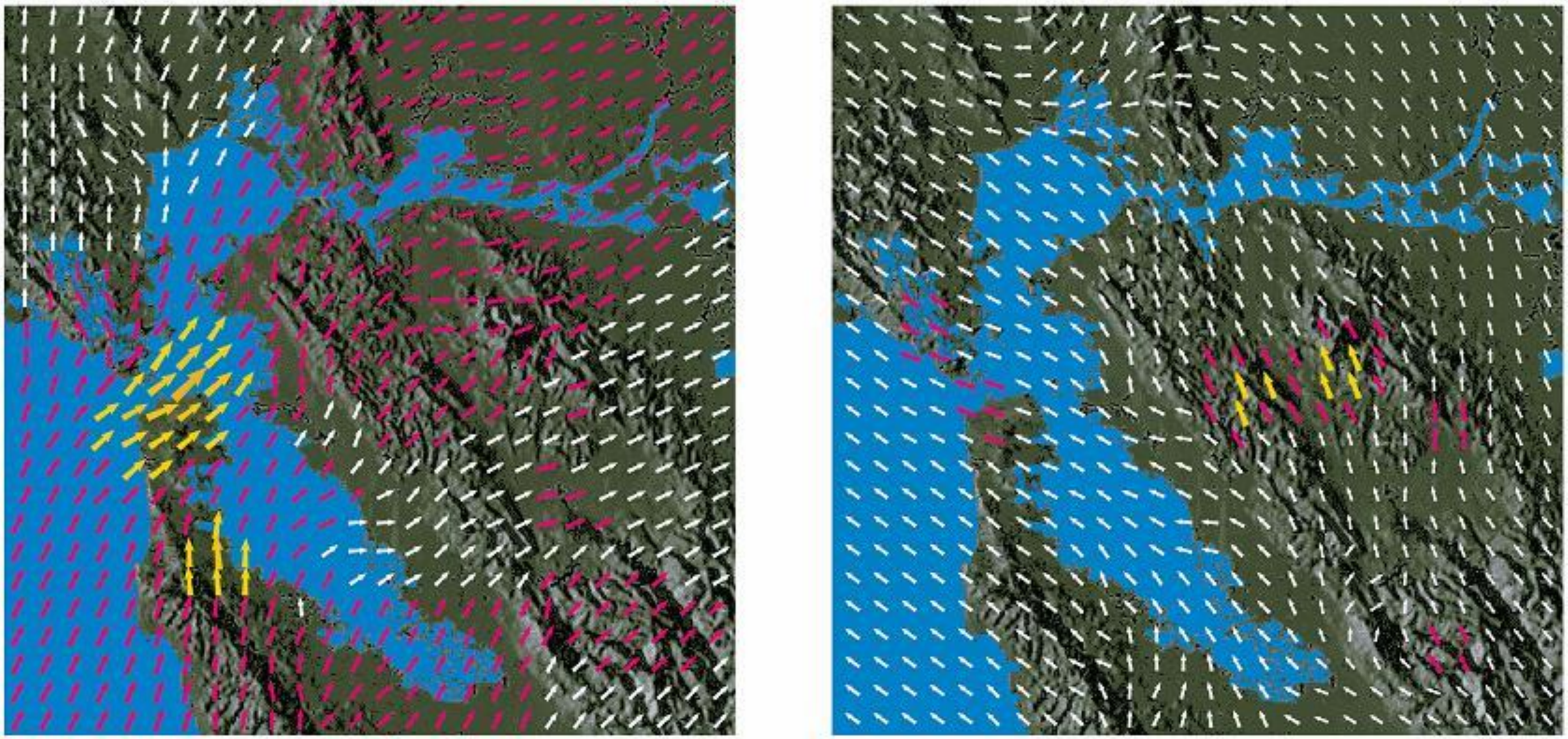
# 16 Vector Calculus

# 16.1 Vector Fields

# Vector Fields



(a) 6:00 PM, March 1, 2010

(b) 6:00 AM, March 1, 2010

Velocity vector fields showing San Francisco Bay wind patterns

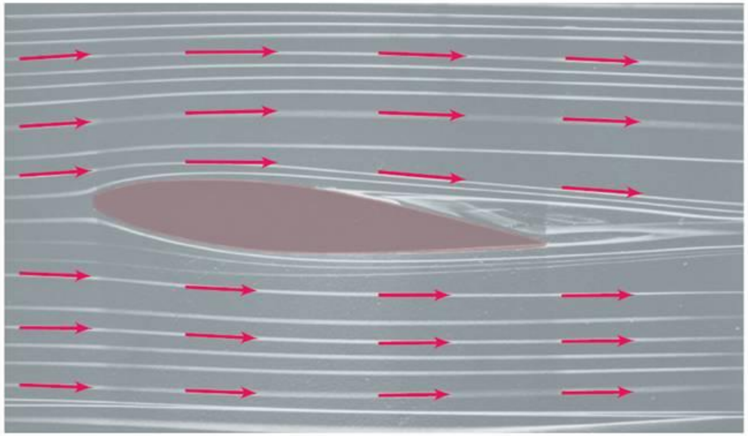
**Figure 1**

# Vector Fields

Associated with every point in the air we can imagine a wind velocity vector. This is an example of a *velocity vector field.*

Other examples of velocity vector fields are illustrated in Figure 2: ocean currents and flow past an airfoil.


(a) Ocean currents off the coast of Nova Scotia



(b) Airflow past an inclined airfoil

Velocity vector fields

**Figure 2**

# Vector Fields

Another type of vector field, called a *force field*, associates a force vector with each point in a region. An example is the gravitational force field.

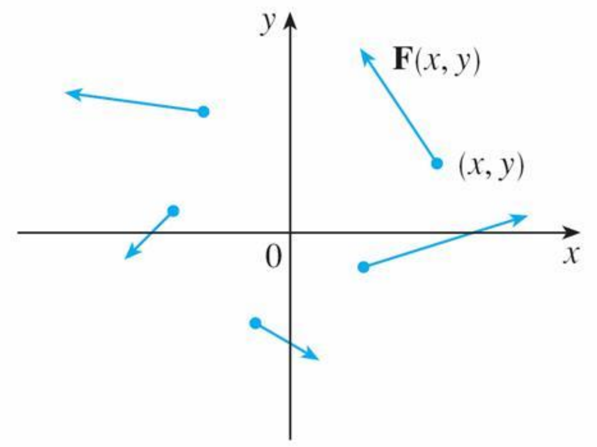
In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in $V_2$ (or $V_3$).

