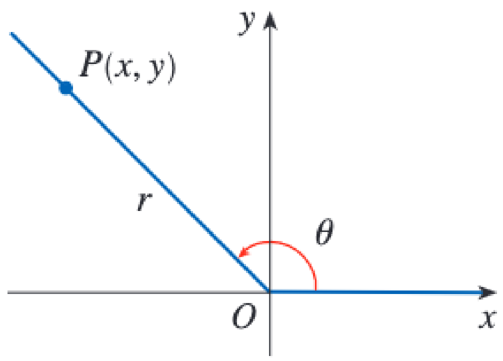
Trigonometric functions

Recall: for a general angle in standard position, we let *P(x,y)* be any point on the terminal side of θ and we let *r* be the distance |*OP*| as in the figure below. Then we define:



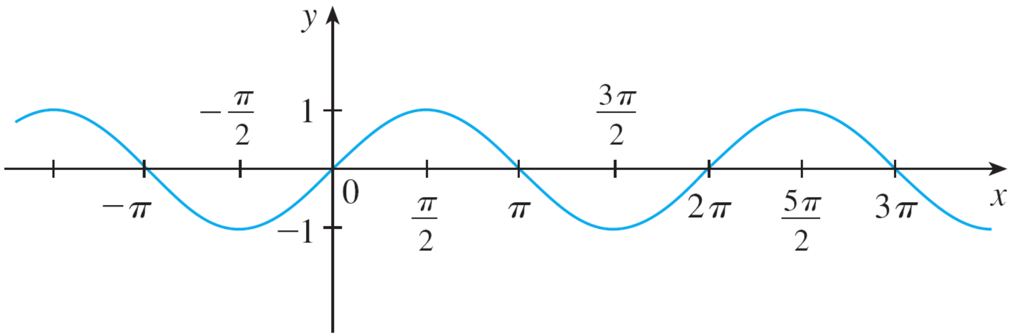
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

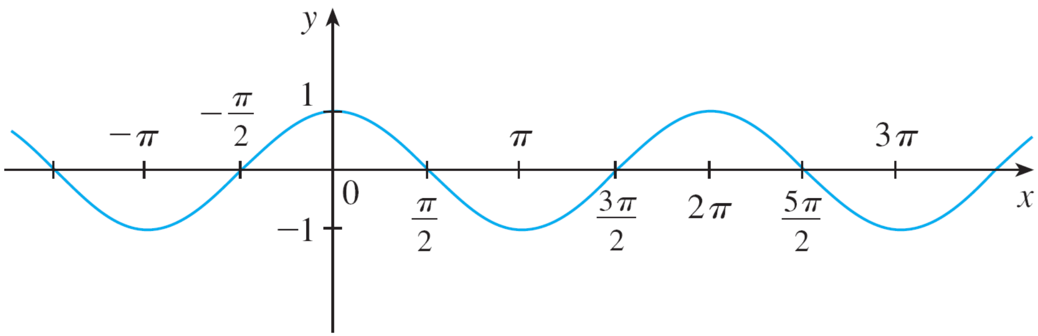
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Trigonometric Functions

Thus, the graphs of the sine and cosine functions are as shown in Figure 18.



(a) $f(x) = \sin x$



(b) $g(x) = \cos x$

Figure 18

Trigonometric Functions

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval [–1, 1].

Thus, for all values of *x*, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

or, in terms of absolute values,

$$|\sin x| \leq 1 \qquad |\cos x| \leq 1$$

Trigonometric Functions

Also, the zeros of the sine and cosine functions:

$\sin x = 0$ when $x = n\pi$ n an integer

$\cos x = 0$ when $x = \pi/2 + n\pi$ n an integer

An important property of the sine and cosine functions is that they are periodic functions and have period 2π.

This means that, for all values of x,

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

Important trigonometric identities

Pythagoraen theorem:

$$\sin^2(x) + \cos^2(x) = 1$$

The sine function is odd, the cosine function is even:

$$\sin(-x) = -\sin x; \quad \cos(-x) = \cos x$$

Addition formulae:

$$\boxed{\begin{aligned} \sin(x + y) &= \sin x \cos y + \cos x \sin y \\ \cos(x + y) &= \cos x \cos y - \sin x \sin y \end{aligned}}$$

Trigonometric formulae

Example: Derive the substraction formulae and double angel formulae for sine and cosine.

