Significance of the Gradient Vectors

Significance of the Gradient Vectors

We now summarize the ways in which the gradient vector is significant.

We first consider a function f of three variables and a point $P(x_0, y_0, z_0)$ in its domain.

On the one hand, we saw that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of f.

On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface S of f through P. (Refer to Figure 10.)

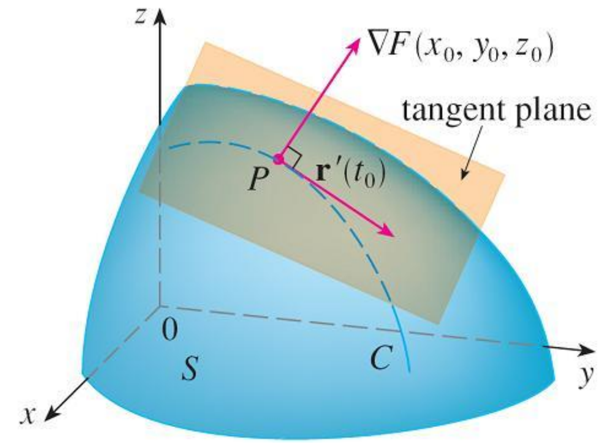


Figure 10

Significance of the Gradient Vectors

These two properties are quite compatible intuitively because as we move away from P on the level surface S, the value of f does not change at all.

So, it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain.

Again, the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f. Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P.

Significance of the Gradient Vectors

Again this is intuitively plausible because the values of *f* remain constant as we move along the curve.
(See Figure 12.)

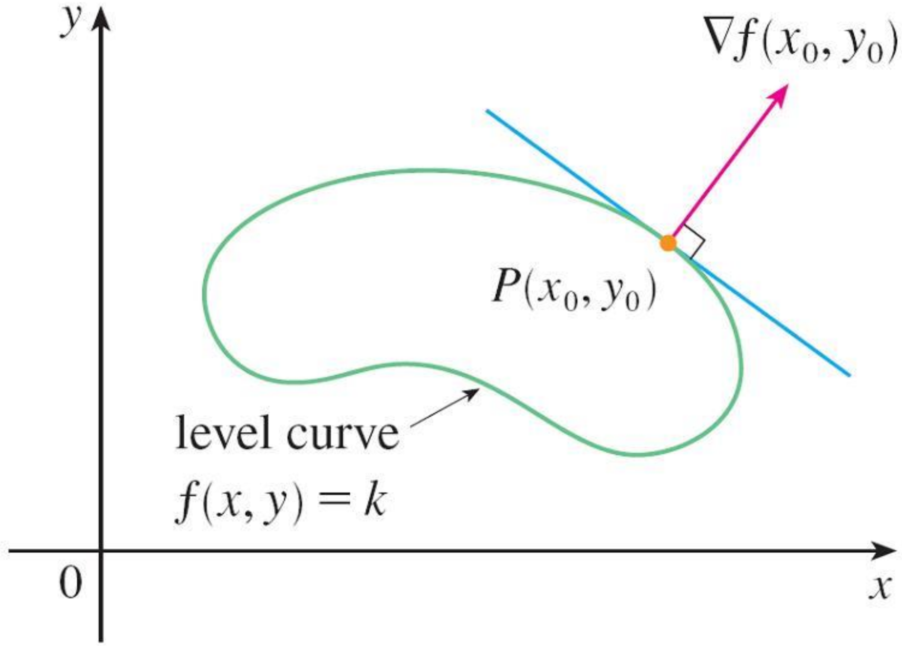


Figure 12

Significance of the Gradient Vectors

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn as in Figure 13 by making it perpendicular to all of the contour lines.

