Infinite Limits

Infinite Limits

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Another notation for

$$\lim_{x \to a} f(x) = \infty$$

is

$$f(x) \to \infty \text{ as } x \to a.$$

Infinite Limits

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

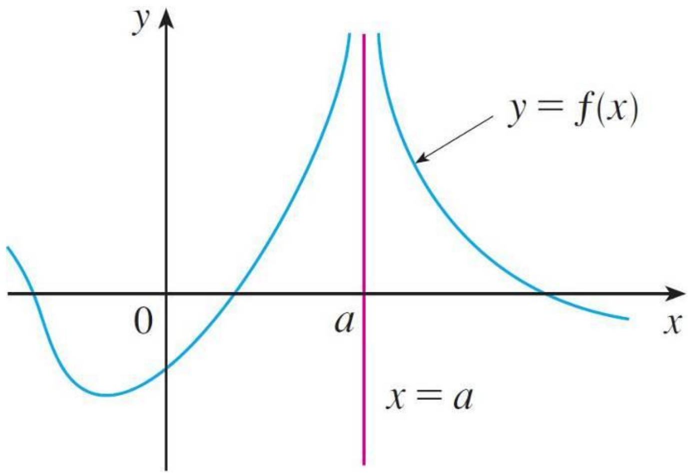
"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

Infinite Limits

This definition is illustrated graphically in Figure 12.

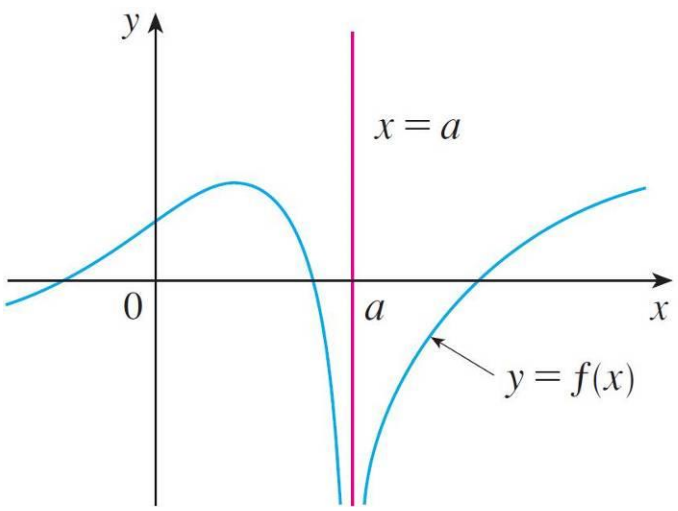


$$\lim_{x \to a} f(x) = \infty$$

Figure 12

Infinite Limits

A similar sort of limit, for functions that become large negative as *x* gets close to *a*, is defined in Definition 4 and is illustrated in Figure 11.



