a.

Representations of Functions

For example, the parabola $x = y^2 - 2$ shown in Figure 14(a) is not the graph of a function of x because, as you can see, there are vertical lines that intersect the parabola twice. The parabola, however, does contain the graphs of *two* functions of x.



$x = y^2 - 2$

Figure 14(a)

Piecewise Defined Functions

Example 7

A function *f* is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Piecewise Defined Functions

The next example of a piecewise defined function is the absolute value function. Recall that the **absolute value** of a number *a*, denoted by $|a|$, is the distance from *a* to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geq 0 \qquad \text{for every number } a$$

For example,

$|3| = 3 \quad |-3| = 3 \quad |0| = 0 \quad |\sqrt{2} - 1| = \sqrt{2} - 1$

$|3 - \pi| = \pi - 3$

Piecewise Defined Functions

In general, we have

$$\boxed{\begin{array}{ll} |a| = a & \text{if } a \geqslant 0 \\ |a| = -a & \text{if } a < 0 \end{array}}$$

(Remember that if *a* is negative, then –*a* is positive.)

Example 8

Sketch the graph of the absolute value function $f(x) = |x|$.

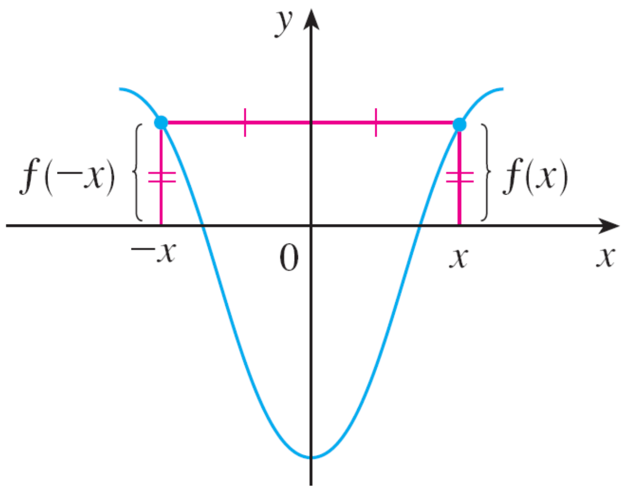
Symmetry

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 19).



An even function

Figure 19

Symmetry

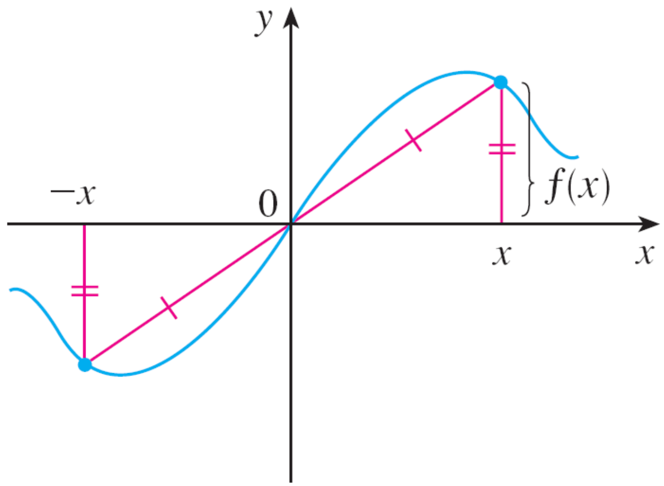
This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Symmetry

The graph of an odd function is symmetric about the origin (see Figure 20).



An odd function

Figure 20

If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin.

Example 11

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ **(b)** $g(x) = 1 - x^4$ **(c)** $h(x) = 2x - x^2$

Increasing and Decreasing Functions

Increasing and Decreasing Functions

The graph shown in Figure 22 rises from *A* to *B*, falls from *B* to *C*, and rises again from *C* to *D*. The function *f* is said to be increasing on the interval [*a*, *b*], decreasing on [*b*, *c*], and increasing again on [*c*, *d*].

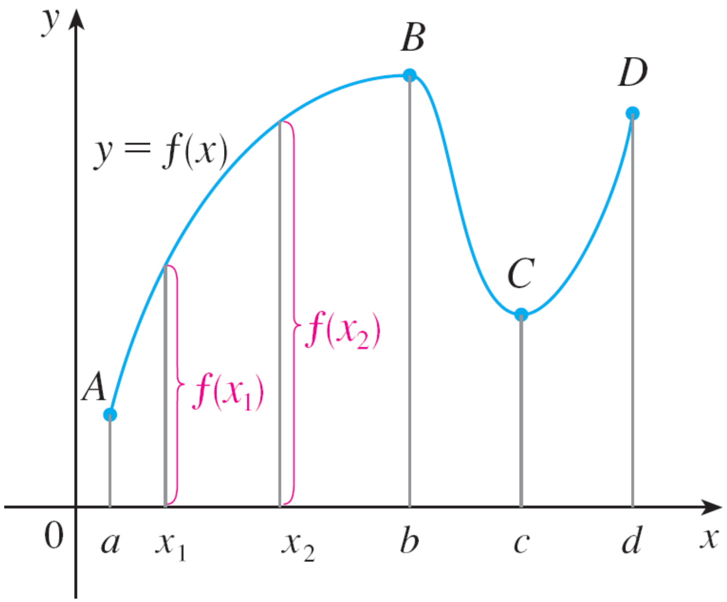


Figure 22

Increasing and Decreasing Functions

Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$.