Example: Derive the half-angle (or linearization) formulae:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Example: Derive the product identities:

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

Example 5

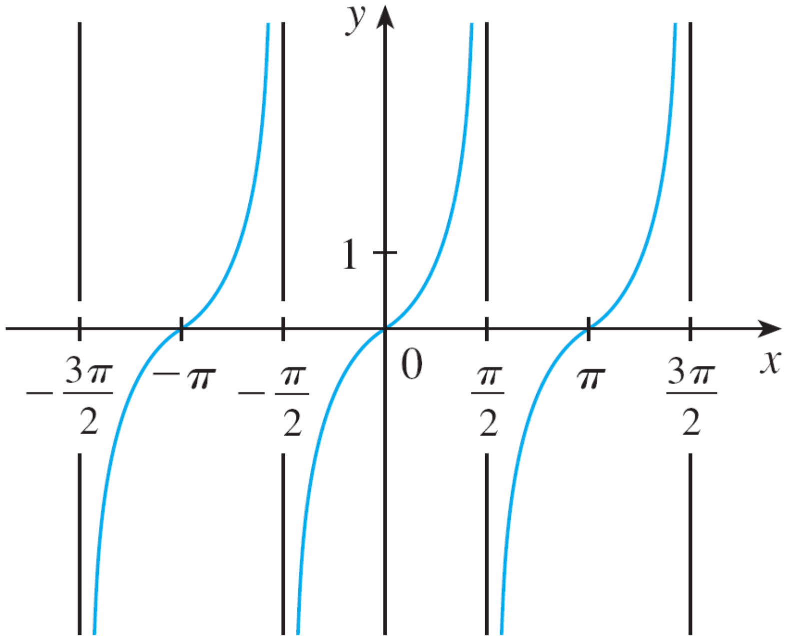
Find the domain of $f(x) = \frac{1}{1-2\cos x}$.

Trigonometric Functions

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure 19. It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm 3\pi/2, \ldots$.



$y = \tan x$

Figure 19

Its range is $(-\infty, \infty)$.