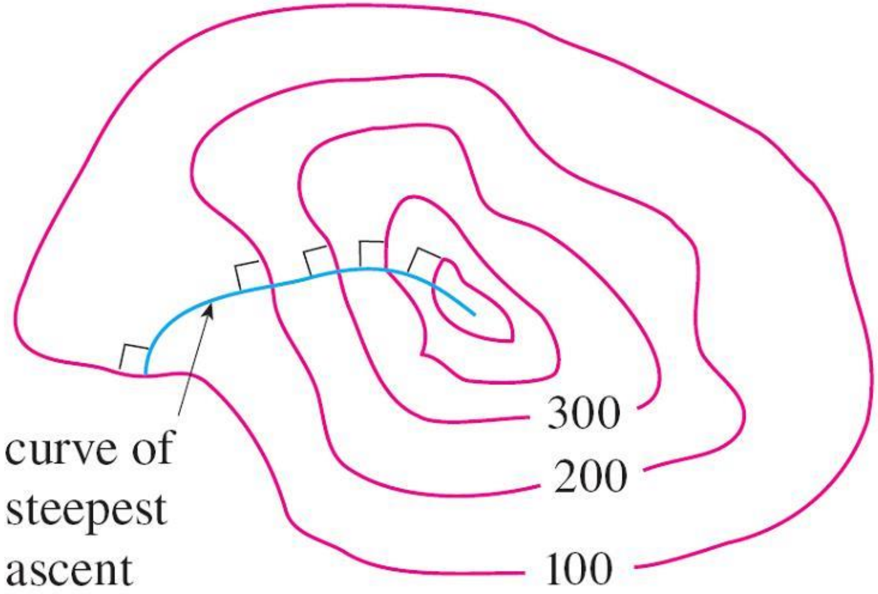


Figure 13

14.7 Maximum and Minimum Values

Maximum and Minimum Values

In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables.

Look at the hills and valleys in the graph of *f* shown in Figure 1.

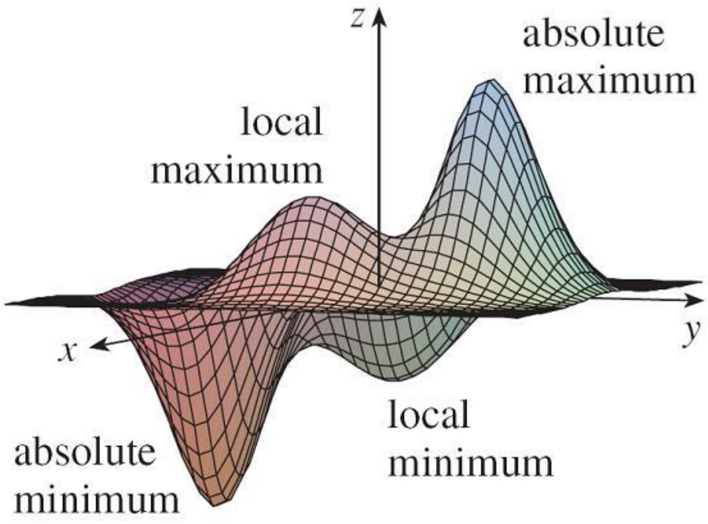


Figure 1

Maximum and Minimum Values

There are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$.

The larger of these two values is the *absolute maximum*.

Likewise, f has two *local minima*, where $f(a, b)$ is smaller than nearby values.

The smaller of these two values is the *absolute minimum*.

Maximum and Minimum Values

$\boxed{1}$ **Definition** A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

$\boxed{2}$ **Fermat's Theorem for Functions of Two Variables** If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Note: in this case, the tangent plane is horizontal at (a,b).

Maximum and Minimum Values

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f.

However, as in single-variable calculus, not all critical points give rise to maxima or minima.

At a critical point, a function could have a local maximum or a local minimum or neither.

Example 1

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Find the local and absolute extreme values of f.

