AMSC/CMSC 460 Dr. Wolfe FINAL EXAM July 24, 2009

1. (8 points) Let N be a positive integer. Consider the following MATLAB script:

$$
\begin{aligned}
&y = 0;\\
&\text{for } i = 1 : N\\
&y = y + (1/N);\\
&\text{end}\\
&y
\end{aligned}
$$

   (a) What would the result of the computation be in exact arithmetic ?
   (b) When the script was actually run with $N = 100,000 (= 10^5)$ the result was $y = .99999999999808$. When it was run with $N = 131,072 (= 2^{17})$ the result was $y = 1$. Explain these results.

2. (17 points) Let

$$A = \begin{pmatrix} 9 & -3 \\ -3 & 5 \end{pmatrix}$$

   (a) Compute the Choleski factorization of A ($A = LL^T$ with L lower triangular) and use it to solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (9, 9)^T$.
   (b) Find an approximate solution to $A\mathbf{x} = \mathbf{b}$ by doing two Jacobi iterations starting at $\mathbf{x}^{(0)} = (1, 2)^T$.
   (c) Find an approximate solution to $A\mathbf{x} = \mathbf{b}$ by doing two Gauss-Seidel iterations starting at $\mathbf{x}^{(0)} = (1, 2)^T$.
   (d) Which method gives a better approximation to the exact solution ?

3. (12 points)
   (b) Find a polynomial $p(x)$ of degree ≤ 2 satisfying $p(0) = 0,\ p(1) = 1,\ p'(1/4) = 2$.
   (b) There is a number $c,\ 0 < c < 1$ such that there is no polynomial $p(x)$ of degree ≤ 2 satisfying $p(0) = 0,\ p(1) = 1,\ p'(c) = 2$. Find c.

4. (12 points)
   (a) Find the node x_1 and the weight w_1 such that the integration rule

$$\int_0^1 \sqrt{x} f(x)\, dx \approx w_1 f(x_1)$$

   is exact if $f(x)$ is any linear polynomial. (This is <u>not</u> a hard computation.)
   (b) Use this rule to compute an approximation to

$$\int_0^1 \sqrt{x} e^{-x}\, dx = 0.3789447.$$

5. (10 points) It is desired to fit a data set $\{(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, , (x_m, y_m, z_m)\}$ (where m may be very large) to a model of the form

$$z = c_1 + c_2 x + c_3 y + c_4 xy$$

in the sense of least-squares. Write a MATLAB script which will find the coefficients c_1, c_2, c_3, c_4.

6. (12 points)
   (a) What are the solutions α, if any, of the equation $x = \sqrt{6x - 8}$?
   (b) Does the iteration $x_{n+1} = \sqrt{6x_n - 8}$ converge to any of these solutions (assuming x_0 is chosen sufficiently close to α)? Explain. (We need some analysis, not just numerical evidence.)

7. (15 points) Consider the system

$$x^2 + y^2 - 25 = 0, \qquad x^2 - y - 3 = 0$$

   (a) By graphing these equations, show that this system has two solutions, one in the first quadrant and one in the second.
   (b) By eliminating x^2 between the two equations, find the solution of the system lying in the first quadrant.
   (c) Let $(x_0, y_0) = (3, 4)$. Apply Newton's method for systems to compute (x_1, y_1). If you do this correctly, you will find that (x_1, y_1) is quite close to the solution found in (b).

8. (14 points) Consider the initial value problem

$$\frac{dy}{dt} = y - t, \qquad y(0) = 2.$$

   (a) Verify that the solution is $Y(t) = e^t + t + 1$.
   Find approximations to $Y(0.2)$ by using
   (b) two steps of the Euler method with $h = .1$.
   (c) one step of the Improved Euler method with $h = .2$.
   In (b) and (c) compare your answers with the exact solution.