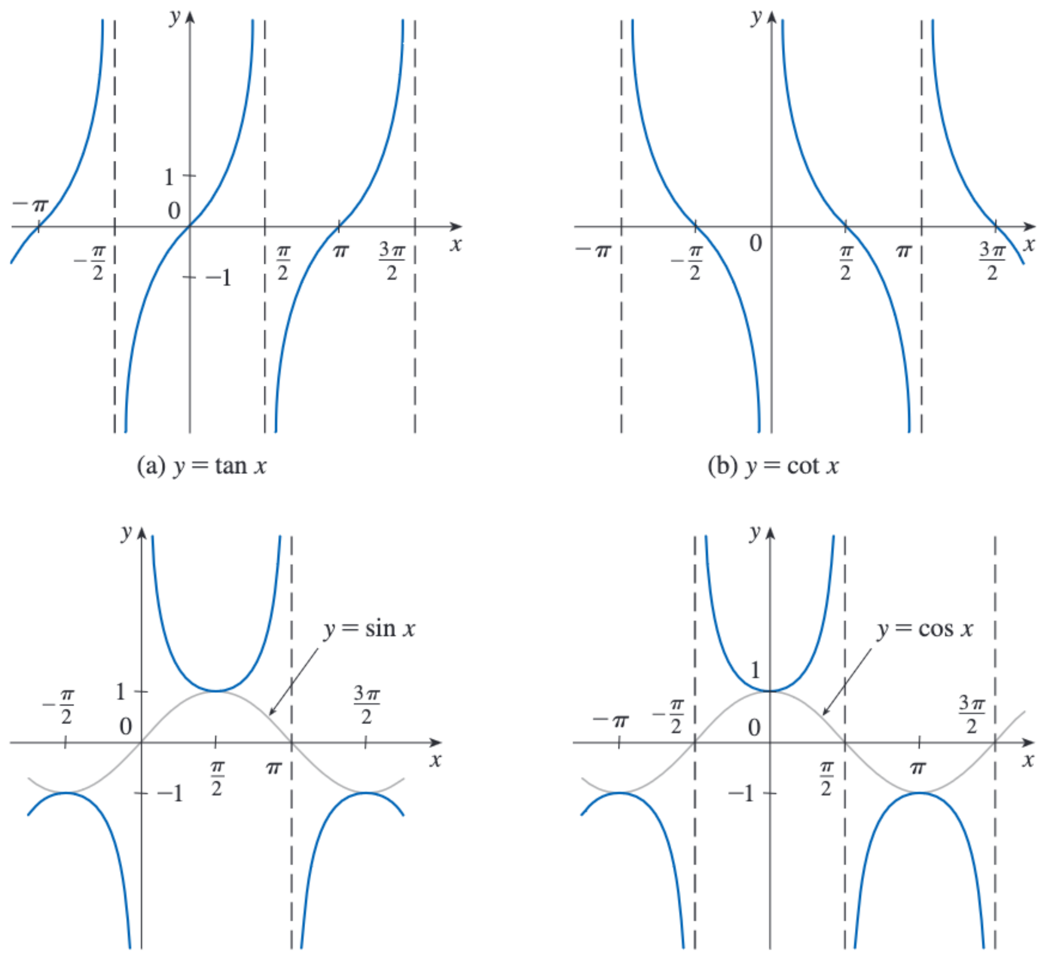
Trigonometric Functions

Notice that the tangent function has period π :

$$\tan (x + \pi) = \tan x \quad \text{for all } x$$

The remaining three trigonometric functions (cosecant, secant, and cotangent) are the reciprocals of the sine, cosine, and tangent functions; their graphs are shown on the next slide.

Trigonometric functions



(a) $y = \tan x$

(b) $y = \cot x$

(c) $y = \csc x$

(d) $y = \sec x$

Exponential Functions

Exponential Functions

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant.

The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure 20. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

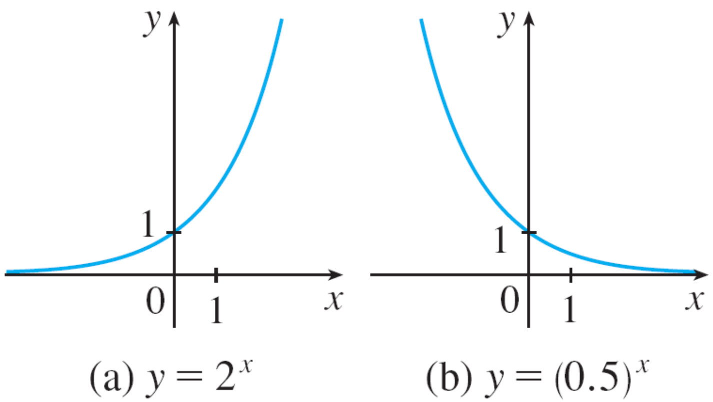


Figure 20

Exponential Functions

Exponential functions are useful for modeling many natural phenomena, such as population growth (if $a > 1$) and radioactive decay (if $a < 1$).

Logarithmic Functions

Logarithmic Functions

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions. Figure 21 shows the graphs of four logarithmic functions with various bases.

In each case the domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the function increases slowly when $x > 1$.

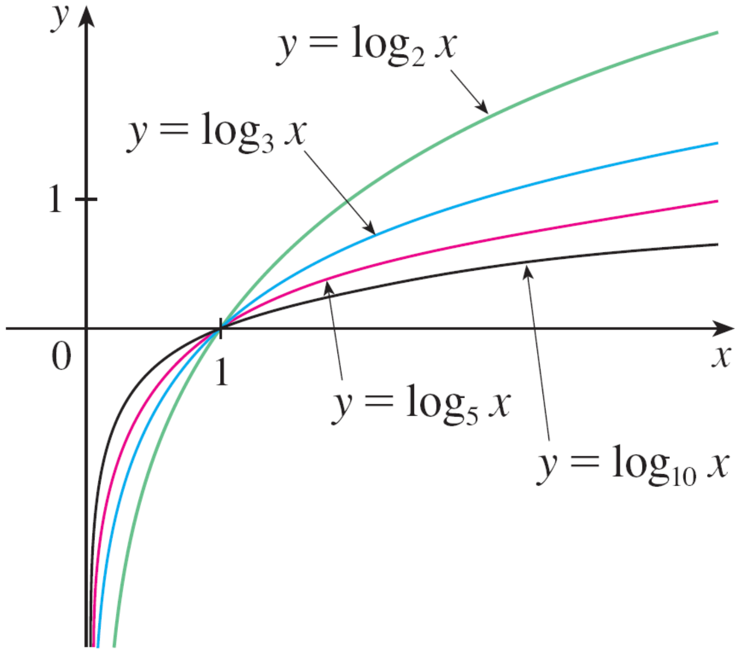


Figure 21

Example 6

Classify the following functions as one of the types of functions that we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1 + x}{1 - \sqrt{x}}$

(d) $u(t) = 1 – t + 5t^4$

1.3 New Functions from Old Functions

Transformations of Functions

Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graphs of certain related functions.

