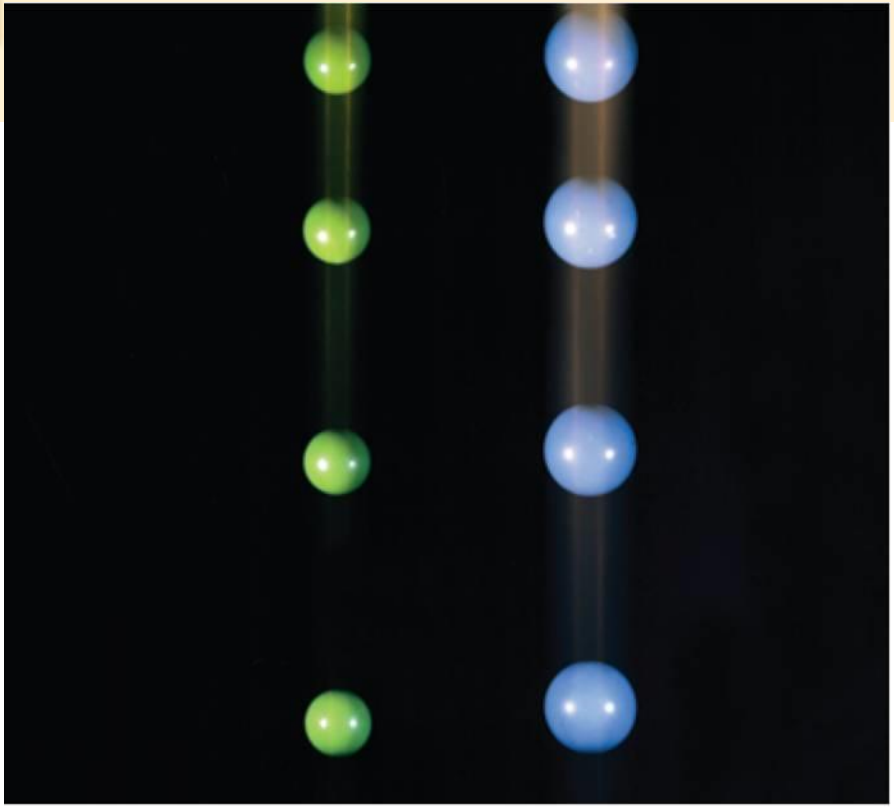
1

Functions and Limits



1.2 Mathematical Models: A Catalog of Essential Functions

Mathematical Models: A Catalog of Essential Functions

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon such as the size of a population, the demand for a product, the speed of a falling object, the concentration of a product in a chemical reaction, the life expectancy of a person at birth, or the cost of emission reductions.

The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior.

Mathematical Models: A Catalog of Essential Functions

Figure 1 illustrates the process of mathematical modeling.

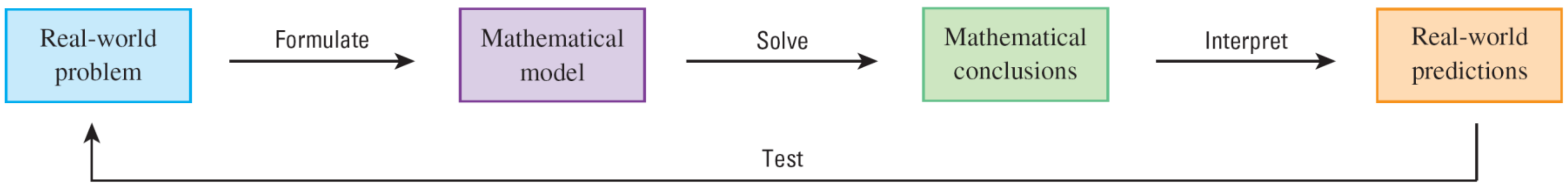


Figure 1

The modeling process

Mathematical Models: A Catalog of Essential Functions

- A mathematical model is never a completely accurate representation of a physical situation—it is an *idealization*.

- A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.

- There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

Linear Models

Linear Models

When we say that *y* is a **linear function** of *x*, we mean that the graph of the function is a line, so we can use the so-called *slope-intercept form* of the equation of a line to write a formula for the function as

