Practice Midterm

Pb1 (i) $T(n) = 2T(n/2) + n^3, \quad T(1) = 1$

n^3

$2 \cdot \left(\frac{n}{2}\right)^3$

$4 \cdot \left(\frac{n}{4}\right)^3$

$\vdots$

$2^{\lg n - 1}\left(\frac{n}{2^{\lg n - 1}}\right)^3$

$n \cdot 1$

$$T(n) = \sum_{i=0}^{\lg n - 1} 2^i \left(\frac{n}{2^i}\right)^3 + n$$

$$= \sum_{i=0}^{\lg n - 1} n^3 \cdot \frac{1}{4^i} + n$$

$$= n^3 \left[\sum_{i=0}^{\lg n - 1} \left(\frac{1}{4}\right)^i\right] + n$$

$$= n^3 \left[\frac{1 - (1/4)^{\lg n}}{1 - 1/4}\right] + n$$

$$= n^3 \frac{4}{3}\left[1 - \frac{1}{n^2}\right] + n$$

$$= \frac{4n^3}{3} - \frac{4}{3}n + n = \frac{4n^3}{3} - \frac{n}{3}$$

$$T(1) = \frac{4}{3} - \frac{1}{3} = 1$$

$T(2) = 2T(1) + 8$
$= 2 + 8$
$= 10$

$T(2) = \frac{4}{3} \cdot 8 - \frac{2}{3}$
$= \frac{30}{3} = 10$

Prac. Mid II

Pb 1 (ii) $T(n) = T(\sqrt{n}) + 1, \quad T(2) = 1$

$$T(n) = T(\sqrt{n}) + 1$$
$$= [T(n^{1/4}) + 1] + 1$$
$$= [T[(n^{1/16}) + 1] + 1] + 1$$
$$= \cdots + 1$$
$$= [T(n^{1/2^k}) + 1 \cdots] + 1$$
$$= \sum_{i=0}^{k-1} 1 + T(2)$$
$$= k + 1$$

$T(n)$
$T(\sqrt{n})$
$T(n^{1/4})$
$T(n^{1/16})$
$\vdots$
$T(n^{1/2^k}) \to 2$

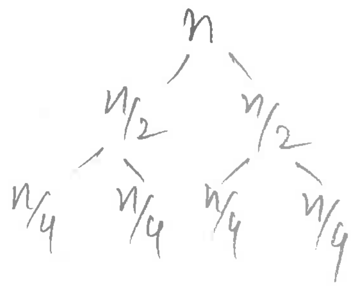
$$n^{\frac{1}{2^k}} = 2$$
$$T(n^{\frac{1}{2^k}}) = T(2) = 1$$

$$\frac{1}{2^k} \lg n = 1$$
$$2^k = \lg n$$
$$k = \lg \lg n$$
$$T(n) = \lg \lg n + 1$$

$$T(2) = \lg \lg 2 + 1 = \lg 1 + 1 = 1$$
$$T(4) = T(2) + 1 = 1 + 1 = 2$$
$$T(4) = \lg \lg 4 + 1 = \lg 2 + 1 = 1 + 1 = 2$$

Prac. $\underline{M\ II}$

Pb. 1 (iii) $T(n) = 2T(n/2) + n\lg n, \quad T(1) = 1$



$n\lg n$

$n\lg\frac{n}{2}$

$n\lg\frac{n}{4}$

$\vdots$

$n\lg\frac{n}{2^{\lg n-1}}$

$n \cdot 1$

$$\sum_{i=0}^{\lg n-1} n \cdot \lg\frac{n}{2^i} + n$$

$$n\left[\sum_{i=0}^{\lg n-1}(\lg n - i)\right] + n$$

$$= n\lg n \sum_{i=0}^{\lg n-1} 1 - n\sum_{i=0}^{\lg n-1} i + n$$

$$= n\lg n\,[\lg n] - n\left[\frac{\lg n(\lg n-1)}{2}\right] + n$$

$$= n(\lg n)^2 - \frac{n}{2}\left[(\lg n)^2 - \lg n\right] + n$$

$$= \frac{n}{2}(\lg n)^2 + \frac{n}{2}\lg n + n$$

$$T(1) = \frac{1}{2}(\lg 1)^2 + \frac{1}{2}\lg 1 + 1 = 1$$

$$T(2) = 2T(1) + 2\cdot\lg 2 = 2+2=4$$

$T(2) = 1+1$
$+2$
$=4$

Prac. M II

Pb. 1 (iv) $T(n) = T(n-3) + 5$, $T(1) = 2$

$$T(n) = T(n-3) + 5$$
$$= [T(n-6) + 5] + 5$$
$$= [[T(n-9) + 5] + 5] + 5$$
$$= \cdots$$
$$= [T(\tfrac{n-1}{3} - 1) \cdots] + 5$$
$$T(\tfrac{n-1}{3}) = T(1) = 2$$

$n - 3k = 1$

$\frac{n-1}{3} = k$

$$\sum_{i=0}^{\frac{n-1}{3}-1} 5 + T(1)$$
$$= 5\left[\frac{n-1}{3}\right] + 2$$

$$T(1) = 2$$
$$T(4) = 5\left(\frac{4-1}{3}\right) + 2$$
$$= 7$$

$$T(n) = T(n-3) + 5$$
$$T(4) = T(4-3) + 5$$
$$= T(1) + 5$$
$$= 2 + 5 = 7$$

Prac. Mid II

Pb. (2a) $$\frac{1}{2} \leq \sum_{j=1}^{\infty} \frac{1}{j2^j} \leq 1$$

$$\sum_{j=1}^{\infty} \frac{1}{j^{2j}} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \cdots$$

lower bound is the smallest term, $\frac{1}{2}$

$$\frac{1}{2} \leq \sum_{j=1}^{\infty} \frac{1}{j2^j}$$

For upper bound. drop the coefficients

$$\sum_{j=1}^{\infty} \frac{1}{j2^j} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \cdots$$

$$\leq \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots = \frac{1}{1-r}$$

