2.9 Linear Approximations and Differentials

Linear Approximations and Differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line.

This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f.

Linear Approximations and Differentials

So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

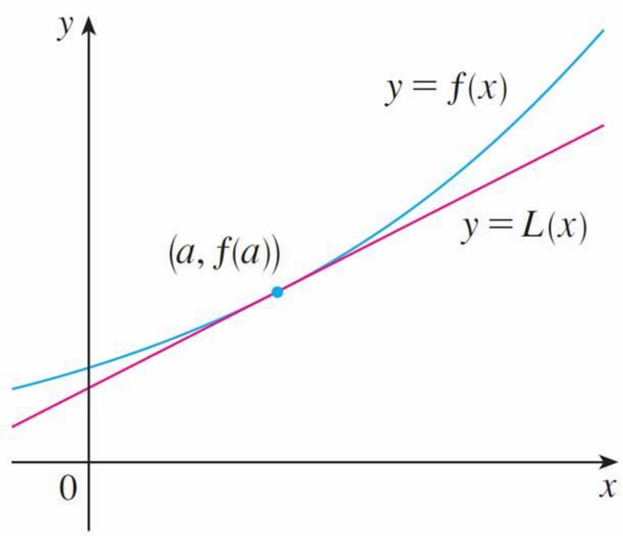

Figure 1

Linear Approximations and Differentials

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a) \tag{1}$$

is called the **linear approximation** or **tangent line approximation** of f at a.

Linear Approximations and Differentials

The linear function whose graph is this tangent line, that is,

$$\boxed{2} \qquad L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Example 1

Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Example 1 – *Solution*

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The linear approximation is illustrated in Figure 2.

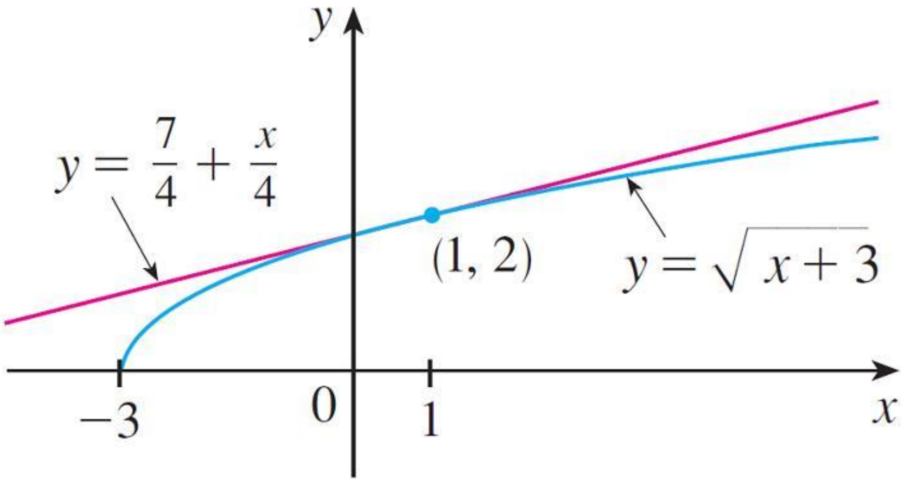


Figure 2

Linear Approximations and Differentials

In the following table we compare the estimates from the linear approximation in Example 1 with the true values.

	x	From $L(x)$	Actual value
$\sqrt{3.9}$	0.9	1.975	1.97484176 . . .
$\sqrt{3.98}$	0.98	1.995	1.99499373 . . .
$\sqrt{4}$	1	2	2.00000000 . . .
$\sqrt{4.05}$	1.05	2.0125	2.01246117 . . .
$\sqrt{4.1}$	1.1	2.025	2.02484567 . . .
$\sqrt{5}$	2	2.25	2.23606797 . . .
$\sqrt{6}$	3	2.5	2.44948974 . . .

