Conservation of Energy

Conservation of Energy

Let's apply the ideas of Chapter 16.3 to a continuous force field $\mathbf{F}$ that moves an object along a path C given by $\mathbf{r}(t)$, $a \le t \le b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C. According to Newton's Second Law of Motion, the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\,dt$$

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$$= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)] \, dt$$

(By formula $\frac{d}{dt}$ [$\mathbf{u}(t) \cdot \mathbf{v}(t)$]
= $\mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$)

$$= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2 \, dt = \frac{m}{2} \left[|\mathbf{r}'(t)|^2 \right]_a^b$$

(Fundamental Theorem of Calculus)

$$= \frac{m}{2} \left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2 \right)$$

Therefore

15 $$W = \tfrac{1}{2} m |\mathbf{v}(b)|^2 - \tfrac{1}{2} m |\mathbf{v}(a)|^2$$

where **v** = **r**′ is the velocity.

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The quantity $\frac{1}{2}m\,|\,\mathbf{v}(t)\,|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore we can rewrite Equation 15 as

$$\boxed{16} \qquad W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that **F** is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$.

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In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$.

Then by the FTC for line integrals we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= -\int_C \nabla P \cdot d\mathbf{r}$$

$$= -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))]$$

$$= P(A) - P(B)$$

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Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant.

This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative.*

16.4 Green's Theorem

Green's Theorem

Green’s Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. (See Figure 1. We assume that D consists of all points inside C as well as all points on C.)

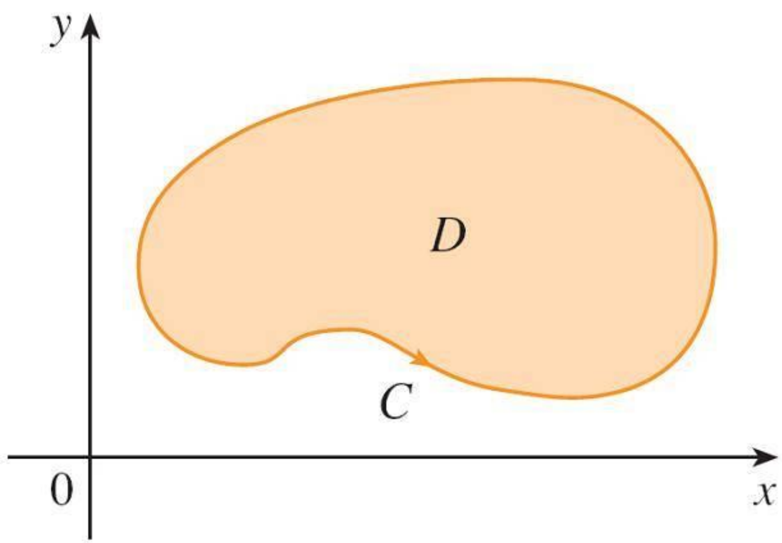

Figure 1

Green's Theorem

In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus, if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C.
(See Figure 2.)

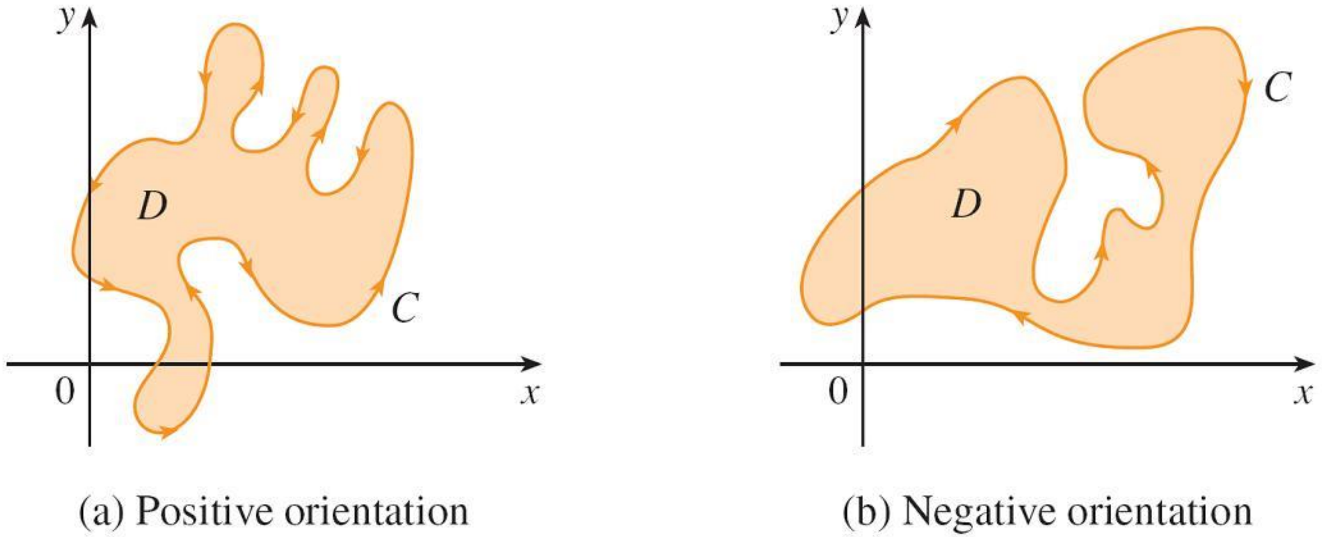

Figure 2

Green's Theorem

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Notation: We sometimes use the notation $\oint_C P\,dx + Q\,dy$ or $\ointctrclockwise_C P\,dx + Q\,dy$ to indicate that the line integral is calculated using the positive orientation of the closed curve C.

