CLP 3

MULTIVARIABLE CALCULUS

FELDMAN RECHNITZER YEAGER

CLP-3 Multivariable Calculus

Joel Feldman Andrew Rechnitzer Elyse Yeager

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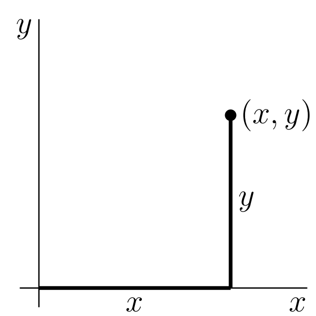
Chapter 1

Vectors and Geometry in Two and Three Dimensions

Before we get started doing calculus in two and three dimensions we need to brush up on some basic geometry, that we will use a lot. We are already familiar with the Cartesian plane[1], but we'll start from the beginning.

1.1 Points

Each point in two dimensions may be labeled by two coordinates[2] (x, y) which specify the position of the point in some units with respect to some axes as in the figure below.



The set of all points in two dimensions is denoted[3] $\mathbb{R}^2$. Observe that

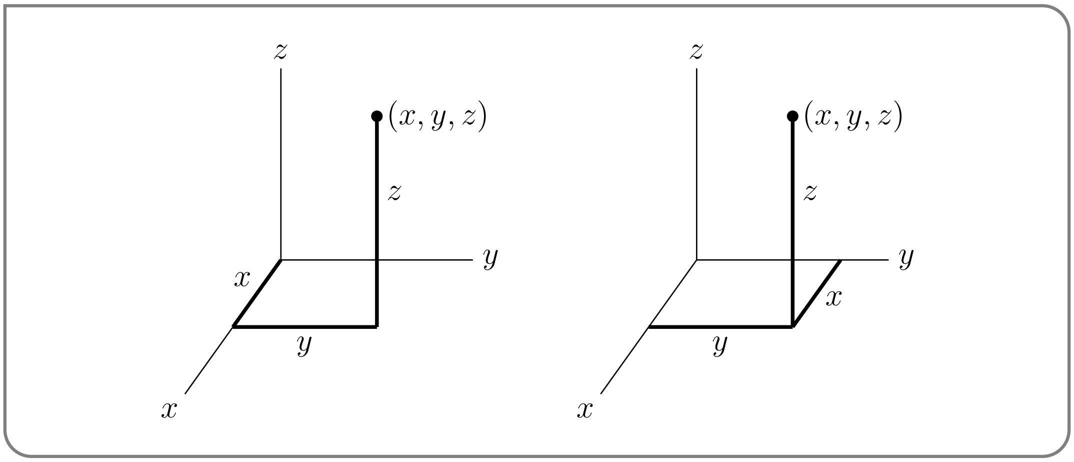
1 René Descartes (1596–1650) was a French scientist and philosopher, who lived in the Dutch Republic for roughly twenty years after serving in the (mercenary) Dutch States Army. He is viewed as the father of analytic geometry, which uses numbers to study geometry.

2 This is why the xy-plane is called "two dimensional" — the name of each point consists of two real numbers.

3 Not surprisingly, the 2 in $\mathbb{R}^2$ signifies that each point is labelled by two numbers and the $\mathbb{R}$ in $\mathbb{R}^2$ signifies that the numbers in question are real numbers. There are more advanced applications (for example in signal analysis and in quantum mechanics) where complex numbers are used. The space of all pairs (z_1, z_2), with z_1 and z_2 complex numbers is denoted $\mathbb{C}^2$.

- the distance from the point (x, y) to the x-axis is $|y|$
- if $y > 0$, then (x, y) is above the x-axis and if $y < 0$, then (x, y) is below the x-axis
- the distance from the point (x, y) to the y-axis is $|x|$
- if $x > 0$, then (x, y) is the right of the y-axis and if $x < 0$, then (x, y) is to the left of the y-axis
- the distance from the point (x, y) to the origin $(0, 0)$ is $\sqrt{x^2 + y^2}$

Similarly, each point in three dimensions may be labeled by three coordinates (x, y, z), as in the two figures below.

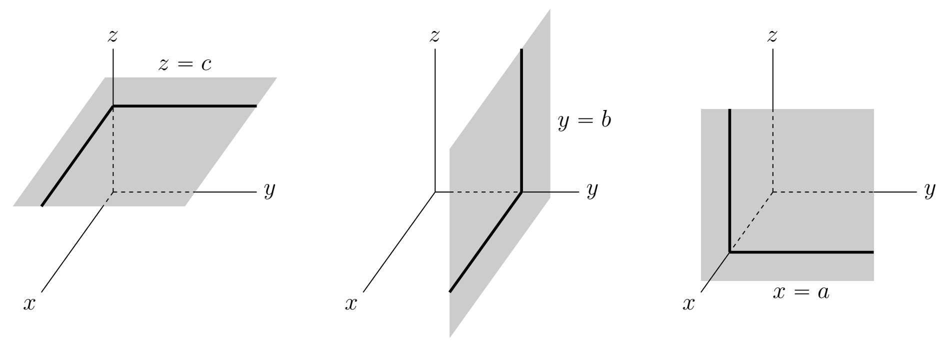


The set of all points in three dimensions is denoted $\mathbb{R}^3$. The plane that contains, for example, the x- and y-axes is called the xy-plane.