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \cdots = \frac{1}{1-r} - 1$$

$$\sum_{j=1}^{\infty} \frac{1}{j2^j} \leq \frac{1}{1-1/2} - 1$$

$$= 2 - 1 = 1$$

$$\frac{1}{2} \leq \sum_{j=1}^{\infty} \frac{1}{j2^j} \leq 1$$

Prac Mid II

Pb 2(b) $$1 \le \sum_{j=1}^{\infty} \frac{1}{j^2} \le 2$$

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

lower bound is the smallest value, 1

$$1 \le \sum_{j=1}^{\infty} \frac{1}{j^2}$$

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = 1 + \sum_{j=2}^{\infty} \frac{1}{j^2}$$

$$\le 1 + \int_1^{\infty} \frac{1}{x^2}\, dx$$

$$= 1 - \frac{1}{x}\Big|_1^{\infty}$$

$$= 1 - (0 - 1)$$

$$= 2$$

$$1 \le \sum_{j=1}^{\infty} \frac{1}{j^2} \le 2$$

Euler found $\frac{\pi^2}{6}$

Prac. M II

Pb 3 : $T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n - 1$

$T(1) = 0$

Guess: $T(n) \le a\, n \lg n$

Base Case: $n = 1$, $T(1) = 0$

$a \cdot 1 \cdot \lg 1 = 0$

Inductive Step:

$$T(n) = a\frac{n}{3}\lg\frac{n}{3} + \frac{2an}{3}\lg\frac{2n}{3} + n - 1$$

$$= \frac{an}{3}\lg n - \frac{an}{3}\lg 3 + \frac{2an}{3}\lg 2n - \frac{2an}{3}\lg 3 + n - 1$$

$$= \frac{an}{3}\lg n + \frac{2an}{3}\lg n + \frac{2an}{3}\lg 2 - an\lg 3 + n - 1$$

$$= an\lg n + \left(\frac{2a}{3} - a\lg 3 + 1\right)n - 1$$

$$\le an\lg n$$

$$\frac{2a}{3} - a\lg 3 + 1 \le 0$$

$$\left(\frac{2}{3} - \lg 3\right)a \le -1$$

$$\left(\lg 3 - \frac{2}{3}\right)a \ge 1 \Rightarrow a \ge \frac{3}{3\lg 3 - 2}$$

$T(n) \le \frac{3}{3\lg 3 - 2} n\lg n$; $T(1) = \frac{3}{3\lg 3 - 2} \cdot 1 \cdot \lg 1 = 0$

M

Pb. 4 (a) There are n/k blocks, each of k elements in input array A.

for $i = 1$ to $\frac{n}{k} - 1$:

$B_i = A[(i-1)k+1 : ik]$

$B_{i+1} = A[ik+1 : (i+1)k]$

Mergesort $\left(A[(i-1)k+1 : (i+1)k]\right)$

end

$$T(n) = \sum_{i=1}^{\frac{n}{k}-1} 2k \lg 2k \approx [2k \lg k + 2k]\left(\frac{n}{k} - 1\right)$$

$$\approx 2n \lg k$$

Minheapsort first 'k' values

for $i = k+1$ to n :

insert $A[i]$ in heap

Minheapify$\left(A[(i-k+1) : i]\right)$

$$2k\lg k + \sum_{i=k+1}^{n} \lg k = 2k\lg k + (n-k)\lg k$$

$$\approx n \lg k$$

(4b)

$$(k!)^{n/k} \leq 2^h$$

$$2^h \geq (k!)^{n/k}$$

$$h \geq \frac{n}{k} \lg(k!)$$

$$h \geq \frac{n}{k} \lg\left(\sqrt{2\pi k} \cdot \left(\frac{k}{e}\right)^k\right)$$

$$= \frac{n}{k}\left[\frac{1}{2}\lg 2\pi + \frac{1}{2}\lg k + k\lg k - k\lg e\right]$$

$$\approx n \lg k$$

Prac. MII

Pb. 5

Sort an array using mergeSort

min_value = ∞, sum = ∞, diff = ∞, output = -1, -1, -1

for i = 1 to n-2:

    j = i + 1

    k = n

    while k $\geq$ j :

        Sum = A[i][1] + A[j][1] + A[k][1]

        diff = abs(sum - 0)

        if sum == 0 :

            return A[i][2], A[j][2], A[k][2]

        if diff $\leq$ min_value :

            min_value = diff

            output = A[i][2], A[j][2], A[k][2]

        if sum < 0 :

            j++

        else :

            k--

return output

2, -3, 1, -1

-3 -1 1 2
i j k

sum = -2

diff = 2

min_value = 2

output = 1, 2, 4

j = 3

Sum = -3 + 1 + 2
= 0

A[(-3, 2), (-1, 4), (1, 3), (2, 1)]

$$\sum_{i=1}^{n-2} \sum_{j=i+1}^{n} 3 = \sum_{i=1}^{n-2} 3(n-i) = 3[n-1 + (n-2) + \cdots + 2 + 1 - 1]$$

$$= 3\left[\frac{n(n-1)}{2} - 1\right]$$

$$= 3\left[\frac{n^2 - n - 2}{2}\right]$$

$O(n^2)$.