How Can a Function Fail to Be Differentiable?

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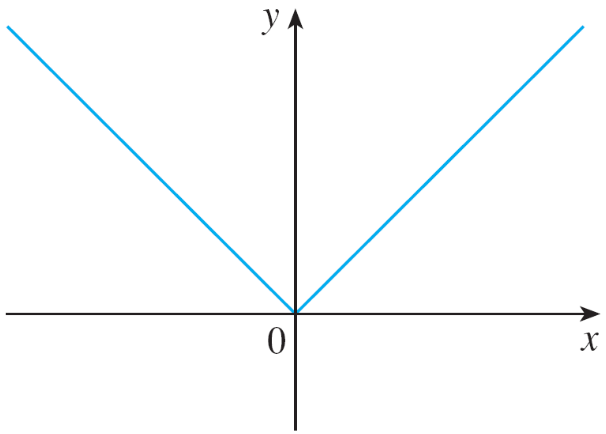
We saw that the function $y = |x|$ is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$.

In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]



$y = f(x) = |x|$

Figure 5(a)

How Can a Function Fail to Be Differentiable?

Theorem 4 gives another way for a function not to have a derivative. It says that if *f* is not continuous at *a*, then *f* is not differentiable at *a*. So, at any discontinuity (for instance, a jump discontinuity) *f* fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, *f* is continuous at *a* and

$$\lim_{h \to 0+} \frac{f(a+h) - f(a)}{h} = \infty \text{ or } -\infty$$

and

$$\lim_{h \to 0-} \frac{f(a+h) - f(a)}{h} = \infty \text{ or } -\infty$$

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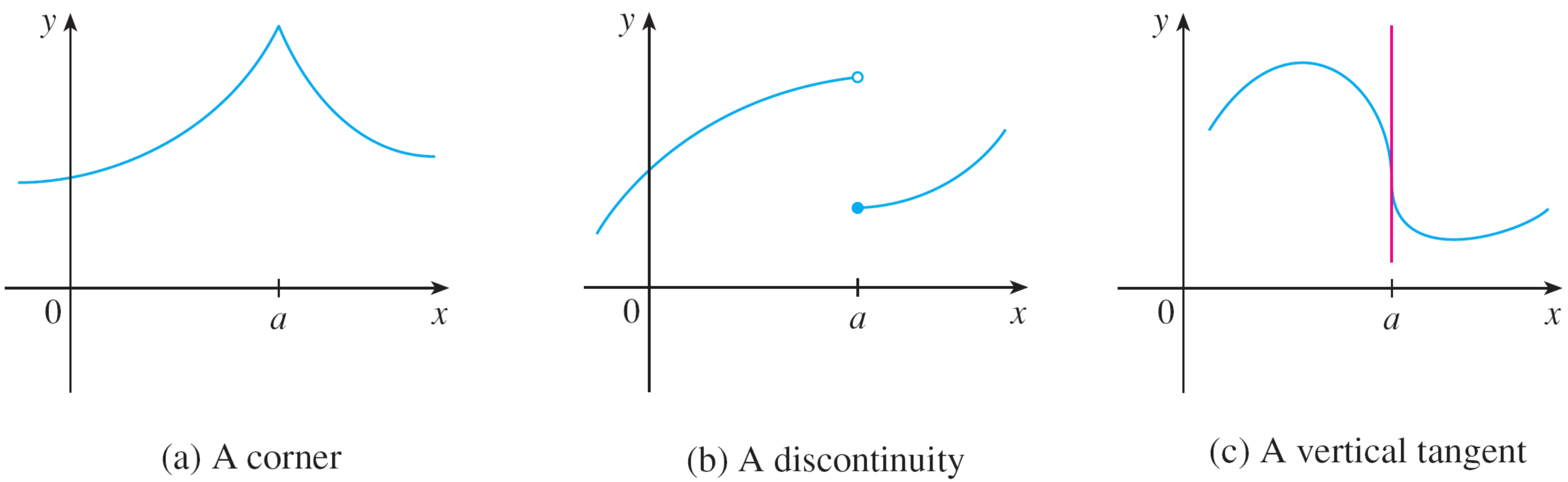
Figure 7 illustrates examples of the three possibilities that we have discussed.



Three ways for *f* not to be differentiable at *a*

Figure 7

Vertical tangent

1. Show that the function $f(x) = \sqrt[3]{x}$ is not differentiable at $a = 0$ (vertical tangent).

2. Show that the function

$$f(x) = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$$

is not differentiable at $a = 0$ (vertical tangent).

Higher Derivatives

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f.

Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Example 6

If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Higher Derivatives

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration,* which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Higher Derivatives

The instantaneous rate of change of velocity with respect to time is called the **acceleration** *a*(*t*) of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Higher Derivatives

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$.

If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Higher Derivatives

The process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$.

In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times.

If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

Higher Derivatives

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line.

Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3 s}{dt^3}$$

Higher Derivatives

Thus, the jerk *j* is the rate of change of acceleration.

It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

Example 7

If $f(x) = x^3 - x$, find $f''(x)$ and $f^{(4)}(x)$.

2.3 Differentiation Formulas

Differentiation Formulas

Let's start with the two simplest functions.

Example. Let $f(x) = c$, for all x, where c is a constant. Then $f'(x) = 0$ for all x.

Example. Let $f(x)=x$. Then $f'(x) = 1$ for all x.

Differentiation Formulas

Theorem. Suppose that *f* and *g* are differentiable at *x* and let *c* be a constant. Then *f* + *g*, *f* - *g*, *cf*, *f*·*g*, *f*/*g* (if g(x) ≠ 0) are all differentiable at x. Furthermore

(a) $(f + g)'(x) = f'(x) + g'(x)$

(b) $(cf)'(x) = cf'(x)$

(c) $(f - g)'(x) = f'(x) - g'(x)$

(d) $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

(e)

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

Sum of multiple functions

Remark: The Sum Rule can be extended to the sum of any number of functions. For instance, using this theorem twice, we get

$$(f + g + h)'(x) = [(f + g) + h]'(x) = (f + g)'(x) + h'(x)$$

$$= f'(x) + g'(x) + h'(x).$$

Power Rule

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 1

(a) If $f(x) = x^6$, find $f'(x)$.

(b) If $y = x^{1000}$, find y'.

(c) If $y = t^4$, find $\dfrac{dy}{dt}$.

(d) Find $\dfrac{d}{dr}(r^3)$.

Example 2

(a) Find $\dfrac{d}{dx}(3x^4)$

(b) Find $\dfrac{d}{dx}(-x)$

Example 3

Find $\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$.

Example 4

Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Example 5

The equation of motion of a particle is

$$s = 2t^3 - 5t^2 + 3t - 4,$$

where *s* is measured in centimeters and *t* in seconds. Find the acceleration as a function of time. What is the acceleration after 2 seconds?