Another notation for the positively oriented boundary curve of D is ∂D. One may replace C by ∂D in Green's Theorem.

Example 1

Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1), and from (0, 1) to (0, 0).

Green's Theorem

In Example 1 we found that the double integral was easier to evaluate than the line integral.

But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction.

For instance, if it is known that *P*(*x*, *y*) = *Q*(*x*, *y*) = 0 on the curve *C*, then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy = 0$$

no matter what values *P* and *Q* assume in the region *D*.

Green's Theorem

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1\,dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$P(x, y) = 0$	$P(x, y) = -y$	$P(x, y) = -\frac{1}{2}y$
$Q(x, y) = x$	$Q(x, y) = 0$	$Q(x, y) = \frac{1}{2}x$

Green's Theorem

Then Green’s Theorem gives the following formulas for the area of *D*:

$$\boxed{5} \qquad \boxed{A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx}$$

Example 3

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Extended Versions of Green’s Theorem

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions.

For example, if D is the region shown in Figure 6, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple.

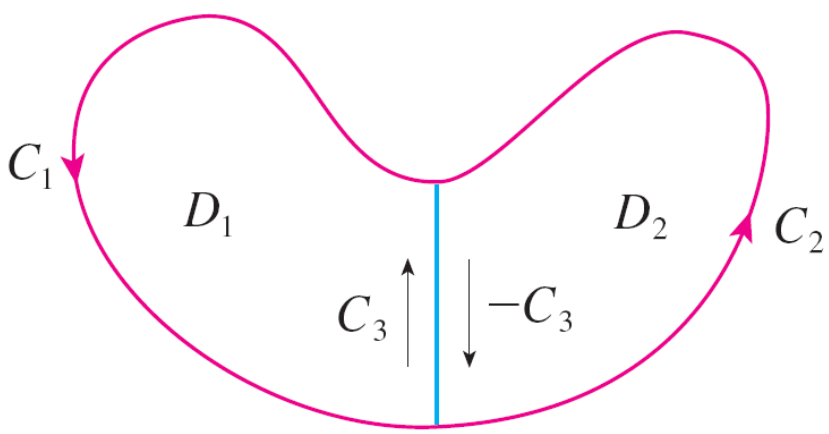

Figure 6

Extended Versions of Green's Theorem

The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup (-C_3)$ so, applying Green's Theorem to D_1 and D_2 separately, we get

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Extended Versions of Green's Theorem

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

The same sort of argument allows us to establish Green's Theorem for any finite union of nonoverlapping simple regions (see Figure 7).

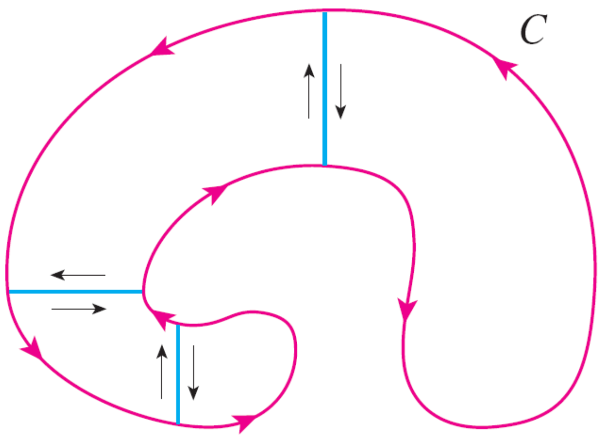

Figure 7

Example 4

Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Figure 8

Extended Versions of Green’s Theorem

Green’s Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected.

Observe that the boundary C of the region D in Figure 9 consists of two simple closed curves C_1 and C_2.

We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed.

Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2.

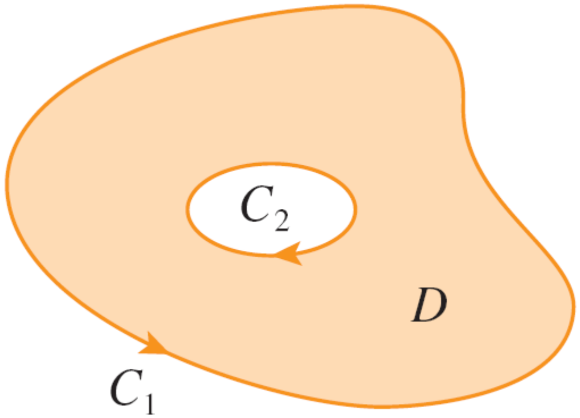

Figure 9

Extended Versions of Green's Theorem

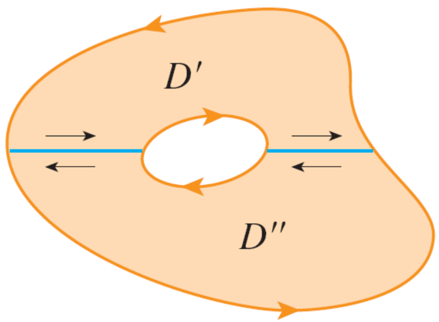

Figure 10

If we divide D into two regions D' and D'' by means of the lines shown in Figure 10 and then apply Green's Theorem to each of D' and D'', we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

Extended Versions of Green's Theorem

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy$$

which is Green's Theorem for the region *D*.

Conservative vector fields again

The proof of:

$\boxed{6}$ **Theorem** Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

from Section 16.3.

Note: From the proof we see why D has to be simply connected. See 16.4/Example 5 in the book for a counterexample where the domain of **F** is missing a point.

Example 5

Let

$$F(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}.$$

(a) Show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on $D = \{ (x,y) \mid (x,y) \neq (0, 0) \}$.

(b) Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any closed path that encloses the origin.