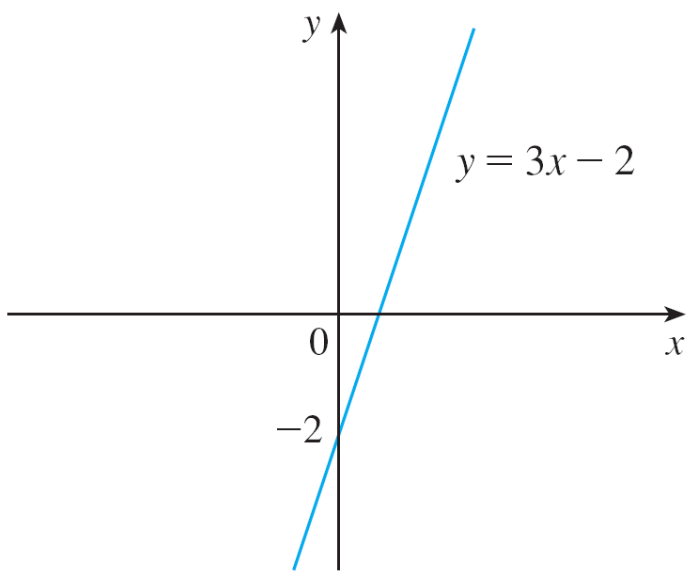
$$y = f(x) = mx + b$$

where *m* is the slope of the line and *b* is the *y*-intercept.

Linear Models

A characteristic feature of linear functions is that they grow at a constant rate.

For instance, Figure 2 shows a graph of the linear function $f(x) = 3x - 2$ and a table of sample values.



x	$f(x) = 3x - 2$
1.0	1.0
1.1	1.3
1.2	1.6
1.3	1.9
1.4	2.2
1.5	2.5

Figure 2

Linear Models

Notice that whenever *x* increases by 0.1, the value of *f*(*x*) increases by 0.3.

So *f*(*x*) increases three times as fast as *x*. Thus the slope of the graph $y = 3x - 2$, namely 3, can be interpreted as the rate of change of *y* with respect to *x*.

In general: if there $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two distinct points on the graph of a linear function, then the slope of the line is given by:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

and thus $\Delta y = m\Delta x$.

Linear Models

The so-called *point-slope form* of the equation of the line through $P(x_1,y_1)$ with slope m is

$$y - y_1 = m(x - x_1)$$

Example: Find both the point-slope and the slope-intercept form of the equation of the line through $P_1(1,2)$ and $P_2(2,4)$.

Example 1

(a) As dry air moves upward, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature *T* (in °C) as a function of the height *h* (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

Polynomials

Polynomials

A function P is called a **polynomial** if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial.

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

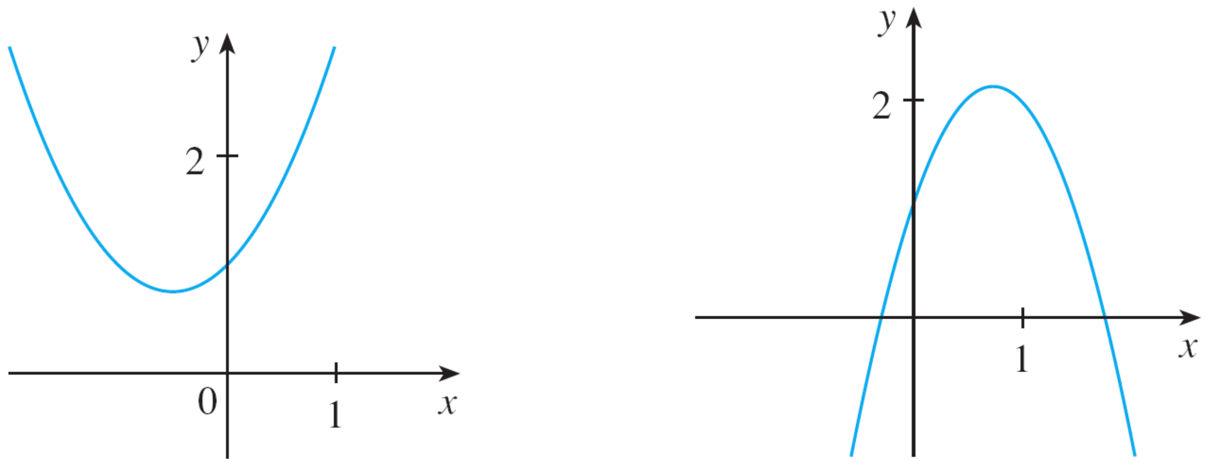
Polynomials

A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function ($m \neq 0$).

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function** ($a \neq 0$).

Polynomials

Its graph is always a parabola obtained by shifting the parabola $y = ax^2$. The parabola opens upward if $a > 0$ and downward if $a < 0$. (See Figure 7.)



(a) $y = x^2 + x + 1$

(b) $y = -2x^2 + 3x + 1$

The graphs of quadratic functions are parabolas.

