11.1 Sequences

Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The number a_1 is called the *first term,* a_2 is the *second term,* and in general a_n is the *nth term.* We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Note: for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers.

Sequences

But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Some sequences can be defined by giving a formula for the nth term.

Example 1

In the following examples give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula for the *n*th term, and a third by writing out the terms of the sequence. Notice that *n* doesn't have to start at 1.

(a) $\left\{\frac{1}{2^n}\right\}$

(b) $\left\{\frac{n}{n+1}\right\}$

(c) $\left\{\sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \ldots\right\}$

Examples 2 and 3

2. Find a_n for $\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \ldots\right\}$.

3. Fibonacci sequence: $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. This is an example of a *recursion*, or *recursive definition,* of a sequence. Write out the first few terms.

Sequences

A sequence such as the one in Example 1(b), $a_n = n/(n + 1)$, can be pictured either by plotting its terms on a number line, as in Figure 2, or by plotting its graph, as in Figure 3.

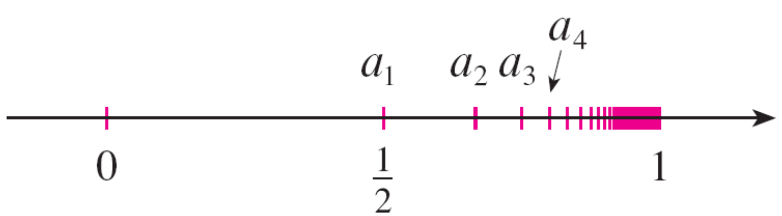


Figure 2

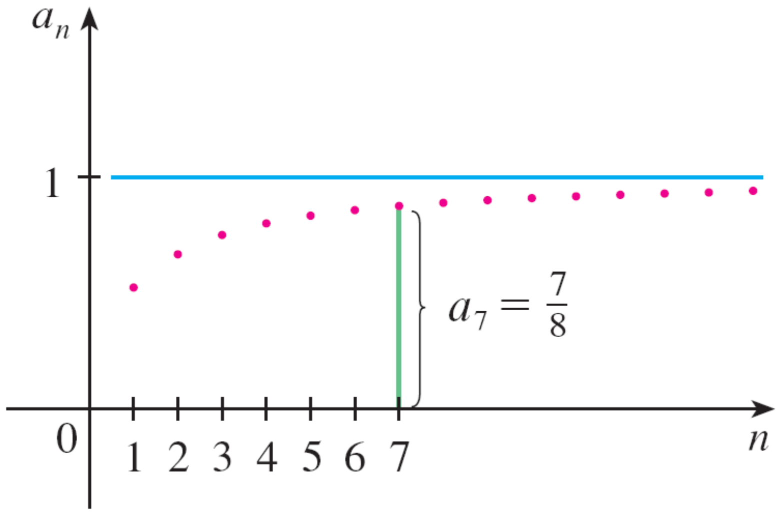


Figure 3

Sequences

Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \quad (2, a_2) \quad (3, a_3) \quad \ldots \quad (n, a_n) \quad \ldots$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n + 1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large.

Sequences

We indicate this by writing

$$\lim_{n \to \infty} \frac{n}{n + 1} = 1$$

In general, the notation

$$\lim_{n \to \infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large.

Sequences

Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity.

1 Intuitive Definition of a Limit of a Sequence A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Note: the sequence $\{a_n\}$ is divergent, if there is no L with the above property.

Sequences

Figure 4 illustrates Definition 1 by showing the graphs of two sequences that have the limit *L*.

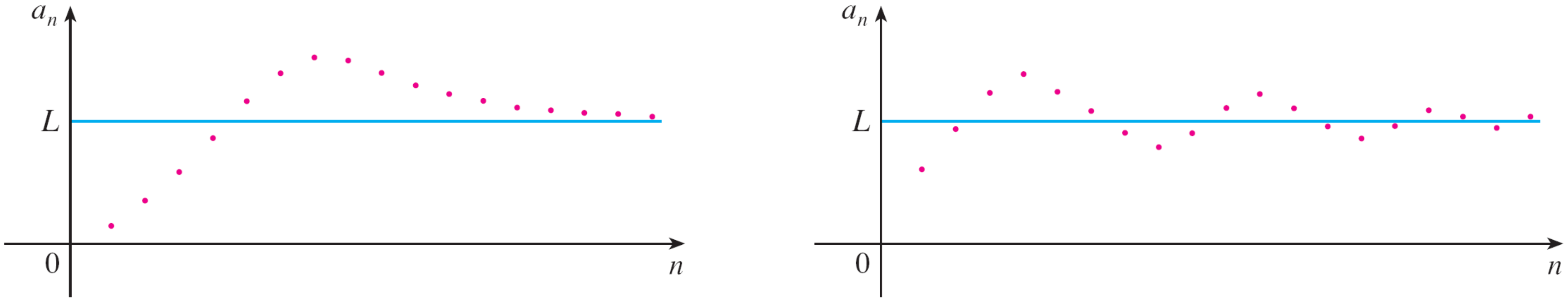


Figure 4

Graphs of two sequences with $\lim_{n \to \infty} a_n = L$

Sequences

A more precise version of Definition 1 is as follows.

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

Note: 1. If a sequence $\{a_n\}$ has the limit *L*, then we can also say that $\{a_n\}$ *converges* to *L* (as *n* approaches ∞).

2. The limit, if it exists, is unique.

3. $\{a_n\}$ is divergent, if there is no *L* with the above property.

Sequences

Definition 2 is illustrated by Figure 5, in which the terms a_1, a_2, a_3 , . . . are plotted on a number line.

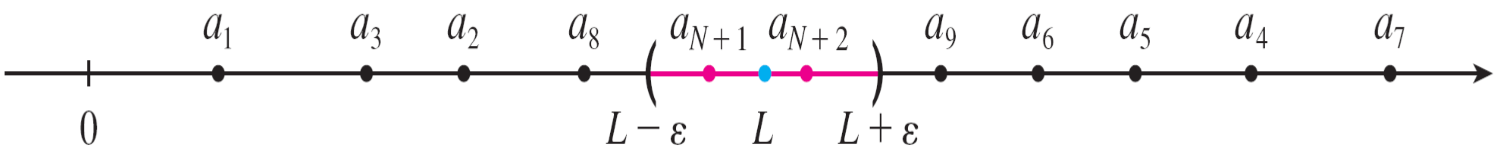


Figure 5

No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

Or, equivalently, no matter how small an interval

$(L - \varepsilon, L + \varepsilon)$ is chosen there are only finitely many terms of the sequence outside of that interval.

Sequences

Another illustration of Definition 2 is given in Figure 6. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.

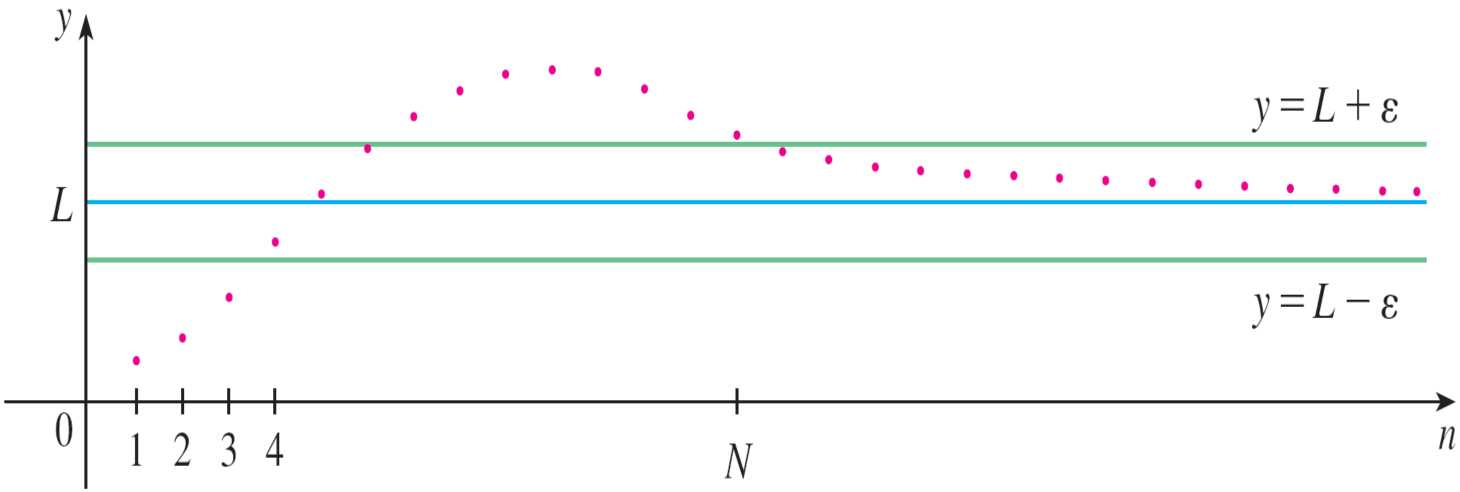
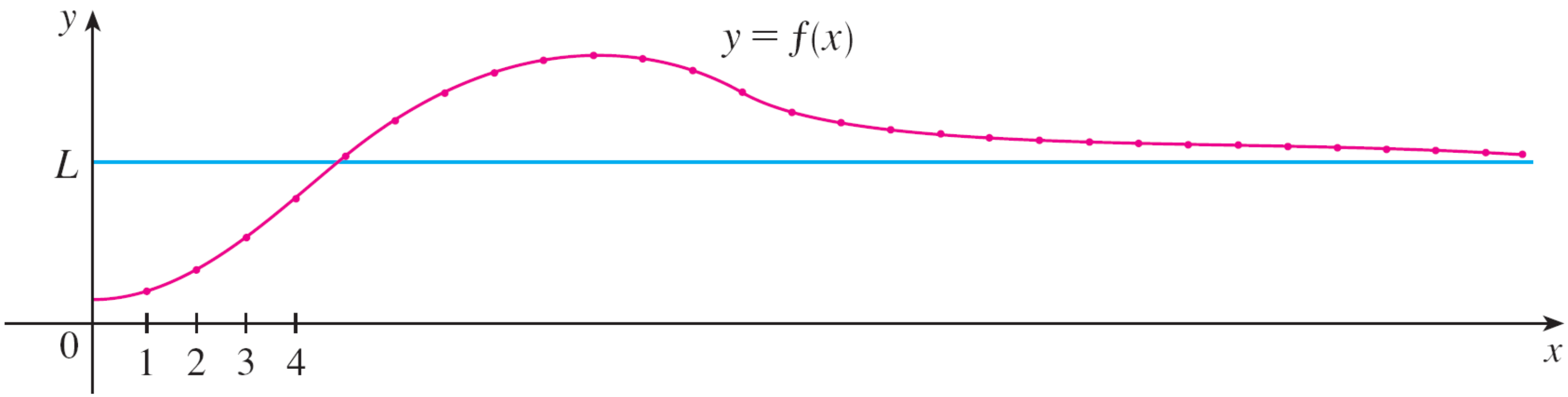


Figure 6

Sequences

The only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem.

Theorem 3. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim a_n = L$**.**



Sequences

Example. Since we know that $\lim_{x\to\infty}(1/x^r) = 0$ when $r > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0.$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$. Consider the definition:

Definition 3. We write $\lim_{n\to\infty} a_n = \infty$ if for every $M > 0$, there is an integer N such that $n > N \implies a_n > M$.

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞. Note that the analogue of Theorem 3 holds in this case too ($L = \infty$).

Sequences

The Limit Laws also hold for the limits of sequences and their proofs are similar.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

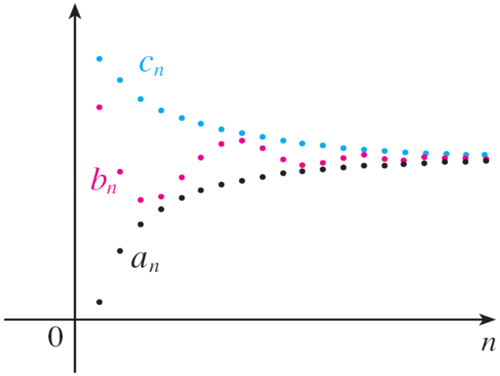
$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Sequences

The Squeeze Theorem can also be adapted for sequences as follows.

Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Note: Similarly, if $a_n \leq b_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} b_n = \infty$.

Sequences

Another useful fact about limits of sequences is given by the following theorem.

Theorem 6. $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$.

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent.

$\boxed{7}$ **Theorem** If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Examples 4 -11

4. Find $\lim_{n\to\infty} \frac{n}{n+1}$ if it exists.

5. Find $\lim_{n\to\infty} \frac{n}{\sqrt{10+n}}$ if it exists.

6. Find $\lim_{n\to\infty} \frac{\ln(n)}{n}$ if it exists.

7. Find $\lim_{n\to\infty} (-1)^n$ if it exists.

8. Find $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

9. Find $\lim_{n\to\infty} \sin\frac{\pi}{n}$ if it exists.

10. Find $\lim_{n\to\infty} \frac{n!}{n^n}$ if it exists.

11. For what values of r is the sequence $\{r^n\}$ convergent ?

Sequences

Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

11 **Definition** A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Example 13: Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

Sequences

For instance, the sequence $a_n = n$ is bounded below ($a_n > 0$) but not above. The sequence $a_n = n/(n + 1)$ is bounded because $0 < a_n < 1$ for all n.

We know that not every bounded sequence is convergent (for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leq a_n \leq 1$ but is divergent). One may show though that every convergent sequence is bounded.

While not every monotonic sequence is convergent ($a_n = n$) but if a sequence is both bounded *and* monotonic, then it must be convergent.

Sequences

This fact is stated in Theorem 12, and intuitively you can understand why it is true by looking at Figure 14.

If $\{a_n\}$ is increasing and $a_n \leq M$ for all n, then the terms are forced to crowd together and approach some number L.

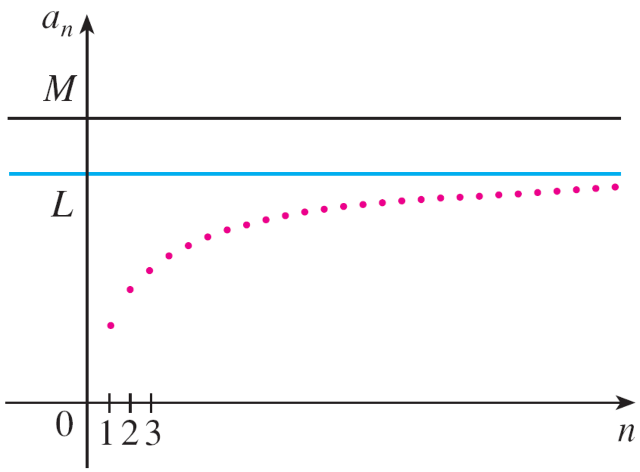


Figure 14

Monotonic Sequence Theorem

Theorem. If a sequence is nondecreasing and bounded above, then it is convergent. Similarly, if If a sequence is nondecreasing and bounded below, then it is convergent.

Note: Stewart only states the theorem for increasing and decreasing sequences, but the statement is true in this slightly greater generality too.

11.2 Series

Series

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

π = 3.14159 26535 89793 23846 26433 83279 50288 . . .

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

where the three dots ($\cdots$) indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π.

Series

In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{1}$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

Series

It would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms we get the cumulative sums 1, 3, 6, 10, 15, 21, . . . and, after the *n*th term, we get $n(n + 1)/2$, which becomes very large as n increases.

However, if we start to add the terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots$$

we get $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots, 1 - 1/2^n, \ldots.$

Series

The table shows that as we add more and more terms, these *partial sums* become closer and closer to 1.

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

Series

In fact, by adding sufficiently many terms of the series we can make the partial sums as close as we like to 1.

So it seems reasonable to say that the sum of this infinite series is 1 and to write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series (1) has a sum.

Series

We consider the **partial sums**

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit.

Series

If $\lim_{n \to \infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\Sigma\, a_n$.

$\boxed{2}$ **Definition** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\Sigma\, a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Series

Thus, the sum of a series is the limit of the sequence of partial sums.

So, when we write $\sum_{n=1}^{\infty} a_n = s$ we mean that by adding sufficiently many terms of the series we can get as close as we like to the number *s*.

Notice that $$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

Example 2

Example 2. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges and find its sum.

Geometric series

An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** *r*.

$\boxed{4}$ The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

Examples 3 and 4

Example 3. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

Example 4. Is the series

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$$

convergent or divergent?

Examples 6 and 7

Example 6. Write the number

$$2.3\overline{17} = 2.3171717\ldots$$

as a ratio for integers.

Example 7. Find the sum of the series

$$\sum_{n=0}^{\infty} x^n,$$

where $|x| < 1$.

A necessary condition for conv.

6 **Theorem** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

Warning: The converse of Theorem 6 is not true in general.

If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\Sigma\, a_n$ is convergent as the following example shows:

Example 8. Show that the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Corollary: test for divergence

7 **The Test for Divergence** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 9. Show that the series

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$$

is divergent.

Series: limit laws

8 Theorem If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Example 10. Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right).$$

11.3 The Integral Test and Estimates of Sums

The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series $\Sigma\ 1/[n(n + 1)]$ because in each of those cases we could find a simple formula for the nth partial sum s_n.

But usually, it isn't easy to discover such a formula. However, in many cases one can determine whether a series converges or diverges without having an explicit formula for the nth partial sum s_n. Moreover, in many cases, one may obtain good estimates for the sum of a convergent series.

The Integral Test

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(a) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Note: It is enough if the function f is ultimately decreasing, that is, it is decreasing for $x > N$ for some positive integer *N*. Similarly, the sum and the integral does not have to start at 1.

Warning: In general, $\int_1^{\infty} f(x)dx \neq \sum_{n=1}^{\infty} a_n$.

The p-series

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the ***p*-series**.

1 The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Example 4

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Estimates of sums

2 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx$$

Corollary:

3

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant s \leqslant s_n + \int_{n}^{\infty} f(x)\,dx$$

Examples 5 and 6

Examples 5 and 6.

(a) Approximate the sum of the series $\Sigma\ 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

(c) Use equation (3) with $n = 10$ to estimate the sum of the series.

11.4 The Comparison Tests

The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

$$\boxed{1} \qquad \sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series $\sum_{n=1}^{\infty} 1/2^n$, which is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Because the series $\boxed{1}$ is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is.

The Comparison Tests

The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series $\boxed{1}$ has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series).

This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

The Comparison Tests

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.

The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

The Comparison Tests

In using the Comparison Test we must, of course, have some known series $\Sigma\, b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A *p*-series [$\Sigma\, 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$]
- A geometric series [$\Sigma\, ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$]

Note: Although the condition $a_n \leq b_n$ or $a_n \geq b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geq N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms.

Examples 1 and 2

Example 1. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

converges or diverges.

Example 2. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$

converges or diverges.

The Comparison Tests

Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because $\Sigma\, b_n = \Sigma \left(\frac{1}{2}\right)^n$ is convergent and $a_n > b_n$.

The Comparison Tests

Nonetheless, we have the feeling that $\Sigma\ 1/(2^n - 1)$ ought to be convergent because it is very similar to the convergent geometric series $\Sigma \left(\frac{1}{2}\right)^n$.

In such cases the following test can be used.

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Examples 3 and 4

Example 3. Test the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

for convergence or divergence.

Example 4. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$$

converges or diverges.

11.5 Alternating Series

Alternating Series

In this section we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series,* whose terms alternate in sign.

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

Alternating Series

We see from these examples that the *n*th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

Remark. As before, one only needs that $b_{n+1} \leq b_n$ for all $n \geq N$.

Examples 1-3

Test the following series for for convergence or divergence.

Example 1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Example 2.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$$

Example 3.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$$

Estimating Sums

A partial sum s_n of any convergent series can be used as an approximation to the total sum s, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$.

The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term.

Estimating Sums

Theorem. If $s = \sum(-1)^{n-1} b_n$ with $b_n > 0, b_{n+1} \leq b_n$ and $\lim\limits_{n\to\infty} b_n = 0$, then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Example 4

Example 4. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

Estimating Sums

Note:

The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. The rule does not apply to other types of series.

Absolute Convergence

Given any series $\Sigma\, a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

$\boxed{1}$ **Definition** A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

Absolute Convergence

Notice that if $\Sigma\, a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

Example 5

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent *p*-series ($p = 2$).

Example 6

We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent, but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (p-series with p = 1) and is therefore divergent.

Absolute Convergence

$\boxed{2}$ **Definition** A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 6 shows that the alternating harmonic series is conditionally convergent. Thus, it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

$\boxed{3}$ **Theorem** If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

Example 7

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Example 8

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(a)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

(b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

(c)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{2n+1}$$

Rearrangements

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series.

Rearrangements

By a **rearrangement** of an infinite series $\Sigma\, a_n$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\Sigma\, a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \ldots$$

It turns out that

if $\Sigma\, a_n$ is absolutely convergent series with sum s,
then any rearrangement of $\Sigma\, a_n$ has the same sum s.

Rearrangements

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots := s$$

Note, that $s \neq 0$ (why?).

If we multiply this series by $\frac{1}{2}$, we get

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{1}{2}s$$

Rearrangements

Inserting zeros between the terms of this series, we have

$$0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\cdots=\frac{1}{2}s$$

Now we add this to the original series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=s$$

we get

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3}{2}s$$

Rearrangements

Notice that the last series in contains the same terms as the original one but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

> if $\Sigma\, a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\Sigma\, a_n$ that has a sum equal to r.

11.6 Ratio and Root Tests

Ratio and Root Tests

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Examples 1 and 2

Example 1. Test the absolute convergence of the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$$

Example 2. Test the convergence of the series

$$\sum_{n=1}^{\infty}\frac{n^n}{n!}.$$

Ratio and Root Tests

The following test is convenient to apply when *n*th powers occur.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Ratio and Root Tests

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series $\Sigma\, a_n$ could converge or diverge.

Note: One can show that if both $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ and $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then the two limits are equal. Thus, if $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too. However, it might happen that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ exists and $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not, but not the other way around. Hence, the root test has, in general, wider scope.

Examples 4 and 5

Example 4. Test the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{2n + 3}{3n + 2} \right)^n.$$

Example 5. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n + 1} \right)^n$$

11.8 Power Series

Power Series

A **power series** is a series of the form

$$\boxed{1} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence.

A power series may converge for some values of x and diverge for other values of x.

Power Series

The sum of the series is a function

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

Power Series

More generally, a series of the form

$$\boxed{2} \qquad \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series in** $(x - a)$ or a **power series centered at *a*** or a **power series about *a***.

Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 2 we have adopted the convention that $(x - a)^0 = 1$ even when $x = a$.

Notice also that when $x = a$ all of the terms are 0 for $n \geq 1$ and so the power series (2) always converges when $x = a$.

Examples 1 - 3

For what values of *x* is the series convergent?

Example 1.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

Example 2.

$$\sum_{n=0}^{\infty} n!\, x^n$$

Example 3.

$$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$$

Power Series

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number *R* in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

Power Series

The **interval of convergence** of a power series is the interval that consists of all values of *x* for which the series converges.

In case (i) the interval consists of just a single point *a*.

In case (ii) the interval is $(-\infty, \infty)$.

In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$.

Power Series

When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints.

Thus in case (iii) there are four possibilities for the interval of convergence:

$$(a - R, a + R) \quad (a - R, a + R] \quad [a - R, a + R) \quad [a - R, a + R]$$

The situation is illustrated in Figure 3.

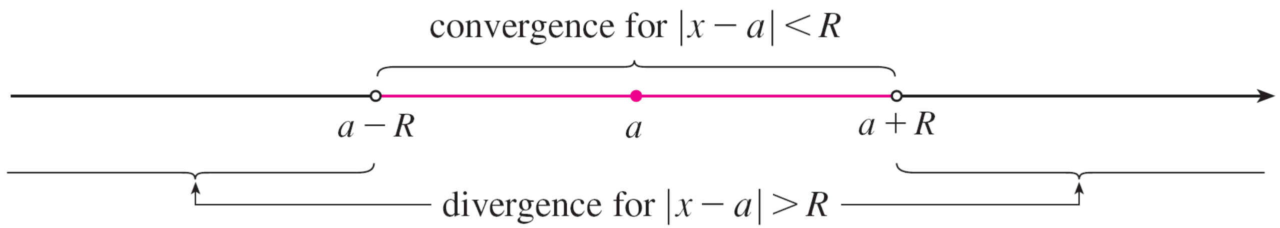


Figure 3

Examples 4 - 5

Find the radius of convergence and interval of convergence of the series.

Example 4.

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Example 5.

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

11.9 Representations of Functions as Power Series

Representations of Functions as Power Series

We start with an equation that we have seen before:

1 $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

We have obtained this equation by observing that the series is a geometric series with *a* = 1 and *r* = *x*.

But here our point of view is different. We now regard Equation 1 as expressing the function $f(x) = 1/(1 - x)$ as a sum of a power series.

Examples 1 - 3

Example 1. Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

Example 2. Find a power series representation for $1/(x + 2)$.

Example 3. Find a power series representation of $x^3/(x + 2)$.

Differentiation and Integration of Power Series

Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$ whose domain is the interval of convergence of the series.

We would like to be able to differentiate and integrate such functions, and the following theorem says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial.

This is called **term-by-term differentiation and integration**.

Differentiation and Integration of Power Series

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

(ii) $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Examples 4 - 6

Example 4. Express $1/(1 - x)^2$ as a power series. What is the radius of convergence?

Example 5. Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

Example 6. Find a power series representation for $\tan^{-1} x$ and its radius of convergence.

Example 8

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry.

In particular, the sum of the power series,

$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, is called a **Bessel function of order 0.**

Example 8.

(a) Find the domain of J_0.

(b) Find the derivative of J_0.

Bessel function

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

as the sum of a series we mean that, for every real number *x*, $J_0(x) = \lim_{n \to \infty} s_n(x)$ where $s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$.

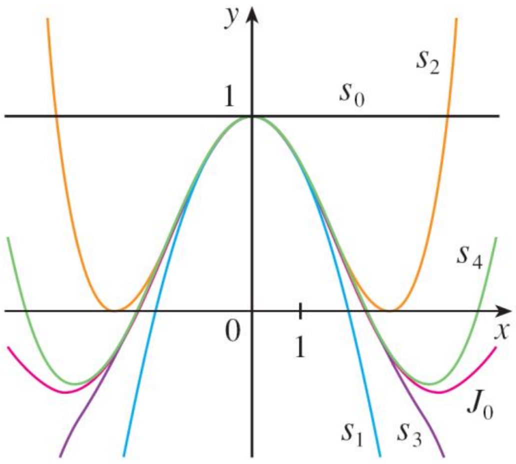
Power Series

The first few partial sums are

$$s_0(x) = 1 \qquad s_1(x) = 1 - \frac{x^2}{4} \qquad s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} \qquad s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$

Figure 2 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function J_0, but notice that the approximations become better when more terms are included.



Partial sums of the Bessel function J_0

Figure 2

Power Series

Figure 3 shows a more complete graph of the Bessel function.

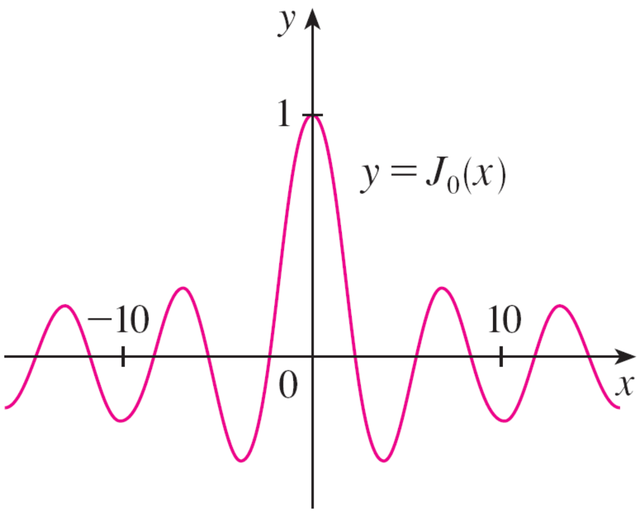


Figure 3

Abel's Theorem

We saw that if a power series $\sum c_n(x-a)^n$ has radius of convergence $R>0$, then the series is differentiable on $(a-R, a+R)$ and hence it is continuous on $(a-R, a+R)$.

Theorem (Abel). Suppose that a power series $f(x)=\sum c_n(x-a)^n$ has radius of convergence $R>0$ and that the series converge for $a-R$ (resp. $a+R$). Then f is continuous from the right at $a-R$ (resp. from the left at $a+R$).

Example. Show Leibniz' formula:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots$$

11.10 Taylor and Maclaurin Series

Taylor and Maclaurin Series

We start by supposing that f is any function that can be represented by a power series

$$\boxed{1} \quad f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + \cdots, \qquad |x - a| < \boldsymbol{R}.$$

Let's try to determine what the coefficients c_n must be in terms of f.

To begin, notice that if we put $x = a$ in Equation 1, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

Taylor and Maclaurin Series

We can differentiate the series in Equation 1 term by term:

$$\boxed{2} \quad f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \cdots, \text{ for } |x - a| < \boldsymbol{R},$$

and substitution of $x = a$ in Equation 2 gives

$$f'(a) = c_1$$

Now we differentiate both sides of Equation 2 and obtain

$$\boxed{3} \quad f''(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \cdots, \text{ for } |x - a| < \boldsymbol{R}$$

Again we put $x = a$ in Equation 3. The result is

$$f''(a) = 2c_2$$

Taylor and Maclaurin Series

Let's apply the procedure one more time. Differentiation of the series in Equation 3 gives

$$\boxed{4} \quad f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + 3 \cdot 4 \cdot 5c_5(x - a)^2 + \cdots, \quad |x - a| < R$$

and substitution of $x = a$ in Equation 4 gives

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

By now you can see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot nc_n = n!c_n$$

Taylor and Maclaurin Series

Solving this equation for the *n*th coefficient c_n, we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)} = f$. Thus we have proved the following theorem.

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \qquad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Taylor and Maclaurin Series

Substituting this formula for c_n back into the series, we see that *if f* has a power series expansion at *a*, then it must be of the following form.

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The series in Equation 6 is called the **Taylor series of the function *f* at *a*** (or **about *a*** or **centered at *a***).

Taylor and Maclaurin Series

For the special case a = 0 the Taylor series becomes

$$\boxed{7} \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

Example 2

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Taylor and Maclaurin Series

The conclusion we can draw from Theorem 5 and Example 2 is that *if* e^x has a power series expansion at 0, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So how can we determine whether e^x *does* have a power series representation?

Let's investigate the more general question: Under what circumstances is a function equal to the sum of its Taylor series? In other words, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor and Maclaurin Series

As with any convergent series, this means that *f*(*x*) is the limit of the sequence of partial sums. In the case of the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that T_n is a polynomial of degree *n* called the ***n*th-degree Taylor polynomial of *f* at *a***.

Taylor and Maclaurin Series

For instance, for the exponential function $f(x) = e^x$, the result of Example 1 shows that the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1, 2$, and 3 are

$$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The graphs of the exponential function and these three Taylor polynomials are drawn in Figure 1.

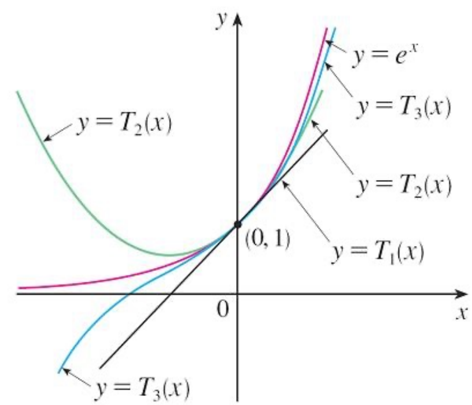


Figure 1

As n increases, $T_n(x)$ appears to approach e^x in Figure 1. This suggests that e^x is equal to the sum of its Taylor series.

Taylor and Maclaurin Series

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n \to \infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n \to \infty} R_n(x) = 0$, then it follows that

$$\lim_{n \to \infty} T_n(x) = \lim_{n \to \infty} [f(x) - R_n(x)] = f(x) - \lim_{n \to \infty} R_n(x) = f(x)$$

Taylor and Maclaurin Series

We have therefore proved the following.

$\boxed{8}$ **Theorem** If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

In trying to show that $\lim_{n \to \infty} R_n(x) = 0$ for a specific function f, we usually use the following Theorem.

$\boxed{9}$ **Taylor's Inequality** If $|f^{(n+1)}(x)| \leqslant M$ for $|x - a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leqslant d$$

Taylor's Theorem

The proof of Taylor's inequality is based on Taylor's Theorem (interestingly, not due to Taylor).

Theorem (Lagrange, 1797). If $f^{(n+1)}$ is continuous on an open interval I that contains a, and $\mathrm{a} \neq x \in I$, then there exists a number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Note. This is the so-called *Lagrange's form* of the remainder term.

Note

There are examples of functions *f* that have derivatives of all order for all x and their Maclaurin series converge for all x, yet the sum of the Maclaurin series of *f* does not equal to *f* except for at *x*=0 (where, of course, it always does).

Example. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Without doing the details it turns out that $f^{(n)}(0) = 0$ for all n and thus the Maclaurin series of f yields the constant 0 function.

Examples 3 and 4

Example 3. Prove that e^x is equal to the sum of its Maclaurin series.

Example 4. Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Taylor and Maclaurin Series

In particular, if we put $x = 1$ in the Maclaurin series, we obtain the following expression for the number e as a sum of an infinite series:

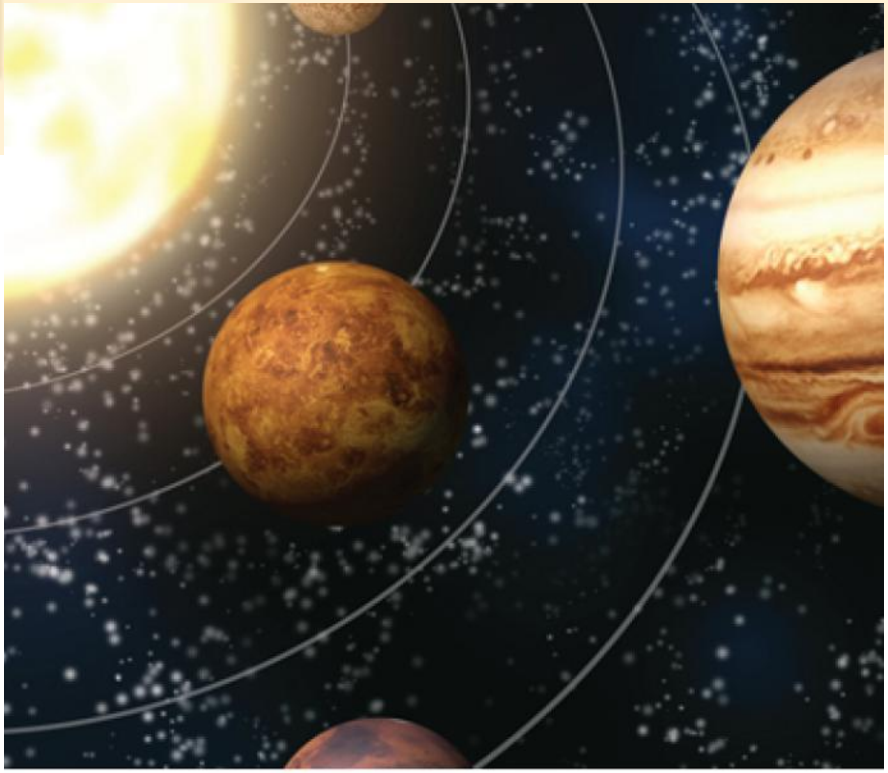
12 $$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Examples 5 and 6

Example 5. Find the Maclaurin series for $f(x)=\sin x$ and prove that it represents $\sin x$ for all x.

Example 6. Find the Maclaurin series for $\cos x$.

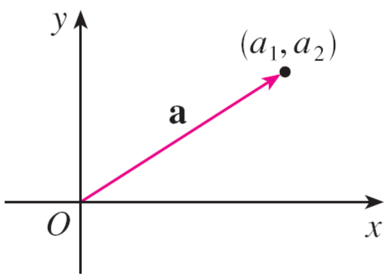
13 Vector Functions



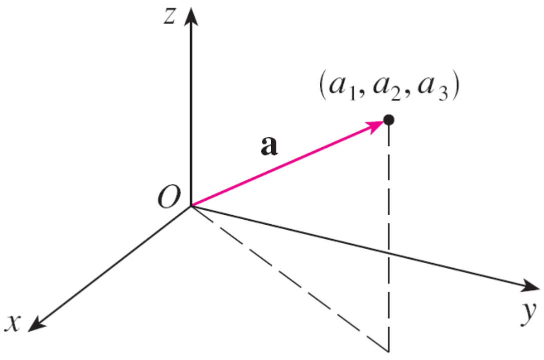
13.1 Vector Functions and Space Curves

Components

If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional.



$\mathbf{a} = \langle a_1, a_2 \rangle$



$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

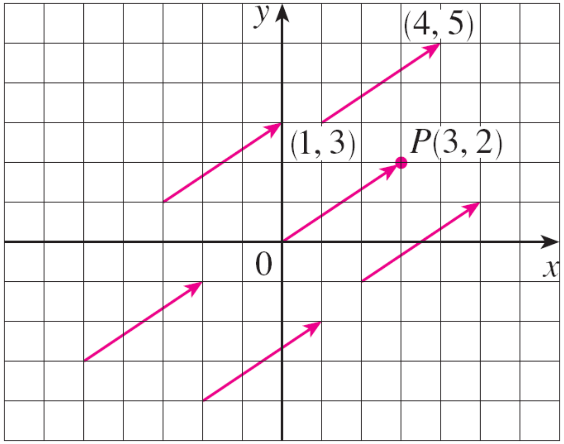
Components

These coordinates are called the **components** of **a** and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

For instance, the vectors shown here are all equivalent to the vector $\overrightarrow{OP} = \langle 3, 2 \rangle$ whose terminal point is $P(3, 2)$.



Representations of the vector $\mathbf{a} = \langle 3, 2 \rangle$

Components

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

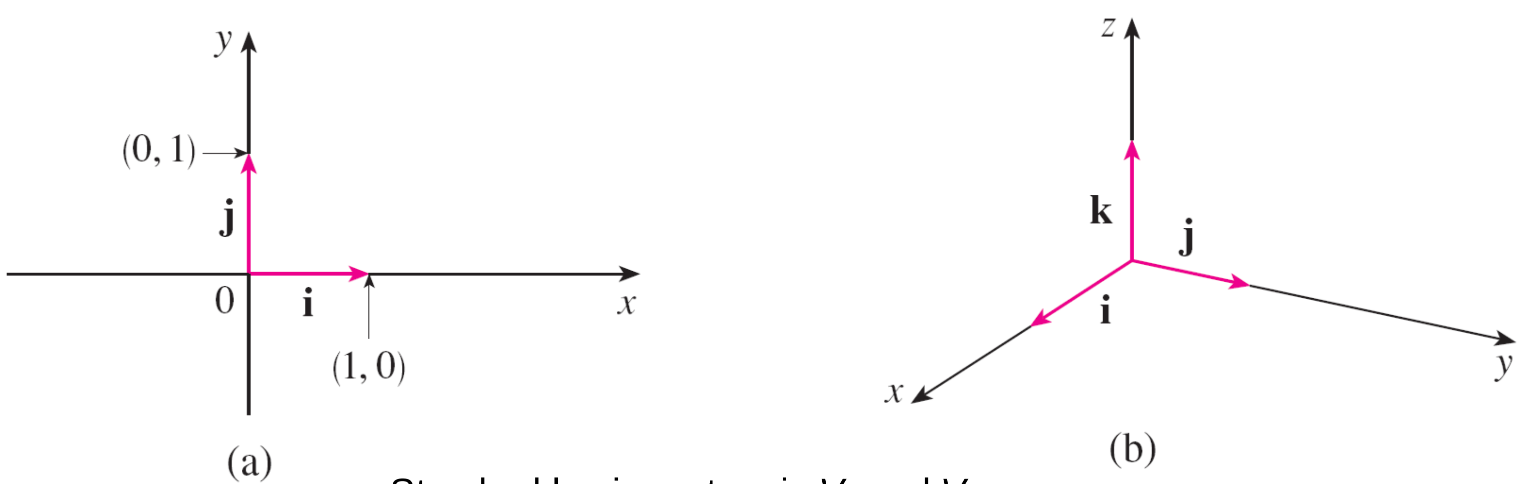
We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three dimensional vectors with the above operations.

