HW2

Yizhan Ao

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1 Question 1

(Must do all computations by hand.) Consider the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ which first reflects through the y-axis, and then rotates counterclockwise about the origin by $\pi/4$ radians (45 degrees). Find the standard 2×2 matrix for this linear transformation in two ways:

1. Find the standard matrix of f by directly determining the images of the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ under f.
2. Find the standard matrix of f by first finding the standard matrices of the reflection and of the rotation, and then use matrix multiplication.

1 a)

Let f_1 be a reflection function that through the y-axis, let f_2 be a rotation counter clockwise $\pi/4$ about the origin we can get

$$f_1(x, y) = (x, -y)$$

$$f_2(x, y) = \begin{bmatrix} x * cos(\pi/4) - y * sin(\pi/4) \\ x * sin(\pi/4) + y * cos(\pi/4) \end{bmatrix}$$

when we are consider the reflection $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ through y-axis it maps (1,0) then rotaton $\pi/4$ and we can get

$$(cos(\pi/4), sin(\pi/4)) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

so $\mathbf{e}_1$ maps to $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ under F_1

Similarly $\mathbf{e}_2$ maps to $(0, -1)$ so F2 maps to $(-1 * x * sin(\pi/4) - 1 * y * cos(\pi/4))$

Therefore, $F(\mathbf{e}_2) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$$F = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

1 b)

$$F_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$F_1 * F_2 = F_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

2 Question 2

Find the point obtained when $(7, 9)$ is rotated counterclockwise about the origin by 50 degrees ($5\pi/18$ radians). (Do the computation in MATLAB, but make it clear what computation you are doing.)

$$\begin{bmatrix} \cos\left(\frac{5\pi}{18}\right) & -\sin\left(\frac{5\pi}{18}\right) & 0 \\ \sin\left(\frac{5\pi}{18}\right) & \cos\left(\frac{5\pi}{18}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.395 \\ 11.147 \\ 1 \end{bmatrix} \tag{1}$$

the point is (-2.395, 11.147)

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Code proof
format long;
x = 7, y =9;
r = sqrt(x^2+y^2)
Q = atan(y/x)
X = 7*sind(50)-9*sind(50)
Y = 9*cosd(50)+7*sind(50)
\item r =  11.401754250991379
\item Q =  0.909753157944210
\item X =  -2.395088886237956
\item Y =  11.147399589011700
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3 Question 3

In two dimensions write down the matrix which rotates around the origin by $\pi/6$ radians and then translates by -3 in the x-direction and 5 in the y-direction. Apply this matrix to the points (0; 3) and (1,1).

$$T(-3,5)R\left(\frac{\pi}{6}\right) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{b}\right) & -\sin\left(\frac{\pi}{6}\right) & 0 \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

3 a)

$$T(-3,5)R\left(\frac{\pi}{6}\right) \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = (-\frac{9}{2}, \frac{10+3\sqrt{2}}{2}) \tag{3}$$

Therefore, (x',y') after rotaion is $(-\frac{9}{2}, \frac{10+3\sqrt{2}}{2})$

3 b)

$$T(-3,5)R\left(\frac{\pi}{6}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = (-\frac{\sqrt{3}-5}{2}, \frac{9-\sqrt{3}}{2}) \tag{4}$$

Therefore, (1,-1) after rotaion is $(-\frac{\sqrt{3}-5}{2}, \frac{9-\sqrt{3}}{2})$