$$\lim_{x \to a} f(x) = -\infty$$

Figure 11

Infinite Limits

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example, we have

$$\lim_{x \to 0} \left(-\frac{1}{x^2}\right) = -\infty$$

Infinite Limits

Similar definitions can be given for the one-sided infinite limits

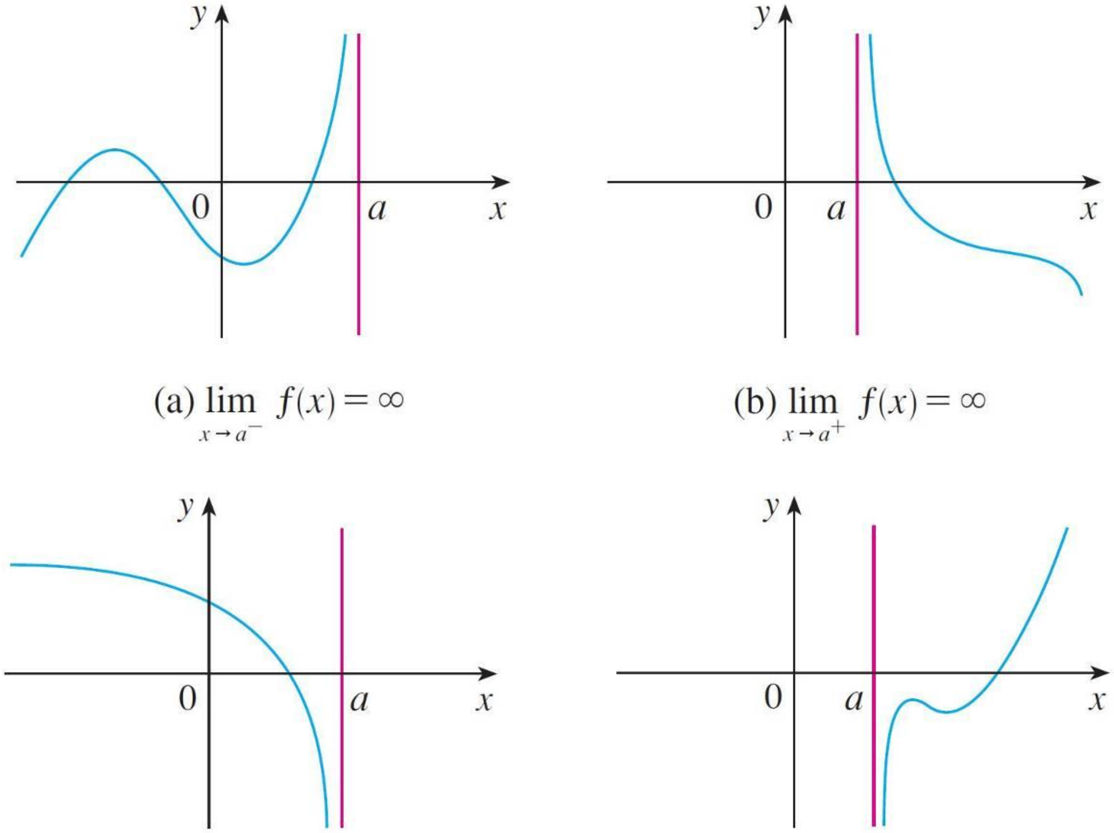
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that "$x \to a^-$" means that we consider only values of that are less than a, and similarly "$x \to a^+$" means that we consider only $x > a$.

Infinite Limits

Illustrations of these four cases are given in Figure 14.



(a) $\lim_{x \to a^-} f(x) = \infty$

(b) $\lim_{x \to a^+} f(x) = \infty$

(c) $\lim_{x \to a^-} f(x) = -\infty$

(d) $\lim_{x \to a^+} f(x) = -\infty$

Figure 14

Infinite Limits

$\boxed{6}$ **Definition** The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Examples 7 and 8

7. Does the curve $y = \frac{2x}{x-3}$ have a vertical asymptote?

8. Find the vertical asymptotes of $f(x) = \tan x$.

1.6 Calculating Limits Using the Limit Laws

Calculating Limits Using the Limit Laws

In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

Calculating Limits Using the Limit Laws

These five laws can be stated verbally as follows:

Sum Law

1. The limit of a sum is the sum of the limits.

Difference Law

2. The limit of a difference is the difference of the limits.

Constant Multiple Law

3. The limit of a constant times a function is the constant times the limit of the function.

Calculating Limits Using the Limit Laws

Product Law

4. The limit of a product is the product of the limits.

Quotient Law

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$.

Remark: These laws hold for one-sided limits, too.

Example 1

Use the Limit Laws and the graphs of *f* and *g* in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

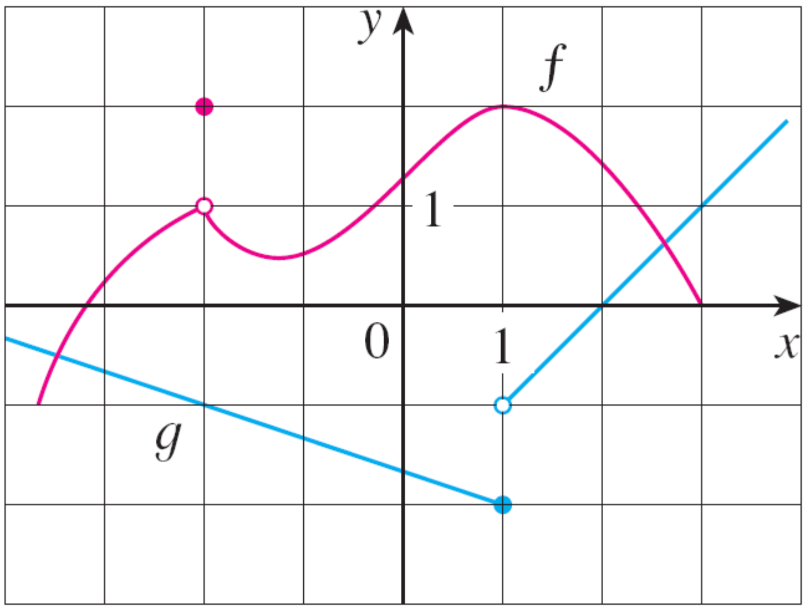


Figure 1

Calculating Limits Using the Limit Laws

If we use the Product Law repeatedly with $g(x) = f(x)$ (or, using induction), we obtain the following law.

