1.7 The Precise Definition of a Limit

The Precise Definition of a Limit

The intuitive definition of a limit is inadequate for some purposes because such phrases as "*x* is close to 2" and "*f*(*x*) gets closer and closer to *L*" are vague.

In order to be able to prove conclusively that

$$\lim_{x\to 0}\left(x^3 + \frac{\cos 5x}{10{,}000}\right) = 0.0001 \qquad \text{or} \qquad \lim_{x\to 0}\frac{\sin x}{x} = 1$$

we must make the definition of a limit precise.

The Precise Definition of a Limit

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x \to 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question:
How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1?

The Precise Definition of a Limit

The distance from x to 3 is $|x - 3|$ and the distance from $f(x)$ to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad |x - 3| < \delta \qquad \text{but } x \neq 3$$

If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation of our problem is to find a number δ such that

$$|f(x) - 5| < 0.1 \quad \text{if} \quad 0 < |x - 3| < \delta$$

The Precise Definition of a Limit

Notice that if $0 < |x - 3| < (0.1)/2 = 0.05$ then

$$|f(x) - 5| = |(2x - 1) - 5| = |2x - 6|$$

$$= 2|x - 3| < 2(0.05) = 0.1$$

that is,

$$|f(x) - 5| < 0.1 \qquad \text{if} \qquad 0 < |x - 3| < 0.05$$

Thus an answer to the problem is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3, then $f(x)$ will be within a distance of 0.1 from 5.

The Precise Definition of a Limit

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that $f(x)$ will differ from 5 by less than 0.01 provided that x differs from 3 by less than $(0.01)/2 = 0.005$:

$$|f(x) - 5| < 0.01 \qquad \text{if} \qquad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \qquad \text{if} \qquad 0 < |x - 3| < 0.0005$$

The numbers 0.1, 0.01 and 0.001 that we have considered are *error tolerances* that we might allow.

The Precise Definition of a Limit

For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below *any* positive number.

And, by the same reasoning, we can! If we write ε (the Greek letter epsilon) for an arbitrary positive number, then we find as before that

$$\boxed{1} \qquad |f(x) - 5| < \varepsilon \qquad \text{if} \qquad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

The Precise Definition of a Limit

This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3 because $\boxed{1}$ says that we can make the values of $f(x)$ within an arbitrary distance ε from 5 by taking the values of x within a distance $\varepsilon/2$ from 3 (but $x \neq 3$).

Note that $\boxed{1}$ can be rewritten as follows: if

$$3 - \delta < x < 3 + \delta \qquad (x \neq 3)$$

then

$$5 - \varepsilon < f(x) < 5 + \varepsilon$$

and this is illustrated in Figure 1.

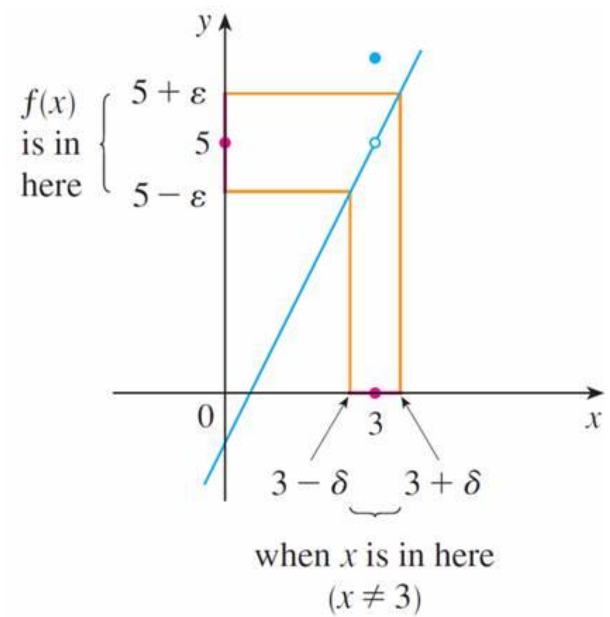

Figure 1

The Precise Definition of a Limit

By taking the values of x ($\neq 3$) to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the values of $f(x)$ lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$.