Applications to Physics

Applications to Physics

Linear approximations are often used in physics. In analyzing the consequences of an equation, a physicist sometimes needs to simplify a function by replacing it with its linear approximation.

For instance, in deriving a formula for the period of a pendulum, physics textbooks obtain the expression $a_T = -g \sin \theta$ for tangential acceleration and then replace $\sin \theta$ by θ with the remark that $\sin \theta$ is very close to θ if θ is not too large.

Applications to Physics

You can verify that the linearization of the function $f(x) = \sin x$ at $a = 0$ is $L(x) = x$ and so the linear approximation at 0 is

$$\sin x \approx x$$

So, in effect, the derivation of the formula for the period of a pendulum uses the tangent line approximation for the sine function.

Applications to Physics

Another example occurs in the theory of optics, where light rays that arrive at shallow angles relative to the optical axis are called *paraxial rays.*

In paraxial (or Gaussian) optics, both $\sin\theta$ and $\cos\theta$ are replaced by their linearizations. In other words, the linear approximations

$$\sin\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1$$

are used because θ is close to 0.

Differentials

Differentials

The ideas behind linear approximations are sometimes formulated in the terminology and notation of *differentials.*

If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number.

The **differential** dy is then defined in terms of dx by the equation

$$\boxed{3} \qquad dy = f'(x)\, dx$$

Differentials

So *dy* is a dependent variable; it depends on the values of *x* and *dx*.

If *dx* is given a specific value and *x* is taken to be some specific number in the domain of *f*, then the numerical value of *dy* is determined.

Differentials

The geometric meaning of differentials is shown in Figure 5.

