6.6 Inverse Trigonometric Functions

Inverse Trigonometric Functions

You can see from Figure 1 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test).

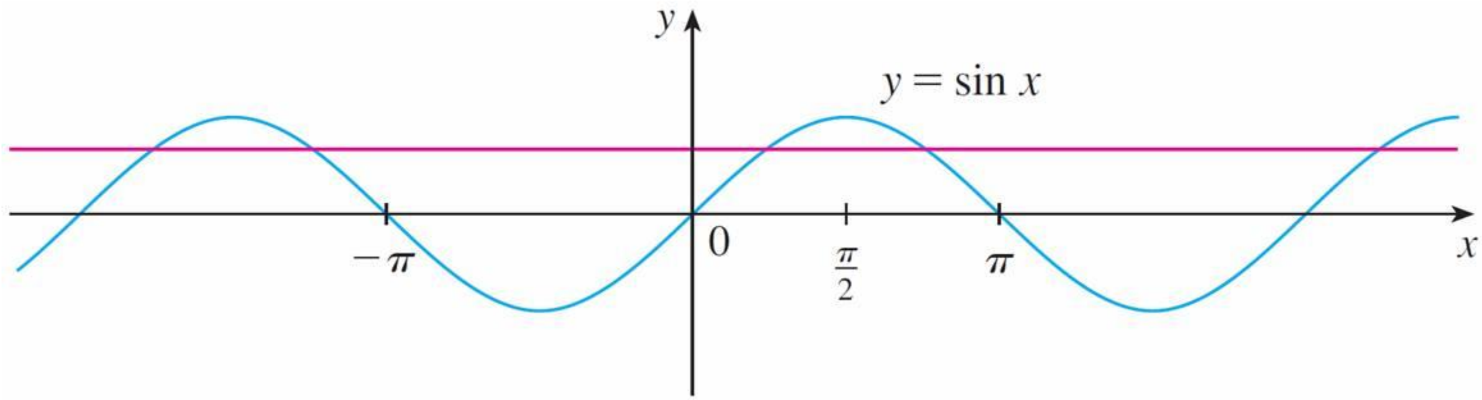
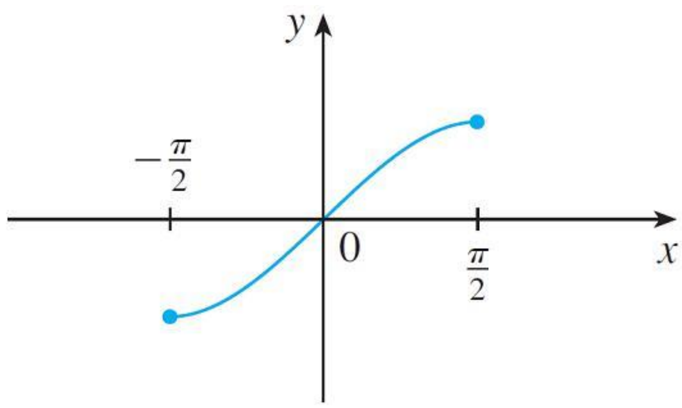


Figure 1

Inverse Trigonometric Functions

But the function $f(x) = \sin x$, $-\pi/2 \leq x \leq \pi/2$ *is* one-to-one (see Figure 2).The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.



$y = \sin x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Figure 2

Inverse Trigonometric Functions

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

1 $$\sin^{-1} x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Thus, if $-1 \leq x \leq 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

Example 1

Evaluate (a) $\sin^{-1}\left(\frac{1}{2}\right)$ and (b) $\tan(\arcsin \frac{1}{3})$.

$\frac{1}{2}$

Inverse Trigonometric Functions

The cancellation equations for inverse functions become, in this case,

$$\boxed{2} \qquad \begin{aligned} &\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ &\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}$$

Example

Evaluate

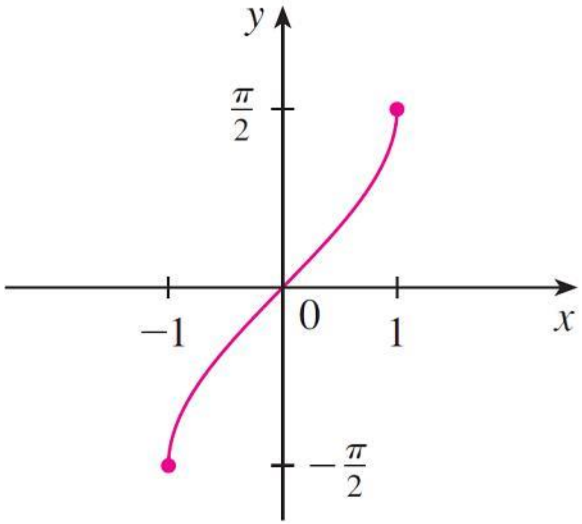
a) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

b) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$

c) $\sin(\sin^{-1} 2)$

Inverse Trigonometric Functions

The inverse sine function, $\sin^{-1}$, has domain [–1, 1] and range [–π/2, π/2], and its graph, shown in Figure 4, is obtained from that of the restricted sine function (Figure 2) by reflection about the line $y = x$.