Components

Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

Then **i**, **j**, and **k** are vectors that have length 1 and point in the directions of the positive *x*-, *y*-, and *z*-axes. Similarly, in two dimensions we define $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.



Standard basis vectors in V_2 and V_3

Components

If **a** = $\langle a_1, a_2, a_3 \rangle$, then we can write

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$
$$= a_1\langle 1, 0, 0 \rangle + a_2\langle 0, 1, 0 \rangle + a_3\langle 0, 0, 1 \rangle$$
$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Thus any vector in V_3 can be expressed in terms of the **standard basis vectors i**, **j**, and **k**. For instance,

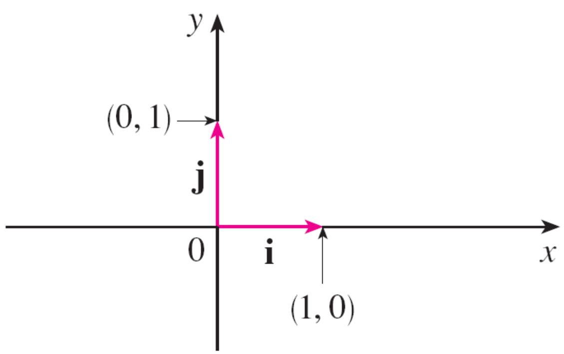
$$\langle 1, -2, 6 \rangle = \mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Similarly, in two dimensions, we can write

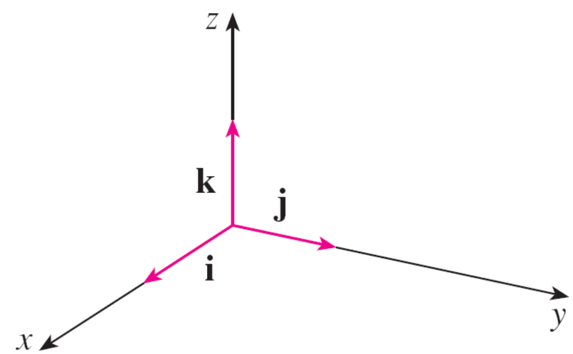
$$\mathbf{a} = \langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

Components

Geometric interpretation:

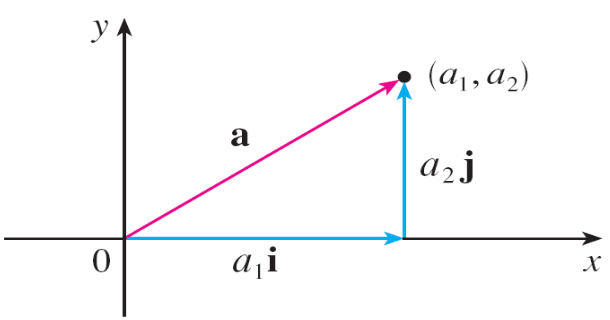


(a)

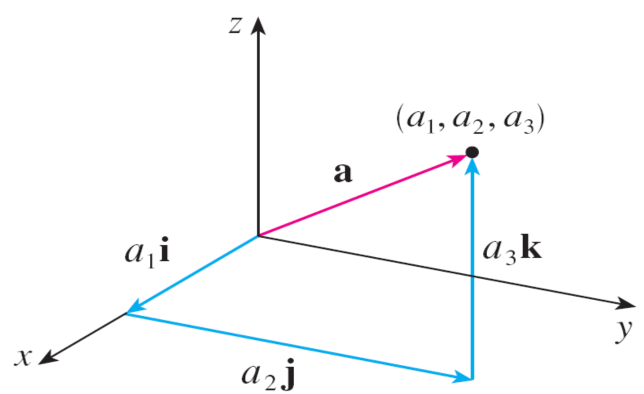


(b)

Standard basis vectors in V_2 and V_3



(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$



(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Length (magnitude)

By using the distance formula to compute the length of a segment *OP*, we obtain the following formulae.

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Components

A **unit vector** is a vector whose length is 1. For instance, **i**, **j**, and **k** are all unit vectors. In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as **a** is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Vector Functions and Space Curves

In general, a function is a rule that assigns to each element in the domain an element in the range.

A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors.

We are most interested in vector functions **r** whose values are three-dimensional vectors.

This means that for every number t in the domain of **r** there is a unique vector in V_3 denoted by $\mathbf{r}(t)$.

Vector Functions and Space Curves

If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g, and h are real-valued functions called the **component functions** of $\mathbf{r}$ and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

Example 1

If $\mathbf{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t} \rangle$, then find the component functions and the domain.

Vector Functions and Space Curves

The **limit** of a vector function **r** is defined by taking the limits of its component functions as follows.

1 If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Limits of vector functions obey the same rules as limits of real-valued functions.

Example 2

Find $\lim_{t \to 0} \mathbf{r}(t)$ where

$$\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$$

Vector Functions and Space Curves

A vector function **r** is **continuous at *a*** if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

In view of Definition 1, we see that **r** is continuous at *a* if and only if its component functions *f*, *g*, and *h* are continuous at *a*.

There is a close connection between continuous vector functions and space curves.

Vector Functions and Space Curves

Suppose that f, g, and h are continuous real-valued functions on an interval I.

Then the set C of all points (x, y, z) in space, where

$$\boxed{2} \qquad x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I, is called a **space curve**.

The equations in $\boxed{2}$ are called **parametric equations of *C*** and t is called a **parameter**.

We can think of C as being traced out by a moving particle whose position at time t is $(f(t), g(t), h(t))$.

Vector Functions and Space Curves

If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C.

Thus any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in Figure 1.

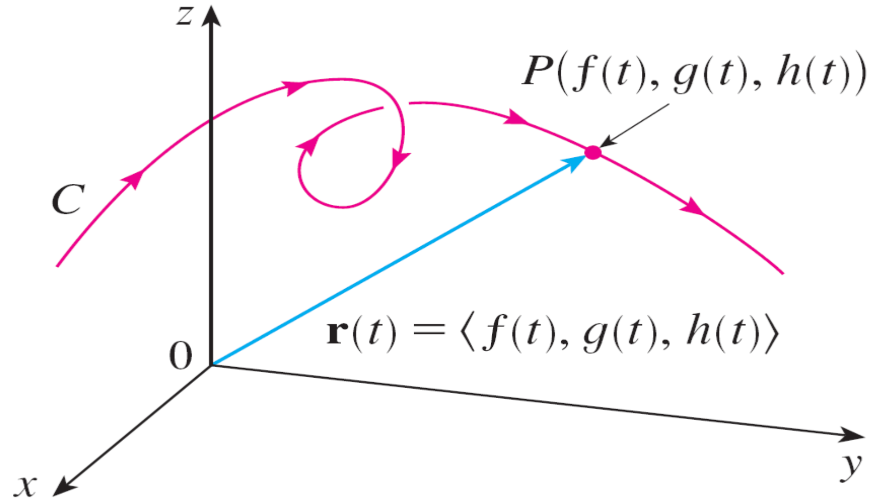


Figure 1

C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.

Example 3

Describe the curve defined by the vector function

$$\mathbf{r}(\mathrm{t}) = \langle 1 + t, 2 + 2t, -1 + 6t \rangle$$

Example 4

Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$$

Example 4

The curve is called a **helix**.

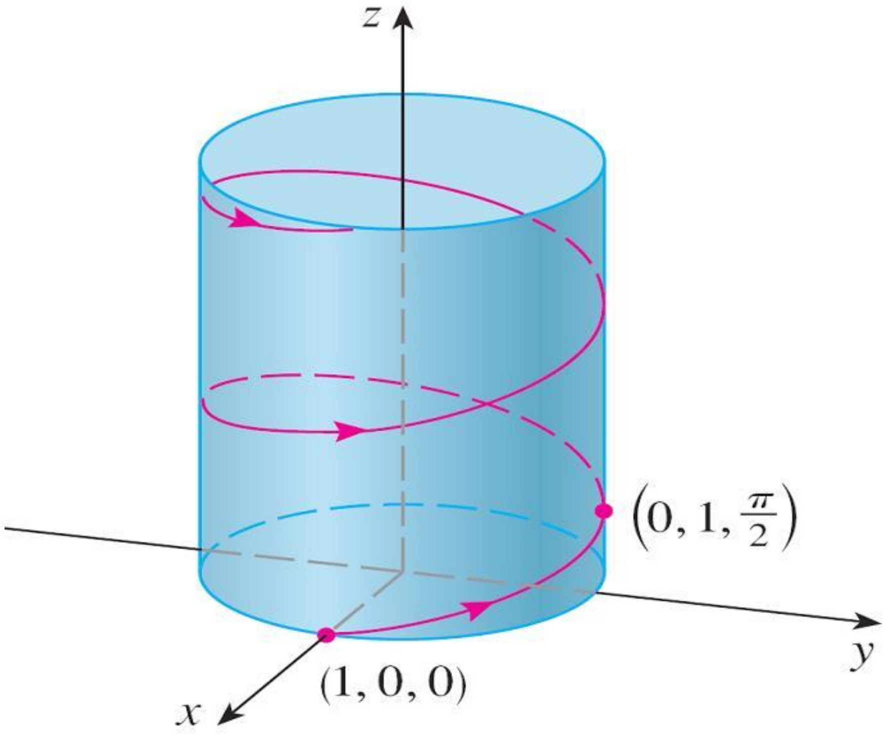


Figure 2

Example 5

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

13.2 Derivatives and Integrals of Vector Functions

Derivatives

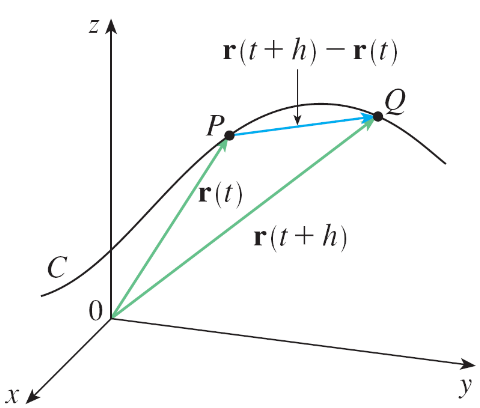
Derivatives

The **derivative r′** of a vector function **r** is defined in much the same way as for real valued functions:

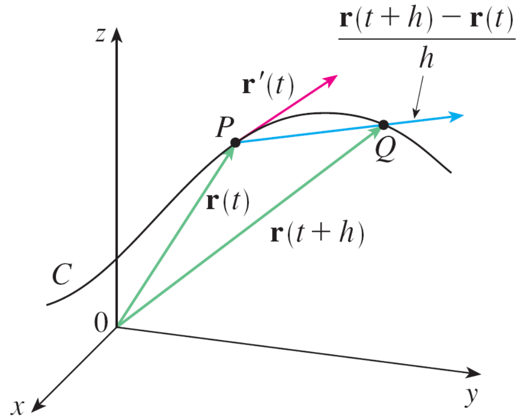
1

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1.



(a) The secant vector $\overrightarrow{PQ}$



(b) The tangent vector $\mathbf{r}'(t)$

Figure 1

Derivatives

If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t + h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector.

If h > 0, the scalar multiple $(1/h)(\mathbf{r}(t + h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t + h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line.

For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$.

Derivatives

We will also have occasions to consider the **unit tangent vector**, which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The following theorem gives us a convenient method for computing the derivative of a vector function **r**: just differentiate each component of **r**.

2 Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

Examples 1 and 2

Example 1.

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

Example 2.

Find the parametric equation for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $\left(0, 1, \frac{\pi}{2}\right)$.

Derivatives

Just as for real-valued functions, the **second derivative** of a vector function **r** is the derivative of **r**′, that is, **r**″ = (**r**′)′.

For instance, the second derivative of the function,

$\mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle$, is

$$\mathbf{r}''(t) = \langle -2 \cos t, -\sin t, 0 \rangle$$

Differentiation Rules

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

$\boxed{3}$ **Theorem** Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Example 4

Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Integrals

Integrals

The **definite integral** of a continuous vector function **r**(t) can be defined in much the same way as for real-valued functions except that the integral is a vector.

But then we can express the integral of **r** in terms of the integrals of its component functions f, g, and h as follows.

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\,\Delta t$$

$$= \lim_{n \to \infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\,\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\,\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\,\Delta t \right) \mathbf{k} \right]$$

Integrals

and so

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

Integrals

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where **R** is an antiderivative of **r**, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (antiderivatives).

Example 5

If $\mathbf{r}(t) = 2 \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + 2t\, \mathbf{k}$, then find $\int \mathbf{r}(t)dt$ and $\int_0^{\pi/2} \mathbf{r}(t)dt$.

13.3 Arc Length and Curvature

Arc Length and Curvature

Suppose that the curve has the vector equation, $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous.

If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$\boxed{2} \quad \begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \end{aligned}$$

Arc Length and Curvature

Notice that the arc length formula (2) can be put into the more compact form

3

$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$

because, for space curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Example 1

Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

Arc Length and Curvature

A single curve C can be represented by more than one vector function. For instance, the twisted cubic

$$\boxed{4} \qquad \mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \qquad 1 \le t \le 2$$

could also be represented by the function

$$\boxed{5} \qquad \mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \qquad 0 \le u \le \ln 2$$

where the connection between the parameters t and u is given by $t = e^u$.

We say that Equations 4 and 5 are **parametrizations** of the curve C.

Arc Length and Curvature

If we were to use Equation 3 to compute the length of *C* using Equations 4 and 5, we would get the same answer.

In general, it can be shown that when Equation 3 is used to compute arc length, the answer is independent of the parametrization that is used.

Now we suppose that *C* is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \qquad a \le t \le b$$

where $\mathbf{r}'$ is continuous and *C* is traversed exactly once as *t* increases from *a* to *b*.

Arc Length and Curvature

We define its **arc length function** *s* by

$$\boxed{6} \qquad s(t) = \int_a^t |\mathbf{r}'(u)|\,du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\,du$$

Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. (See Figure 3.)

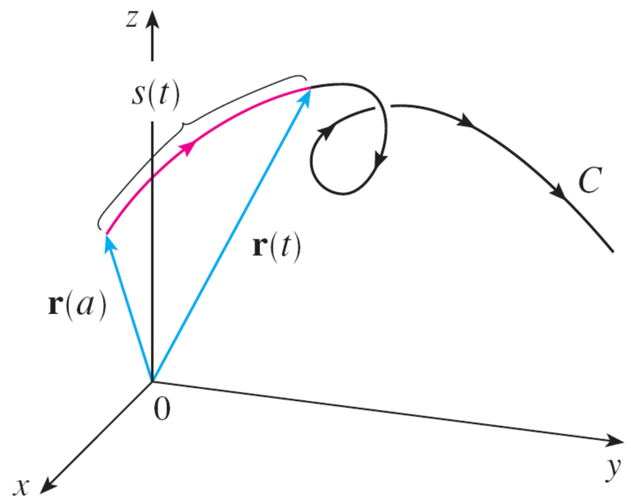


Figure 3

Arc Length and Curvature

If we differentiate both sides of Equation 6 using Part 1 of the Fundamental Theorem of Calculus, we obtain

$$\boxed{7} \qquad \frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system.

Example 2

Reparametrize the helix

$$r(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$$

with respect to arc length measured from (1, 0, 0) in the direction of increasing t.

Curvature

Curvatures

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I.

A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

If C is a smooth curve defined by the vector function $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicates the direction of the curve.

Curvatures

From Figure 4 you can see that **T**(t) changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.

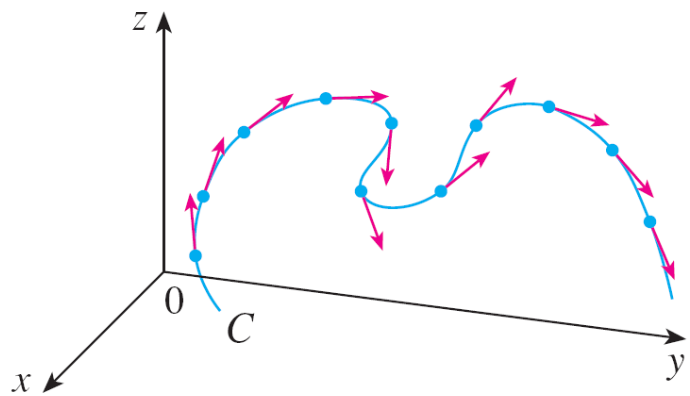


Figure 4

Unit tangent vectors at equally spaced points on C

Curvatures

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point.

Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.)

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

Curvatures

The curvature is easier to compute if it is expressed in terms of the parameter *t* instead of *s*, so we use the Chain Rule to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \text{and} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|$$

But $ds/dt = |\mathbf{r}'(t)|$ from Equation 7, so

9

$$\boxed{\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}}$$

Example 3

Show that the curvature of a circle of radius a is $1/a$.

Curvature

The result of Example 3 shows that small circles have large curvature and large circles have small curvature, in accordance with our intuition.

We can see directly from the definition of curvature that the curvature of a straight line is always 0 because the tangent vector is constant.

Although Formula 9 can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

10 Theorem The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Example 4

Find the curvature of the twisted cubic

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

at a general point and at (0, 0, 0).

The Normal and Binormal Vectors

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$.

We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t, we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$.

Note that $\mathbf{T}'(t)$ is itself not a unit vector.

But at any point where $\kappa \neq 0$ we can define the **principal unit normal vector** $\mathbf{N}(t)$ (or simply **unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

The Normal and Binormal Vectors

The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector**.

It is perpendicular to both **T** and **N** and is also a unit vector. (See Figure 6.)

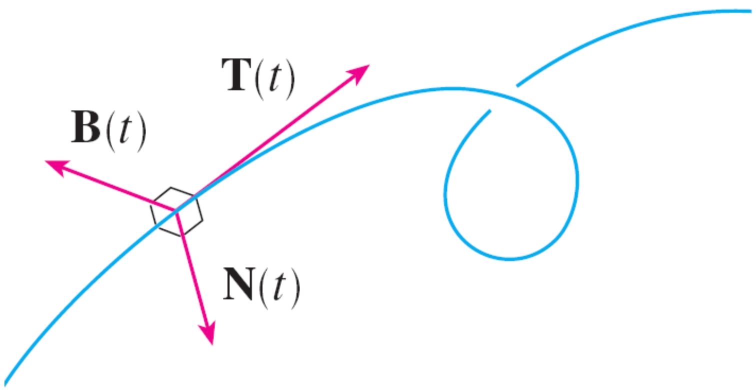


Figure 6

The Normal and Binormal Vectors

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

14 Partial Derivatives

14.1 Functions of Several Variables

Functions of Two Variables

Functions of Several Variables

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point.

We can think of T as being a function of the two variables x and y , or as a function of the pair (x, y). We indicate this functional dependence by writing $T = f(x, y)$.

The volume of V a circular cylinder depends on its radius r and its height h. In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h, and we write $V(r, h) = \pi r^2 h$.

Functions of Several Variables

Definition A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**.

[Compare this with the notation $y = f(x)$ for functions of a single variable.]

Example 1

For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

(a)

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

(b)

$$f(x, y) = x \ln(y^2 - x)$$

Graphs

Graphs

A way of visualizing the behavior of a function of two variables is to consider its graph.

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Just as the graph of a function f of one variable is a curve c with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$.

Graphs

We can visualize the graph *S* of *f* as lying directly above or below its domain *D* in the *xy*-plane (see Figure 5).

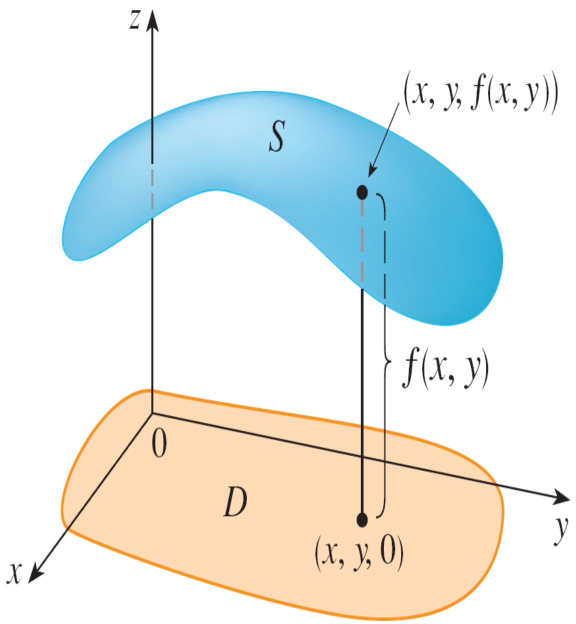


Figure 5

Examples 5 and 6

Example 5. Sketch the graph of the function

$$f(x, y) = 6 - 3x - 2y.$$

Example 6. Find the domain and range of the function

$$z = \sqrt{9 - x^2 - y^2},$$

and sketch its graph.

Level Curves

Level Curves

An important method for visualizing functions, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour lines*, or *level curves*.

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k.

In other words, it shows where the graph of f has height k.

Level Curves

You can see from Figure 11 the relation between level curves and horizontal traces.

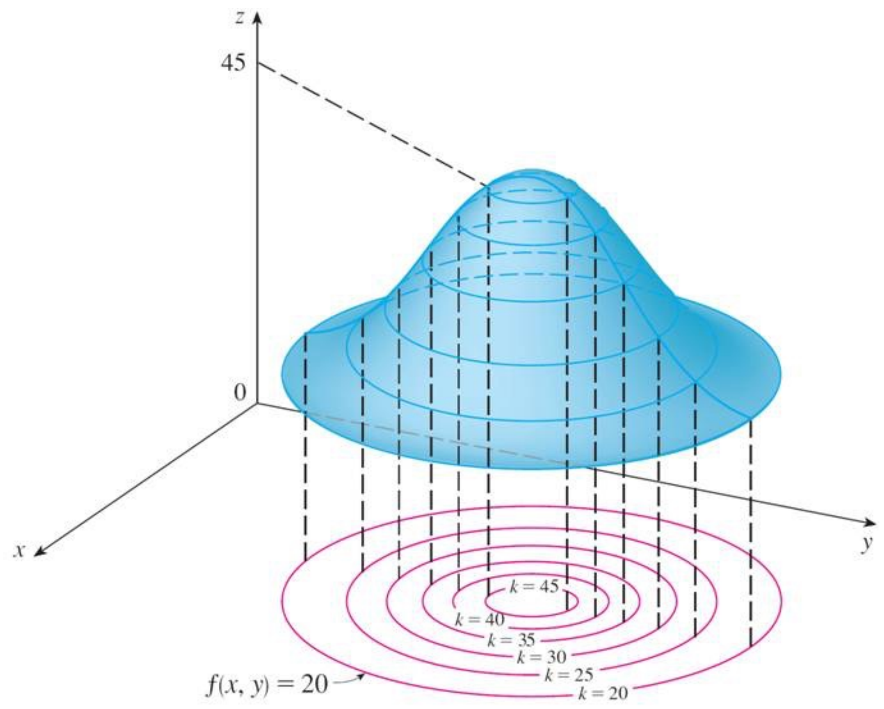


Figure 11

Level Curves

The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.

So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph.

The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

Level Curves

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in Figure 12.

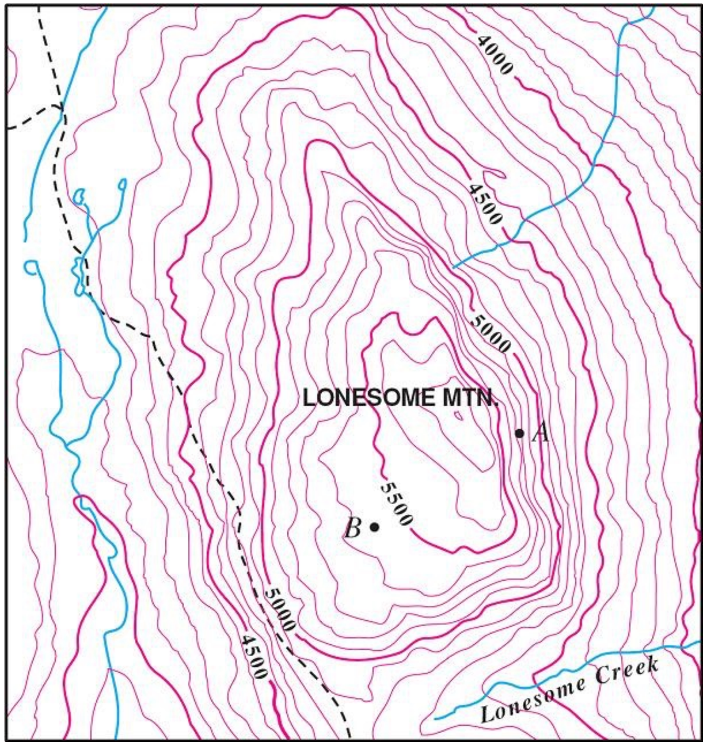


Figure 12

Level Curves

The level curves are curves of constant elevation above sea level.

If you walk along one of these contour lines, you neither ascend nor descend.

Another common example is the temperature at locations (x, y) with longitude x and latitude y.

Here the level curves are called **isothermals** and join locations with the same temperature.

Level Curves

Figure 13 shows a weather map of the world indicating the average January temperatures. The isothermals are the curves that separate the colored bands.

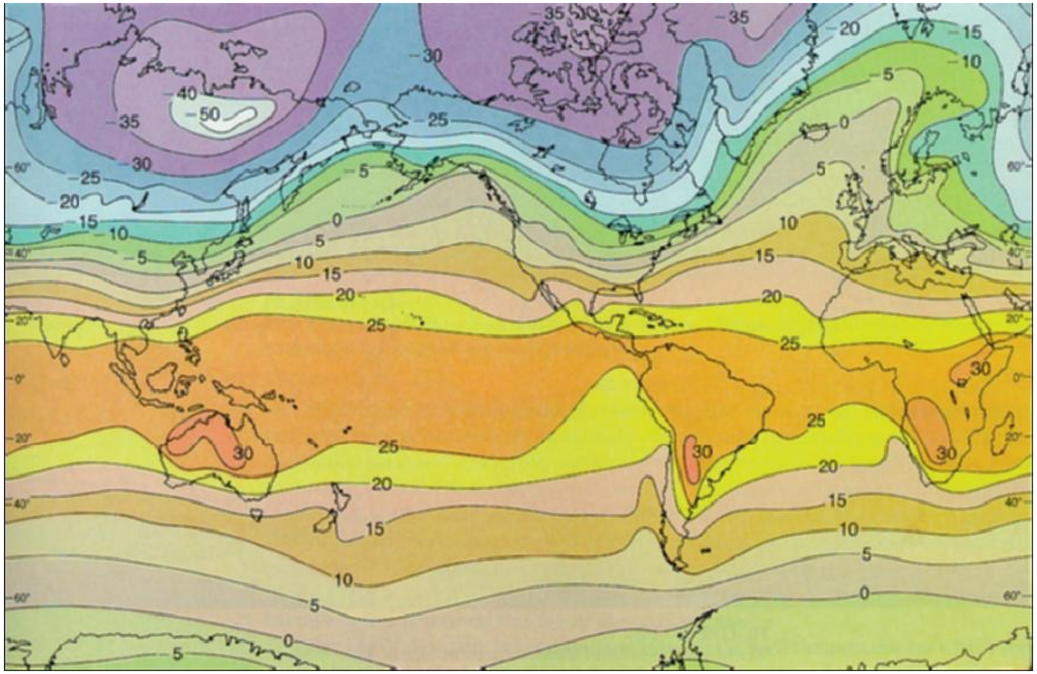


Figure 13

World mean sea-level temperatures in January in degrees Celsius

Example 11

Sketch the level curves and horizontal traces of the function

$$z = \sqrt{9 - x^2 - y^2}$$

for $k = 0, 1, 2, 3$.

Functions of Three or More Variables

Functions of Three or More Variables

A **function of three variables**, *f*, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

Example 14

Find the domain of *f* if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

Functions of Three or More Variables

It's very difficult to visualize a function *f* of three variables by its graph, since that would lie in a four-dimensional space.

However, we do gain some insight into *f* by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

Example 15. Find the level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

Functions of Three or More Variables

Functions of any number of variables can be considered.

A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.

Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$.

14.2 Limits and Continuity

Limits and Continuity

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and

$$g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as x and y both approach 0 [and therefore the point (x, y) approaches the origin].

Limits and Continuity

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Values of $f(x, y)$

Table 1

Limits and Continuity

cont'd

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Values of $g(x, y)$

Table 2

Limits and Continuity

It appears that as (x, y) approaches (0, 0), the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y)\to(0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1$$

and

$$\lim_{(x, y)\to(0, 0)} \frac{x^2 - y^2}{x^2 + y^2}$$ does not exist.

Limits and Continuity

In general, we use the notation

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f.

$\boxed{1}$ **Definition** Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Limits and Continuity

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L$$

and $f(x, y) \to L$ as $(x, y) \to (a, b)$

For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right.

We recall that if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

Limits and Continuity

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see Figure 3) as long as (x, y) stays within the domain of f.

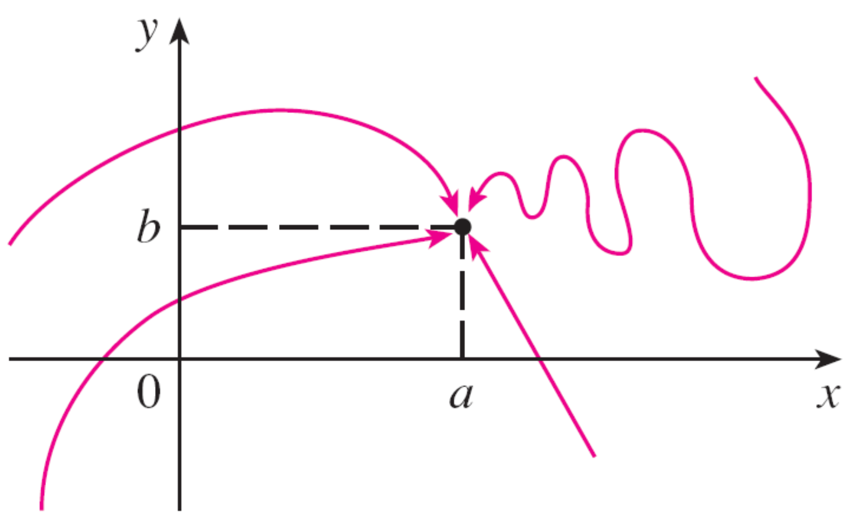


Figure 3

Limits and Continuity

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0).

The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach.

Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b).

Limits and Continuity

Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist.

Example 1

Show that $\lim_{(x, y) \to (0, 0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Limits and Continuity

Now let's look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits.

The Limit Laws can be extended to functions of two variables: The limit of a sum/difference is the sum/difference of the limits, the limit of a product/quotient is the product/quotient of the limits, and so on.

In particular, the following equations are true.

2 $$\lim_{(x, y) \to (a, b)} x = a \qquad \lim_{(x, y) \to (a, b)} y = b \qquad \lim_{(x, y) \to (a, b)} c = c$$

The Squeeze Theorem also holds.

Polynomials and rational functions

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers.

A **rational function** is a ratio of polynomials.

For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function.

Polynomials and rational functions

Since any polynomial can be built up out of the simple functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ by multiplication and addition, it follows from the limit laws and the limits in $\boxed{2}$ that if $p(x,y)$ is a **polynomial** or a **rational function** then for all (a,b) in the domain of p:

$$\lim_{(x,y)\to(a,b)} p(x,y) = p(a,b)$$

Examples 4 and 5

Example 4. Evaluate the limit if it exists.

$$\lim_{(x,\,y)\to(1,\,2)} (x^2y^3 - x^3y^2 + 3x + 2y).$$

Example 5. Evaluate the limit if it exists.

$$\lim_{(x,y)\to(-2,3)} \frac{x^2y + 1}{x^3y^2 - 2x}$$

Continuity

Continuity

Recall that evaluating limits of *continuous* functions of a single variable is easy.

It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x \to a} f(x) = f(a)$.

Continuous functions of two variables are also defined by the direct substitution property.

4 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

Continuity

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount.

This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains.

In particular, polynomials and rational functions are continuous on their domains.

Continuity

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third.

In fact, it can be shown that if *f* is a continuous function of two variables and *g* is a continuous function of a single variable that is defined on the range of *f*, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

Example 10. Where is the function $h(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ continuous?

Functions of Three or More Variables

Functions of Three or More Variables

Everything that we have done in this section can be extended to functions of three or more variables.

The notation

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f.

The function f is **continuous** at (a, b, c) if

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = f(a, b, c)$$

Functions of Three or More Variables

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$.

In other words, it is discontinuous on the sphere with center the origin and radius 1.

Functions of Three or More Variables

We can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

5 If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Here, similarly to the two variable case, we implicitly assume that the domain *D* of *f* contains points arbitrarily close to **a**.

14.3 Partial Derivatives

Partial Derivatives

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant.

Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of f with respect to x at (a, b)** and denote it by $f_x(a, b)$. Thus

1

$$f_x(a, b) = g'(a) \qquad \text{where} \qquad g(x) = f(x, b)$$

Partial Derivatives

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

and so Equation 1 becomes

2
$$\boxed{f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}}$$

Partial Derivatives

Similarly, the **partial derivative of *f* with respect to *y* at (*a*, *b*)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3

$$\boxed{f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}}$$

Partial Derivatives

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

$\boxed{4}$ If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Partial Derivatives

There are many alternative notations for partial derivatives.

For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f / \partial x$.

But here $\partial f / \partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Partial Derivatives

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed.

Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Examples 1 and 2

Example 1. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Example 2. If

$$f(x, y) = \sin \frac{x}{1+y},$$

calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

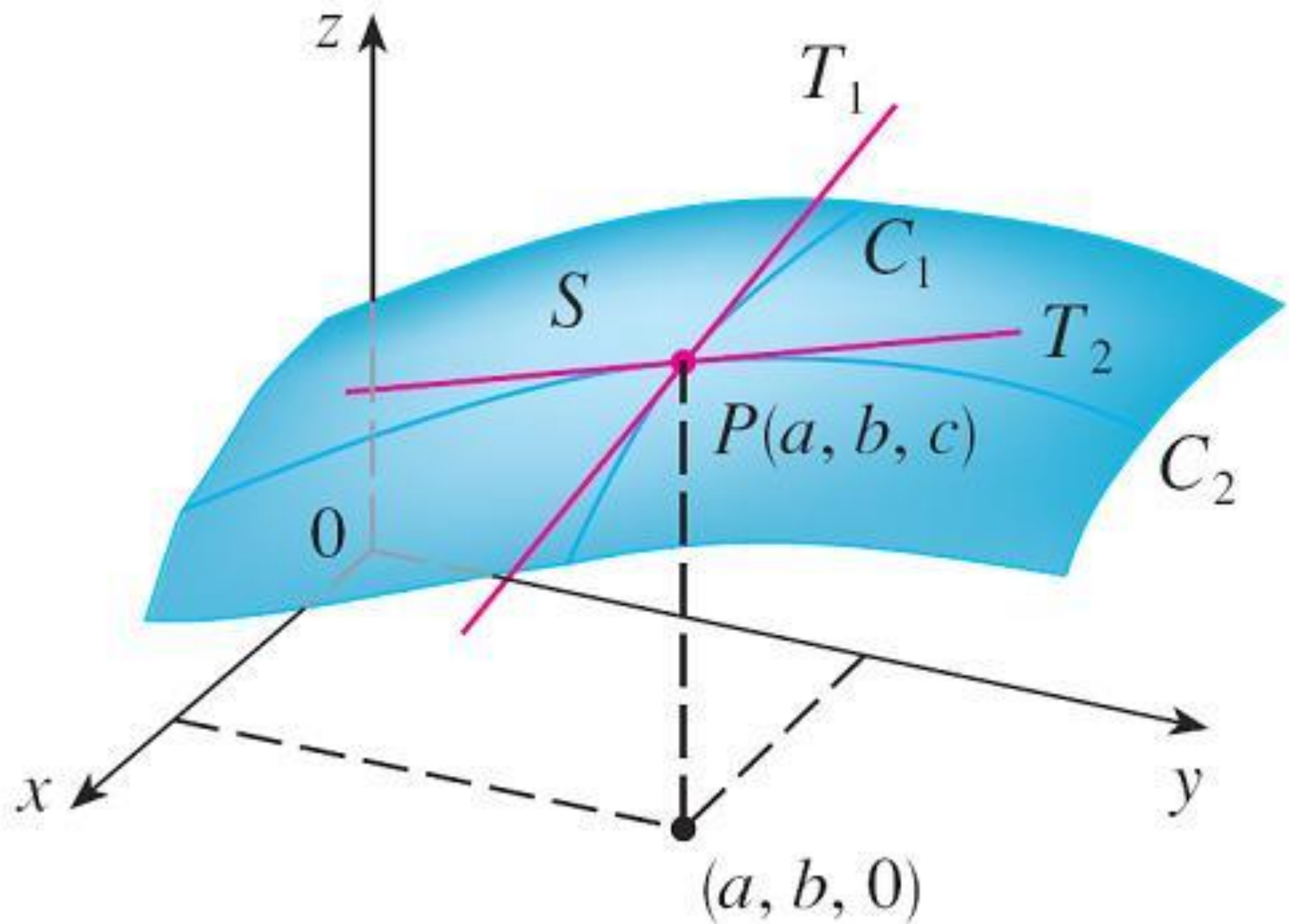
Interpretations of Partial Derivatives

Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S.

By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.)

Interpretations of Partial Derivatives

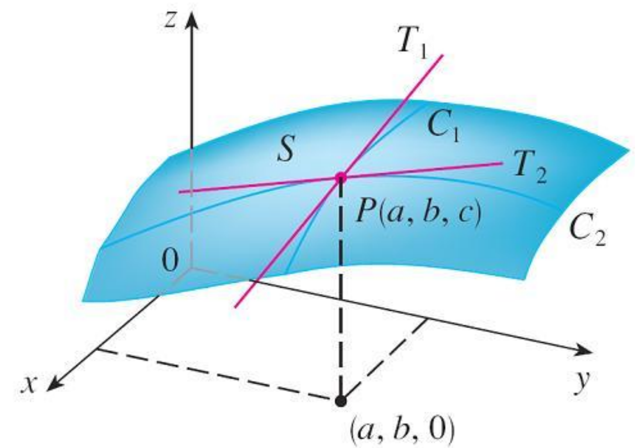


Interpretations of Partial Derivatives

Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$.

The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.



