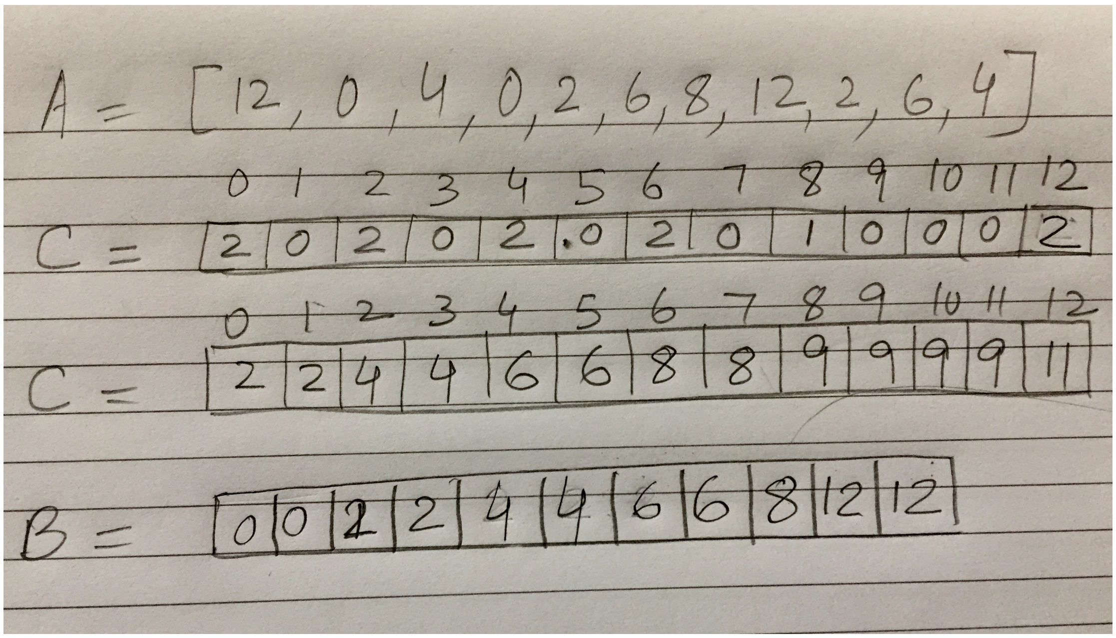
Problem 1. Illustrate the operation of counting sort algorithm on the array, $A = [12, 0, 4, 0, 2, 6, 8, 12, 2, 6, 4]$.

Solution:



Problem 2. Assume you are given a set of seemingly identical keys to a treasure chest and a two pan weighing scale. One of the keys is lighter than the rest and is the one you need to open the chest. The treasure can be all yours if you can find the lightest key in the least number of weighings. If you exceed the optimal number of weighings, you will lose the treasure. Find the least number of weighings for 2, 4, 8, 16 and 32 keys. Explain your strategy.

Solution:

32 keys:

Weigh 16 keys against 16 keys.
Whichever pan is lighter, the key is one of those 16 keys.
Now, take those 16 keys.
We have shown it takes 3 weighings to find the lighter key among 16 keys.
Therefore we need a total of 4 weighings to find the lighter key amongst 32 keys.

Summary

Keys	Weighings
2	1
4	2
8	2
16	3
32	4

The pattern that I see is that the number of weighings depends on the power of 3.
The findings are summarized in the table below:

8 keys. Label them

1 2 3 4 5 6 7 8

Weigh keys 1, 2, 3 vs 4, 5, 6 — 1

If 1, 2, 3 is the lighter side

Weigh 1 vs. 2 — 2

whiever is lighter is the key you're looking for.

If they are equal, then 3 is the lighter key.

Therefore, in two weighings we can find the lighter key amongst 8 keys.

16 keys Label them

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Weigh 1, 2, 3, 4, 5, 6, 7, 8 vs. 9, 10, 11, 12, 13, 14, 15, 16

Whichever pan goes higher, the lighter key is in that group of eight keys.

We already showed it takes 2 weighings to find the lighter key amongst ~~eight~~ eight keys. Therefore, we can find the lighter key in 3 weighings amongst 16 keys.

Two keys. Label them 1 and 2

1 weighing (Trivial)

Whichever pan goes higher contains the lighter key.

<u>4 keys</u>: Label them 1, 2, 3, 4

Weigh 12 against 34

Suppose, 1,2 are lighter

Weigh 1 against 3 (the normal one)

If the weights are equal

2 is the lighter key.

