14.6 Directional Derivatives and the Gradient Vector

Directional Derivatives and the Gradient Vector

In this section we introduce a type of derivative, called a *directional derivative,* that enables us to find the rate of change of a function of two or more variables in any direction.

Directional Derivatives

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$\boxed{1} \qquad f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors **i** and **j**.

Directional Derivatives

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.)

To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S.

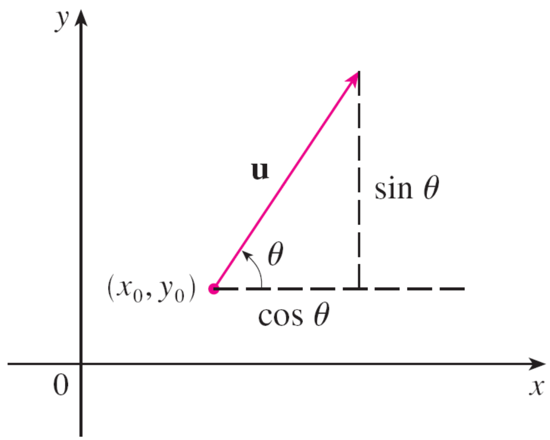

Figure 2

A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$

Directional Derivatives

The vertical plane that passes through P in the direction of **u** intersects S in a curve C. (See Figure 3.)

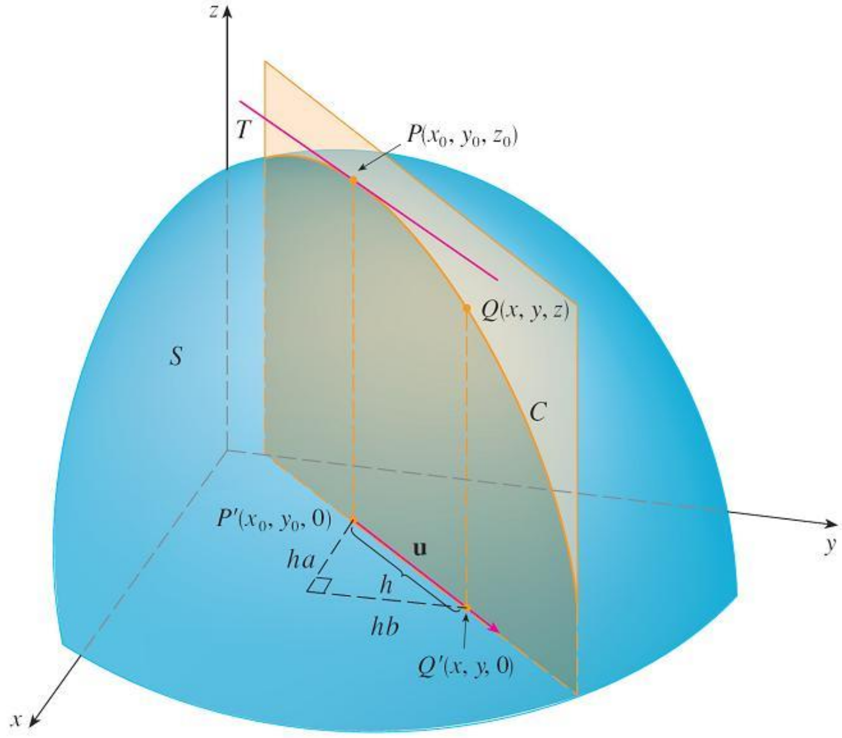

Figure 3

Directional Derivatives

The slope of the tangent line *T* to *C* at the point *P* is the rate of change of *z* in the direction of **u**.

The parametric equations of the line in the *xy*-plane through the point (x_0, y_0) with direction vector $\mathbf{u} = \langle a, b \rangle$ is given by $x = x_0 + ta$ and $y = y_0 + tb$ with t being a real parameter.

Then C has parameterization

$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = f(x_0 + ta, y_0 + tb)$$

and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Directional Derivatives

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of $\mathbf{u}$, which is called the directional derivative of f in the direction of $\mathbf{u}$.

$\boxed{2}$ **Definition** The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Directional Derivatives

By comparing Definition 2 with Equations $\boxed{1}$, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}}f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}}f = f_y$.