Using $\boxed{1}$ as a model, we give a precise definition of a limit.

$\boxed{2}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

The Precise Definition of a Limit

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from $f(x)$ to L, and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

$$\lim_{x \to a} f(x) = L$$

means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0).

The Precise Definition of a Limit

Alternatively,

$$\lim_{x \to a} f(x) = L$$

the values of $f(x)$ can be made as close as we please to L by taking x close enough to a (but not equal to a).

The Precise Definition of a Limit

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as $a - \delta < x < a + \delta$.

Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$.

The Precise Definition of a Limit

Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

$$\lim_{x \to a} f(x) = L$$

means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

The Precise Definition of a Limit

We interpret this statement geometrically by representing a function by an arrow diagram as in Figure 2, where *f* maps a subset of $\mathbb{R}$ onto another subset of $\mathbb{R}$.

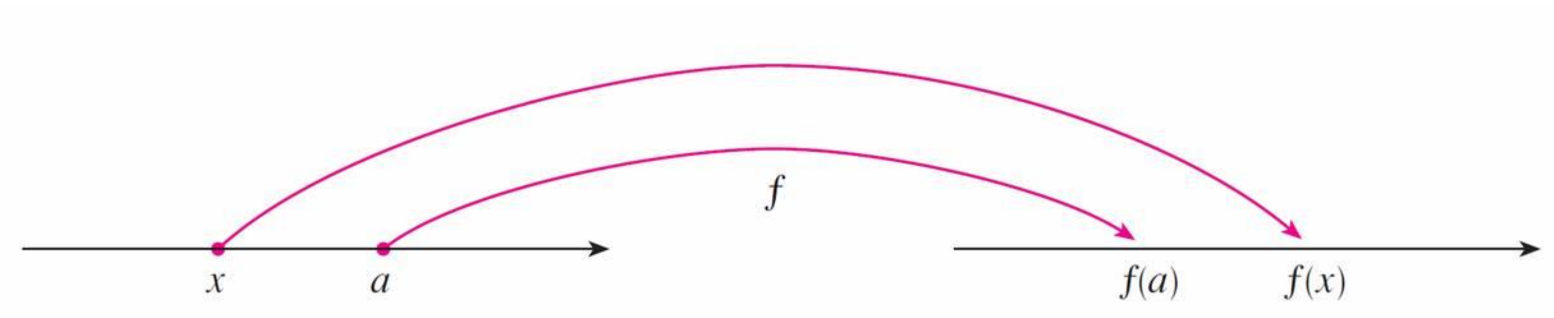

Figure 2

The Precise Definition of a Limit

The definition of limit says that if any small interval ($L - \varepsilon$, $L + \varepsilon$) is given around L, then we can find an interval ($a - \delta$, $a + \delta$) around a such that f maps all the points in ($a - \delta$, $a + \delta$) (except possibly a) into the interval ($L - \varepsilon$, $L + \varepsilon$). (See Figure 3.)