We use this as the defining property of an increasing function.

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

Increasing and Decreasing Functions

In the definition of an increasing function it is important to realize that the inequality $f(x_1) < f(x_2)$ must be satisfied for *every* pair of numbers x_1 and x_2 in I with $x_1 < x_2$.

You can see from Figure 23 that the function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.

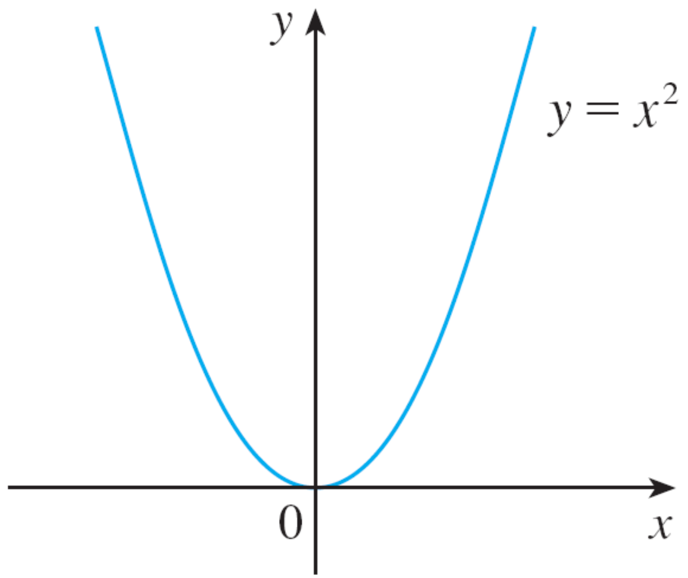


Figure 23

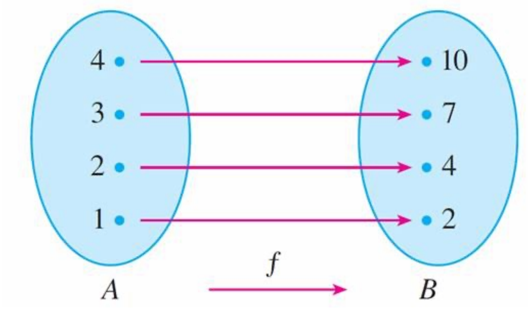
Inverse functions (Chapter 6.1)

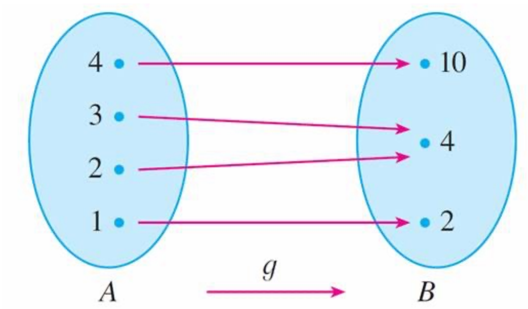
Inverse Functions

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Q: Which of the following functions is one-to one?

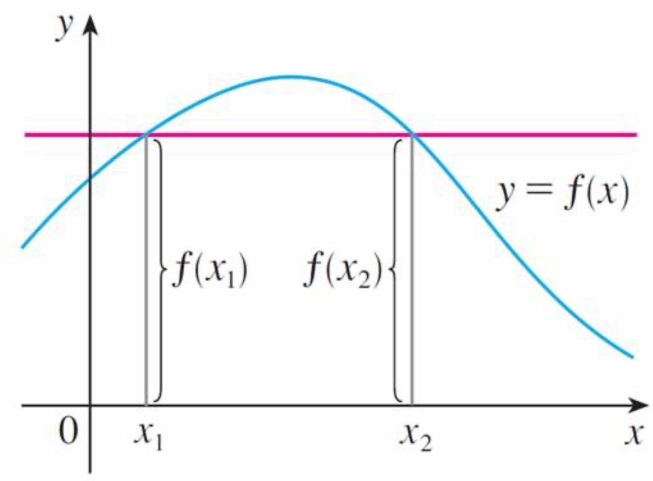




Note: increasing/decreasing functions are one-to-one.

Inverse Functions

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 2 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one.



This function is not one-to-one because $f(x_1) = f(x_2)$.

Figure 2

Inverse Functions

Therefore, we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1 and 2

Example 1. Is the function $f(x) = x^3$ one-to-one?

Example 2. Is the function $f(x) = x^2$ one-to-one?

