Example 2

Find all functions *g* such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

Antiderivatives

An equation that involves the derivatives of a function is called a **differential equation**.

The general solution of a differential equation involves an arbitrary constant (or constants).

However, there may be some extra conditions given that will determine the constants and therefore uniquely specify the solution.

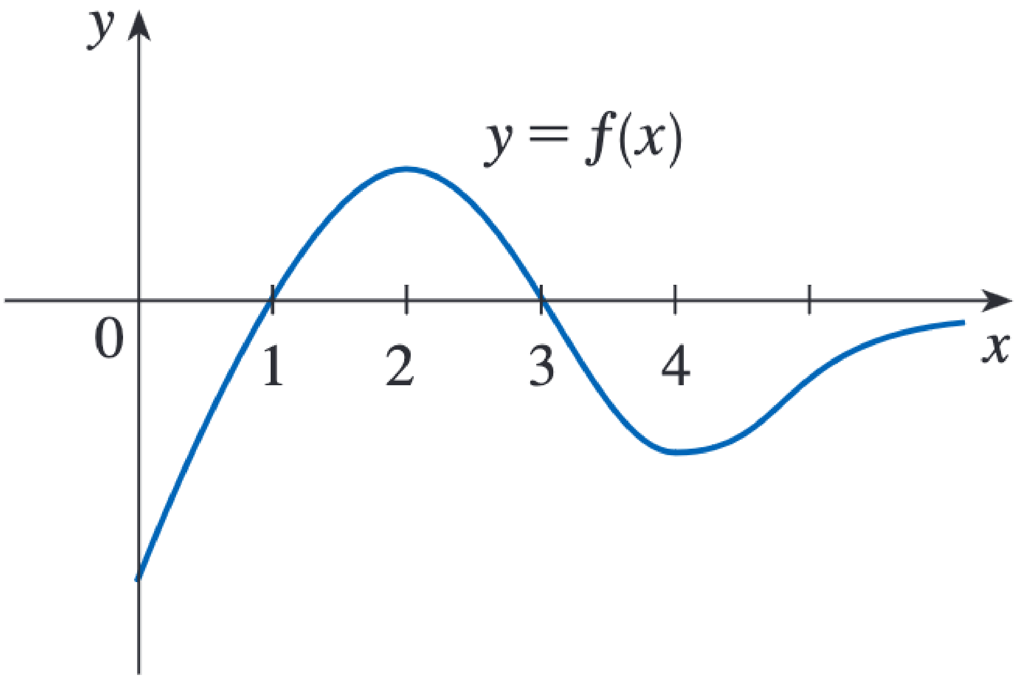
Example 3 and 4

Example 3. Find f if $f'(x) = x\sqrt{x}$ and $f(1) = 2$.

Example 4. Find f if $f''(x) = 12x^2 + 6x - 4, f(0) = 4$ and $f(1) = 1$.

Example 5

The graph of a function *f* is given below. Make a rough sketch of an antiderivative *F*, given that *F*(0) = 2.



Rectilinear Motion

Rectilinear Motion

Antidifferentiation is particularly useful in analyzing the motion of an object moving in a straight line. Recall that if the object has position function $s = f(t)$, then the velocity function is $v(t) = s'(t)$.

This means that the position function is an antiderivative of the velocity function.

Likewise, the acceleration function is $a(t) = v'(t)$, so the velocity function is an antiderivative of the acceleration.

If the acceleration and the initial values $s(0)$ and $v(0)$ are known, then the position function can be found by antidifferentiating twice.