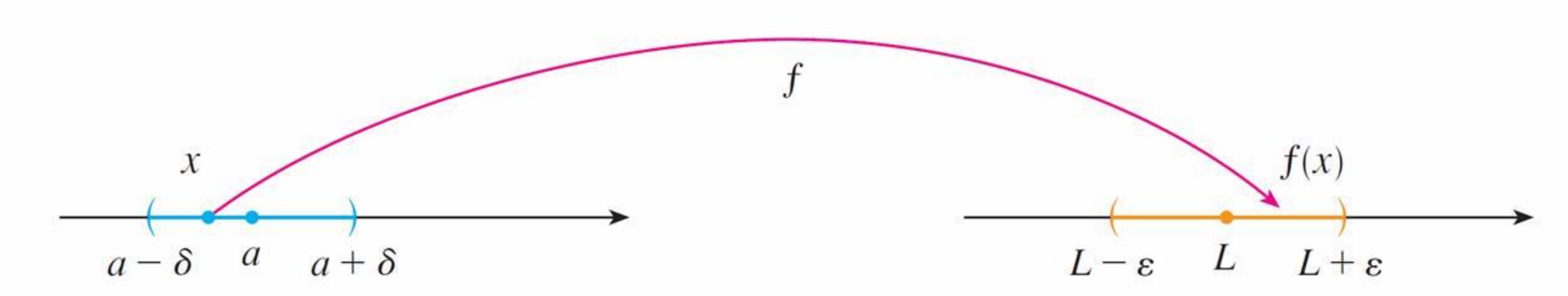

Figure 3

The Precise Definition of a Limit

Another geometric interpretation of limits can be given in terms of the graph of a function. If $\varepsilon > 0$ is given, then we draw the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ and the graph of f. (See Figure 4.)

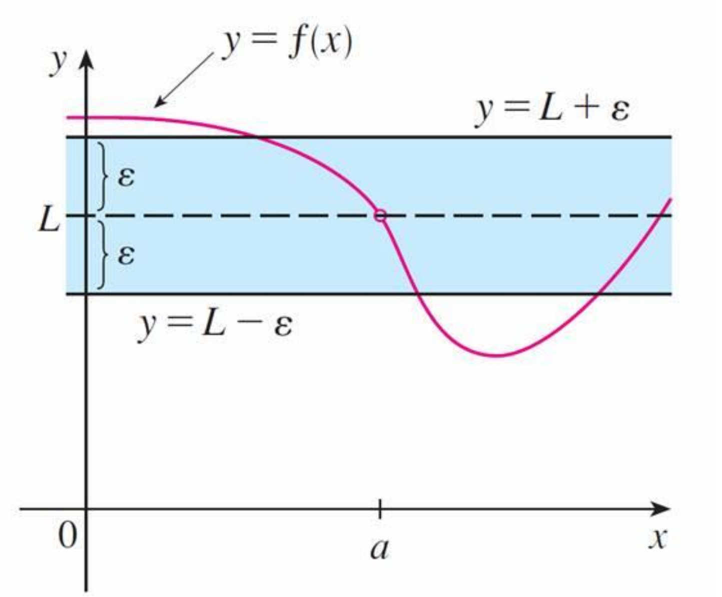

Figure 4

The Precise Definition of a Limit

If $\lim_{x \to a} f(x) = L$, then we can find a number $\delta > 0$ such that if we restrict x to lie in the interval $(a - \delta, a + \delta)$ and take $x \neq a$, then the curve $y = f(x)$ lies between the lines $y = L - \varepsilon$ and $y = L + \varepsilon$ (See Figure 5.) You can see that if such a δ has been found, then any smaller δ will also work.

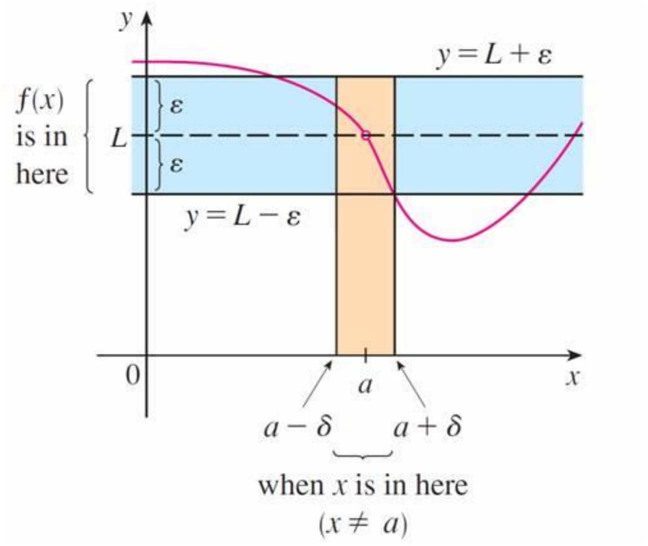

Figure 5

The Precise Definition of a Limit

It is important to realize that the process illustrated in Figures 4 and 5 must work for *every* positive number ε, no matter how small it is chosen. Figure 6 shows that if a smaller ε is chosen, then a smaller δ may be required.