The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Figure 1

Interpretations of Partial Derivatives

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

Thus, partial derivatives can also be interpreted as *rates of change*.

If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

Example 3

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Example 3 – *Solution*

cont'd

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.)

The slope of the tangent line to this parabola at the point (1, 1, 1) is $f_x(1, 1) = -2$.

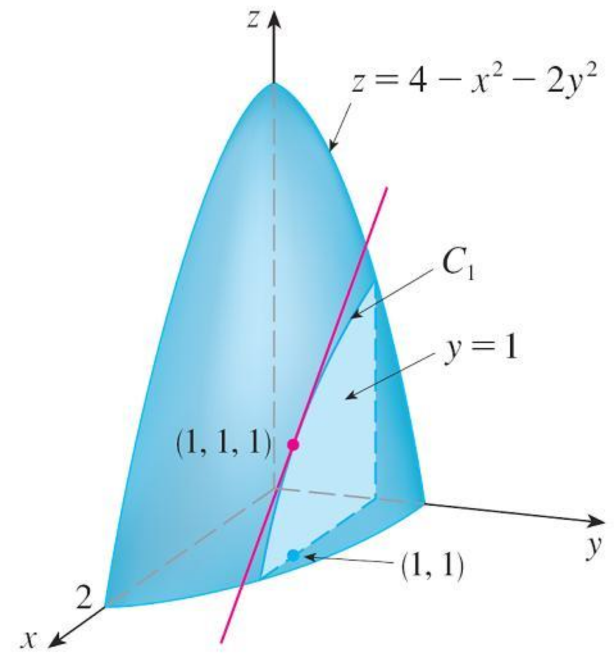


Figure 2

Example 3 – *Solution*

cont'd

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at (1, 1, 1) is $f_y(1, 1) = -4$. (See Figure 3.)

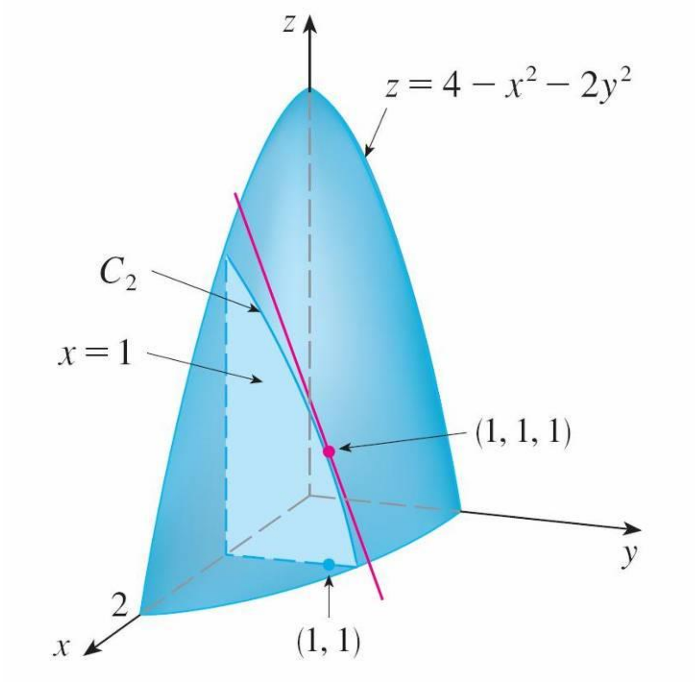


Figure 3

Example 5

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz + 4 = 0.$$

Then evaluate these partial derivatives at the point (-1,1,2).

Functions of More Than Two Variables

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if *f* is a function of three variables *x*, *y*, and *z*, then its partial derivative with respect to *x* is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding *y* and *z* as constants and differentiating *f*(*x*, *y*, *z*) with respect to *x*.

Functions of More Than Two Variables

If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Example 6

Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Higher Derivatives

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f.

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

Higher Derivatives

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y\,\partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example 7

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Higher Derivatives

Notice that $f_{xy} = f_{yx}$ in Example 6. This is not just a coincidence.

It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice.

The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Higher Derivatives

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial y^2\,\partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Partial Differential Equations

Partial Differential Equations

Partial derivatives occur in *partial differential equations* that express certain physical laws.

For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** after Pierre Laplace (1749–1827).

Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

Example 9

Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

Partial Differential Equations

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

Partial Differential Equations

For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as in Figure 8), then $u(x, t)$ satisfies the wave equation.

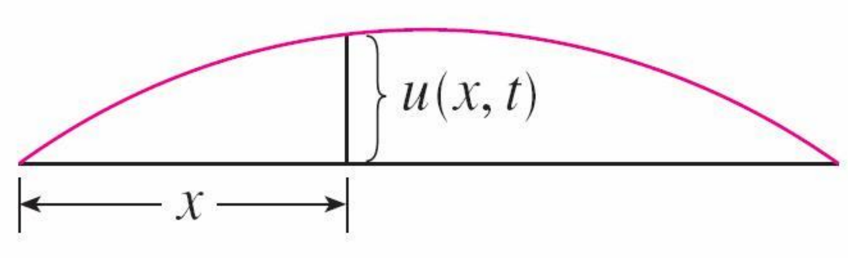


Figure 8

Here the constant a depends on the density of the string and on the tension in the string.

Example 10

Verify that the function

$$u(x, t) = \sin(x - at)$$

satisfies the wave equation.

14.4 Tangent Planes and Linear Approximations

Tangent Planes

Tangent Planes

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_o)$ be a point on S.

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2.

Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

Tangent Planes

Then the **tangent plane** to the surface *S* at the point *P* is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)

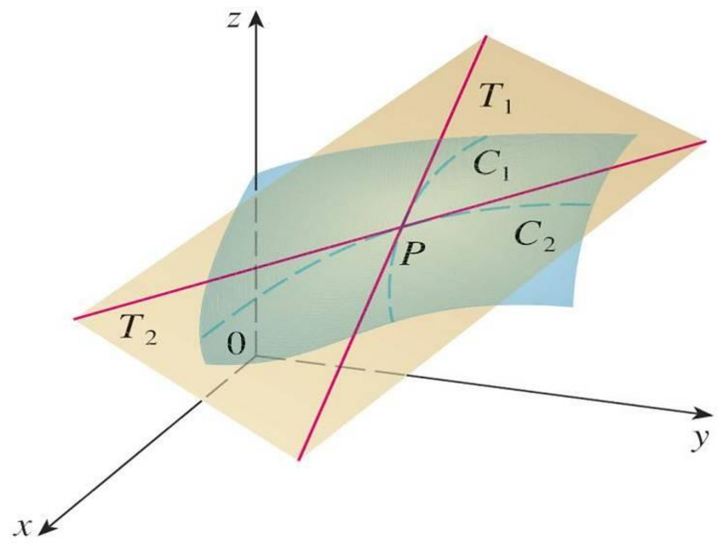


Figure 1

The tangent plane contains the tangent lines T_1 and T_2.

Tangent Planes

If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane.

Therefore, you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Tangent Planes

Here the vector <u>*n*</u>=(*A*,*B*,*C*) is a normal vector of the plane.

We may parameterize the curve C_1 using the vector function

$$\mathbf{r}_1(x) = \langle x, y_0, f(x, y_0)\rangle$$

and parameterize the curve C_2 using the vector function

$$\mathbf{r}_2(y) = \langle x_0, y, f(x_0, y)\rangle$$

Then the tangent vector of $\boldsymbol{r}_1$ at the point $(x_0, y_0, f(x_0, y_0)$ is

$$\mathbf{r}_1'(x_0) = \langle 1,0, f_x(x_0, y_0)\rangle$$

and the tangent vector of $\boldsymbol{r}_1$ at the point $(x_0, y_0, f(x_0, y_0)$ is

$$\mathbf{r}_2'(y_0) = \left\langle 0,1, f_y(x_0, y_0)\right\rangle$$

Tangent Planes

Hence a normal vector of the tangent plane is given by

$$\mathbf{n} = \mathbf{r}_1'(x_0) \times \mathbf{r}_2'(y_0) = \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \rangle$$

Hence we arrive at the following formula.

$\boxed{2}$ Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

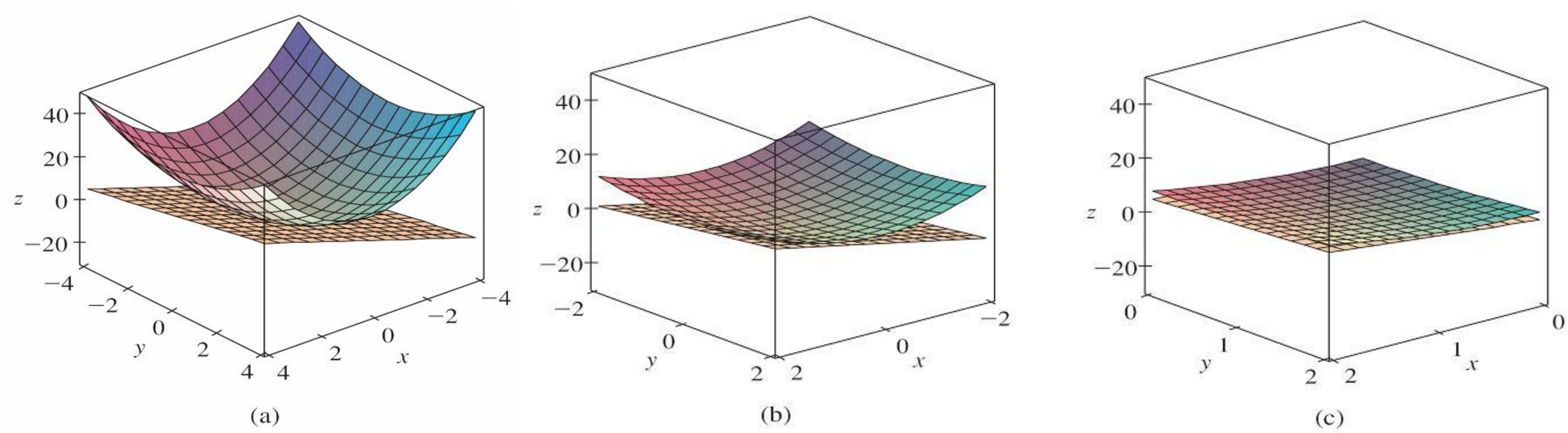
$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

Tangent Planes

Figure 2(a) shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1. In parts (b) and (c) we zoom in toward the point (1, 1, 3) by restricting the domain of the function $f(x, y) = 2x^2 + y^2$.



The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

Figure 2

Linear Approximations

Linear Approximations

In Example 1 we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is $z = 4x + 2y - 3$. Therefore, the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when (x, y) is near (1, 1). The function L is called the *linearization* of f at (1, 1) and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the *linear approximation* or *tangent plane approximation* of f at (1, 1).

Linear Approximations

For instance, at the point (1.1, 0.95) the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

which is quite close to the true value of

$$f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225.$$

But if we take a point farther away from (1, 1), such as (2, 3), we no longer get a good approximation.

In fact, $L(2, 3) = 11$ whereas $f(2, 3) = 17$.

Linear Approximations

In general, we know from $\boxed{2}$ that an equation of the tangent plane to the graph of a function *f* of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$$\boxed{3} \qquad L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of *f* at (a, b) and the approximation

$$\boxed{4} \qquad f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of *f* at (a, b).

Linear Approximations

We have defined tangent planes for surfaces $z = f(x, y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous? Figure 4 pictures such a function; its equation is

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

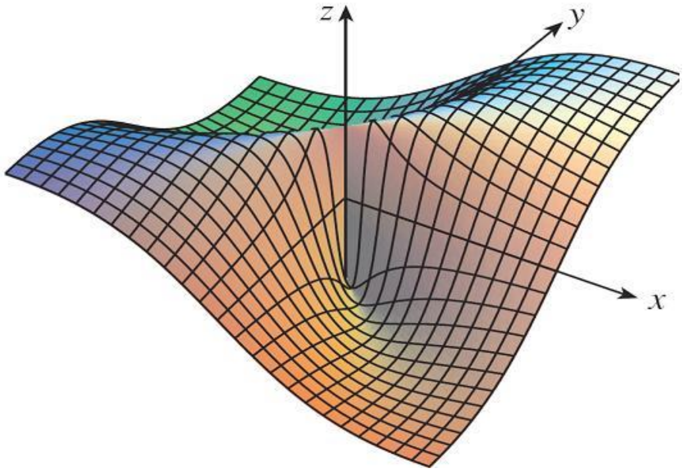


Figure 4

$f(x, y) = \dfrac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$,
$f(0, 0) = 0$

You can verify that its partial derivatives exist at the origin and, in fact, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous (even f is not continuous!).

Linear Approximations

The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$.

So a function of two variables can behave badly even though both of its partial derivatives exist.

To rule out such behavior, we formulate the idea of a differentiable function of two variables.

Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximations

If f is differentiable at a, then

5 $$\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

6 $$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Thus, the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Linear Approximations

By analogy with $\boxed{5}$ we define the differentiability of a function of two variables as follows.

$\boxed{7}$ **Definition** If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\, \Delta x + f_y(a, b)\, \Delta y + \varepsilon_1\, \Delta x + \varepsilon_2\, \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \ \Delta y) \to (0, 0)$.

Definition 7 says that a differentiable function is one for which the linear approximation $\boxed{4}$ is a good approximation when (x, y) is near (a, b).

In other words, the tangent plane approximates the graph of f well near the point of tangency.

Linear Approximations

It's sometimes hard to use Definition 7 directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example 2

Show that $f(x, y) = xe^{xy}$ is differentiable at (1,0) and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Differentials

Differentials

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable; that is, dx can be given the value of any real number.

The differential of y is then defined as

$$\boxed{9} \qquad dy = f'(x)\,dx$$

Differentials

Figure 6 shows the relationship between the increment Δy and the differential dy: Δy represents the change in height of the curve $y = f(x)$ and dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

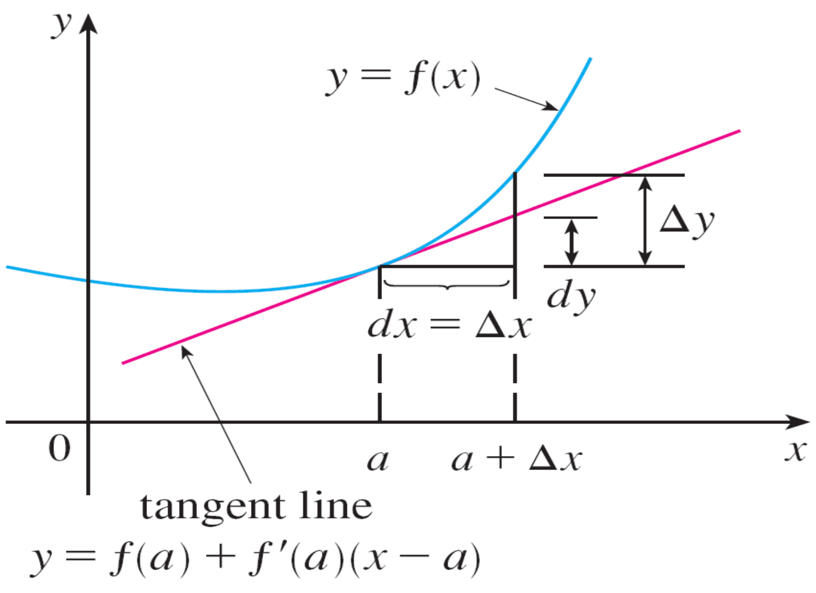


Figure 6

Differentials

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

10

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

Sometimes the notation df is used in place of dz.

Differentials

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in Equation 10, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation $\boxed{4}$ can be written as

$$f(x, y) \approx f(a, b) + dz$$

Differentials

Figure 7 is the three-dimensional counterpart of Figure 6 and shows the geometric interpretation of the differential dz and the increment Δz: dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

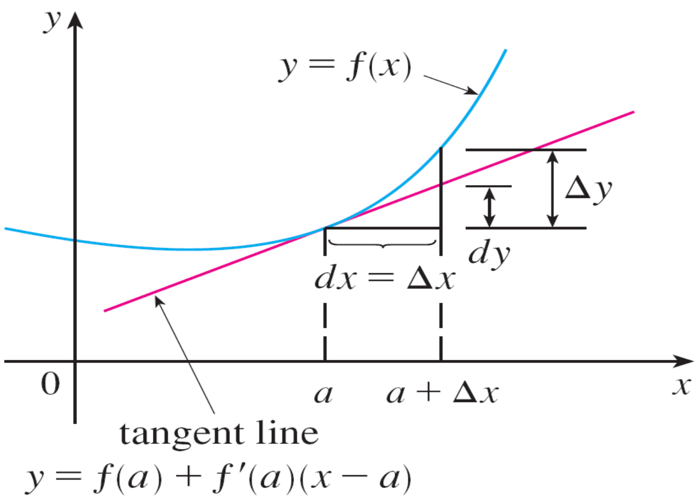


Figure 6

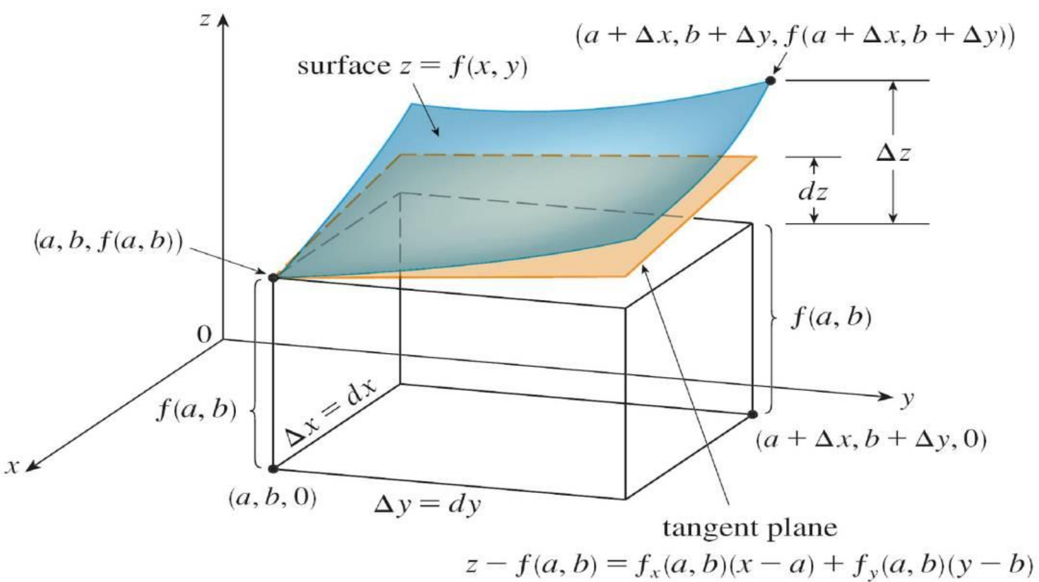
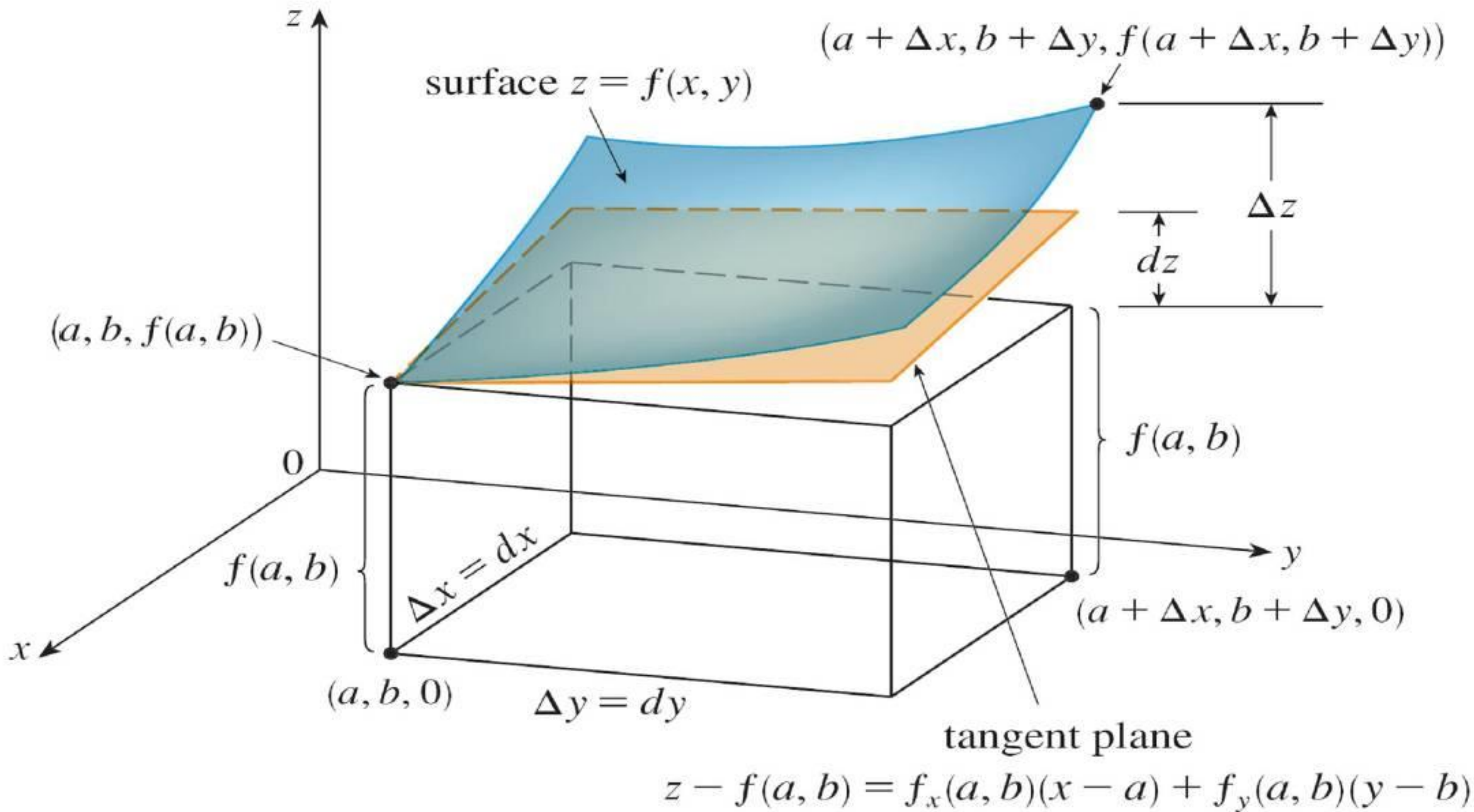


Figure 7

Differentials



Example 4

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Functions of Three or More Variables

Functions of Three or More Variables

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables. A differentiable function is defined by an expression similar to the one for functions of two variables.

Differentiability

Definition. If $w = f(x, y, z)$ is defined at and near (a, b, c), then f is **differentiable** at (a, b, c) if the increment

$\Delta w = f(a + \Delta x, b + \Delta y, c + \Delta z) - f(a, b, c)$

can be expressed near (a, b, c) in the form

$$\Delta w = f_x(a, b, c)\Delta x + f_y(a, b, c)\Delta y + f_z(a, b, c)\Delta z + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are functions of $\Delta x, \Delta y, \Delta z$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0,0,0)$.

Note: Similarly to the two variable case if the partial derivatives f_x, f_y, f_z exist near (a, b, c) and are continuous at (a, b, c), then f is differentiable at (a, b, c).

Functions of Three or More Variables

For such functions the **linear approximation** is

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

The **differential** dw is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$$

Example 6

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

14.5 The Chain Rule

The Chain Rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function:
If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\boxed{1} \qquad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

The Chain Rule

The first version (Theorem 1) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t. This means that z is indirectly a function of t, $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t.

Theorem 1 (Chain rule, case 1). Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The Chain Rule

We assume that f is differentiable. Recall that this is the case when f_x and f_y are continuous.

Since we often write $\partial z/\partial x$ in place of $\partial f/\partial x$, we can rewrite the Chain Rule in the form

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Example 1

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

A multivariable Taylor's formula

Suppose that f is a function of 2 variables and suppose that aall second order partial derivatives exist in an open disc D with center $\mathbf{a} = (a_1, a_2)$ and are continuous at **a**. If $\mathbf{x} = (x_1, x_2) \in D,$ then there is **b** on the line segment joining **a** and **x** such that

$$f(\mathbf{x}) = f(\mathbf{a}) + f_x(\mathbf{a})(x_1 - a_1) + f_y(\mathbf{a})(x_2 - a_2)$$
$$+\frac{1}{2}\left(f_{xx}(\mathbf{b})(x_1 - a_1)^2 + f_{yy}(\mathbf{b})(x_2 - a_2)^2 + 2f_{xy}(\mathbf{b})(x_1 - a_1)(x_2 - a_2)\right).$$

Remark. The above can be naturally extended to functions of n variables.

The Chain Rule

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t:

$$x = g(s, t), \quad y = h(s, t).$$

Then z is indirectly a function of s and t and we wish to find $\partial z/\partial s$ and $\partial z/\partial t$.

Recall that in computing $\partial z/\partial t$ we hold s fixed and compute the ordinary derivative of z with respect to t.

Therefore, we can apply Theorem 1 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The Chain Rule

A similar argument holds for $\partial z/\partial s$ and so we have proved the following version of the Chain Rule.

Theorem 2 (Chain rule, case 2). Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Case 2 of the Chain Rule contains three types of variables: *s* and *t* are **independent** variables, *x* and *y* are called **intermediate** variables, and *z* is the **dependent** variable.

The Chain Rule

Notice that Theorem 2 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule.

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2.

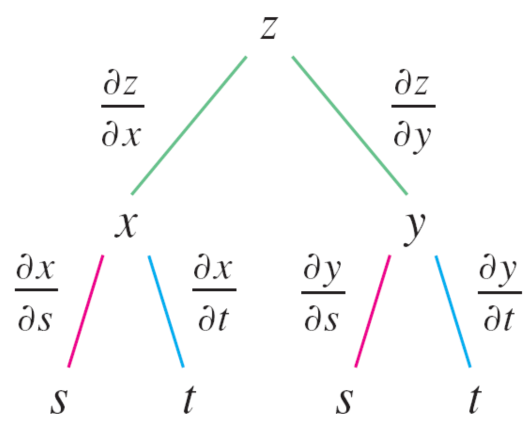


Figure 2

The Chain Rule

We draw branches from the dependent variable z to the intermediate variables x and y to indicate that z is a function of x and y. Then we draw branches from x and y to the independent variables s and t.

On each branch we write the corresponding partial derivative. To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from z to s and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Similarly, we find $\partial z/\partial t$ by using the paths from z to t.

Example 3

If $z = e^x \sin y$ where $x = st^2$ and $y = s^2t$ find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

The Chain Rule

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, \ldots, x_n$, each of which is, in turn, a function of m independent variables $t_1, \ldots, t_m$. Notice that there are n terms in the Theorem, one for each intermediate variable.

Theorem (Chain rule, general version). Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Examples 4 and 5

Example 4. Write out the chain rule for the case where and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

Example 5. If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$ and $t = 0$.

Implicit Differentiation

Implicit Differentiation

The Chain Rule can be used to give a more complete description of the process of implicit differentiation. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f.

If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x.

Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Implicit Differentiation

But *dx*/*dx* = 1, so if $\partial F/\partial y \neq 0$ we solve for *dy*/*dx* and obtain

$$\boxed{\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

To derive this equation, we assumed that $F(x, y) = 0$ defines *y* implicitly as a function of *x*.

Implicit Function Theorem

The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid:

It states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a differentiable function of x near the point (a, b) and the derivative of this function shown in the previous slide.

Example 8

Find y' if $x^3 + y^3 = 6xy$.

Implicit Differentiation

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Implicit Differentiation

But $\dfrac{\partial}{\partial x}(x) = 1$ and $\dfrac{\partial}{\partial x}(y) = 0$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} = -\frac{F_x}{F_z}.$$

Implicit Function Theorem

Similarly

$$\frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Again, a version of the **Implicit Function Theorem** gives conditions under which our assumption is valid:

If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given as above.

Example 9

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz + 4 = 0$.

14.6 Directional Derivatives and the Gradient Vector

Directional Derivatives and the Gradient Vector

In this section we introduce a type of derivative, called a *directional derivative,* that enables us to find the rate of change of a function of two or more variables in any direction.

Directional Derivatives

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$\boxed{1} \qquad \begin{aligned} f_x(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\ f_y(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \end{aligned}$$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors **i** and **j**.

Directional Derivatives

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.)

To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S.

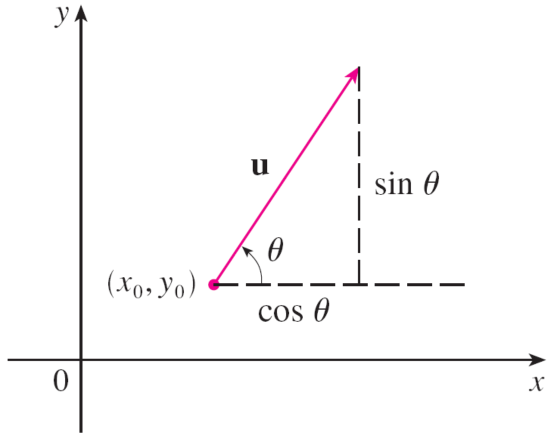


Figure 2

A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$

Directional Derivatives

The vertical plane that passes through *P* in the direction of **u** intersects *S* in a curve *C*. (See Figure 3.)

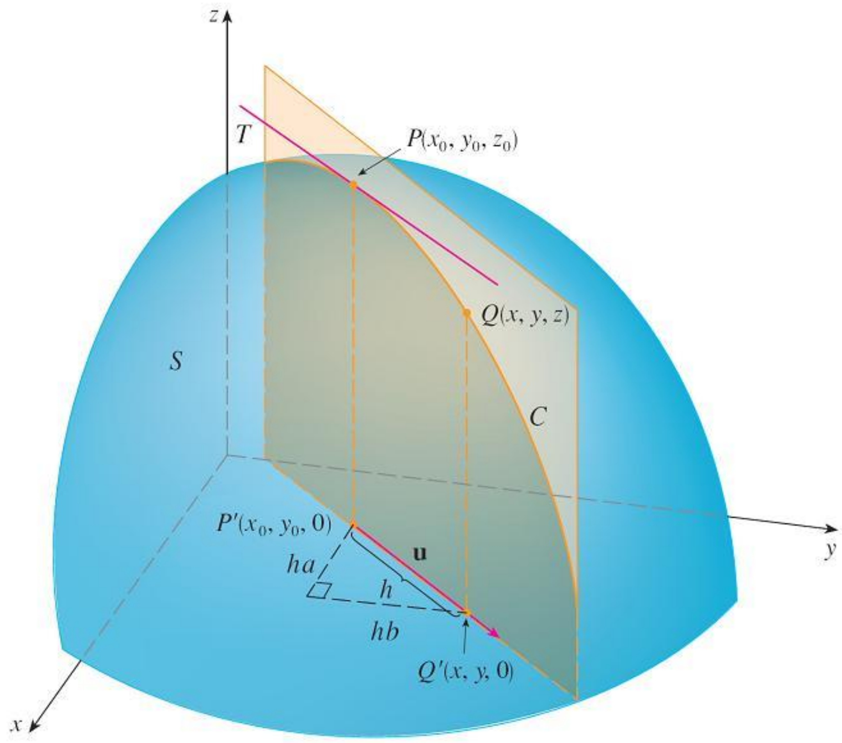


Figure 3

Directional Derivatives

The slope of the tangent line *T* to *C* at the point *P* is the rate of change of *z* in the direction of **u**.

The parametric equations of the line in the *xy*-plane through the point (x_0, y_0) with direction vector $\mathbf{u} = \langle a, b \rangle$ is given by $x = x_0 + ta$ and $y = y_0 + tb$ with t being a real parameter.

Then C has parameterization

$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = f(x_0 + ta, y_0 + tb)$$

and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Directional Derivatives

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of **u**, which is called the directional derivative of f in the direction of **u**.

$\boxed{2}$ **Definition** The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Directional Derivatives

By comparing Definition 2 with Equations $\boxed{1}$, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}}f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}}f = f_y$.

In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

Directional Derivatives

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

$\boxed{3}$ **Theorem** If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Directional Derivatives

If the unit vector **u** makes an angle θ with the positive x-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

$$\boxed{6} \qquad D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

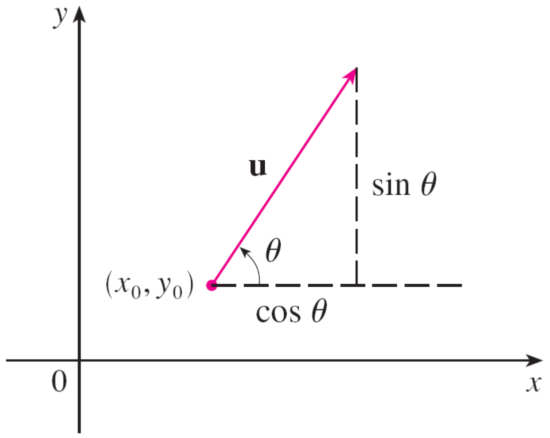


Figure 2

A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$

Example 2

Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^4$$

and u is the unit vector in the direction given by the angle $\theta = \frac{\pi}{6}$, measured from the positive *x*-axis. What is $D_{\mathbf{u}}f(1, 2)$?

The Gradient Vectors

The Gradient Vectors

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\boxed{7} \qquad \begin{aligned} D_{\mathbf{u}}f(x, y) &= f_x(x, y)a + f_y(x, y)b \\ &= \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle \\ &= \langle f_x(x, y), f_y(x, y)\rangle \cdot \mathbf{u} \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well.

So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f, which is read "del f").

The Gradient Vectors

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example 3

If $f(x, y) = \sin x + e^{xy}$, then find $\nabla f(x, y)$ and $\nabla f(0, 1)$.

The Gradient Vectors

With this notation for the gradient vector, we can rewrite the expression for the directional derivative of a differentiable function as

$$\boxed{9} \qquad \boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$$

This expresses the directional derivative in the direction of **u** as the scalar projection of the gradient vector onto **u**.

Functions of Three Variables

Functions of Three Variables

For functions of three variables we can define directional derivatives in a similar manner.

Again $D_{\mathbf{u}}f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector **u**.

$\boxed{10}$ **Definition** The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Functions of Three Variables

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

11

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$.

This is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + h\,\mathbf{u}$ and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

Functions of Three Variables

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then

12 $$D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

13 $$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}}$$

Functions of Three Variables

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

$$\boxed{14} \qquad \boxed{D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}}$$

Note. Both Definition 11 and formula (14) can be gene-ralized to functions of *n* variables in a straightforward fashion.

Example 5

If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at (1, 3, 0) in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Maximizing the Directional Derivatives

Maximizing the Directional Derivatives

Suppose we have a function *f* of two or three variables and we consider all possible directional derivatives of *f* at a given point.

These give the rates of change of *f* in all possible directions.

We can then ask the questions: In which of these directions does *f* change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

$\boxed{15}$ **Theorem** Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Example 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Tangent Planes to Level Surfaces

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S.

Let C be any curve that lies on the surface S and passes through the point P. Recall that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$\boxed{16} \qquad F(x(t), y(t), z(t)) = k$$

Tangent Planes to Level Surfaces

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

$$\boxed{17} \qquad \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

Tangent Planes to Level Surfaces

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

$$\boxed{18} \qquad \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

Equation 18 says that *the gradient vector* $\nabla F(x_0, y_0, z_0)$ *at* P *is perpendicular to the tangent vector* $\mathbf{r}'(t_0)$ *to any curve* C *on* S *that passes through* P. (See Figure 9.)

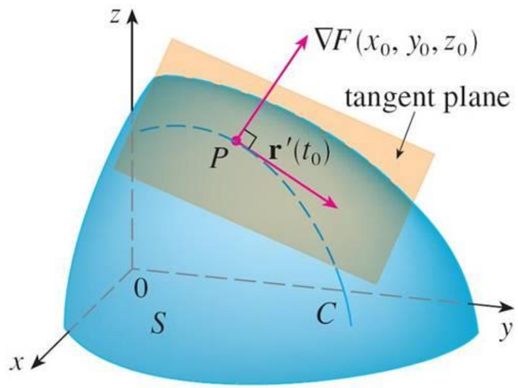


Figure 9

Tangent Planes to Level Surfaces

If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

Using the standard equation of a plane, we can write the equation of this tangent plane as

$$\boxed{19} \quad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Tangent Planes to Level Surfaces

The **normal line** to *S* at *P* is the line passing through *P* and perpendicular to the tangent plane.

The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, its symmetric equations are

$$\boxed{20} \qquad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Tangent Planes to Level Surfaces

Note: In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$
$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$
$$F_z(x_0, y_0, z_0) = -1$$

so we get the familiar equation of the tangent plane

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Example 8

Find the equations of the tangent plane and normal line at the point (–2, 1, –3) to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Significance of the Gradient Vectors

Significance of the Gradient Vectors

We now summarize the ways in which the gradient vector is significant.

We first consider a function *f* of three variables and a point $P(x_0, y_0, z_0)$ in its domain.

On the one hand, we saw that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of *f*.

On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface *S* of *f* through *P*. (Refer to Figure 10.)

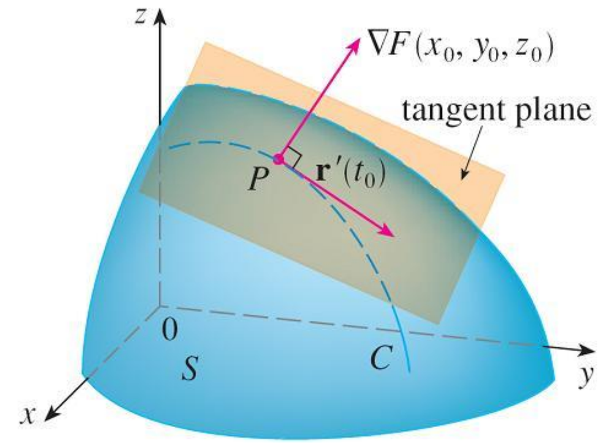


Figure 10

Significance of the Gradient Vectors

These two properties are quite compatible intuitively because as we move away from P on the level surface S, the value of f does not change at all.

So, it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain.

Again, the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f. Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P.

Significance of the Gradient Vectors

Again this is intuitively plausible because the values of *f* remain constant as we move along the curve.
(See Figure 12.)

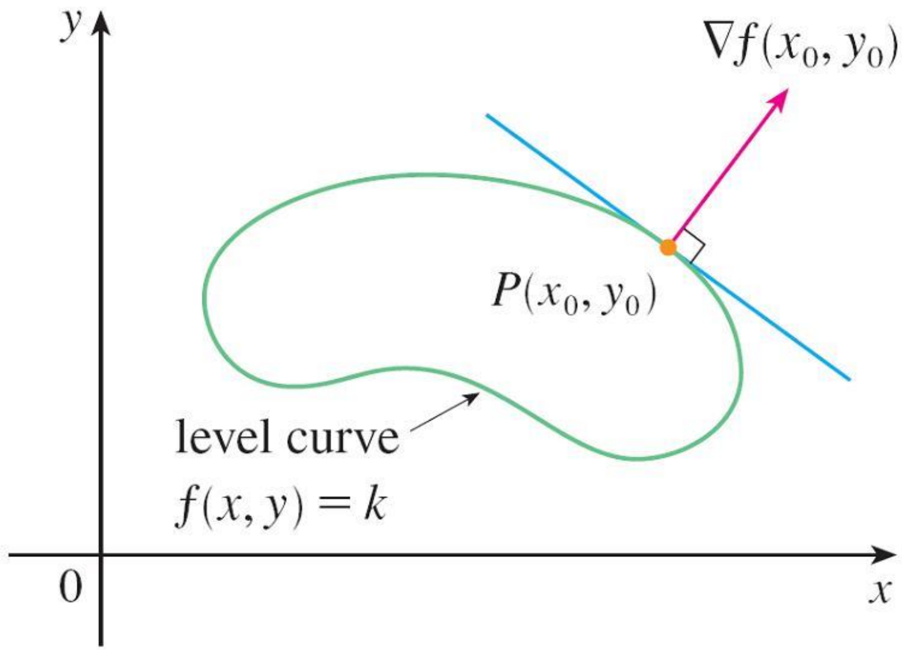


Figure 12

Significance of the Gradient Vectors

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn as in Figure 13 by making it perpendicular to all of the contour lines.

