(1)

$$f(x) = f(x) + hf'(x) + \frac{1}{2}f''(x) + \frac{1}{6}f'''(x) + O(h^4)$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}h'''(x) + O(h^4)$$

$$f(x+2h) = f(x) + 2hf'(x) + 2\cdot\frac{h^2}{2!}f''(x) + 2\frac{h^3}{3!}f'''(x) + O(h^4)$$

$$f(x+3h) = f(x) + 3hf'(x) + 3\cdot\frac{h^2}{3!}f''(x) + 3\cdot\frac{h^3}{3!}h'''(x) + O(h^4)$$

Find $F''(x)$ Assume $F'(x) = 0$, $F'''(x) = 0$

$$af(x+h) + bf(x+2h) + cf(x+3h) = 0$$

$$\begin{cases} a + 2b + 3c = 0 \\ a + 4b + 9c = 2 \\ a + 8b + 27c = 0 \end{cases} \Rightarrow \begin{cases} a = -5 \\ b = 4 \\ c = -1 \end{cases}$$

$$-5f(x+h) + 4f(x+2h) - f(x+3h) = -2f(x) + h^2 f''(x) + O(h^4)$$

$$\cancel{-2f(x+h^2)+}$$

$$f''(x) = \frac{2f(x) - 5f(x+h) + 4f(x+2h) - f(x+3h)}{h^2} + O(h^4)$$

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$$F(h) = \frac{f(x_0+h) - f(x_0)}{h}$$

2.

① $$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(x_0) - \frac{h^2}{6} f'''(x_0) + O(h^3)$$

Given Richardsons extrapolation replacing ① with $h \Rightarrow 2h$ we got

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{2h} - h f''(x_0) - \frac{2h^2}{3} f'''(h_0) + O(h^3) \quad ②$$

~~2 * ②~~ 2 * ① − ② we got

$$f'(x_0) = \frac{2}{h}\left(f(x_0+h) - f(x_0)\right) - \frac{1}{2h}\left(f(x_0+2h) - f(x_0)\right) - \frac{h^2}{3} f'''(x_0) + \frac{2h}{3} f'''(x_0) + O(h^3)$$

$$= \frac{2}{h}\left(f(x_0+h) - f(x_0)\right) - \frac{1}{2h}\left(f(x_0+2h) - f(x_0)\right) + \frac{h^3}{3} f'''(x_0) + O(h^3)$$

$$= \frac{-f(x_0+2h) + 4f(x_0+h) - 3f(x_0)}{2h} + \frac{h^2}{3} f'''(x_0) + O(h^3) \quad ③$$

Replace ③ $h \Rightarrow 2h$

$$f'(x_0) = \frac{-f(x_0+4h) + 4f(x_0+2h) - 3f(x_0)}{4h} + \frac{4h^2}{3} f'''(x_0) + O(h^3) \quad ④$$

4 * ③ − ④ by calculation

$$f'(x_0) = \frac{f(x_0+4h) - 12f(x_0+2h) + 32f(x_0+h) - 21f(x_0)}{12h} + O(h^3)$$