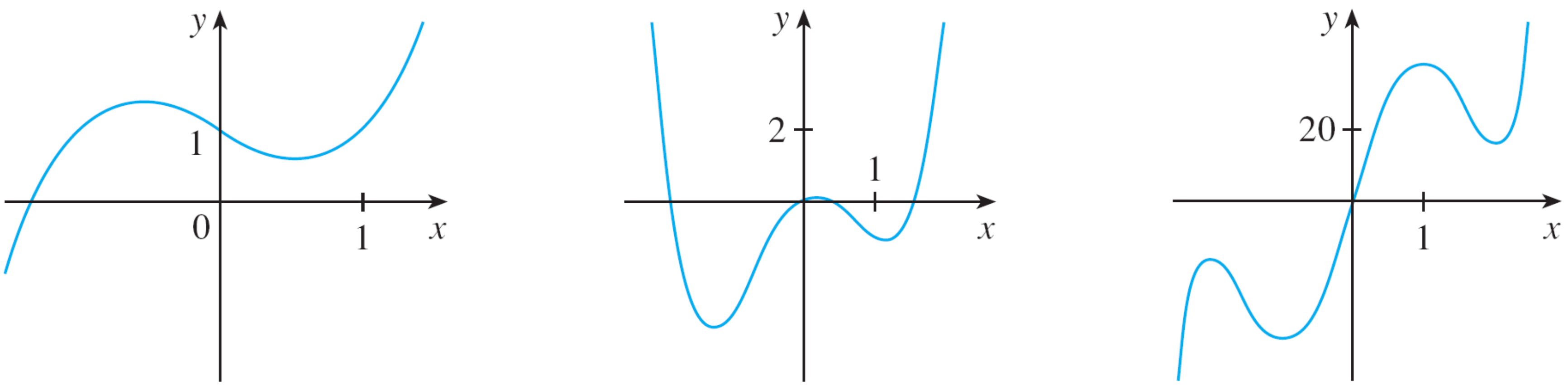
Figure 7

Polynomials

A polynomial of degree 3 is of the form

$$P(x) = ax^3 + bx^2 + cx + d \qquad a \neq 0$$

and is called a **cubic function**. Figure 8 shows the graph of a cubic function in part (a) and graphs of polynomials of degrees 4 and 5 in parts (b) and (c).



(a) $y = x^3 - x + 1$ (b) $y = x^4 - 3x^2 + x$ (c) $y = 3x^5 - 25x^3 + 60x$

Figure 8

Power Functions

Power Functions

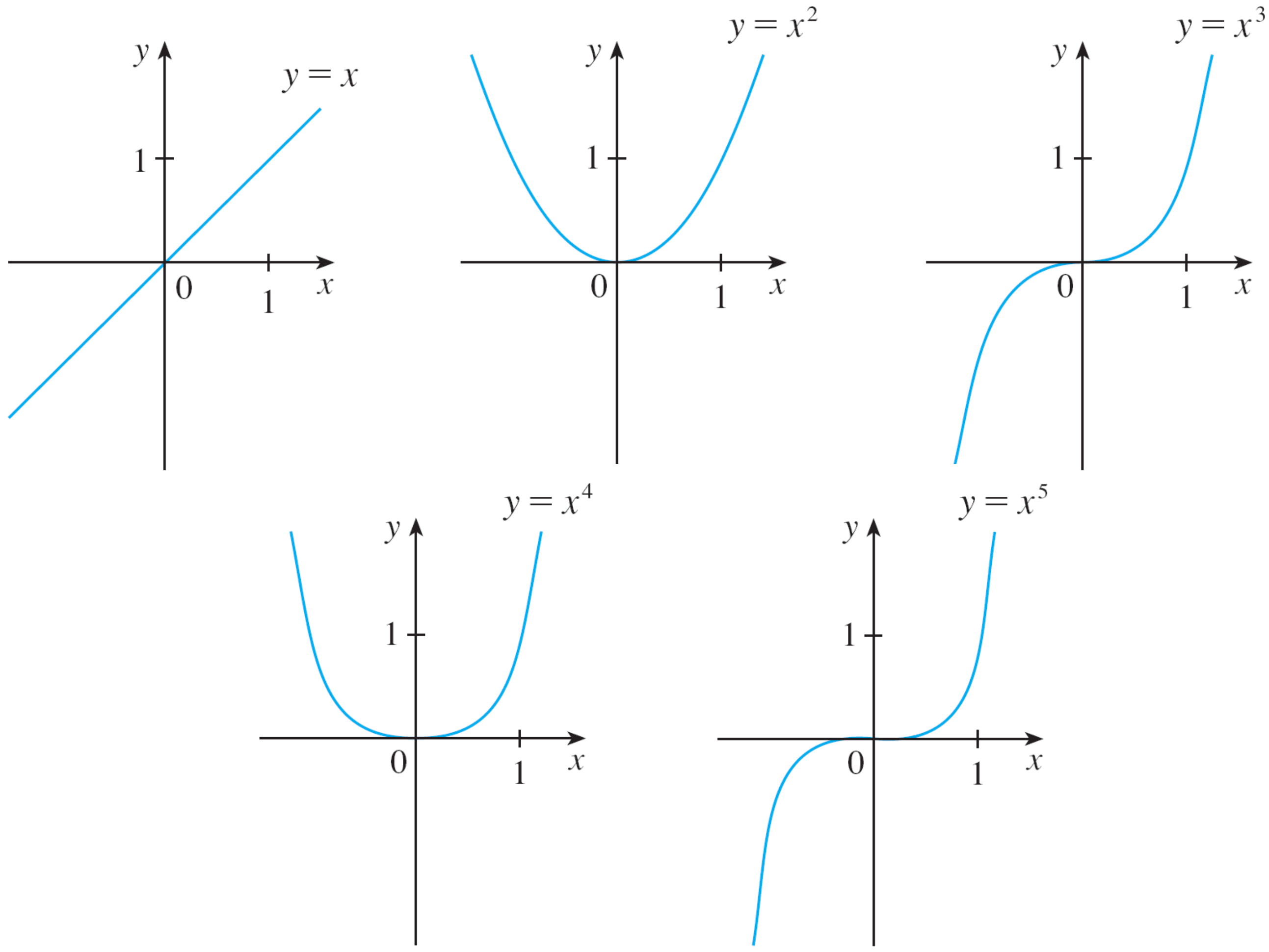
A function of the form $f(x) = x^a$, where a is a constant, is called a **power function**. We consider several cases.