Examples 6 and 7

Example 6. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Example 7. A ball is thrown upward with a speed of 15 m/s from the edge of a cliff, 130 m above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

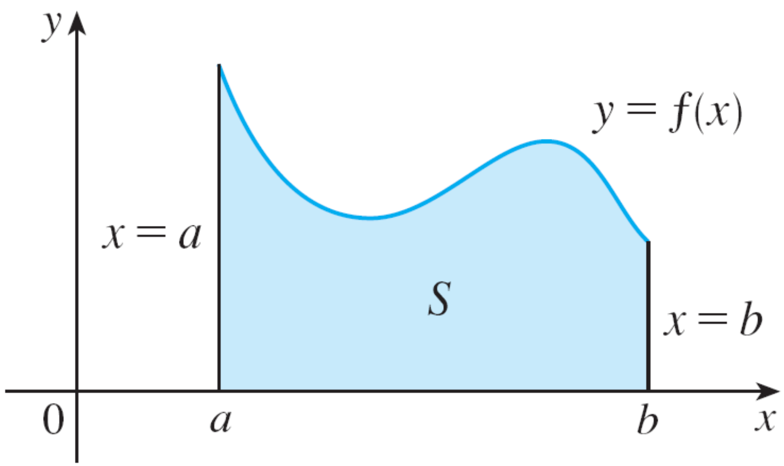
4.1 Areas and Distances

The Area Problem

The Area Problems

We begin by attempting to solve the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.



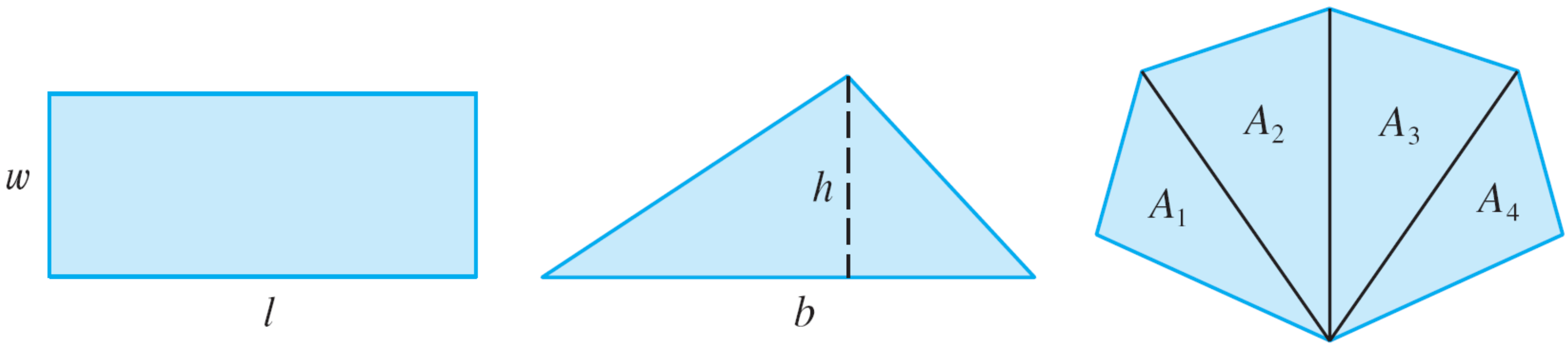
$S = \{(x, y) \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$

Figure 1

The Area Problems

For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height.

The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.



$A = lw$ $\qquad$ $A = \frac{1}{2}bh$ $\qquad$ $A = A_1 + A_2 + A_3 + A_4$

Figure 2

The Area Problems

However, it isn’t so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations.

We pursue a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

Examples 1 and 2

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to t, where t>0 fixed (the parabolic region S illustrated in Figure 3 for t=1).

