14.8 Lagrange Multipliers

Lagrange Multipliers

In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$ (both f and g are differentiable).

It's easier to explain the geometric basis of Lagrange's method for functions of two variables.

So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$.

In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$.

Lagrange Multipliers

Figure 1 shows this curve together with several level curves of *f*.

These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$.

To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.

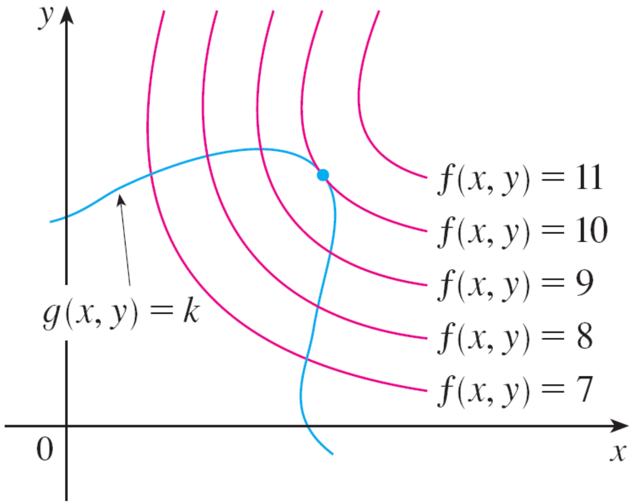

Figure 1

It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.)

Lagrange Multipliers

As gradient vectors are perpendicular to the tangent lines of level curves and f and g has a common tangent line to a corresponding level curve at (x_0, y_0), it follows that the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$.

Lagrange Multipliers

Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

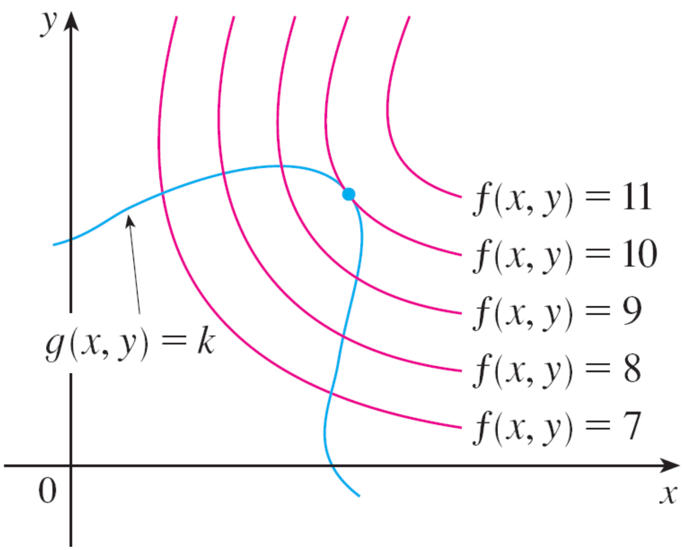

Figure 1

Lagrange Multipliers

This intuitive argument can be made precise as follows. Suppose that a differentiable function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a smooth curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ that lies on S and passes through P.

If t_0 is the parameter value corresponding to the point P, then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$.

The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C.

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Since f has an extreme value at (x_0, y_0, z_0), it follows that h has an extreme value at t_0, so $h'(t_0) = 0$. But f is differentiable, an **r** is smooth so we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

Thus, the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C and hence orthogonal to the tangent plane of S at (x_0, y_0, z_0).

But we already know that the gradient vector of g, $\nabla g(x_0, y_0, z_0)$, is also orthogonal to the tangent plane of S at (x_0, y_0, z_0).

Lagrange Multipliers

This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

$$\boxed{1} \qquad \boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)}$$

The number λ in Equation 1 is called a **Lagrange multiplier**.

Note: if $\nabla f(x_0, y_0, z_0) = 0$, then (1) holds with $\lambda = 0$ and the point (x_0, y_0, z_0) is a possible local extreme value of f.

Lagrange Multipliers

The procedure based on Equation 1 is as follows.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and $$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Note: This method provides a necessary condition only because we assume that the extreme values exist!

Lagrange Multipliers

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z \quad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.

For functions of two variables the method of Lagrange multipliers is similar to the method just described.

Lagrange Multipliers

To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

Example 1

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $g(x, y) = x^2 + y^2 = 1$.

Example 2

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Example 3

Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are the closest to and farthest from the point $(3,1,-1)$.

Two Constraints

Two Constraints

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$ (f, g and h are differentiable).

Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.)

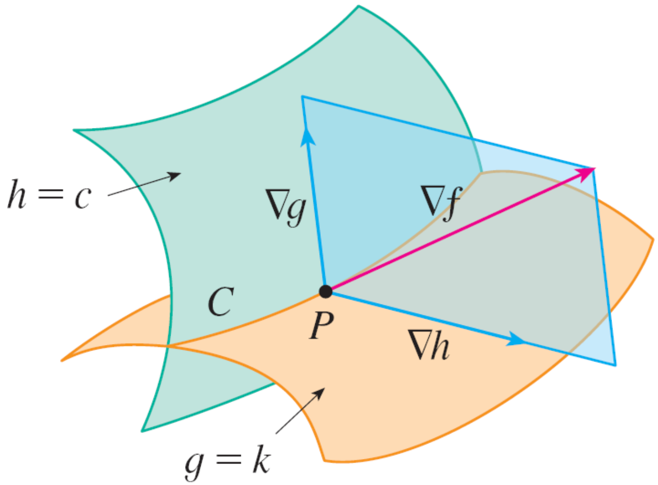

Figure 5

Two Constraints

Suppose f has such an extreme value on C at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to (the tangent vector of) C at P.

But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to (the tangent vector of) C, as C lies in both level surfaces.

This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are nonzero and non-parallel.)

Two Constraints

So there are numbers λ and μ (called Lagrange multipliers) such that

$$\boxed{16} \qquad \boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)}$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ, and μ.

Two Constraints

These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

Example 5

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.