(i) $a = n$, where n is a positive integer

The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 11. (These are polynomials with only one term.)

We already know the shape of the graphs of $y = x$ (a line through the origin with slope 1) and $y = x^2$ (a parabola).

Power Functions



Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

Figure 11

Power Functions

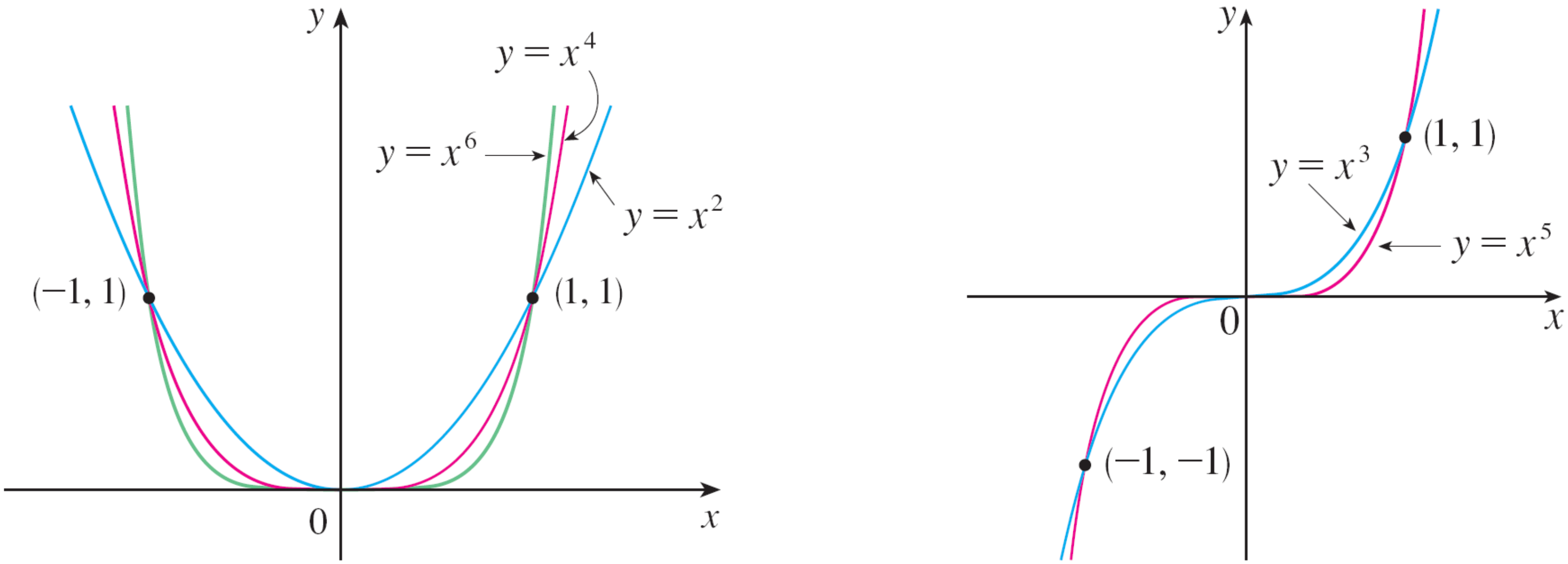
The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.

If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$.

If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$.

Power Functions

Notice from Figure 12, however, that as n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| \geq 1$. (If x is small, then x^2 is smaller, x^3 is even smaller, x^4 is smaller still, and so on.)