Example 1

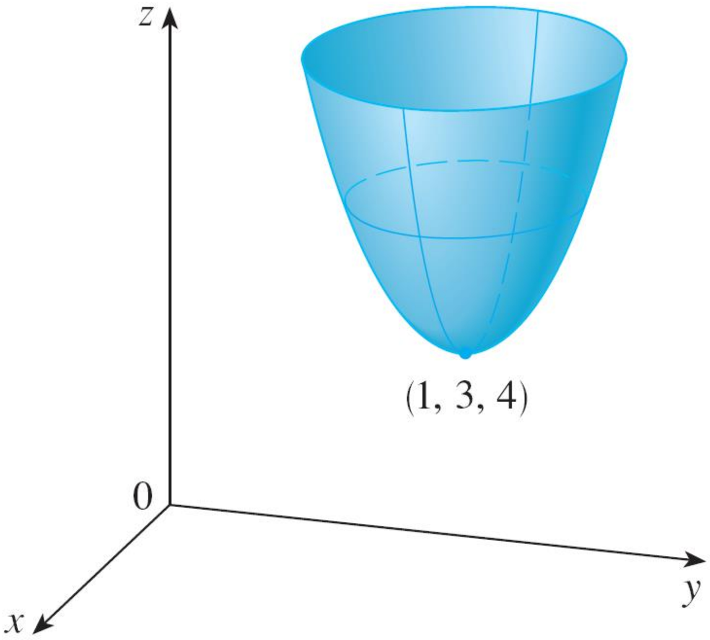


Figure 2

$$z = x^2 + y^2 - 2x - 6y + 14$$

Maximum and Minimum Values

The following test, is analogous to the Second Derivative Test for functions of one variable.

$\boxed{3}$ **Second Derivatives Test** Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

In case (c) the point (*a*, *b*) is called a **saddle point** of *f* and the graph of *f* crosses its tangent plane at (*a*, *b*).

Second derivative test

To remember de formula for *D*, it is helpful to rewrite it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

Second derivative test

Note. If we introduce the symmetric matrix (so-called Hessian matrix of *f*)

$$A = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix},$$

then condition (a) is equivalent of the matrix A being positive definite, (b) is equivalent of the matrix A being negative definite, while (c) is equivalent of the matrix A being indefinite.

With these notions a similar statement holds in *n* dimensions considering the corresponding *nxn* Hessian matrix.

Functions of *n*-variables

More precisely, analogously to the case of functions of two variables, the notion of local/global extrema can be defined for functions of *n*-variables. Fermat's Theorem still holds: if *f* has a local extreme value at $(a_1, a_2, \ldots, a_n)$ and the first order partial derivatives of *f* exist there, then

$$f_1(a_1, \ldots, a_n) = \cdots = f_n\,(a_1, \ldots, a_n) = 0.$$

If we introduce the Hessian of *f* :

$$A = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix},$$

then the second derivative test still applies, if we formulate it in terms of the definiteness of *A* (see previous slide).

Example 3

Find the local minimum and maximum values and sadle points of $f(x, y) = x^4 + y^4 - 4xy + 1$

Example 6

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Absolute Maximum and Minimum Values on bounded and closed sets

Absolute Maximum and Minimum Values

For a function *f* of one variable, the Extreme Value Theorem says that if *f* is continuous on a closed interval [*a*, *b*], then *f* has an absolute minimum value and an absolute maximum value.

According to the Closed Interval Method, we found these by evaluating *f* not only at the critical numbers but also at the endpoints *a* and *b*.

There is a similar situation for functions of two variables.

Absolute Maximum and Minimum Values

Just as a closed interval contains its endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

[A boundary point of *D* is a point (a, b) such that every disk with center (a, b) contains points in *D* and also points not in *D*.]

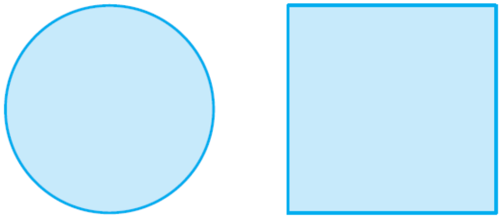
For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

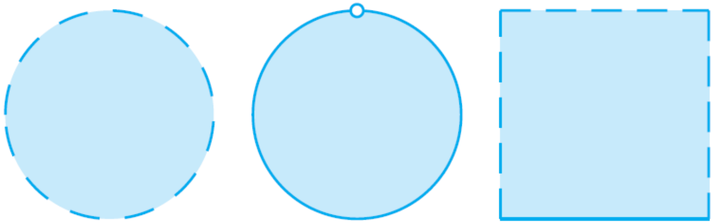
which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$).

Absolute Maximum and Minimum Values

But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)



(a) Closed sets



(b) Sets that are not closed

Figure 11

Absolute Maximum and Minimum Values

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk.

In other words, it is finite in extent.

Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Note. An analogous statement holds in $\mathbb{R}^n$ with the appropriate definitions of boundedness and closedness.

Absolute Maximum and Minimum Values

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value in D at (x_1, y_1), then (x_1, y_1) is either a critical point of f or a boundary point of D.

Thus, we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle

$D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.