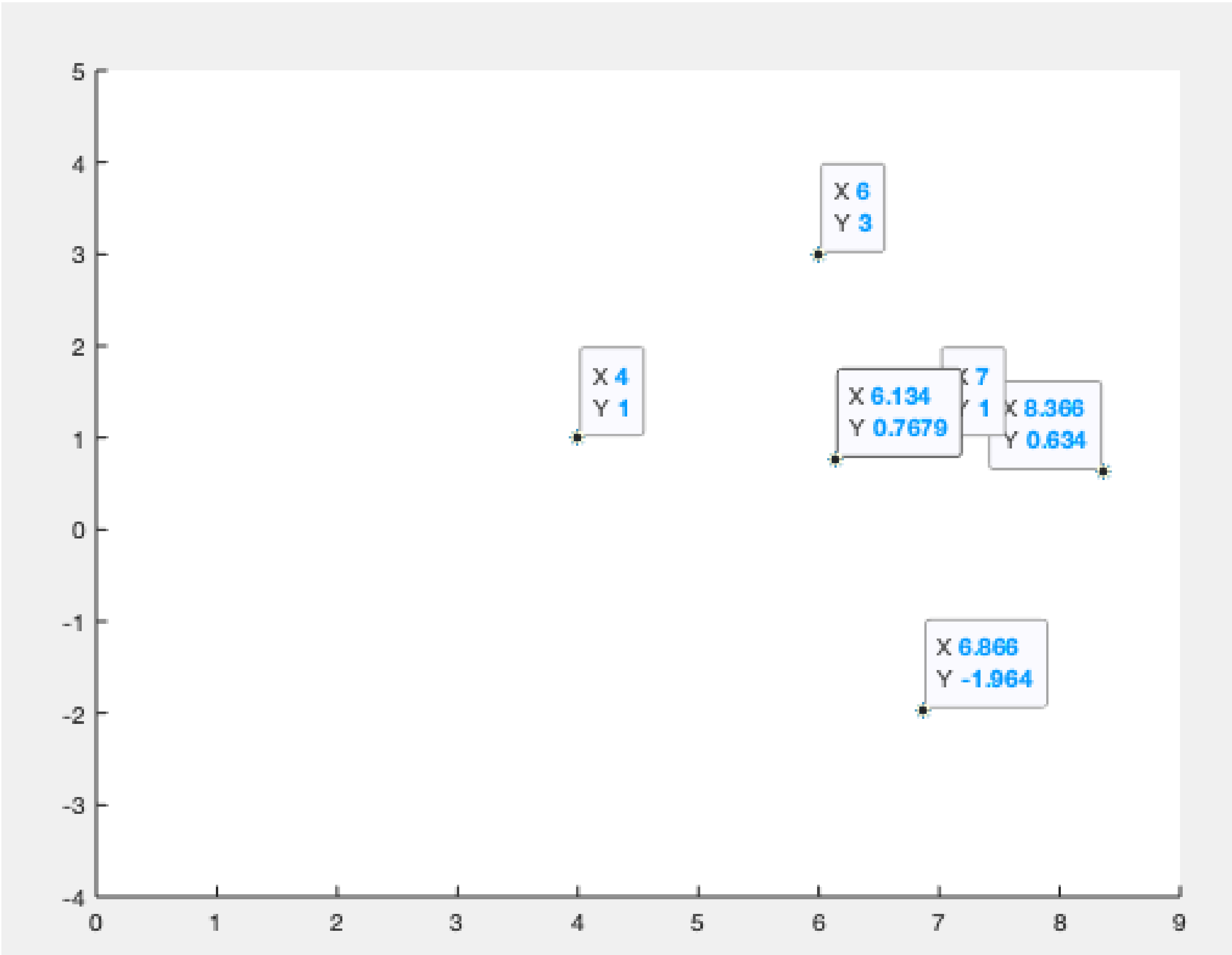
4 Question 4

1. In two dimensions find the image of the three points (6; 3); (4; 1); (7; 1) under rotation around the point (8; 2) by $\pi = 3$ radians. Sketch the original points and the images.

$$T(8,2)R(\pi/3)T(-8,2) = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\pi/3) & -sin(\pi/3) & 0 \\ sin(\pi/3) & cos(\pi/3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5000 & -0.8660 & 5.7321 \\ 0.8660 & 0.5000 & -5.9282 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.5000 & -0.8660 & 5.7321 \\ 0.8660 & 0.5000 & -5.9282 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 4 & 7 \\ 3 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 6.1340 & 6.8660 & 8.3660 \\ 0.7679 & -1.9641 & 0.6340 \\ 1 & 1 & 1 \end{bmatrix}$$



5 Question 5

1. In two dimensions write down the matrix (simplified) for rotation around the point (a; b) by θ radians.

$$T(a,b)R(\theta)T(-a,-b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

6 Question 6

Find the 3 3 matrix (for homogeneous coordinates) for the transformation which rst rotates counterclockwise about the point (4; 8) by =5 radians, and then rotates counterclockwise about the point (7; 1) by 4=11 radians.

$$T_1 = \begin{bmatrix} cos(\pi/5) & -sin(\pi/5) & 4 \\ sin(\pi/5) & cos(\pi/5) & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} cos(\pi/5) & -sin(\pi/5) & 4 \\ sin(\pi/5) & cos(\pi/5) & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_1 * T_2 = \begin{bmatrix} -0.507 & -1.122 & 1.394 \\ 1.122 & -0.507 & 7.969 \\ 0 & 0 & 1 \end{bmatrix}$$

7 Question 7

1. Let L denote a line in the plane that does **not** go through the origin. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ denote the transformation which reflects through the line L. Give a reason why f is not a linear transformation.
2. Reflection through a line which does not go through the origin can be represented as a 3×3 matrix (which operates on homogeneous coordinates). Give the matrix for reflection through the vertical line $x = 5$.
3. Give a formula for the image of the generic point (x, y) under reflection through the line $x = 5$.

1)

let L be a line within the coordinates that doesnt pass through the origin, let ax + by = c and c not equal to 0

so we can be the perpendicular line to L that f(0,0) = 2*A that doesnt equal to (0,0)

therefore f is not a line of transformation

2)

let line L be x = 5and y coordinate will remain the same but at the x coordinate

x -¿10-x

So f(x,y) = (10-x, y) which can also be (10, 0) + (-x, y)

so the matrix will be A = $\begin{bmatrix} 10 & 0 & 0 \end{bmatrix}$

$+ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ f(x,y) = $\begin{bmatrix} 10 & 0 & 0x \end{bmatrix} + \begin{bmatrix} x & y & 0 \end{bmatrix}$

3)

c) from the above we can have $f(x, y) = (10 - x, y)$

8 Question 8

(Must do all computations by hand.) Let L_θ denote the line in the plane through the origin that makes an angle of θ radians with the positive x-axis. In class, we saw that reflection through the line L_θ has standard matrix $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$. Now let θ and φ be two angles. Show that the composition of reflection through L_φ followed by reflection through L_θ is equal to a rotation about the origin, and find the angle of rotation. (Multiply the matrices and use trig identities to recognize the result as a rotation matrix.)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \operatorname{ars}\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\theta + \sin\theta\cos\alpha & \cos\theta\sin\alpha - \sin\theta\cos\alpha \\ \sin\theta\cos 2d - \cos\theta \operatorname{cin}\alpha & \sin\theta\sin\alpha + \cos\theta\cos\alpha \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \cos(\theta-\alpha) & -\sin(\theta-\alpha) \\ \sin(\theta-a) & \cos(\theta-\alpha) \end{bmatrix} (7)$$

Therefore we can comeup to a conclusion that the angle is $\theta - \alpha$