6.7 Hyperbolic Functions

Hyperbolic Functions

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names.

In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle.

Hyperbolic Functions

For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Graphs of hyperbolic functions

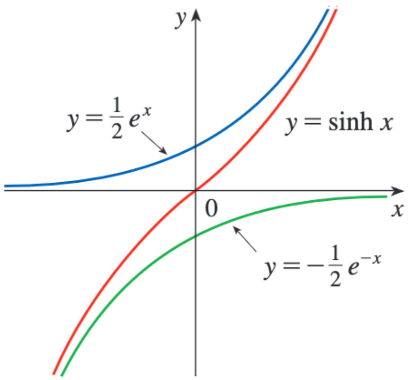


FIGURE 1
$y = \sinh x = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$

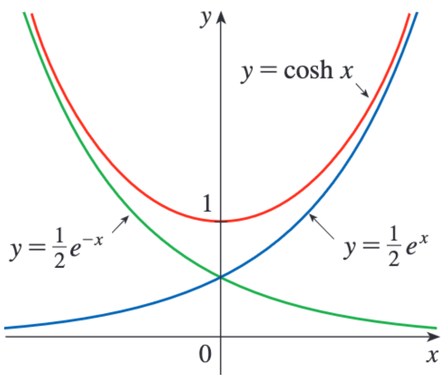


FIGURE 2
$y = \cosh x = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

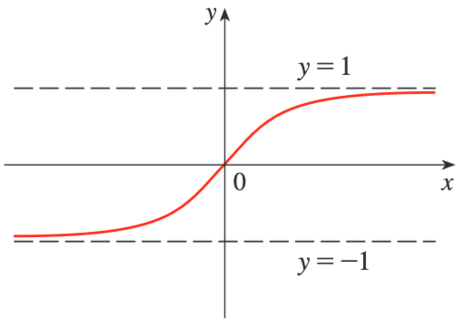
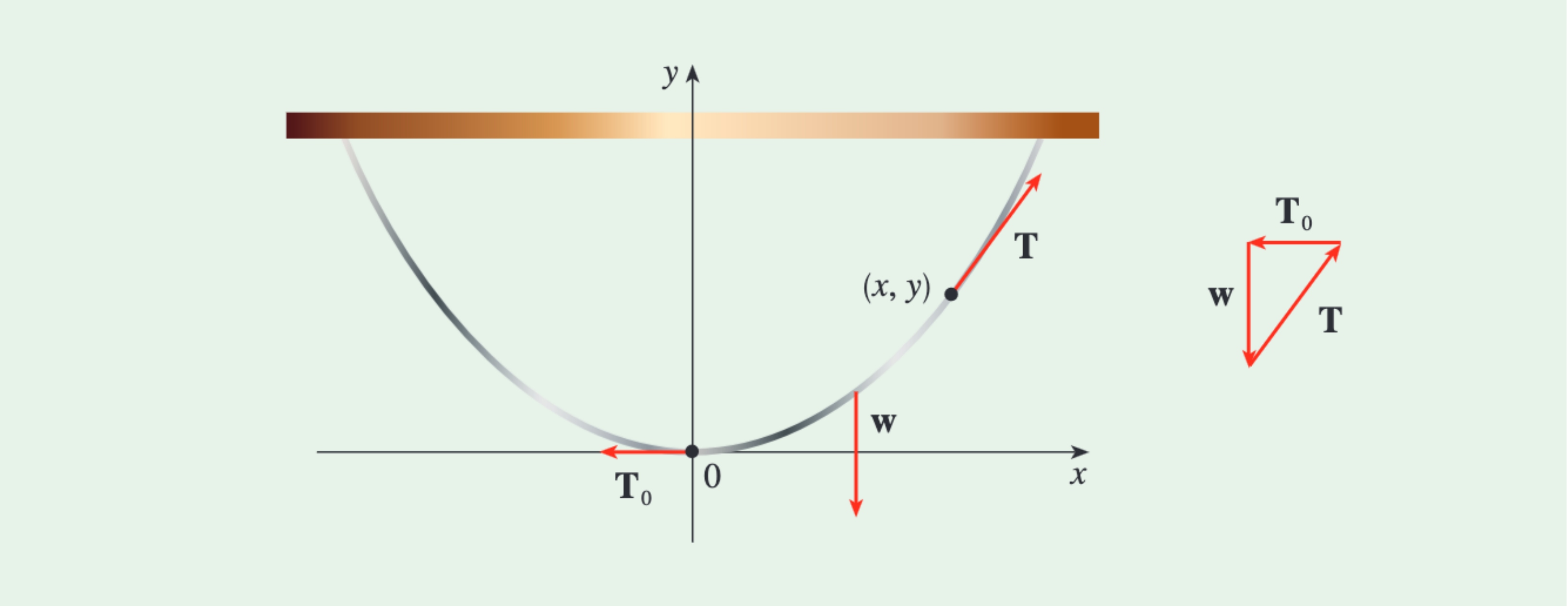


