7 Techniques of Integration

7.1 Integration by Parts

Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called the rule for *integration by parts.*

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Integration by Parts

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\, dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\, dx + \int g(x)f'(x)\, dx = f(x)g(x)$$

We can rearrange this equation as

$$\boxed{1} \qquad \boxed{\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx}$$

Formula 1 is called the **formula for integration by parts**.

Integration by Parts

It is perhaps easier to remember in the following notation.

Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

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$$\int u\,dv = uv - \int v\,du$$

Examples

1. Find $\int x \sin x \, dx$.

2. Evaluate $\int \ln x \, dx$.

3. Evaluate $\int t^2 e^t \, dt$.

4. Evaluate $\int e^x \sin x \, dx$.

Integration by Parts

If we combine the formula for integration by parts with Part 2 of Fundamental Theorem of Calculus, we can evaluate definite integrals by parts.

Evaluating both sides of Formula 1 between *a* and *b*, assuming f' and g' are continuous, and using the Fundamental Theorem, we obtain

$$\boxed{6} \qquad \boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx}$$

Definite integrals

Example 5. Calculate $\int_0^1 \tan^{-1} x \, dx.$

Reduction formulas

Example 6. Prove the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

7.2 Trigonometric Integrals

Trigonometric Integrals

In this section we use trigonometric identities to integrate certain combinations of trigonometric functions.

We start with powers of sine and cosine.

Examples 1- 2

Example 1. Find $\int \sin^5 x \cos^2 x \, dx$.

Example 2. Evaluate $\int \cos^3 x \, dx$.

Examples 3 and 4

Example 3. Evaluate $\int_0^{\pi} \sin^2 x \, dx$.

Example 4. Find $\int \sin^4 x \ \ dx$.

Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Trigonometric Integrals

We can use a similar strategy to evaluate integrals of the form $\int \tan^m x \sec^n x \, dx$.

Since $(d/dx) \tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining (even) power of secant to an expression involving tangent using the identity $\sec^2 x = 1 + \tan^2 x$.

Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining (even) power of tangent to secant.

Examples 5 - 6

Example 5. Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Example 6. Evaluate $\int \tan^5 \theta \sec^7 \theta \, d\theta$.

Trigonometric Integrals

The preceding examples demonstrate strategies for evaluating integrals of the form $\int \tan^m x \sec^n x \, dx$ for two cases, which we summarize here.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geq 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

Then substitute $u = \sec x$.

Trigonometric Integrals

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity.

We will sometimes need to be able to integrate tan *x* by using the formula given below:

$$\boxed{\int \tan x \, dx = \ln |\sec x| + C}$$

Trigonometric Integrals

We will also need the indefinite integral of secant:

$$\boxed{1} \qquad \boxed{\int \sec x \, dx = \ln |\sec x + \tan x| + C}$$

We could verify Formula 1 by, for example, differentiating the right side.

Examples 7 - 8

Example 7. Find $\int \tan^3 x \, dx$.

Example 8. Find $\int \sec^3 x \, dx$.

Trigonometric Integrals

Finally, we can make use of another set of trigonometric identities:

2 To evaluate the integrals (a) $\int \sin mx \cos nx\, dx$, (b) $\int \sin mx \sin nx\, dx$, or (c) $\int \cos mx \cos nx\, dx$, use the corresponding identity:

$$\text{(a)} \quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)} \quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)} \quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Example 9

Evaluate $\int \sin 4x \cos 5x \, dx$.