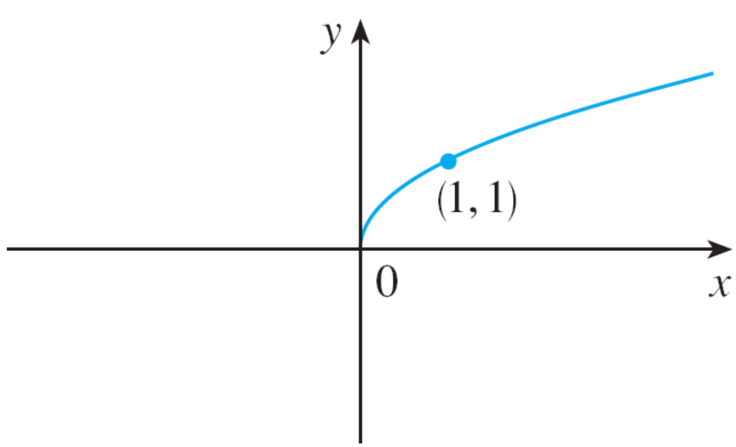
Families of power functions

Figure 12

Power Functions

(ii) *a* = 1/*n*, where *n* is a positive integer

The function $f(x) = x^{1/n} = \sqrt[n]{x}$ is a **root function**. For *n* = 2 it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 13(a).]



$f(x) = \sqrt{x}$

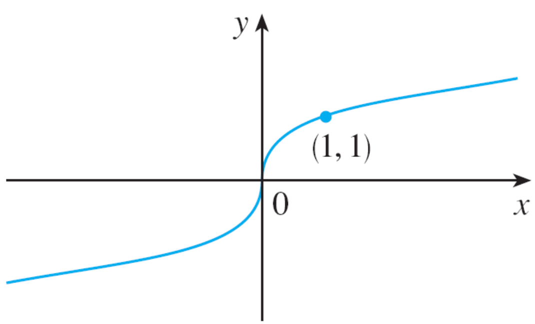
Graph of root function

Figure 13(a)

Power Functions

For other even values of *n*, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.

For *n* = 3 we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ (recall that every real number has a cube root) and whose graph is shown in Figure 13(b). The graph of $y = \sqrt[n]{x}$ for *n* odd (*n* > 3) is similar to that of $y = \sqrt[3]{x}$.



$f(x) = \sqrt[3]{x}$

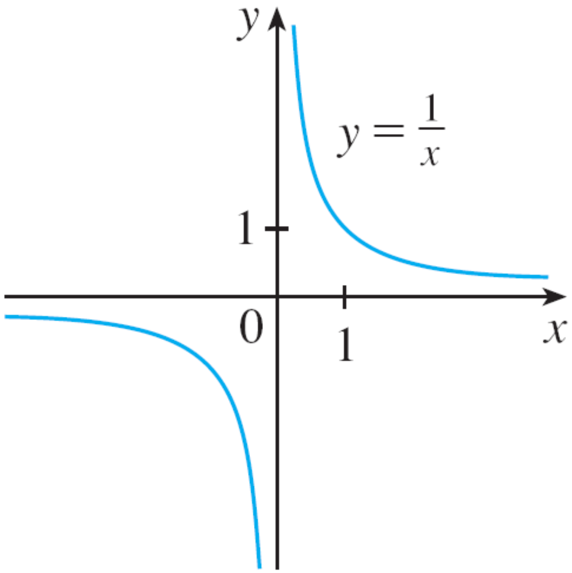
Graph of root function

Figure 13(b)

Power Functions

(iii) *a* = –1

The graph of the **reciprocal function** $f(x) = x^{-1} = 1/x$ is shown in Figure 14. Its graph has the equation $y = 1/x$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes.



The reciprocal function

Figure 14

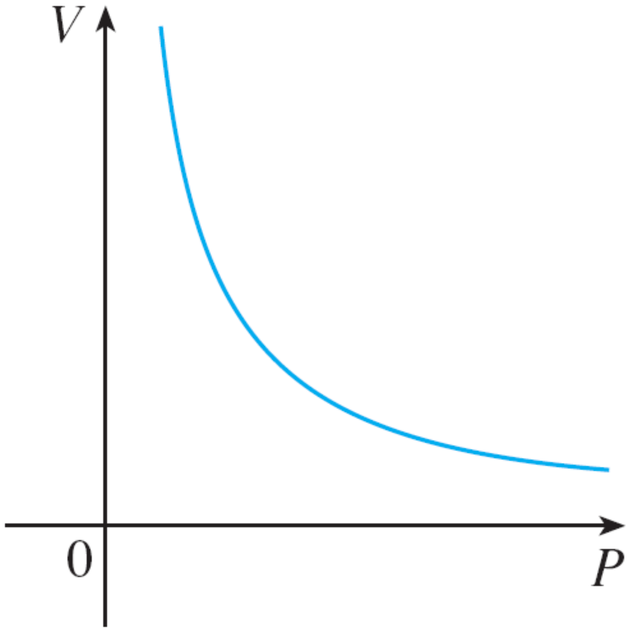
Power Functions

This function arises in physics and chemistry in connection with Boyle's Law, which says that, when the temperature is constant, the volume *V* of a gas is inversely proportional to the pressure *P*:

$$V = \frac{C}{P}$$

where *C* is a constant.