FIGURE 3
$y = \tanh x$

Famous application: catenary curve

Suppose that a chain (or cable) of uniform linear mass density ρ is hanging between two points, as shown in the figure. We place the origin at the vertex of the catenary, and let (x, y) be any point on the curve.



Famous application: catenary curve

The chain is in equlipbrium so one has

$$\boldsymbol{T_0} + \boldsymbol{T} + \boldsymbol{w} = \boldsymbol{0}.$$

One can then show that the shape of the chain is given by:

$$y = a\cosh\frac{x}{a} - a,$$

where $a = \frac{|\boldsymbol{T_0}|}{g\rho}$.

For more details: see Stewart Section 12.3 (p. 884), Discovery Project

Hyperbolic Functions

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

We list some of them here.

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Example 1

Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and

(b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Hyperbolic Functions

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x + e^{-x}}{2}$$

$$= \cosh x$$

Hyperbolic Functions

We list the differentiation formulas for the hyperbolic functions as Table 1.

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Example 2

Find $\frac{d}{dx}\cosh\sqrt{x}$.

Inverse Hyperbolic Functions

Inverse Hyperbolic Functions

The sinh and tanh are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. The cosh is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one.

The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

2

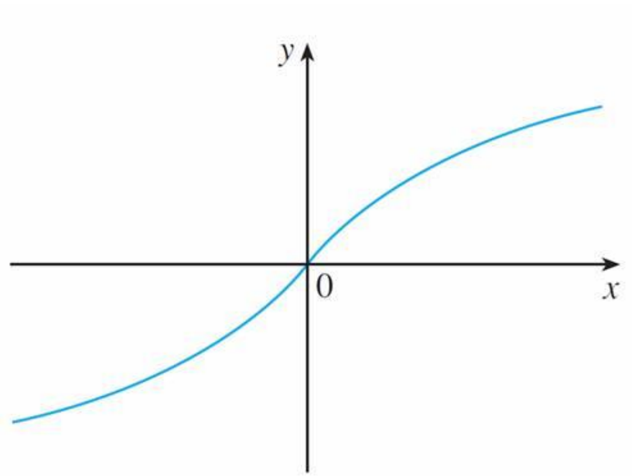
$$y = \sinh^{-1} x \iff \sinh y = x$$

$$y = \cosh^{-1} x \iff \cosh y = x \quad \text{and} \quad y \geqslant 0$$

$$y = \tanh^{-1} x \iff \tanh y = x$$

Inverse Hyperbolic Functions

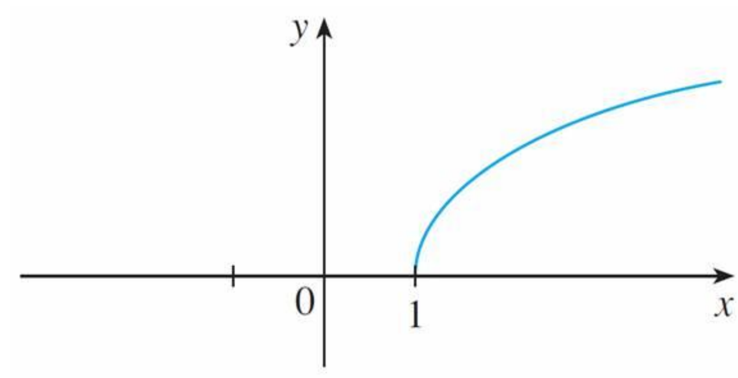
We can sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ in Figures 8, 9, and 10.



