15 Multiple Integrals

15.1 Double Integrals over Rectangles

Review of the Definite Integrals

Review of the Definite Integrals

First let's recall the basic facts concerning definite integrals of functions of a single variable.

If $f(x)$ is defined for $a \le x \le b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

1 $$\sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

2 $$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

Review of the Definite Integrals

In the special case where $f(x) \geq 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b.

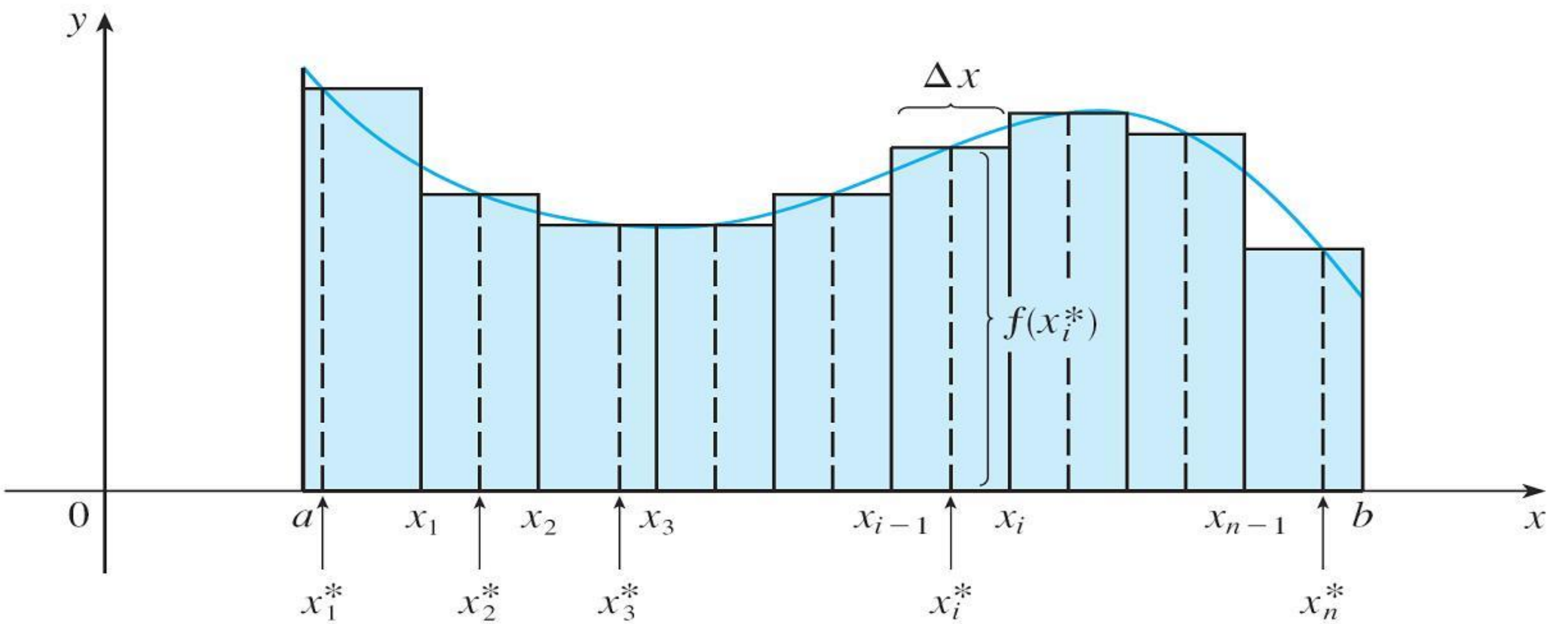


Figure 1

Volumes and Double Integrals

Volumes and Double Integrals

In a similar manner we consider a function *f* of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

and we first suppose that $f(x, y) \geq 0$.

The graph of *f* is a surface with equation $z = f(x, y)$.

