Combinations of Functions

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Examples: composition

8. Find $f \circ g \circ h$ if $f(x) = \frac{x}{x+1}$, $g(x) = x^{10}$ and $h(x) = x + 3$.

9. Given $F(x) = \cos^2(x + 9)$, find functions f, g, h such that

$$F = f \circ g \circ h.$$

1.4 The Tangent and Velocity Problems

Tangent and velocity problems

Before introducing the notion of the limit of a function we consider two motivating problems: finding equations of *tangent lines* to curves and calculating the *instantaneous velocity* of a moving object knowing its position.

The Tangent Problem

The Tangent Problems

The word *tangent* is derived from the Latin word *tangens*, which means “touching.”

Thus, a tangent to a curve is a line that touches the curve.

In other words, a tangent line should have the same direction as the curve at the point of contact.

The Tangent Problems

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once, as in Figure 1(a).

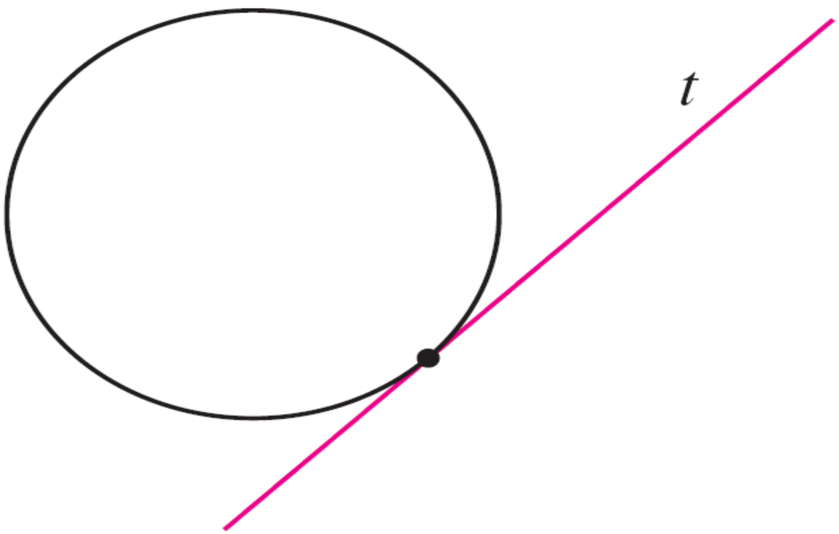


Figure 1(a)

The Tangent Problems

For more complicated curves this definition is inadequate. Figure 1(b) shows two lines *l* and *t* passing through a point *P* on a curve *C*.

The line *l* intersects *C* only once, but it certainly, does not look like what we think of as a tangent.

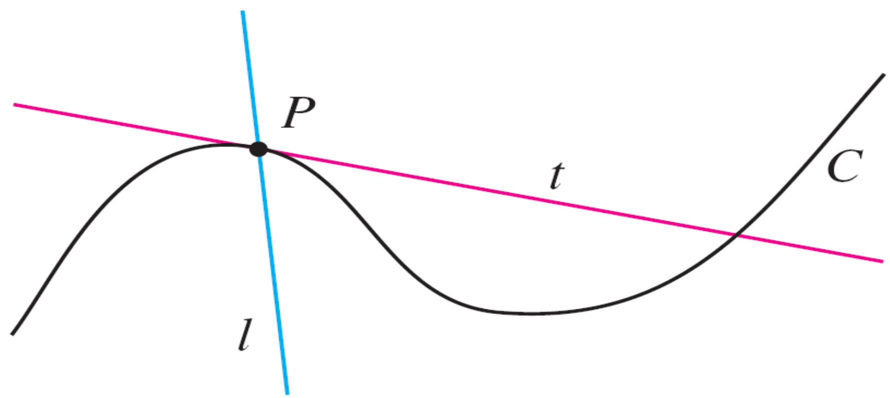


Figure 1(b)

The line *t*, on the other hand, looks like a tangent but it intersects *C* twice.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution:

We will be able to find an equation of the tangent line t as soon as we know its slope m.

The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope.

