AMSC/CMSC 460 Dr. Wolfe FINAL EXAM July 26, 2002

1. (10 points) Suppose you have a computer which carries only 4 decimal digits and rounds.
   (a) Find a number ϵ which is about as large as possible such that on this computer $\cos x = 1$ for $|x| < \epsilon$.
   (b) For values of x with $0 < |x| < \epsilon$ what is a good way of computing $f(x) = \frac{1-\cos x}{x^2}$ on this computer ?

2. (15 points) Let

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

   (a) Solve $A\mathbf{x} = \mathbf{b}$, using the LU factorization, forward elimination and back substitution.
   (b) Find an approximate solution to $A\mathbf{x} = \mathbf{b}$ by doing three Jacobi iterations starting at $\mathbf{x}^{(0)} = (0,0)^T$.
   (c) Find an approximate solution to $A\mathbf{x} = \mathbf{b}$ by doing three Gauss-Seidel iterations starting at $\mathbf{x}^{(0)} = (0,0)^T$.

3. (15 points) Let

$$I = \int_0^1 \sqrt{x+2}\, dx = 1.578483532$$

   (a) Compute T_2, the 2-panel trapezoid rule approximation to I. Compare your answer with the exact value of I.
   (b) Compute S_2, the 2-panel Simpson's rule appoximation to I. Compare your answer with the exact value of I.
   (c) Compute CT_2, the 2-panel corrected trapeziod rule approximation to I. Compare your answer with the exact value of I.
   (d) Which method gives the best result ?

4. (12 points) Let

$$Q = \frac{1}{7}\begin{pmatrix} -6 & 2 & 3 \\ 2 & -3 & 6 \\ 3 & 6 & 2 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

   (a) Complete the following definition:
   An $n \times n$ matrix Q is *orthogonal* if ______________
   (b) Show that Q is an orthogonal matrix.
   (c) Let $A = QR$. Find the least squares solution to $A\mathbf{x} = \mathbf{b}$ <u>as efficiently as possible</u>. (Recall that an orthogonal matrix preserves lengths: $\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x}$.)

5. (15 points) The equation $4x^2 - e^x = 0$ has a root in the interval $[4,5]$.
   (a) Show that we <u>cannot</u> find this root by using the iteration scheme

$$x_{n+1} = \frac{1}{2}e^{x_n/2}.$$

(b) Find an iteration scheme which will converge to the root. Prove that your scheme converges.

6. (15 points) Consider the nonlinear system

$$x^2 + y^2 = 1, \quad y = x^3.$$

(a) How many solutions does this system have ? (It might help to draw a picture.)
(b) We wish to find a solution to the system by Newton's method. If $(x_0, y_0) = (1, 1)$ what is (x_1, y_1) ? Do not reduce to a single equation.

7. (5 points) Consider the initial value problem for the second order differential equation

$$\frac{d^2u}{dt^2} + 2\frac{du}{dt} + u^2 - 3u = 0, \quad u(0) = 1, \quad \frac{du}{dt}(0) = -1. \tag{1}$$

Write (1) as an equivalent first order system.

8. (15 points) Let $Y(t)$ be the solution of the initial value problem

$$\frac{dy}{dx} = x + y + 1 \qquad y(0) = 0.$$

Find approximations to $Y(0.2)$ by using
(a) two steps of the Euler method with $h = .1$.
(b) one step of the Improved Euler method with $h = .2$.
For comparison, the exact value is $Y(0.2) = .24281$. Which method gives the best results ?