Recap: The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We often use the notation

$$F(b) - F(a) = F(x)]_a^b$$

Other common notations are:

$$[F(x)]_a^b$$

or

$$F(x)|_a^b.$$

Examples

Example 4. Find

$$\frac{d}{dx}\int_{1}^{x^4} \sec t \, dt \, .$$

Example 5. Evaluate

$$\int_{-2}^{1} x^3 dx.$$

Example 6. Find the area under the parabola $y = x^2$ for $0 \le x \le 1$.

Examples

Example 7. Find the area under the cosine curve from $x = 0$ to $x = b$, where $0 \leq b \leq \frac{\pi}{2}$.

Example 8. What is wrong with the following calculation:

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\bigg]_{-1}^{3} = -\left(\frac{1}{3} - (-1)\right) = -\frac{4}{3}.$$

4.4 Indefinite Integrals and the Net Change Theorem

Indefinite Integrals and the Net Change Theorem

In this section we introduce a notation for antiderivatives, review the formulas for antiderivatives, and use them to evaluate definite integrals.

We also reformulate FTC2 in a way that makes it easier to apply to science and engineering problems.

Indefinite Integrals

Indefinite Integrals

Both parts of the Fundamental Theorem establish connections between antiderivatives and definite integrals. Part 1 says that if *f* is continuous, then $\int_a^x f(t)\,dt$ is an antiderivative of *f*. Part 2 says that $\int_a^b f(x)\,dx$ can be found by evaluating *F*(*b*) – *F*(*a*), where *F* is an antiderivative of *f*.

We need a convenient notation for antiderivatives that makes them easy to work with.

Because of the relation given by the Fundamental Theorem between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of *f* and is called an **indefinite integral.**

Indefinite Integrals

Thus

$$\boxed{\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)}$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

So we can regard an indefinite integral as representing an entire *family* of functions (one antiderivative for each value of the constant *C*).

Indefinite Integrals

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*, whereas an indefinite integral $\int f(x)\,dx$ is a *function* (or family of functions).

The connection between them is given by Part 2 of the Fundamental Theorem:

If *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions.

Indefinite Integrals

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

The most general antiderivative *on a given interval* is obtained by adding a constant to a particular antiderivative.

Indefinite Integrals

Any formula can be verified by differentiating the function on the right side and obtaining the integrand.

For instance,

$$\int \sec^2 x \, dx = \tan x + C \qquad \text{because} \qquad \frac{d}{dx}(\tan x + C) = \sec^2 x$$

Indefinite Integrals

We adopt the convention that when a formula for a general indefinite integral is given, it is valid only on an interval.

Thus we write

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

with the understanding that it is valid on the interval $(0, \infty)$ or on the interval $(-\infty, 0)$.

Indefinite Integrals

This is true despite the fact that the general antiderivative of the function $f(x) = 1/x^2$, $x \neq 0$, is

$$F(x) = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x < 0 \\ -\dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

Examples

Example 1. Find the most general indefinite integral

$$\int 10x^4 - 2\sec^2 x \, dx.$$

Example 2. Evaluate $\int \frac{\cos\theta}{\sin^2\theta} d\theta$.

Example 3. Evaluate

$$\int_0^3 (x^3 - 6x)dx.$$

Examples

Example 4. Find

$$\int_{0}^{12} (x - 12\sin x)\,dx\,.$$

Example 5. Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt.$

Applications

Applications

Part 2 of the Fundamental Theorem says that if *f* is continuous on [*a*, *b*], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where *F* is any antiderivative of *f*. This means that $F' = f$, so the equation can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Applications

We know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) - F(a)$ is the change in y when x changes from a to b.

[Note that y could, for instance, increase, then decrease, then increase again.

Although y might change in both directions, $F(b) - F(a)$ represents the *net* change in y.]

Applications

So we can reformulate FTC2 in words as follows.

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in the natural and social sciences. Here are a few instances of this idea:

- If $V(t)$ is the volume of water in a reservoir at time t, then its derivative $V'(t)$ is the rate at which water flows into (or out of) the reservoir at time t.

Applications

So

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If [C](t) is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $d[\mathrm{C}]/dt$.

Applications

So

$$\int_{t_1}^{t_2} \frac{d[\mathrm{C}]}{dt}\, dt = [\mathrm{C}](t_2) - [\mathrm{C}](t_1)$$

is the change in the concentration of C from time t_1 to time t_2.

- If the mass of a rod measured from the left end to a point x is $m(x)$, then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\, dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between $x = a$ and $x = b$.