In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

Directional Derivatives

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

$\boxed{3}$ **Theorem** If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Directional Derivatives

If the unit vector **u** makes an angle θ with the positive x-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

$$\boxed{6} \qquad D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

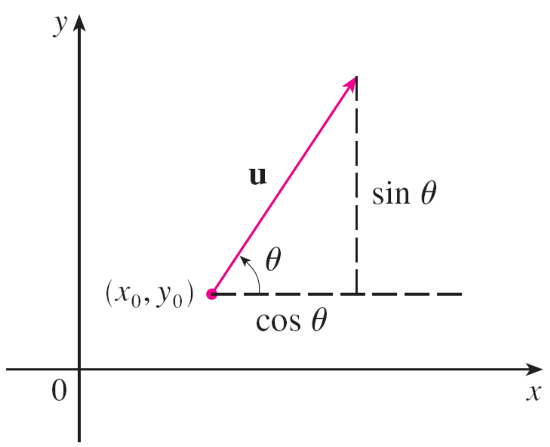

Figure 2
A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$

Example 2

Find the directional derivative $D_{\mathbf{u}}f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^4$$

and u is the unit vector in the direction given by the angle $\theta = \frac{\pi}{6}$, measured from the positive *x*-axis. What is $D_{\mathbf{u}}f(1, 2)$?

The Gradient Vectors

The Gradient Vectors

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\boxed{7} \qquad \begin{aligned} D_{\mathbf{u}}f(x, y) &= f_x(x, y)a + f_y(x, y)b \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u} \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well.

So we give it a special name (the *gradient* of f) and a special notation (**grad** f or ∇f, which is read "del f").

The Gradient Vectors

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example 3

If $f(x, y) = \sin x + e^{xy}$, then find $\nabla f(x, y)$ and $\nabla f(0, 1)$.

The Gradient Vectors

With this notation for the gradient vector, we can rewrite the expression for the directional derivative of a differentiable function as

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$$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$$

This expresses the directional derivative in the direction of **u** as the scalar projection of the gradient vector onto **u**.

Functions of Three Variables

Functions of Three Variables

For functions of three variables we can define directional derivatives in a similar manner.

Again $D_{\mathbf{u}}f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector **u**.

$\boxed{10}$ **Definition** The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Functions of Three Variables

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

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$$\boxed{D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$.

This is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + h\,\mathbf{u}$ and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

Functions of Three Variables

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then

12 $$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

13 $$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}}$$

Functions of Three Variables

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

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$$\boxed{D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}}$$

Note. Both Definition 11 and formula (14) can be generalized to functions of *n* variables in a straightforward fashion.

Example 5

If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Maximizing the Directional Derivatives

Maximizing the Directional Derivatives

Suppose we have a function *f* of two or three variables and we consider all possible directional derivatives of *f* at a given point.

These give the rates of change of *f* in all possible directions.

We can then ask the questions: In which of these directions does *f* change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

$\boxed{15}$ **Theorem** Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Example 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Tangent Planes to Level Surfaces

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S.

Let C be any curve that lies on the surface S and passes through the point P. Recall that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$\boxed{16} \qquad F(x(t), y(t), z(t)) = k$$

Tangent Planes to Level Surfaces

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

$$\boxed{17} \qquad \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

Tangent Planes to Level Surfaces

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

$$\boxed{18} \qquad \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$$

Equation 18 says that *the gradient vector* $\nabla F(x_0, y_0, z_0)$ *at* P *is perpendicular to the tangent vector* $\mathbf{r}'(t_0)$ *to any curve* C *on* S *that passes through* P. (See Figure 9.)

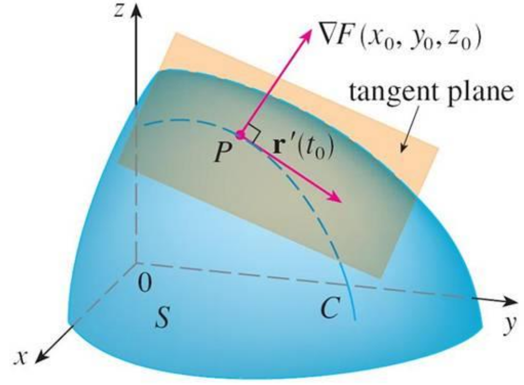

Figure 9

Tangent Planes to Level Surfaces

If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

Using the standard equation of a plane, we can write the equation of this tangent plane as

$$\boxed{19} \quad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Tangent Planes to Level Surfaces

The **normal line** to *S* at *P* is the line passing through *P* and perpendicular to the tangent plane.

The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, its symmetric equations are

$$\boxed{20} \qquad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Tangent Planes to Level Surfaces

Note: In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$
$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$
$$F_z(x_0, y_0, z_0) = -1$$

so we get the familiar equation of the tangent plane

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Example 8

Find the equations of the tangent plane and normal line at the point (–2, 1, –3) to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$