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AMSC/CMSC 460 Section 0201 (Fall 2018)

FINAL EXAM – QUIZ

Please remember: you have to hand in both your assignment and your quiz.

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1. (9 pts) In the setting of problem 1(a) of the take-home assignment, we consider $n = 3$ points with spacing $h = 1$. In such a case, the matrix and the right-hand side vector involved in the slopes computation (in the `splineslopes` function) become

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 4 & 1 \\ 0 & 2 & 4 \end{bmatrix}, \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 3(\delta_1+\delta_2) \\ 3(\delta_1+\delta_2) \\ 3(\delta_1+\delta_2) \end{bmatrix} = 3(\delta_1+\delta_2)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

   (a) Find the LU factorization of $\mathbf{A}$.

   (b) Use this factorization to solve $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   (c) Suppose you have a code to solve $\mathbf{Ax} = \mathbf{b}$ for any $\mathbf{b}$. How would you use the code to find $\mathbf{A}^{-1}$?

2. (9 pts) Suppose that you want to estimate

$$I := \int_{-1}^{1} \frac{1}{1+25x^2}\,dx$$

   using $n = 51$ equally spaced points (including the endpoints) on the interval $[-1, 1]$.

   (a) What problems would arise when using your program `NewtonCotes` for that purpose? You may want to consider the assembly of the system, its numerical solution, and the quality of the resulting interpolating polynomial.

   (b) How can be piecewise linear interpolation be used to obtain a more reliable approximation of I? What integration method do you obtain in doing so? Why is this more reliable than using your program `NewtonCotes`?

   (c) Find a bound for the error using the approximation strategy from (b).
   It may be useful to recall that the following theorem we discussed in class: if f has continuous derivatives up to order n in $[a, b]$, the points $a \le x_1 < \ldots < x_n \le b$ are different from each other and p_{n-1} is the interpolating polynomial of degree n through $(x_1, f(x_1)), \ldots (x_n, f(x_n))$, then for every $x \in [a, b]$ there is some $t \in [a, b]$ such that

$$f(x) - p_{n-1}(x) = \frac{1}{n!} f^{(n)}(t)\,(x - x_1)\ldots(x - x_n).$$

3. (9 pts) Consider the first four data points from the take-home exercise 3:

t	y
24.41	0.591
34.82	1.547
44.09	2.902
45.07	2.894

   Suppose we want to make a least squares fitting of these data using a *linear* model, $y(t) = \beta_1 t + \beta_2$.

   (a) Explain how you would compute the parameters $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$. Namely, if β is the least squares solution of the system $X\beta \approx y$, what are the matrix X and the right-hand side vector y? What quantity does such β minimize?

   (b) Describe an efficient algorithm to compute the QR factorization of the matrix X (you can use a geometric argument). **You don't need to find β nor compute the QR factorization of X.**

4. (9 pts) We consider the IVP

$$\begin{cases} \dot{y} = y^2 - 1 \\ y(0) = 0. \end{cases} \tag{1}$$

(a) We seek the solution to (1) in the interval $[0, 100]$ with a tolerance of 10^{-4}, and use two different Matlab solvers for that purpose (see figures below). Solver A requires 90 steps, while solver B takes 28 steps. What feature of the problem makes the solver B to perform so much better than solver A? Which solvers would you guess that are being used?

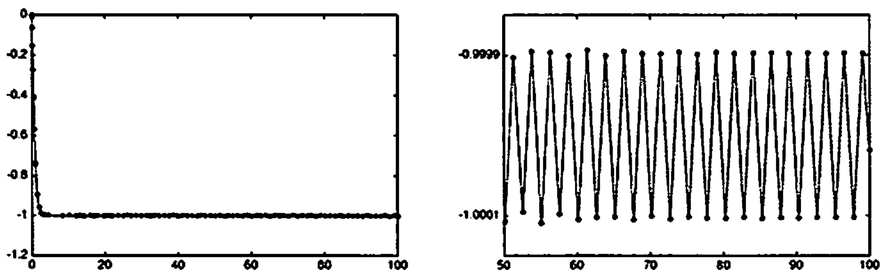

FIGURE 1. Result with Solver A. Left: plot of the solution in $[0, 100]$. Right: detail of the behavior for $50 \le t \le 100$.

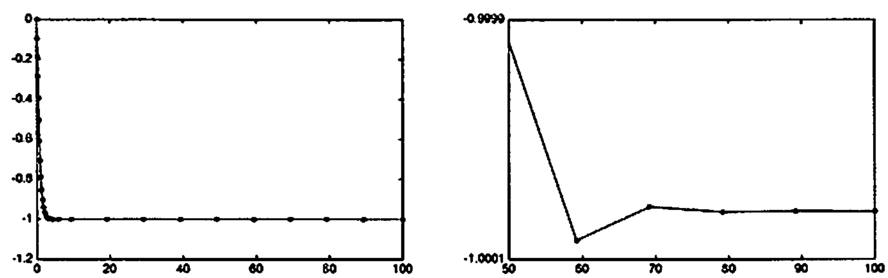

FIGURE 2. Result with Solver B. Left: plot of the solution in $[0, 100]$. Right: detail of the behavior for $50 \le t \le 100$.

(b) We solve IVP (1) using Euler's method and backward Euler's method with a fixed step-size h. Given $y_0 = 0$, write the equations needed to compute y_1 using both methods.

5. (9 pts) In exercise 4, parts (c) and (d), we have considered the finite difference approximation

$$y''(x_i) \simeq \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}.$$

Assuming that y is smooth enough, give a bound for the error

$$\left| y''(x_i) - \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \right|.$$