$y = \sin^{-1} x = \arcsin x$

Figure 4

Derivative of $\sin^{-1} x$

Theorem. The function $f(x) = \sin^{-1} x$ is differentiable on $(-1, 1)$ and its derivative is given by

$$\boxed{3} \qquad \boxed{\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad -1 < x < 1}$$

Example 2

If $f(x) = \sin^{-1}(x^2 - 1)$, find (a) the domain of f, (b) $f'(x)$, and (c) the domain of f'.

Inverse Trigonometric Functions

The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x$, $0 \leq x \leq \pi$ is one-to-one (see Figure 6) and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

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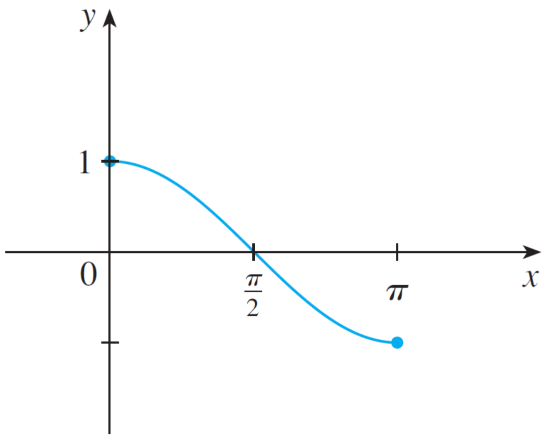
$$\boxed{\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi}$$



$y = \cos x$, $0 \leq x \leq \pi$

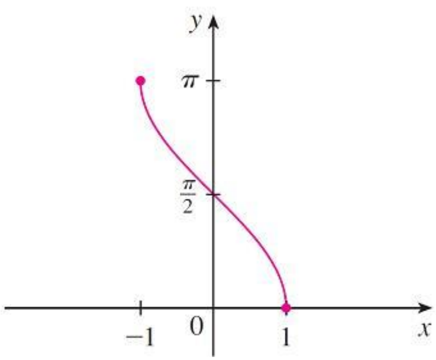
Figure 6

Inverse Trigonometric Functions

The cancellation equations are

$$\boxed{5} \qquad \begin{aligned} \cos^{-1}(\cos x) &= x \quad \text{for } 0 \leqslant x \leqslant \pi \\ \cos(\cos^{-1}x) &= x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}$$

The inverse cosine function, $\cos^{-1}$, has domain [–1, 1] and range [0, π] and is a continuous function whose graph is shown in Figure 7.



$y = \cos^{-1}x = \arccos x$

Figure 7

Inverse Trigonometric Functions

Its derivative is given by

$$\boxed{6} \qquad \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

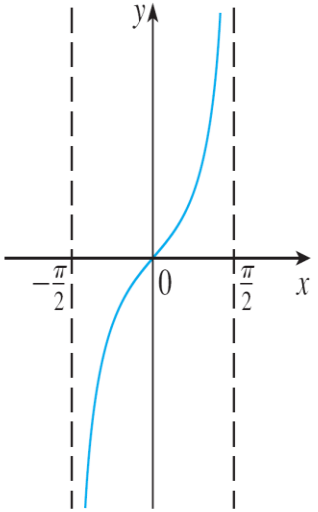
The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2\ ,\ \pi/2)$.

Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$.

Inverse Trigonometric Functions

It is denoted by $\tan^{-1}$ or arctan. (See Figure 8.)