$\boxed{1}$ **Definition** Let $D$ be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point $(x, y)$ in $D$ a two-dimensional vector $\mathbf{F}(x, y)$.

# Vector Fields

The best way to picture a vector field is to draw the arrow representing the vector **F**($x$, $y$) starting at the point ($x$, $y$).

Of course, it's impossible to do this for all points ($x$, $y$), but we can gain a reasonable impression of **F** by doing it for a few representative points in $D$ as in Figure 3.




Vector field on $\mathbb{R}^2$

**Figure 3**

# Vector Fields

Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** $P$ and $Q$ as follows:

$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y) \rangle$$

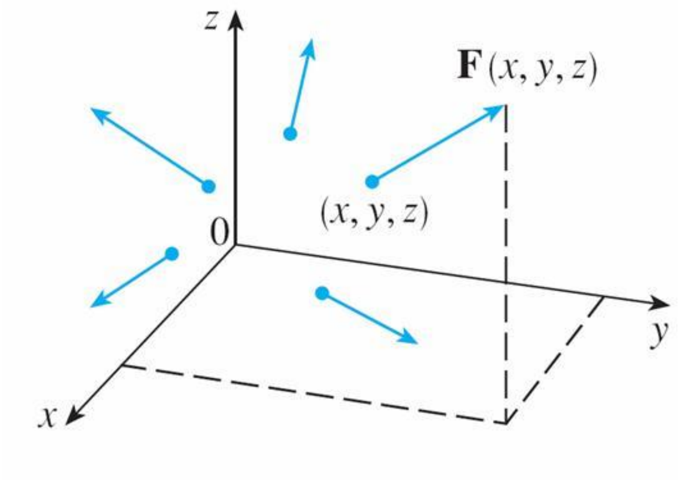
or, for short, $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$

Notice that $P$ and $Q$ are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

**2 Definition** Let $E$ be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point $(x, y, z)$ in $E$ a three-dimensional vector $\mathbf{F}(x, y, z)$.