Thus the graph of *V* as a function of *P* (see Figure 15) has the same general shape as the right half of Figure 14.



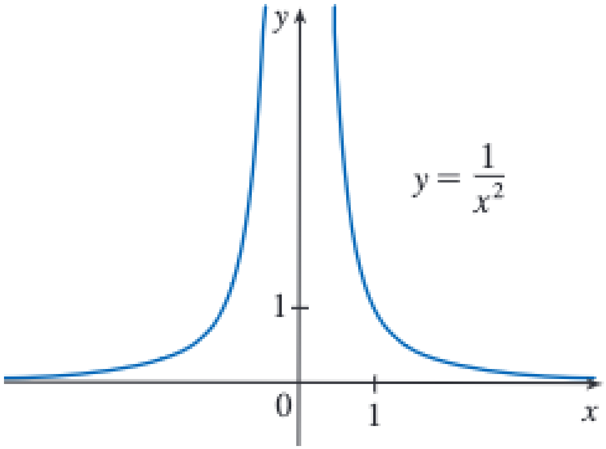
Volume as a function of pressure at constant temperature

Figure 15

Power Functions

(iv) n = – 2: $f(x) = \frac{1}{x^2}$

Many natural laws state that one quantity is inversely proportional to the square of another quantity. In other words, the first quantity is modeled by a function of the form $f(x) = C/x^2$ and we refer to this as an **inverse square law.**



Inverse square laws model gravitational force, loudness of sound, and electrostatic force between two charged particles

Rational Functions

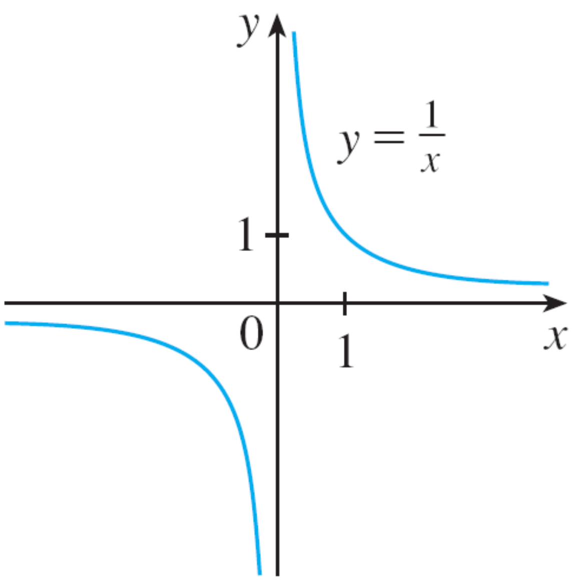
Rational Functions

A **rational function** *f* is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where *P* and *Q* are polynomials. The domain consists of all values of *x* such that $Q(x) \neq 0$.

A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14.



The reciprocal function

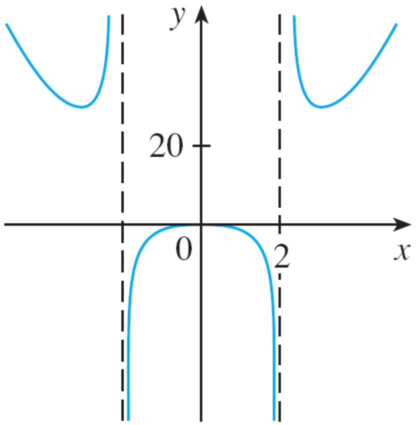
Figure 14

Rational Functions

The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 16.



$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

Figure 16

Algebraic Functions

Algebraic Functions

A function *f* is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function.

Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

Algebraic Functions

The graphs of algebraic functions can assume a variety of shapes. Figure 17 illustrates some of the possibilities.

