Problem 1. Assume you run quicksort on the following sequence of numbers, [70, 50, 110, 60, 90, 40, 100, 80] using the algorithm shown in the class.

1. Show the array after the first call to partition routine. How many comparisons and exchanges were carried out?

Solution: $70, 50, 60, 40, 80, 110, 90, 100$
$7, 5$

2. How many comparisons and exchanges are carried out in the complete Quicksort algorithm?

Solution: $16, 14$

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[70, 50, 110, 60, 90, 40, 100, 80]
Comparisons: 7, Exchanges: 5
[70, 50, 60, 40, 80, 110, 100, 90]
Comparisons: 3, Exchanges: 1
[40, 50, 60, 70, 80, 110, 100, 90]
Comparisons: 2, Exchanges: 3
[40, 50, 60, 70, 80, 110, 100, 90]
Comparisons: 1, Exchanges: 2
[40, 50, 60, 70, 80, 110, 100, 90]
Comparisons: 2, Exchanges: 1
[40, 50, 60, 70, 80, 90, 100, 110]
Comparisons: 1, Exchanges: 2
[40, 50, 60, 70, 80, 90, 100, 110]
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Problem 2. Assume, we pick a pivot for Quicksort that always gives us a 9-to-1 split.

1. Write the recurrence equation.

Solution: $T(n) = T\left(\frac{n}{10}\right) + T\left(\frac{9n}{10}\right) + n - 1$

2. What is the height of the recursion tree?

Solution: $lg_{\frac{10}{9}} n \sim lgn$

3. Solve the recurrence equation using constructive induction.

Solution:

$$T(n) = T\left(\frac{n}{10}\right) + T\left(\frac{9n}{10}\right) + n - 1$$

Guess: $T(n) \le an \lg n$

Base Case: $T(1) = a \cdot 1 \cdot \lg 1 = 0$

Inductive Step:

$$T(n) = \frac{an}{10} \lg \frac{n}{10} + \frac{9an}{10} \lg \frac{9n}{10} + n - 1$$

$$= \frac{an}{10} \lg n - \frac{an}{10} \lg 10 + \frac{9an}{10} \lg 9n - \frac{9an}{10} \lg 10 + n - 1$$

$$= \frac{an}{10} \lg n + \frac{9an}{10} \lg 9 + \frac{9an}{10} \lg n - \frac{an}{10} \lg 10 - \frac{9an}{10} \lg 10 + n - 1$$

$$= an \lg n + \frac{9an}{10} \lg 9 - an \lg 10 + n - 1$$

$$= an \lg n + \left(\frac{9a}{10} \lg 9 - a \lg 10 + 1\right) n - 1$$

$$\le an \lg n$$

$$\Rightarrow \frac{9a}{10} \lg 9 - a \lg 10 + 1 \le 0$$

$$\left(\frac{9}{10} \lg 9 - \lg 10\right) a \le -1$$

$$a \left(\lg 10 - \frac{9}{10} \lg 9\right) \ge 1$$

$$a \ge \frac{10}{10 \lg 10 - 9 \lg 9}$$

$$\therefore T(n) \le \frac{10}{10 \lg 10 - 9 \lg 9} n \lg n$$

$$T(1) = 0$$

Problem 3. The Quicksort algorithm described in class, is called recursively until a base case of leaves (elements) of size 1. However, on small input sequences, insertion sort works really well.

For this question you would modify the Quicksort algorithm so that after some threshold, m, the sorting is performed by using insertion sort (without sentinel) instead of Quicksort all the way to the leaves. In other words, instead of going all the way up to $r-p>0$ in the algorithm, you would use Quicksort up to some threshold, m, such that $r-p>m$ and insertion sort (without sentinel) after that.

1. Write the pseudo code for the modified algorithm.

Solution:

```
ModifiedQuicksort(A,p,r)
    if r - p > m:
        q = partition(A,p,r)
        ModifiedQuicksort(A,p,q-1)
        ModifiedQuicksort(A,q+1,r)
    else:
        InsertionSort(A[p:r])
```

2. Write the recurrence equation for an average case of comparisons for this algorithm.

Solution:

$$T(n) = \begin{cases} \sum\limits_{q=m+1}^{n} \frac{1}{n-m}[T(q-m-1)+T(n-q)] + n-m-1 & n>m \\ \frac{m(m+3)}{4} - H_m & n \le m \end{cases}$$

3. What is the average number of comparisons? Show your work.

Solution:

$$T(n) = \begin{cases} \sum_{q=m+1}^{n} \frac{1}{(n-m)} \left[T(q-m-1) + T(n-q)\right] + n-m-1 & n > m \\ \frac{m(m+3)}{4} - H_m & n \le m \end{cases} \quad \text{— } 1$$

$$T(QS) = \sum_{q=m+1}^{n} \frac{1}{(n-m)} \left[T(q-m-1) + T(n-q)\right] + n-m-1$$

$$= \frac{1}{(n-m)} \left[\sum_{q=m+1}^{n} T(q-m-1) + \sum_{q=m+1}^{n} T(n-q)\right] + n-m-1$$

$$= \frac{1}{n-m}\left[\left[T(0) + T(1) + \dots + T(n-m-1)\right] + \left[T(n-m-1) + T(n-m-2) + \dots + T(0)\right]\right] + n-m-1$$

$$T(0) = 0$$

$$T(QS) = \frac{1}{(n-m)} \left[\sum_{q=1}^{n-m-1} T(q) + \sum_{q=1}^{n-m-1} T(q)\right] + n-m-1$$

$$= \frac{2}{(n-m)} \left[\sum_{q=1}^{n-m-1} T(q)\right] + n-m-1 \quad \text{— } 2$$