# Vector Fields

A vector field **F** on $\mathbb{R}^3$ is pictured in Figure 4.




Vector field on $\mathbb{R}^3$

**Figure 4**

We can express it in terms of its component functions *P*, *Q*, and *R* as

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

# Vector Fields

As with the vector functions, we can define continuity of vector fields and show that **F** is continuous if and only if its component functions $P$, $Q$, and $R$ are continuous.

We sometimes identify a point $(x, y, z)$ with its position vector $\mathbf{x} = \langle x, y, z \rangle$ and write $\mathbf{F}(\mathbf{x})$ instead of $\mathbf{F}(x, y, z)$.

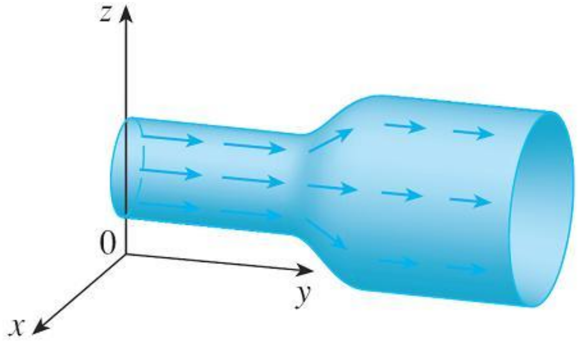
Then **F** becomes a function that assigns a vector $\mathbf{F}(\mathbf{x})$ to a vector $\mathbf{x}$.

# Example 3

Imagine a fluid flowing steadily along a pipe and let $\mathbf{V}(x, y, z)$ be the velocity vector at a point $(x, y, z)$.

Then $\mathbf{V}$ assigns a vector to each point $(x, y, z)$ in a certain domain $E$ (the interior of the pipe) and so $\mathbf{V}$ is a vector field on $\mathbb{R}^3$ called a **velocity field**.

A possible velocity field is illustrated in Figure 13.




Velocity field in fluid flow

**Figure 13**

# Example 3

cont'd

The speed at any given point is indicated by the length of the arrow.

Velocity fields also occur in other areas of physics.

# Example 4

Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses *m* and *M* is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where *r* is the distance between the objects and *G* is the gravitational constant.
(This is an example of an inverse square law.)

Let's assume that the object with mass *M* is located at the origin in $\mathbb{R}^3$. (For instance, *M* could be the mass of the earth and the origin would be at its center.)

# Example 4

cont'd

Let the position vector of the object with mass $m$ be $\mathbf{x} = \langle x, y, z \rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |\mathbf{x}|^2$.

The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}$$

Therefore the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is

$$\boxed{3} \qquad \mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

[Physicists often use the notation **r** instead of **x** for the position vector, so you may see Formula 3 written in the form $\mathbf{F} = -(mMG/r^3)\mathbf{r}$.]

# Example 4

cont'd

The function given by Equation 3 is an example of a vector field, called the **gravitational field**, because it associates a vector [the force **F**(**x**)] with every point **x** in space.