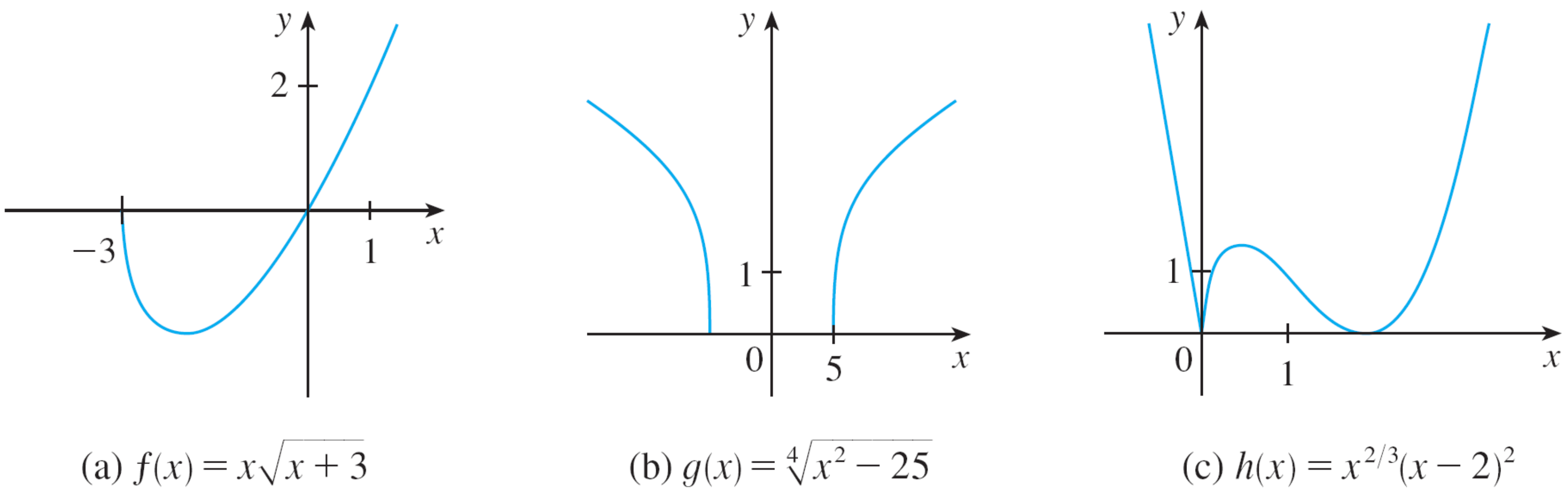


Figure 17

Algebraic Functions

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity *v* is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

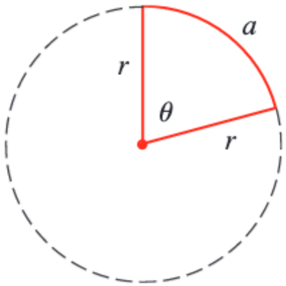
where m_0 is the rest mass of the particle and
$c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

Trigonometric Functions

Radian measure

The radian measure of an angle θ is given by

$$\theta = \frac{a}{r}$$



Note: when *r = 1* then the radian measure of θ equals to the arc length of the corresponding arc.

Example: (a) Find the radian measure of 60° (b) Express $\frac{5\pi}{4}$ rad in degrees.

Radian measure

In calculus we use radians to measure angles except when otherwise indicated. The following table gives the correspondence between degree and radian measures of some common angles.

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	$270°$	$360°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Standard position

- The standard position of an angle occurs when we place its vertex at the origin of a rectangular coordinate system and its initial side on the positive x-axis as in Figure 3.
- A positive angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side.
- Likewise, negative angles are obtained by clockwise rotation as in Figure 4

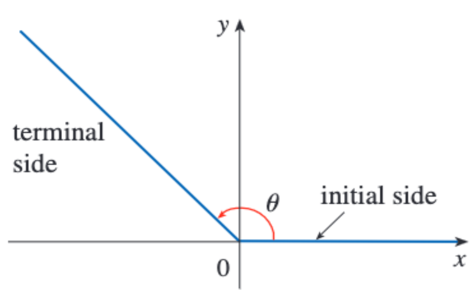


FIGURE 3 $\theta \geqslant 0$

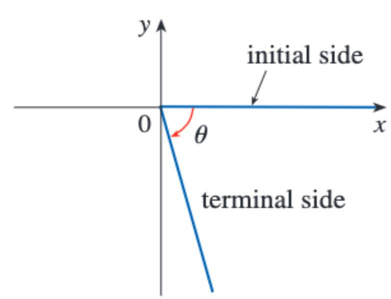
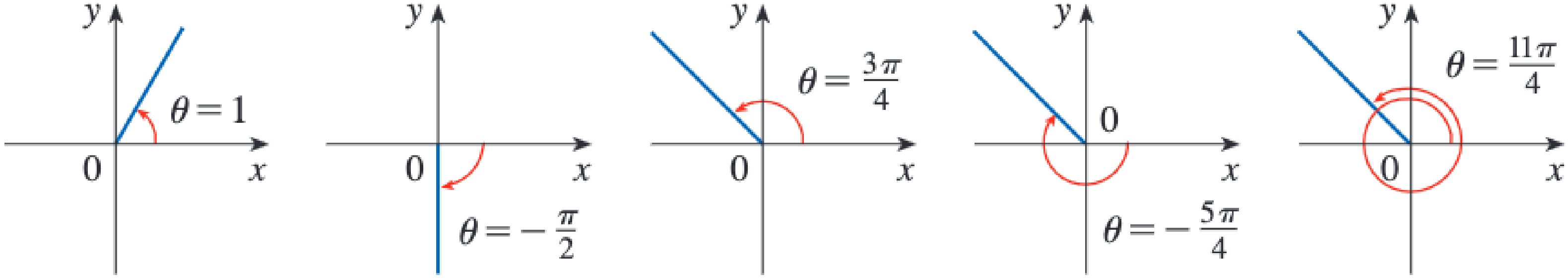


