AMSC/CMSC 460 FINAL EXAM H. GLAZ

TIME: 1:30–3:30 PM, MAY 21, 2003

Answer each question on one answer sheet, using the back if necessary (indicating on the front that your work continues on the back). Write your name, your instructor's name, and the question number on each answer sheet. Carefully **show all the steps in your solutions**, explaining your arguments in complete English sentences. Cross out any material that you do not wish graded. Where appropriate, circle or box your final answer. Good luck!

(1.1) (30 pts) Let

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 7 & 2 \\ 4 & 8 & 16 \end{pmatrix}.$$

Assuming partial pivoting, determine the matrices P, L, U with $A = PLU$ which would be computed in a Gaussian elimination algorithm. Solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = (2, 0, 12)^T$; indicate clearly the back substitution and forward elimination processes when computing the solution, and do not forget the permutation matrix.

(1.2) (25 pts) Determine whether the natural cubic spline that interpolates the table

x	0	1	2	3
y	1	1	0	10

is or is not the (piecewise cubic) function

$$f(x) = \begin{cases} 1 + x - x^3, & 0 \le x \le 1 \\ 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3, & 1 \le x \le 2 \\ 4(x-2) + 9(x-2)^2 - 3(x-2)^3, & 2 \le x \le 3 \end{cases}$$

If not, is $f(x)$ a cubic spline? And, does it interpolate the table?

(1.3) (25 pts) Let

$$I = \int_0^4 \exp(2x)dx.$$

Obtain approximate solutions for the value of I **(Note: it is not necessary to evaluate exponentials or actually perform the summations. Put away your calculators.)** by using the compound trapezoid rule with

a. (5 pts) 2 panels (equally spaced) .

b. (5 pts) 4 panels (equally spaced) .

c. (10 pts) Now, determine how many panels are required to insure that the quadrature error is no greater than 10^{-6}. Your answer should be an integer; also, it should be the smallest possible integer satisfying the tolerance condition. Your answer need not assume any information or ideas other than the estimate $|I - CT(h)| \le \frac{1}{12}M(b-a)h^2$, where M is the maximum over [a,b] of $f''(x)$, and CT(h) denotes compound trapezoid rule with panel size h.

d. (5 pts) Reduce the required number of panels by deriving a better error estimate. Do so by subdividing the interval and obtaining estimates as above over the subintervals.

(1.4) (20 pts) Consider the IVP

$$\begin{cases} y' = y^2 + ty \\ y(-1) = 3\,. \end{cases}$$

Approximate the value of $y(0)$ by using **two** (equal) steps of the Euler Method.

(2.1) (50 pts) Consider the system of nonlinear equations

$$\begin{cases} x^2 + y^2 = 4 \\ x^2 + \frac{1}{16}y^2 = 1\,, \end{cases}$$

and verify (e.g., a rough sketch) that there are four solutions.

a. For this part, ignore the specifics of the system above. Write a routine of the form [xout, numiter] = newt('F','DF',x0,tol) where the inputs F, DF represent the (2-vector) function of (x,y) to be solved, and it's Jacobian matrix (also a function of (x,y)); x0 is a (2-vector) initial guess, and 'tol' is an error tolerance. The outputs are the (2-vector) approximate solution xout and the number of iterations required, numiter.

b. Write a calling program and function subroutines F, DF appropriate for the stated example problem and x0 = (4,5).

c. Write a calling program which selects $N = 10,000$ initial guesses randomly from [-10,10] x [-10,10] and tabulates the output in a useful and/or interesting fashion.

(2.2) (50 pts)

(2.2) (pts) Consider using Newton's method to find a root of the equation $g(y) = 0$ where

$$g(y) = (y - \frac{1}{3})^2 - 1$$

and define the function $f(x)$ to be the result (i.e., $\lim_{n\to\infty} y_n$ where y_0, y_1 , ... are the iterates) of (analytically) converging the above Newton iteration with initial guess $y_0 = x$.

a. Write a program, using a generic quadrature routine in the usual form fint = quad('ff', a, b, tol), which will compute an approximation to

$$I = \int_0^1 f(x)dx$$

with an error no greater than 10^{-3}. Note: a user-defined function must be supplied for 'ff'.

b. (pts) What is the exact value of I?

(2.3) (50 pts) Let $f(x, y) = x^2 + y^2 - \frac{1}{4}$, and define $C = f^{-1}(0)$. Consider the following algorithm for constructing an approximation to C : let $\Omega =$ [-1,1] x [-1,1] and construct an appropriate collection of rays from the origin to the boundary of Ω. Along each section, use the bisection method to find a point on C. Plot the points.

a. Verify that the algorithm works. In particular, show that each use of the bisection method leads to a useful result.

b. Write pseudocode which implements the approach.

c. (Harder example) Explain the issues involved for the example on the board.