vonPetersdorff Exam

Final Exam AMSC/CMSC 460 Fall 2011

You must show all your work to get credit. NO CALCULATORS ALLOWED.

1. (35 pts) Let $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

- **(a)** (10 pts) Consider the linear system $A = b$ with A from (a) and $b = [2.01, 1.99, 2.02]^\top$. We guess that $\hat{x} = [3, 4, 3]^\top$ is very close to x. Find an upper bound for $\|\hat{x} - x\|_1 / \|x\|_1$ if we know that $\|A^{-1}\|_1 \leq 2$. Use the residual. DO NOT SOLVE ANY LINEAR SYSTEM.
- **(b)** (10 pts) For the matrix A use Gaussian elimination without pivoting to find L, U.
- **(c)** (15 pts) We want to solve the nonlinear system

$$x_1^2 - x_2 = 1, \qquad -x_1 + x_2^2 - x_3 = 0, \qquad -x_2 + x_3^2 = 2$$

  using the Newton method with initial guess $x^{(0)} = [1, 1, 1]^\top$. Show that we have to solve a linear system with the matrix A. Use the LU decomposition from (b) to solve this linear system and find $x^{(1)}$.

2. (15 pts) Let $y = [0, 2, 2, 4]^\top$. The matrix A is given as $A = PS$ with

$$P = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the columns of P are orthogonal on each other. Use this to find $c \in \mathbb{R}^3$ so that $\|Ac - y\|_2$ is minimal. DO NOT FIND THE ENTRIES OF A. DO NOT USE THE NORMAL EQUATIONS FOR A.

3. (25 pts) Consider the function $f(x) = \frac{1}{x}$ on the interval $[\frac{1}{2}, 2]$.

- **(a)** (9 pts) We divide the interval into N subintervals of equal length. We approximate $f(x)$ by the piecewise linear function $p(x)$ using the values of f on the endpoints of the subintervals. Use the error formula to find an upper bound $\max_{x \in [\frac{1}{2}, 2]} |f(x) - p(x)| \leq CN^{-\alpha}$.
- **(b)** (8 pts) We are interested in the integral $I = \int_{\frac{1}{2}}^{2} f(x)dx$. We divide the interval $[\frac{1}{2}, 2]$ into the subintervals $[\frac{1}{2}, 1]$ and $[1, 2]$. Find the (simple) trapezoid rule approximations Q_1, Q_2 for the two subintervals and find $Q = Q_1 + Q_2$.
- **(c)** (8 pts) Use the error formula to find an upper bounds for the error $|I - Q|$ from (b). *Hint:* First find bounds for the errors $|I_1 - Q_1|$, $|I_2 - Q_2|$ where I_1, I_2 are the exact integrals over the two subintervals.

4. (25 pts) Consider the initial value problem

$$y'' + 2y' + 4y = t, \qquad y(1) = 2, \quad y'(1) = 3.$$

- **(a)** (10 pts) Perform one step of the Euler method with $h = 1$.
- **(b)** (10 pts) Perform one step of the improved Euler method with $h = 1$.
- **(c)** (5 pts) Assume that we want to find $y(t)$ for $t \in [1, 2]$. We pick a large integer N and use the Euler method with step size $h = 1/N$. Let $E_N^{(j)}$ denote the error after j steps. Then the errors satisfy $\left|E_N^{(1)}\right| \leq CN^{-\alpha}$ and $\left|E_N^{(N)}\right| \leq C'N^{-\beta}$. What are the values of α and β?