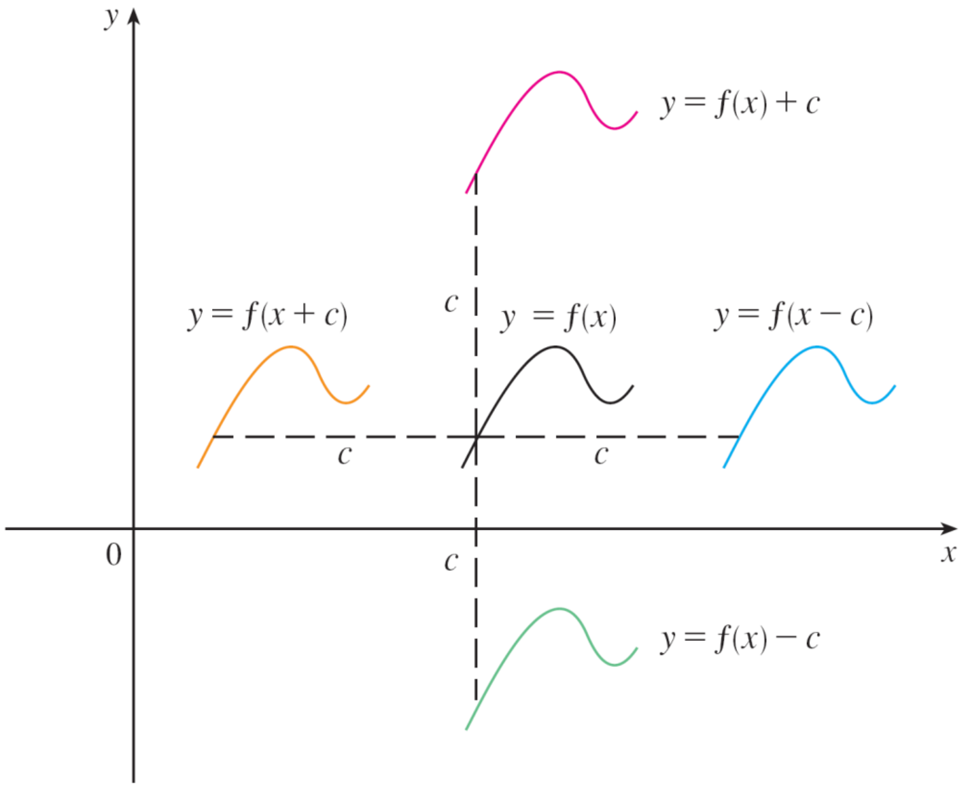
This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

Let's first consider **translations**. If c is a positive number, then the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ shifted upward a distance of c units (because each y-coordinate is increased by the same number c).

Transformations of Functions

Likewise, if $g(x) = f(x - c)$, where $c > 0$, then the value of g at x is the same as the value of f at $x - c$ (c units to the left of x).

Therefore, the graph of $y = f(x - c)$, is just the graph of $y = f(x)$ shifted c units to the right (see Figure 1).



Translating the graph of f

Figure 1

Transformations of Functions

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Example: Show that the graph of the cosine function is the same as the graph of the sine function shifted to the left by $\pi/2$; that is, $\cos x = \sin(x + \frac{\pi}{2})$.