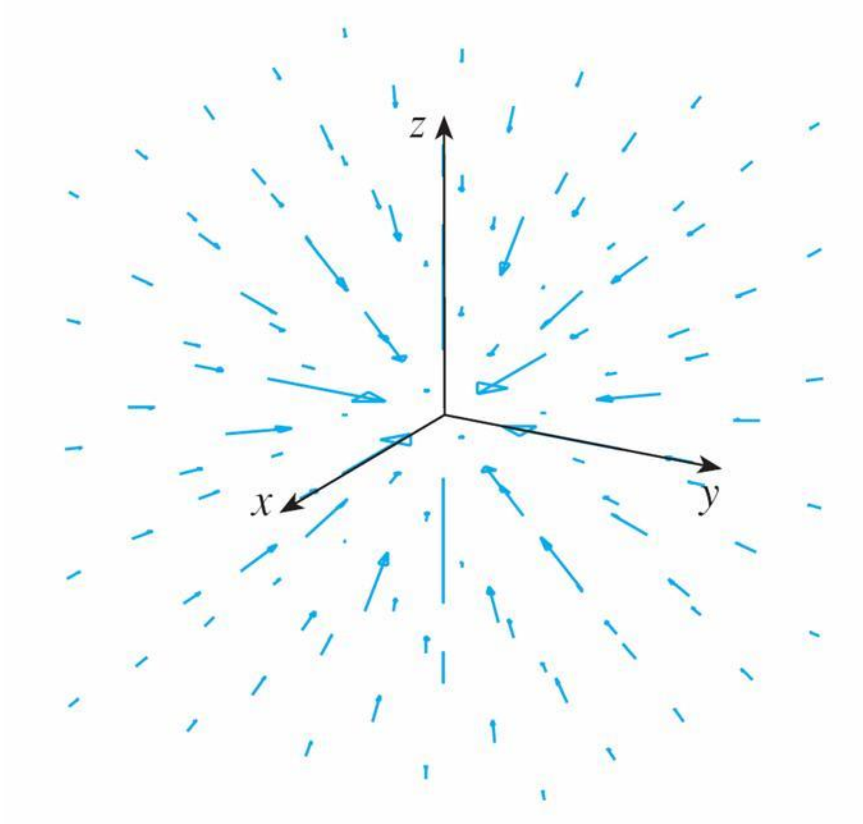
Formula 3 is a compact way of writing the gravitational field, but we can also write it in terms of its component functions by using the facts that **x** = *x* **i** + *y* **j** + *z* **k** and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$:

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}$$

# Example 4

cont'd

The gravitational field **F** is pictured in Figure 14.




Gravitational force field

**Figure 14**

# Example 5

Suppose an electric charge $Q$ is located at the origin. According to Coulomb's Law, the electric force $\mathbf{F}(\mathbf{x})$ exerted by this charge on a charge $q$ located at a point $(x, y, z)$ with position vector $\mathbf{x} = \langle x, y, z \rangle$ is

$$\boxed{4} \qquad \mathbf{F}(\mathbf{x}) = \frac{\varepsilon q Q}{|\mathbf{x}|^3}\,\mathbf{x}$$

where $\varepsilon$ is a constant (that depends on the units used).

For like charges, we have $qQ > 0$ and the force is repulsive; for unlike charges, we have $qQ < 0$ and the force is attractive.

# Example 5

cont'd

Notice the similarity between Formulas 3 and 4. Both vector fields are examples of **force fields**.

Instead of considering the electric force **F**, physicists often consider the force per unit charge:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{q}\mathbf{F}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$$

Then **E** is a vector field on $\mathbb{R}^3$ called the **electric field** of *Q*.

# Gradient Fields

# Gradient Fields

If $f$ is a scalar function of two variables, recall that its gradient $\nabla f$ (or grad $f$) is defined by

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Therefore $\nabla f$ is really a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**.

Likewise, if $f$ is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$

# Example 6

Find the gradient vector field of $f(x, y) = x^2y - y^3$.

# Gradient Fields

A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function $f$ such that $\mathbf{F} = \nabla f$.

In this situation $f$ is called a **potential function** for **F**.

Not all vector fields are conservative, but such fields do arise frequently in physics.

# Gradient Fields

For example, the gravitational field **F** in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}$$

$$= \mathbf{F}(x, y, z)$$

# 16.2 Line Integrals

# Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve $C$.

Such integrals are called *line integrals,* although "curve integrals" would be better terminology.

They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

# Line Integrals

We start with a plane curve $C$ given by the parametric equations

$$\boxed{1} \qquad x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that $C$ is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.]

# Line Integrals

If we divide the parameter interval $[a, b]$ into $n$ subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$, and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide $C$ into $n$ subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 1.)