Example 1 – *Solution*

cont'd

But observe that we can compute an approximation to m by choosing a near by point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ. [A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

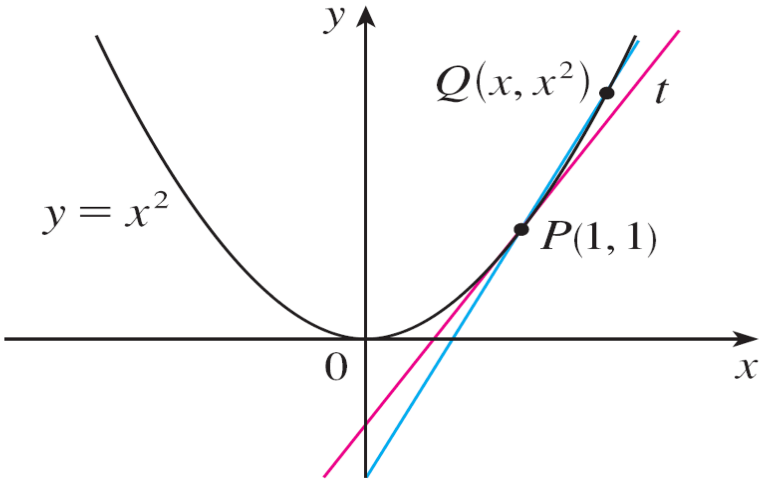


Figure 2

Example 1 – *Solution*

cont'd

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1}$$

$$= \frac{1.25}{0.5}$$

$$= 2.5$$

Example 1 – *Solution*

cont'd

The tables in the margin show the values of m_{PQ} for several values of x close to 1.

The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2.

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

Example 1 – *Solution*

cont'd

This suggests that the slope of the tangent line *t* should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line to write the equation of the tangent line through (1, 1) as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Example 1 – *Solution*

cont'd

Figure 3 illustrates the limiting process that occurs in this example.

