Computational Methods Summer 2021

HOMEWORK 18

Due Date: Thursday, July 1

Homework should be handed in *individually*, though you may work with others and collaboration is encouraged. For MATLAB problems please follow the guidelines specified in ELMS.

1. Verify that the following two formulas are difference quotients that approximate the third derivative of f and find the errors for both by Taylor expanding. Also give an upper bound on both errors in the form $|f'''(x) - D_h f(x)| \leq Ch^p \|f^{(n)}\|_\infty$ (i.e. find C, p, n).

   (a) $f'''(x) \approx \dfrac{1}{h^3}(f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x))$,

   (b) $f'''(x) \approx \dfrac{1}{2h^3}(f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h))$.

2. (Optional, not graded) Use a quadratic function to interpolate the values of $f(x)$ at $x_0 - h, x_0$, and $x_0 + 2h$. Use your interpolant to find a finite difference formula approximating $f'(x_0 + h/2)$.

3. (Optional, not graded) Consider the 3-point centered difference formula

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2).$$

   In theory, the convergence to $f'(x)$ is guaranteed as $h \to 0$ since the $O(h^2)$ error term goes to zero. In practice this is not true and there is a limit to how small h can be before rounding error takes over and pollutes the computation. Following the argument from class used to find the optimal h value for the 2-point forward difference formula, show for the centered difference formula that

$$\left| f'(x) - \frac{f(x+h) - f(x-h)}{2h} \right| \leq \frac{Mh^2}{6} + \frac{\varepsilon}{h},$$

   where $M = \|f'''\|_\infty$ and $\varepsilon = \varepsilon_{mach}$. Then minimize the upper bound over all h to determine the optimal value of h.