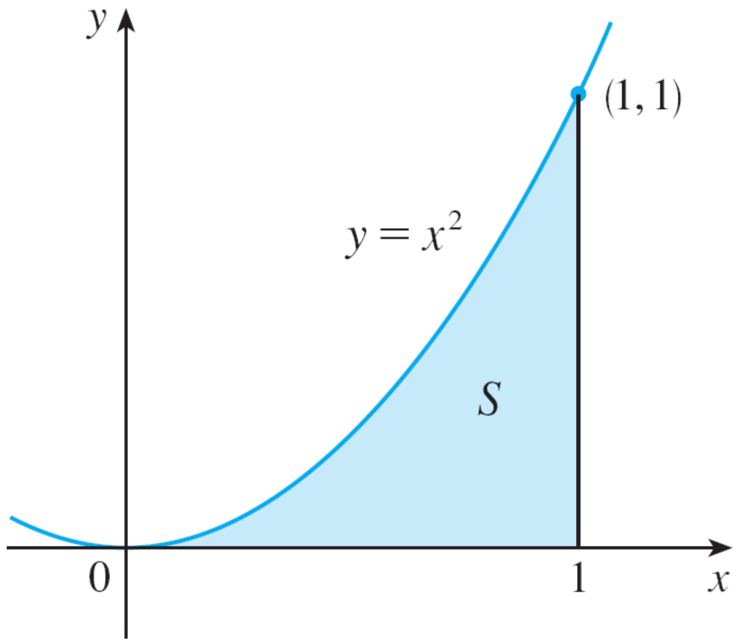
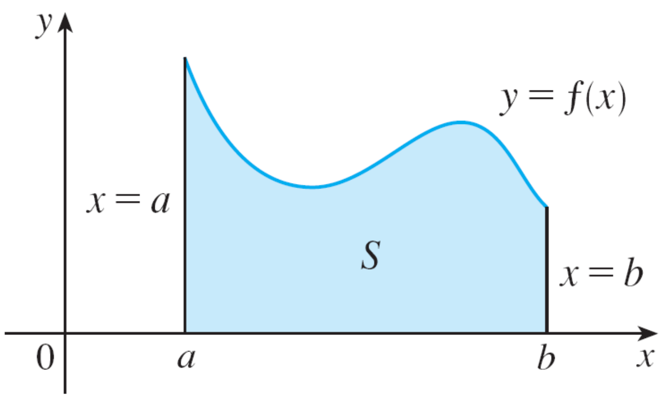


Figure 3

The Area Problems

Let's apply the idea of Example 1 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.



$S = \{(x, y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}$

Figure 1

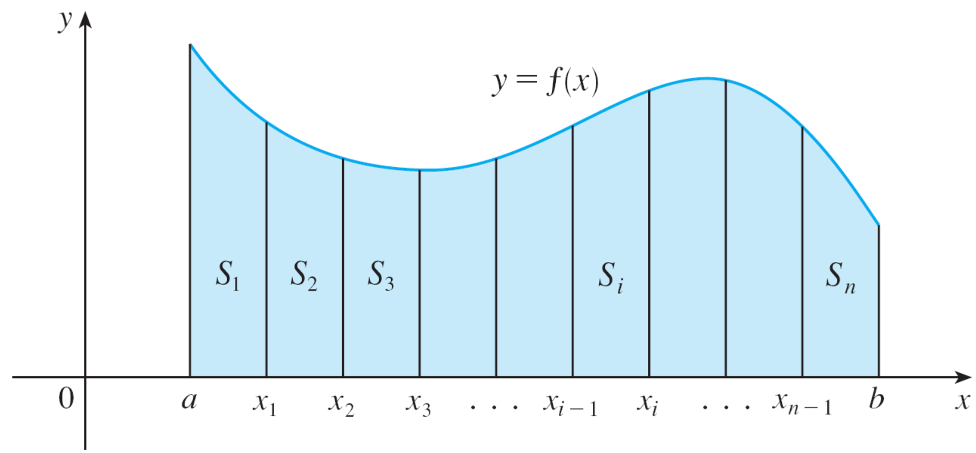


Figure 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

The Area Problems

These strips divide the interval [a, b] into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x,$$

$$x_2 = a + 2\,\Delta x,$$

$$x_3 = a + 3\,\Delta x,$$

$$\vdots$$

The Area Problems

Let's approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 11).

