Computational Methods Summer 2021

HOMEWORK 16

Due Date: Monday, June 28

1. For an $m \times n$ matrix A, with $m > n$, the system $Ax = b$ is overdetermined for any $b \in \mathbb{R}^m$. The least squares solution satisfies the normal equations
$$A^T Ax = A^T b,$$
and has a unique solution if A has linearly independent columns. However the matrix $A^T A$ can be badly conditioned and solving this way is often unstable. Alternatively, we may factor $A = QR$ using QR factorization, where Q is $m \times m$ orthogonal and R is an $m \times n$ matrix of the form $R = \begin{bmatrix} \widehat{R} \\ 0 \end{bmatrix}$, where $\widehat{R}$ is an upper $n \times n$ upper triangular matrix. As shown in class, the least squares solution also satisfies the triangular system
$$\widehat{R}x = (Q^T b)_{1:n},$$
where the vector on the right only includes the first n components. Consider the problem
$$Ax = b, \quad A = \begin{bmatrix} 1 + 10^{-8} & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$
   (a) Using MATLAB's backslash command, find the solution to the normal equations $A^T Ax = A^T b$.
   (b) Using MATLAB's `qr` command and backslash, find the solution to the triangular system $\widehat{R}x = (Q^T b)_{1:n}$. Compare this to the solution to the normal equations. How far apart are the answers? Compute the distance in a norm of your choice.
   (c) Using MATLAB's `cond` command, what are the condition numbers of A, $A^T A$, and $\widehat{R}$? Which condition number should we worry about in double precision floating point arithmetic? Which computed answer is more accurate?

2. (Optional, not graded) Find the QR decomposition for
$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}$$
by hand using Gram-Schmidt orthogonalization.

   Note: There are two versions of QR, depending on the size of Q. To obtain a 3×3 matrix for Q as done in the lecture notes, add a third vector to the set of columns of A, that is linearly independent, say $(1, 0, 0)^T$. Then orthogonalize all three vectors using Gram-Schmidt. By construction the 3rd column of Q will be hit by the last row of zeros in R when multiplying QR, and so recomputing A is no problem. But this way we get the full orthonormal basis for $\mathbb{R}^3$ which allows us to get the orthogonal matrix Q.