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AMSC/CMSC 460 Section 0201

Show all steps. No credit for solution only. Please circle your answer.

Final Exam May 16, 2016

Problem 1 (10 pts). Given the system of floating point number x with normalized representation

$$x = \pm(1.d_1d_2d_3d_4)_2 \cdot 2^{\pm(e_1e_2e_3e_4)_2}, \quad d_i, e_i = 0, 1.$$

(a) Find the floating point representation $x = \mathrm{fl}(x^*)$ of the binary number

$$x^* = 2^{-3} + 2^{-5} + 2^{-7} + 2^{-9}.$$

b) Find the absolute and relative errors in rounding x^*.

Problem 2 (25 pts). Given the LU decomposition of $A \in \mathcal{R}^3$, $PA = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) Find the solution of $Av = (6, 2, -2)^T$.
b) Compute $det(A)$ efficiently.
c) The matrix

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & -1 & 1 \\ 2 & -2 & 0 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 3/2 & -1 \\ -1/2 & 3/2 & -3/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}.$$

The ∞-norm of A is given by

$$\|A\|_\infty = \max_{i=1,2,3} \sum_{j=1,2,3} |a_{ij}|.$$

Find the condition number of A in ∞-norm.
d) Suppose x_* is the computed solution of $Ax = b$ obtained using Gaussian elimination with partial pivoting. Assume the computation is done with MATLAB, whose machine epsilon is 2×10^{-16}. Estimate the error $\|x - x_*\|_\infty$.

Problem 3 (20 pts). Suppose the value $\sin(0.3)$, $\sin(0.4)$ and $\sin(0.5)$ are known. Let $p(x)$ be the quadratic polynomials interpolating $\sin x$ at 0.3, 0.4 and 0.5.
a) Find an approximation to $\sin(0.45)$ by evaluating $p(x)$ at 0.45. Write the approximation in terms of $\sin(0.3)$, $\sin(0.4)$ and $\sin(0.5)$.
b) Find an bound for the interpolation error. Note that

$$\max_{s \in (0.3, 0.5)} |\cos(s)| \le 1.$$

Problem 4 (20 pts). Consider the initial value problem

$$y' = -y^3, \quad y(0) = a$$

for some constant $a > 0$.
(a) Write down the implicit Euler method for this problem, using stepsize $h = 1$.
(b) To solve for the first step of implicit Euler method, we note that y_1 is defined implicitly in terms of y_0. Write a Newton iteration for solving y_1.

Problem 5 (20 pts). a) Given the set of data point, find the normal equation of the least square problem for fitting the data with a linear function $q(x) = a + b(\frac{x-1950}{10})$ with unknown coefficients a, b. Notice the affine transformation $\frac{x-1950}{10}$ in the definition of $q(x)$. DO NOT solve for the problem.

1930	1940	1950	1960	1970
1	2	5	8	10

b) Consider the following matrix $A \in \mathcal{R}^{3\times 2}$, its QR decomposition $A = QR$, and a vector $b \in \mathcal{R}^3$:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 4 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = QR, \quad b = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}.$$

Find the least squares solution of $Ax = b$ using the QR factorization of A.

Problem 6 (30 pts).
(a) Let $I = [-r, r]$ and $x_0 = 0$, $x_1 = \sqrt{\frac{3}{5}}r$ and $x_2 = -\sqrt{\frac{3}{5}}r$. Determine the weights ω_0, ω_1 and ω_2 for the following integral rule to be exact for quadratic polynomials

$$I(f; -r, r) = \int_{-r}^{r} f(x)dx \approx Q(f; -r, r) = \omega_0 f(x_0) + \omega_1 f(x_1) + \omega_2 f(x_2)$$

(b) Suppose on interval $[a, b]$, the error of quadrature rule

$$|I - Q| \le \frac{1}{6!}\left(\frac{2}{5}\right)^2 \max_{s\in(a,b)} \left|f^{(6)}(s)\right| (b-a)^7.$$

Let $a = x_0 < x_1 < \cdots < x_{n+1} = b$ be a uniform partition of interval $[a, b]$. Show an error estimate for the composite rule in terms of n.

Problem 7 (25 pts). The (explicit) Euler method and the implicit Euler method with step-size $h = 0.1$ form z to 8 are used to approximate an initial value problem. The following MATLAB script performs this computation

```
h = 0.1;
x = zeros(80,1);
yEE = zeros(80,1);
yIE = zeros(80,1);
yEE(1) = 1;
yIE(1) = 1;
for n = 2:80
x(n) = x(n-1) + h;
yEE(n) = yEE(n-1)*(1-x(n-1)2̂*h);
yIE(n) = yIE(n-1)/(1+h*x(n)2̂);
end
plot(x, yEE, x, yIE)
```

(a) What is the initial value problem (both ODE and initial condition) being approximated

(b) The plot shows that one method produces oscillations and the other does not. Which method does not produce oscillations? Explain the oscillation of the other method in terms of stability for each method.

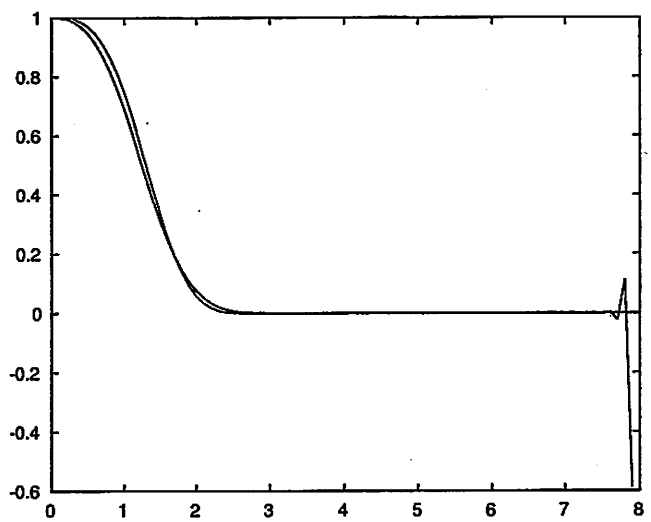

Figure 1: Approximate solution of explicit and implicit Euler method