14 Partial Derivatives

14.1 Functions of Several Variables

Functions of Two Variables

Functions of Several Variables

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point.

We can think of T as being a function of the two variables x and y , or as a function of the pair (x, y). We indicate this functional dependence by writing $T = f(x, y)$.

The volume of V a circular cylinder depends on its radius r and its height h. In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h, and we write $V(r, h) = \pi r^2 h$.

Functions of Several Variables

Definition A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**.

[Compare this with the notation $y = f(x)$ for functions of a single variable.]

Example 1

For each of the following functions, evaluate *f*(3, 2) and find and sketch the domain.

(a)

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

(b)

$$f(x, y) = x \ln(y^2 - x)$$

Graphs

Graphs

A way of visualizing the behavior of a function of two variables is to consider its graph.

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Just as the graph of a function f of one variable is a curve c with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$.

Graphs

We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane (see Figure 5).

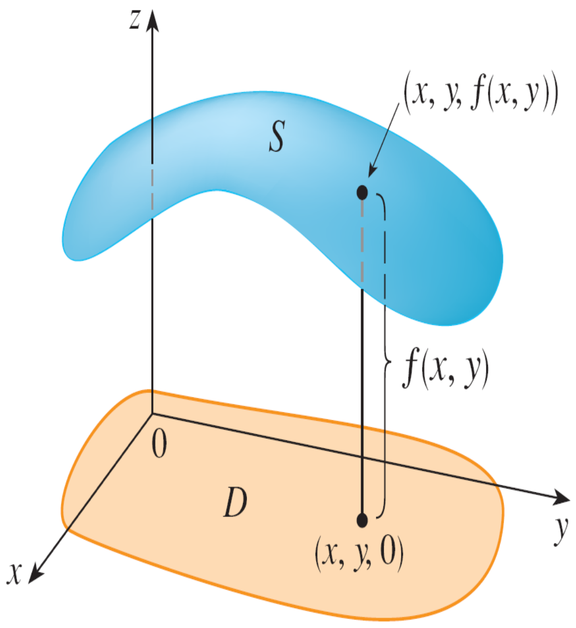

Figure 5

Examples 5 and 6

Example 5. Sketch the graph of the function

$$f(x, y) = 6 - 3x - 2y.$$

Example 6. Find the domain and range of the function

$$z = \sqrt{9 - x^2 - y^2},$$

and sketch its graph.

Level Curves

Level Curves

An important method for visualizing functions, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour lines*, or *level curves*.

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k.

In other words, it shows where the graph of f has height k.

Level Curves

You can see from Figure 11 the relation between level curves and horizontal traces.

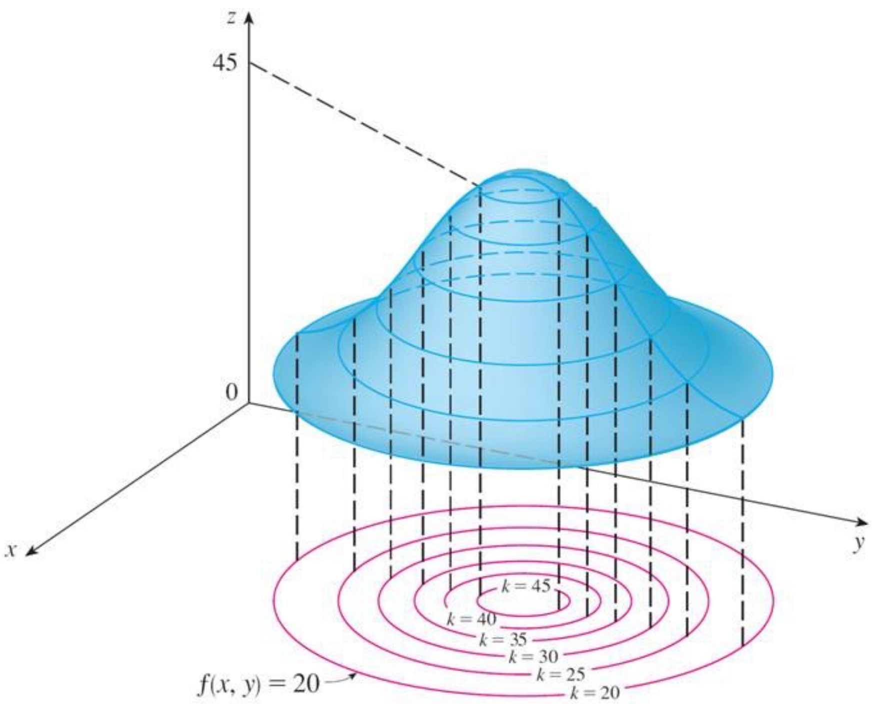

Figure 11

Level Curves

The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.