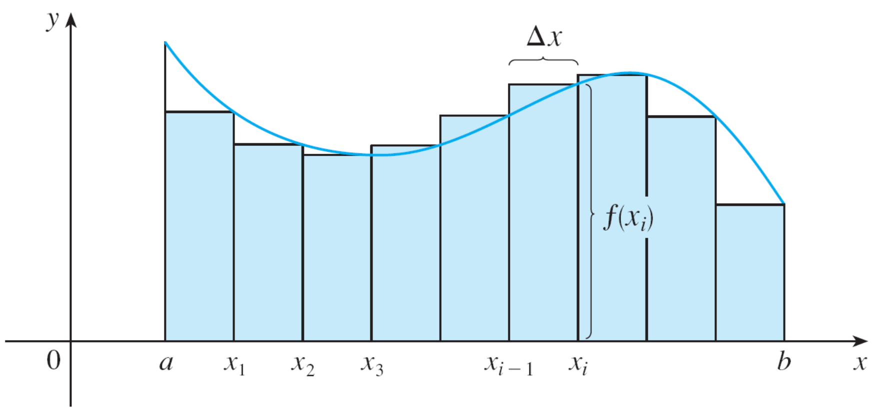


Figure 11

Then the area of the ith rectangle is $f(x_i)\ \Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_n)\,\Delta x$$

The Area Problems

Figure 12 shows this approximation for n = 2, 4, 8, and 12. Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

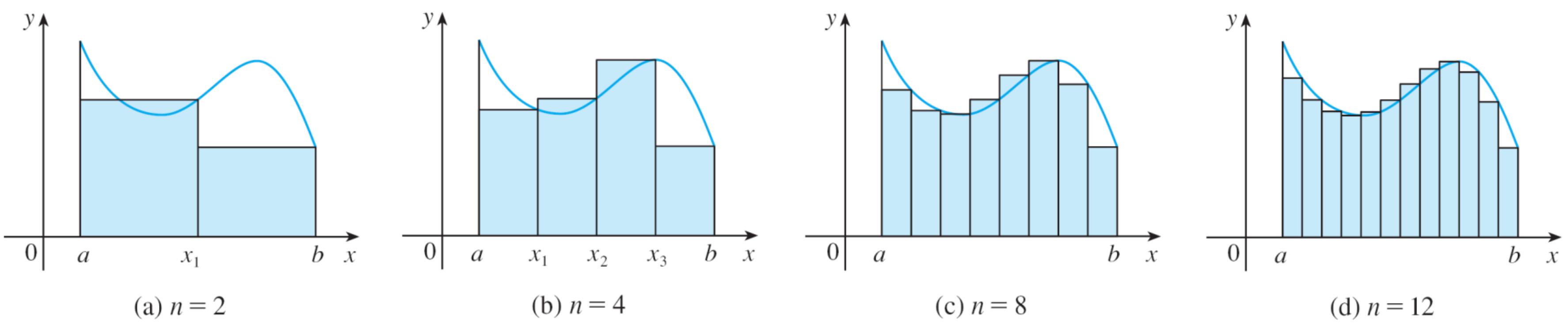


Figure 12

The Area Problems

Therefore, we define the area *A* of the region *S* in the following way.

2 Definition The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

It can be proved that the limit in Definition 2 always exists, since we are assuming that *f* is continuous. It can also be shown that we get the same value if we use left endpoints:

3

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + \cdots + f(x_{n-1})\,\Delta x]$$

The Area Problems

In fact, instead of using left endpoints or right endpoints, we could take the height of the *i*th rectangle to be the value of *f* at *any* number x_i* in the *i*th subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1*, x_2*, \ldots, x_n*$ the **sample points**.

Figure 13 shows approximating rectangles when the sample points are not chosen to be endpoints.

