3.7 Optimization Problems

Example 1

A farmer has 1200 m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Example 2

A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Optimization Problems

First Derivative Test for Absolute Extreme Values Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Example 3

Find the point on the parabola $y^2 = 2x$ that is closest to the point (1, 4).

Example 4

A woman launches her boat from point *A* on a bank of a straight river, 3 km wide, and wants to reach point *B*, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). She could row her boat directly across the river to point *C* and then run to *B*, or she could row directly to B, or she could row to some point *D* between *C* and *B* and then run to *B*. If she can row 6 km/h and run 8 km/h, where should she land to reach *B* as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the woman rows.)

Example 4

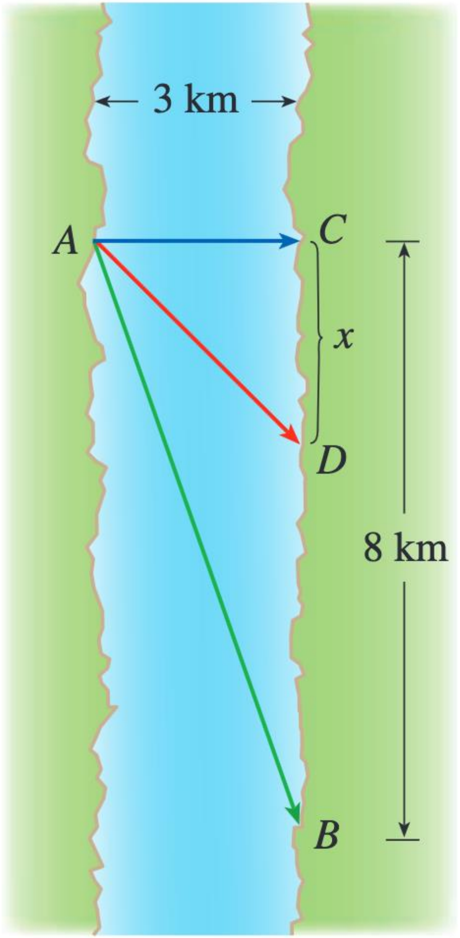


FIGURE 7

3.8 Newton's Method

Newton's Method

We aim to find a root of a function f; that is find a solution of

$$f(x) = 0.$$

There are a variety of methods, but probably the most of famous one is called **Newton's method**, also called the **Newton-Raphson method**. Numerical rootfinders usually use this or a variant of this method.

We will explain how this method works, partly to show what happens inside a calculator or computer, and partly as an application of the idea of linear approximation.

Newton's Method

The geometry behind Newton's method is shown in Figure 2, where the root that we are trying to find is labeled *r*.

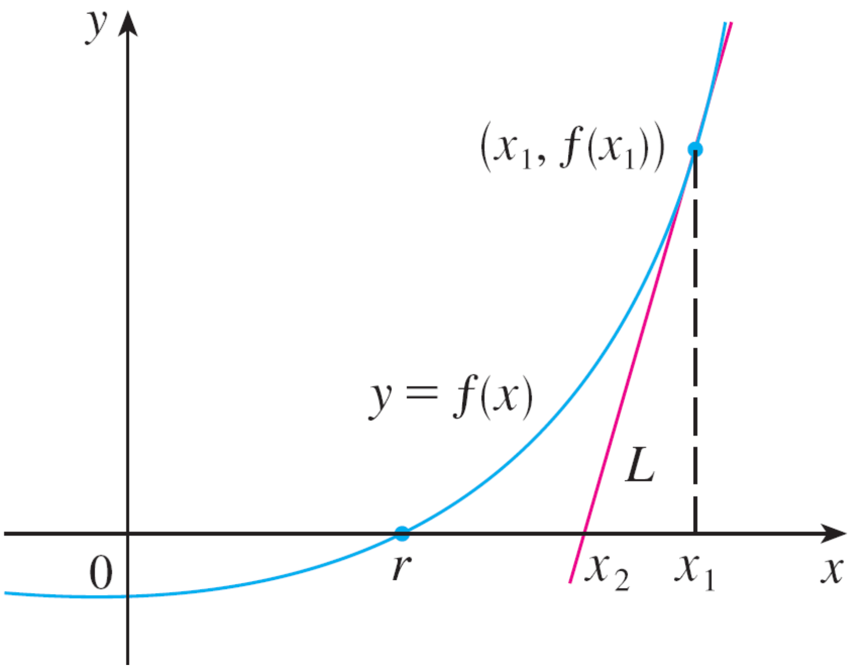


Figure 2

Newton's Method

In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

$$\boxed{2} \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence *converges* to r and we write

$$\lim_{n \to \infty} x_n = r$$

Warning

Although the sequence of successive approximations converges to the desired solution for functions of the type illustrated in Figure 3, in certain circumstances the sequence may not converge.

