# 2.1 Derivatives and Rates of Change

# Derivatives and Rates of Change

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit.

This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

# Tangents

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If a curve $C$ has equation $y = f(x)$ and we want to find the tangent line to $C$ at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line $PQ$:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let $Q$ approach $P$ along the curve $C$ by letting $x$ approach $a$.

# Tangents

If $m_{PQ}$ approaches a number $m$, then we define the *tangent* $t$ to be the line through $P$ with slope $m$. (This amounts to saying that the tangent line is the limiting position of the secant line $PQ$ as $Q$ approaches $P$. See Figure 1.)

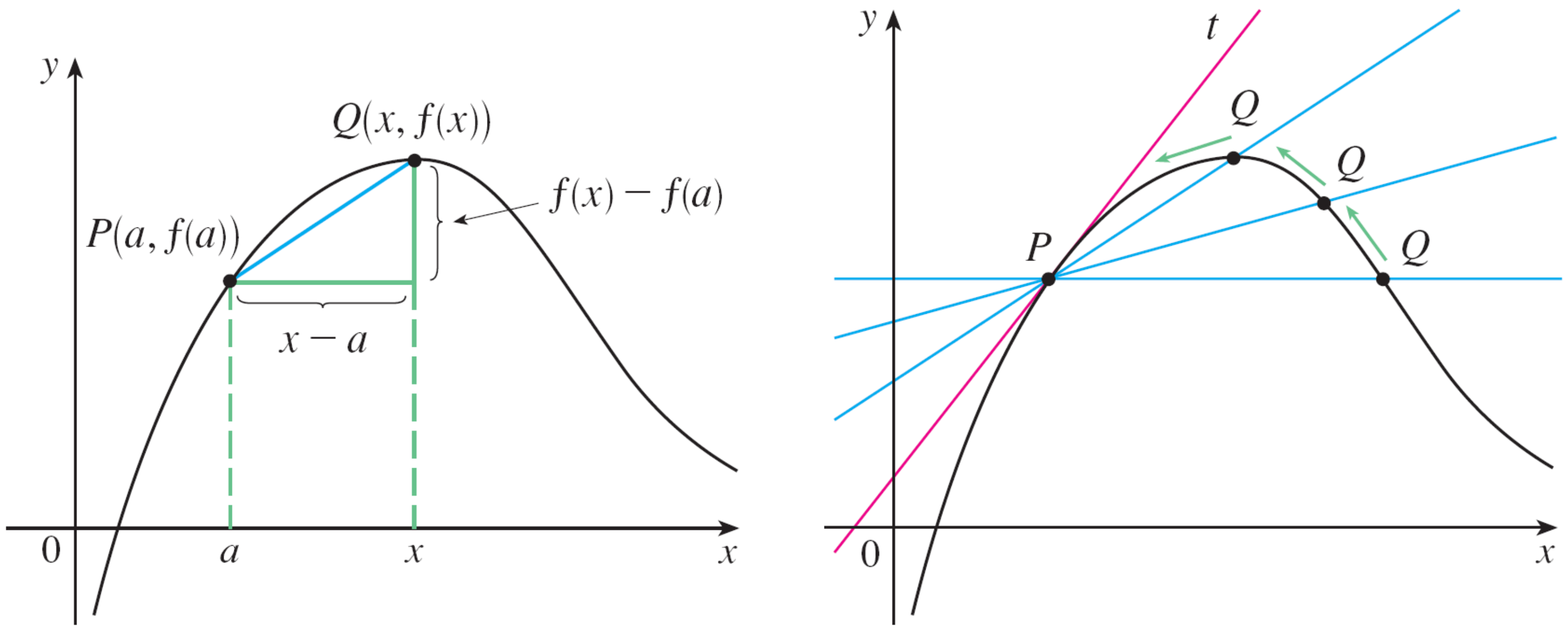



**Figure 1**

# Tangents

**1 Definition** The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through $P$ with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

# Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

# Tangents

If $h = x - a$, then $x = a + h$ and so the slope of the secant line $PQ$ is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} = \frac{f(a+h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and $Q$ is to the right of $P$. If it happened that $h < 0$, however, $Q$ would be to the left of $P$.)