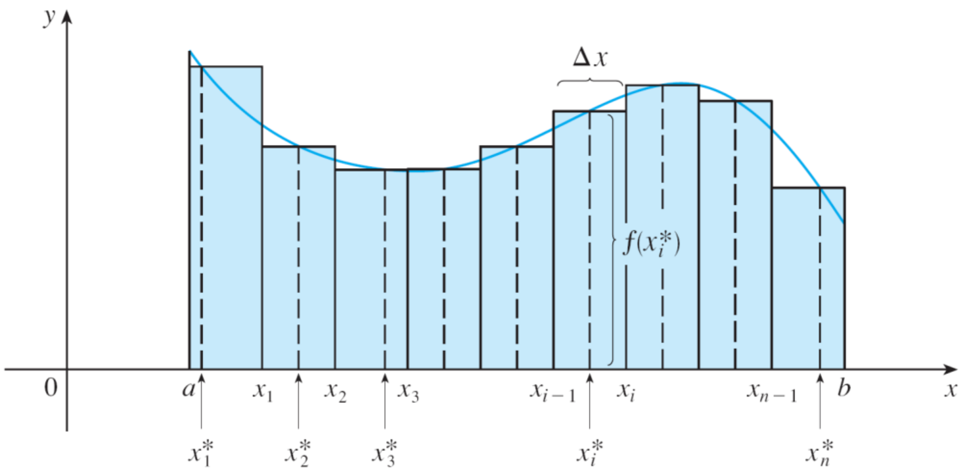


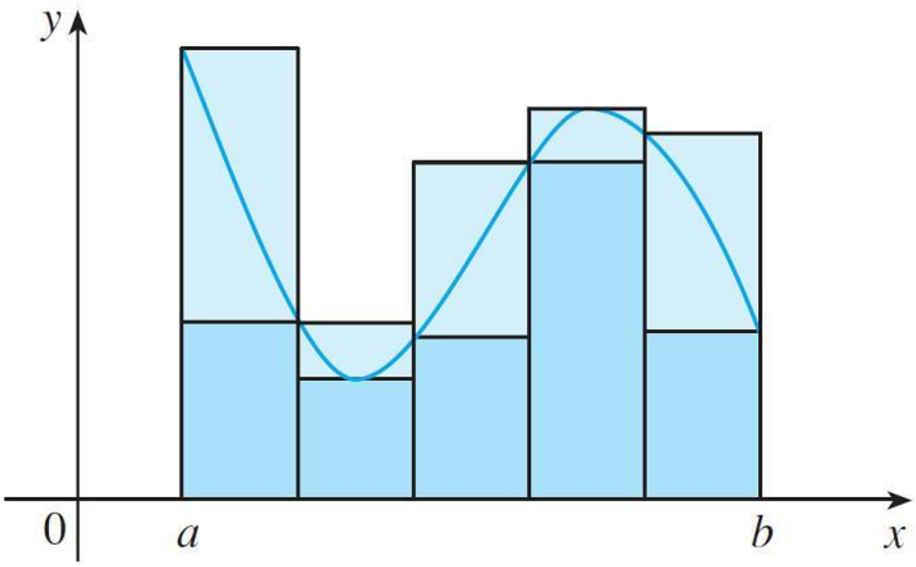
Figure 13

So a more general expression for the area of *S* is

$$\boxed{4} \qquad A = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

The Area Problems

Note: In general, we can form the so-called **lower** (and **upper**) **sums** by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (and maximum) value of *f* on the *i*th subinterval. (See Figure 14)



Lower sums (short rectangles) and upper sums (tall rectangles)

Figure 14

The Area Problems

It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums for every n*.

The Area Problems

We often use **sigma notation** to write sums with many terms more compactly. For instance,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

So the expressions for area in Equations 2, 3, and 4 can be written as follows:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

The Distance Problem

The Distance Problems

Suppose an object moves with velocity $v = f(t)$, where $a \leq t \leq b$ and $f(t) \geq 0$ (so the object always moves in the positive direction).

We take velocity readings at times t_0 (= a), t_1, t_2,. . ., t_n (= b) so that the velocity is approximately constant on each subinterval.

If these times are equally spaced, then the time between consecutive readings is $\Delta t = (b - a)/n$. During the first time interval the velocity is approximately $f(t_0)$ and so the distance traveled is approximately $f(t_0)\,\Delta t$.

The Distance Problems

Similarly, the distance traveled during the second time interval is about $f(t_1)\,\Delta t$ and the total distance traveled during the time interval $[a, b]$ is approximately

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \cdots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

If we use the velocity at right endpoints instead of left endpoints, our estimate for the total distance becomes

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The Distance Problems

The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the *exact* distance *d* traveled is the *limit* of such expressions:

$$\boxed{5} \qquad d = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

4.2 The Definite Integral

The Definite Integral

We have seen that a limit of the form

$$\boxed{1} \quad \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n \to \infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

arises when we compute an area.