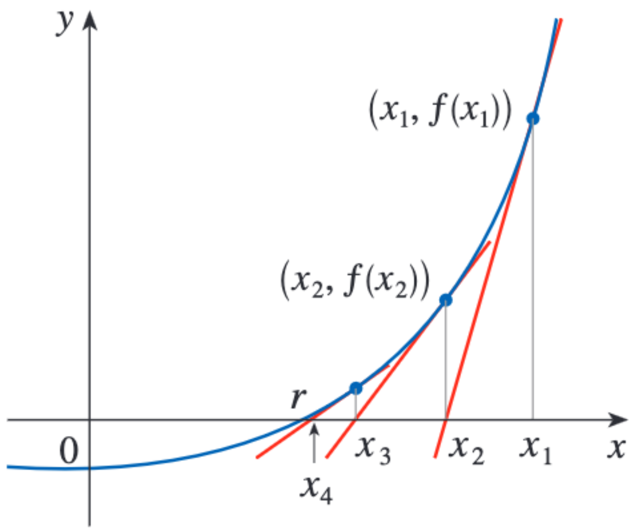


FIGURE 3

Warning

For example, consider the situation shown in Figure 4. You can see that x_2 is a worse approximation than x_1. This is likely to be the case when $f'(x_1)$ is close to 0. It might even happen that an approximation (such as x_3 in Figure 4) falls outside the domain of f. Then Newton's method fails and a better initial approximation x_1 should be chosen.

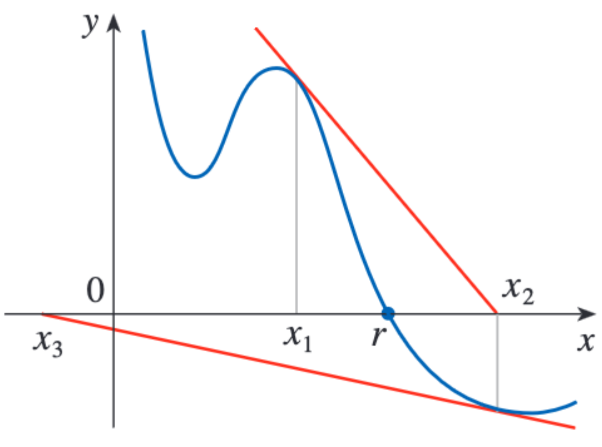


FIGURE 4

Example 1

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 – 2x – 5 = 0$. (Newton himself used this equation to illustrate his method)

Example 2

Use Newton’s method to find $\sqrt[6]{2}$ correct to eight decimal places.

3.9 Antiderivatives

Antiderivatives

A physicist who knows the velocity of a particle might wish to know its position at a given time.

An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period.

A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time.

Antiderivatives

In each case, the problem is to find a function *F* whose derivative is a known function *f*. If such a function *F* exists, it is called an *antiderivative* of *f*.

Definition A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Antiderivatives

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$.

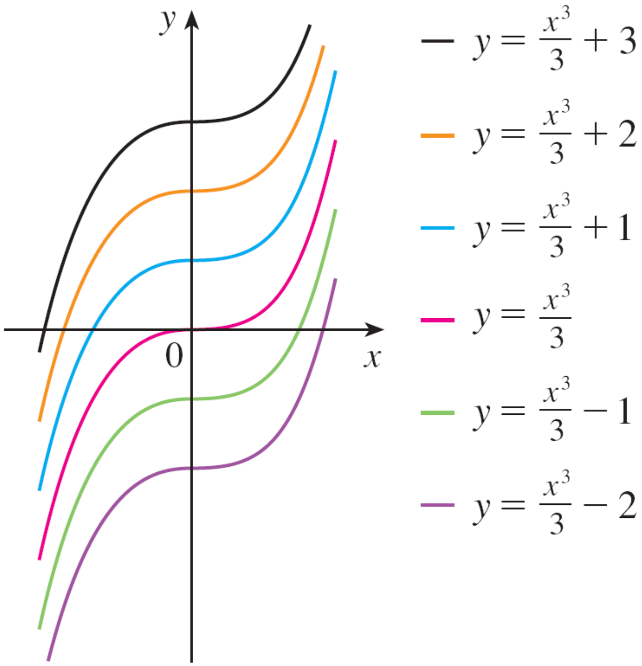
But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore, both F and G are antiderivatives of f.

Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f.

Antiderivatives

By assigning specific values to the constant *C*, we obtain a family of functions whose graphs are vertical translates of one another (see Figure 1).

This makes sense because each curve must have the same slope at any given value of *x*.



Members of the family of antiderivatives of $f(x) = x^2$

Figure 1

Antiderivatives

1 **Theorem** If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example 1

Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ **(b)** $f(x) = x^n$, $n \geq 0$ **(c)** $f(x) = x^{-3}$

Antiderivatives

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives.

2 Table of Antidifferentiation Formulas

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
		$\sec x \tan x$	$\sec x$

To obtain the most general antiderivative from the particular ones in Table 2, we have to add a constant (or constants), as in Example 1.

Antiderivatives

Each formula in the table is true because the derivative of the function in the right column appears in the left column.

In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function.

The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation $F' = f$, $G' = g$.)