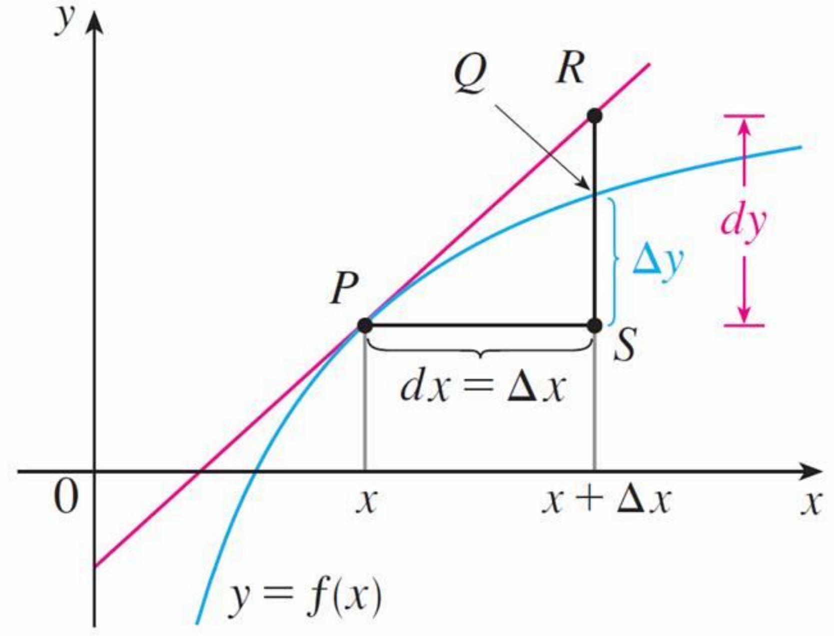


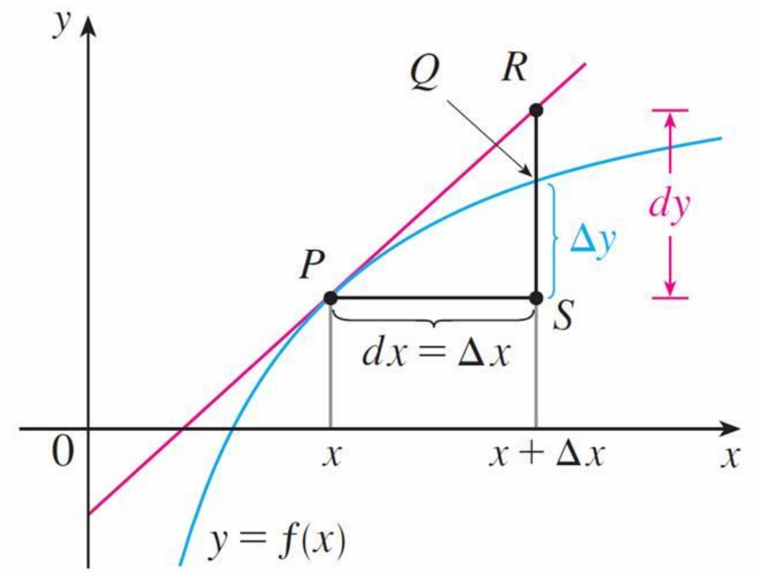
Figure 5

Differentials

Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)\, dx = dy$.



Example 3

Therefore, dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Example 3. Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes (a) from 2 to 2.05 and (b) from 2 to 2.01.

Example 4

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

3.1 Maximum and Minimum Values

Maximum and Minimum Values

Some of the most important applications of differential calculus are *optimization problems*, in which we are required to find the optimal (best) way of doing something. These can be done by finding the maximum or minimum values of a function.

Let's first explain exactly what we mean by maximum and minimum values.

We see that the highest point on the graph of the function *f* shown in Figure 1 is the point (3, 5).

In other words, the largest value of *f* is *f*(3) = 5. Likewise, the smallest value is *f*(6) = 2.

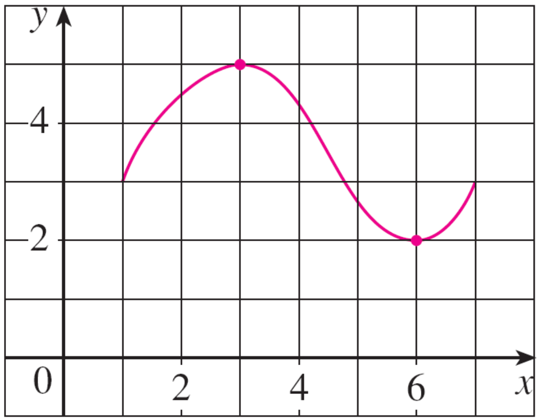


Figure 1

Maximum and Minimum Values

We say that $f(3) = 5$ is the *absolute maximum* of f and $f(6) = 2$ is the *absolute minimum*. In general, we use the following definition.

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

An absolute maximum or minimum is sometimes called a **global** maximum or minimum.

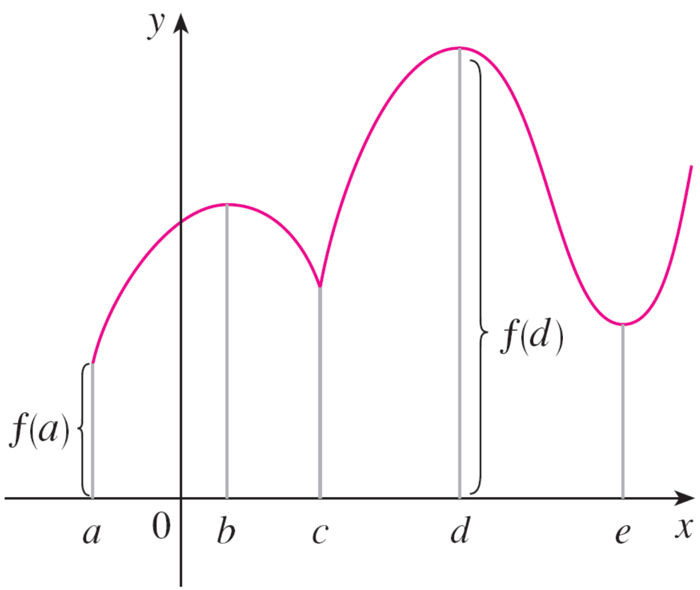
The maximum and minimum values of f are called **extreme values** of f.