We also have seen that it arises when we try to find the distance traveled by an object.

It turns out that this same type of limit occurs in a wide variety of situations even when *f* is not necessarily a positive function.

The Definite Integral

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let x_0 $(= a)$, $x_1, x_2, \ldots, x_n$ $(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Note 1: The symbol $\int$ was introduced by Leibniz and is called an **integral sign**.

It is an elongated *S* and was chosen because an integral is a limit of sums.

Definite integral: precise definition

A function *f* defined on $[a, b]$ is integrable on $[a, b]$ if there is a number, denoted by

$$\int_a^b f(x)\,dx,$$

such that for every $\varepsilon > 0$ there is an integer N such that

$$\left|\int_a^b f(x)dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x\right| < \varepsilon$$

for all $n > N$ and for every choice of $x_i^* \in [x_{i-1}, x_i]$. In this case $\int_a^b f(x)\,dx$ is called the definite integral of *f* from *a* to *b*.

The Definite Integral

In the notation $\int_a^b f(x)\,dx$, *f*(*x*) is called the **integrand** and *a* and *b* are called the **limits of integration**; *a* is the **lower limit** and *b* is the **upper limit**.

For now, the symbol *dx* has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol.

The *dx* simply indicates that the independent variable is *x*. The procedure of calculating an integral is called **integration**.

The Definite Integral

Note 2: The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on *x*. In fact, we could use any letter in place of *x* without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

Note 3: The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866).

The Definite Integral

So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

We know that if f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1).

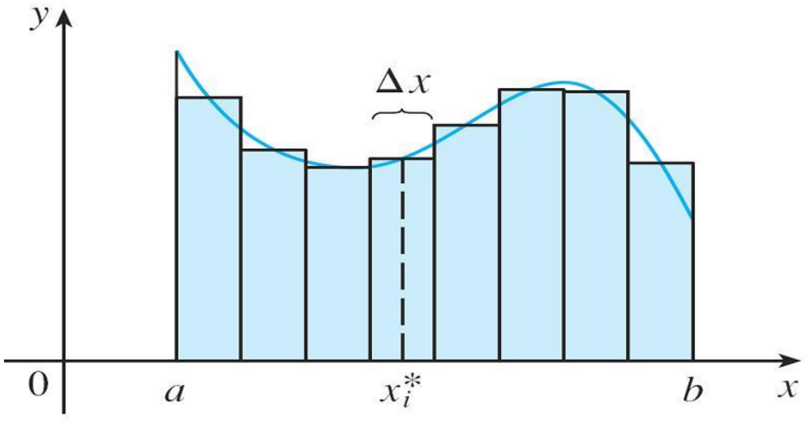


Figure 1

If $f(x) \geq 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

The Definite Integral

We see that the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve *y* = *f*(*x*) from *a* to *b*. (See Figure 2.)

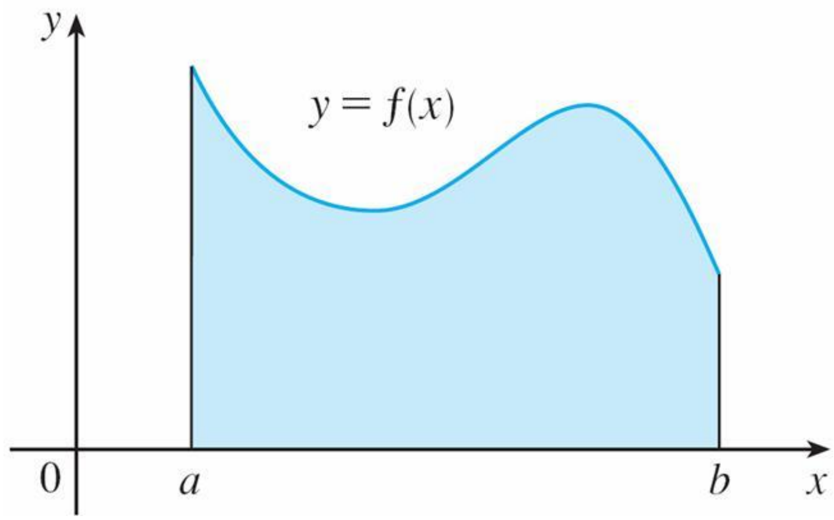


Figure 2

If *f*(*x*) ≥ 0, the integral $\int_a^b f(x)\,dx$ is the area under the curve *y* = *f*(*x*) from *a* to *b*.

