AMSC/CMSC 460 – Computational Methods 29 January 2010

FINAL EXAM

Lecturer: H. Glaz *Time: 2 Hours*

Instructions and Hints, etc:

Answer each question on one answer sheet, using the back if necessary (indicating on the front that your work continues on the back). Write your name and the question number on each answer sheet. **Show all the steps in your solutions**, explaining your arguments in complete English sentences. Cross out any material that you do not wish graded.

Where appropriate, circle or box your final answer. Good luck!

- Calculators: your calculator should **NOT** be preprogrammed with anything directly relevant to the course.
- Show your work!!
- Problems are graded as a whole unless otherwise indicated. Each of the six (6) problems counts equally.

---

(1) (40 pts) — NO CALCULATORS —

Let

$$A = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \, , \, b = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \, ,$$

and observe (do **not** prove/verify) that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2/3 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -4/3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix} .$$

(a) (8 pts) Write down a lower triangular matrix L, an upper triangular matrix U, and a permutation matrix P such that $LU = PA$.

(b) (11 pts) Find a vector $\boldsymbol{z}$ so that solving (i) $L\boldsymbol{y} = \boldsymbol{z}$, and then (ii) $U\boldsymbol{x} = \boldsymbol{y}$ is equivalent to solving the problem $A\boldsymbol{x} = \boldsymbol{b}$.

(c) (20 pts) Solve for $\boldsymbol{y}$ and $\boldsymbol{x}$ (in that order, *showing all steps*) as indicated in (b) above.

(d) (1 pt) CHECK that $A\boldsymbol{x} = \boldsymbol{b}$. Show a few steps.

(2) (40 pts) Let

$$I = \int_0^1 e^{-x^2} \, \mathrm{dx} = \frac{\sqrt{\pi}}{2} \, \mathrm{erf}(1) = \frac{\sqrt{\pi}}{2} * 0.84270079 = 0.74682413 \, .$$

We are concerned here with approximating I using the *Composite Trapezoid* rule with N equally spaced *intervals* of size h (i.e., $h * N = 1 - 0 = 1$). Notation: T_h OR T_N will represent the approximate value (and should be used in your solution as needed; either is fine).

(a) (5 pts) In terms of a *sum of exponentials*, h, etc., what is the approximate value for $N = 2$, equivalently, $h = 1/2$? NO CALCULATORS.

(b) (10 pts) As above, what is T_4? NO CALCULATORS.

(c) (25 pts) Here is a table of (unsigned) errors:

N	Error
2	1.54538810e-02
4	3.84003501e-03
8	9.58517967e-04
16	2.39536024e-04

Obviously, *accuracy* is increasing as N increases; equivalently, T_N is *converging* to I as N increases. What can you say about the *rate* of convergence (or, *order* of accuracy) **based on these results and error ratios**. USE YOUR CALCULATOR (BUT – don't recompute the table values; just assume that they are correct).

(3) (40 pts) You are given data

t	0	1	2	3
y	1	2	4	3

and need to find the least-squares fit of the form

$$y(t) \sim \beta_1 + \beta_2 * t\,.$$

(a) Write the problem in the form

$$X\beta \sim \boldsymbol{y}\,,$$

i.e., define $X \in \mathbb{R}^{m \times n}$ and $\boldsymbol{y} \in \mathbb{R}^m$.

(b) Derive the *normal equations* for this problem.
Hint: Recall $X^T X\beta = X^T \boldsymbol{y}$.

(c) *SOLVE* the normal equations for β.

(4) (40 pts) Let

$$f(x) = x^2 - a\,.$$

(a) *Prove* that one iteration of Newton's method for $f(x) = 0$ is equivalent to one fixed point iteration

$$x_{k+1} = \frac{1}{2}(x_k + \frac{x_k}{a})\,.$$

(b) Let $a = 4$ and $x_0 = 1$. Compute three (3) iterates and demonstrate quadratic convergence for this example.

(5) (40 pts) Given $c = 201.125$ (base 10):

(a) Derive the (double precision) *normalized floating point* representation of c :

$$c = (-1)^s \; 1.\mathrm{f} * 2^e .$$

(b) In MATLAB –

```
>> format hex
>> c = 201.125;
>> c
ans =  ____________
```

fill in the blank.