Maximum and Minimum Values

Figure 2 shows the graph of a function *f* with absolute maximum at *d* and absolute minimum at *a*.

Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point.

In Figure 2, if we consider only values of *x* near *b* [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a *local maximum value* of *f*.



Abs min $f(a)$, abs max $f(d)$
loc min $f(c)$, $f(e)$, loc max $f(b)$, $f(d)$

Figure 2

Maximum and Minimum Values

Likewise, $f(c)$ is called a *local minimum value* of f because $f(c) \le f(x)$ for x near c [in the interval (b, d), for instance].

The function f also has a local minimum at e. In general, we have the following definition.

$\boxed{2}$ **Definition** The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

In Definition 2 (and elsewhere), if we say that something is true **near** c, we mean that it is true on some open interval containing c.

Maximum and Minimum Values

For instance, in Figure 3 we see that $f(4) = 5$ is a local minimum because it's the smallest value of f on the interval I.

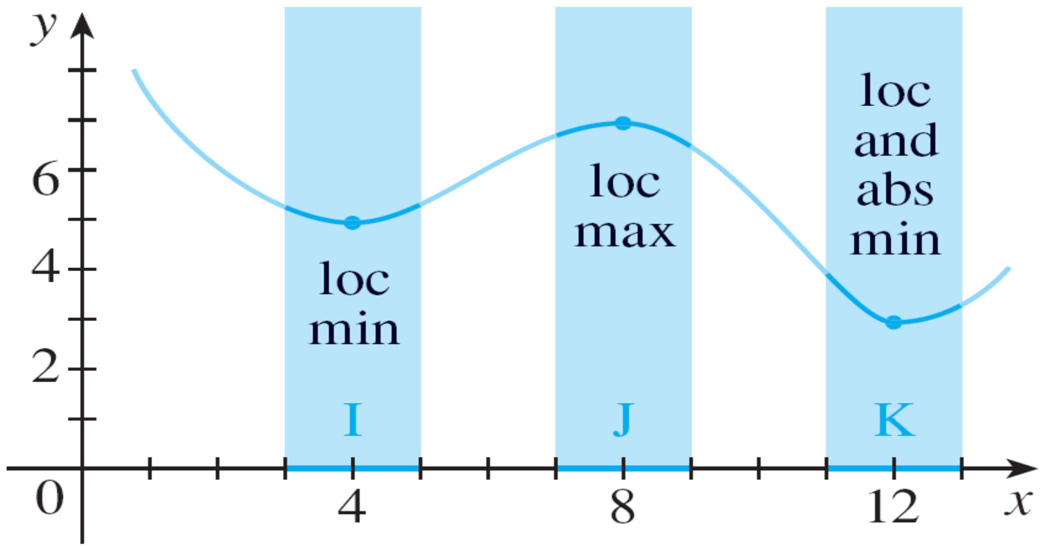


Figure 3

Maximum and Minimum Values

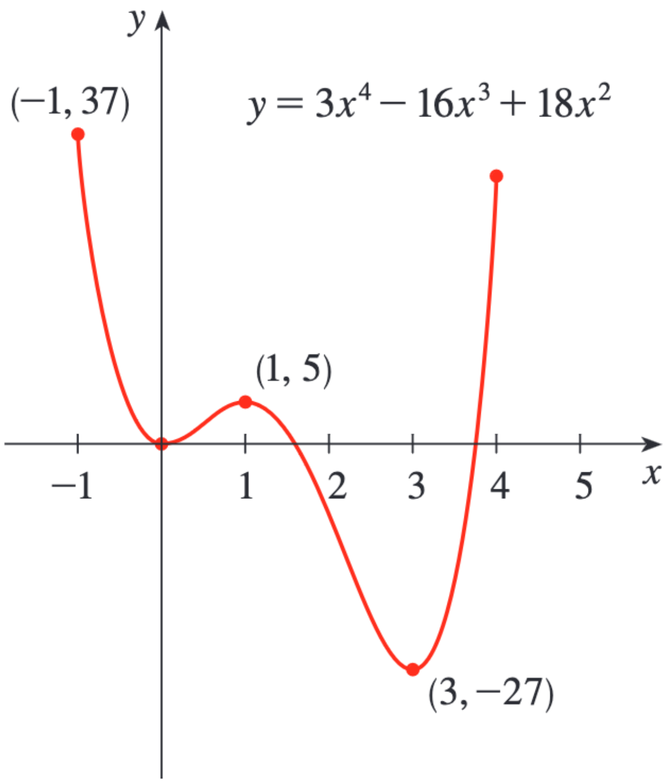
It's not the absolute minimum because $f(x)$ takes smaller values when x is near 12 (in the interval K, for instance).

In fact $f(12) = 3$ is both a local minimum and the absolute minimum.

Similarly, $f(8) = 7$ is a local maximum, but not the absolute maximum because f takes larger values near 1.

Example 1

Consider the function $f(x) = 3x^4 - 16x^3 + 12x^2$.



Examples 2 - 4

Discuss $f(x) = \cos x, f(x) = x^2, f(x) = x^3$.

Maximum and Minimum Values

The following theorem gives conditions under which a function is guaranteed to possess extreme values.

$\boxed{3}$ **The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Maximum and Minimum Values

The Extreme Value Theorem is illustrated in Figure 7.

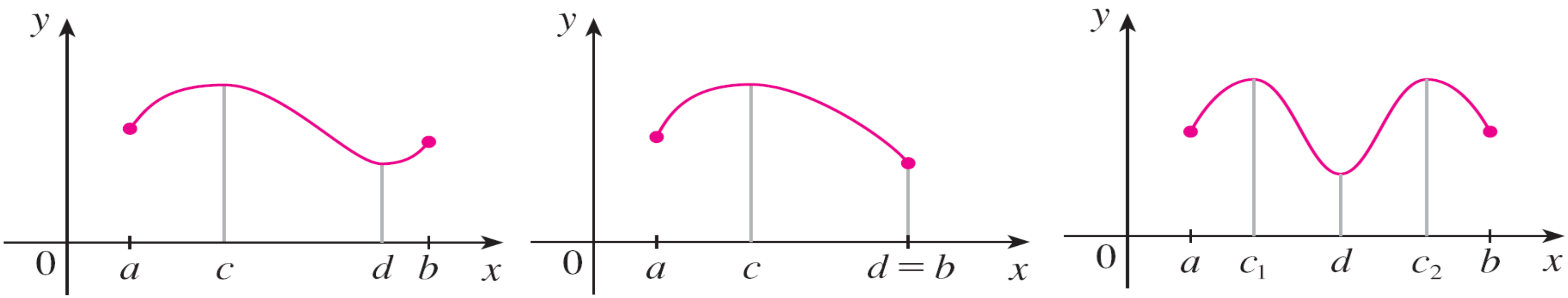


Figure 7

Note that an extreme value can be taken on more than once.

