Computational Methods Summer 2021

HOMEWORK 8

Due Date: Monday, June 14

1. Let $A \in \mathbb{R}^{2\times 2}$ be given by
$$A = \begin{bmatrix} 1 & 1+\epsilon \\ 1-\epsilon & 1 \end{bmatrix}.$$
Find $\|A\|_\infty$ and $\|A^{-1}\|_\infty$. What is $\mathrm{cond}_\infty(A)$? Is this matrix well-conditioned or ill-conditioned?

   Let $\mathbf{b} = [1 + (1+\epsilon)\epsilon,\ 1]^T$. Find the exact solution of $A\mathbf{x} = \mathbf{b}$. Use MATLAB's backslash command to solve $A\mathbf{x} = \mathbf{b}$ for progressively smaller values of $\epsilon = 10^k$ for $k = -5, -6, \ldots$. At which value of ϵ does the computed solution no longer accurately represent the true solution?

2. (MATLAB) Consider the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [.254\ \ .127]^T$ and
$$A = \begin{bmatrix} .913 & .659 \\ .457 & .330 \end{bmatrix}.$$
Use the MATLAB backslash command to find the exact solution $\mathbf{x}$. Use the command `cond` to find the 2-norm condition number of A.

   Consider the two approximate solutions
$$\mathbf{x}_1 = [-0.0827\ \ 0.5]^T, \quad \text{and} \quad \mathbf{x}_2 = [0.999\ \ -1.001]^T.$$
Using the `norm` command, compute

   (a) the relative forward errors for $\mathbf{x}_1$ and $\mathbf{x}_2$ using the 2-norm

   (b) the relative backward errors for $\mathbf{x}_1$ and $\mathbf{x}_2$

   Comment on the size of your backward and forward errors. Does a small backward error imply an approximate solution is accurate? How do your observations relate to the condition number of A?

3. (Optional, not graded) Let $\mathbf{D} \in \mathbb{R}^{n\times n}$ be the diagonal matrix with $d_1, d_2, \ldots, d_n$ on the diagonal. Show that the ∞-condition number is given by
$$\mathrm{cond}_\infty(\mathbf{D}) = \frac{\max_i |d_i|}{\min_i |d_i|}.$$

4. (Optional, not graded) Draw the unit sphere of $\mathbb{R}^2$ in the norm $\|\cdot\|_1$, i.e. the set of points satisfying $\|\mathbf{x}\|_1 = 1$. Let A be the matrix
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Sketch the image of your unit sphere under the transformation A, i.e. sketch the set of points $\{A\mathbf{x} : \|\mathbf{x}\|_1 = 1\}$. Which points in the image are distance $\|A\|_1$ from the origin?

(You may want to compare your answer to the definition of the operator norm $\|A\|_1$, which characterizes the norm as the maximum 'stretching' that occurs under A.)