7.3 Trigonometric Substitution

Trigonometric Substitution

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$.

If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult.

Trigonometric Substitution

If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta}$$

$$= \sqrt{a^2(1 - \sin^2\theta)}$$

$$= \sqrt{a^2 \cos^2\theta}$$

$$= a\left|\cos \theta\right|$$

Trigonometric Substitution

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin\theta$ (the old variable is a function of the new one).

In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse.

To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one.

Trigonometric Substitution

In this case, if we replace *u* by *x* and *x* by *t* in the Substitution Rule, we obtain

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt$$

This kind of substitution is called *inverse substitution*.

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function.

Trigonometric Substitution

This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Trigonometric Substitution

In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one.

Example 1

Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2}\, dx.$

Example 2

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Trigonometric Substitution

Note:

Since the integral in Example 2 was a definite integral, we changed the limits of integration and did not have to convert back to the original variable *x*.

Example 3

Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$

Example 5

Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

Trigonometric Substitution

Note:
As Example 5 illustrates, hyperbolic substitutions can be used in place of trigonometric substitutions and sometimes they lead to simpler answers.

But we usually use trigonometric substitutions because trigonometric identities are more familiar than hyperbolic identities.

Examples

Example 6. Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\, dx$.

Example 7. Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\, dx$.