So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph.

The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

Level Curves

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12.

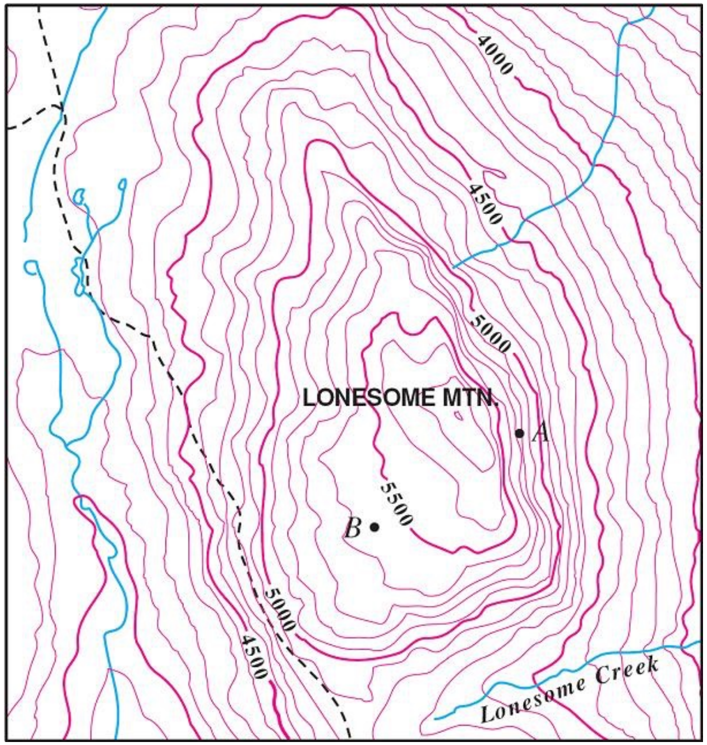

Figure 12

Level Curves

The level curves are curves of constant elevation above sea level.

If you walk along one of these contour lines, you neither ascend nor descend.

Another common example is the temperature at locations (x, y) with longitude x and latitude y.

Here the level curves are called **isothermals** and join locations with the same temperature.

Level Curves

Figure 13 shows a weather map of the world indicating the average January temperatures. The isothermals are the curves that separate the colored bands.

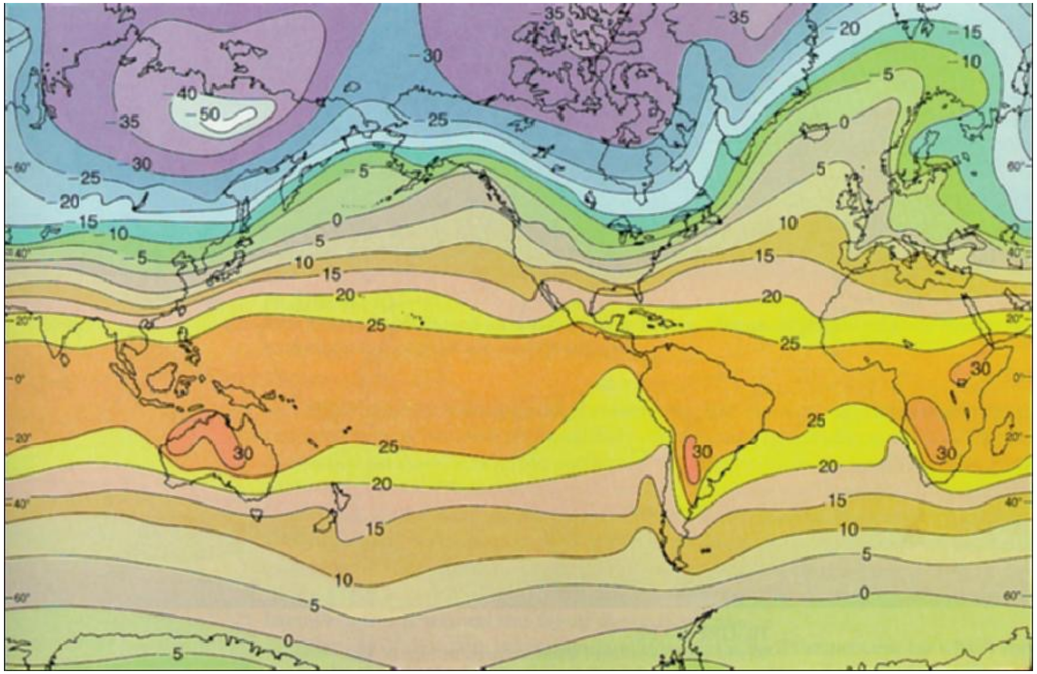

Figure 13

World mean sea-level temperatures in January in degrees Celsius

Example 11

Sketch the level curves and horizontal traces of the function

$$z = \sqrt{9 - x^2 - y^2}$$

for $k = 0, 1, 2, 3$.

