Contents

AMSC 460 - EXAM 1

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clear all; format rational; close all; syms f(x) x y
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Problem 1 (finished)

$$\phi_0(x) = 1, \phi_1(x) = x - \alpha_1$$

$$\alpha_1 = \frac{\int_{-1}^{1} w(x) * x \, dx}{\int_{-1}^{1} w(x) \, dx} = \frac{\int_{-1}^{0} x \, dx + \int_{0}^{1} 3x \, dx}{4} = \frac{1}{4}$$

$$(\phi_0, \phi_0) = 2, (f, y_0) = \int_{-1}^{1} w(x) x^3 \, d = \frac{1}{2}$$

$$(\phi_1, \phi_1) = \int_{-1}^{1} w(x)(x - 1/2)^2 \, dx = \int_{-1}^{0} (x - 1/2)^2 \, dx \ + \int_{0}^{1} 3(x - 1/2)^2 \, dx = \frac{4}{3}$$

$$(f_1, \phi_1) = \int_{-1}^{1} w(x) x^3 (x - 1/2)^2 \, dx = \int_{-1}^{0} x^4 - (1/2) x^3 \, dx + \int_{0}^{1} 3(x^4 - (1/2) x^3 \, dx = \frac{11}{20}$$

$$\phi(x) = \frac{(\phi_1, \phi_0)}{(\phi_0, \phi_0)} * \phi_0(x) + \frac{(f_1, \phi)}{(\phi_1, \phi_1)} * \phi_1(x) = \frac{1/4}{2} + \frac{\frac{11}{20}}{\frac{4}{3}}(x - \frac{1}{2}) = \frac{33x}{80} - \frac{13}{160}$$

Problem 2 (Yup)

$f(x) = 1 * \int_{1}^{-1} 1 \, dx$=2

$$w_1 * f(-1) + w_2 * f'(-1) + w_3 * f'(1) + w_4 * f(1) = 2$$

which means:

$$w_1 + w_2 + w_3 + w_4 = 2$$

$$w_1 + w_4 = 2$$

$$when f(x) = x, f'(x) = 1, -w_1 + w_2 + w_3 + w_4 = 0$$

$$when f(x) = x^2, f'(x) = 2x, w_1 - 2 * w_2 + 2 * w_3 + w_4 = 2/3$$

$$when f(x) = x^3, f'(x) = 3x^2, -w_1 + 3 * w_2 + 3 * w_3 + w_4 = 0$$

$$when f(x) = x^4, f'(x) = 4x^3, -w_1 - 4 * w_2 + 4 * w_3 + w_4 = 2/5$$

We dont want to consider the forth term since the precision is quadrature. The matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ -1 & 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 1 & 2/3 \\ -1 & 3 & 3 & 1 & 0 \end{bmatrix}$$

solve by matlab shown below:

```
A = [1 0 0 1 2; -1 1 1 1 0; 1 -2 2 1 2/3; -1 3 3 1 0]
rref(A)
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A =
          1           0           0           1           2
         -1           1           1           1           0
          1          -2           2           1         2/3
         -1           3           3           1           0
   ans =
          1           0           0           0           1
          0           1           0           0         1/3
          0           0           1           0        -1/3
          0           0           0           1           1
```

so we have

$$w_1 = 1$$

$$w_2 = 1/3$$

$$w_3 = -1/3$$

$$w_4 = 1$$

Problem 3 (Yup)

(a) first we can write two equations base on $y(t) = xt$

$$3 = x$$

$$2 = 2x$$

so write the two equations in matrix form

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix}$$

so we can get that the matrix $A = \begin{pmatrix} 1, & 0 \\ 0, & 2 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$r = \begin{pmatrix} 1, & 0 \\ 0, & 2 \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} x-3 \\ 2x-2 \end{pmatrix}$$

Therefore:

$$||r||_2^2 = (x-3)^2 + (2x-2)^2 = 5x^2 - 14x + 3$$
$$f(x) = ||r||_2^2 = 5x^2 - 14x + 3$$

$$f^{'}(x) = 10x - 14$$

$$f'(x) = 0 = 10x - 14, x = \frac{7}{5}$$

$$||r||_2^2$$
$$x = \frac{7}{5}$$

((b)

$$r = \begin{pmatrix} x-3 \\ 2x-2 \end{pmatrix}$$

so

$$||r||_4^4 = (x-3)^4 + (2x-2)^4 = 17x^4 - 76x^3 + 150x^2 - 172x + 97$$

Problem 4

Compute the order of the local truncation error for this method.

$$f_{i+1} = f(t_i + h, y(t_i + h))$$

$$y_{i+1} = y(t_i + h) = y_i + hf_i + \frac{h^2}{2}y^{''} + \frac{h^3}{6}y^{'''} + O(h^4)$$

$$y_{i-1} = y(t_i - h) = y_i - hf_i + \frac{h^2}{2}y^{''} - \frac{h^3}{6}y^{'''} + O(h^4)$$

$$y^{''} = \frac{d^2y}{dt^2} = \frac{d}{dt}(\frac{dy}{dt}) = \frac{df}{dt} = f_t + ff_y$$

$$f(t_i{+}h, y(t_i{+}h)) = f_i{+}h(\frac{\partial}{\partial t}{+}\frac{\partial y}{\partial t}\frac{\partial}{\partial y})f{+}\frac{h^2}{2}(\frac{\partial}{\partial t}{+}\frac{\partial y}{\partial t}\frac{\partial}{\partial y})^2 f{+}\frac{h^3}{6}(\frac{\partial}{\partial t}{+}\frac{\partial y}{\partial t}\frac{\partial}{\partial y})^3 f{+}O(h^4)$$

so we can get:

$$y'' = \frac{df}{dt}, y''' = \frac{d^2 f}{dt^2}$$

$$f(t_i + h, y(t_i + h)) = f_i + hy'' + \frac{h^2}{2}y''' + \frac{h^3}{6}y^{(4)} + O(h^4)$$

so we can get that:

$$3y_{i+1} - 4y_i + y_{i-1}$$

$$= 3y_i + 3hf_i + \frac{3h^2}{2}y'' + \frac{h^3}{2}y''' + O(h^4) - 4y_i$$

$$+y_i - hf_i + \frac{h^2}{2}y'' - \frac{h^3}{6}y''' + O(h^4)$$

$$= 2hf_i + 2h^2 y'' + \frac{h^3}{3}y''' + O(h^4)$$

then:

$$2hf_{i+1} = 2hf_i + 2h^2 y'' + h^3 y''' + \frac{h^4}{3}y^{(4)} + O(h^5)$$

so we can get

$$3y_{i+1} - 4y_i + y_{i-1} - 2hf_{i+1}$$

$$= 2hf_i + 2h^2 y'' + \frac{h^3}{3}y''' + O(h^4) - (2hf_i + 2h^2 y'' + h^3 y''' + \frac{h^4}{3}y^{(4)} + O(h^5))$$

$$= -\frac{2h^3}{3}y''' + O(h^4)$$

The truncation error of the equation below:

$$w_{i+1} = \frac{1}{3}(4w_i - w_{i-1} + 2hf_{i+1})$$

is:

$$LTE = \frac{2}{9}h^3 y''' + O(h^4)$$