AMSC/CMSC 460: Final Exam

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 120 minutes
- Good luck!

Problems:

1. **(10 points)** Consider the following 3 values of $f(x)$: $f(x-h)$, $f(x)$, and $f(x+2h)$.

   (a) Use the method of undetermined coefficients to find the best approximation for $f'(x)$. What is the order of this approximation?

   (b) Use the method of undetermined coefficients to find the best approximation for $f''(x)$. What is the order of this approximation?

2. **(10 points)** Let $w(x) = x^2$, $\forall x \in [-1, 1]$.

   (a) Use the Gram-Schmidt process to find the first two orthogonal polynomials with respect to the inner product

$$\langle f(x), g(x) \rangle_w = \int_{-1}^{1} f(x)g(x)w(x)dx.$$

   Do not normalize the polynomials.

(b) Use the least squares theory to find the polynomial of degree 0, $Q_0(x)$, that minimizes

$$\int_{-1}^{1} x^2(x^2 - Q_0(x))^2 dx.$$

3. **(10 points)**

(a) Use the Lagrange interpolation polynomial to derive a formula of the form $\int_{-1}^{1} f(x)dx \approx Af(0) + Bf(1)$.

(b) Find a formula of the form $\int_{0}^{1} f(x)dx \approx Af(0) + Bf(1)$, that is exact for all functions of the form $f(x) = ax + b\sin(\pi x)$. (Note that this problem has different boundaries of integration than in part (a)).

4. **(10 points)** Consider the initial-value problem: $y'(t) = f(t, y(t)), \quad y(0) = a$.

(a) Explain how to obtain Euler's method for approximating solutions of this initial-value problem, by using the rectangular quadrature rule on the integral form of the ODE.

(b) Perform two steps of Euler's method, assuming that $f(t, y(t)) = t + y$, $y(0) = 1$, and $h = 0.5$.

5. **(10 points)**

(a) Let $f(x) = x^3 + x - 3$. Explain why $f(x)$ has at least one positive root. Explain how to use Newton's method for approximating a root of $f(x)$. Compute two iterations of Newton's method, starting from $x_0 = 1$.

(b) Consider the same polynomial from part (a): $f(x) = x^3 + x - 3$.
Consider the values of $f(x)$ at $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$. Let $Q_3(x)$ denote the interpolation polynomial through these four points. Find $Q_3(x)$.