Estimating Sums

A partial sum s_n of any convergent series can be used as an approximation to the total sum s, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$.

The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term.

Estimating Sums

Theorem. If $s = \sum(-1)^{n-1} b_n$ with $b_n > 0, b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Example 4

Example 4. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

Estimating Sums

Note:

The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. The rule does not apply to other types of series.

Absolute Convergence

Given any series $\Sigma\, a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

$\boxed{1}$ **Definition** A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

Absolute Convergence

Notice that if Σa_n is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

Example 5

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent *p*-series ($p = 2$).

Example 6

We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent, but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (p-series with $p = 1$) and is therefore divergent.

Absolute Convergence

2 Definition A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 6 shows that the alternating harmonic series is conditionally convergent. Thus, it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

3 Theorem If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

Example 7

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Example 8

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(a)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

(b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

(c)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+1}$$

Rearrangements

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series.

Rearrangements

By a **rearrangement** of an infinite series $\Sigma\, a_n$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\Sigma\, a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \ldots$$

It turns out that

if $\Sigma\, a_n$ is absolutely convergent series with sum s,
then any rearrangement of $\Sigma\, a_n$ has the same sum s.

Rearrangements

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots := s$$

Note, that $s \neq 0$ (why?).

If we multiply this series by $\frac{1}{2}$, we get

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots = \frac{1}{2}s$$

Rearrangements

Inserting zeros between the terms of this series, we have

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2}s$$

Now we add this to the original series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = s$$

we get

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}s$$

Rearrangements

Notice that the last series in contains the same terms as the original one but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

> if $\Sigma\, a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\Sigma\, a_n$ that has a sum equal to r.

11.6 Ratio and Root Tests

Ratio and Root Tests

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

Examples 1 and 2

Example 1. Test the absolute convergence of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$

Example 2. Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}.$$

Ratio and Root Tests

The following test is convenient to apply when nth powers occur.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Ratio and Root Tests

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series $\Sigma\, a_n$ could converge or diverge.

Note: One can show that if both $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ and $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then the two limits are equal. Thus, if $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too. However, it might happen that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ exists and $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not, but not the other way around. Hence, the root test has, in general, wider scope.

Examples 4 and 5

Example 4. Test the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{2n + 3}{3n + 2} \right)^n.$$

Example 5. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^n$$