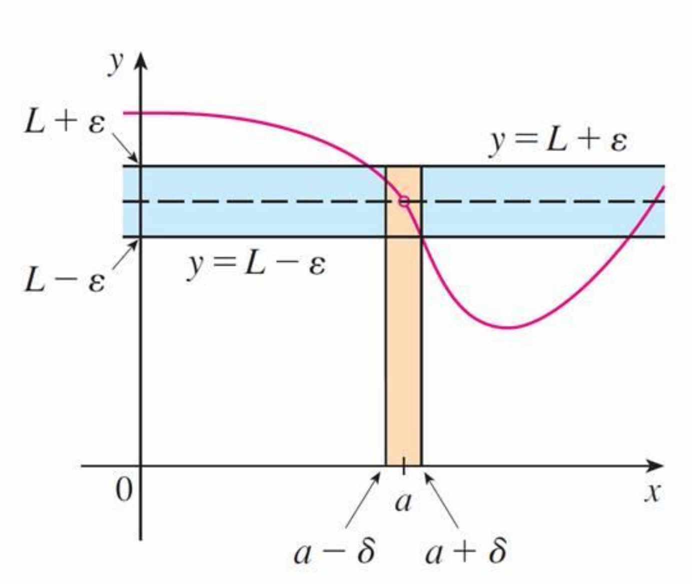

Figure 6

Examples 2 and 3

Example 2. Prove that $\lim_{x \to 3} (4x - 5) = 7$.

Example 3. Prove that

$$\lim_{x \to 3} x^2 = 9.$$

The Precise Definition of a Limit

The intuitive definitions of one-sided limits can be precisely reformulated as follows.

3 Definition of Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a - \delta < x < a \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

The Precise Definition of a Limit

4 Definition of Right-Hand Limit

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 3

Use Definition 4 to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Uniqeness of the limit

Prove that the limit, if it exists, is unique.

Limit laws. Proof of the sum law.

Prove that if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist, then

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Infinite Limits

Infinite Limits

Infinite limits can also be defined in a precise way.

$\boxed{6}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M$$

Infinite Limits

This says that the values of $f(x)$ can be made arbitrarily large (larger than any given number M) by taking x close enough to a (within a distance δ, where δ depends on M, but with $x \neq a$). A geometric illustration is shown in Figure 10.

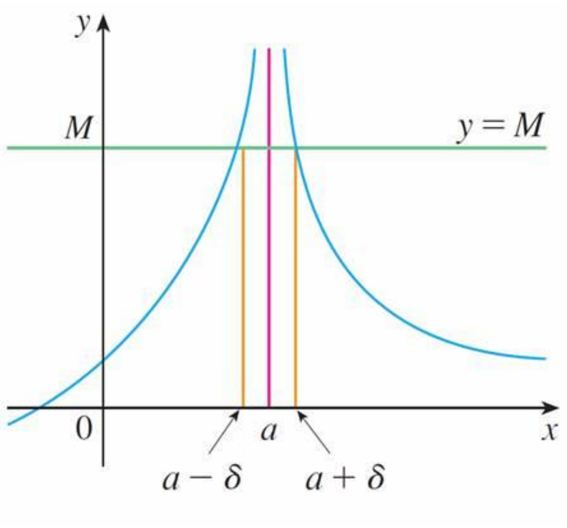

Figure 10

Infinite Limits

Given any horizontal line $y = M$, we can find a number $\delta > 0$ such that if we restrict to x lie in the interval $(a - \delta, a + \delta)$ but $x \neq a$, then the curve $y = f(x)$ lies above the line $y = M$.

You can see that if a larger M is chosen, then a smaller δ may be required.

Example 5

Use Definition 6 to prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Infinite Limits

$\boxed{7}$ **Definition** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) < N$$