FIGURE 4 $\theta < 0$

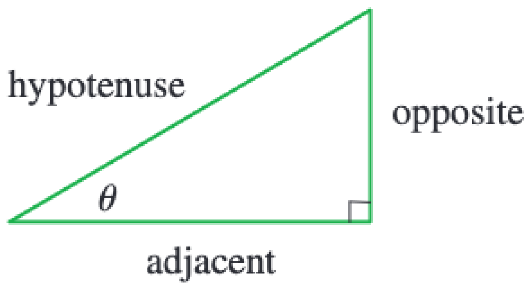
Standard position

For example:



Trigonometric functions

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows:



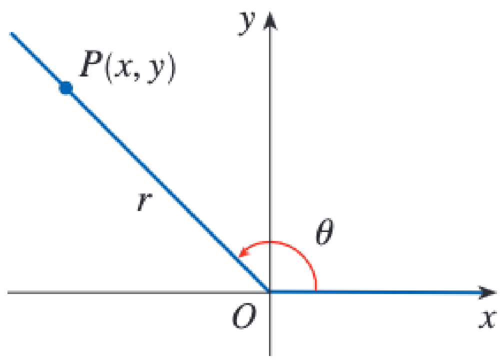
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Trigonometric functions

This definition doesn't apply to obtuse or negative angles, so for a general angle in standard position we let *P(x,y)* be any point on the terminal side of θ and we let *r* be the distance |*OP*| as in the figure below. Then we define:



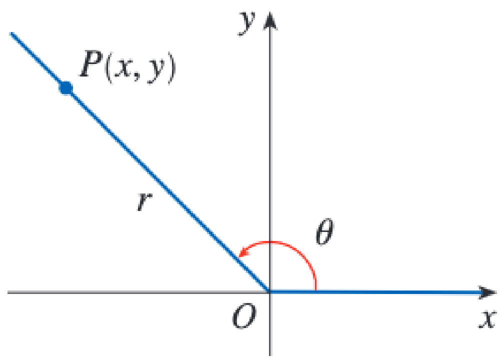
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Trigonometric functions

Since division by 0 is not defined, $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$ and $\cot\theta$ and $\csc\theta$ are undefined when $y = 0$. Notice that these previous definitions are consistent when θ is an acute angle.



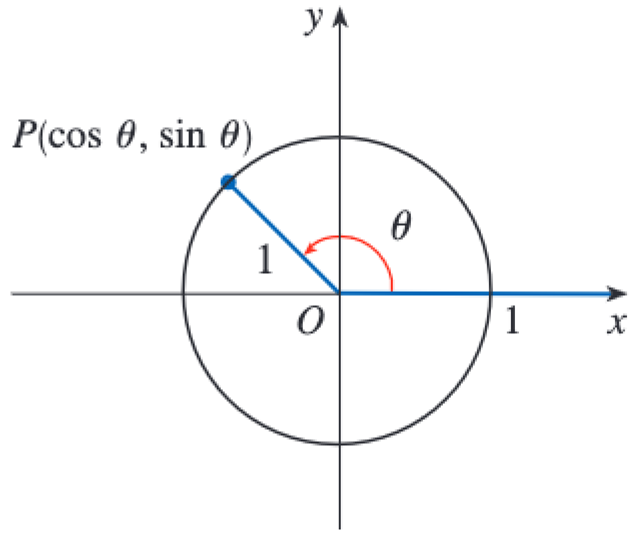
$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

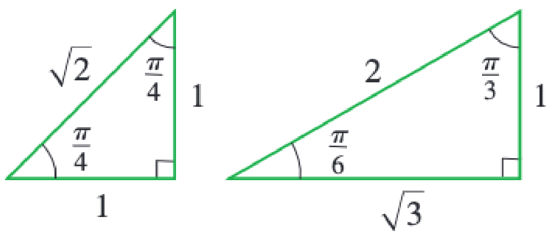
Trigonometric functions

If we put r = 1 in the previous definition and draw a unit circle with center the origin and label θ as in the figure below, then the coordinates of *P* are $(\cos\theta, \sin\theta)$.



Trigonometric functions

The exact trigonometric ratios for certain angles can be read from the triangles in the following figure:



For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

Trigonometric functions

Example: Find the exact trigonometric ratios for $\theta = \frac{2\pi}{3}$.

Example: If $\cos\theta = \frac{2}{5}$, find the other five trigonometric functions of θ.

Trigonometric functions

The following table gives some values of found by the same method as before:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1