Inverse Functions

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

$\boxed{2}$ **Definition** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if *f* maps *x* into *y*, then f^{-1} maps *y* back into *x*. (If were not one-to-one, then f^{-1} would not be uniquely defined.)

Inverse Functions

The arrow diagram in Figure 5 indicates that f^{-1} reverses the effect of f.

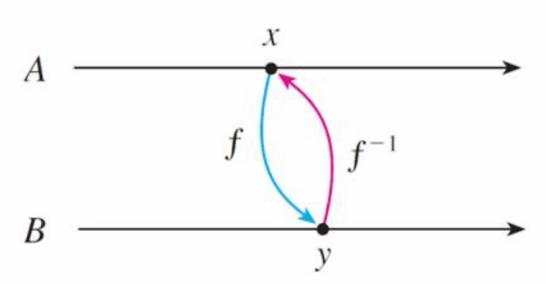


Figure 5

Note that

$$\boxed{\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned}}$$

Inverse Functions

For example, the inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$ because if $y = x^3$, then

$$f^{-1}(y) = f^{-1}(x^3) = (x^3)^{1/3} = x$$

Warning: This is different from the reciprocal $1/f(x)$ which could, however, be written as $[f(x)]^{-1}$.

Example 3

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

Inverse Functions

The letter *x* is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on *f*, we usually reverse the roles of *x* and *y* in Definition 2 and write

3

$$f^{-1}(x) = y \iff f(y) = x$$

By substituting for *y* in Definition 2 and substituting for *x* in $\boxed{3}$, we get the following **cancellation equations**:

4

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Inverse Functions

The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (see the machine diagram in Figure 7).

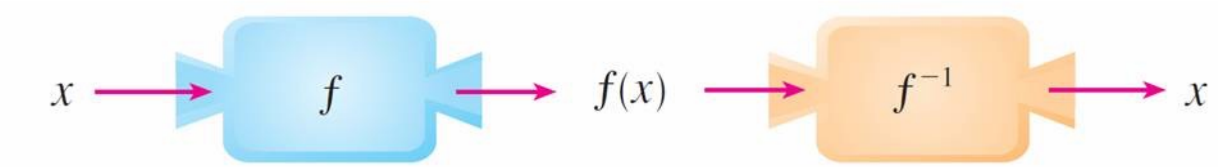


Figure 7

Thus f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does.

Inverse Functions

For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so the cancellation equations become

$$f^{-1}(f(x)) = (x^3)^{1/3} = x$$

$$f(f^{-1}(x)) = (x^{1/3})^3 = x$$

These equations simply say that the cube function and the cube root function cancel each other when applied in succession.

Inverse Functions

Now let's see how to compute inverse functions. If we have a function $y = f(x)$ and are able to solve this equation for x in terms of y, then according to Definition 2 we must have $x = f^{-1}(y)$.

If we want to call the independent variable x, we then interchange x and y and arrive at the equation $y = f^{-1}(x)$.

5 How to Find the Inverse Function of a One-to-One Function f

Step 1 Write $y = f(x)$.

Step 2 Solve this equation for x in terms of y (if possible).

Step 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 4

Find the inverse function of $f(x) = x^3 + 2$.

Inverse Functions

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f.

Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.

But we get the point (b, a) from (a, b) by reflecting about the line $y = x$. (See Figure 8.)

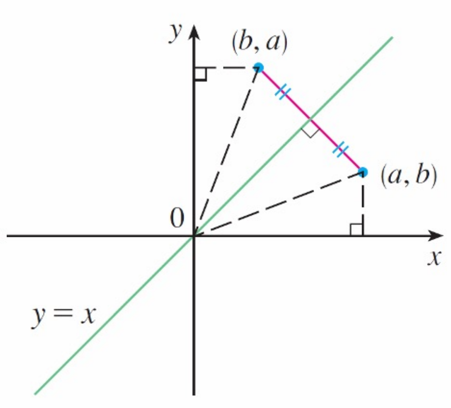


Figure 8

Inverse Functions

Therefore, as illustrated by Figure 9:

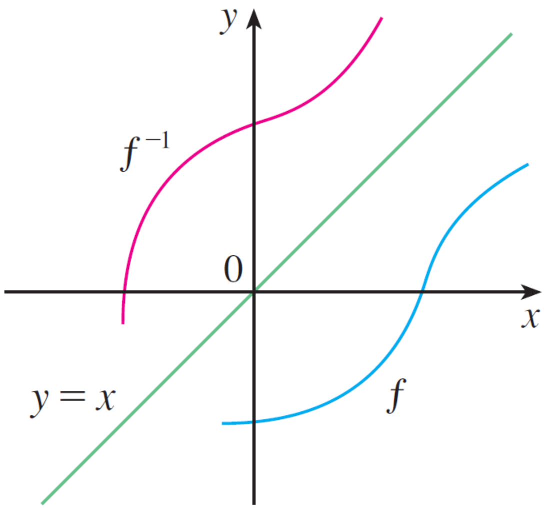


Figure 9

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.