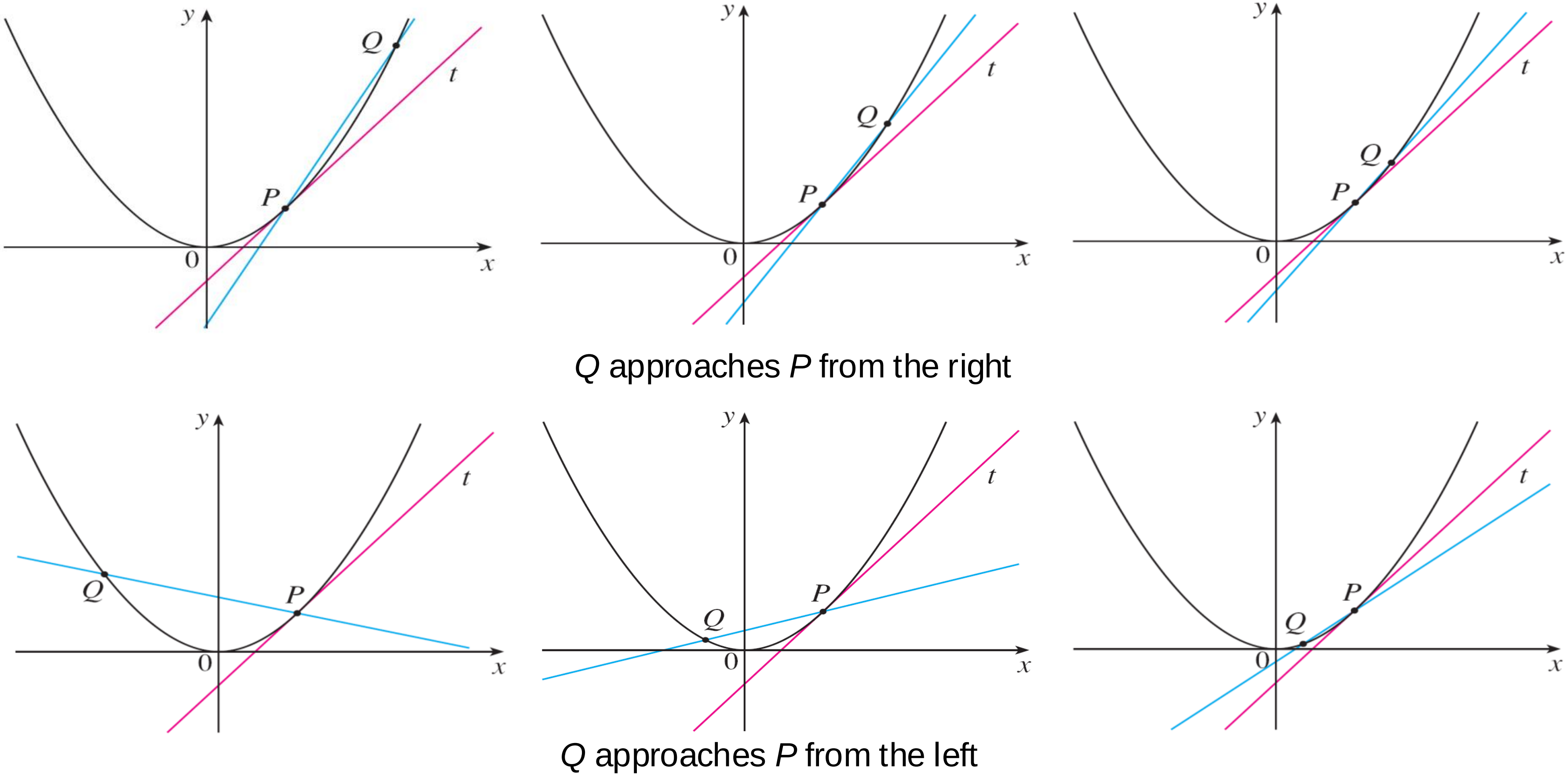


Figure 3

Example 1 – *Solution*

cont'd

As *Q* approaches *P* along the parabola, the corresponding secant lines rotate about *P* and approach the tangent line *t*.

The Velocity Problem

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Solution:

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.)

If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation $s(t) = 4.9t^2$

Example 3 – *Solution*

cont'd

The difficulty in finding the velocity after 5 s is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved.

However, we can approximate the desired quantity by computing the average velocity over the brief time interval of length Δt from $t = 5$ to $t = 5 + \Delta t$ and calculate:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

Example 3 – *Solution*

cont'd

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

Example 3 – *Solution*

cont'd

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s.

The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$.

Thus the (instantaneous) velocity after 5 s is

$$v = 49 \text{ m/s}$$

1.5 The Limit of a Function

The Limit of a Function

In this section we will introduce an intuitive definition of the limit of a function and discuss means and pitfalls of finding a limit graphically/numerically.

To find, for example, the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Let's investigate the behavior of the function *f* defined by

$f(x) = \frac{x-1}{x^2-1}$ for values of *x* near 1.

The Limit of a Function

$$f(x) = \frac{x-1}{x^2-1}$$

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

The Limit of a Function

In fact, it appears that we can make the values of *f*(*x*) as close as we like to 1/2 by taking *x* sufficiently close to 1.

We express this by saying "the limit of the function $f(x) = \frac{x-1}{x^2-1}$ as *x* approaches 1 is equal to 1/2."

The notation for this is

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

The Limit of a Function

We can illustrate what is happening by considering the graph of *f* in Figure 3.

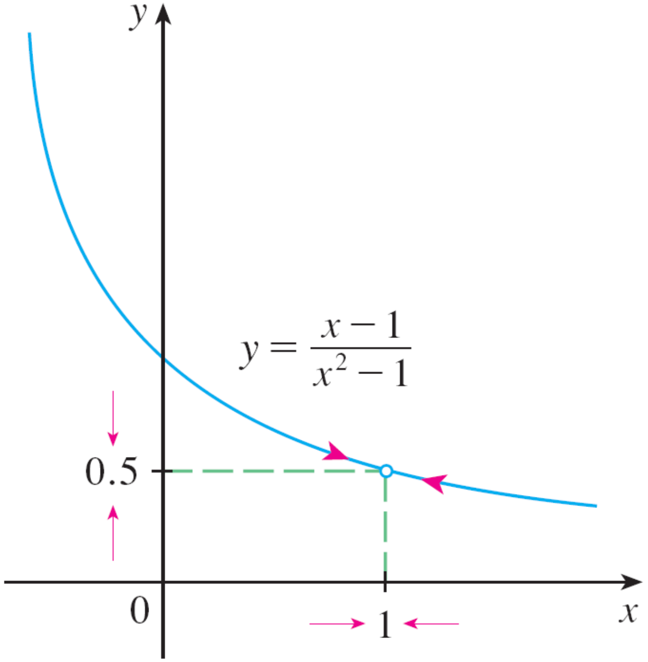


Figure 3

The Limit of a Function

In general, we use the following notation.