Maximum and Minimum Values

Figures 8 and 9 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

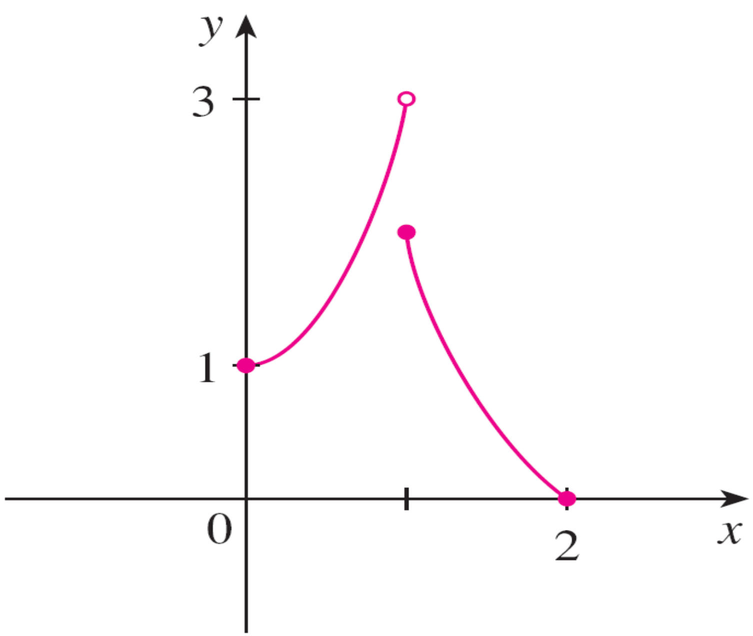


Figure 8 This function has minimum value $f(2) = 0$, but no maximum value.

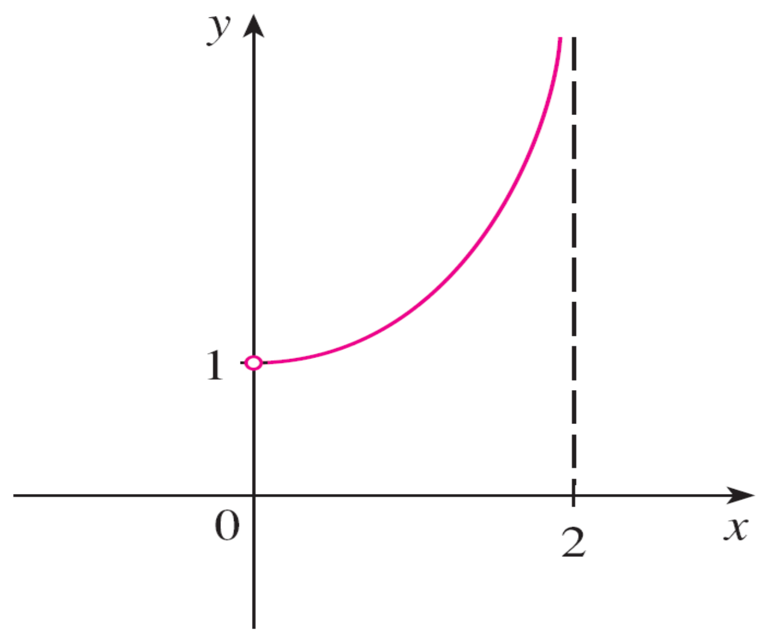


Figure 9 This continuous function g has no maximum or minimum.

Proof of the EVT

Definition. Given $A \subset \mathbb{R}$ we say that A is **bounded above** if there is a real number m such that $a \le m$ for all $a \in A$. Such an m is called an **upper bound** of A.

Similar definition can be given for a set A to be **bounded below** and a **lower bound** of A.

Definition. A set $A \subset \mathbb{R}$ is **bounded** if it is both bounded above and below. A function f defined on [*a*, *b*] is bounded, bounded above/below on [*a*, *b*], if the set

$$A = \{f(x) |\, x \in [a, b]\}$$

is bounded, bounded above/below.

Proof of the EVT

The **completeness axiom** of the real numbers assert that if $A \subset \mathbb{R}$ is bounded above, then A has a **least upper bound** M, called the **supremum** of A, denoted by $M = \sup A$. Thus, (i) $\sup A$ is an upper bound of A (ii) if m is an upper bound of A, then $\sup A \leq m$.

Similarly, $A \subset \mathbb{R}$ is bounded below, then A has a **greatest lower bound** K, called the **infimum** of A, denoted by $K = \inf A$. Thus, (i) $\inf A$ is a lower bound of A (ii) if k is a lower bound of A, then $\inf A \geq k$.

Examples: (0, 1), (0,1], [0,1).