Functions of Three or More Variables

Functions of Three or More Variables

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

Example 14

Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

Functions of Three or More Variables

It's very difficult to visualize a function *f* of three variables by its graph, since that would lie in a four-dimensional space.

However, we do gain some insight into *f* by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

Example 15. Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

Functions of Three or More Variables

Functions of any number of variables can be considered.

A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2,\ldots, x_n)$ to an n-tuple $(x_1, x_2,\ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.

Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$.

14.2 Limits and Continuity

Limits and Continuity

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and

$$g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

Limits and Continuity

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Values of $f(x, y)$

Table 1

Limits and Continuity

cont'd

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Values of $g(x, y)$

Table 2

Limits and Continuity

It appears that as (x, y) approaches (0, 0), the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y)\to(0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1$$

and

$$\lim_{(x, y)\to(0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$$ does not exist.

Limits and Continuity

In general, we use the notation

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f.

$\boxed{1}$ **Definition** Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Limits and Continuity

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L$$

and $f(x, y) \to L$ as $(x, y) \to (a, b)$

For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right.

We recall that if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

Limits and Continuity

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f.

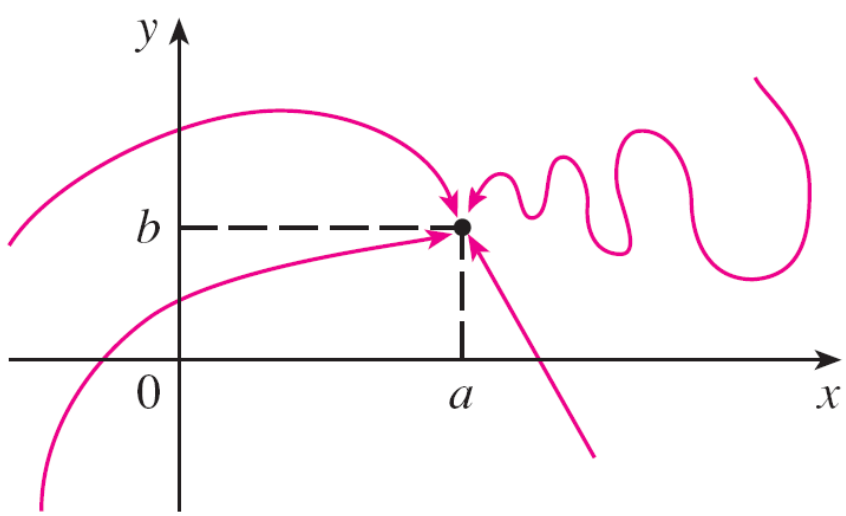

Figure 3

Limits and Continuity

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0).

The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach.

Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b).

Limits and Continuity

Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

Example 1

Show that $\lim_{(x, y) \to (0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Limits and Continuity

Now let's look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits.

The Limit Laws can be extended to functions of two variables: The limit of a sum/difference is the sum/difference of the limits, the limit of a product/quotient is the product/quotient of the limits, and so on.

In particular, the following equations are true.

$$\boxed{2} \qquad \lim_{(x,\, y) \to (a,\, b)} x = a \qquad \lim_{(x,\, y) \to (a,\, b)} y = b \qquad \lim_{(x,\, y) \to (a,\, b)} c = c$$

The Squeeze Theorem also holds.

Polynomials and rational functions

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers.

A **rational function** is a ratio of polynomials.

For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

Polynomials and rational functions

Since any polynomial can be built up out of the simple functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ by multiplication and addition, it follows from the limit laws and the limits in $\boxed{2}$ that if $p(x,y)$ is a **polynomial** or a **rational function** then for all (a,b) in the domain of p:

$$\lim_{(x,y)\to(a,b)} p(x,y) = p(a,b)$$

Examples 4 and 5

Example 4. Evaluate the limit if it exists.

$$\lim_{(x,\, y)\to(1,\, 2)} (x^2y^3 - x^3y^2 + 3x + 2y).$$

Example 5. Evaluate the limit if it exists.

$$\lim_{(x,y)\to(-2,3)} \frac{x^2y+1}{x^3y^2-2x}$$

Continuity

Continuity

Recall that evaluating limits of *continuous* functions of a single variable is easy.

It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x \to a} f(x) = f(a)$.

Continuous functions of two variables are also defined by the direct substitution property.

$\boxed{4}$ **Definition** A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

Continuity

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount.

This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains.

In particular, polynomials and rational functions are continuous on their domains.

Continuity

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third.

In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

Example 10. Where is the function $h(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ continuous?