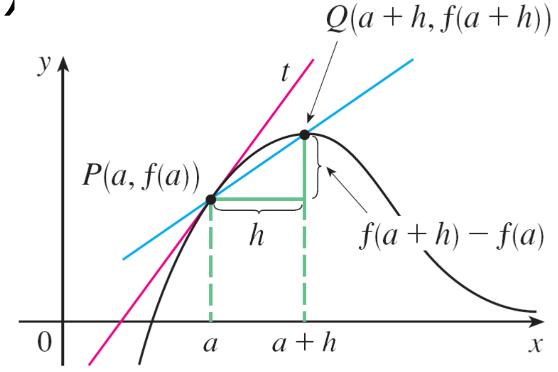



Figure 3

# Tangents

Notice that as $x$ approaches $a$, $h$ approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

$$\boxed{2} \qquad \boxed{m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}}$$

# Example 2

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point (3, 1)

Velocities

# Velocities

In general, suppose an object moves along a straight line (a.k.a rectilinear motion) according to an equation of motion $s = f(t)$, where $s$ is the displacement (directed distance) of the object from the origin at time $t$.

The function $f$ that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) – f(a)$. (See Figure 5.)

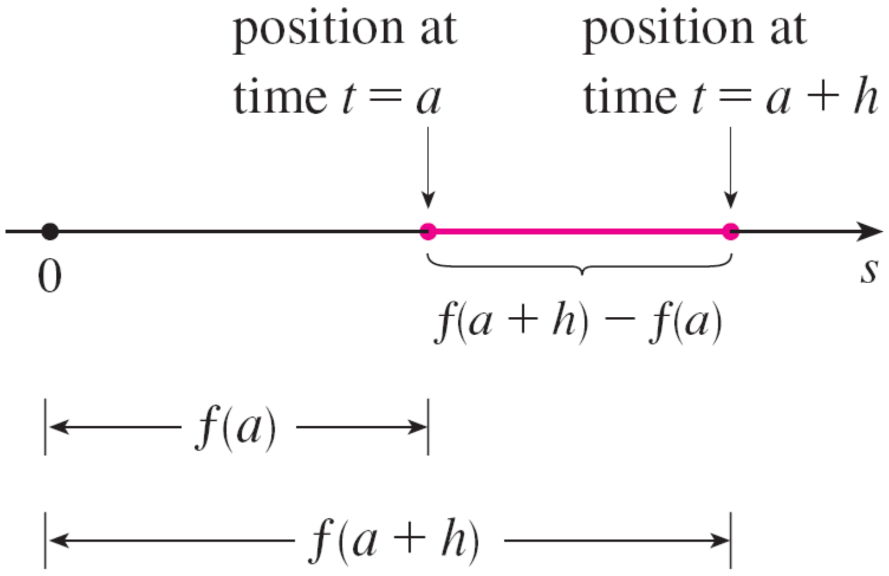



Figure 5

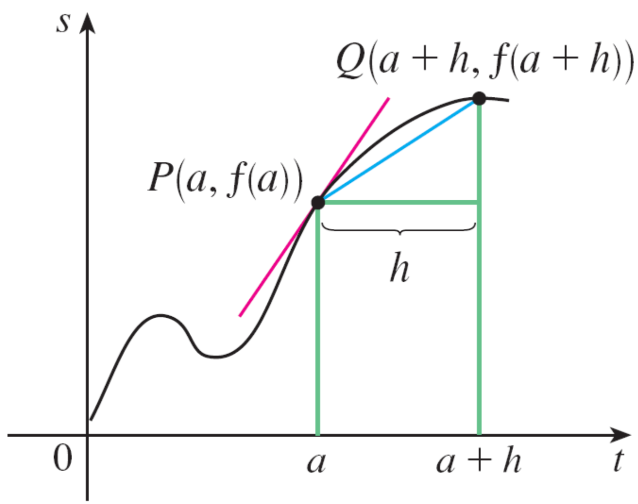
# Velocities

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a + h) - f(a)}{h}$$

# Velocities

which is the same as the slope of the secant line *PQ* in Figure 6.




$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$
$$= \text{average velocity}$$

**Figure 6**

# Velocities

Now suppose we compute the average velocities over shorter and shorter time intervals [*a*, *a* + *h*].