Solve $T(q)$ using constructive induction

Guess: $T(q) \le a q \lg q$

$$T(QS) \le \frac{2}{(n-m)} \sum_{q=1}^{n-m-1} a q \lg q + n-m-1$$

$$= \frac{2a}{n-m} \sum_{q=1}^{n-m-1} q \lg q + n-m-1 \quad \text{— } 3$$

Apply integral approximation for upper bound

$$\sum_{q=1}^{n-m-1} q \lg q \le \int_1^{n-m} x \lg x \, dx = \frac{1}{\ln 2} \int_1^{n-m} x \ln x \, dx$$

$$\sum_{q=1}^{n-m-1} q\lg q \leq \frac{1}{\ln 2}\int_1^{n-m} x\ln x \, dx$$

$$= \frac{1}{\ln 2}\left[\ln x \int_1^{n-m} x\,dx - \int_1^{n-m} \frac{d}{dx}\ln x \int x\,dx\right]$$

$$= \frac{1}{\ln 2}\left[\frac{x^2 \ln x}{2}\Big|_1^{n-m} - \int_1^{n-m} \frac{1}{x}\cdot\frac{x^2}{2}\,dx\right]$$

$$= \frac{1}{\ln 2}\left[\frac{(n-m)^2}{2}\ln(n-m) - \frac{x^2}{4}\Big|_1^{n-m}\right]$$

$$= \frac{1}{\ln 2}\left[\frac{(n-m)^2}{2}\ln(n-m) - \frac{(n-m)^2}{4} + \frac{1}{4}\right]$$

$$\sum_{q=1}^{n-m-1} q\lg q \leq \frac{(n-m)^2}{2}\lg(n-m) - \frac{(n-m)^2}{4}\lg e + \frac{\lg e}{4}$$

$$\boxed{\ln 2 = \frac{\lg 2}{\lg e}}$$

Substitute this in eq. 3, represented as

$$T(QS) \leq \frac{2a}{n-m}\sum_{q=1}^{n-m-1} q\lg q + n-m-1$$

$$\leq \frac{2a}{(n-m)}\left[\frac{(n-m)^2}{2}\lg(n-m) - \frac{(n-m)^2}{4}\lg e + \frac{\lg e}{4}\right] + n-m-1$$

$$T(QS) \leq a(n-m)\lg(n-m) - \frac{a(n-m)\lg e}{2} + \frac{a\lg e}{2(n-m)} + n-m-1$$

Using constructive induction, guess an upper bound for $T(QS)$ $a(n-m)\lg(n-m)$

$$T(QS) \leq a(n-m)\lg(n-m) - \frac{a(n-m)}{2}\lg e + \frac{a}{2(n-m)}\lg e + n-m-1$$

$$\leq a(n-m)\lg(n-m) \quad \text{— } (4)$$

$$\Rightarrow -\frac{a}{2}\lg e + 1 \leq 0 \quad \text{— } (4a)$$

$$\frac{a}{2(n-m)}\lg e - 1 \leq 0 \quad \text{— } (4b)$$

Solving 4a

$$-\frac{a}{2}\lg e + 1 \le 0$$

$$\Rightarrow \quad a \ge \frac{2}{\lg e}$$

Substitute this value in (4b)

$$\frac{a}{2(n-m)}\lg e - 1 = \frac{2}{\lg e}\cdot\frac{\lg e}{2(n-m)} - 1 \le 0$$

$$\Rightarrow \quad \frac{1}{n-m} - 1 \le 0$$

Since $n \ge m+1$, this equation is always true.

Therefore, From eq. (4) with $a = \frac{2}{\lg e}$

$$T(QS) \le a(n-m)\lg(n-m)$$

$$= \frac{2}{\lg e}(n-m)\lg(n-m)$$

$$\Rightarrow \quad T(QS) \le 2(n-m)\ln(n-m) \quad - (5)$$

From eq. (1)

$$T(n) = \begin{cases} \sum_{q=m+1}^{n} \frac{1}{n-m}\left[T(q-m-1)+T(n-q)\right] + n-m-1 & n > m \\ \frac{m(m+3)}{4} - H_m & n \le m \end{cases}$$

$$\Rightarrow \quad T(n) = \begin{cases} T(QS) & \\ \frac{m(m+3)}{4} - H_m & n > m \end{cases}$$

For the overall algorithm,

$$T(n) \leq T(QS) + \frac{m(m+3)}{4} - H_m$$

$$H_m = \sum_{i=1}^{m} \frac{1}{i} \leq \ln(m+1)$$

$$\Rightarrow T(n) \leq T(QS) + \frac{m(m+3)}{4} - \ln(m+1) \quad \text{--- (6)}$$

Substitute, $T(QS) \leq 2(n-m)\ln(n-m)$
from eq. (5) in eq. (6), we get

$$\boxed{T(n) \leq 2(n-m)\ln(n-m) + \frac{m(m+3)}{4} - \ln(m+1)}$$

This is the upper bound on the number of comparisons for the modified quicksort with a base case of insertion sort.

When $m = 0$, $T(n) \approx 2n\ln n$
just like in regular quicksort without insertion sort.