There are twelve problems. Within reason, you should show your work. Problems 0 is due Tuesday, September 1. Problems 1-7 are due Friday, September 4. The other problems will be due later.

Problem 0. Memorize the following summation formulas. Do not hand in.

(a)

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(b)

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

(c)

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}, \qquad |r| < 1$$

Problem 1. Evaluate the following sums (by hand).

(a)

$$\sum_{i=1}^{4} i(i+1)$$

(b)

$$\sum_{i=0}^{4} 2^i$$

Problem 2. Write

$$3\sum_{i=1}^{n}(5i^2 - 4) - 2\sum_{i=1}^{n}(3i^2 - 1)$$

as a single summation.

Problem 3.

(a) Assume $b^x = a$. What is x (in terms of a and b)?

(b) Using only part (a), show that $\log_c(ab) = \log_c a + \log_c b$.

(c) Show that $a^{\log_b n} = n^{\log_b a}$.

Problem 4. Differentiate the following functions:

(a) $\ln(x^2 + 5)$

(b) $\lg(x^2 + 5)$ [NOTE: In Computer Science we use $\lg x$ to mean $\log_2 x$.]

(c) $\frac{1}{\ln(x^2+5)}$

Problem 5. Integrate the following functions:

(a) $\frac{1}{x}$

(b) $\frac{1}{7x+3}$

(c) $\ln x$ [HINT: Use integration by parts.]

(d) $x \ln x$ [HINT: Use integration by parts.]

(e) $x \lg x$

Problem 6. Assume that you have a random list of n distinct numbers. For example here is a list of nine numbers.

$$50,\ 20,\ 90,\ 60,\ 40,\ 70,\ 10,\ 80,\ 30$$

The first number is 50, the second number is 20, the third number is 90, etc. The smallest number is 10, the second smallest number is 20, the third smallest number is 30, etc.

Let $1 \leq i \neq j \leq n$.

Do NOT show your work.

(a) What is the probability that the ith number is the smallest number?

(b) What is the probability that the ith number is one of the two smallest numbers?

(c) What is the probability that the ith number is the smallest of the first i numbers?

(d) What is the probability that the ith number is one of the i smallest numbers?

(e) What is the probability that the ith and jth numbers are the two smallest numbers (in either order)?

Problem 7. Consider the formula $3n^4 + 7n^3 \log n + 2n^2$.

(a) What is the high order term?

(b) What is the second order term?

(c) Write the formula in Θ notation (in simplest form).

Problem 8. Use mathematical induction to show the following:

(a)

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

(b)

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Problem 9. Assume that you guess that

$$\sum_{i=0}^{n} 2^i = a2^n + b$$

for constants a and b. Use constructive induction to verify the formula and derive a and b.

Problem 10. You play the following game: Randomly pick two different numbers from a list of n distinct numbers (just like above). You win if at least one of them is among the i smallest numbers.

(a) What is the probabilty that you win?

(b) Assume you discover that you won the game. What is the probability that both numbers are among the i smallest numbers. Do NOT simplify.

(c) Assuming that n is even, what is the value in Part (b) when $i = n/2$? Simplify.

Problem 11. You take an oral exam where you keep answering questions until you miss one.

(a) Your probability of getting each question right is p for some constant $0 \leq p < 1$. Write a summation for the average number of questions you are asked. Do NOT simplify.

(b) Simplify your sum. You may look up the answer, for example on Wolfram Alpha.

(c) What if the questions keep getting harder? Your probability of getting the ith question right is r^i for some constant $0 \leq r < 1$. Write a summation for the average number of questions you are asked. Do NOT simplify (much).