Applications

- If the rate of growth of a population is dn/dt, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\, dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2.

(The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

Applications

- If $C(x)$ is the cost of producing x units of a commodity, then the marginal cost is the derivative $C'(x)$.

  So

$$\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$$

  is the increase in cost when production is increased from x_1 units to x_2 units.

Applications

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\boxed{2} \qquad \int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2.

This was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

Applications

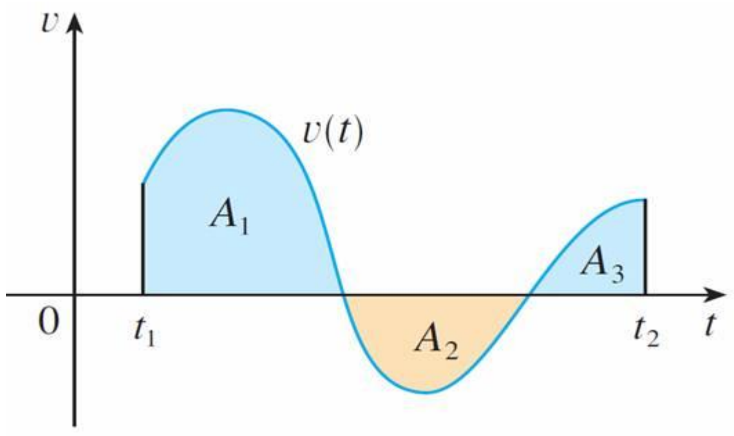
- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left).

  In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

$$\boxed{3} \qquad \int_{t_1}^{t_2} |v(t)|\, dt = \text{total distance traveled}$$

Applications

Figure 3 shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\, dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\, dt = A_1 + A_2 + A_3$$

Figure 3

Applications

- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2.

Example 6

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.

(b) Find the distance traveled during this time period.

4.5 The Substitution Rule

The Substitution Rule

Because of the Fundamental Theorem, it's important to be able to find antiderivatives.

But our antidifferentiation formulas don't tell us how to evaluate integrals such as

$$\boxed{1} \qquad \int 2x\sqrt{1 + x^2}\, dx$$

To find this integral we use the problem-solving strategy of *introducing a new variable*; we change from the variable *x* to a new variable *u*.

The Substitution Rule

Suppose that we let u be the quantity under the root sign in $\boxed{1}$, $u = 1 + x^2$. Then the differential of u is $du = 2x\,dx$.

Notice that if the dx in the notation for an integral were to be interpreted as a differential, then the differential $2x\,dx$ would occur in $\boxed{1}$ and so, formally, without justifying our calculation, we could write

$$\boxed{2} \qquad \int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du$$

$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(x^2+1)^{3/2} + C$$

The Substitution Rule

But now we can check that we have the correct answer by using the Chain Rule to differentiate the final function of Equation 2:

$$\frac{d}{dx}\left[\tfrac{2}{3}(x^2+1)^{3/2}+C\right]=\tfrac{2}{3}\cdot\tfrac{3}{2}(x^2+1)^{1/2}\cdot 2x=2x\sqrt{x^2+1}$$

In general, this method works whenever we have an integral that we can write in the form $\int f(g(x))g'(x)\,dx$.

The Substitution Rule

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Notice that the Substitution Rule for integration was proved using the Chain Rule for differentiation.

Notice also that if $u = g(x)$, then $du = g'(x)\,dx$, so a way to remember the Substitution Rule is to think of dx and du in $\boxed{4}$ as differentials.

The Substitution Rule

Thus the Substitution Rule says: **It is permissible to operate with *dx* and *du* after integral signs as if they were differentials.**

Examples

Example 1. Find $\int x^3 \cos(x^4 + 2)\, dx$.

Example 2. Evaluate $\int \sqrt{2x+1}\, dx$.

Example 3. Find

$$\int \frac{x}{\sqrt{1-4x^2}}\, dx.$$

Example 4. Evaluate $\int \cos 5x \; dx$.

Example 5. Evaluate $\int \sqrt{1+x^2}\, x^5\, dx$.

Definite Integrals

Definite Integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem.

For example,

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4 = \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4$$

$$= \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2} = \tfrac{1}{3}(27-1) = \tfrac{26}{3}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

Definite Integrals

5 **The Substitution Rule for Definite Integrals** If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Examples

Example 6. Evaluate $\int_0^4 \sqrt{2x+1}\,dx$.

Example 7. Evaluate

$$\int_1^2 \frac{dx}{(3-5x)^2}.$$