Let *S* be the solid that lies above *R* and under the graph of *f*, that is,

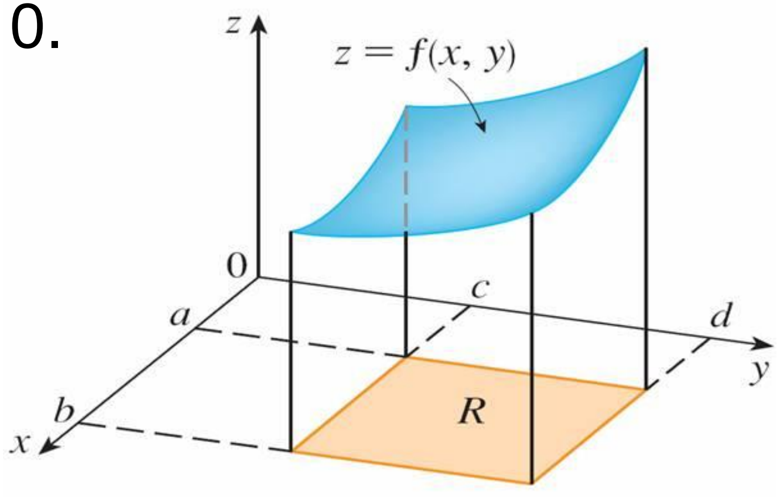


Figure 2

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y),\ (x, y) \in R\}$$

(See Figure 2.)

Volumes and Double Integrals

Our goal is to find the volume of S.

The first step is to divide the rectangle R into subrectangles.

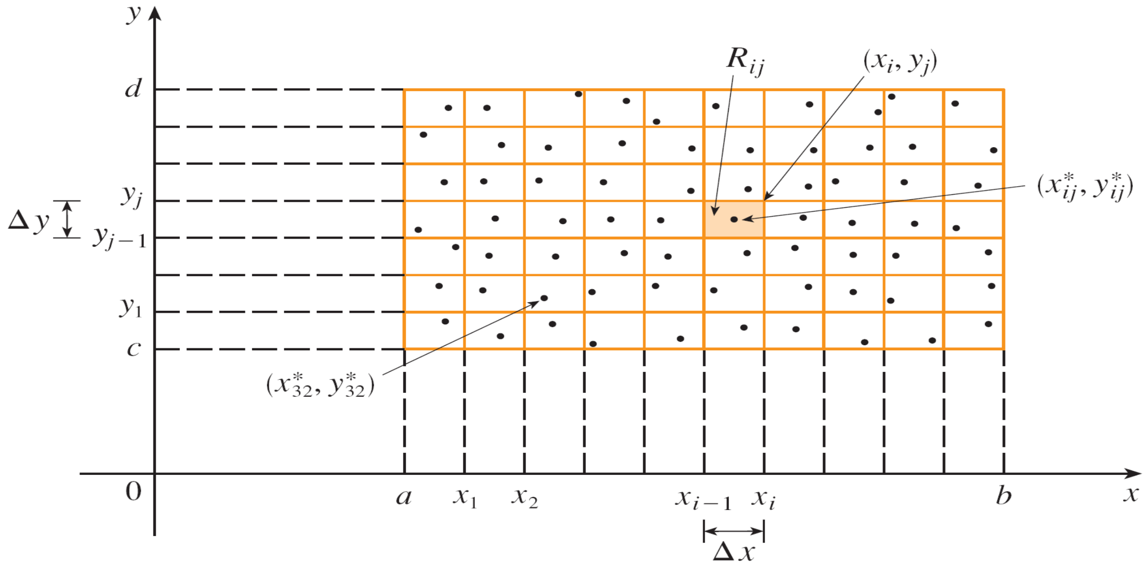
We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$.

Volumes and Double Integrals

By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.



Dividing R into subrectangles

Figure 3

Volumes and Double Integrals

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4.

The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

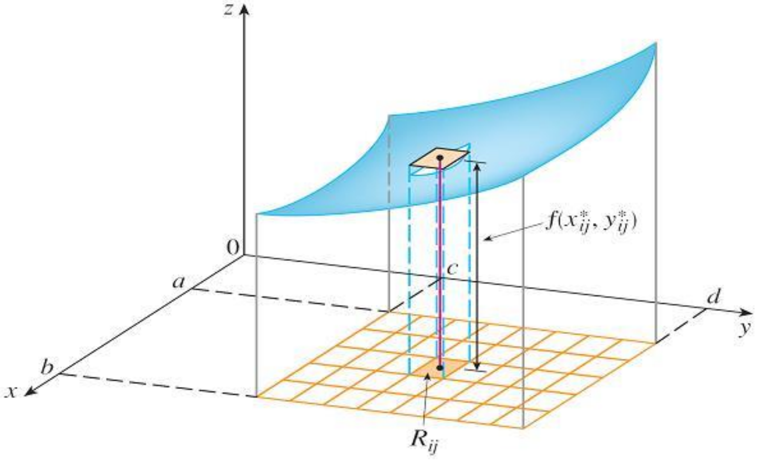


Figure 4

Volumes and Double Integrals

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of *S*:

$$\boxed{3} \qquad V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate *f* at the chosen point and multiply by the area of the subrectangle, and then we add the results.

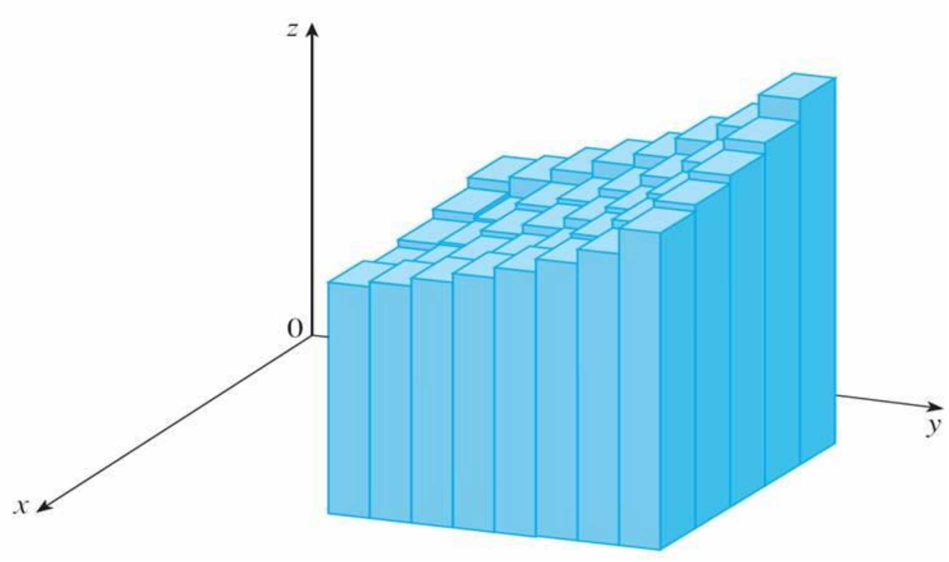


Figure 5

Volumes and Double Integrals

Our intuition tells us that the approximation given in (3) becomes better as m and n become larger and so we would expect that

$$\boxed{4} \qquad V = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

We use the expression in Equation 4 to define the volume of the solid *S* that lies under the graph of *f* and above the rectangle *R*.

Volumes and Double Integrals

Limits of the type that appear in Equation 4 occur frequently, not just in finding volumes but in a variety of other situations even when *f* is not a positive function. So we make the following definition.

$\boxed{5}$ **Definition** The **double integral** of f over the rectangle R is

$$\iint\limits_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

A function *f* is called **integrable** if the limit in Definition 5 exists.

Note: There is a precise definition of the limit on the RHS.

Precise Definition

The limit in Definition 5 means, that for all $\varepsilon > 0$ there is an integer *N* such that

$$\left| \iint_R f(x, y)\, dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A \right| < \varepsilon$$

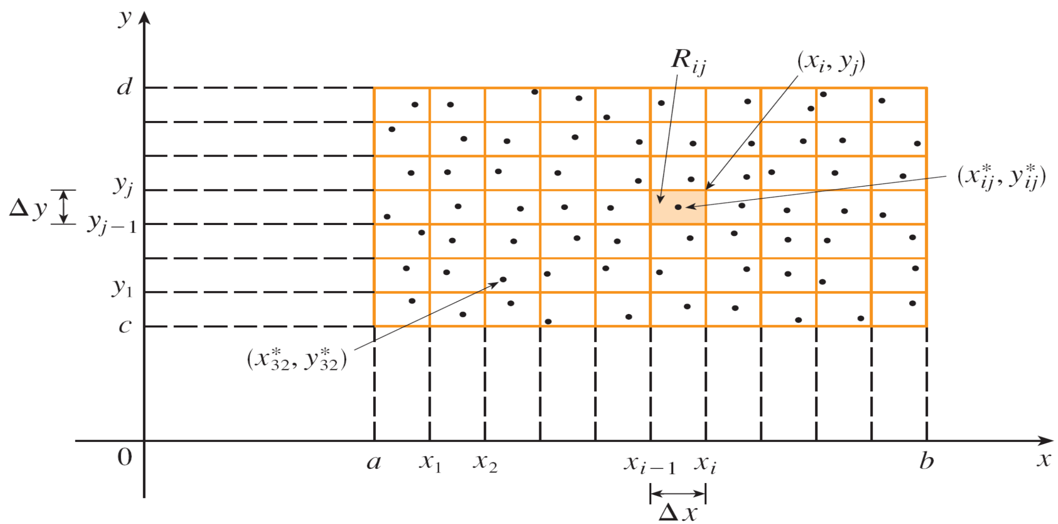
for all *m*,*n* > *N* and arbitrary (x_{ij}^*, y_{ij}^*) sample points.

Volumes and Double Integrals

It is shown in courses on advanced calculus that all continuous functions are integrable. In fact, the double integral of f exists provided that *f* is "not too discontinuous."

In particular, if *f* is bounded [that is, there is a constant *M* such that $|f(x, y)| \leq M$ for all (x, y) in *R*], and *f* is continuous there, except on a finite number of smooth curves, then *f* is integrable over *R*.

Volumes and Double Integrals



Dividing *R* into subrectangles

Figure 3

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij}, but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j), see Figure 3], then the expression for the double integral looks simpler:

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$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

Volumes and Double Integrals

By comparing Definitions 4 and 5, we see that a volume can be written as a double integral:

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

Volumes and Double Integrals

The sum in Definition 5,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in $\boxed{1}$ for a function of a single variable.]

Volumes and Double Integrals

If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume under the graph of f.

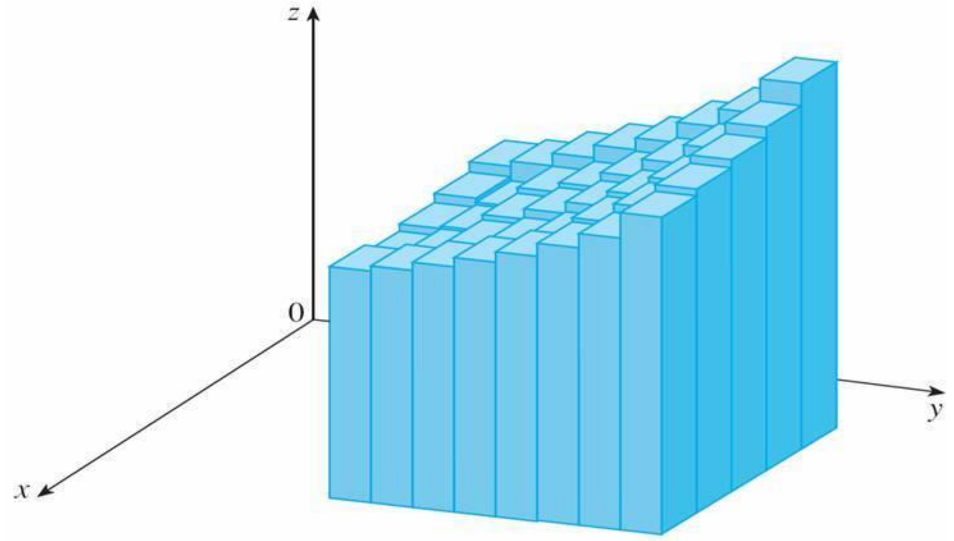

Figure 5

Example 1

Estimate the volume of the solid that lies above the square R = [0, 2] × [0, 2] and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}.

Example 1

This is the volume of the approximating rectangular boxes shown in Figure 7.

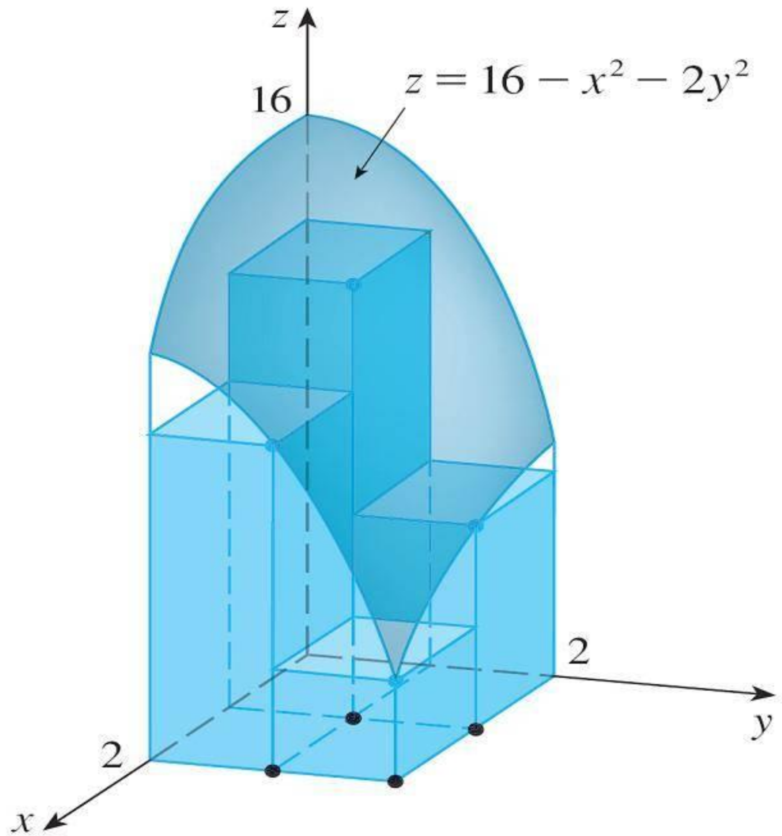

Figure 7

The Midpoint Rule

The Midpoint Rule

The methods that can be used for approximating single integrals (such as the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) all have counterparts for double integrals. Here we consider only the Midpoint Rule for double integrals.

This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center $(\overline{x}_i, \overline{y}_j)$ of R_{ij}. In other words, $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

The Midpoint Rule

Midpoint Rule for Double Integrals

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Iterated Integrals

Iterated Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

We use the notation $\int_c^d f(x, y)\,dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$ (if it is integrable as a one variable function). This procedure is called *partial integration with respect to y*.

Now $\int_c^d f(x, y)\,dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\,dy$$

Iterated Integrals

If we now integrate the function *A* with respect to *x* from $x = a$ to $x = b$ (if it is integrable), we get

(7) $$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx$$

The integral on the right side of Equation 7 is called an **iterated integral**. Usually the brackets are omitted. Thus

(8) $$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx$$

means that we first integrate with respect to *y* from *c* to *d* and then with respect to *x* from *a* to *b*.

Iterated Integrals

Similarly, the iterated integral

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy \tag{9}$$

means that we first integrate with respect to *x* (holding *y* fixed) from *x* = *a* to *x* = *b* and then we integrate the resulting function of *y* with respect to *y* from *y* = *c* to *y* = *d*.

Notice that in both Equations 8 and 9 we work *from the inside out*.

Example 4

Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Iterated Integrals

Notice that in Example 4 we obtained the same answer whether we integrated with respect to y or x first.