The Definite Integral

If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles *minus* the areas of the gold rectangles).

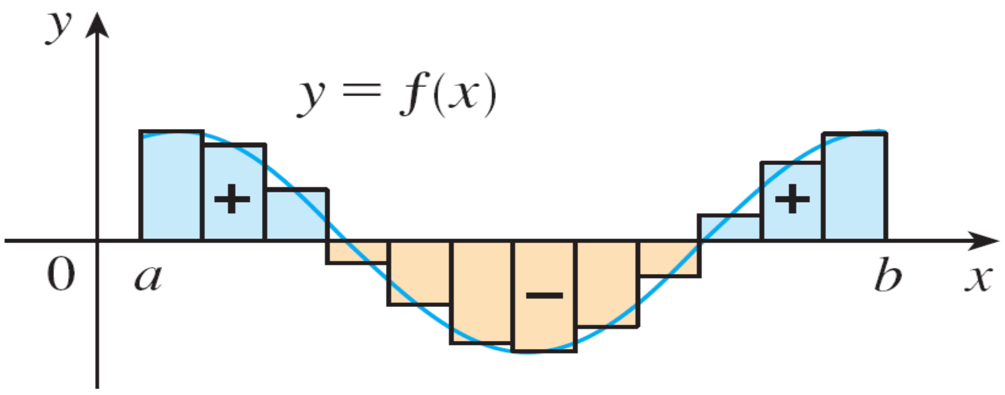


Figure 3

$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area.

The Definite Integral

When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

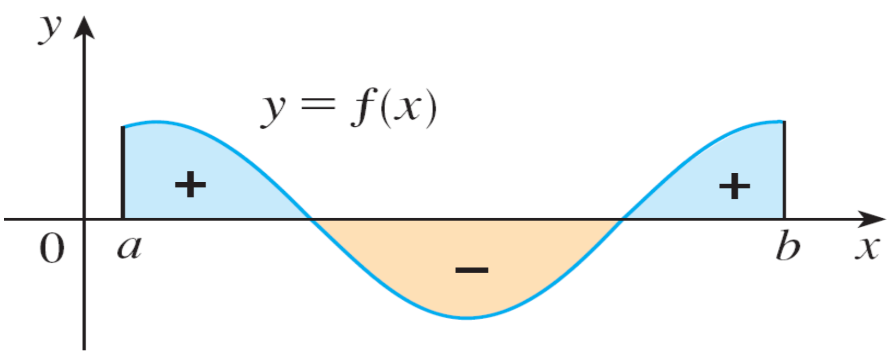


Figure 4

$\int_a^b f(x)\,dx$ is the net area.

The Definite Integral

Note 4: Although we have defined $\int_a^b f(x)\,dx$ by dividing [*a*, *b*] into subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width.

If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we have to ensure that all these widths approach 0 in the limiting process. This happens if the largest width, $\max \Delta x_i$, approaches 0. So in this case the definition of a definite integral becomes

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The Definite Integral

Note 5: Not all functions are integrable. The following theorem shows that the most commonly occurring functions are in fact integrable. It is proved in more advanced courses.

$\boxed{3}$ **Theorem** If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is integrable on $[a, b]$, then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points x_i^*.

The Definite Integral

To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

$\boxed{4}$ **Theorem** If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\qquad \Delta x = \dfrac{b - a}{n} \qquad$ and $\qquad x_i = a + i\,\Delta x$

Note: The converse is not true. It can happen that the particular limit above exists, but *f* is not integrable on [a,b].