16.5 Curl and Divergence

Curl

Curl

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{1} \qquad \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

Curl

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field **F** as follows:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Curl

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \operatorname{curl} \mathbf{F}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

$$\boxed{2} \qquad \boxed{\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}}$$

Example 1

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

Curl

Recall that the gradient of a function *f* of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl.

The following theorem says that the curl of a gradient vector field is **0**.

3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

Curl

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$.

This gives us a way of verifying that a vector field is not conservative.

Example 2

Show that the vector field

$$\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$$

is not conservative.

Curl

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if **F** is defined everywhere. (More generally it is true if the domain is simply-connected, that is, "has no hole.")

$\boxed{4}$ **Theorem** If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Example 3

(a) Show that the vector field

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

(b) Find a function *f* such that $\mathbf{F} = \nabla f$.

Curl

The reason for the name *curl* is that the curl vector is associated with rotations.

Suppose that **F** represents the velocity field in fluid flow. Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1).

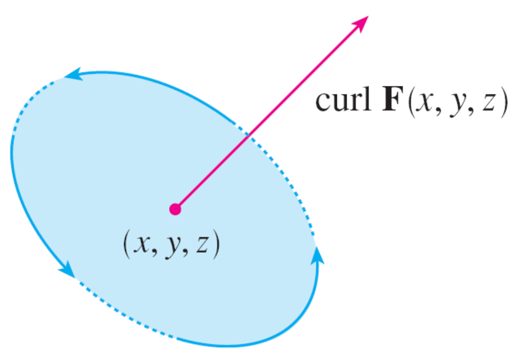

Figure 1

Curl

If curl $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and $\mathbf{F}$ is called **irrotational** at P.

In other words, there is no whirlpool or eddy at P.

If curl $\mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis.

If curl $\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis.

Paddle wheel

Divergence

Divergence

If **F** = P **i** + Q **j** + R **k** is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

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$$\boxed{\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Observe that curl **F** is a vector field but div **F** is a scalar field.

Divergence

In terms of the gradient operator

$\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of **F** can be written symbolically as the dot product of ∇ and **F**:

$$\boxed{10} \qquad \boxed{\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}}$$

Example 4

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div $\mathbf{F}$.

Divergence

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence.

The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Again, the reason for the name *divergence* can be understood in the context of fluid flow.

Divergence

If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then div $\mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.

In other words, div $\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z).

If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f.

Divergence

If *f* is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Divergence

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

in terms of its components:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$

Note: Sometimes, when it causes no confusion with the increment of a function, one uses the following notation for the Laplace operator: $\nabla^2 = \Delta$.

Example 5

Show that $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ cannot be written as the curl of another vector field.

Vector Forms of Green’s Theorem

Vector Forms of Green’s Theorem

The curl and divergence operators allow us to rewrite Green’s Theorem in versions that will be useful in our later work.

We suppose that the plane region *D*, its boundary curve *C*, and the functions *P* and *Q* satisfy the hypotheses of Green’s Theorem.

Then we consider the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

Vector Forms of Green's Theorem

Its line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$

and, regarding **F** as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Vector Forms of Green's Theorem

Therefore

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

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$$\boxed{\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA}$$

Vector Forms of Green's Theorem

Equation 12 expresses the line integral of the tangential component of **F** along *C* as the double integral of the vertical component of curl **F** over the region *D* enclosed by *C*. We now derive a similar formula involving the *normal* component of **F**.

If *C* is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad a \leq t \leq b$$

then the unit tangent vector is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

Vector Forms of Green's Theorem

You can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

(See Figure 4.)

