Math 240-03**

Final Exam

Fall 2020
Dr. Manning

The use of calculators and notes is prohibited, as is seeking help from any other outside source during the exam. True/False questions require no justification. All other answers must be justified in order to receive credit. The maximum possible score is 200 points. Good luck!

---

1. (a) (10pt) Let $A = \begin{bmatrix} 3 & 2 & 0 & 30 \\ 17 & 234 & -3 & -23 \\ 1 & -1 & 0 & 42 \\ 0 & 0 & 0 & 10 \end{bmatrix}$. Calculate $\det(A)$. Show all steps.

   (b) (8pt) Explain how the three elementary row operations change the determinant of a square matrix.

   (c) (8pt) Let $A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix}$. Find A^{-1}, showing your work.

   (d) (3pt) `TRUE` or `FALSE`: The set of all 2×2 matrices with determinant 1 is a vector subspace of the vector space of all real 2×2 matrices.

   (e) (3pt) `TRUE` or `FALSE`: Every square matrix with a negative determinant is invertible.

2. Consider the linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ given by the formula $S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ -x_3 \\ 4x_1 + 4x_2 + 9x_3 \end{bmatrix}$.

   (a) (4pt) What is the standard matrix for S?

   (b) (6pt) What is the standard matrix for the transformation $S \circ S$ mapping $\mathbf{x}$ to $S(S(\mathbf{x}))$?

   (c) (6pt) Is S 1-1? Is S onto? Justify.

   (d) (3pt) Give an example of a linear transformation that is onto but not 1-1.

   (e) (3pt) Give an example of a linear transformation that is both 1-1 and onto.

   (f) (3pt) `TRUE` or `FALSE`: An $m \times n$ matrix is invertible if and only if its columns are linearly independent.

   (g) (3pt) `TRUE` or `FALSE`: If $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a spanning set for a vector space V and $T : V \to W$ is a linear transformation, then $\{T(\mathbf{b}_1), \ldots, T(\mathbf{b}_n)\}$ is a spanning set for W.

3. Consider the matrices $A = \begin{bmatrix} 1 & -2 & -4 & -18 \\ -2 & 3 & 8 & 30 \\ 0 & -4 & -1 & -25 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 3 & 8 \\ 0 & -4 & -1 \end{bmatrix}$. Let $\mathbf{b} = \begin{bmatrix} 7 \\ 15 \\ -2 \end{bmatrix}$.

   (a) (14pt) Write the linear system associated to the equation $A\mathbf{x} = \mathbf{b}$ and express the solution $\mathbf{x}$ in parametric vector form.

   (b) (10pt) Are the columns of B linearly independent or linearly dependent? If they are linearly independent, explain why. If they are linearly dependent, find a relation of linear dependence between them.

   (c) (3pt) Suppose C is an $m \times n$ matrix and let $\mathbf{y} \in \mathbb{R}^m$ be a nonzero vector. `TRUE` or `FALSE`: The solution set of $C\mathbf{x} = \mathbf{y}$ is a vector subspace of $\mathbb{R}^n$.

4. (a) (12pt) Let $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (you do not have to compute P^{-1}). Show your work.

   (b) (3pt) `TRUE` or `FALSE`: If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$ are eigenvectors for T with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$) then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set in $\mathbb{R}^n$.

CONTINUE

5. Given below is a matrix A and its reduced row echelon form:

$$A = \begin{bmatrix} 4 & 12 & -4 & 4 & 16 \\ -1 & -3 & 7 & 23 & -16 \\ 2 & 6 & 10 & 50 & -16 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) (4pt) Determine the rank of A.

   (b) Find a basis for each of the following subspaces, and state its dimension. No justification is necessary for this part.

      i. (4pt) $\operatorname{Nul} A$ ii. (4pt) $\operatorname{Col} A$ iii. (4pt) $\operatorname{Row} A$

   (c) (3pt) `TRUE` or `FALSE`: If X and Y are row-equivalent matrices, then they have the same (identical) null space.

   (d) (3pt) `TRUE` or `FALSE`: If X and Y are row-equivalent matrices, then they have the same (identical) column space.

6. (a) (10pt) Consider the set $S \subseteq \mathbb{R}^4$ consisting of the two vectors $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 11 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$. Find a basis of $\mathbb{R}^4$ containing S. (Hint: there are many ways to do this, but one idea is to think about $W^\perp$, where $W = \operatorname{Span}(S)$.)

   (b) Consider the two bases $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -8 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \end{bmatrix} \right\}$ for $\mathbb{R}^2$.

      i. (4pt) Determine the $\mathcal{B}$-coordinates of the vector $\mathbf{x} = \begin{bmatrix} 10 & 10 \end{bmatrix}^T$.

      ii. (4pt) Determine which vector $\mathbf{y} \in \mathbb{R}^2$ has $\mathcal{C}$-coordinates $[\mathbf{y}]_{\mathcal{C}} = \begin{bmatrix} -3 & 2 \end{bmatrix}^T$.

      iii. (5pt) Compute the change-of-coordinates matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

   (c) (3pt) `TRUE` or `FALSE`: Two square matrices with the same eigenvalues (with the same algebraic multiplicities) must be similar.

   (d) (3pt) `TRUE` or `FALSE`: Two square matrices that are similar must have the same eigenvalues (with the same algebraic multiplicities).

7. (a) (14pt) Let $W = \operatorname{Span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -1 \\ -17 \end{bmatrix} \right\}$. Find an orthonormal basis for W.

   (b) (8pt) Find a least-squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 4 \\ 5 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 15 & -39 & 0 & 0 \end{bmatrix}^T$. You may use your work from part (a), but you don't have to.

   (c) (3pt) `TRUE` or `FALSE`: Every real $m \times n$ matrix A having linearly independent columns has a QR decomposition $A = QR$, where Q has orthonormal columns and R is upper-triangular and invertible.

   (d) (3pt) `TRUE` or `FALSE`: For a subspace W of $\mathbb{R}^n$, the only vector $\mathbf{v}$ such that $\mathbf{v} \in W$ and $\mathbf{v} \in W^\perp$ is the zero vector.

   (e) (3pt) `TRUE` or `FALSE`: Every symmetric real $n \times n$ matrix is orthogonally diagonalizable.

8. (16pt) Let $A = \begin{bmatrix} 6 & -5 \\ 5 & -2 \end{bmatrix}$. Find both eigenvalues of A and a corresponding eigenvector in $\mathbb{C}^2$ for each eigenvalue. Show your work.