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

This says that the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

The Limit of a Function

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is $\quad f(x) \to L \quad$ as $\quad x \to a$

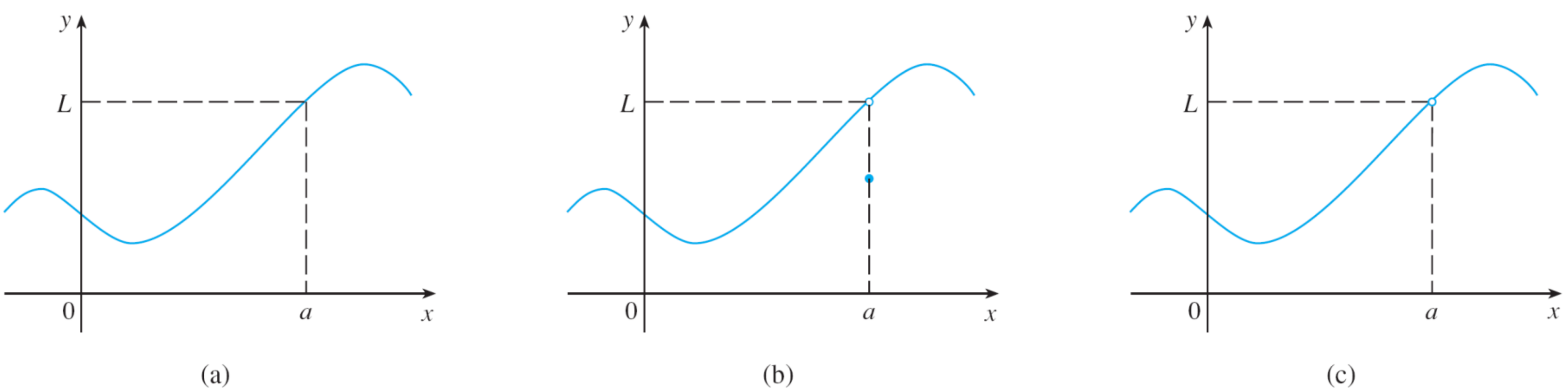
which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near a*.

The Limit of a Function

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$.

But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.



$\lim_{x \to a} f(x) = L$ in all three cases

Figure 2

The Limit of a Function

For example, if we change *f* in the first example slightly by giving it the value 2 when *x* = 1 and calling the resulting function *g*, then *g* still has the same limit as *f* as *x* approaches 1 :

$$g(x) = \begin{cases} \dfrac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

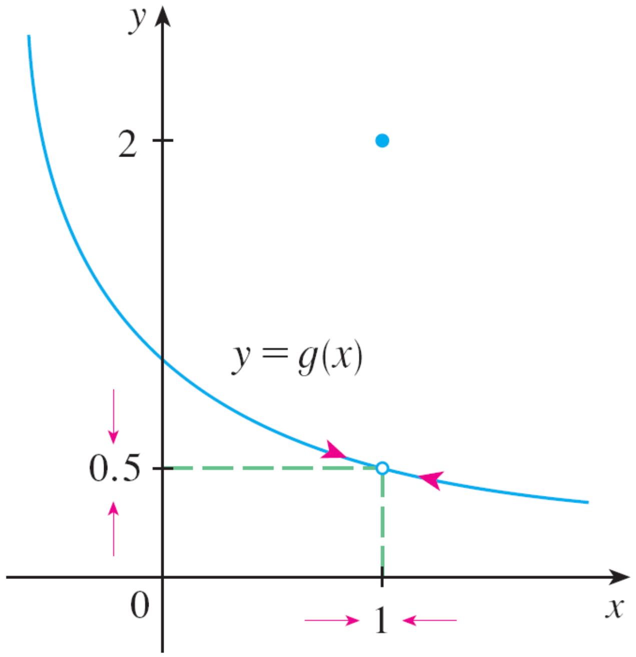


Figure 4

Example 1

Guess the value of

$$\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}.$$

Example 2

Guess the value

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

One-Sided Limits

One-Sided Limits

The function *H* is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

H(*t*) approaches 0 as *t* approaches 0 from the left and *H*(*t*) approaches 1 as *t* approaches 0 from the right.