If the pan with key 1 goes higher

1 is the lighter key.

The same strategy would apply if

3, 4 were lighter.

Therefore, we need atleast two weighings to guarantee that we find the lightest key.

# coins (n)	weighings (w)
≤ 3	1
$3 < n \leq 9 \; (3^2)$	2
$9 < n \leq 27 \; (3^3)$	3
$27 < n \leq 81 \; (3^4)$	4
$\vdots$	
$3^{w-1} < n \leq 3^w$	w (general formula).

Problem 3. Continuing with the key weighing of Problem 2, assume you are given 12 keys, except this time, instead of knowing that one of them is lighter than the rest, you are told that one key is different. That means it may be lighter or heavier than the rest.

1. What is the minimum number of weighings needed to find the different key? Show your work.

Solution:

label the keys 1 — 12

1 2 3 4 5 6 7 8 9 10 11 12

Weigh {1, 2, 3, 4} vs {5, 6, 7, 8} (First weighing)

Since the different key is either heavier or lighter, we have two outcomes here:

① They are equal weight.

Now weigh three keys from 1 2 3 4 5 6 7 8

let us say, 1 2 3 against three keys from 9, 10, 11, 12 i.e.,

{1, 2, 3} vs {9, 10, 11} (Second weighing)

If they are balanced (equal weight), then 12 is the different key. To determine whether it is heavier or lighter, weigh

{1} vs. {12} — probable (Third weighing)

If 1 goes up 12 is heavier.

If 1 goes down 12 is lighter.

However, if {1, 2, 3} vs {9, 10, 11} is not equal weight then, one of the keys in 9, 10 and 11 is lighter or heavier.

Weigh {9} vs. {10}

if equal

weigh {1} vs. {11}

We can tell whether 11 is lighter or heavier.

Mark $\{1, 2, 3, 4\}$ as heavy.
and $\{5, 6, 7, 8\}$ as light
$\{9, 10, 11, 12\}$ are neither and marked normal
Now, weigh

$$\{1, 2, 5, 6\} \text{ vs. } \{3, 7, 9, 10\}$$

If they are the same weight then
the key is either $\{4\}$ or $\{8\}$
Weigh $\{9\}$ vs. $\{4\}$
if it is not equal then 4 is heavy,
if they are the same weight 8 is the
key and it is light.

If $\{1, 2, 5, 6\}$ vs. $\{3, 7, 9, 10\}$
are not the same weight, then
the key is either 1, 2 or 7 if $\{1, 2, 5, 6\}$ is
heavier or if $\{3, 7, 9, 10\}$ is heavier
then 3 is heavier or 5, 6 is lighter
In either case, there are three keys,
2 marked heavy and 1 marked light
or 2 marked light and 1 marked heavy

If {9} vs. {10} is not equal weight
then one of these keys is lighter or
heavier. let us say 9 went down
Then assume 9 is heavier and 10 is
lighter, we need to verify this.
Weigh {1} vs. {9}
If 9 goes down, then it is the key
and it is heavier.
If 9 goes up, then it is the key
and it is lighter.
If they are balanced then
10 is the lighter key and the
key we are looking for.
Hence we need Three (3) weighings

2 {1,2,3,4} vs. {5,6,7,8}
are not equal weight.
either one of {1, 2, 3, 4} is lighter and {5, 6, 7, 8}
is heavier or vice versa
let us assume {1, 2, 3, 4} side goes
down and the other side goes up.

Now, weigh two keys that are marked the same on the heavier side

If they are the same, the third one is the key and we know whether it is lighter or heavier.

If they are not the same, then the key is the one that goes up if the two keys are light or the one that goes down if the two keys are heavy.

2. Use the decision tree model to find the lower bound on the number of weighings. Show your work.

Solution:

Decision tree model

Number of leaves = Number of keys = 12

Each Comparison between keys, results in three comparisons: $<$, $>$, or $=$

If 'h' is the height of the longest branch of the decision tree with each internal node representing a comparison we have

$$12 < 3^h$$

Applying $\lg_3$ on both sides

$$\lg_3 12 < \lg_3 3^h$$
$$= h \lg_3 3$$
$$= h$$

$$\Rightarrow h > \lg_3 12$$
$$\text{or } h > 2.26$$

Since we can't weigh 2.26 times, we will have to use a higher value.

Thus $h \approx 3$

Therefore, we will need atleast 3 weighings to guarantee that we find the faulty key and that sets the lower bound on the number of weighings.