Spring, 2019 Name: ____________________

AMSC/CMSC 460 final exam

Instructor: Heyrim Cho

∘ Use exactly ONE answer sheet per question (use reverse side of the sheet if needed). ∘ Put your name and the question number on EACH page. ∘ Show enough work that I can follow your thinking. You must show all appropriate work in order to receive full credit for an answer. ∘ Copy the pledge and sign on your FIRST answer sheet only.

1. (*30pts*) Consider the initial value problem (IVP) $y' = 2y - t^2$ on $t \in [0, 2]$, with $y(0) = 1$.

   (a) Derive the 2nd-order Taylor's method with stepsize h to solve IVP.

   (b) Write the Runge-Kutta method corresponding to the Butcher table given on the right.

   $$\begin{array}{c|ccc} 0 & 0 & & \\ 1 & 1 & 0 & \\ 1/2 & 1/4 & 1/4 & 0 \\ \hline & 1/6 & 1/6 & 2/3 \end{array}$$

   (c) Compare the convergence of method (a) and (b).

2. (*30pts*) Consider a root finding problem $g(x) = -x^2 + 1 - \cos(2\pi x) = 0$.

   (a) Make an argument that $g(x) = 0$ has at least one root on interval $[\frac{1}{2}, \frac{5}{2}]$, and write one step of bisection method with initial interval $[\frac{1}{2}, \frac{5}{2}]$. How many steps of bisection method do you need to achieve accuracy 10^{-3}?

   (b) Compute one iteration of Newton's method with $x_0 = 1$.

   (c) Compare the convergence of bisection and Newton's method near each root $x = 0.8070$ and $x = 0$.

3. (*50pts*) Consider $f(x) = e^{-2x}$.

   (a) Define the least square (L^2) error between $f(x)$ and polynomial $P(x)$ on $[1, 3]$, then find the linear system solving the linear least square approximation $P_a(x)$ of $f(x)$ on $[1, 3]$.

   (b) Write the linear least square approximation $P_b(x)$ of $f(x)$ on $[1, 3]$ in terms of the orthogonal polynomials given as $Q_0(x) = 1$, $Q_1(x) = x - 2$, $Q_2(x) = \frac{1}{2}(3x^2 - 12x + 11)$, ...

   (c) Write the linear interpolation polynomial $P_c(x)$ of $f(x)$ on $[1, 3]$ at interpolation points $x_0 = 1$ and $x_1 = 3$, and estimate the interpolation error.

   (d) Denote the least square error defined in (a) as $E_2(f(x), P(x))$. Compare the magnitude of $E_2(f(x), P_a(x))$, $E_2(f(x), P_b(x))$, and $E_2(f(x), P_c(x))$.

   (e) Compute the LU decomposition of the matrix in (a) with row pivoting.

4. (*50pts*) (a) Find the most accurate difference formula in the following form and compute error term.

   $$f'(x) = Af(x) + Bf(x + h) + Cf(x + 2h)$$

   (b) Use Richardson extrapolation to find a higher-order method using the formula in (a).

   (c) Find the linear system solving the boundary value problem $y''(x) - 4y'(x) = 1$ and $y(0) = 1$, $y(1) = 0$ using $f''(x) = \frac{1}{h^2}(f(x+h) - 2f(x) + f(x-h))$ and $f'(x) = \frac{1}{h}(f(x+h) - f(x))$ with $h = 1/5$. Comment about the accuracy of this numerical approximation. What happens if you approximate y' with formula (a)?

   (d) Assume that it took 10^{-4} seconds with your laptop to find the LU decomposition of the 4×4 matrix from (c). How long will it take to compute the LU decomposition of BVP matrix with $h = 1/201$ (200×200)? How many BVP with different right-hand-side can you solve using the LU decomposition in ten seconds. (For matrix size n, operation count of LU decomposition: $\frac{2}{3}n^3$, backward substitution: n^2.)

5. (*40pts*) (a) Compute the integration rule in the following form

$$\int_0^1 f(x)dx = A\,f(0) + B\,f(\frac{2}{3}).$$

(b) Explain how one can achieve the highest degree of polynomial exactness in the following form $\int_0^1 f(x)dx = A\,f(x_0) + B\,f(x_1)$. (Gaussian quadrature)

(c) Derive an explicit IVP solver of $y' = f(t,y)$ with $y(0) = y_0$ using the integration rule in (a). (Hint. Integral formation of IVP)

(d) Represent $1/15 = (0.000100010001....)_2 = (0.\overline{0001})_2$ in the IEEE single format by rounding to the nearest, and estimate the relative rounding error.

	(in decimal)																							

6. Have a good summer break!