In other words, we let *h* approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

3

$$\boxed{v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}}$$

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at *P*.

# Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Derivatives

# Derivatives

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Equation 3).

In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics.

Since this type of limit occurs so widely, it is given a special name and notation.

# Derivatives

**4 Definition** The **derivative of a function $f$ at a number $a$**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ and $h$ approaches 0 if and only if $x$ approaches $a$. Therefore, an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

**5**
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

# Example 4

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number $a$.

# Derivatives

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through $P$ and has slope $m$ given by Equation 1 or 2.

Since, by Definition 4, this is the same as the derivative $f'(a)$, we can now say the following.

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of $f$ at $a$.

# Derivatives

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

# Rates of Change

# Rates of Change

Suppose $y$ is a quantity that depends on another quantity $x$. Thus, $y$ is a function of $x$ and we write $y = f(x)$. If $x$ changes from $x_1$ to $x_2$, then the change in $x$ (also called the **increment** of $x$) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in $y$ is

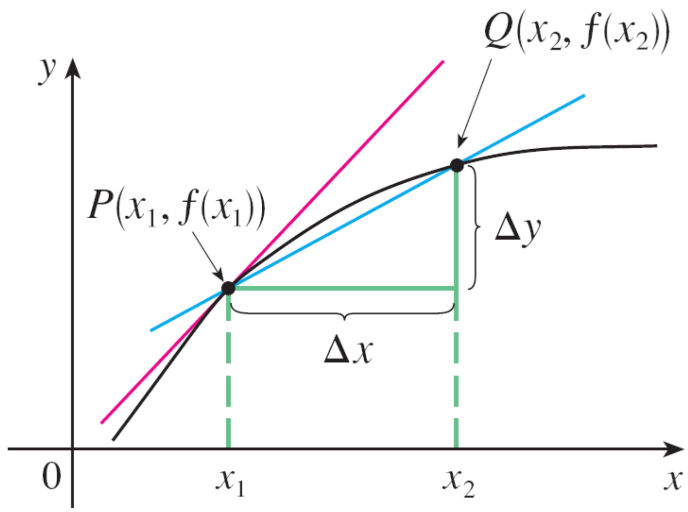
$$\Delta y = f(x_2) - f(x_1)$$

# Rates of Change

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line *PQ* in Figure 8.




average rate of change $= m_{PQ}$

instantaneous rate of change $=$ slope of tangent at $P$

**Figure 8**

# Rates of Change

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting $x_2$ approach $x_1$ and therefore letting $\Delta x$ approach 0.

The limit of these average rates of change is called the (**instantaneous**) **rate of change of *y* with respect to *x*** at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

$$\boxed{6} \qquad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

# Rates of Change

We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

> The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to $x$ when $x = a$.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$.

# Rates of Change

This means that when the derivative is large (and therefore the curve is steep, as at the point *P* in Figure 9), the *y*-values change rapidly.

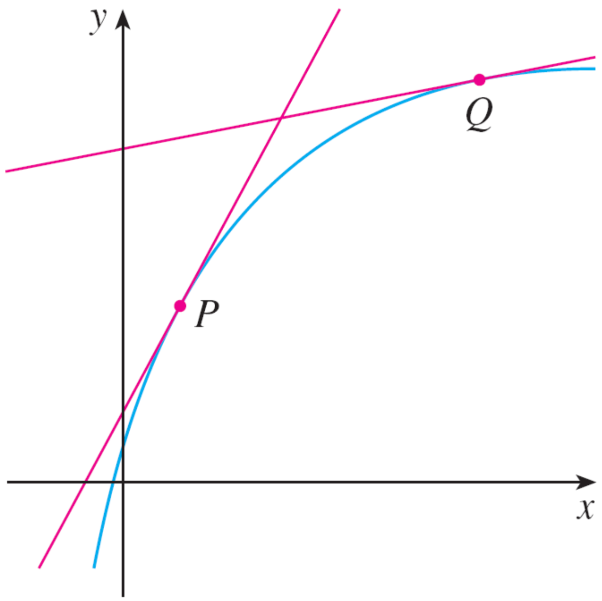



**Figure 9**

The *y*-values are changing rapidly at *P* and slowly at *Q*.