We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

One-Sided Limits

The symbol "$t \to 0^-$" indicates that we consider only values of t that are less than 0.

Likewise, "$t \to 0^+$" indicates that we consider only values of t that are greater than 0.

One-Sided Limits

2 Intuitive Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the left*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *less than* a.

We write

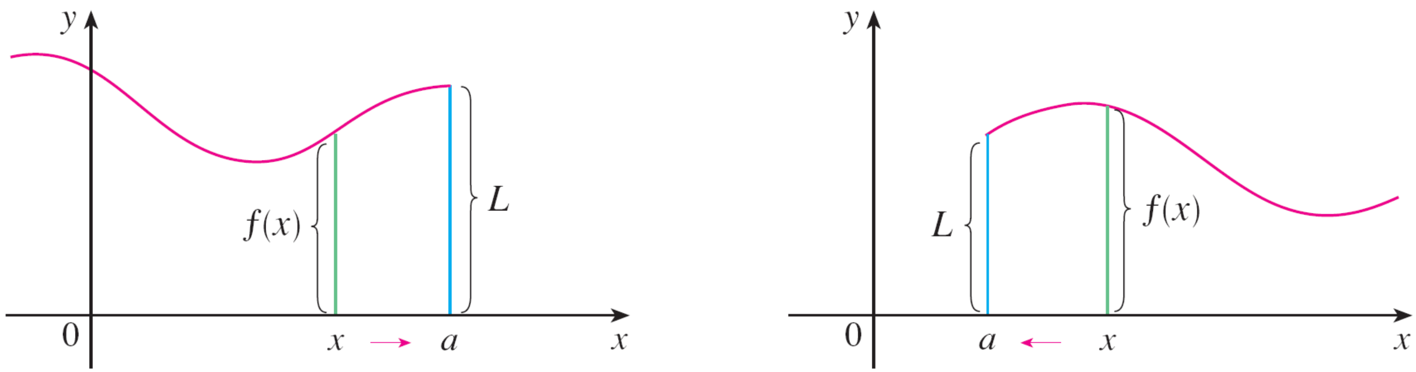
$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the right*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *greater than* a.

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a.

One-Sided Limits

These definitions are illustrated in Figure 6.



(a) $\lim_{x \to a^-} f(x) = L$

(b) $\lim_{x \to a^+} f(x) = L$

Figure 6

One-Sided Limits

By comparing Definition 1 with the definitions of one-sided limits, we see that the following is true.

$$\boxed{3} \quad \lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

Example 4

The graph of a function g is shown in Figure 7. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

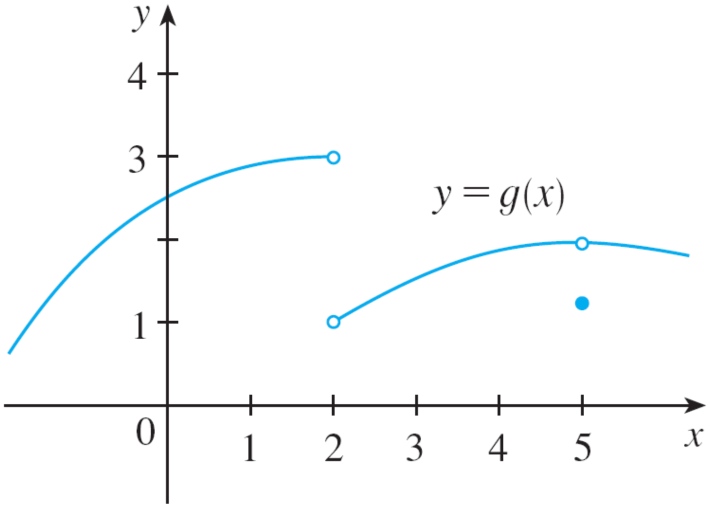


Figure 7

How can a limit fail to exist?

We have seen that a limit fails to exist at a number a if the left- and right-hand limits are not equal (as in Example 4). The next two examples illustrate additional ways that a limit can fail to exist.

Example 5. Investigate

$$\lim_{x \to 0} \sin \frac{\pi}{x}.$$

Example 6. Find

$$\lim_{x \to 0} \frac{1}{x^2}$$

if it exists.