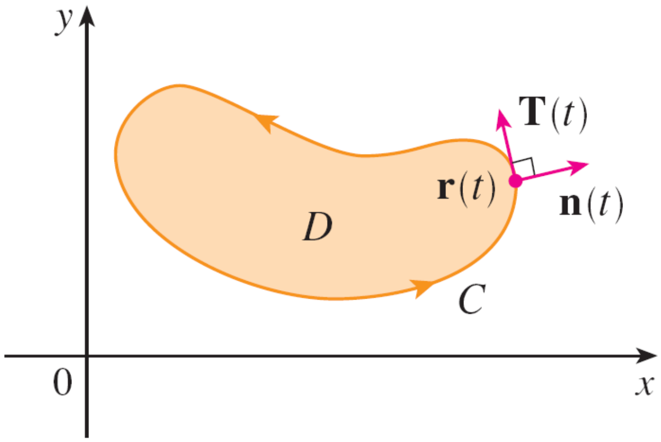


Figure 4

Vector Forms of Green's Theorem

Then, from equation

$$\int_C f(x, y)\, ds = \int_a^b f\big(x(t), y(t)\big) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{n}\, ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t)\, |\mathbf{r}'(t)|\, dt \\
&= \int_a^b \left[\frac{P\big(x(t), y(t)\big)\, y'(t)}{|\mathbf{r}'(t)|} - \frac{Q\big(x(t), y(t)\big)\, x'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\, dt \\
&= \int_a^b P\big(x(t), y(t)\big)\, y'(t)\, dt - Q\big(x(t), y(t)\big)\, x'(t)\, dt \\
&= \int_C P\, dy - Q\, dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA
\end{aligned}$$

by Green's Theorem.

Vector Forms of Green's Theorem

But the integrand in this double integral is just the divergence of **F**. So we have a second vector form of Green's Theorem.

$$\boxed{13} \qquad \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

This version says that the line integral of the normal component of **F** along *C* is equal to the double integral of the divergence of **F** over the region *D* enclosed by *C*.

16.6 Parametric Surfaces and Their Areas

Parametric Surfaces and Their Areas

Here we use vector functions to describe more general surfaces, called *parametric surfaces*, and compute their areas.

Then we take the general surface area formula and see how it applies to special surfaces.

Parametric Surfaces

Parametric Surfaces

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v.

We suppose that

$$\boxed{1} \qquad \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane.

Parametric Surfaces

So x, y, and, z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D.

The set of all points (x, y, z) in $\mathbb{R}^3$ such that

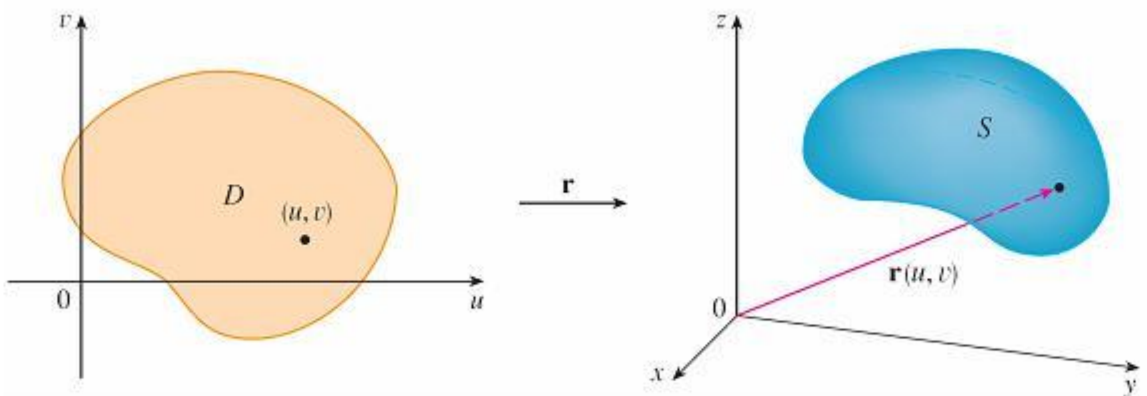
$$\boxed{2} \qquad x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies throughout D, is called a **parametric surface** S and Equations 2 are called **parametric equations** of S.

Parametric Surfaces

Each choice of *u* and *v* gives a point on *S*; by making all choices, we get all of *S*.

In other words, the surface is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region *D*. (See Figure 1.)



A parametric surface

Figure 1

Example 1

Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u \, \mathbf{i} + v \, \mathbf{j} + 2 \sin u \, \mathbf{k}$$

Example 3

Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors **a** and **b**.

Parametric Surfaces

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant.

These families correspond to vertical and horizontal lines in the uv-plane.

Parametric Surfaces

If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S. (See Figure 4.)

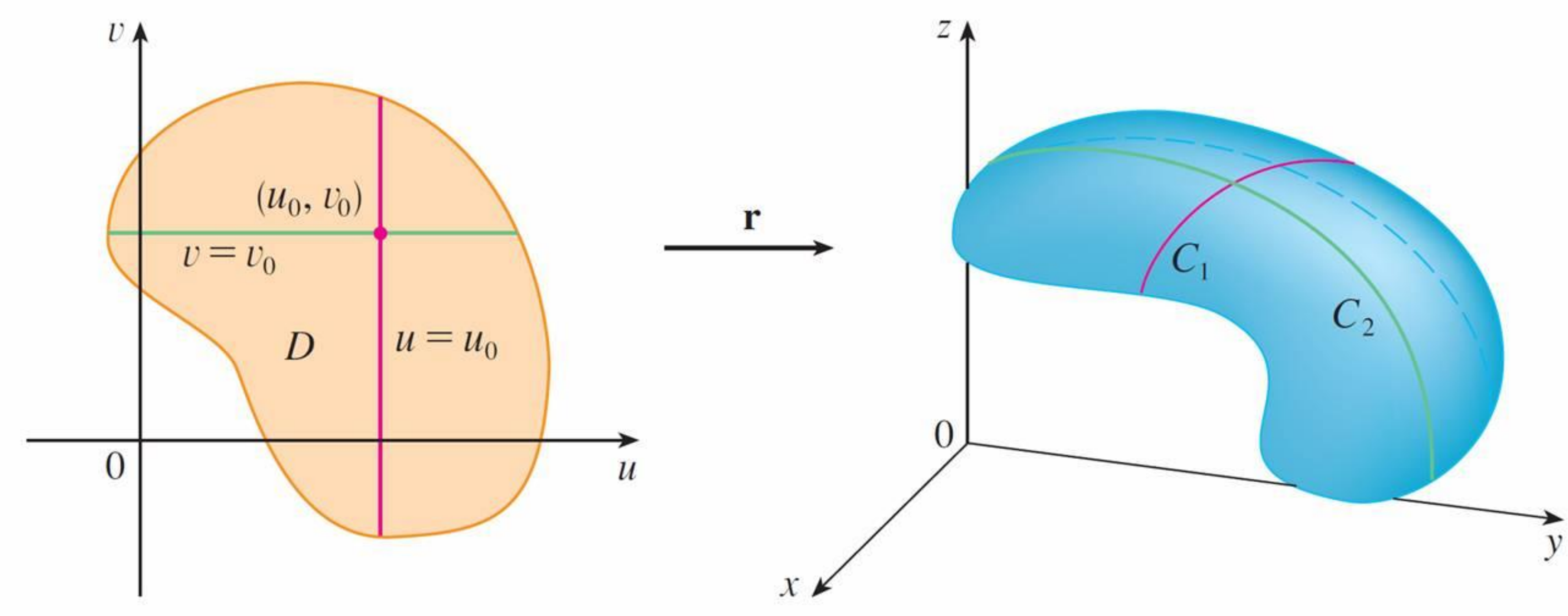


Figure 4

Parametric Surfaces

Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S.

We call these curves **grid curves**. (In Example 1, for instance, the grid curves obtained by letting u be constant are horizontal lines whereas the grid curves with v constant are circles.)

In fact, when a computer graphs a parametric surface, it usually depicts the surface by plotting these grid curves.

Example 4

Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

and find the grid curves.

Parametric Surfaces

Note:
We saw in Example 4 that the grid curves for a sphere are curves of constant latitude and longitude.

For a general parametric surface we are really making a map and the grid curves are similar to lines of latitude and longitude.

Describing a point on a parametric surface by giving specific values of u and v is like giving the latitude and longitude of a point.