# Rates of Change

When the derivative is small, the curve is relatively flat (as at point $Q$) and the $y$-values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement $s$ with respect to the time $t$.

In other words, $f'(a)$ *is the velocity of the particle at time* $t = a$.

The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

# Example 6

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing $x$ meters of this fabric is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

# The Derivative as a Function

We have considered the derivative of a function *f* at a fixed number *a*:

**1** $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here we change our point of view and let the number *a* vary. If we replace *a* in Equation 1 by a variable *x*, we obtain

**2** $$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

# The Derivative as a Function

Given any number $x$ for which this limit exists, we assign to $x$ the number $f'(x)$. So we can regard $f'$ as a new function, called the **derivative of *f*** and defined by Equation 2.

We know that the value of $f'$ at $x$, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of $f$ at the point $(x, f(x))$.

The function $f'$ is called the derivative of $f$ because it has been "derived" from $f$ by the limiting operation in Equation 2. The domain of $f'$ is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of $f$.

# Example 1 – Derivative of a Function given by a Graph

The graph of a function $f$ is given in Figure 1. Use it to sketch the graph of the derivative $f'$.

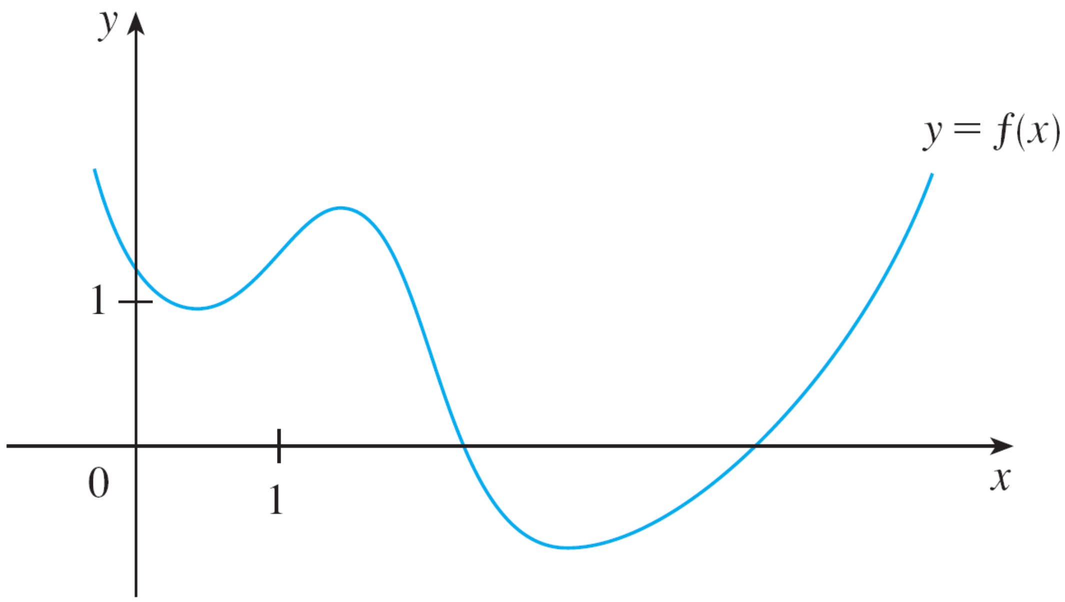



**Figure 1**

# Example 3

If $f(x) = \sqrt{x}$, find the derivative of *f*. State the domain of *f'*.

# Homework: read Example 4 in the book

Find *f'* if $f(x) = \frac{1-x}{2+x}$.

# Other Notations

# Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is $x$ and the dependent variable is $y$, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols $D$ and $d/dx$ are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

# Other Notations

The symbol *dy*/*dx*, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation.

We can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

# Other Notations

If we want to indicate the value of a derivative *dy*/*dx* in Leibniz notation at a specific number *a*, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$.

$\boxed{3}$ **Definition** A function $f$ is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** $(a, b)$ [or $(a, \infty)$ or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

# Example 5

Where is the function $f(x) = |x|$ differentiable?

# Continuity vs differentiability

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

**4 Theorem** If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable.