AMSC/CMSC 460 Dr. Wolfe FINAL EXAM July 25, 2008

1. (10 points) Recall that in any computation the relative error is defined as

$$\text{rel. error} = \frac{|\text{error}|}{|\text{true value}|}.$$

We are concerned with the evaluation of a function $f(x)$.

(a) Show that if the error in x is small

$$\text{rel. error in } f(x) \approx \frac{|xf'(x)|}{|f(x)|} \cdot \text{rel. error in } x.$$

The quantity $\kappa = |xf'(x)/f(x)|$ can be considered the condition number of f at x. If κ is large we say f is ill-conditioned at x. If κ is small we say that f is well-conditioned at x.

(b) Show that for any $x > 0$, $f(x) = \sqrt{x}$ is well-conditioned while for x near $\pi/2$, $f(x) = \cos x$ is ill-conditioned. What about $f(x) = \sin x$ near $x = 0$?

2. (5 points) The following statements pertain to the solution of $A\mathbf{x} = \mathbf{b}$ in floating point arithmetic. Complete them so they are meaningful and true.

(a) Gauss elimination with partial pivoting almost always gives good results if A ________________

(b) For any A, Guass elimination with partial pivoting is virtually guaranteed to produce ________________

3. (17 points) Let

$$A = \begin{pmatrix} 9 & -3 \\ -3 & 5 \end{pmatrix}$$

(a) Compute the Choleski factorization of A ($A = LL^T$ with L lower triangular) and use it to solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (9, 9)^T$.

(b) Find an approximate solution to $A\mathbf{x} = \mathbf{b}$ by doing two Jacobi iterations starting at $\mathbf{x}^{(0)} = (1, 2)^T$.

(c) Find an approximate solution to $A\mathbf{x} = \mathbf{b}$ by doing two Gauss-Seidel iterations starting at $\mathbf{x}^{(0)} = (1, 2)^T$.

(d) Which method gives a better approximation to the exact solution ?

4. (14 points) Given the data points $(0, 2)$, $(0.5, 5)$, $(1, 4)$

(a) Find the quadratic polynomial $p_2(x)$ interpolating the data.

(b) Find the function $P(x) = a + b\cos(\pi x) + c\sin(\pi x)$, which interpolates the data.

5. (12 points) Given the data points $(-2, 2)$, $(-1, 1)$, $(0, 2)$, $(1, 2)$, find the function $g(x)$ of the form $g(x) = c_1|x| + c_2x^2$ which best fits this data in the sense of least squares.

6. (15 points) Let

$$I = \int_{-1}^{1} \frac{1}{x+4}\, dx = .5108256238$$

Compute approximations to I using
(a) The 4 panel trapezoid rule.
(b) The 4 panel Simpson's rule.
(c) The two point Gauss-Legendre rule. (Recall that the nodes for this are $\pm\frac{1}{\sqrt{3}}$.)
Which method gives the best result ?

7. (12 points)
   (a) What are the solutions α, if any, of the equation $x = \sqrt{3x-2}$?
   (b) Does the iteration $x_{n+1} = \sqrt{3x_n - 2}$ converge to any of these solutions (assuming x_0 is chosen sufficiently close to α)? Explain. (We need some analysis, not just numerical evidence.)

8. (15 points) Consider the initial value problem

$$\frac{dy}{dt} = ty^2, \qquad y(1) = 2.$$

(a) Verify that the solution is $Y(t) = \frac{2}{2-t^2}$.
Find approximations to $Y(1.2)$ by using
(b) two steps of the Euler method with $h = .1$.
(c) one step of the Improved Euler method with $h = .2$.
In (b) and (c) compare your answers with the exact solution.