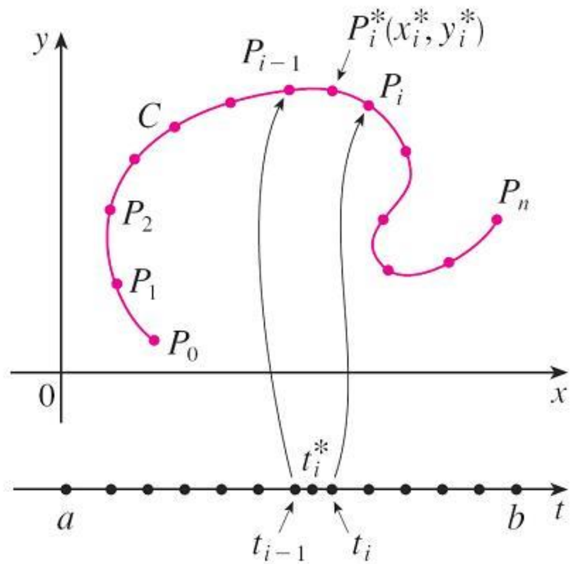



**Figure 1**

# Line Integrals

We choose any point $P_i^*(x_i^*, y_i^*)$ in the *i*th subarc. (This corresponds to a point $t_i^*$ in $[t_{i-1}, t_i]$.)

Now if $f$ is any function of two variables whose domain includes the curve $C$, we evaluate $f$ at the point $(x_i^*, y_i^*)$, multiply by the length $\Delta s_i$ of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

which is similar to a Riemann sum.

# Line Integrals

Then we take the limit of these sums and make the following definition by analogy with a single integral.

**2 Definition** If $f$ is defined on a smooth curve $C$ given by Equations 1, then the **line integral of $f$ along $C$** is

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

if this limit exists.

We have found that the length of $C$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

# Line Integrals

A similar type of argument can be used to show that if *f* is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

$$\boxed{3} \qquad \int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as *t* increases from *a* to *b*.

# Line Integrals

If $s(t)$ is the length of $C$ between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.$$

So the way to remember Formula 3 is to express everything in terms of the parameter $t$: Use the parametric equations to express $x$ and $y$ in terms of $t$ and write $ds$ as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

# Line Integrals

In the special case where $C$ is the line segment that joins $(a, 0)$ to $(b, 0)$, using $x$ as the parameter, we can write the parametric equations of $C$ as follows: $x = x$, $y = 0$, $a \le x \le b$.

Formula 3 then becomes

$$\int_C f(x, y)\, ds = \int_a^b f(x, 0)\, dx$$

and so the line integral reduces to an ordinary single integral in this case.

# Line Integrals

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area.

In fact, if $f(x, y) \geq 0$, $\int_C f(x, y)\, ds$ represents the area of one side of the "fence" or "curtain" in Figure 2, whose base is $C$ and whose height above the point $(x, y)$ is $f(x, y)$.

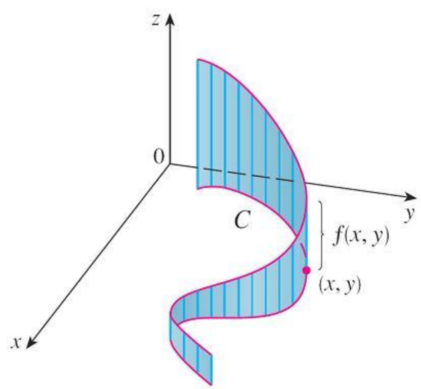



**Figure 2**

# Example 1

Evaluate $\int_C (2 + x^2y)\, ds$, where $C$ is the upper half of the unit circle $x^2 + y^2 = 1$.

# Line Integrals

Suppose now that $C$ is a **piecewise-smooth curve**; that is, $C$ is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in Figure 4, the initial point of $C_{i+1}$ is the terminal point of $C_i$.