Transformations of Functions

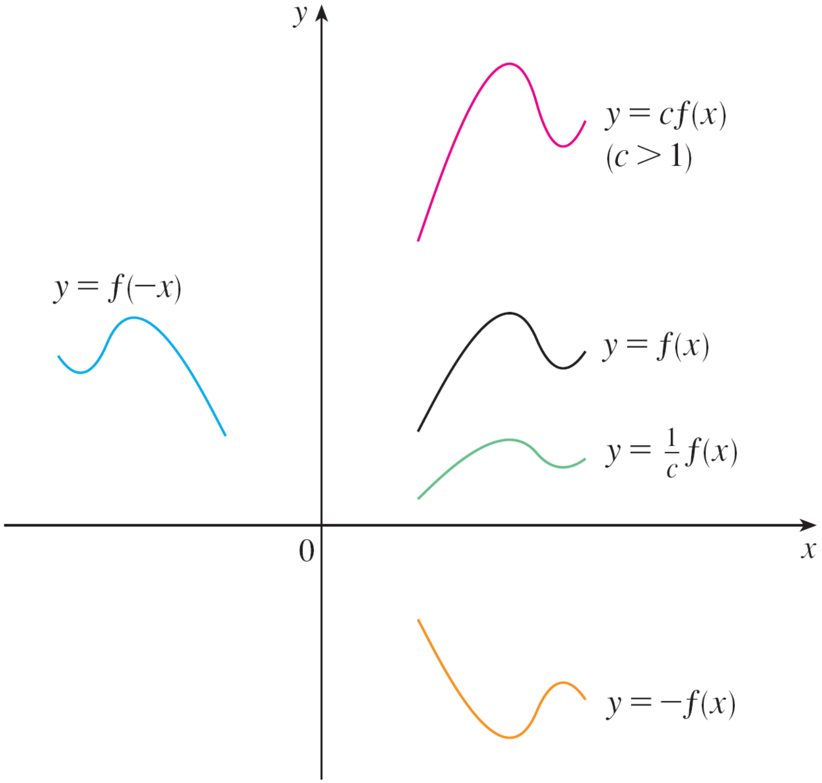
Now let's consider the **stretching** and **reflecting** transformations.

If $c > 1$, then the graph of $y = cf(x)$ is the graph of $y = f(x)$ stretched by a factor of c in the vertical direction (because each y-coordinate is multiplied by the same number c).

Transformations of Functions

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis because the point (x, y) is replaced by the point $(x, -y)$.

(See Figure 2 and the following chart, where the results of other stretching, shrinking, and reflecting transformations are also given.)



Stretching and reflecting the graph of *f*

Figure 2

Transformations of Functions

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Transformations of Functions

Figure 3 illustrates these stretching transformations when applied to the cosine function with c = 2.

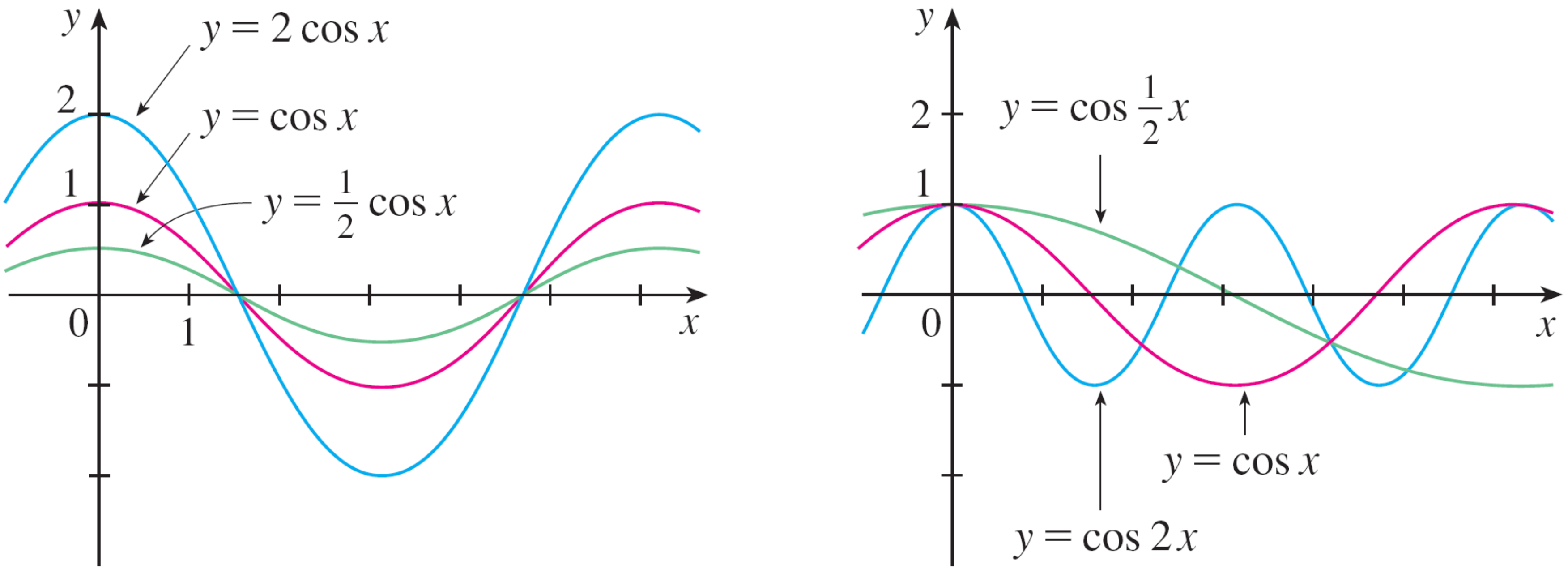


Figure 3

Transformations of Functions

For instance, in order to get the graph of $y = 2 \cos x$ we multiply the y-coordinate of each point on the graph of $y = \cos x$ by 2.

This means that the graph of $y = \cos x$ gets stretched vertically by a factor of 2.

Examples

1. Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x - 2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

2. Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

3. Sketch the graph of (a) $y = \sin(2x)$ and (b) $y = 1 - \sin x$

Transformations of Functions

Another transformation of some interest is taking the *absolute value* of a function. If $y = |f(x)|$, then according to the definition of absolute value, $y = f(x)$ when $f(x) \geq 0$ and $y = -f(x)$ when $f(x) < 0$.

This tells us how to get the graph of $y = |f(x)|$ from the graph of $y = f(x)$: The part of the graph that lies above the x-axis remains the same; the part that lies below the x-axis is reflected about the x-axis.

Example 5. Sketch the graph of the function $y = |x^2 - 1|$.

Combinations of Functions

Combinations of Functions

Two functions *f* and *g* can be combined to form new functions *f* + *g*, *f* – *g*, *fg*, and *f*/*g* in a manner similar to the way we add, subtract, multiply, and divide real numbers. The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \qquad\qquad (f - g)(x) = f(x) - g(x)$$

If the domain of *f* is *A* and the domain of *g* is *B*, then the domain of *f* + *g* is the intersection $A \cap B$ because both *f*(*x*) and *g*(*x*) have to be defined.

For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2 - x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}$ is $A \cap B = [0, 2]$.

Combinations of Functions

Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of fg is $A \cap B$, but we can't divide by 0 and so the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$.

For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $(f/g)(x) = x^2/(x - 1)$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

Combinations of Functions

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$.

Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

Combinations of Functions

In general, given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, then we can calculate the value of $f(g(x))$.

The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g *and* is denoted by $f \circ g$ ("f circle g").

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

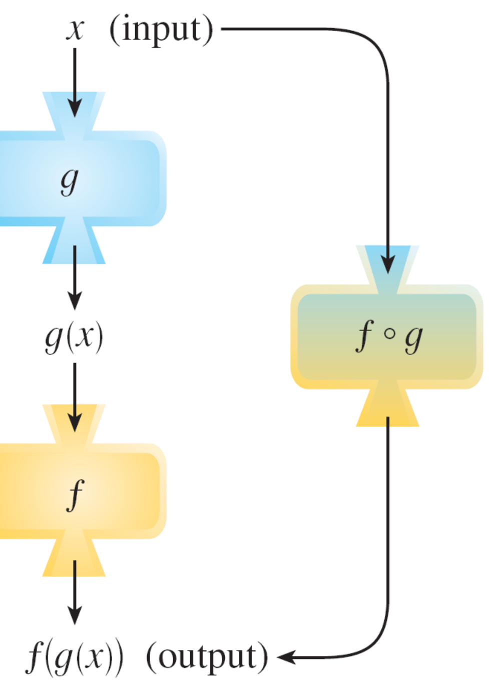
$$(f \circ g)(x) = f(g(x))$$

Combinations of Functions

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

Figure 11 shows how to picture $f \circ g$ in terms of machines.



The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Figure 11

Examples

6. If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

7. If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each function and its domain. (a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$