In general, it turns out (see Theorem 4) that the two iterated integrals in Equations 8 and 9 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

Iterated Integrals

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

Fubini's Theorem. If f is continuous on the rectangle $R = \{(x, y) \mid a \le x \le b, c \le y \le d\}$, then both iterated integrals $\int_a^b \int_c^d f(x, y)\, dy\, dx$ and $\int_c^d \int_a^b f(x, y)\, dx\, dy$ exist and

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

Fubini's Theorem

Note: More generally, if we assume that f is bounded on *R*, *f* is discontinuous only at a finite number of smooth curves and the iterated integrals exits then the equality of the three integrals hold.

Iterated Integrals

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form.

To be specific, suppose that $f(x, y) = g(x)h(y)$ is continuous and $R = [a, b] \times [c, d]$.

Then Fubini's Theorem gives

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

Iterated Integrals

In the inner integral, *y* is a constant, so *h*(*y*) is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy = \int_c^d \left[h(y) \left(\int_a^b g(x)\, dx \right) \right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

since $\int_a^b g(x)\, dx$ is a constant.

Therefore, in this case, the double integral of *f* can be written as the product of two single integrals:

$$\iint_R f(x, y) dA = \int_a^b g(x)\, dx \int_c^d h(y) dy, \;\; R = [a, b] \times [c, d].$$

Examples 5 - 8

Example 5. Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$.

Example 6. Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1,2] \times [0, \pi]$.

Example 7. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$ and the three coordinate planes.

Example 8. Evaluate $\iint_R \sin x \cos y\, dA$, $R = [0, \pi/2] \times [0, \pi/2]$.

Average Values

Average Values

Recall that the average value of a function *f* of one variable defined on an interval [*a*, *b*] is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

In a similar fashion we define the **average value** of a function *f* of two variables defined on a rectangle *R* to be

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where *A*(*R*) is the area of *R*.

Average Values

If $f(x, y) \geq 0$, the equation

$$A(R) \times f_{\text{ave}} = \iint_R f(x, y)\, dA$$

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

[If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{ave}, then you can use them to fill in the valleys so that the region becomes completely flat. See Figure 11.]

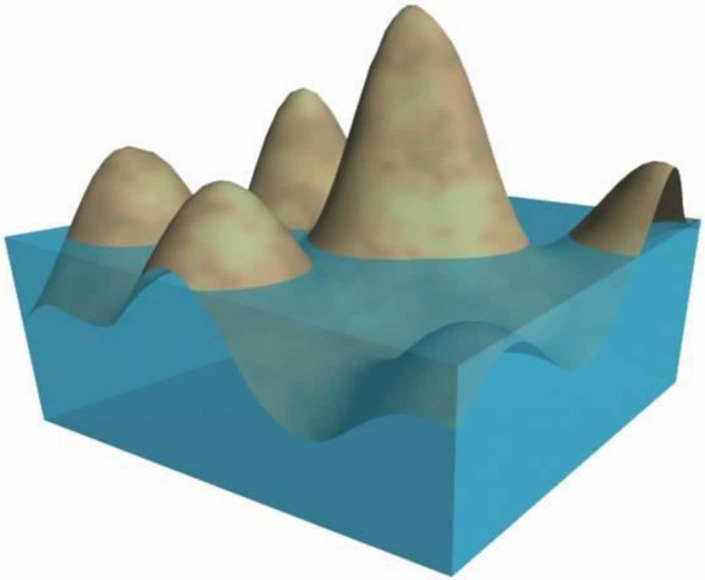

Figure 11

Properties of Double Integrals

Properties of Double Integrals

We list here three properties of double integrals. We assume that *f* and *g* are integrable on *R*. Properties 7 and 8 are referred to as the *linearity* of the integral, while Property 9 is called the *monotonicity* of the integral.

$$\boxed{7} \qquad \iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

$$\boxed{8} \qquad \iint_R c\, f(x, y)\, dA = c \iint_R f(x, y)\, dA \quad \text{where } c \text{ is a constant}$$

If $f(x, y) \geq g(x, y)$ for all (x, y), then

$$\boxed{9} \qquad \iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$$