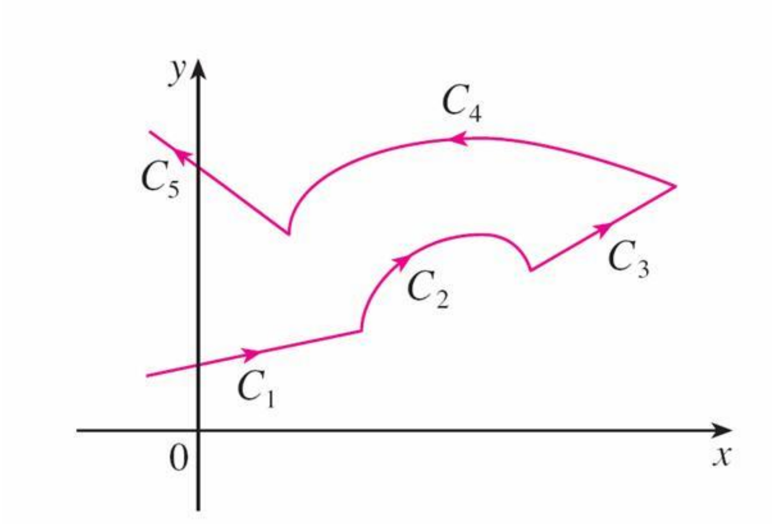



**Figure 4**

A piecewise-smooth curve

# Line Integrals

Then we define the integral of *f* along *C* as the sum of the integrals of *f* along each of the smooth pieces of *C*:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds$$

# Example 2

Evaluate $\int_C 2x\, ds$, where $C$ consists of the arc $C_1$ of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment $C_2$ from (1, 1) to (1, 2).

# Line Integrals

Any physical interpretation of a line integral $\int_C f(x, y)\, ds$ depends on the physical interpretation of the function $f$.

Suppose that $\rho(x, y)$ represents the linear density at a point $(x, y)$ of a thin wire shaped like a curve $C$.

Then the mass of the part of the wire from $P_{i-1}$ to $P_i$ in Figure1 is approximately $\rho(x_i^*, y_i^*)\, \Delta s_i$ and so the total mass of the wire is approximately $\sum \rho(x_i^*, y_i^*)\, \Delta s_i$.

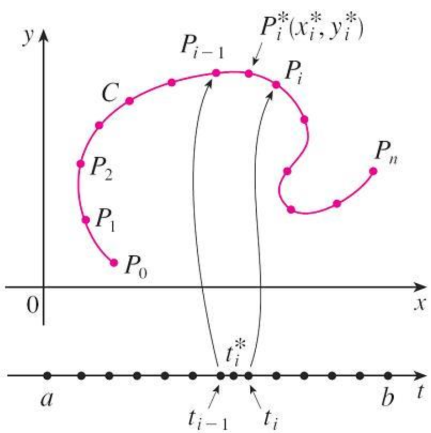



**Figure 1**

# Line Integrals

By taking more and more points on the curve, we obtain the **mass** *m* of the wire as the limiting value of these approximations:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\, \Delta s_i = \int_C \rho(x, y)\, ds$$

[For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.]

# Line Integrals

The **center of mass** of the wire with density function $\rho$ is located at the point $(\bar{x}, \bar{y})$, where

$$\boxed{4} \qquad \bar{x} = \frac{1}{m} \int_C x\,\rho(x, y)\,ds \qquad \bar{y} = \frac{1}{m} \int_C y\,\rho(x, y)\,ds$$

# Line Integrals

Two other line integrals are obtained by replacing $\Delta s_i$ by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2.

They are called the **line integrals of *f* along *C* with respect to *x* and *y***:

$$\boxed{5} \qquad \int_C f(x, y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta y_i$$

$$\boxed{6} \qquad \int_C f(x, y)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

# Line Integrals

When we want to distinguish the original line integral $\int_C f(x, y)\, ds$ from those in Equations 5 and 6, we call it the **line integral with respect to arc length**.

The following formulas say that line integrals with respect to $x$ and $y$ can also be evaluated by expressing everything in terms of $t$: $x = x(t)$, $y = y(t)$, $dx = x'(t)\, dt$, $dy = y'(t)\, dt$.