$y = \sinh^{-1} x$

domain = $\mathbb{R}$ range = $\mathbb{R}$

Figure 8



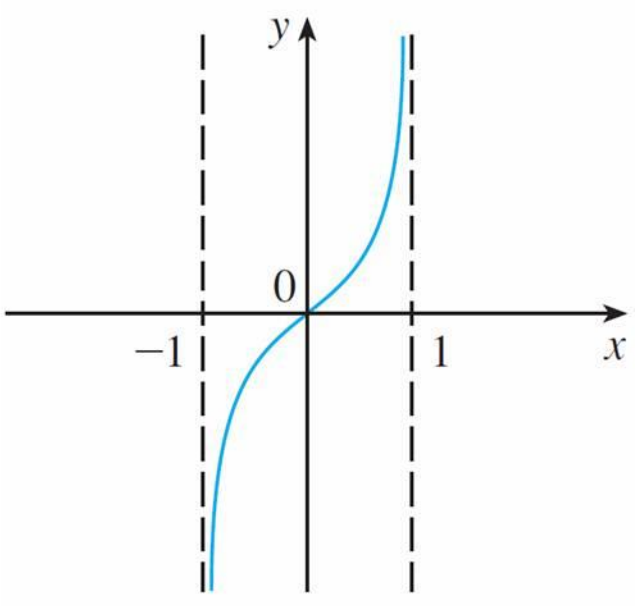
$y = \cosh^{-1} x$

domain = $[1, \infty)$ range = $[0, \infty)$

Figure 9

Inverse Hyperbolic Functions

cont'd



$y = \tanh^{-1} x$

domain = (–1, 1)   range = $\mathbb{R}$

Figure 10

Inverse Hyperbolic Functions

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms.

In particular, we have:

3 $$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

4 $$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geqslant 1$$

5 $$\tanh^{-1}x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad -1 < x < 1$$

Example 3

Show that $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

Inverse Hyperbolic Functions

6 Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1}x) = -\frac{1}{|x|\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\coth^{-1}x) = \frac{1}{1 - x^2}$$

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable.

Example 4

Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1 + x^2}}$.

Examples 5 and 6

Example 5. Find $\frac{d}{dx}[\tanh^{-1}(\sin x)]$.

Example 6. Evaluate

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}}.$$

6.8

Indeterminate Forms and l'Hospital's Rule

Indeterminate Forms and l'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although *F* is not defined when *x* = 1, we need to know how *F* behaves *near* 1. In particular, we would like to know the value of the limit

$$\boxed{1} \qquad \lim_{x \to 1} \frac{\ln x}{x - 1}$$

Indeterminate Forms and l'Hospital's Rule

In computing this limit we can't apply Law 5 of limits because the limit of the denominator is 0. In fact, although the limit in $\boxed{1}$ exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type $\mathbf{\frac{0}{0}}$**.

Indeterminate Forms and l'Hospital's Rule

For rational functions, we can cancel common factors:

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x - 1)}{(x + 1)(x - 1)}$$

$$= \lim_{x \to 1} \frac{x}{x + 1}$$

$$= \frac{1}{2}$$

We used a geometric argument to show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Indeterminate Forms and l'Hospital's Rule

But these methods do not work for limits such as $\boxed{1}$, so in this section we introduce a systematic method, known as *l'Hospital's Rule,* for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of *F* and need to evaluate its limit at infinity:

$$\boxed{2} \qquad \lim_{x \to \infty} \frac{\ln x}{x - 1}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$.

Indeterminate Forms and l'Hospital's Rule

There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞; if the denominator wins, the answer will be 0. Or there may be some compromise, in which case the answer will be some finite positive number.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then the limit may or may not exist and is called an **indeterminate form of type** ∞ / ∞.

Indeterminate Forms and l'Hospital's Rule

L'Hospital's Rule applies to this type of indeterminate form.

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Indeterminate Forms and l'Hospital's Rule

Note 1:

L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of *f* and *g* before using l'Hospital's Rule.

Note 2:

L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

Indeterminate Forms and l'Hospital's Rule

Proof of a special case: in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$.

Examples

Example 1. Find $\lim_{x \to 1} \dfrac{\ln x}{x - 1}$.

Example 2. Find $\lim_{x \to \infty} \dfrac{e^x}{x^2}$.

Example 3. Find $\lim_{x \to 0} \dfrac{\tan x - x}{x^3}$

Example 4. Find $\lim_{x \to \pi-} \dfrac{\sin x}{1 - \cos x}$

Indeterminate Products

Indeterminate Products

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x \to a} [f(x)\, g(x)]$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$).

Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**.

Indeterminate Products

We can deal with it by writing the product *fg* as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞ / ∞ so that we can use l'Hospital's Rule.

Example 6

Evaluate $\lim_{x \to 0^+} x \ln x$.

Indeterminate Differences

Indeterminate Differences

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is called an **indeterminate form of type** $\infty - \infty$.

Example 8

Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$.

Indeterminate Powers

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0

2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0

3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Indeterminate Powers

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

Example 11

Find $\lim_{x \to 0+} x^x$.