AMSC/CMSC 460 – Computational Methods	18 December 2012
FINAL EXAM	
Lecturer: H. Glaz	*Soln: 1:30 – 3:30*

Instructions and Hints, etc:

Problems are graded as a whole unless otherwise indicated.

Calculators: your calculator should **NOT** be preprogrammed with anything directly relevant to the course.

Show your work!

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(1) (20 pts) — NO CALCULATORS —

Let

$$A = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix},$$

and observe (do **not** prove/verify) that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2/3 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -4/3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix}.$$

(a) Write down a lower triangular matrix L, an upper triangular matrix U, and a permutation matrix P such that $LU = PA$.

(b) Find a vector $\boldsymbol{z}$ so that solving (i) $L\boldsymbol{y} = \boldsymbol{z}$, and then (ii) $U\boldsymbol{x} = \boldsymbol{y}$ is equivalent to solving the problem $A\boldsymbol{x} = \boldsymbol{b}$.

(c) Solve for $\boldsymbol{y}$ and $\boldsymbol{x}$ (in that order, *showing all steps*) as indicated in (b) above.

(d) (1 pt) CHECK that $A\boldsymbol{x} = \boldsymbol{b}$. Show a few steps.

(2) (20 pts) Consider the problem of floating point evaluation for positive, but very small (relative to 1) x, of the function

$$\sqrt{1+x} - \sqrt{1-x}.$$

(a) Show that – over the real numbers – this is the same as

$$\frac{2x}{\sqrt{1+x} + \sqrt{1-x}}.$$

(b) Consider the m-file and MATLAB output

```
kk = 1:12; x = 10.^(-kk);
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%% -- note that f1 & f2 below are correct.

f1 = sqrt(1 + x) - sqrt(1 - x);
f2 = 2.*x./(sqrt(1 + x) + sqrt(1 - x));

for l = 1:12
  disp(sprintf('%3.2e  %17.14e  %17.14e', x(l),f1(l),f2(l)))
end
------------------------------------------------------------------
>> Prob2

1.00e-01  1.00125550119638e-01  1.00125550119638e-01
1.00e-02  1.00001250054690e-02  1.00001250054691e-02
1.00e-03  1.00000012500001e-03  1.00000012500005e-03
1.00e-04  1.00000000124889e-04  1.00000000125000e-04
1.00e-05  1.00000000000655e-05  1.00000000001250e-05
1.00e-06  1.00000000002876e-06  1.00000000000012e-06
1.00e-07  1.00000000058387e-07  1.00000000000000e-07
1.00e-08  1.00000000502476e-08  1.00000000000000e-08
1.00e-09  1.00000008274037e-09  1.00000000000000e-09
1.00e-10  1.00000008274037e-10  1.00000000000000e-10
1.00e-11  1.00000008274037e-11  1.00000000000000e-11
1.00e-12  1.00008890058234e-12  1.00000000000000e-12
```

Based on the 2 formulas, is one of these methods better than the other (or not) at avoiding floating point pitfalls? Does the output help directly?

(c) Denote $f(x) = \sqrt{1+x} - \sqrt{1-x}$. Show that the Taylor series expansion around $x = 0$ leads to

$$f(x) = x + \frac{1}{8}x^3 + \text{h.o.t.}.$$

Use this result to determine which of the two methods is more accurate. Include comparisons between the series expansion and the output above.

(3) (20 pts) Find the node x_1 and weight w_1 so that the Quadrature Rule

$$\int_0^1 \sqrt{x}\, f(x)\, \mathrm{dx} \sim w_1 * f(x_1)$$

is *exact* for any linear polynomial. **Note:** The integrals are very easy!

What is the polynomial degree of the rule so derived?

Use the Rule derived above to compute an approximation to

$$\int_0^1 \sqrt{x}\, e^{-x}\, \mathrm{dx} \sim 0.3789447\,.$$

(4) (20 pts) Consider the rule –

$$\int_0^1 f(x)\,\mathrm{dx} \sim w_1 * f(0) + w_2 * f(1) + w_3 * f'(0) + w_4 * f'(1)\,.$$

(a) Find values of the weights so that the rule is exact for polynomials of degree 3 or less.

(b) What is the polynomial degree of the rule derived in Part (a)? Justify your result.

(c) (1 pt) The rule is **not** a Newton–Cotes Rule. Why?

(d) (2 pts) **Check** your result by using it to approximate

$$\int_0^1 x + x^3\,\mathrm{dx}\,.$$

(5) (45 pts) The equation $3\ln x = x$ has 2 real roots, and this is to be **assumed**.

(a) Rewrite the equation in the form $f(x) = 0$.
Consider using Newton's Method to solve the problem. Write down the iteration formula to obtain x_{k+1} in terms of x_k. Simplify by expressing the result as a single fraction.

(b) Let $x_0 = 2$. Compute enough Newton iterates with this initial guess so that one of the roots is found to 'machine precision' (here, the machine is your calculator). Record the iterates.

(c) **Convincingly demonstrate** that the iteration in Part (b) converged quadratically.

(d) Use Newton's Method to find the *other* solution. Record your initial guess, the next 2 iterates, and the solution.

(6) (35 pts) Consider the nonlinear system

$$x^2 + y^2 = 1$$
$$\frac{1}{4}x^2 + 9y^2 = 1$$

(a) (1 pt) Find all solutions to the system. Hint: Easy – draw a sketch.

(b) (2 pts) Write the system in the form

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}\,,$$

where $\mathbf{x} = (x\ y)^t$, and $\mathbf{F}(\mathbf{x}) = (f(\mathbf{x})\ g(\mathbf{x}))^t$.

(c) Let $\mathbf{x}_0 = (1,1)^t$. Compute the first three iterates $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$.

(d) Which root is the iteration approaching?

(e) Is there evidence of quadratic convergence? Compute error ratios **in the 1 – norm.**

(f) Find another initial guess which likely converges to a different root.

(7) (20 pts) Consider the IVP

$$(*)\quad \begin{cases} y' = \sin(y) \\ y(0) = 1\,. \end{cases}$$

Use Euler's method to find y_1 and y_2, assuming the step size $h = 0.5$.

(8) (20 pts) Assume that

$$S(x) = \begin{cases} 1 + B(x-1) - D(x-1)^2 & \text{if} 1 \le x < 2 \\ 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3 & \text{if} \quad 2 \le x \le 3 \end{cases}$$

interpolates thate data (1,1), (2,1), (3,0). Find B, D, b, d so that $S(x)$ is a cubic spline. Is there a choice so that it is a natural cubic spline?