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

In applying these six limit laws, we need to use two special limits:

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$).

Calculating Limits Using the Limit Laws

If we now put *f*(*x*) = *x* in Law 6 and use Law 8, we get another useful special limit.

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

A similar limit holds for roots as follows.

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

More generally, we have the following law.

Root Law

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

$\left[\text{If } n \text{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

Example 2

Evaluate the following limit and justify each step.

(a)

$$\lim_{x \to 5} (2x^2 - 3x + 4)$$

(b)

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

Calculating Limits Using the Limit Laws

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a.*

In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

Example 3

Find

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

Example 4

Find

$$\lim_{x \to 1} g(x)$$

where

$$g(x) = \begin{cases} x + 1 \text{ if } x \neq 1 \\ \pi \qquad \text{ if } x = 1. \end{cases}$$

Example 5

Evaluate

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}.$$

Calculating Limits Using the Limit Laws

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

Example 7 and 8

Example 7. Show that

$$\lim_{x \to 0} |x| = 0.$$

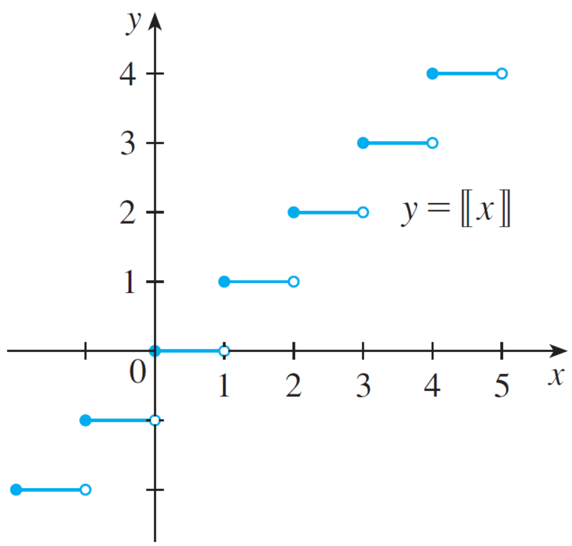
Example 8. Prove that

$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist.

Example 10

The **greatest integer function** is defined $[\![x]\!]$ = the largest integer that is less than or equal to *x*. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}\,]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x\to 3} [\![x]\!]$ does not exist.



Greatest integer function

Calculating Limits Using the Limit Laws

The next two theorems give two additional properties of limits.

$\boxed{2}$ **Theorem** If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

$\boxed{3}$ **The Squeeze Theorem** If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Calculating Limits Using the Limit Laws

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7.

It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

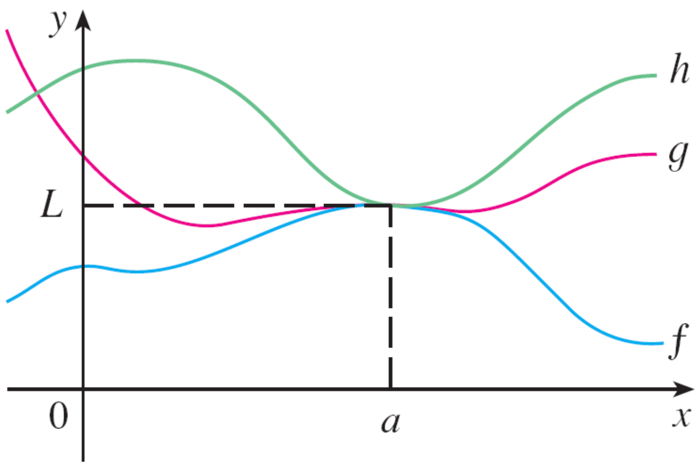


Figure 7

Example 11

Show that

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$$