**7**

$$\int_C f(x, y)\, dx = \int_a^b f\big(x(t), y(t)\big)\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f\big(x(t), y(t)\big)\, y'(t)\, dt$$

# Line Integrals

It frequently happens that line integrals with respect to $x$ and $y$ occur together.

When this happens, it's customary to abbreviate by writing

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

When we are setting up a line integral, sometimes the most difficult thing is to think of a parametric representation for a curve whose geometric description is given.

# Line Integrals

In particular, we often need to parametrize a line segment, so it's useful to remember that a vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\boxed{8} \qquad \boxed{\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\,\mathbf{r}_1 \qquad 0 \leq t \leq 1}$$

# Example 4

Evaluate $\int_C y^2\, dx + x\, dy$ in two different ways.

(a) $C = C_1$ is the line segment from (-5, -3) to (0, 2).

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from (-5, -3) to (0, 2).

# Line Integrals

In general, a given parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an **orientation** of a curve $C$, with the positive direction corresponding to increasing values of the parameter $t$. (See Figure 8, where the initial point $A$ corresponds to the parameter value $a$ and the terminal point $B$ corresponds to $t = b$.)

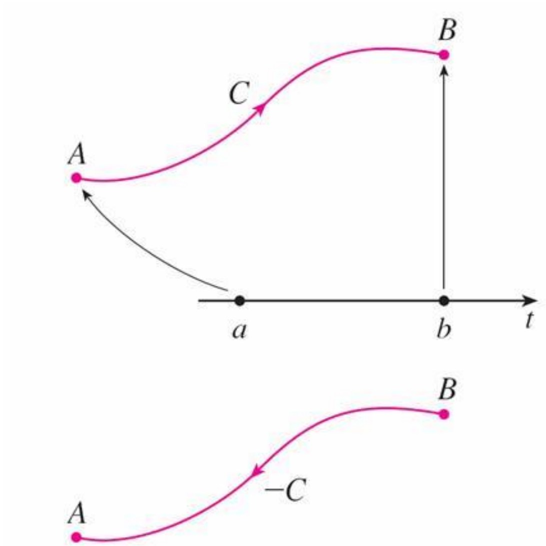



**Figure 8**

# Line Integrals

If $-C$ denotes the curve consisting of the same points as $C$ but with the opposite orientation (from initial point $B$ to terminal point $A$ in Figure 8), then we have

$$\int_{-C} f(x, y)\,dx = -\int_C f(x, y)\,dx \qquad \int_{-C} f(x, y)\,dy = -\int_C f(x, y)\,dy$$

But if we integrate with respect to arc length, the value of the line integral does *not* change when we reverse the orientation of the curve:

$$\int_{-C} f(x, y)\,ds = \int_C f(x, y)\,ds$$

This is because $\Delta s_i$ is always positive, whereas $\Delta x_i$ and $\Delta y_i$ change sign when we reverse the orientation of $C$.

# Line Integrals in Space

# Line Integrals in Space

We now suppose that $C$ is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

or by a vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$.

If $f$ is a function of three variables that is continuous on some region containing $C$, then we define the **line integral of *f* along *C*** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta s_i$$

# Line Integrals in Space

We evaluate it using a formula similar to Formula 3:

$$\boxed{9} \qquad \int_C f(x, y, z)\, ds = \int_a^b f\big(x(t), y(t), z(t)\big) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Observe that the integrals in both Formulas 3 and 9 can be written in the more compact vector notation

$$\int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$$

For the special case $f(x, y, z) = 1$, we get

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\, dt = L$$

where $L$ is the length of the curve $C$.

# Line Integrals in Space

Line integrals along $C$ with respect to $x$, $y$, and $z$ can also be defined. For example,

$$\int_C f(x, y, z)\, dz = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta z_i$$

$$= \int_a^b f\big(x(t), y(t), z(t)\big)\, z'(t)\, dt$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\boxed{10} \qquad \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

by expressing everything ($x$, $y$, $z$, $dx$, $dy$, $dz$) in terms of the parameter $t$.