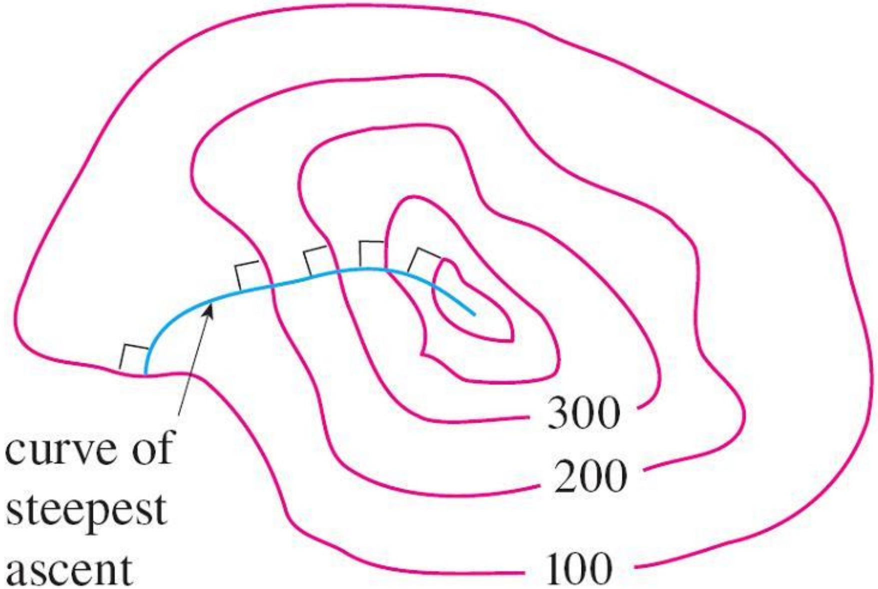


Figure 13

14.7 Maximum and Minimum Values

Maximum and Minimum Values

In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables.

Look at the hills and valleys in the graph of *f* shown in Figure 1.

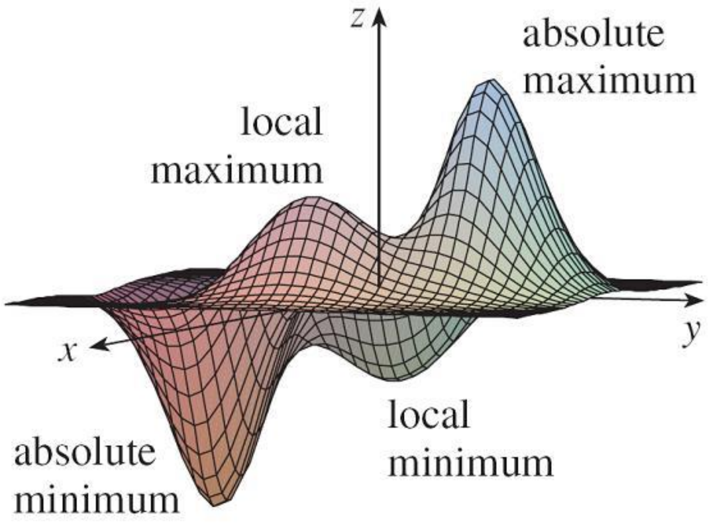


Figure 1

Maximum and Minimum Values

There are two points (a, b) where f has a *local maximum*, that is, where f(a, b) is larger than nearby values of f(x, y).

The larger of these two values is the *absolute maximum*.

Likewise, f has two *local minima*, where f(a, b) is smaller than nearby values.

The smaller of these two values is the *absolute minimum*.

Maximum and Minimum Values

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

2 Fermat's Theorem for Functions of Two Variables If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Note: in this case, the tangent plane is horizontal at (a,b).

Maximum and Minimum Values

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f.

However, as in single-variable calculus, not all critical points give rise to maxima or minima.

At a critical point, a function could have a local maximum or a local minimum or neither.

Example 1

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Find the local and absolute extreme values of f.

Example 1

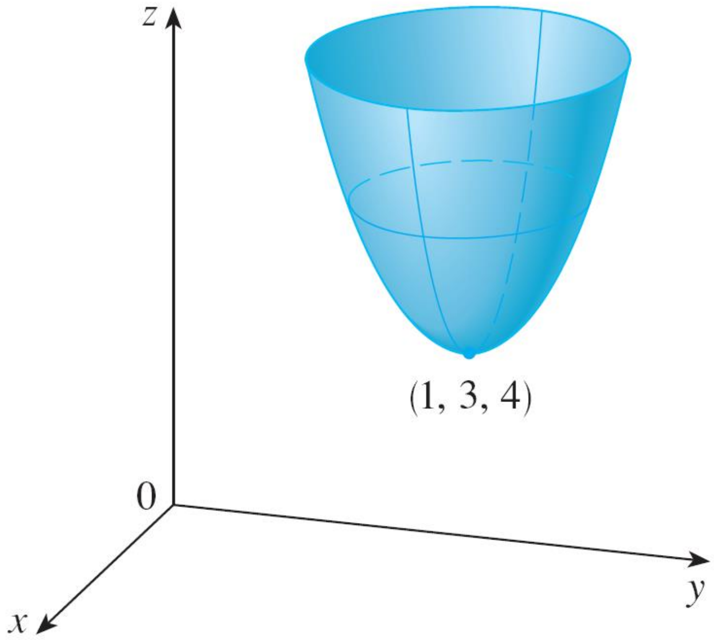


Figure 2

$$z = x^2 + y^2 - 2x - 6y + 14$$

Maximum and Minimum Values

The following test, is analogous to the Second Derivative Test for functions of one variable.

$\boxed{3}$ **Second Derivatives Test** Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

Second derivative test

To remember de formula for *D,* it is helpful to rewrite it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

Second derivative test

Note. If we introduce the symmetric matrix (so-called Hessian matrix of *f*)

$$A = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix},$$

then condition (a) is equivalent of the matrix A being positive definite, (b) is equivalent of the matrix A being negative definite, while (c) is equivalent of the matrix A being indefinite.

With these notions a similar statement holds in *n* dimensions considering the corresponding *n*x*n* Hessian matrix.

Functions of *n*-variables

More precisely, analogously to the case of functions of two variables, the notion of local/global extrema can be defined for functions of *n*-variables. Fermat's Theorem still holds: if *f* has a local extreme value at $(a_1, a_2, \ldots, a_n)$ and the first order partial derivatives of *f* exist there, then

$$f_1(a_1, \ldots, a_n) = \cdots = f_n\,(a_1, \ldots, a_n) = 0.$$

If we introduce the Hessian of *f* :

$$A = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix},$$

then the second derivative test still applies, if we formulate it in terms of the definiteness of *A* (see previous slide).

Example 3

Find the local minimum and maximum values and sadle points of $f(x, y) = x^4 + y^4 - 4xy + 1$

Example 6

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Absolute Maximum and Minimum Values on bounded and closed sets

Absolute Maximum and Minimum Values

For a function *f* of one variable, the Extreme Value Theorem says that if *f* is continuous on a closed interval [*a*, *b*], then *f* has an absolute minimum value and an absolute maximum value.

According to the Closed Interval Method, we found these by evaluating *f* not only at the critical numbers but also at the endpoints *a* and *b*.

There is a similar situation for functions of two variables.

Absolute Maximum and Minimum Values

Just as a closed interval contains its endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

[A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.]

For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$).

Absolute Maximum and Minimum Values

But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)

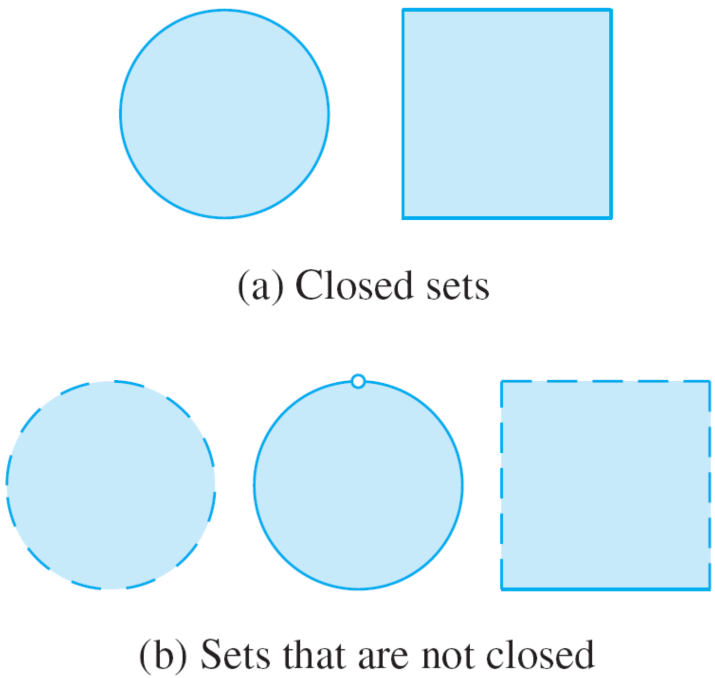


Figure 11

Absolute Maximum and Minimum Values

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk.

In other words, it is finite in extent.

Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Note. An analogous statement holds in $\mathbb{R}^n$ with the appropriate definitions of boundedness and closedness.

Absolute Maximum and Minimum Values

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value in D at (x_1, y_1), then (x_1, y_1) is either a critical point of f or a boundary point of D.

Thus, we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.

14.8 Lagrange Multipliers

Lagrange Multipliers

In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$ (both f and g are differentiable).

It's easier to explain the geometric basis of Lagrange's method for functions of two variables.

So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$.

In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$.

Lagrange Multipliers

Figure 1 shows this curve together with several level curves of f.

These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$.

To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.

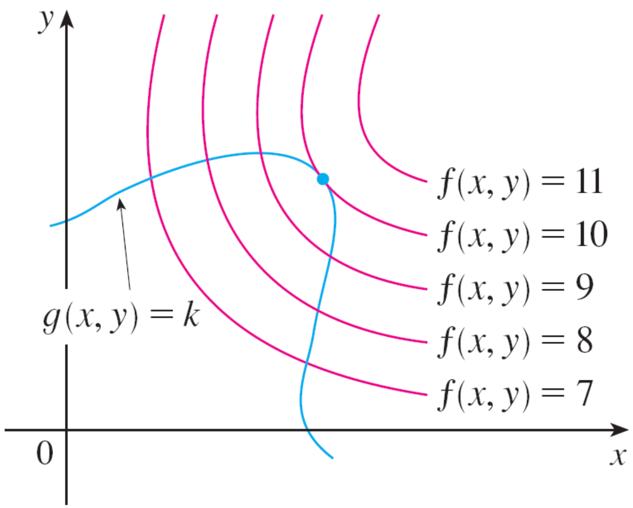


Figure 1

It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.)

Lagrange Multipliers

As gradient vectors are perpendicular to the tangent lines of level curves and f and g has a common tangent line to a corresponding level curve at (x_0, y_0), it follows that the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$.

Lagrange Multipliers

Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

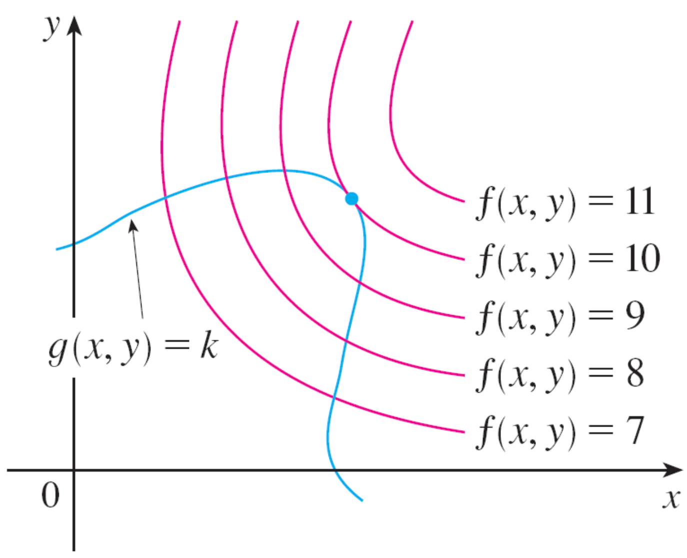


Figure 1

Lagrange Multipliers

This intuitive argument can be made precise as follows. Suppose that a differentiable function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a smooth curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on S and passes through P.

If t_0 is the parameter value corresponding to the point P, then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$.

The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C.

Lagrange Multipliers

Since f has an extreme value at (x_0, y_0, z_0), it follows that h has an extreme value at t_0, so $h'(t_0) = 0$. But f is differentiable, an $\mathbf{r}$ is smooth so we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

Thus, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C and hence orthogonal to the tangent plane of S at (x_0, y_0, z_0).

But we already know that the gradient vector of g, $\nabla g(x_0, y_0, z_0)$, is also orthogonal to the tangent plane of S at (x_0, y_0, z_0).

Lagrange Multipliers

This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

$$\boxed{1} \qquad \boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)}$$

The number λ in Equation 1 is called a **Lagrange multiplier**.

Note: if $\nabla f(x_0, y_0, z_0) = 0$, then (1) holds with $\lambda = 0$ and the point (x_0, y_0, z_0) is a possible local extreme value of f.

Lagrange Multipliers

The procedure based on Equation 1 is as follows.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Note: This method provides a necessary condition only because we assume that the extreme values exist!

Lagrange Multipliers

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z \quad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.

For functions of two variables the method of Lagrange multipliers is similar to the method just described.

Lagrange Multipliers

To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

Example 1

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $g(x, y) = x^2 + y^2 = 1$.

Example 2

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Example 3

Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are the closest to and farthest from the point $(3,1,-1)$.

Two Constraints

Two Constraints

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$ (f ,g and h are differentiable).

Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.)

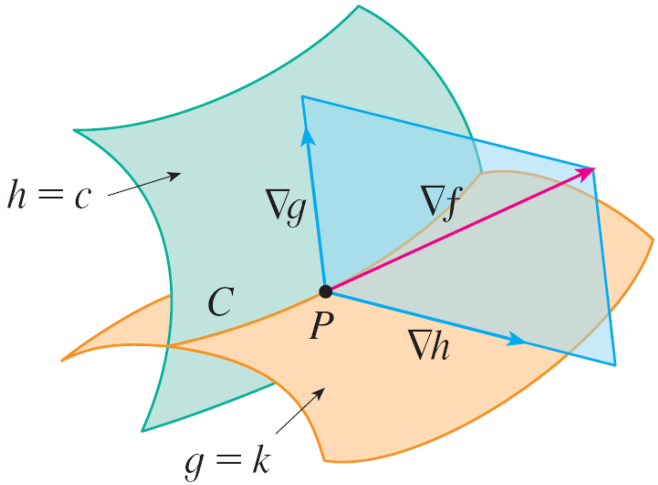


Figure 5

Two Constraints

Suppose f has such an extreme value on C at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to (the tangent vector of) C at P.

But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to (the tangent vector of) C, as C lies in both level surfaces.

This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are nonzero and non-parallel.)

Two Constraints

So there are numbers λ and μ (called Lagrange multipliers) such that

$$\boxed{16} \qquad \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns *x*, *y*, *z*, λ, and μ.

Two Constraints

These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

Example 5

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

15 Multiple Integrals

15.1 Double Integrals over Rectangles

Review of the Definite Integrals

Review of the Definite Integrals

First let's recall the basic facts concerning definite integrals of functions of a single variable.

If $f(x)$ is defined for $a \le x \le b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

1 $$\sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

2 $$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

Review of the Definite Integrals

In the special case where $f(x) \geq 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b.

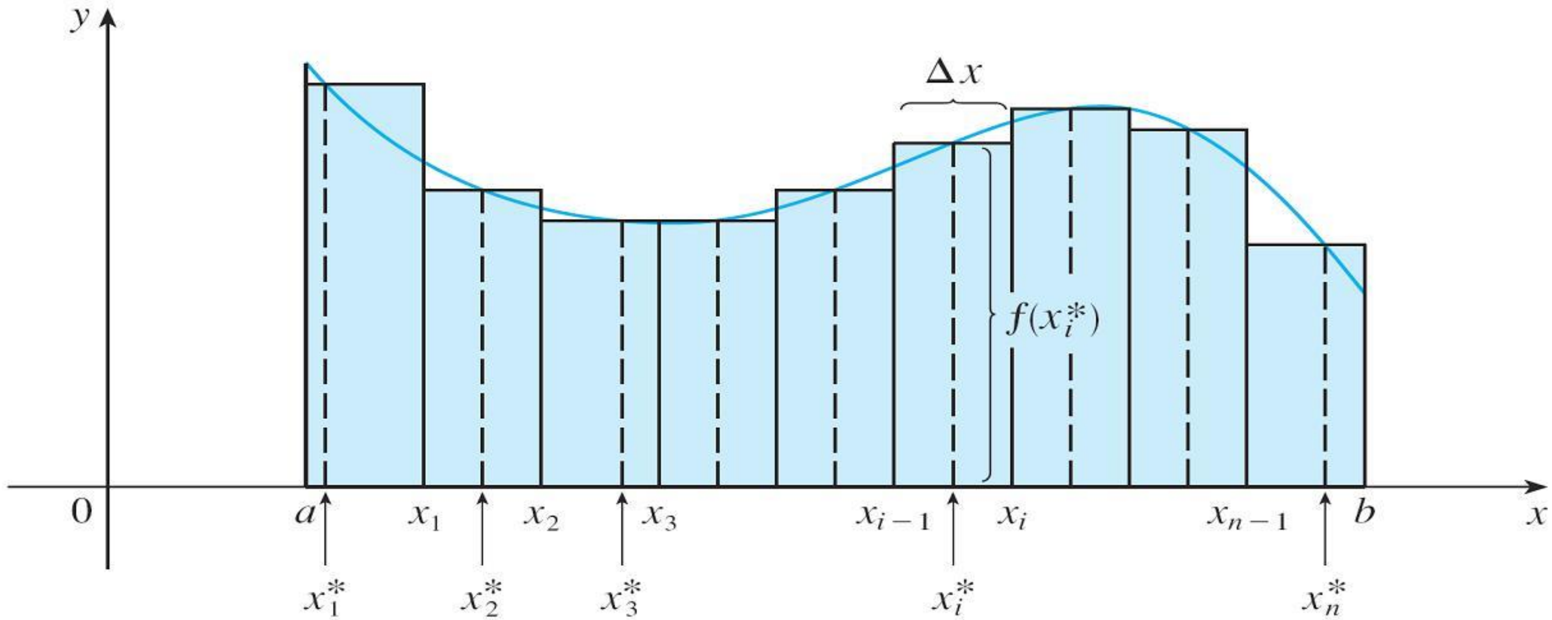


Figure 1

Volumes and Double Integrals

Volumes and Double Integrals

In a similar manner we consider a function *f* of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$$

and we first suppose that $f(x, y) \ge 0$.

The graph of *f* is a surface with equation $z = f(x, y)$.

Let *S* be the solid that lies above *R* and under the graph of *f*, that is,

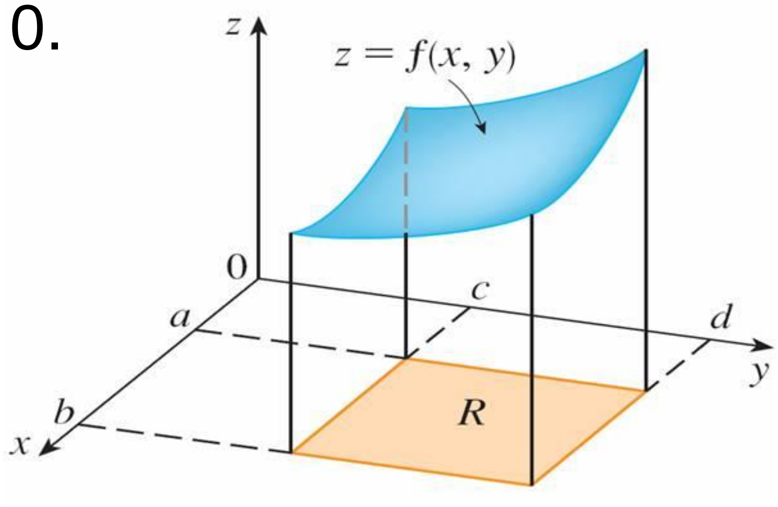


Figure 2

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le f(x, y),\ (x, y) \in R\}$$

(See Figure 2.)

Volumes and Double Integrals

Our goal is to find the volume of *S*.

The first step is to divide the rectangle *R* into subrectangles.

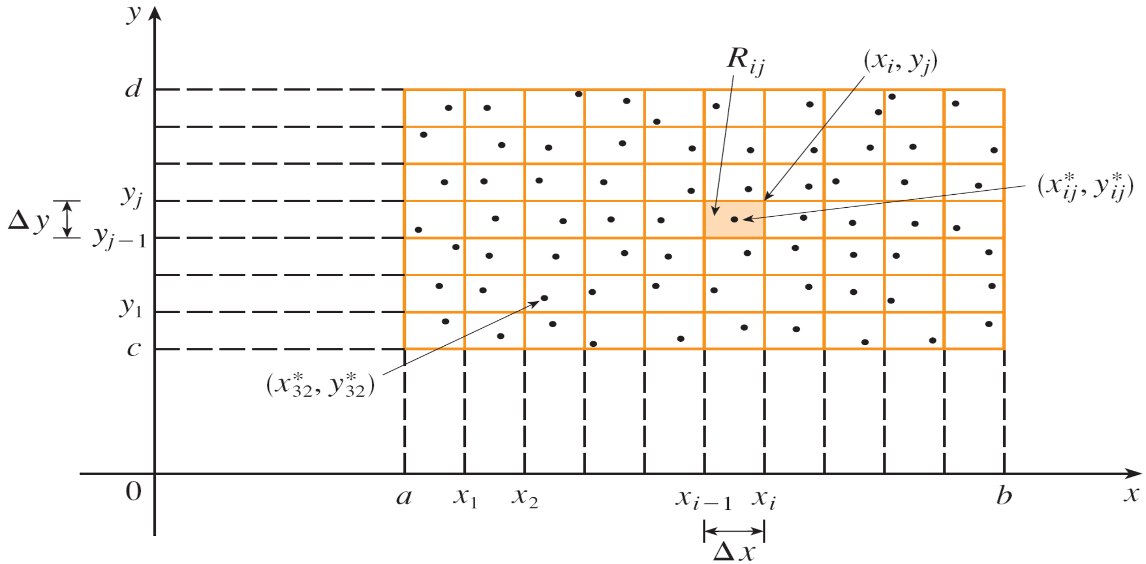
We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$.

Volumes and Double Integrals

By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.



Dividing *R* into subrectangles

Figure 3

Volumes and Double Integrals

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4.

The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\ \Delta A$$

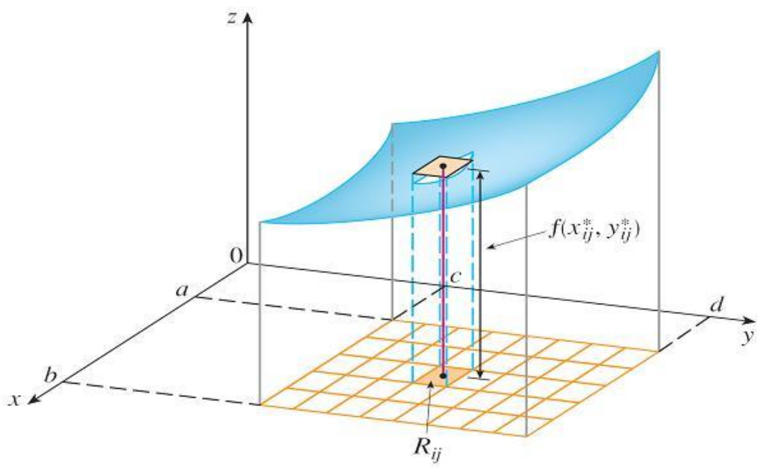


Figure 4

Volumes and Double Integrals

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of *S*:

$$\boxed{3} \qquad V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate *f* at the chosen point and multiply by the area of the subrectangle, and then we add the results.

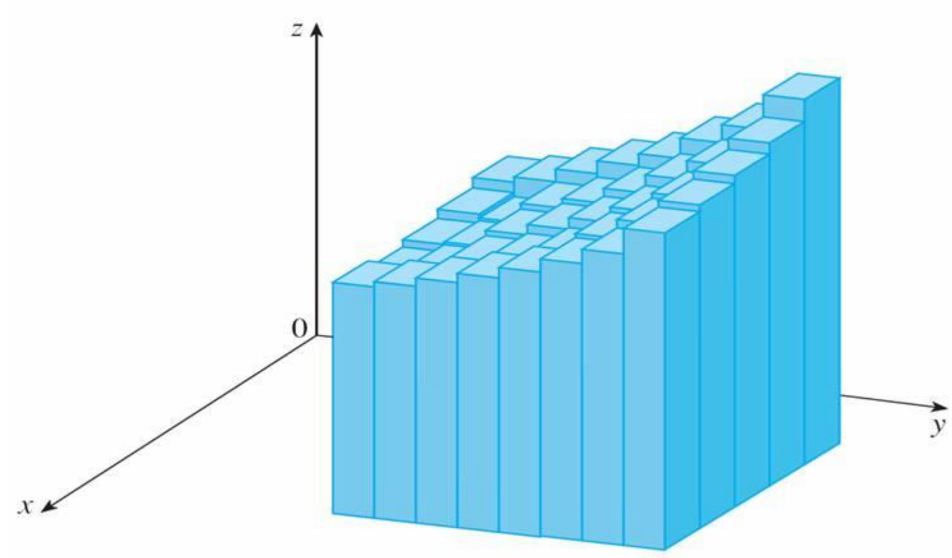


Figure 5

Volumes and Double Integrals

Our intuition tells us that the approximation given in (3) becomes better as m and n become larger and so we would expect that

$$\boxed{4} \qquad V = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

We use the expression in Equation 4 to define the volume of the solid *S* that lies under the graph of *f* and above the rectangle *R*.

Volumes and Double Integrals

Limits of the type that appear in Equation 4 occur frequently, not just in finding volumes but in a variety of other situations even when *f* is not a positive function. So we make the following definition.

5 Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

A function *f* is called **integrable** if the limit in Definition 5 exists.

Note: There is a precise definition of the limit on the RHS.

Precise Definition

The limit in Definition 5 means, that for all $\varepsilon > 0$ there is an integer *N* such that

$$\left| \iint_R f(x, y)\, dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A \right| < \varepsilon$$

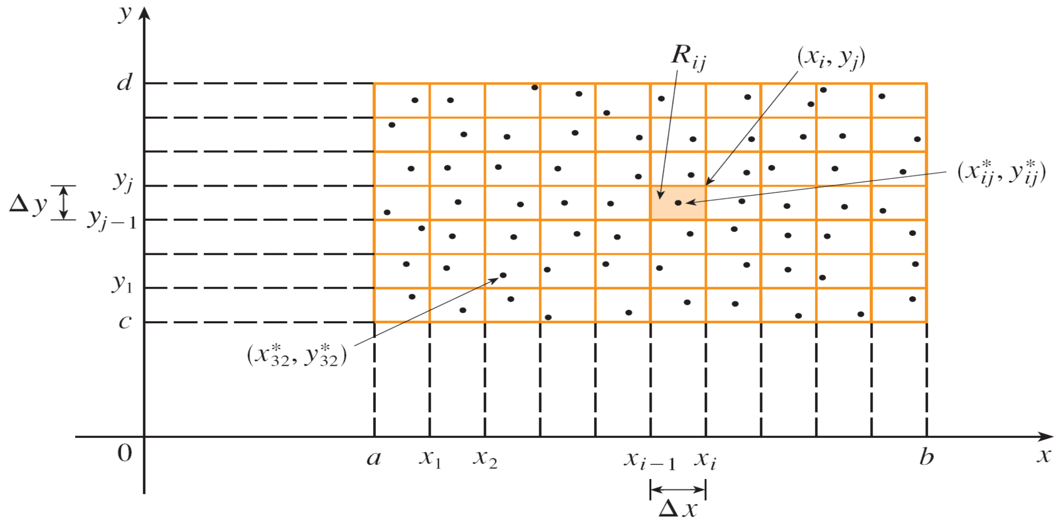
for all *m*,*n* > *N* and arbitrary (x_{ij}^*, y_{ij}^*) sample points.

Volumes and Double Integrals

It is shown in courses on advanced calculus that all continuous functions are integrable. In fact, the double integral of f exists provided that f is “not too discontinuous.”

In particular, if f is bounded [that is, there is a constant M such that $|f(x, y)| \leq M$ for all (x, y) in R], and f is continuous there, except on a finite number of smooth curves, then f is integrable over R.

Volumes and Double Integrals



Dividing *R* into subrectangles

Figure 3

The sample point (x_{ij}^*, y_{ij}^*) can be chosen to be any point in the subrectangle R_{ij}, but if we choose it to be the upper right-hand corner of R_{ij} [namely (x_i, y_j), see Figure 3], then the expression for the double integral looks simpler:

6

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

Volumes and Double Integrals

By comparing Definitions 4 and 5, we see that a volume can be written as a double integral:

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

Volumes and Double Integrals

The sum in Definition 5,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in $\boxed{1}$ for a function of a single variable.]

Volumes and Double Integrals

If *f* happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume under the graph of *f*.

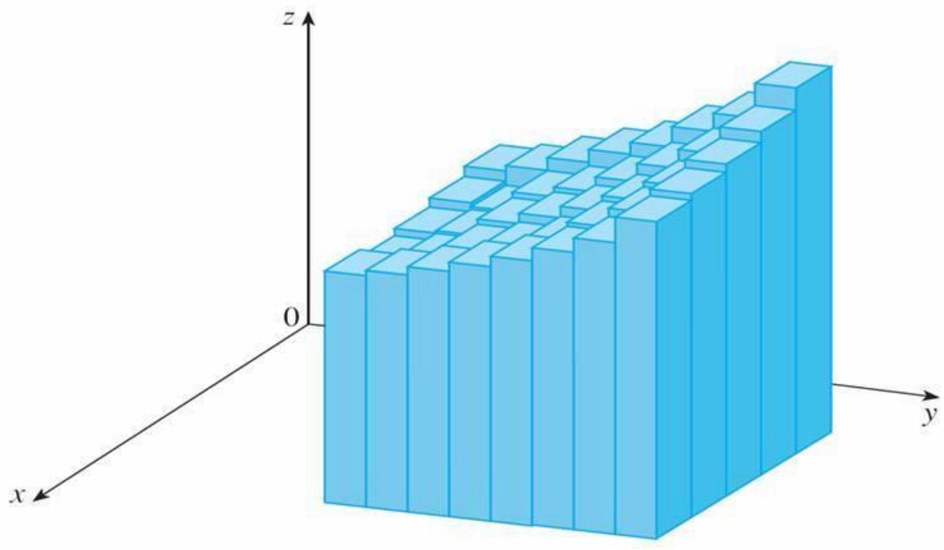


Figure 5

Example 1

Estimate the volume of the solid that lies above the square R = [0, 2] × [0, 2] and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}.

Example 1

This is the volume of the approximating rectangular boxes shown in Figure 7.

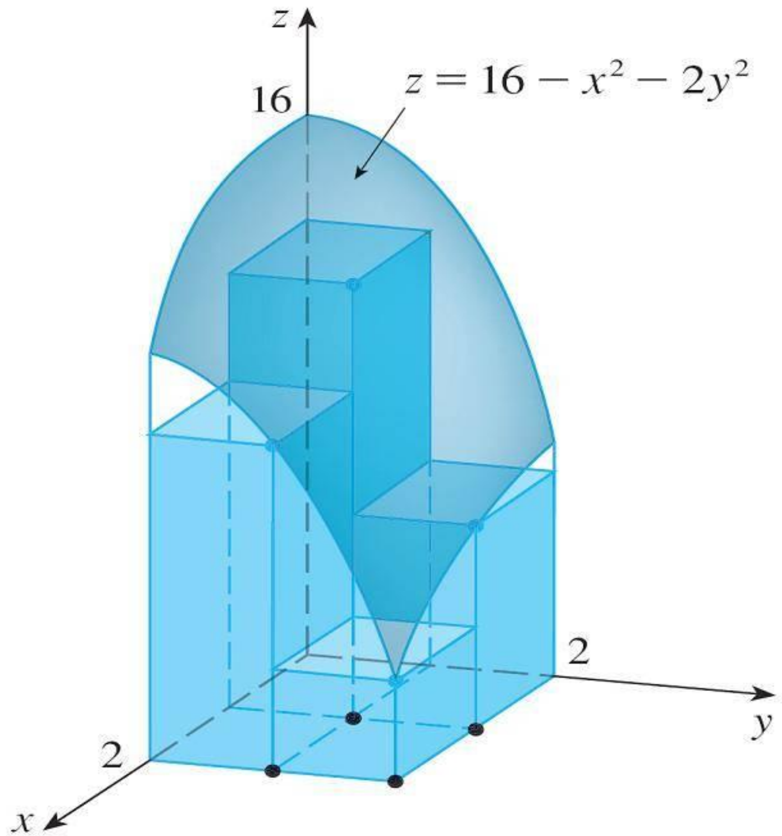


Figure 7

The Midpoint Rule

The Midpoint Rule

The methods that can be used for approximating single integrals (such as the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) all have counterparts for double integrals. Here we consider only the Midpoint Rule for double integrals.

This means that we use a double Riemann sum to approximate the double integral, where the sample point (x_{ij}^*, y_{ij}^*) in R_{ij} is chosen to be the center $(\overline{x}_i, \overline{y}_j)$ of R_{ij}. In other words, $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

The Midpoint Rule

Midpoint Rule for Double Integrals

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\, \Delta A$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Iterated Integrals

Iterated Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

We use the notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$ (if it is integrable as a one variable function). This procedure is called *partial integration with respect to y*.

Now $\int_c^d f(x, y)\, dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\, dy$$

Iterated Integrals

If we now integrate the function *A* with respect to *x* from $x = a$ to $x = b$ (if it is integrable), we get

(7) $$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$$

The integral on the right side of Equation 7 is called an **iterated integral**. Usually the brackets are omitted. Thus

(8) $$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$$

means that we first integrate with respect to *y* from *c* to *d* and then with respect to *x* from *a* to *b*.

Iterated Integrals

Similarly, the iterated integral

(9) $$\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx \right] dy$$

means that we first integrate with respect to *x* (holding *y* fixed) from *x* = *a* to *x* = *b* and then we integrate the resulting function of *y* with respect to *y* from *y* = *c* to *y* = *d*.

Notice that in both Equations 8 and 9 we work *from the inside out*.

Example 4

Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

(b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Iterated Integrals

Notice that in Example 4 we obtained the same answer whether we integrated with respect to y or x first.

In general, it turns out (see Theorem 4) that the two iterated integrals in Equations 8 and 9 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

Iterated Integrals

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

Fubini's Theorem. If f is continuous on the rectangle $R = \{(x, y) \mid a \le x \le b, c \le y \le d\}$, then both iterated integrals $\int_a^b \int_c^d f(x, y)\, dy\, dx$ and $\int_c^d \int_a^b f(x, y)\, dx\, dy$ exist and

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

Fubini’s Theorem

Note: More generally, if we assume that f is bounded on *R*, *f* is discontinuous only at a finite number of smooth curves and the iterated integrals exits then the equality of the three integrals hold.

Iterated Integrals

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form.

To be specific, suppose that $f(x, y) = g(x)h(y)$ is continuous and $R = [a, b] \times [c, d]$.

Then Fubini's Theorem gives

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

Iterated Integrals

In the inner integral, *y* is a constant, so *h*(*y*) is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)\,h(y)\,dx \right] dy = \int_c^d \left[h(y)\left(\int_a^b g(x)\,dx \right) \right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant.

Therefore, in this case, the double integral of *f* can be written as the product of two single integrals:

$$\iint_R f(x,y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy, \quad R = [a,b]\times[c,d].$$

Examples 5 - 8

Example 5. Evaluate the double integral $\iint_R (x - 3y^2)\, dA$, where $R = \{(x, y) \mid 0 \leq x \leq 2,\ 1 \leq y \leq 2\}$.

Example 6. Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1,2]\times[0,\pi]$.

Example 7. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$ and the three coordinate planes.

Example 8. Evaluate $\iint_R \sin x \cos y\, dA$, $R = [0,\pi/2] \times [0,\pi/2]$.

Average Values

Average Values

Recall that the average value of a function *f* of one variable defined on an interval [*a*, *b*] is

$$f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

In a similar fashion we define the **average value** of a function *f* of two variables defined on a rectangle *R* to be

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where *A*(*R*) is the area of *R*.

Average Values

If $f(x, y) \geq 0$, the equation

$$A(R) \times f_{\text{ave}} = \iint_R f(x, y)\, dA$$

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

[If $z = f(x, y)$ describes a mountainous region and you chop off the tops of the mountains at height f_{ave}, then you can use them to fill in the valleys so that the region becomes completely flat. See Figure 11.]

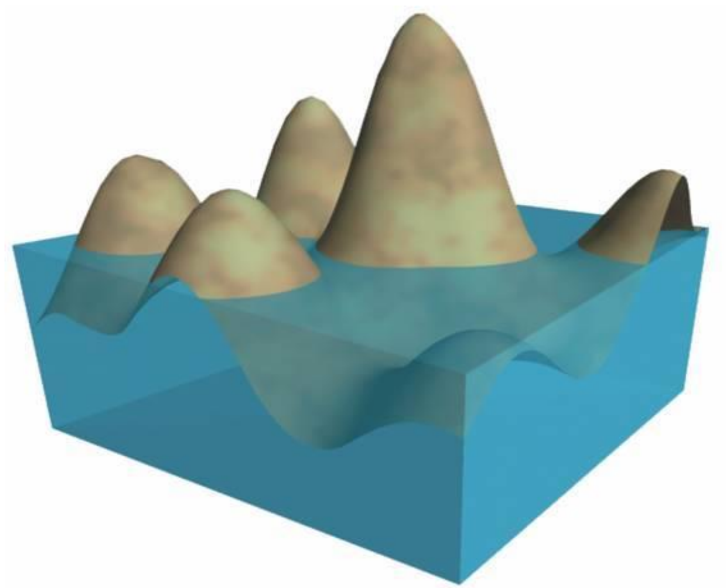


Figure 11

Properties of Double Integrals

Properties of Double Integrals

We list here three properties of double integrals. We assume that *f* and *g* are integrable on *R*. Properties 7 and 8 are referred to as the *linearity* of the integral, while Property 9 is called the *monotonicity* of the integral.

7 $$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

8 $$\iint_R c\, f(x, y)\, dA = c \iint_R f(x, y)\, dA \quad \text{where } c \text{ is a constant}$$

If $f(x, y) \geq g(x, y)$ for all (x, y), then

9 $$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$$

15.6 Triple Integrals

Triple Integrals

We have defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.

Let's first deal with the simplest case where f is defined on a rectangular box:

1 $$B = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz.

Triple Integrals

The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1.

Each sub-box has volume $\Delta V = \Delta x\, \Delta y\, \Delta z$.

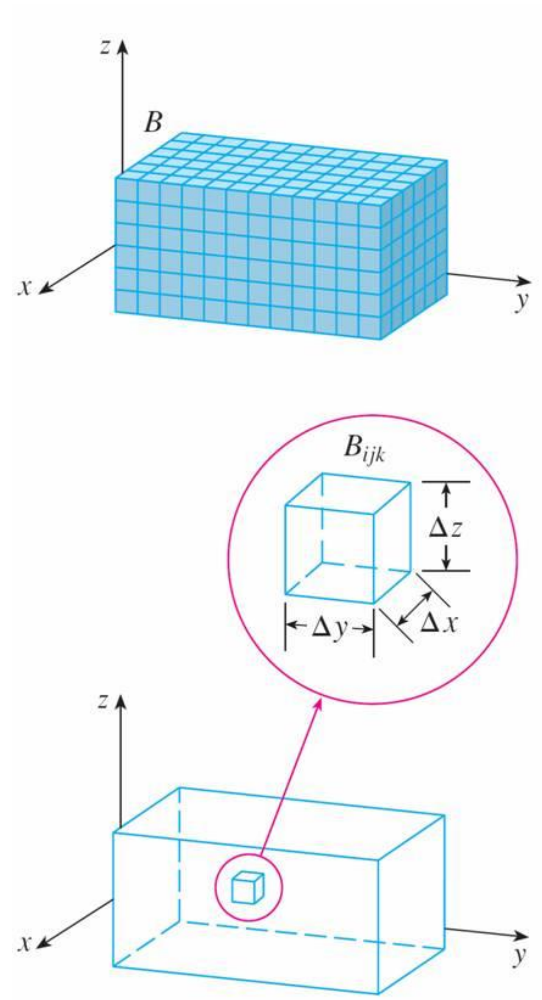


Figure 1

Triple Integrals

Then we form the **triple Riemann sum**

$$\boxed{2} \qquad \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}.

By analogy with the definition of a double integral, we define the triple integral as the limit of the triple Riemann sums in (2).

Triple Integrals

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, \Delta V$$

Triple Integrals

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

$\boxed{4}$ **Fubini's Theorem for Triple Integrals** If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to *x* (keeping *y* and *z* fixed), then we integrate with respect to *y* (keeping *z* fixed), and finally we integrate with respect to *z*.

Triple Integrals

There are five other possible orders in which we can integrate, all of which give the same value.

For instance, if we integrate with respect to *y*, then *z*, and then *x*, we have

$$\iiint_B f(x, y, z)\, dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx$$

Example 1

Example 1. Evaluate the triple integral $\iiint_B xyz^2 \, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

Triple Integrals

Now we define the **triple integral over a general bounded region *E*** in three-dimensional space (a solid) by much the same procedure that we used for double integrals.

We enclose *E* in a box *B* of the type given by Equation 1. Then we define a function *F* so that it agrees with *f* on *E* but is 0 for points in *B* that are outside *E*.

By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

This integral exists if *f* is continuous and the boundary of *E* is "reasonably smooth."

Triple Integrals

The triple integral has essentially the same properties as the double integral.

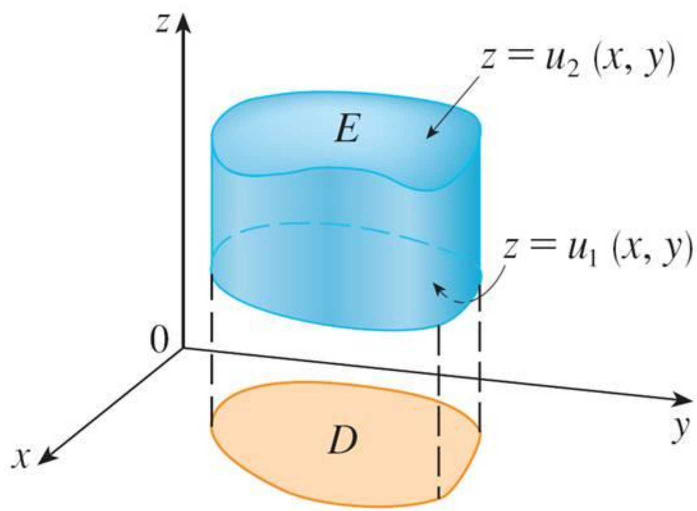
We restrict our attention to continuous functions *f* and to certain simple types of regions.

Triple Integrals

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$\boxed{5} \qquad E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 2.



A type 1 solid region

Figure 2

Triple Integrals

Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument as for double integrals, it can be shown that if E is a type 1 region given by Eq. 5, then

6 $$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

Triple Integrals

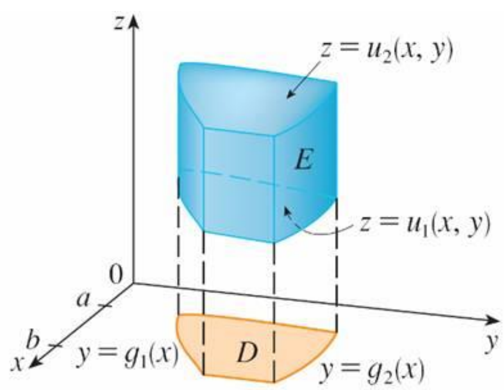
In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3), then

$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}$

and Equation 6 becomes

7

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$



A type 1 solid region where the projection D is a type I plane region

Figure 3

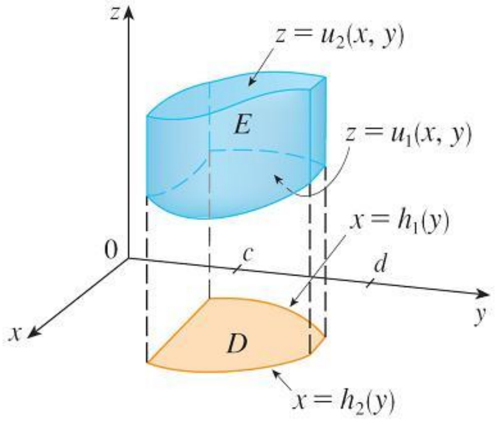
Triple Integrals

If, on the other hand, D is a type II plane region (as in Figure 4), then

$E = \{(x, y, z) \mid c \le y \le d, h_1(y) \le x \le h_2(y), u_1(x, y) \le z \le u_2(x, y)\}$

and Equation 6 becomes

$$\boxed{8} \qquad \iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$



A type 1 solid region with a type II projection

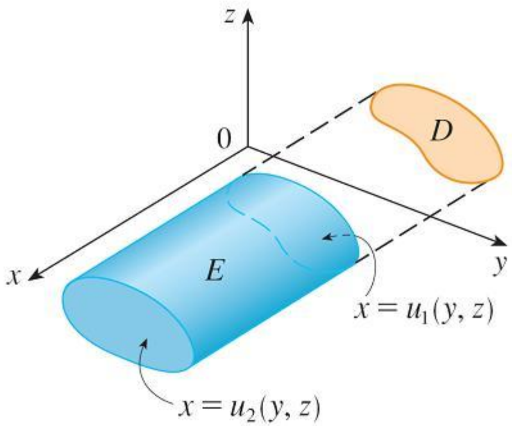
Figure 4

Triple Integrals

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 7).



A type 2 region

Figure 7

The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

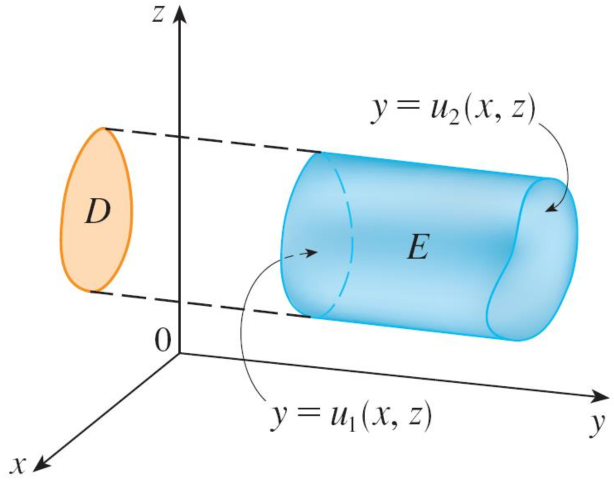
$$\boxed{10} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx \right] dA$$

Triple Integrals

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 8).



A type 3 region

Figure 8

Triple Integrals

For this type of region we have

$$\boxed{11} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether *D* is a type I or type II plane region (and corresponding to Equations 7 and 8).

Examples 2 - 3

Example 2. Evaluate $\iiint_E z\, dV$ where E is the solid in the first octant bounded by the surface $z=12xy$, and the planes $y=x$ and $x=1$.

Example 3. Set up the iterated integral in two different ways to evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$ where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Applications of Triple Integrals

Applications of Triple Integrals

Recall that if $f(x) \geq 0$, then the single integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x, y) \geq 0$, then the double integral $\iint_D f(x, y)\,dA$ represents the volume under the surface $z = f(x, y)$ and above D.

The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\,dV$, where $f(x, y, z) \geq 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f; the graph of f lies in four-dimensional space.)

Applications of Triple Integrals

Nonetheless, the triple integral $\iiint_E f(x, y, z)\, dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

12

$$\boxed{V(E) = \iiint_E dV}$$

Applications of Triple Integrals

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1 \, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} dz \right] dA = \iint_D [u_2(x, y) - u_1(x, y)] \, dA$$

and we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

Example 5

Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Applications of Triple Integrals

For example, if the density function of a solid object that occupies the region E is $\rho(x, y, z)$, in units of mass per unit volume, at any given point (x, y, z), then its **mass** is

13
$$m = \iiint_E \rho(x, y, z)\, dV$$

and its **moments** about the three coordinate planes are

14
$$M_{yz} = \iiint_E x\,\rho(x, y, z)\, dV \qquad M_{xz} = \iiint_E y\,\rho(x, y, z)\, dV$$

$$M_{xy} = \iiint_E z\,\rho(x, y, z)\, dV$$

Applications of Triple Integrals

The **center of mass** is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

15 $$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the **centroid** of *E*.

The **moments of inertia** about the three coordinate axes are

16 $$I_x = \iiint_E (y^2 + z^2)\,\rho(x, y, z)\,dV \qquad I_y = \iiint_E (x^2 + z^2)\,\rho(x, y, z)\,dV$$

$$I_z = \iiint_E (x^2 + y^2)\,\rho(x, y, z)\,dV$$

Applications of Triple Integrals

The total **electric charge** on a solid object occupying a region E and having charge density $\sigma(x, y, z)$ is

$$Q = \iiint_E \sigma(x, y, z)\, dV$$

If we have three continuous random variables X, Y, and Z, their **joint density function** is a function of three variables such that the probability that (X, Y, Z) lies in E is

$$P\big((X, Y, Z) \in E\big) = \iiint_E f(x, y, z)\, dV$$

Applications of Triple Integrals

In particular,

$$P(a \leqslant X \leqslant b,\ c \leqslant Y \leqslant d,\ r \leqslant Z \leqslant s) = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx$$

The joint density function satisfies

$$f(x, y, z) \geqslant 0 \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dz\, dy\, dx = 1$$

15.2 Double Integrals over General Regions

Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval.

But for double integrals, we want to be able to integrate a function *f* not just over rectangles but also over regions *D* of more general shape, such as the one illustrated in Figure 1.

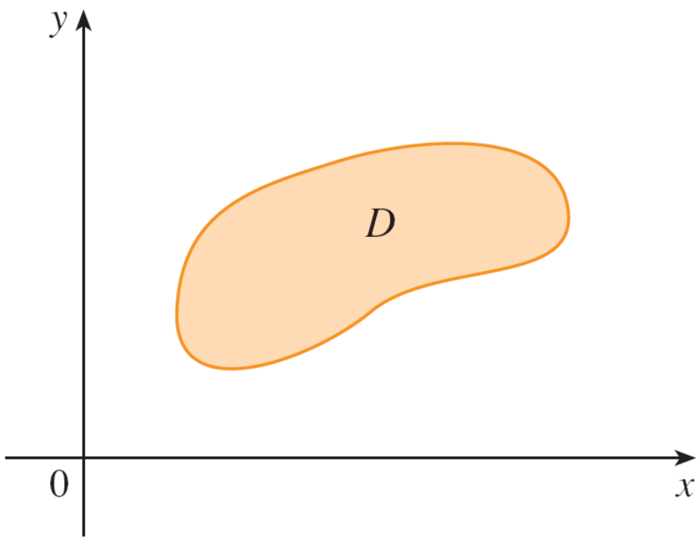


Figure 1

Double Integrals over General Regions

We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2.

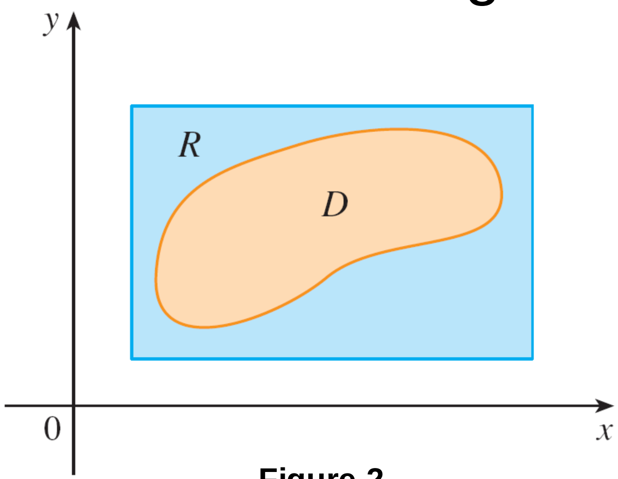


Figure 2

Then we define a new function F with domain R by

$$\boxed{1} \qquad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Double Integrals over General Regions

If F is integrable over R, then we define the **double integral of *f* over *D*** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because R is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined.

Double Integrals over General Regions

The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral.

This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geq 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f).

Double Integrals over General Regions

You can see that this is reasonable by comparing the graphs of *f* and *F* in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of *F*.

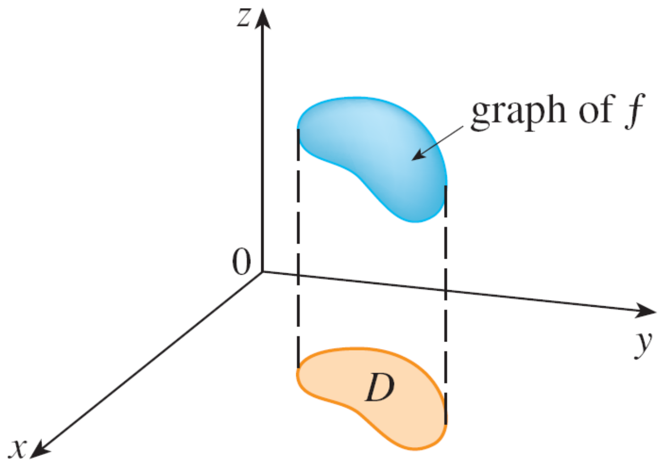


Figure 3

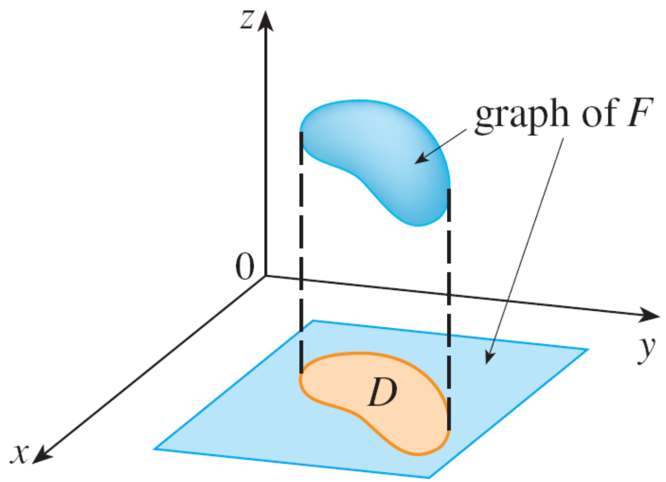


Figure 4

Double Integrals over General Regions

Figure 4 also shows that F is likely to have discontinuities at the boundary points of D.

Nonetheless, if f is continuous on D and the boundary curve of D is “well behaved”, then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists.

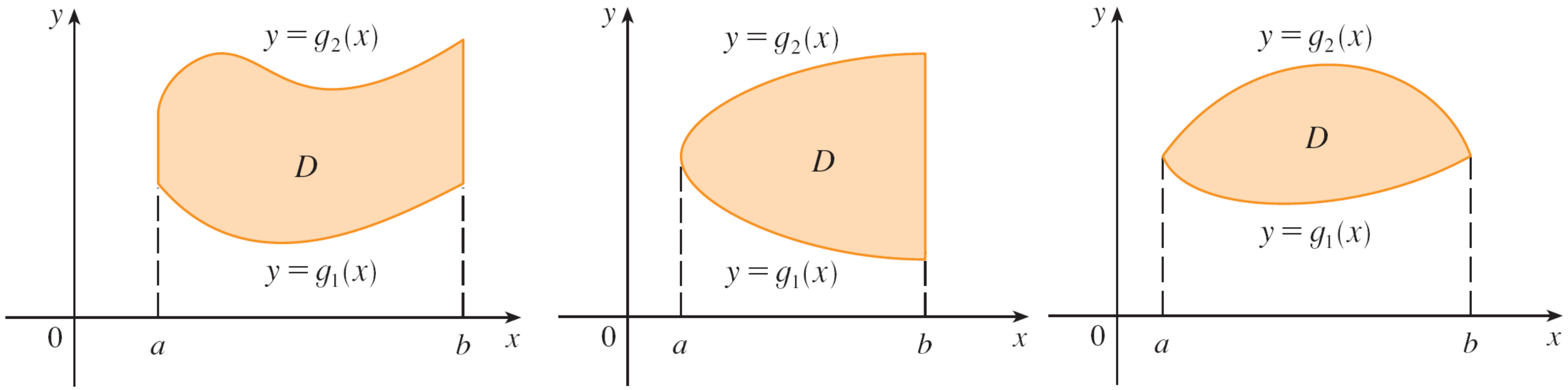
In particular, this is the case for **type I** and **type II** regions.

Double Integrals over General Regions

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.



Some type I regions

Figure 5

Double Integrals over General Regions

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D.

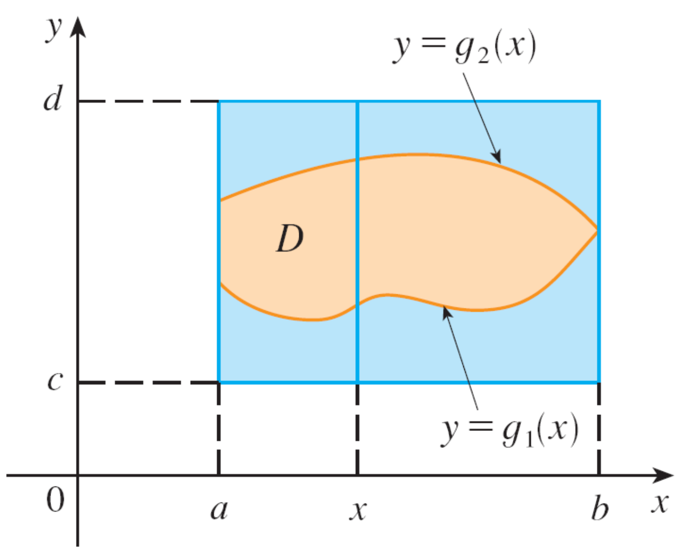


Figure 6

Double Integrals over General Regions

Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \le y \le g_2(x)$.

Double Integrals over General Regions

Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

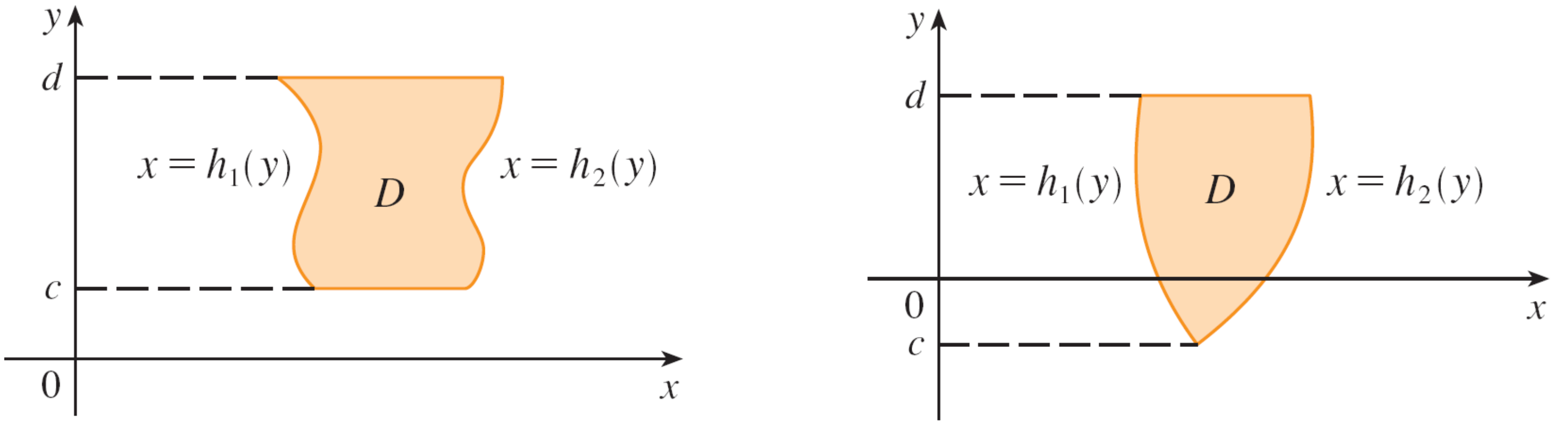
The integral on the right side of $\boxed{3}$ is an iterated integral, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

Double Integrals over General Regions

We also consider plane regions of **type II**, which can be expressed as

4 $$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.



Some type II regions

Figure 7

Double Integrals over General Regions

Using the same methods that were used in establishing $\boxed{3}$, we can show that

$$\boxed{5} \qquad \iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

Examples 1 - 4

Example 1. Evaluate $\iint_D (x+2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Example 2. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example 3. (different from the one in the book). Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = 2x$ and the parabola $y^2 = 2x + 6$.

Example 4. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 0$, $x = 2y$, and $z = 0$.

Changing the order of integration

Example 5. Evaluate the interated integral
$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

Properties of Double Integrals

Properties of Double Integrals

We assume that all of the following integrals exist. The first three properties of double integrals over a region *D* follow immediately from Definition 2.

6 $$\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

7 $$\iint_D c f(x, y)\, dA = c \iint_D f(x, y)\, dA$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then

8 $$\iint_D f(x, y)\, dA \geqslant \iint_D g(x, y)\, dA$$

Properties of Double Integrals

The next property of double integrals is similar to the property of single integrals given by the equation

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

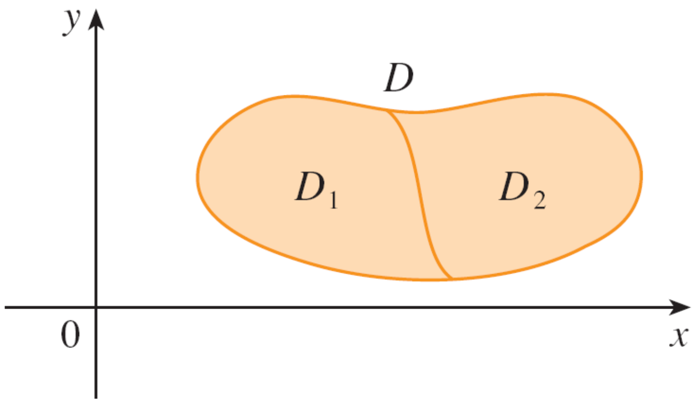


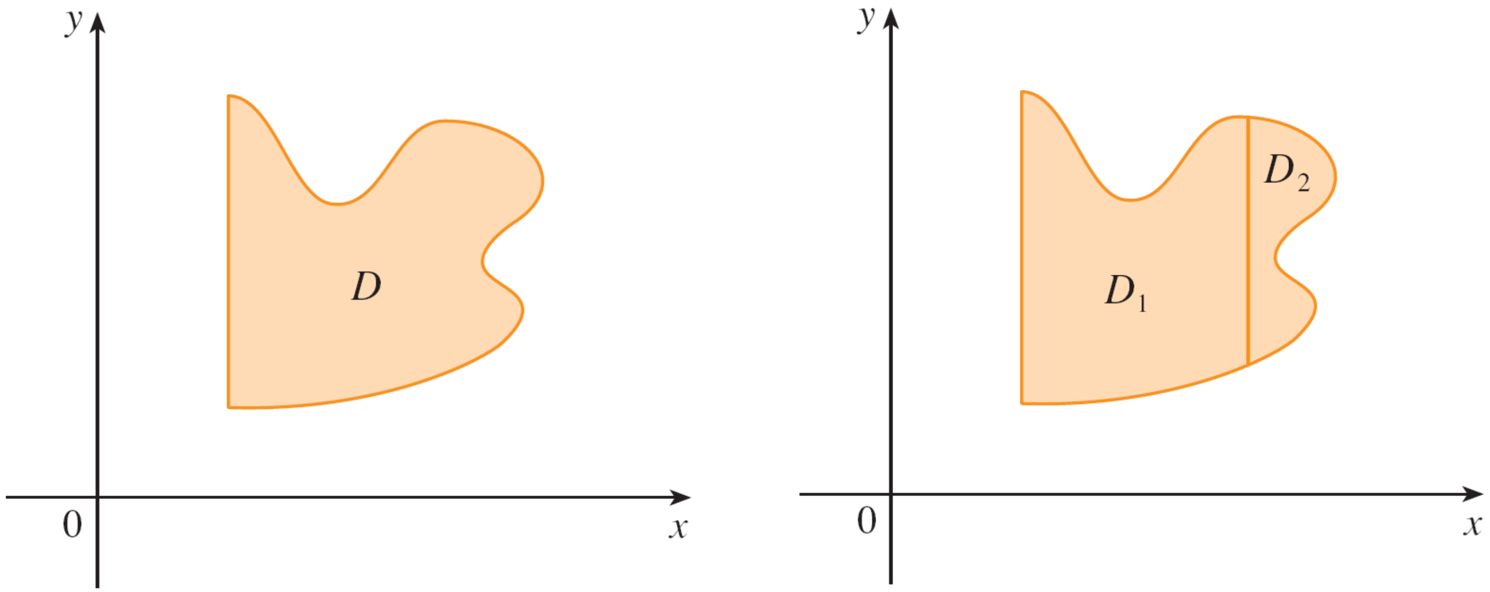
Figure 17

9

$$\boxed{\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA}$$

Properties of Double Integrals

Property 9 can be used to evaluate double integrals over regions *D* that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure.



(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$; D_1 is type I, D_2 is type II.

Figure 18

Properties of Double Integrals

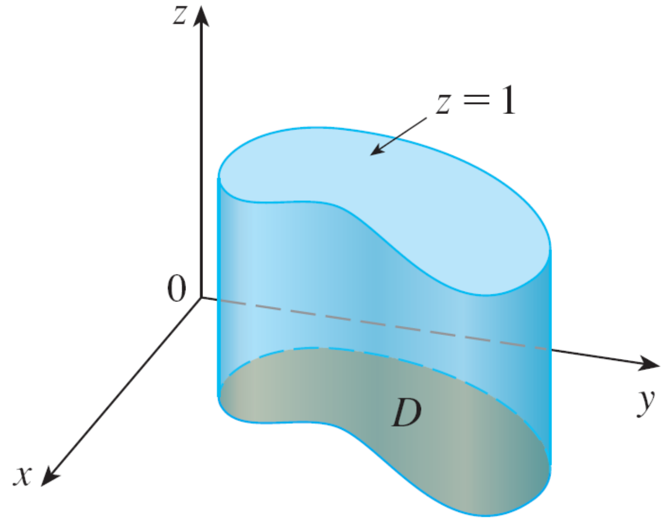
The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

10

$$\boxed{\iint_D 1 \, dA = A(D)}$$

Properties of Double Integrals

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.



Cylinder with base D and height 1

Figure 19

Properties of Double Integrals

Finally, we can combine Properties 7, 8, and 10 to prove the following property.

11 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$

15.9 Change of Variables in Multiple Integrals

Change of Variables in Multiple Integrals

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. By reversing the roles of x and u, we can write

$$\boxed{1} \qquad \int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Another way of writing Formula 1 is as follows:

$$\boxed{2} \qquad \int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

Change of Variables in Multiple Integrals

We consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$\boxed{3} \qquad x = g(u, v) \qquad y = h(u, v)$$

or, as we sometimes write,

$$x = x(u, v) \qquad y = y(u, v)$$

Change of Variables in Multiple Integrals

We usually assume that T is a $\boldsymbol{C^1}$ **transformation**, which means that g and h have continuous first-order partial derivatives.

A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$.

If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1).

If no two points have the same image, T is called **one-to-one**.

Change of Variables in Multiple Integrals

Figure 1 shows the effect of a transformation *T* on a region *S* in the *uv*-plane.

T transforms *S* into a region *R* in the *xy*-plane called the **image of *S***, consisting of the images of all points in *S*.

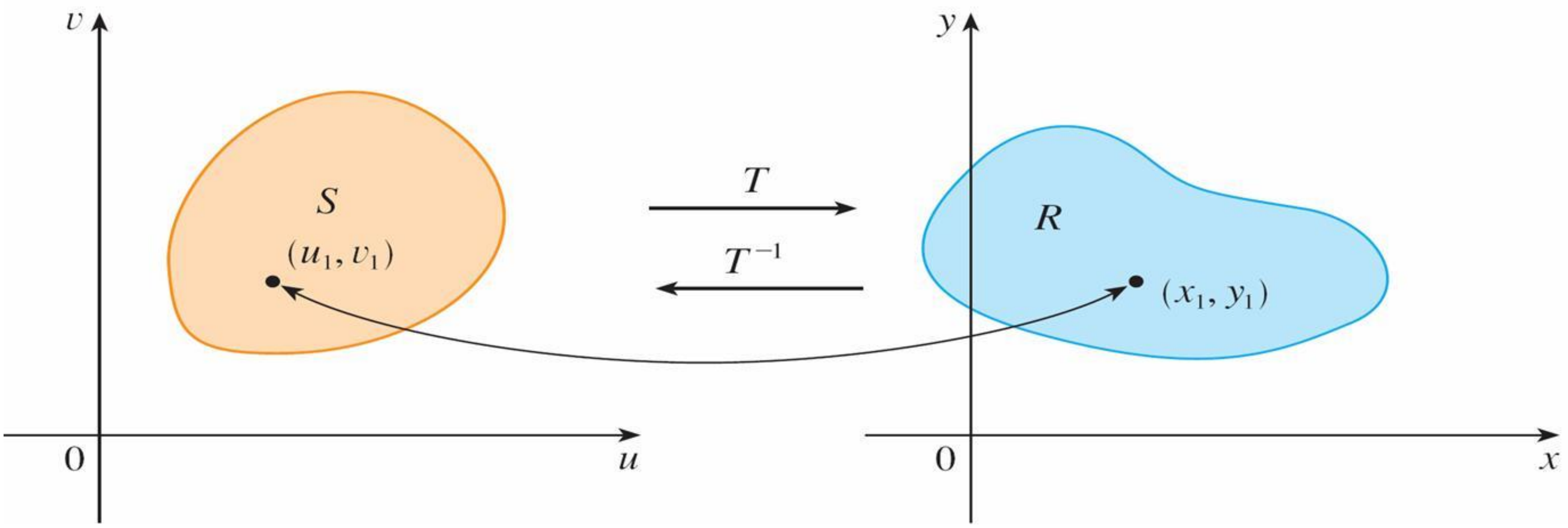


Figure 1

Change of Variables in Multiple Integrals

If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane and it may be possible to solve Equations 3 for u and v in terms of x and y:

$$u = G(x, y) \qquad\qquad v = H(x, y)$$

Example 1

A transformation is defined by the equations

$$x = u^2 - v^2 \qquad\qquad y = 2uv$$

Find the image of the square

$$S = \{(u, v) \mid 0 \leq u \leq 1,\ 0 \leq v \leq 1\}.$$

Example 1

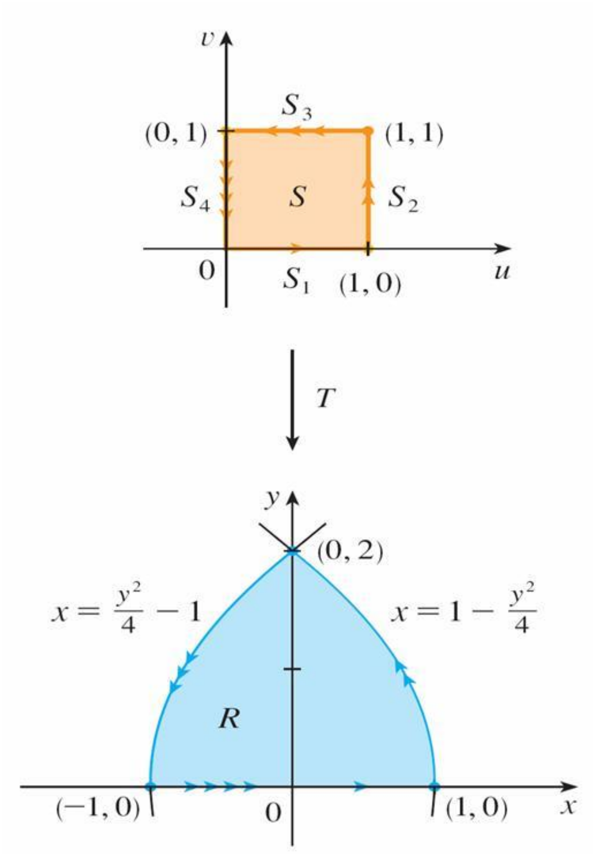


Figure 2

Change of Variables in Multiple Integrals

Now let's see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. (See Figure 3.)

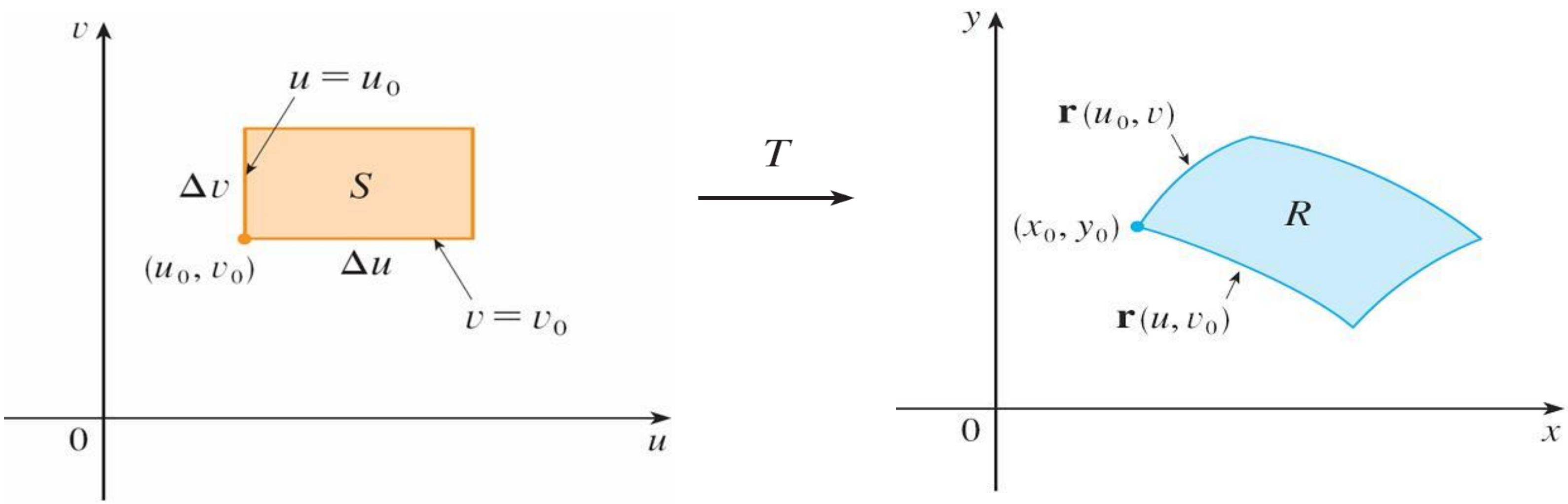


Figure 3

Change of Variables in Multiple Integrals

The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$.

The vector

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

is the position vector of the image of the point (u, v).

The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$.

Change of Variables in Multiple Integrals

The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\,\mathbf{i} + h_u(u_0, v_0)\,\mathbf{j}$$

$$= \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j}$$

Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\mathbf{r}_v = g_v(u_0, v_0)\,\mathbf{i} + h_v(u_0, v_0)\,\mathbf{j}$$

$$= \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j}$$

Change of Variables in Multiple Integrals

We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$$

shown in Figure 4.

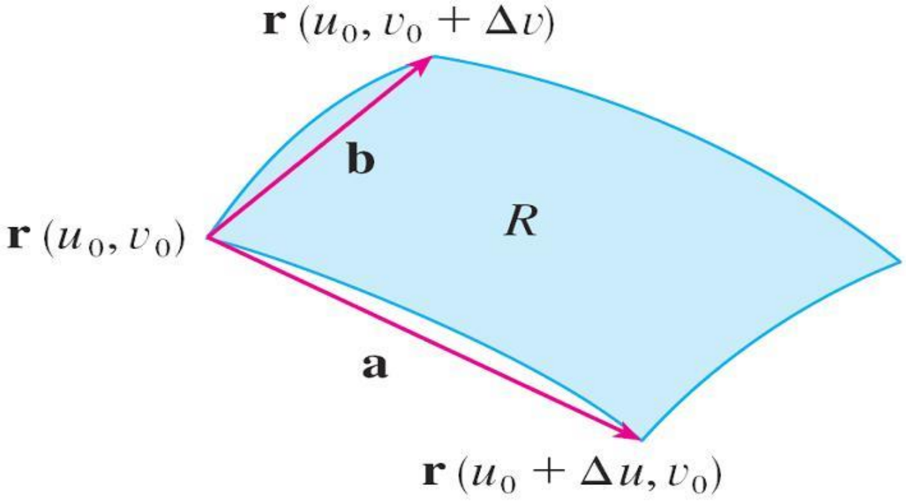


Figure 4

Change of Variables in Multiple Integrals

But

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and so $\qquad \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \, \mathbf{r}_u$

Similarly $\qquad \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \, \mathbf{r}_v$

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u \, \mathbf{r}_u$ and $\Delta v \, \mathbf{r}_v$. (See Figure 5.)

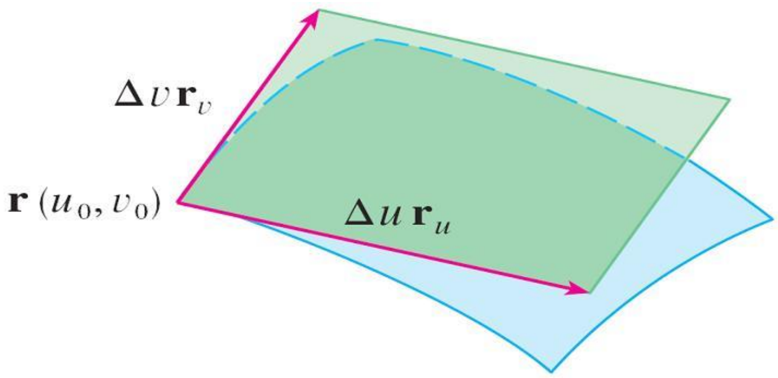


Figure 5

Change of Variables in Multiple Integrals

Therefore, we can approximate the area of R by the area of this parallelogram, which is

$$\boxed{6} \qquad |(\Delta u\, \mathbf{r}_u) \times (\Delta v\, \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$$

Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

Change of Variables in Multiple Integrals

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

$\boxed{7}$ **Definition** The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area ΔA of R:

$$\boxed{8} \qquad \Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \, \Delta v$$

where the Jacobian is evaluated at (u_0, v_0).

Change of Variables in Multiple Integrals

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}. (See Figure 6.)

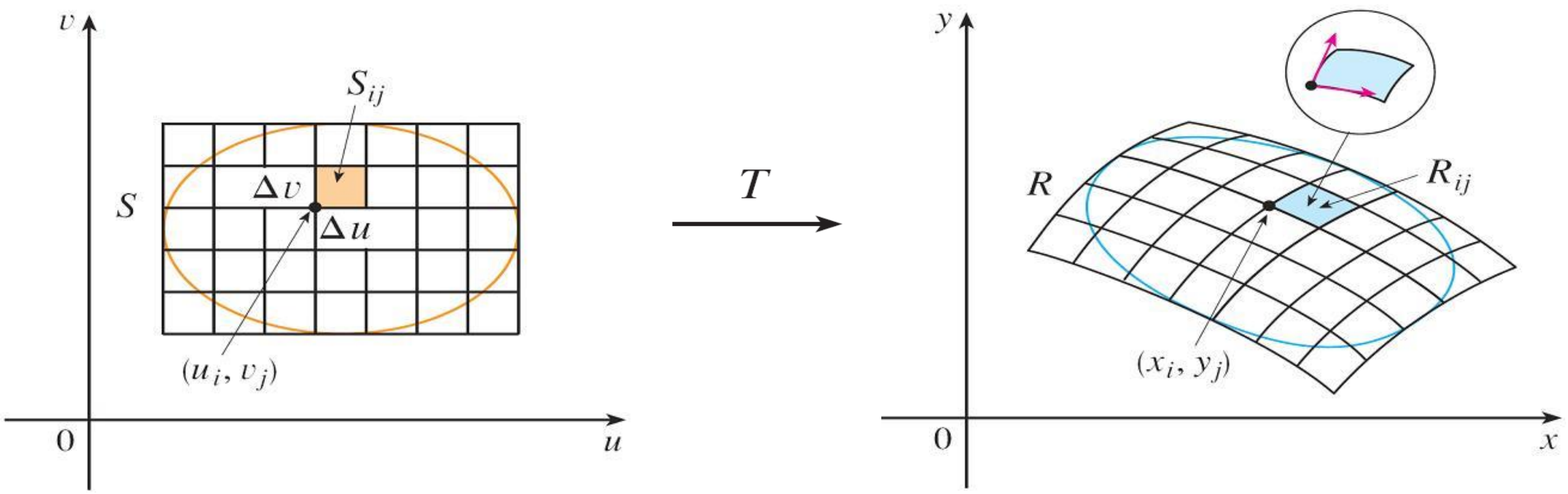


Figure 6

Change of Variables in Multiple Integrals

Applying the approximation (8) to each R_{ij}, we approximate the double integral of f over R as follows:

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\big(g(u_i, v_j), h(u_i, v_j)\big) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f\big(g(u, v), h(u, v)\big) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Change of Variables in Multiple Integrals

The foregoing argument suggests that the following theorem is true.

9 Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Theorem 9 says that we change from an integral in *x* and *y* to an integral in *u* and *v* by expressing *x* and *y* in terms of *u* and *v* and writing

$$dA = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Change of Variables in Multiple Integrals

The requirements about the type of R and S are not essential.

Notice the similarity between Theorem 9 and the one-dimensional formula in Equation 2.

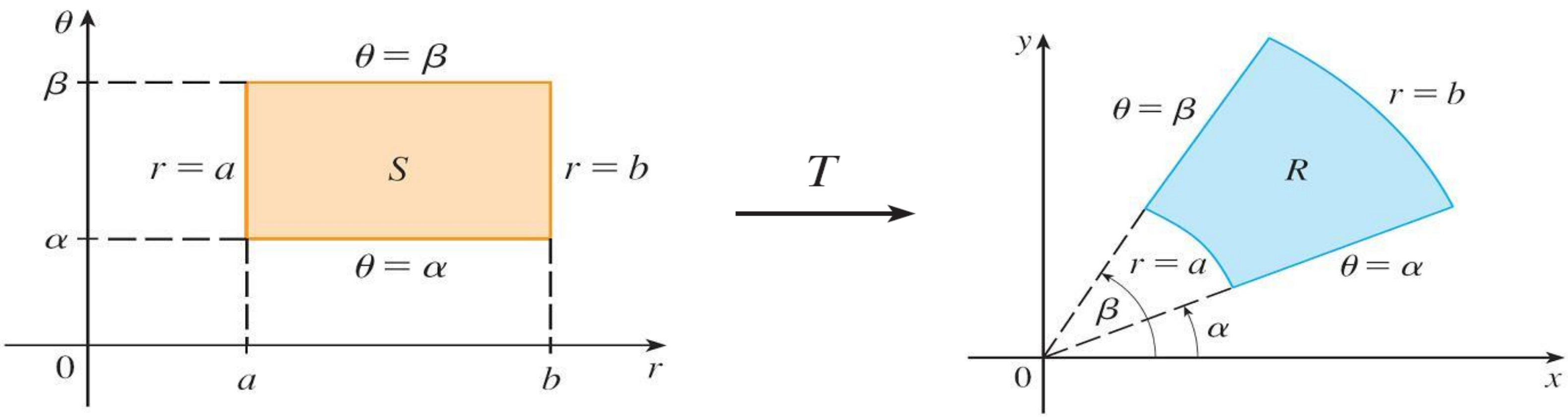
Instead of the derivative dx/du, we have the absolute value of the Jacobian, that is, $|\partial(x, y)/\partial(u, v)|$.

As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates.

Example: Polar coordinates

Derive the formula for double integration in polar coordinates

Change of Variables in Multiple Integrals



The polar coordinate transformation

Triple Integrals

Triple Integrals

There is a similar change of variables formula for triple integrals.

Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

Triple Integrals

The **Jacobian** of *T* is the following 3 × 3 determinant:

$$\boxed{12} \qquad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Under hypotheses similar to those in Theorem 9, we have the following formula for triple integrals:

$$\boxed{13} \qquad \iiint\limits_R f(x, y, z)\, dV = \iiint\limits_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

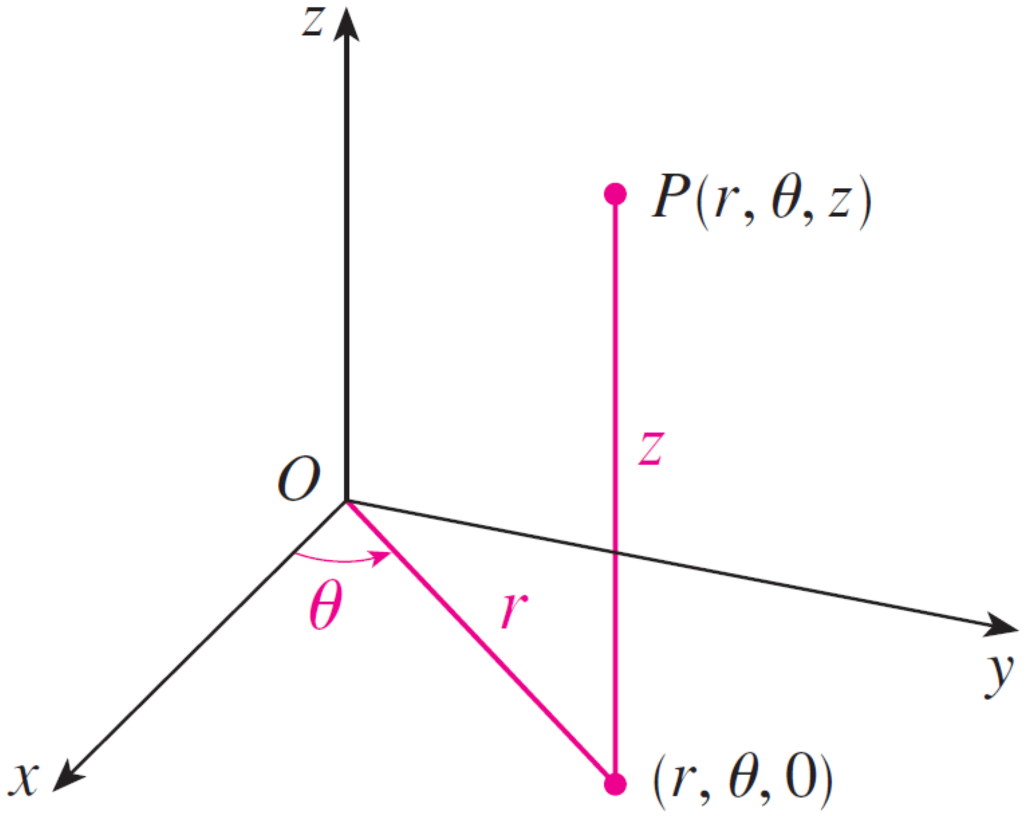
Example

Use Formula 13 to derive the formula for triple integration in

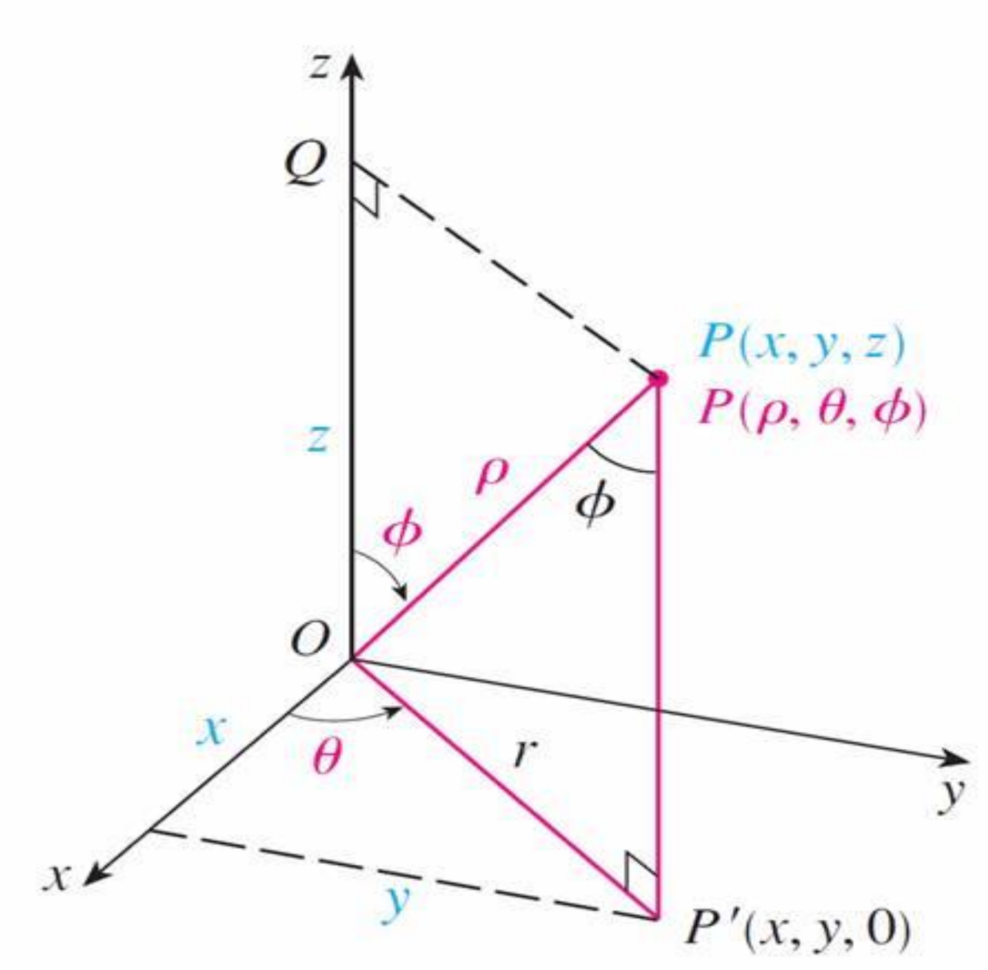
(a) cylindrical coordinates about the z-axis;

(b) spherical coordinates.

Cylindrical coordinates



Spherical coordinates



Example (15.8/Example 4)

Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 9.)

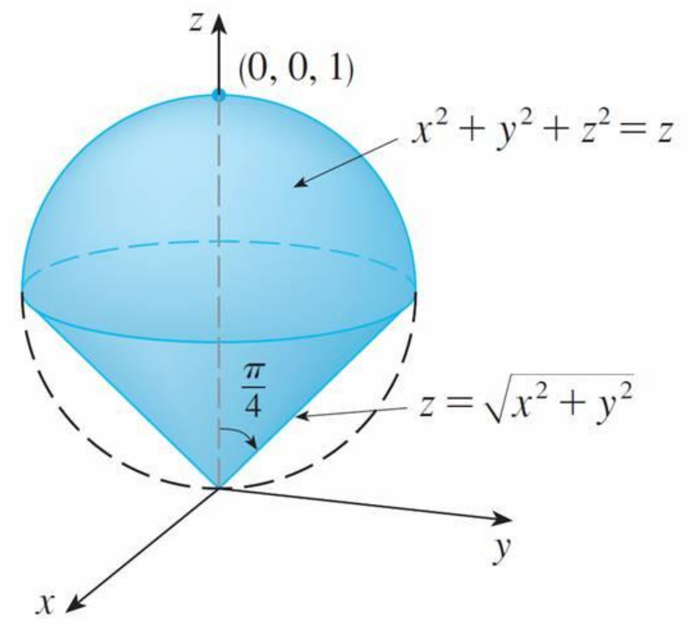
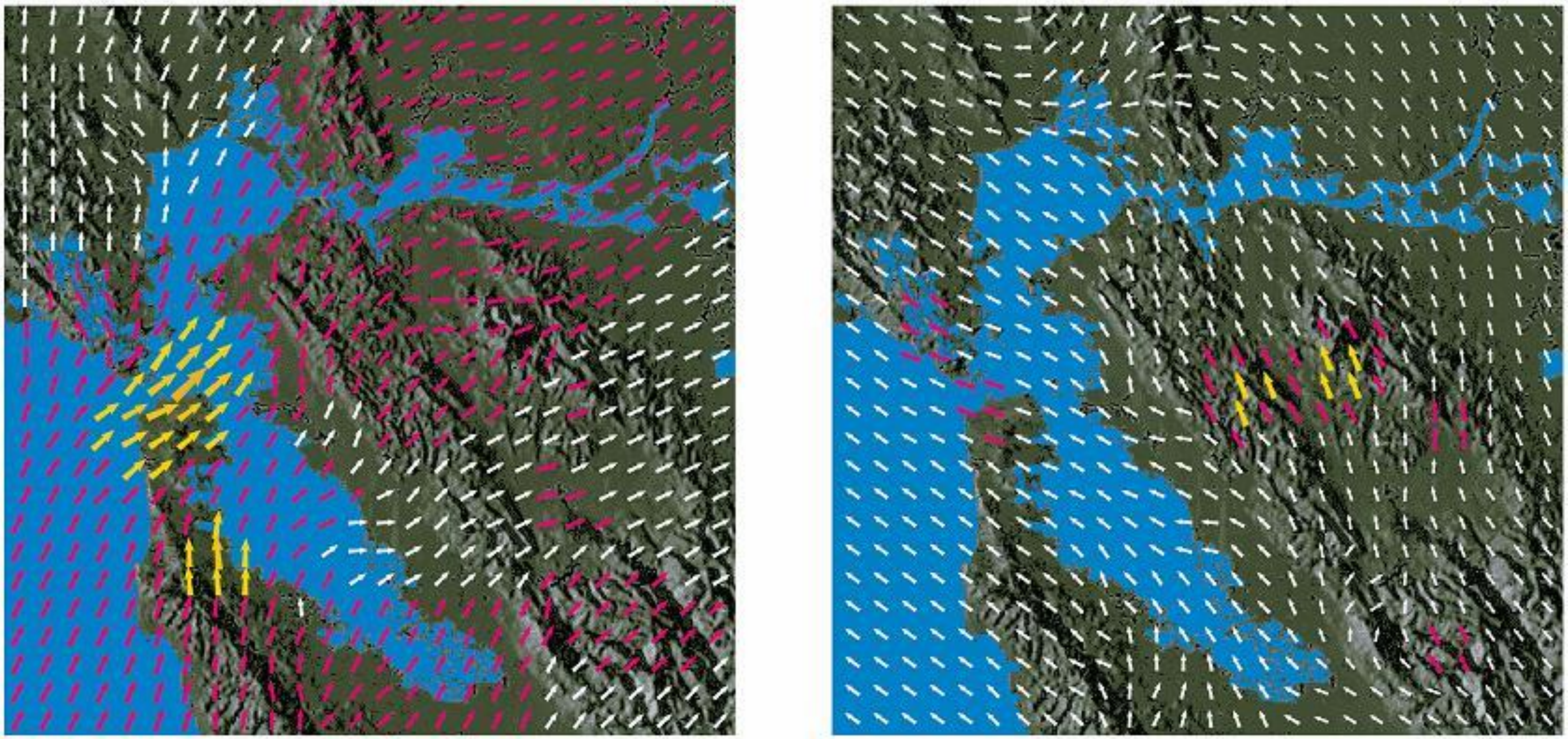


Figure 9

16 Vector Calculus

16.1 Vector Fields

Vector Fields



(a) 6:00 PM, March 1, 2010 (b) 6:00 AM, March 1, 2010

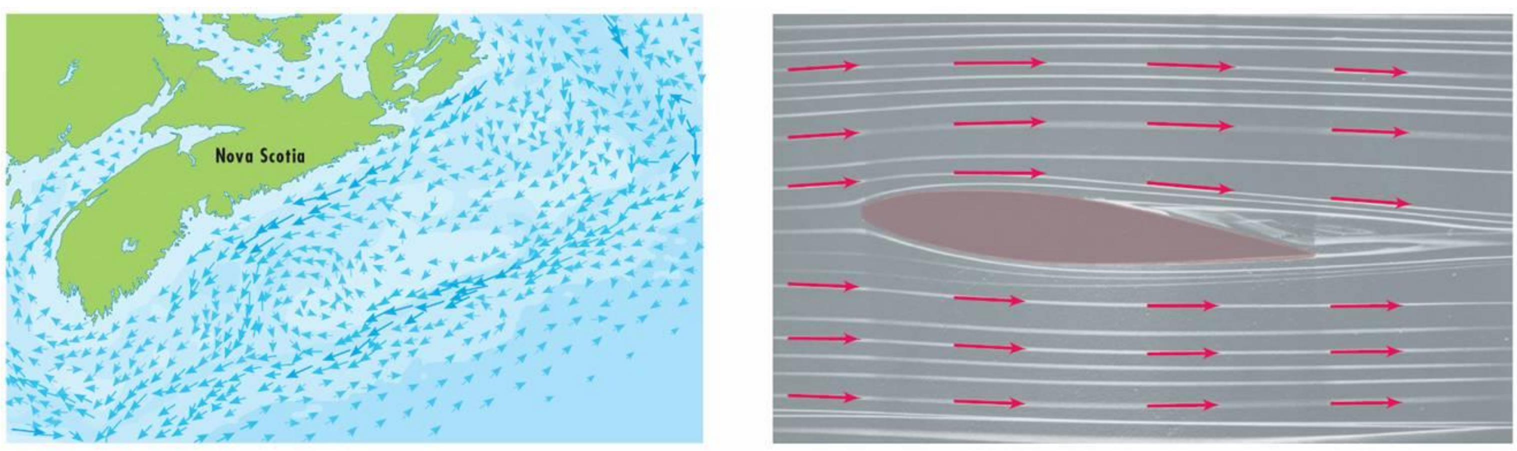
Velocity vector fields showing San Francisco Bay wind patterns

Figure 1

Vector Fields

Associated with every point in the air we can imagine a wind velocity vector. This is an example of a *velocity vector field.*

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.



(a) Ocean currents off the coast of Nova Scotia

(b) Airflow past an inclined airfoil

Velocity vector fields

Figure 2

Vector Fields

Another type of vector field, called a *force field*, associates a force vector with each point in a region. An example is the gravitational force field.

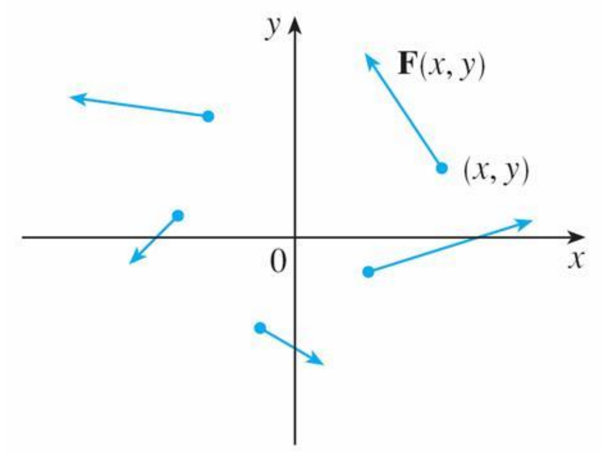
In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3).

1 Definition Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

Vector Fields

The best way to picture a vector field is to draw the arrow representing the vector **F**(x, y) starting at the point (x, y).

Of course, it's impossible to do this for all points (x, y), but we can gain a reasonable impression of **F** by doing it for a few representative points in D as in Figure 3.



Vector field on $\mathbb{R}^2$

Figure 3

Vector Fields

Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

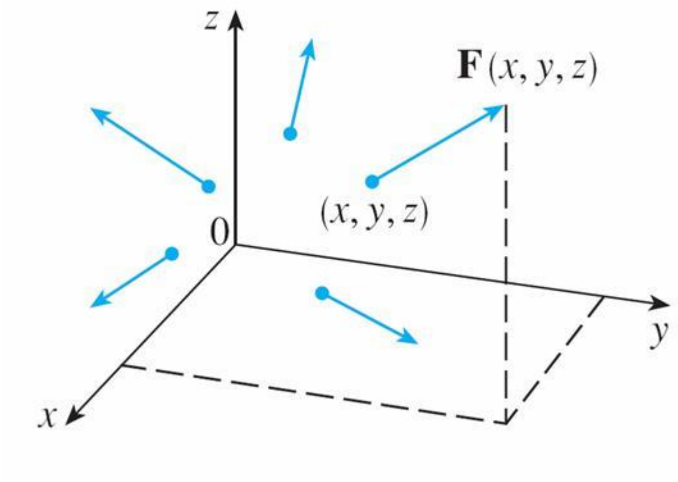
or, for short, $$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

2 Definition Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Vector Fields

A vector field **F** on $\mathbb{R}^3$ is pictured in Figure 4.



Vector field on $\mathbb{R}^3$

Figure 4

We can express it in terms of its component functions P, Q, and R as

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

Vector Fields

As with the vector functions, we can define continuity of vector fields and show that **F** is continuous if and only if its component functions P, Q, and R are continuous.

We sometimes identify a point (x, y, z) with its position vector $\mathbf{x} = \langle x, y, z \rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$.

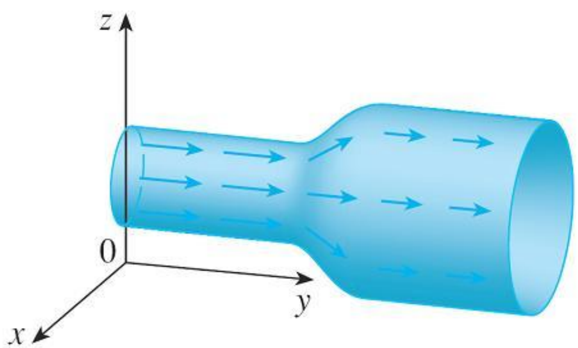
Then **F** becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector **x**.

Example 3

Imagine a fluid flowing steadily along a pipe and let **V**(x, y, z) be the velocity vector at a point (x, y, z).

Then **V** assigns a vector to each point (x, y, z) in a certain domain E (the interior of the pipe) and so **V** is a vector field on $\mathbb{R}^3$ called a **velocity field**.

A possible velocity field is illustrated in Figure 13.



Velocity field in fluid flow

Figure 13

Example 3

cont'd

The speed at any given point is indicated by the length of the arrow.

Velocity fields also occur in other areas of physics.

Example 4

Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses *m* and *M* is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where *r* is the distance between the objects and *G* is the gravitational constant.
(This is an example of an inverse square law.)

Let's assume that the object with mass *M* is located at the origin in $\mathbb{R}^3$. (For instance, *M* could be the mass of the earth and the origin would be at its center.)

Example 4

cont'd

Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z \rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |\mathbf{x}|^2$.

The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is

$$\boxed{3} \qquad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

[Physicists often use the notation **r** instead of **x** for the position vector, so you may see Formula 3 written in the form $\mathbf{F} = -(mMG/r^3)\mathbf{r}$.]

Example 4

cont'd

The function given by Equation 3 is an example of a vector field, called the **gravitational field**, because it associates a vector [the force **F**(**x**)] with every point **x** in space.

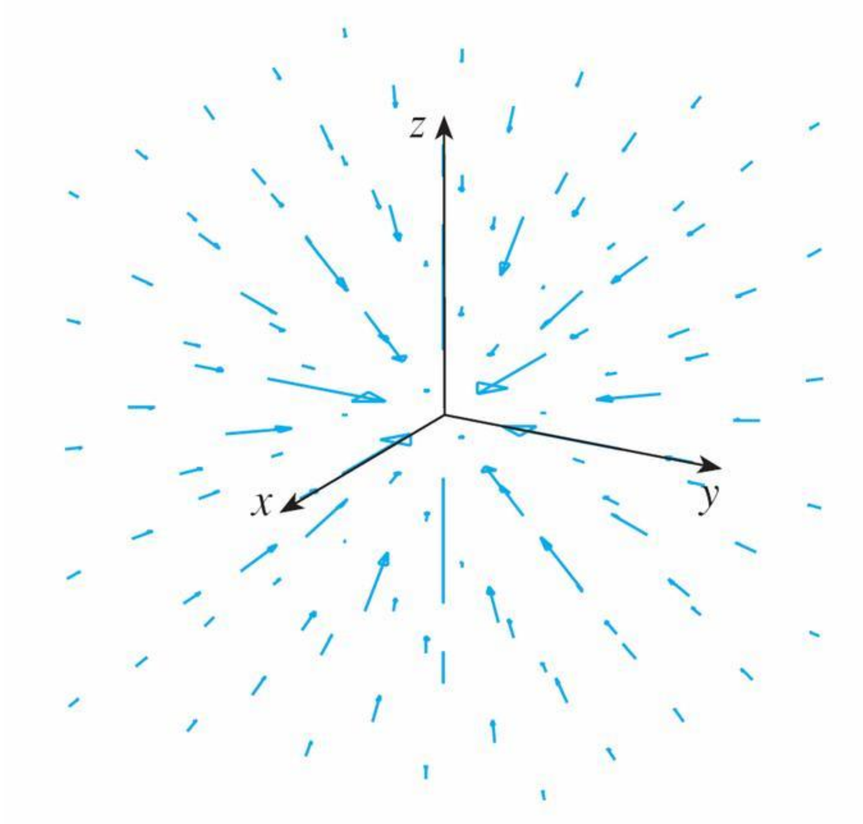
Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that **x** = x **i** + y **j** + z **k** and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

Example 4

cont'd

The gravitational field **F** is pictured in Figure 14.



Gravitational force field

Figure 14

Example 5

Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge q located at a point (x, y, z) with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

$$\boxed{4} \qquad \mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3}\mathbf{x}$$

where ε is a constant (that depends on the units used).

For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive.

Example 5

cont'd

Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Instead of considering the electric force **F**, physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

Then **E** is a vector field on $\mathbb{R}^3$ called the **electric field** of Q.

Gradient Fields

Gradient Fields

If f is a scalar function of two variables, recall that its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore ∇f is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**.

Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

Example 6

Find the gradient vector field of $f(x, y) = x^2y - y^3$.

Gradient Fields

A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$.

In this situation f is called a **potential function** for **F**.

Not all vector fields are conservative, but such fields do arise frequently in physics.

Gradient Fields

For example, the gravitational field **F** in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

$$= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

$$= \mathbf{F}(x, y, z)$$

16.2 Line Integrals

Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval [*a*, *b*], we integrate over a curve *C*.

Such integrals are called *line integrals,* although “curve integrals” would be better terminology.

They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

Line Integrals

We start with a plane curve C given by the parametric equations

$$\boxed{1} \qquad x = x(t) \qquad y = y(t) \qquad a \leq t \leq b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.]

Line Integrals

If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$, and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 1.)

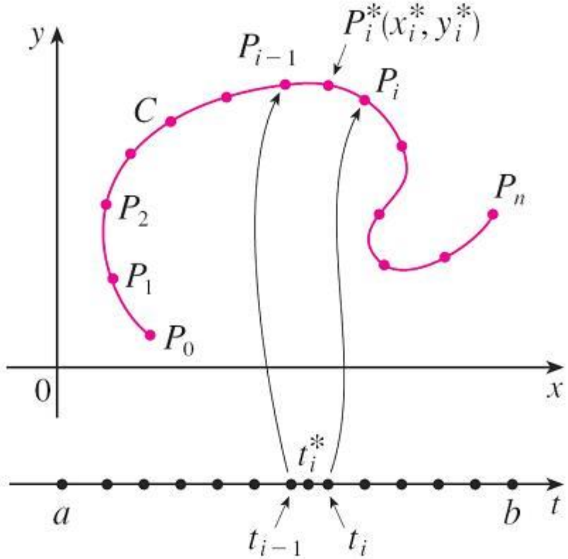


Figure 1

Line Integrals

We choose any point $P_i^*(x_i^*, y_i^*)$ in the *i*th subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.)

Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

which is similar to a Riemann sum.

Line Integrals

Then we take the limit of these sums and make the following definition by analogy with a single integral.

2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

We have found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Line Integrals

A similar type of argument can be used to show that if *f* is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

$$\boxed{3} \qquad \int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as *t* increases from *a* to *b*.

Line Integrals

If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

So the way to remember Formula 3 is to express everything in terms of the parameter t: Use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Line Integrals

In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \leq x \leq b$.

Formula 3 then becomes

$$\int_C f(x, y)\, ds = \int_a^b f(x, 0)\, dx$$

and so the line integral reduces to an ordinary single integral in this case.

Line Integrals

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area.

In fact, if $f(x, y) \geq 0$, $\int_C f(x, y)\, ds$ represents the area of one side of the "fence" or "curtain" in Figure 2, whose base is C and whose height above the point (x, y) is $f(x, y)$.

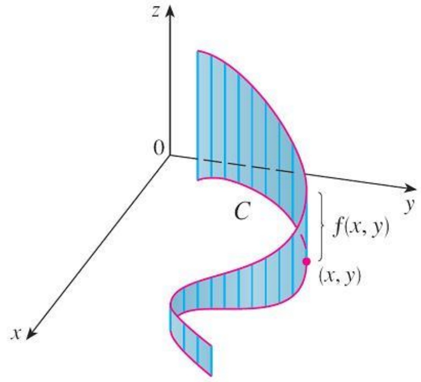


Figure 2

Example 1

Evaluate $\int_C (2 + x^2y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Line Integrals

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i.

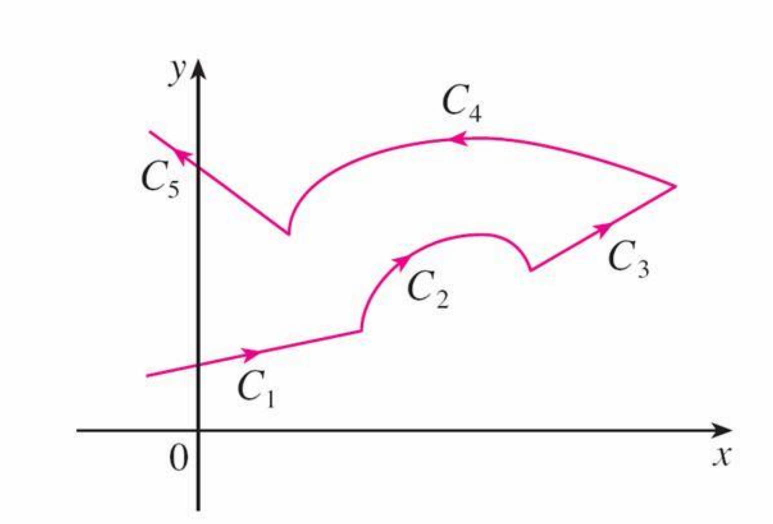


Figure 4

A piecewise-smooth curve

Line Integrals

Then we define the integral of *f* along *C* as the sum of the integrals of *f* along each of the smooth pieces of *C*:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds$$

Example 2

Evaluate $\int_C 2x\, ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2).

Line Integrals

Any physical interpretation of a line integral $\int_C f(x, y)\, ds$ depends on the physical interpretation of the function f.

Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C.

Then the mass of the part of the wire from P_{i-1} to P_i in Figure1 is approximately $\rho(x_i^*, y_i^*)\,\Delta s_i$ and so the total mass of the wire is approximately $\sum \rho(x_i^*, y_i^*)\,\Delta s_i$.

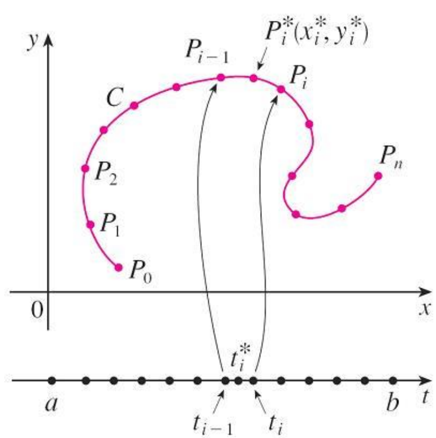


Figure 1

Line Integrals

By taking more and more points on the curve, we obtain the **mass** *m* of the wire as the limiting value of these approximations:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*) \, \Delta s_i = \int_C \rho(x, y) \, ds$$

[For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.]

Line Integrals

The **center of mass** of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

$$\boxed{4} \qquad \bar{x} = \frac{1}{m} \int_C x\,\rho(x, y)\,ds \qquad \bar{y} = \frac{1}{m} \int_C y\,\rho(x, y)\,ds$$

Line Integrals

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2.

They are called the **line integrals of *f* along *C* with respect to *x* and *y***:

$$\boxed{5} \qquad \int_C f(x, y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

$$\boxed{6} \qquad \int_C f(x, y)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

Line Integrals

When we want to distinguish the original line integral $\int_C f(x, y)\, ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t)$, $y = y(t)$, $dx = x'(t)\, dt$, $dy = y'(t)\, dt$.

$$\boxed{7} \qquad \int_C f(x, y)\, dx = \int_a^b f\big(x(t), y(t)\big)\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f\big(x(t), y(t)\big)\, y'(t)\, dt$$

Line Integrals

It frequently happens that line integrals with respect to x and y occur together.

When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given.

Line Integrals

In particular, we often need to parametrize a line segment, so it's useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

8

$$\boxed{\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \qquad 0 \leq t \leq 1}$$

Example 4

Evaluate $\int_C y^2\,dx + x\,dy$ in two different ways.

(a) $C = C_1$ is the line segment from (-5, -3) to (0, 2).

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from (-5, -3) to (0, 2).

Line Integrals

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing values of the parameter t. (See Figure 8, where the initial point A corresponds to the parameter value a and the terminal point B corresponds to $t = b$.)

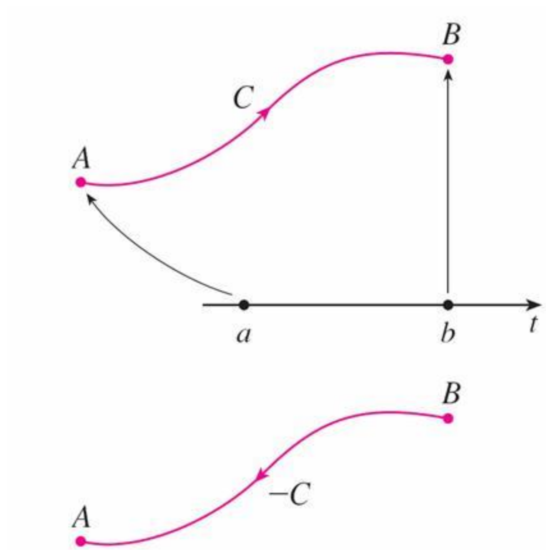


Figure 8

Line Integrals

If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point B to terminal point A in Figure 8), then we have

$$\int_{-C} f(x, y)\, dx = -\int_C f(x, y)\, dx \qquad \int_{-C} f(x, y)\, dy = -\int_C f(x, y)\, dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\, ds = \int_C f(x, y)\, ds$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

Line Integrals in Space

Line Integrals in Space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

or by a vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$.

If f is a function of three variables that is continuous on some region containing C, then we define the **line integral of *f* along *C*** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z)\,ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$$

Line Integrals in Space

We evaluate it using a formula similar to Formula 3:

$$\boxed{9} \qquad \int_C f(x, y, z)\, ds = \int_a^b f\big(x(t), y(t), z(t)\big) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\, dt = L$$

where L is the length of the curve C.

Line Integrals in Space

Line integrals along *C* with respect to *x*, *y*, and *z* can also be defined. For example,

$$\int_C f(x, y, z)\, dz = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta z_i$$

$$= \int_a^b f\big(x(t), y(t), z(t)\big)\, z'(t)\, dt$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\boxed{10} \qquad \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

by expressing everything (*x*, *y*, *z*, *dx*, *dy*, *dz*) in terms of the parameter *t*.

Line Integrals of Vector Fields

Line Integrals of Vector Fields

Recall that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \int_a^b f(x)\, dx$.

It is known from physics that the work done by a constant force $\mathbf{F}$ in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$, where $\mathbf{D} = \overrightarrow{PQ}$ is the displacement vector.

Now suppose that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$. (A force field on $\mathbb{R}^2$ could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.)

We wish to compute the work done by this force in moving a particle along a smooth curve C.

Line Integrals of Vector Fields

We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width. (See Figure 1 for the two-dimensional case or Figure 11 for the three-dimensional case.)

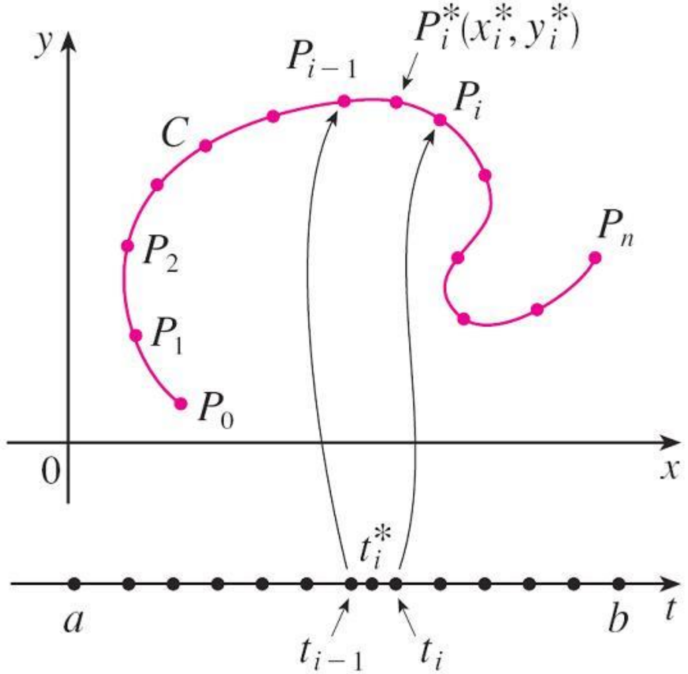


Figure 1

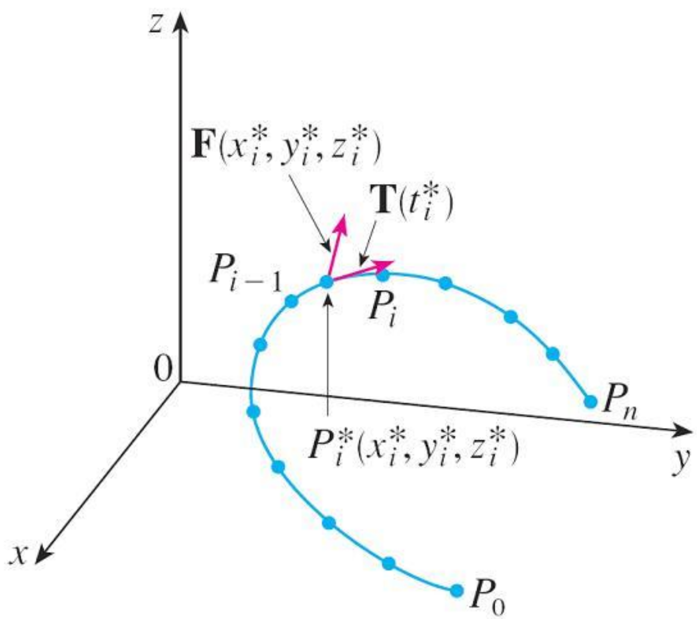


Figure 11

Line Integrals of Vector Fields

Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the *i*th subarc corresponding to the parameter value t_i^*.

If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\mathbf{T}(t_i^*)$, the unit tangent vector at P_i^*.

Line Integrals of Vector Fields

Thus the work done by the force **F** in moving the particle from P_{i-1} to P_i is approximately

$$\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)] \Delta s_i$$

and the total work done in moving the particle along C is approximately

$$\boxed{11} \qquad \sum_{i=1}^{n} [\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(x_i^*, y_i^*, z_i^*)] \Delta s_i$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C.

Line Integrals of Vector Fields

Intuitively, we see that these approximations ought to become better as *n* becomes larger.

Therefore we define the **work** *W* done by the force field **F** as the limit of the Riemann sums in $\boxed{11}$, namely,

$\boxed{12}$
$$W = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Equation 12 says that *work is the line integral with respect to arc length of the tangential component of the force.*

Line Integrals of Vector Fields

If the curve C is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},$$

then $\mathbf{T}(t) = \mathbf{r}'(t)/\,|\mathbf{r}'(t)|$, so using Equation 9 we can rewrite Equation 12 in the form

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

This integral is often abbreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well.

Line Integrals of Vector Fields

Therefore we make the following definition for the line integral of *any* continuous vector field.

$\boxed{13}$ **Definition** Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

When using Definition 13, remember that $\mathbf{F}(\mathbf{r}(t))$ is just an abbreviation for $\mathbf{F}(x(t), y(t), z(t))$, so we evaluate $\mathbf{F}(\mathbf{r}(t))$ simply by putting $x = x(t)$, $y = y(t)$, and $z = z(t)$ in the expression for $\mathbf{F}(x, y, z)$.

Notice also that we can formally write $d\mathbf{r} = \mathbf{r}'(t)\, dt$.

Example 7

Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le \pi/2$.

Line Integrals of Vector Fields

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field **F** on $\mathbb{R}^3$ is given in component form by the equation **F** = P **i** + Q **j** + R **k**.

We use Definition 13 to compute its line integral along C:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \big(x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} + z'(t)\,\mathbf{k}\big)\, dt$$

$$= \int_a^b \Big[P\big(x(t), y(t), z(t)\big)\, x'(t) + Q\big(x(t), y(t), z(t)\big)\, y'(t)$$

$$+ R\big(x(t), y(t), z(t)\big)\, z'(t)\Big]\, dt$$

Line Integrals of Vector Fields

Therefore, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz \qquad \text{where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

For example, the integral $\int_C y\,dx + z\,dy + x\,dz$ could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$

16.3 The Fundamental Theorem for Line Integrals

The Fundamental Theorem for Line Integrals

Recall that Part 2 of the Fundamental Theorem of Calculus can be written as

$$\boxed{1} \qquad \int_a^b F'(x)\,dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$.

We also called Equation 1 the Net Change Theorem: The integral of a rate of change is the net change.

The Fundamental Theorem for Line Integrals

If we think of the gradient vector ∇f of a function f of two or three variables as a sort of derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

$\boxed{2}$ **Theorem** Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Example 1

Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass *m* from the point (3, 4, 12) to the point (2, 2, 0) along a piecewise-smooth curve *C*.

Independence of Path

Independence of Paths

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and terminal point B.

We know that, in general, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. But one implication of Theorem 2 is that $\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$

whenever ∇f is continuous.

In other words, the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve.

Independence of Paths

In general, if **F** is a continuous vector field with domain *D*, we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in *D* that have the same initial and terminal points.

With this terminology we can say that *line integrals of conservative vector fields are independent of path.*

A curve is called **closed** if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$. (See Figure 2.)

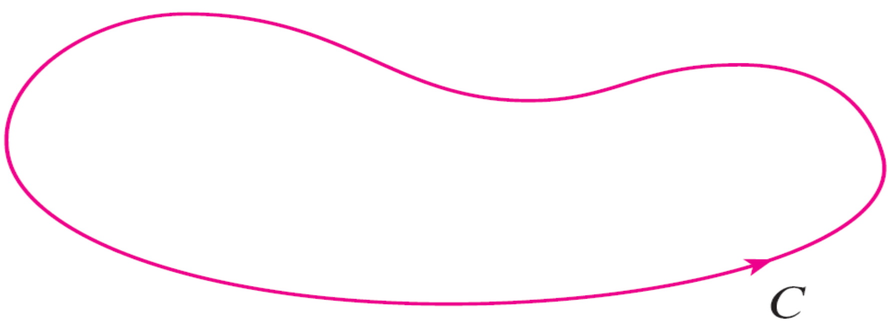


Figure 2
A closed curve

Independence of Paths

Let $\int_C \mathbf{F} \cdot d\mathbf{r}$ be independent of path in D and let C be any closed path in D. Choose any two distinct points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A. (See Figure 3.)

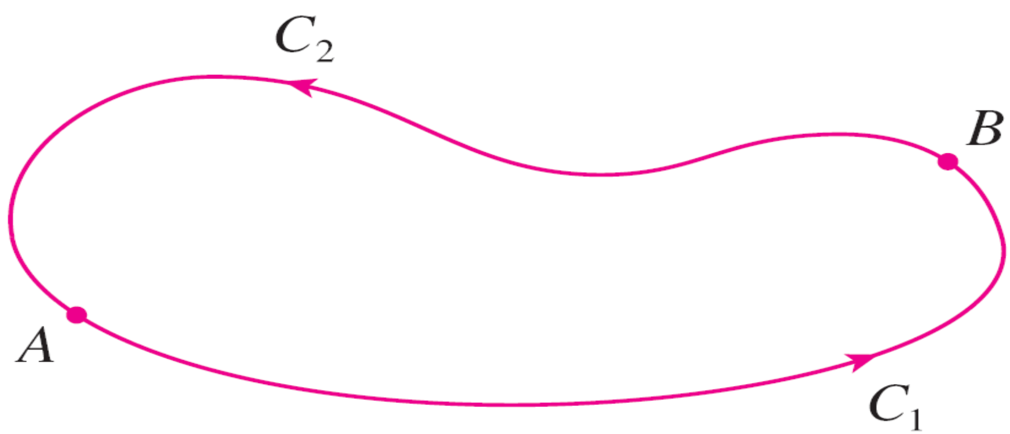


Figure 3

Independence of Paths

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.

Conversely, Suppose that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then we demonstrate independence of path as follows.

Take any two paths C_1 and C_2 from A to B in D and define C to be the curve consisting of C_1 followed by $-C_2$.

Independence of Paths

Then

$$0 = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

and so $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

Thus we have proved the following theorem.

3 Theorem $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Since we know that the line integral of any conservative vector field **F** is independent of path, it follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path.

Independence of Paths

The physical interpretation is that the work done by a conservative force field as it moves an object around a closed path is 0.

Independence of Paths

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves.

We assume that D is **open**, which means that for every point P in D there is a disk with center P that lies entirely in D. (So D doesn't contain any of its boundary points.)

In addition, we assume that D is **connected**: this means that any two points in D can be joined by a path that lies in D.

4 Theorem Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Independence of Paths

The question remains: How is it possible to determine whether or not a vector field **F** is conservative? Suppose it is known that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative, where P and Q have continuous first-order partial derivatives.

Independence of Paths

Then there is a function *f* such that **F** = ∇*f*, that is, $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$. Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial Q}{\partial x}$$

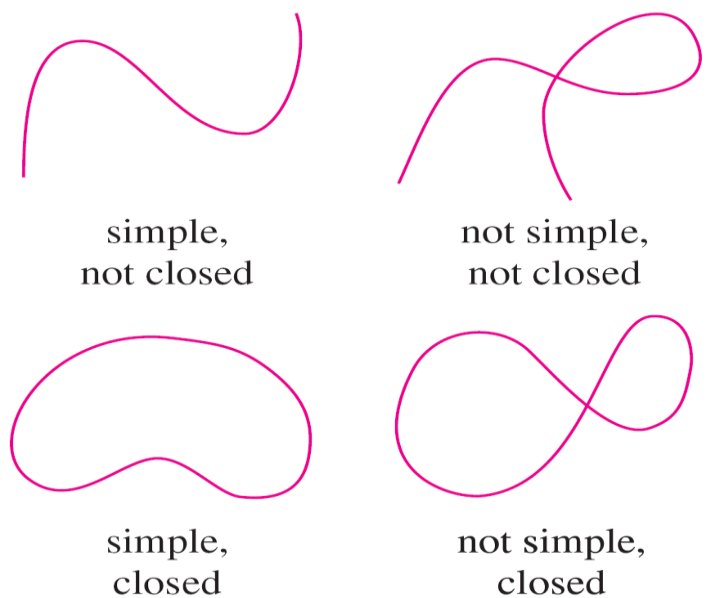
$\boxed{5}$ **Theorem** If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The converse of Theorem 5 is true only for a special type of region.

Independence of Paths

To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints. [See Figure 6; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.]



Types of curves

Figure 6

Independence of Paths

In Theorem 4 we needed an open connected region. For the next theorem we need a stronger condition.

A **simply-connected region** in the plane is a connected region *D* such that every simple closed curve in *D* encloses only points that are in *D*.

Notice from Figure 7 that, intuitively speaking, a simply-connected region contains no hole and can't consist of two separate pieces.

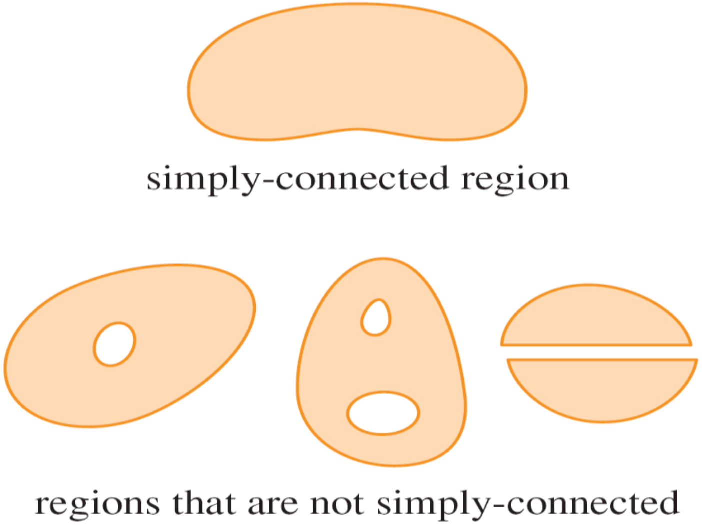


Figure 7

Independence of Paths

In terms of simply-connected regions, we can now state a partial converse to Theorem 5 that gives a convenient method for verifying that a vector field on $\mathbb{R}^2$ is conservative.

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

Example 2

Determine whether or not the vector field is conservative.

(a) $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$

(b) $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$

Examples 3 and 4

Example 3. If $\mathbf{F}(x, y) = (3+2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\nabla f = \mathbf{F}$.

Example 4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where **F** is as in Example 3 and $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j}$, $0 \le t \le \pi$.

Conservation of Energy

Conservation of Energy

Let's apply the ideas of Chapter 16.3 to a continuous force field **F** that moves an object along a path *C* given by $\mathbf{r}(t)$, $a \le t \le b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of *C*. According to Newton's Second Law of Motion, the force $\mathbf{F}(\mathbf{r}(t))$ at a point on *C* is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt$$

Conservation of Energy

$$= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)] \, dt$$

(By formula $\frac{d}{dt}$ [$\mathbf{u}(t) \cdot \mathbf{v}(t)$]
$= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$)

$$= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2 \, dt = \frac{m}{2} \Big[|\mathbf{r}'(t)|^2 \Big]_a^b$$

(Fundamental Theorem of Calculus)

$$= \frac{m}{2} \left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2 \right)$$

Therefore

$\boxed{15}$ $$W = \tfrac{1}{2} m |\mathbf{v}(b)|^2 - \tfrac{1}{2} m |\mathbf{v}(a)|^2$$

where **v** = **r**′ is the velocity.

Conservation of Energy

The quantity $\frac{1}{2}m\,|\,\mathbf{v}(t)\,|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore we can rewrite Equation 15 as

$$\boxed{16} \qquad W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that **F** is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$.

Conservation of Energy

In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$.

Then by the FTC for line integrals we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= -\int_C \nabla P \cdot d\mathbf{r}$$

$$= -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))]$$

$$= P(A) - P(B)$$

Conservation of Energy

Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant.

This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative.*

16.4 Green's Theorem

Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 1. We assume that D consists of all points inside C as well as all points on C.)

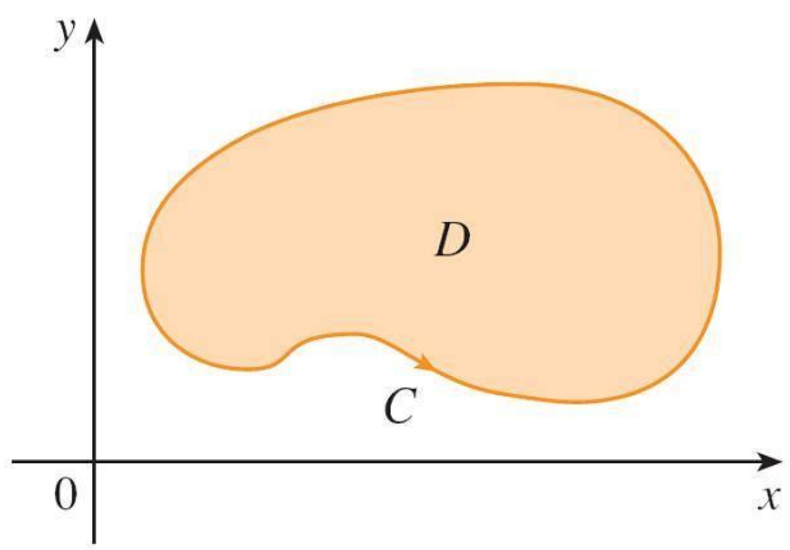
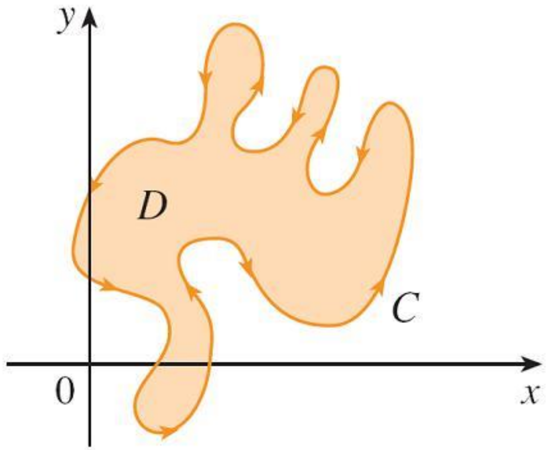


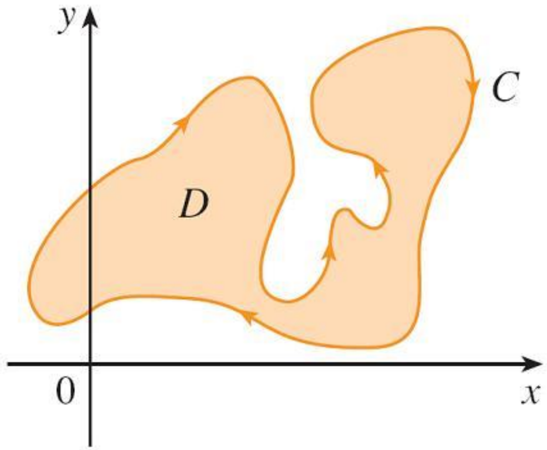
Figure 1

Green's Theorem

In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus, if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C.
(See Figure 2.)



(a) Positive orientation



(b) Negative orientation

Figure 2

Green's Theorem

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Notation: We sometimes use the notation $\oint_C P\,dx + Q\,dy$ or $\ointctrclockwise_C P\,dx + Q\,dy$ to indicate that the line integral is calculated using the positive orientation of the closed curve C.

Another notation for the positively oriented boundary curve of D is ∂D. One may replace C by ∂D in Green's Theorem.

Example 1

Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1), and from (0, 1) to (0, 0).

Green's Theorem

In Example 1 we found that the double integral was easier to evaluate than the line integral.

But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction.

For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C, then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy = 0$$

no matter what values P and Q assume in the region D.

Green's Theorem

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1\, dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$P(x, y) = 0$	$P(x, y) = -y$	$P(x, y) = -\frac{1}{2} y$
$Q(x, y) = x$	$Q(x, y) = 0$	$Q(x, y) = \frac{1}{2} x$

Green's Theorem

Then Green’s Theorem gives the following formulas for the area of *D*:

5

$$A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx$$

Example 3

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Extended Versions of Green’s Theorem

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions.

For example, if D is the region shown in Figure 6, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple.

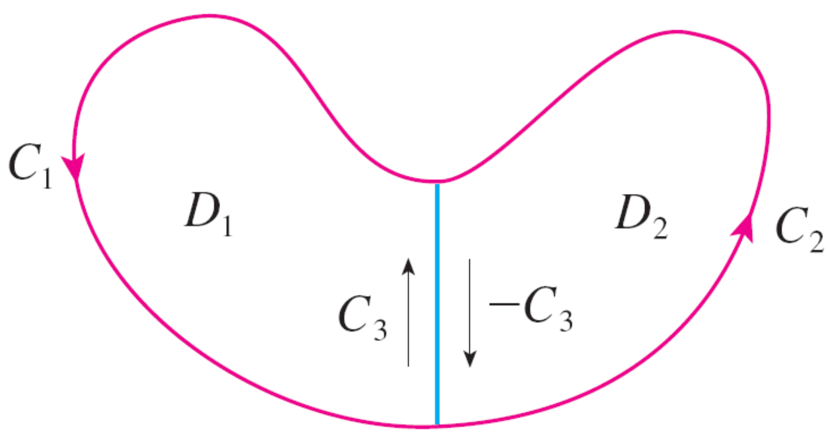


Figure 6

Extended Versions of Green's Theorem

The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup (-C_3)$ so, applying Green's Theorem to D_1 and D_2 separately, we get

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Extended Versions of Green's Theorem

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

The same sort of argument allows us to establish Green's Theorem for any finite union of nonoverlapping simple regions (see Figure 7).

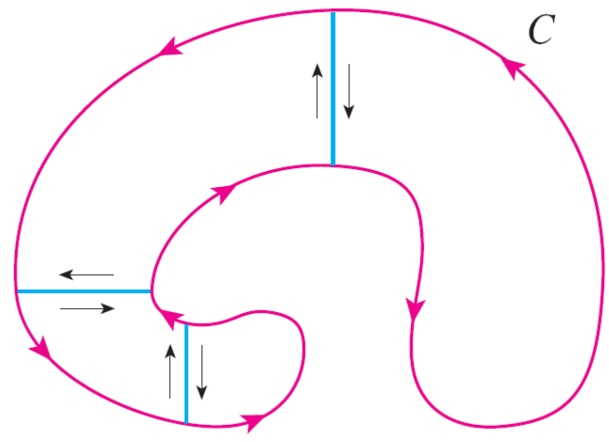


Figure 7

Example 4

Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Figure 8

Extended Versions of Green's Theorem

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected.

Observe that the boundary C of the region D in Figure 9 consists of two simple closed curves C_1 and C_2.

We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed.

Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2.

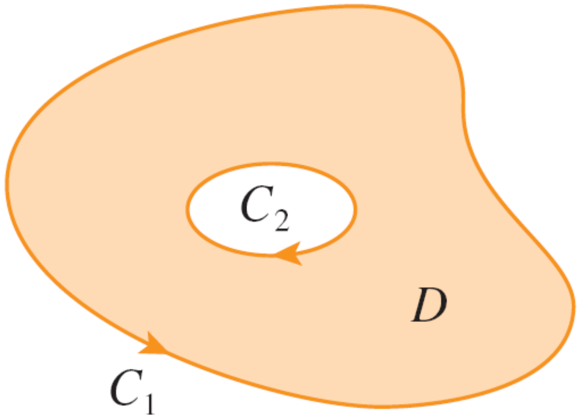


Figure 9

Extended Versions of Green's Theorem

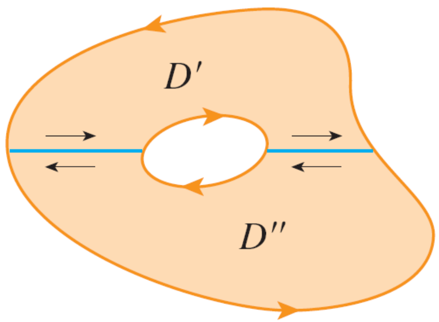


Figure 10

If we divide D into two regions D' and D'' by means of the lines shown in Figure 10 and then apply Green's Theorem to each of D' and D'', we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

Extended Versions of Green's Theorem

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy$$

which is Green's Theorem for the region *D*.

Conservative vector fields again

The proof of:

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

from Section 16.3.

Note: From the proof we see why D has to be simply connected. See 16.4/Example 5 in the book for a counterexample where the domain of **F** is missing a point.

Example 5

Let

$$F(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}.$$

(a) Show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on $D = \{ (x,y) \mid (x,y) \neq (0, 0) \}$.

(b) Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any closed path that encloses the origin.

16.5 Curl and Divergence

Curl

Curl

If **F** = P **i** + Q **j** + R **k** is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of **F** is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{1} \qquad \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Let’s rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ (“del”) as

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

Curl

It has meaning when it operates on a scalar function to produce the gradient of *f*:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field **F** as follows:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Curl

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \text{curl } \mathbf{F}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

2 $$\boxed{\text{curl } \mathbf{F} = \nabla \times \mathbf{F}}$$

Example 1

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

Curl

Recall that the gradient of a function *f* of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl.

The following theorem says that the curl of a gradient vector field is **0**.

$\boxed{3}$ **Theorem** If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

Curl

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$.

This gives us a way of verifying that a vector field is not conservative.

Example 2

Show that the vector field

$$\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$$

is not conservative.

Curl

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if **F** is defined everywhere. (More generally it is true if the domain is simply-connected, that is, “has no hole.”)

$\boxed{4}$ **Theorem** If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Example 3

(a) Show that the vector field

$$\mathbf{F}(x, y, z) = y^2z^3\ \mathbf{i} + 2xyz^3\ \mathbf{j} + 3xy^2z^2\ \mathbf{k}$$

is a conservative vector field.

(b) Find a function *f* such that $\mathbf{F} = \nabla f$.

Curl

The reason for the name *curl* is that the curl vector is associated with rotations.

Suppose that **F** represents the velocity field in fluid flow. Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1).

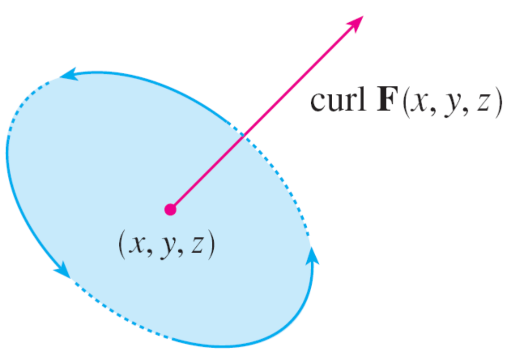


Figure 1

Curl

If curl **F** = **0** at a point *P*, then the fluid is free from rotations at *P* and **F** is called **irrotational** at *P*.

In other words, there is no whirlpool or eddy at *P*.

If curl **F** = **0**, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis.

If curl **F** ≠ **0**, the paddle wheel rotates about its axis.

Paddle wheel

Divergence

Divergence

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

9

$$\boxed{\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Observe that curl **F** is a vector field but div **F** is a scalar field.

Divergence

In terms of the gradient operator $\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of $\mathbf{F}$ can be written symbolically as the dot product of ∇ and $\mathbf{F}$:

$$\boxed{10} \qquad \boxed{\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}}$$

Example 4

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div $\mathbf{F}$.

Divergence

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence.

The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Again, the reason for the name *divergence* can be understood in the context of fluid flow.

Divergence

If **F**(x, y, z) is the velocity of a fluid (or gas), then div **F**(x, y, z) represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.

In other words, div **F**(x, y, z) measures the tendency of the fluid to diverge from the point (x, y, z).

If div **F** = 0, then **F** is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f.

Divergence

If f is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Divergence

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

in terms of its components:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$

Note: Sometimes, when it causes no confusion with the increment of a function, one uses the following notation for the Laplace operator: $\nabla^2 = \Delta$.

Example 5

Show that $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ cannot be written as the curl of another vector field.

Vector Forms of Green’s Theorem

Vector Forms of Green's Theorem

The curl and divergence operators allow us to rewrite Green's Theorem in versions that will be useful in our later work.

We suppose that the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem.

Then we consider the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

Vector Forms of Green's Theorem

Its line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$

and, regarding **F** as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Vector Forms of Green's Theorem

Therefore

$$(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

12

$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA}$$

Vector Forms of Green's Theorem

Equation 12 expresses the line integral of the tangential component of **F** along C as the double integral of the vertical component of curl **F** over the region D enclosed by C. We now derive a similar formula involving the *normal* component of **F**.

If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad a \le t \le b$$

then the unit tangent vector is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

Vector Forms of Green's Theorem

You can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

(See Figure 4.)

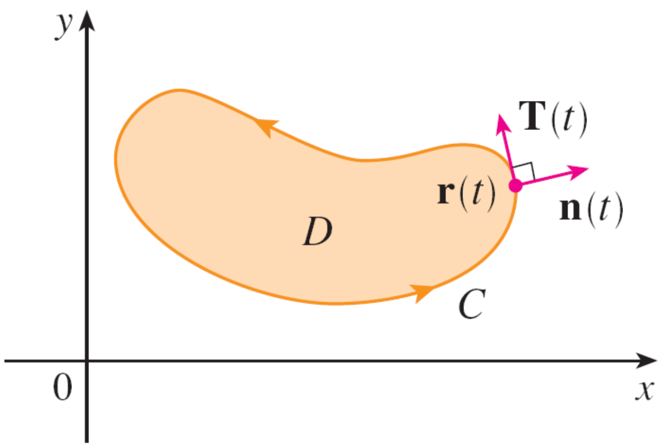


Figure 4

Vector Forms of Green's Theorem

Then, from equation

$$\int_C f(x, y)\, ds = \int_a^b f\big(x(t), y(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{n}\, ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t)\, |\mathbf{r}'(t)|\, dt \\
&= \int_a^b \left[\frac{P\big(x(t), y(t)\big)\, y'(t)}{|\mathbf{r}'(t)|} - \frac{Q\big(x(t), y(t)\big)\, x'(t)}{|\mathbf{r}'(t)|}\right] |\mathbf{r}'(t)|\, dt \\
&= \int_a^b P\big(x(t), y(t)\big)\, y'(t)\, dt - Q\big(x(t), y(t)\big)\, x'(t)\, dt \\
&= \int_C P\, dy - Q\, dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA
\end{aligned}$$

by Green's Theorem.

Vector Forms of Green's Theorem

But the integrand in this double integral is just the divergence of **F**. So we have a second vector form of Green's Theorem.

13
$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

This version says that the line integral of the normal component of **F** along *C* is equal to the double integral of the divergence of **F** over the region *D* enclosed by *C*.

16.6 Parametric Surfaces and Their Areas

Parametric Surfaces and Their Areas

Here we use vector functions to describe more general surfaces, called *parametric surfaces*, and compute their areas.

Then we take the general surface area formula and see how it applies to special surfaces.

Parametric Surfaces

Parametric Surfaces

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v.

We suppose that

$$\boxed{1} \qquad \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane.

Parametric Surfaces

So x, y, and, z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D.

The set of all points (x, y, z) in $\mathbb{R}^3$ such that

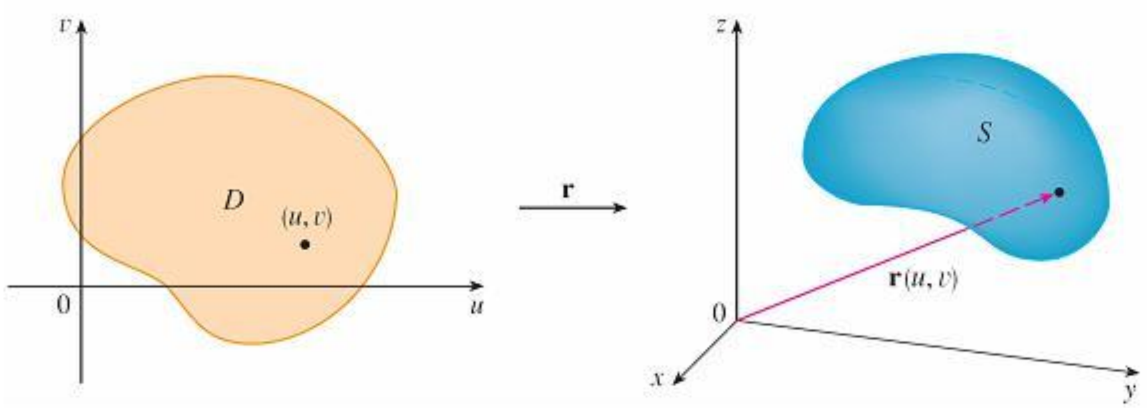
$$\boxed{2} \qquad x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies throughout D, is called a **parametric surface** S and Equations 2 are called **parametric equations** of S.

Parametric Surfaces

Each choice of *u* and *v* gives a point on *S*; by making all choices, we get all of *S*.

In other words, the surface is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region *D*. (See Figure 1.)



A parametric surface

Figure 1

Example 1

Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u \, \mathbf{i} + v \, \mathbf{j} + 2 \sin u \, \mathbf{k}$$

Example 3

Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors **a** and **b**.

Parametric Surfaces

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant.

These families correspond to vertical and horizontal lines in the uv-plane.

Parametric Surfaces

If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S. (See Figure 4.)

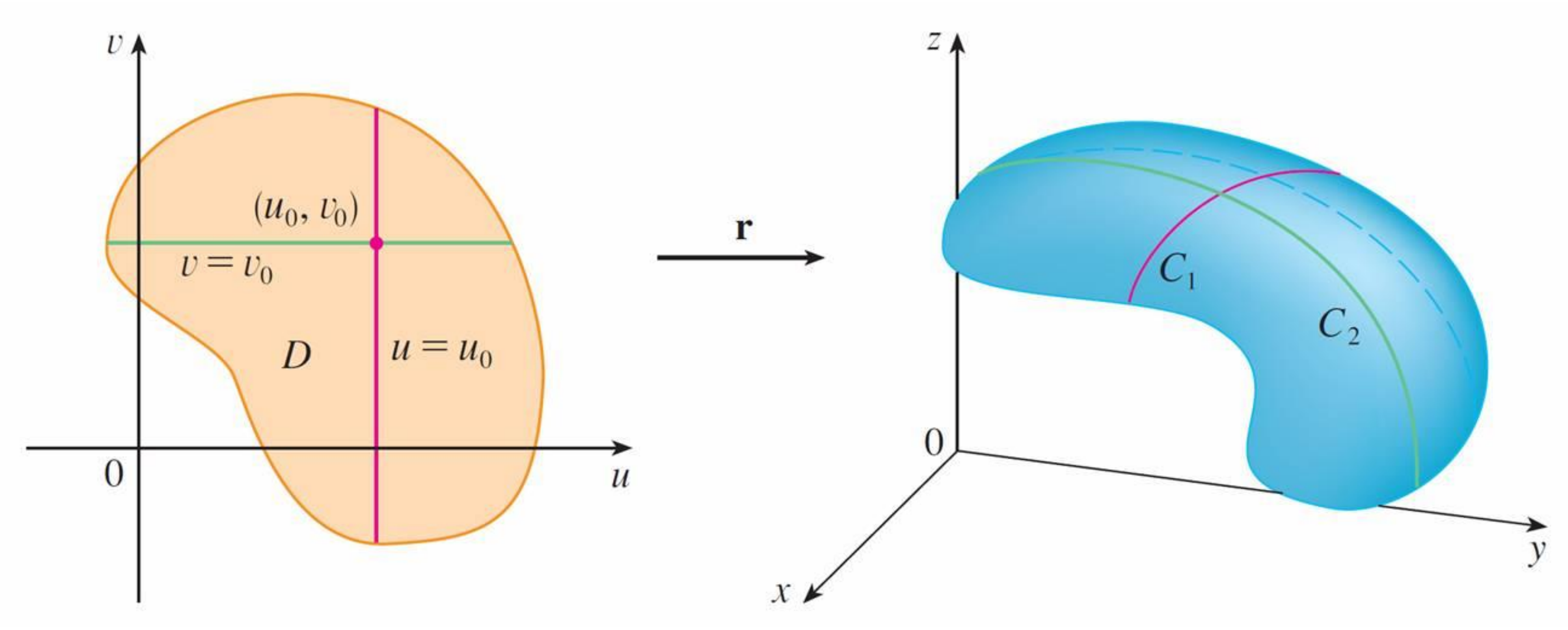


Figure 4

Parametric Surfaces

Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S.

We call these curves **grid curves**. (In Example 1, for instance, the grid curves obtained by letting u be constant are horizontal lines whereas the grid curves with v constant are circles.)

In fact, when a computer graphs a parametric surface, it usually depicts the surface by plotting these grid curves.

Example 4

Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

and find the grid curves.

Parametric Surfaces

Note:
We saw in Example 4 that the grid curves for a sphere are curves of constant latitude and longitude.

For a general parametric surface we are really making a map and the grid curves are similar to lines of latitude and longitude.

Describing a point on a parametric surface by giving specific values of u and v is like giving the latitude and longitude of a point.

Surfaces of Revolution

Surfaces of Revolution

Surfaces of revolution can be represented parametrically and thus graphed using a computer. For instance, let's consider the surface S obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis, where $f(x) \geq 0$.

Let θ be the angle of rotation as shown in Figure 10.

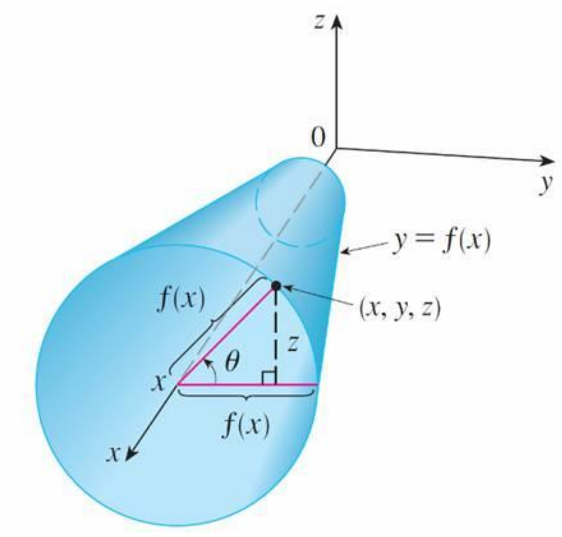


Figure 10

Surfaces of Revolution

If (x, y, z) is a point on S, then

$$\boxed{3} \qquad x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta$$

Therefore, we take x and θ as parameters and regard Equations 3 as parametric equations of S.

The parameter domain is given by $a \leq x \leq b$, $0 \leq \theta \leq 2\pi$.

Graph of a function

If a surface S is given by the equation $z = f(x,y)$ where $(x,y) \in D$, then a simple choice for parametric equations describing the surface is:

$$x = x \qquad y = y \qquad z = f(x,y) \qquad (x,y) \in D.$$

Note: Parametrizations of surfaces are not unique so this is just one simple way of writing the surface using parametric equations in this case.

Tangent Planes

Tangent Planes

We now find the tangent plane to a parametric surface S traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.

Tangent Planes

If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S. (See Figure 12.) The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

$$\boxed{4} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\,\mathbf{k}$$

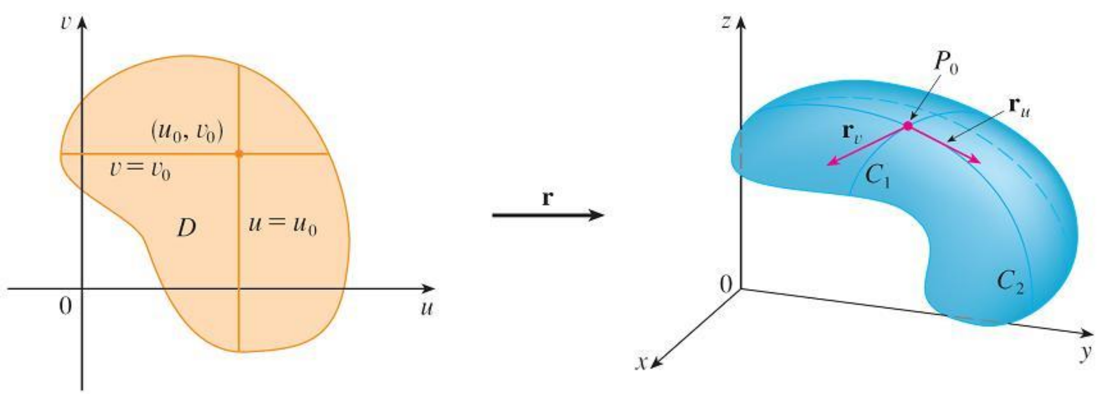


Figure 12

Tangent Planes

Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S, and its tangent vector at P_0 is

$$\boxed{5} \qquad \mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\,\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\,\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\,\mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called **smooth** (it has no "corners").

For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and hence the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Example 9

Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point (1, 1, 3).

Surface Area

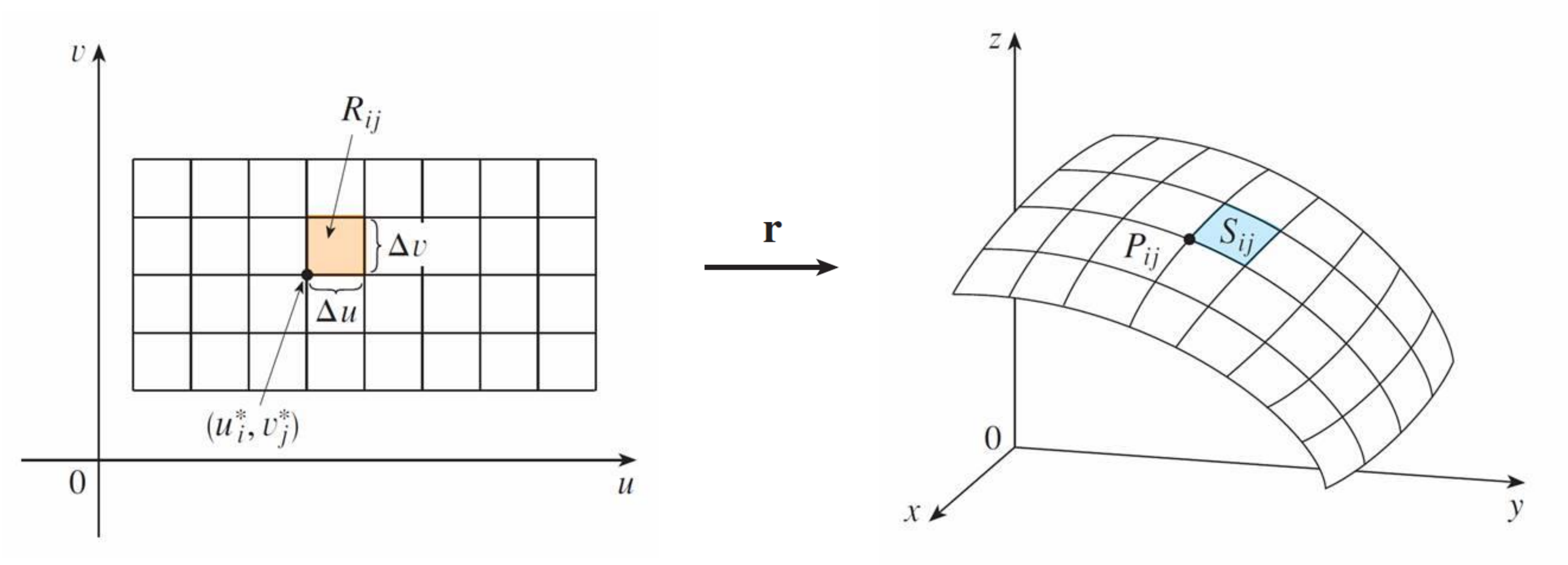
Surface Area

Now we define the surface area of a general parametric surface.

For simplicity we start by considering a surface whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij}.

Surface Area

Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}. (See Figure 14.)



The image of the subrectangle R_{ij} is the patch S_{ij}.

Figure 14

Surface Area

The part S_{ij} of the surface that corresponds to R_{ij} is called a *patch* and has the point P_{ij} with position $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners.

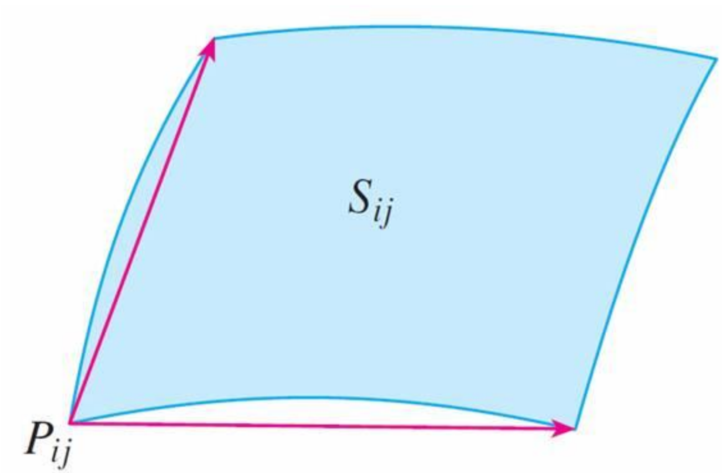
Let

$$\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*) \quad \text{and} \quad \mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij} as given by Equations 5 and 4.

Surface Area

Figure 15(a) shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors. These vectors, in turn, can be approximated by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$ because partial derivatives can be approximated by difference quotients.



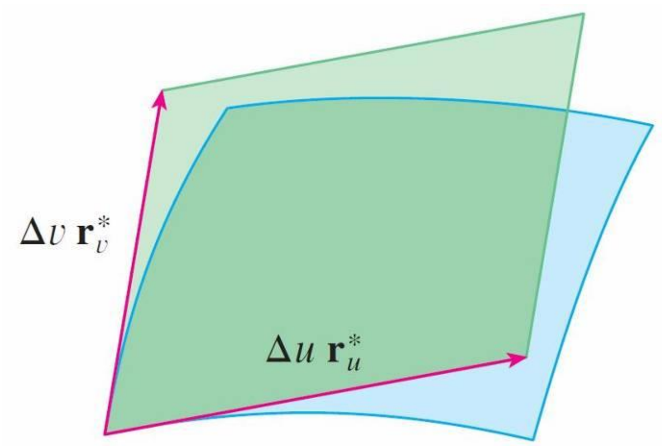
Approximating a patch by a parallelogram.

Figure 15(a)

Surface Area

So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u\, \mathbf{r}_u^*$ and $\Delta v\, \mathbf{r}_v^*$.

This parallelogram is shown in Figure 15(b) and lies in the tangent plane to S at P_{ij}.



Approximating a patch by a parallelogram.

Figure 15(b)

Surface Area

The area of this parallelogram is

$$|(\Delta u\, \mathbf{r}_u^*) \times (\Delta v\, \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

and so an approximation to the area of *S* is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*|\, \Delta u\, \Delta v$$

Surface Area

Our intuition tells us that this approximation gets better as we increase the number of subrectangles, and we recognize the double sum as a Riemann sum for the double integral $\iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$.

Surface Area

This motivates the following definition.

$\boxed{6}$ **Definition** If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where $\quad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$

Example 10

Find the surface area of a sphere of radius a.

Surface Area of the Graph of a function

Surface Area of the Graph of a Function

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters.

The parametric equations are

$$x = x \qquad y = y \qquad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

and

$$\boxed{7} \qquad \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$$

Surface Area of the Graph of a Function

Thus we have

8 $$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

and the surface area formula in Definition 6 becomes

9 $$\boxed{A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA}$$

Example 11

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

16.7 Surface Integrals

Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length.

Suppose f is a function of three variables whose domain includes a surface S.

We will define the surface integral of f over S in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal to the surface area of S.

We start with parametric surfaces and then deal with the special case where S is the graph of a function of two variables.

Parametric Surfaces

Parametric Surfaces

Suppose that a surface has a vector equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

We first assume that the parameter domain D is a rectangle and we divide it into subrectangles R_{ij} with dimensions Δu and Δv.

Then the surface S is divided into corresponding patches S_{ij} as in Figure 1.

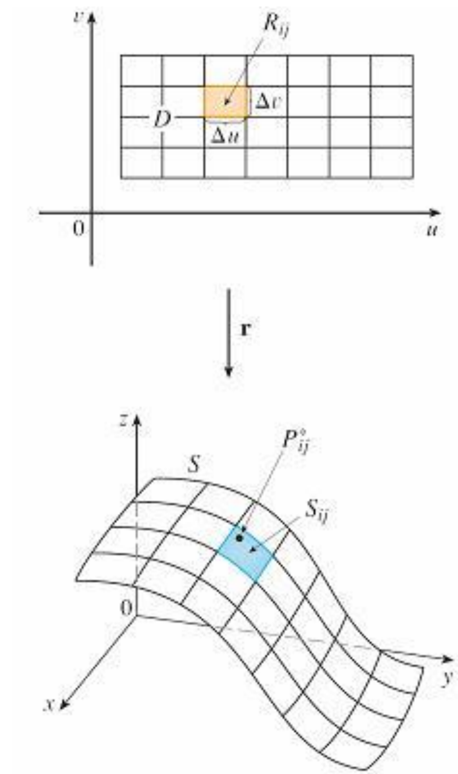


Figure 1

Parametric Surfaces

We evaluate *f* at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \, \Delta S_{ij}$$

Then we take the limit as the number of patches increases and define the **surface integral of *f* over the surface *S*** as

$$\boxed{1} \qquad \iint_S f(x, y, z) \, dS = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \, \Delta S_{ij}$$

Notice the analogy with the definition of a line integral and also the analogy with the definition of a double integral.

Parametric Surfaces

To evaluate the surface integral in Equation 1 we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane.

In our discussion of surface area we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \, \Delta u \, \Delta v$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

are the tangent vectors at a corner of S_{ij}.

Parametric Surfaces

If the components are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of *D*, it can be shown from Definition 1, even when *D* is not a rectangle, that

$$\boxed{2} \qquad \boxed{\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\,|\mathbf{r}_u \times \mathbf{r}_v|\, dA}$$

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\, dt$$

Parametric Surfaces

Observe also that

$$\iint_S 1\, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = A(S)$$

Formula 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain D.

When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the formula for $f(x, y, z)$.

Example 1

Compute the surface integral $\iint_S x^2 \, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Parametric Surfaces

If a thin sheet (say, of aluminum foil) has the shape of a surface *S* and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z)\, dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m} \iint_S x\,\rho(x, y, z)\, dS \qquad \bar{y} = \frac{1}{m} \iint_S y\,\rho(x, y, z)\, dS \qquad \bar{z} = \frac{1}{m} \iint_S z\,\rho(x, y, z)\, dS$$

Graphs

Graphs

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

Thus

$$\boxed{3} \qquad \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Graphs

Therefore, in this case, Formula 2 becomes

$$\boxed{4} \qquad \iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Similar formulas apply when it is more convenient to project *S* onto the *yz*-plane or *xz*-plane. For instance, if *S* is a surface with equation $y = h(x, z)$ and *D* is its projection onto the *xz*-plane, then

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, h(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

Example 2

Evaluate $\iint_S y\, dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$. (See Figure 2.)

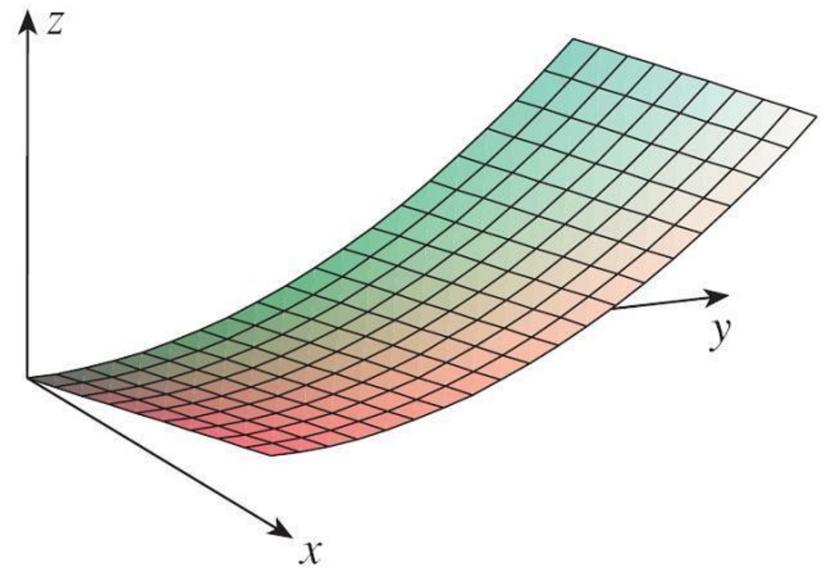


Figure 2

Graphs

If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces $S_1, S_2, \ldots, S_n$ that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x, y, z)\, dS = \iint_{S_1} f(x, y, z)\, dS + \cdots + \iint_{S_n} f(x, y, z)\, dS$$

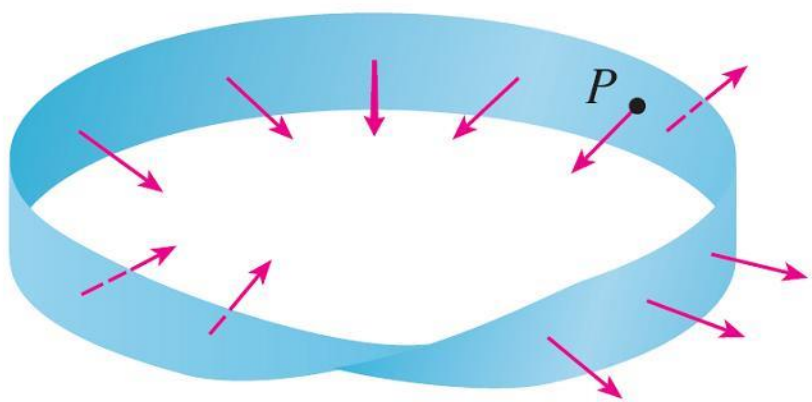
Example 3

Evaluate $\iint_S z\, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose botton S_2 is the disc $x^2 + y^2 \leq 1$ in the plane $z = 0$, and whose top S_3 is part of the plane $z = 1 + x$ above S_2.

Oriented Surfaces

Oriented Surfaces

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).]

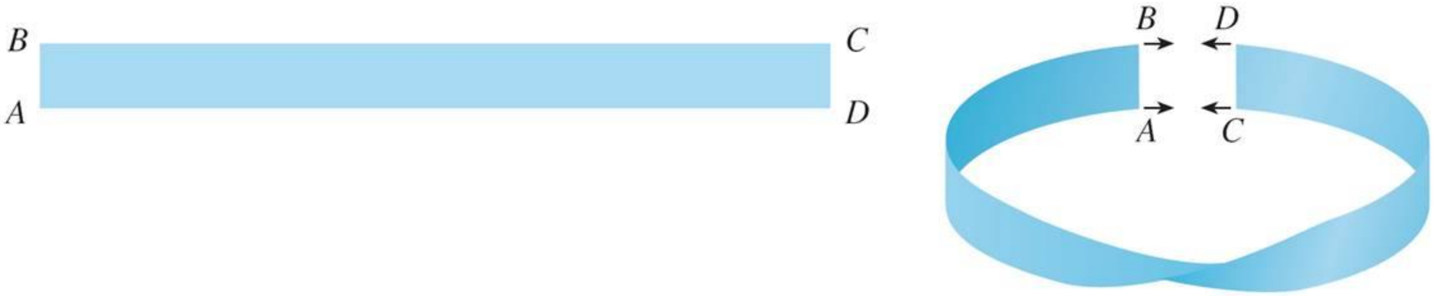


A Möbius strip

Figure 4

Oriented Surfaces

You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5.



Constructing a Möbius strip

Figure 5

Oriented Surfaces

If an ant were to crawl along the Möbius strip starting at a point *P*, it would end up on the "other side" of the strip (that is, with its upper side pointing in the opposite direction).

Then, if the ant continued to crawl in the same direction, it would end up back at the same point *P* without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.)

Therefore, a Möbius strip really has only one side.

From now on we consider only orientable (two-sided) surfaces.

Oriented Surfaces

We start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point).

There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z). (See Figure 6.)

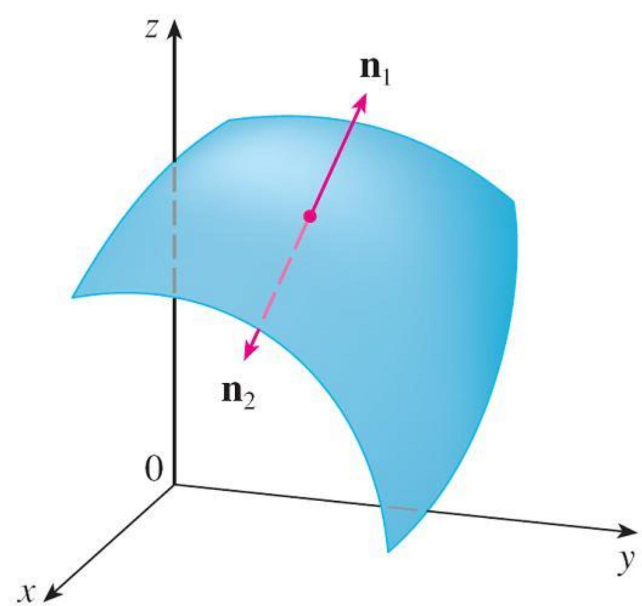
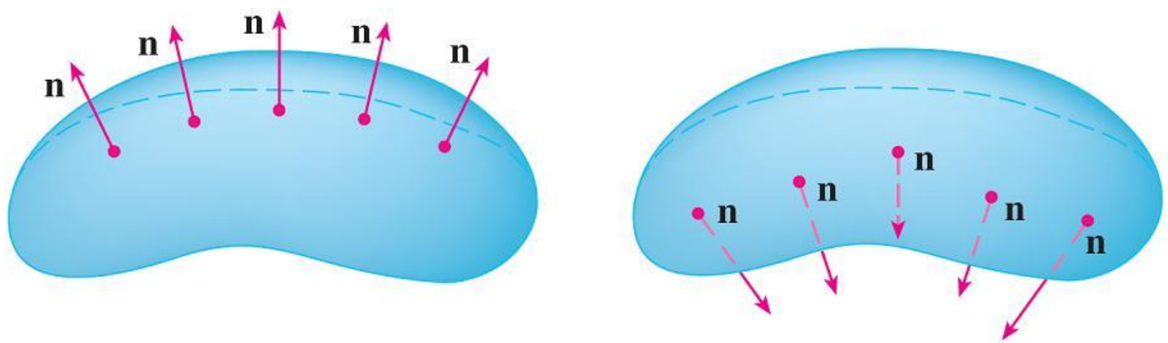


Figure 6

Oriented Surfaces

If it is possible to choose a unit normal vector **n** at every such point (x, y, z) so that **n** varies continuously over S, then S is called an **oriented surface** and the given choice of **n** provides S with an **orientation**.

There are two possible orientations for any orientable surface (see Figure 7).



The two orientations of an orientable surface

Figure 7

Recap: Graphs

We saw: a surface S with equation $z = g(x, y)$ can be seen as a parametric surface with parametric equations

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

Thus

$$\boxed{3} \qquad \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Oriented Surfaces

Thus, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\boxed{5} \qquad \mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\,\mathbf{i} - \dfrac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$$

Since the **k**-component is positive, this gives the *upward* orientation of the surface.

Oriented Surfaces

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

6
$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and the opposite orientation is given by $-\mathbf{n}$.

For instance, the parametric representation is

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \, \mathbf{i} + a \sin \phi \sin \theta \, \mathbf{j} + a \cos \phi \, \mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$.

Oriented Surfaces

We saw last time that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2 \phi \cos \theta\, \mathbf{i} + a^2 \sin^2 \phi \sin \theta\, \mathbf{j} + a^2 \sin \phi \cos \phi\, \mathbf{k}$$

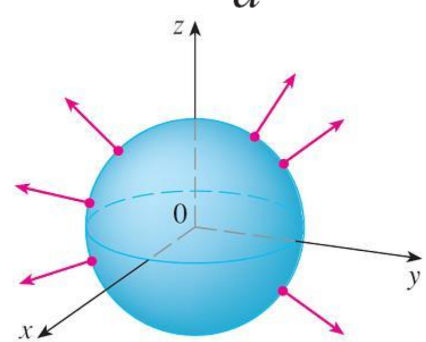
and

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin \phi$$

So the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin\phi \cos\theta\, \mathbf{i} + \sin\phi \sin\theta\, \mathbf{j} + \cos\phi\, \mathbf{k} = \frac{1}{a}\mathbf{r}(\phi, \theta)$$

Observe that $\mathbf{n}$ points in the same direction as the position vector, that is, outward from the sphere (see Figure 8).

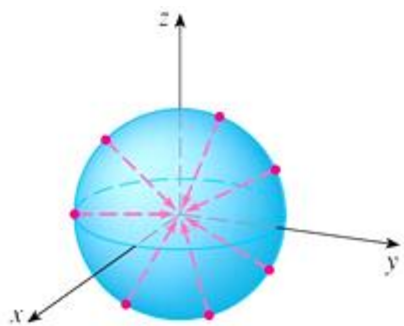


Positive orientation

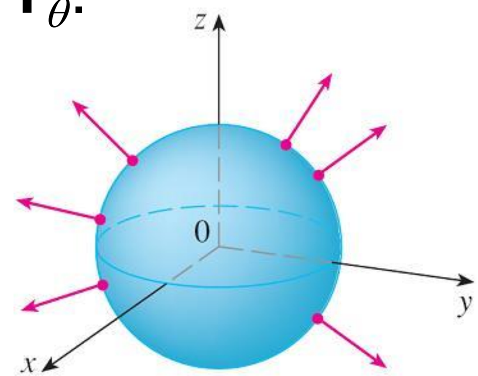
Figure 8

Oriented Surfaces

The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.



Negative orientation
Figure 9



Positive orientation
Figure 8

For a **closed surface**, that is, a surface that is the boundary of a solid region E, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E, and inward-pointing normals give the negative orientation (see Figures 8 and 9).

Surface Integrals of Vector Fields

Surface Integrals of Vector Fields

Suppose that S is an oriented surface with unit normal vector $\mathbf{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing *through* S. (Think of S as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.)

Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$.

Surface Integrals of Vector Fields

We divide S into small patches S_{ij}, as in Figure 10 (compare with Figure 1).

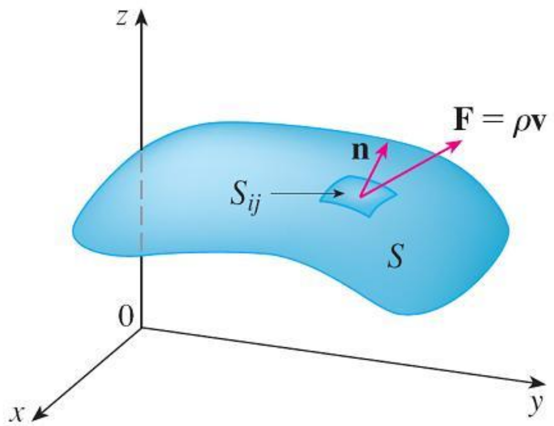


Figure 10

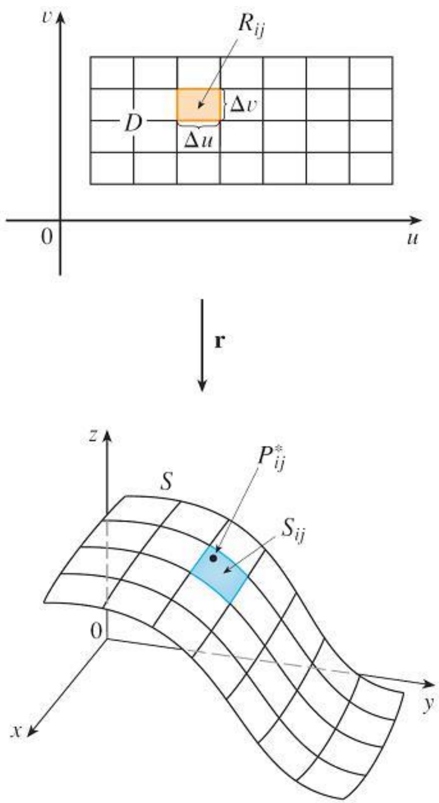


Figure 1

Surface Integrals of Vector Fields

Then S_{ij} is nearly planar and so we can approximate the mass of fluid per unit time crossing S_{ij} in the direction of the normal **n** by the quantity

$$(\rho \mathbf{v} \cdot \mathbf{n}) A(S_{ij})$$

where ρ, **v**, and **n** are evaluated at some point on S_{ij}. (Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector **n** is $\rho\mathbf{v} \cdot \mathbf{n}$.)

Surface Integrals of Vector Fields

By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over S:

$$\boxed{7} \qquad \iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS$$

and this is interpreted physically as the rate of flow through S.

If we write $\mathbf{F} = \rho\mathbf{v}$, then $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$ and the integral in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

Surface Integrals of Vector Fields

A surface integral of this form occurs frequently in physics, even when **F** is not $\rho\mathbf{v}$, and is called the *surface integral* (or *flux integral*) of **F** over *S*.

8 Definition If **F** is a continuous vector field defined on an oriented surface S with unit normal vector **n**, then the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of **F** across S.

In words, Definition 8 says that the surface integral of a vector field over *S* is equal to the surface integral of its normal component over *S*.

Surface Integrals of Vector Fields

If S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where D is the parameter domain. Thus we have

9

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

Example 4

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Surface Integrals of Vector Fields

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface *S* given by a graph $z = g(x, y)$, we can think of *x* and *y* as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\,\mathbf{i} - \frac{\partial g}{\partial y}\,\mathbf{j} + \mathbf{k}\right)$$

Surface Integrals of Vector Fields

Thus Formula 9 becomes

10

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

This formula assumes the upward orientation of S; for a downward orientation we multiply by –1.

Similar formulas can be worked out if S is given by $y = h(x, z)$ or $x = k(y, z)$.

Example 5

Example 5. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations.

Surface Integrals of Vector Fields

For instance, if **E** is an electric field, then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of **E** through the surface *S*. One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface *S* is

$$\boxed{11} \qquad Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ C^2/N \cdot m^2$.)

Surface Integrals of Vector Fields

Therefore, if the vector field **F** in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = \frac{4}{3}\pi\varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\,\nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance. This is called Fourier's law of heat conduction.

Surface Integrals of Vector Fields

The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

16.8 Stokes' Theorem

Stokes' Theorem

Stokes’ Theorem can be regarded as a higher-dimensional version of Green’s Theorem.

Whereas Green’s Theorem relates a double integral over a plane region *D* to a line integral around its plane boundary curve, Stokes’ Theorem relates a surface integral over a surface *S* to a line integral around the boundary curve of *S* (which is a space curve).

Figure 1 shows an oriented surface with unit normal vector **n**.

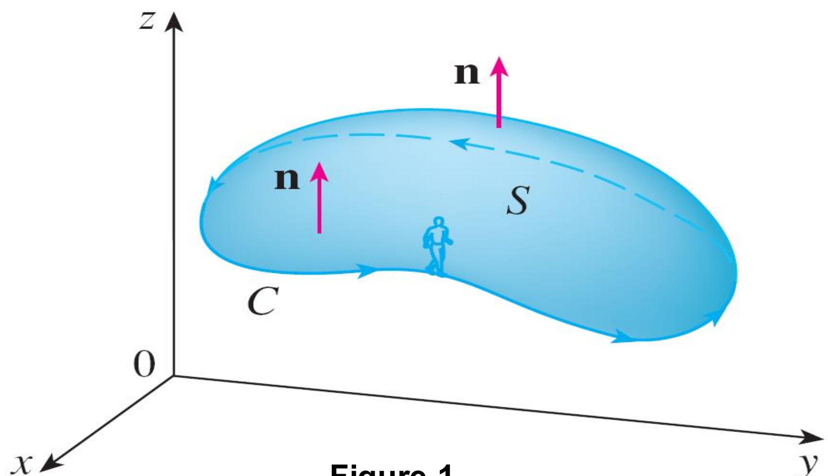


Figure 1

Stokes' Theorem

The orientation of S induces the **positive orientation of the boundary curve C** shown in the figure.

This means that if you walk in the positive direction around C with your head pointing in the direction of **n**, then the surface will always be on your left.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Stokes' Theorem

Recall Stokes' Theorem:

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Stokes' Theorem

Since

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ and $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$

Stokes' Theorem says that the line integral around the boundary curve of *S* of the tangential component of **F** is equal to the surface integral over *S* of the normal component of the curl of **F**.

The positively oriented boundary curve of the oriented surface *S* is often written as ∂S, so Stokes' Theorem can be expressed as

$$\boxed{1} \qquad \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

Stokes' Theorem

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus.

As before, there is an integral involving derivatives on the left side of Equation 1 (recall that curl **F** is a sort of derivative of **F**) and the right side involves the values of **F** only on the *boundary* of *S*.

Stokes' Theorem

In fact, in the special case where the surface S is flat and lies in the xy-plane with upward orientation, the unit normal is **k**, the surface integral becomes a double integral, and Stokes' Theorem becomes

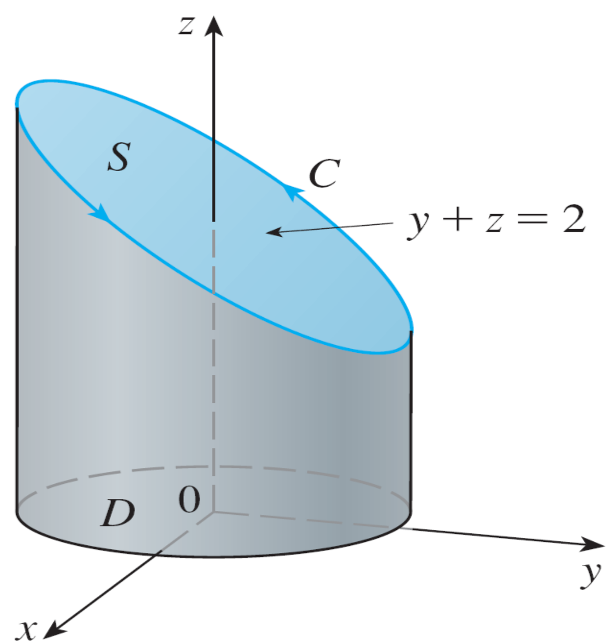
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA$$

This is precisely the vector form of Green's Theorem.

Thus, we see that Green's Theorem is really a special case of Stokes' Theorem.

Example 1

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)



Stokes' Theorem

In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\boxed{3} \qquad \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

Example 2. Use Stokes' Theorem to compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy - plane.

Stokes' Theorem

We now use Stokes' Theorem to throw some light on the meaning of the curl vector.

Suppose that C is an oriented closed curve and **v** represents the velocity field in fluid flow.

Consider the line integral

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot \mathbf{T}\, ds$$

and recall that $\mathbf{v} \cdot \mathbf{T}$ is the component of **v** in the direction of the unit tangent vector **T**.

Stokes' Theorem

This means that the closer the direction of **v** is to the direction of **T**, the larger the value of **v** · **T**.

Thus $\int_C \mathbf{v} \cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around *C* and is called the **circulation** of **v** around *C*. (See Figure 5.)

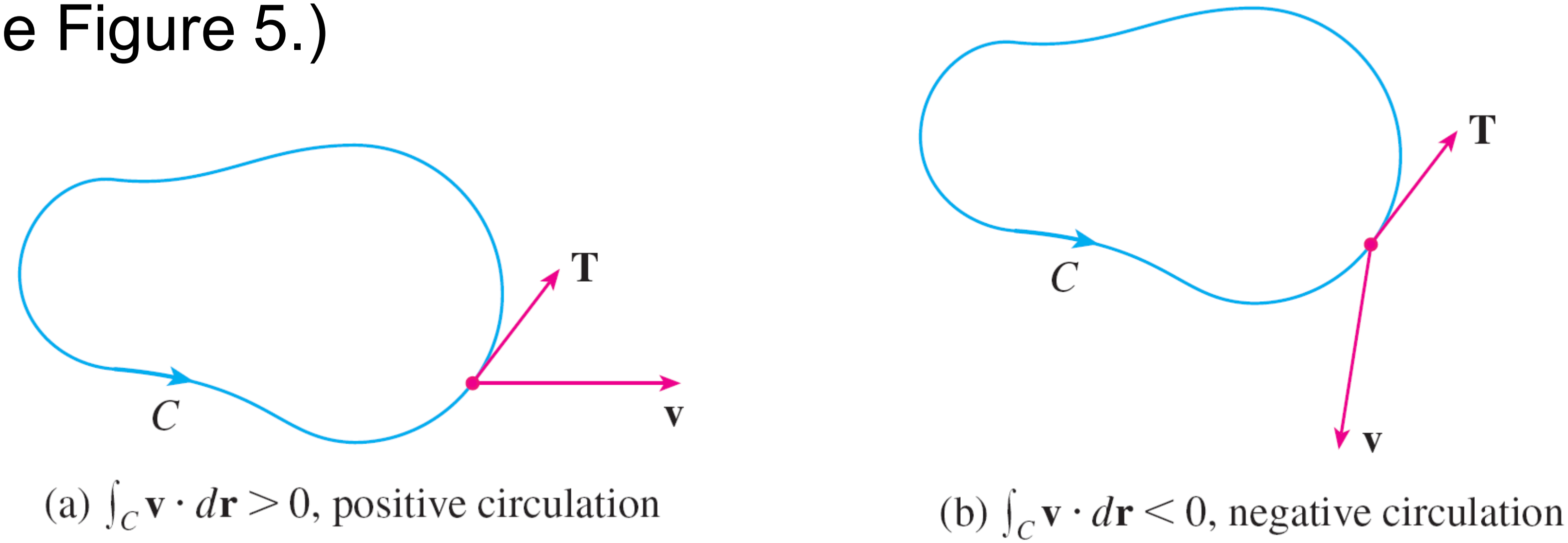


Figure 5

Stokes' Theorem

Now let $P_0(x_0, y_0, z_0)$ be a point in the fluid and let S_a be a small disk with radius a, center P_0, unit normal vector **n**.

Then (curl **F**)(P) ≈ (curl **F**)(P_0) and **n**(P) ≈ **n**(P_0) for all points P on S_a because curl **F** and **n** are continuous.

Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a:

$$\int_{C_a} \mathbf{v} \cdot d\mathbf{r} = \iint_{S_a} \operatorname{curl} \mathbf{v} \cdot d\mathbf{S} = \iint_{S_a} \operatorname{curl} \mathbf{v} \cdot \mathbf{n}\, dS$$

$$\approx \iint_{S_a} \operatorname{curl} \mathbf{v}(P_0) \cdot \mathbf{n}(P_0)\, dS = \operatorname{curl} \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) \pi a^2$$

Stokes' Theorem

This approximation becomes better as $a \to 0$ and we have

$$\boxed{4} \qquad \operatorname{curl} \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r}$$

Equation 4 gives the relationship between the curl and the circulation.

It shows that curl $\mathbf{v} \cdot \mathbf{n}$ is a measure of the rotating effect of the fluid about the axis $\mathbf{n}$.

The curling effect is greatest about the axis parallel to curl $\mathbf{v}$.

Sketch of proof of Theorem 16.5.4

> **4 Theorem** If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Given *C* simple closed curve, suppose we can find an orientable surface *S* whose boundary is *C* (this is nontrivial).

Then Stokes' Theorem gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0. Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve *C* and hence **F** is conservative.

16.9 The Divergence Theorem

The Divergence Theorem

We write Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

where C is the positively oriented boundary curve of the plane region D.

If we were seeking to extend this theorem to vector fields on $\mathbb{R}^3$, we might make the guess that

$$\boxed{1} \qquad \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z)\, dV$$

where S is the boundary surface of the solid region E.

The Divergence Theorem

It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem.

Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function (div **F** in this case) over a region to the integral of the original function **F** over the boundary of the region.

We state the Divergence Theorem for regions *E* that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangles are simple solid regions.)

The Divergence Theorem

The boundary of E is a closed surface, and we use the convention, that the positive orientation is outward; that is, the unit normal vector **n** is directed outward from E.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

Thus the Divergence Theorem states that, under the given conditions, the flux of **F** across the boundary surface of E is equal to the triple integral of the divergence of **F** over E.

Example 1

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

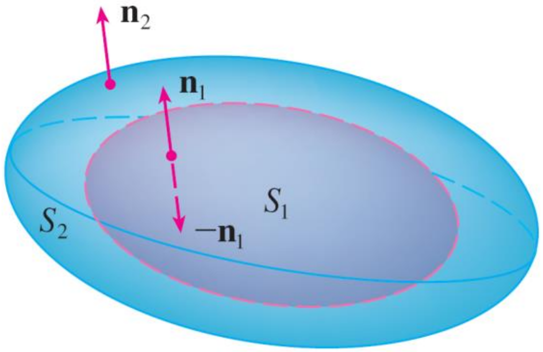
The Divergence Theorem

The divergence theorem can be applied to more general regions being a finite union of simple regions (The procedure is similar to the one we used in Section 16.4 to extend Green's Theorem).

The divergence Theorem

For example, it holds for the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2.

Then the boundary surface of E is $S = S_1 \cup S_2$ and its outward normal $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2.



The Divergence Theorem

Applying the Divergence Theorem to S, we get

$$\boxed{7} \qquad \iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

$$= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

$$= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1)\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS$$

$$= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

Example 3

We considered the electric field:

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z \rangle$ is a position vector.

Use the Divergence Theorem to show that the electric flux of **E** through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 4\pi\varepsilon Q$$

The Divergence Theorem

Another application of the Divergence Theorem occurs in fluid flow. Let **v**(x, y, z) be the velocity field of a fluid with constant density ρ. Then **F** = ρ**v** is the rate of flow per unit area.

The Divergence Theorem

If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then div $\mathbf{F}(P) \approx$ div $\mathbf{F}(P_0)$ for all points in B_a since div $\mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B_a} \operatorname{div} \mathbf{F} \, dV \approx \iiint_{B_a} \operatorname{div} \mathbf{F}(P_0) \, dV = \operatorname{div} \mathbf{F}(P_0) V(B_a)$$

This approximation becomes better as $a \to 0$ and suggests that

$$\boxed{8} \qquad \operatorname{div} \mathbf{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F} \cdot d\mathbf{S}$$

The Divergence Theorem

Equation 8 says that div **F**(P_0) is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence*.)

If div **F**(P) > 0, the net flow is outward near P and P is called a **source**.

If div **F**(P) < 0, the net flow is inward near P and P is called a **sink**.

The Divergence Theorem

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1.

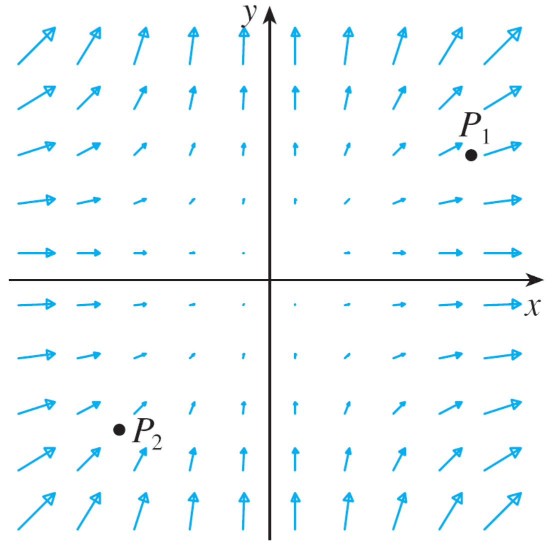


Figure 4

The vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

The Divergence Theorem

Thus the net flow is outward near P_1, so div $\mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows.

Here the net flow is inward, so div $\mathbf{F}(P_2) < 0$ and P_2 is a sink.

We can use the formula for **F** to confirm this impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, we have div $\mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.