7 $$\boxed{\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}}$$



$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

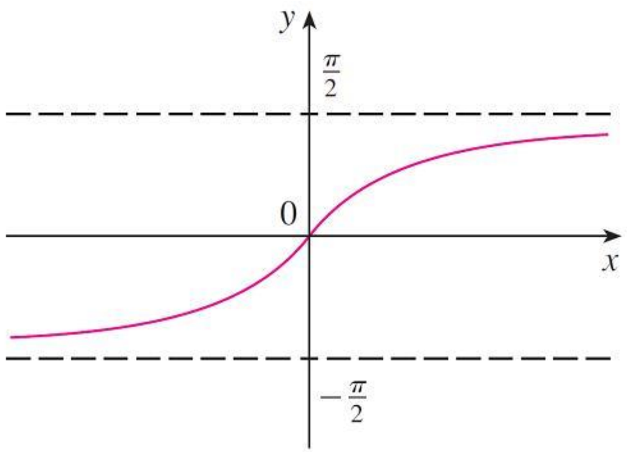
Figure 8

Example 3

Simplify the expression $\cos(\tan^{-1} x)$.

Inverse Trigonometric Functions

The inverse tangent function, $\tan^{-1}x = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 10.



$y = \tan^{-1}x = \arctan x$

Figure 10

Inverse Trigonometric Functions

We know that

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to -(\pi/2)^+} \tan x = -\infty$$

and so the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan.

Since the graph of $\tan^{-1}$ is obtained by reflecting the graph of the restricted tangent function about the line $y = x$, it follows that the lines $y = \pi/2$ and $y = -\pi/2$ are horizontal asymptotes of the graph of $\tan^{-1}$.

Inverse Trigonometric Functions

This fact is expressed by the following limits:

8 $$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2} \qquad \lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Example 4

Evaluate $\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right)$.

The derivative of $\tan^{-1} x$

Theorem. The function $f(x) = \tan^{-1} x$ is differentaiable on $(-\infty, \infty)$ and

$$\boxed{9} \qquad \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

Inverse Trigonometric Functions

The remaining inverse trigonometric functions are not used as frequently and are summarized here.

$$\boxed{10} \quad y = \csc^{-1}x \;\; (|x| \geqslant 1) \iff \csc y = x \;\; \text{and} \;\; y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1}x \;\; (|x| \geqslant 1) \iff \sec y = x \;\; \text{and} \;\; y \in [0, \pi/2) \cup [\pi, 3\pi/2)$$

$$y = \cot^{-1}x \;\; (x \in \mathbb{R}) \iff \cot y = x \;\; \text{and} \;\; y \in (0, \pi)$$

Inverse trigonometric functions

For example, in case of the secant function:

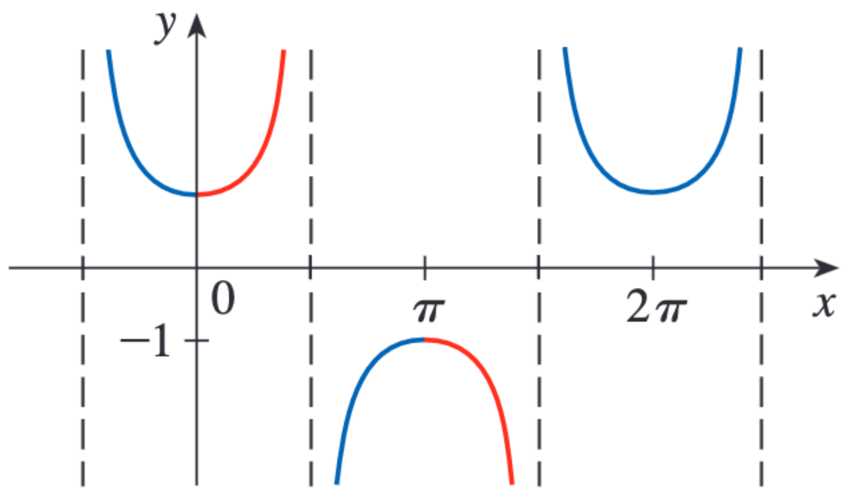


FIGURE 11

$y = \sec x$

Inverse Trigonometric Functions

We collect in Table 11 the differentiation formulas for all of the inverse trigonometric functions.

$\boxed{11}$ **Table of Derivatives of Inverse Trigonometric Functions**

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Example 5

Differentiate (a) $y = \dfrac{1}{\sin^{-1} x}$ and (b) $f(x) = x \arctan \sqrt{x}$.

Integrals od Inverse Trigonometric Functions

12 $$\boxed{\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C}$$

13 $$\boxed{\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C}$$

Example 7

Find

$$\int_0^{1/4} \frac{1}{\sqrt{1 - 4x^2}}\, dx.$$

Integrals of Inverse Trigonometric Functions

Example 8. Show that

14 $$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Example 9

Find $\int \frac{x}{x^4 + 9}\, dx.$