- The xy-plane is the set of all points (x, y, z) that satisfy $z = 0$.
- The xz-plane is the set of all points (x, y, z) that satisfy $y = 0$.
- The yz-plane is the set of all points (x, y, z) that satisfy $x = 0$.

More generally,

- The set of all points (x, y, z) that obey $z = c$ is a plane that is parallel to the xy-plane and is a distance $|c|$ from it. If $c > 0$, the plane $z = c$ is above the xy-plane. If $c < 0$, the plane $z = c$ is below the xy-plane. We say that the plane $z = c$ is a signed distance c from the xy-plane.
- The set of all points (x, y, z) that obey $y = b$ is a plane that is parallel to the xz-plane and is a signed distance b from it.
- The set of all points (x, y, z) that obey $x = a$ is a plane that is parallel to the yz-plane and is a signed distance a from it.

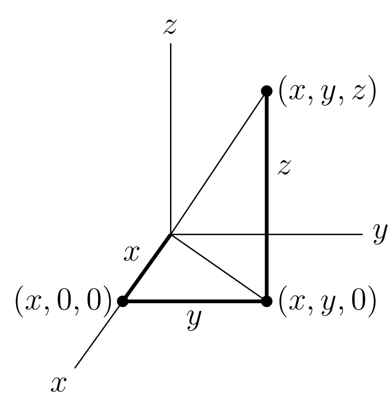


Observe that our 2d distances extend quite easily to 3d.

- the distance from the point (x, y, z) to the xy-plane is $|z|$
- the distance from the point (x, y, z) to the xz-plane is $|y|$
- the distance from the point (x, y, z) to the yz-plane is $|x|$
- the distance from the point (x, y, z) to the origin $(0, 0, 0)$ is $\sqrt{x^2 + y^2 + z^2}$

To see that the distance from the point (x, y, z) to the origin $(0, 0, 0)$ is indeed $\sqrt{x^2 + y^2 + z^2}$,

- apply Pythagoras to the right-angled triangle with vertices $(0, 0, 0)$, $(x, 0, 0)$ and $(x, y, 0)$ to see that the distance from $(0, 0, 0)$ to $(x, y, 0)$ is $\sqrt{x^2 + y^2}$ and then
- apply Pythagoras to the right-angled triangle with vertices $(0, 0, 0)$, $(x, y, 0)$ and (x, y, z) to see that the distance from $(0, 0, 0)$ to (x, y, z) is $\sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$.



More generally, the distance from the point (x, y, z) to the point (x', y', z') is

$$\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

Notice that this gives us the equation for a sphere quite directly. All the points on a sphere are equidistant from the centre of the sphere. So, for example, the equation of the sphere centered on $(1, 2, 3)$ with radius 4, that is, the set of all points (x, y, z) whose distance from $(1, 2, 3)$ is 4, is

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 16$$

Here is an example in which we sketch a region in the xy-plane that is specified using inequalities.

Example 1.1.1

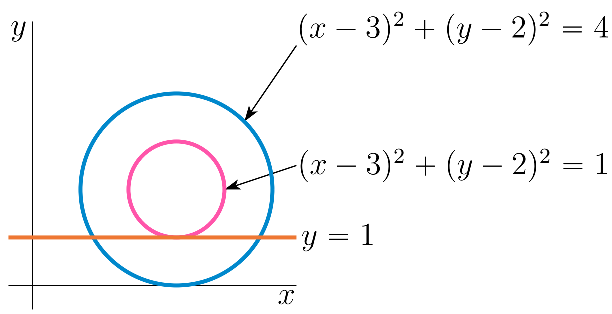
In this example, we sketch the region

$$\{\ (x,y)\ \big|\ -12 \leqslant x^2-6x+y^2-4y \leqslant -9,\ \ y \geqslant 1\ \}$$

in the xy-plane.

We do so in two steps. In the first step, we sketch the curves $x^2-6x+y^2-4y=-12$, $x^2-6x+y^2-4y=-9$, and $y=1$.

- By completing squares, we see that the equation $x^2-6x+y^2-4y=-12$ is equivalent to $(x-3)^2+(y-2)^2=1$, which is the circle of radius 1 centred on $(3,2)$. It is sketched in the figure below.
- By completing squares, we see that the equation $x^2-6x+y^2-4y=-9$ is equivalent to $(x-3)^2+(y-2)^2=4$, which is the circle of radius 2 centred on $(3,2)$. It is sketched in the figure below.
- The point (x,y) obeys $y=1$ if and only if it is a distance 1 vertically above the x-axis. So $y=1$ is the line that is parallel to the x-axis and is one unit above it. This line is also sketched in the figure below.



In the second step we determine the impact that the inequalities have.

- The inequality $x^2-6x+y^2-4y \geqslant -12$ is equivalent to $(x-3)^2+(y-2)^2 \geqslant 1$ and hence is equivalent to $\sqrt{(x-3)^2+(y-2)^2} \geqslant 1$. So the point (x,y) satisfies $x^2-6x+y^2-4y \geqslant -12$ if and only if the distance from (x,y) to $(3,2)$ is at least 1, i.e. if and only if (x,y) is outside (or on) the circle $(x-3)^2+(y-2)^2=1$.
- The inequality $x^2-6x+y^2-4y \leqslant -9$ is equivalent to $(x-3)^2+(y-2)^2 \leqslant 4$ and hence is equivalent to $\sqrt{(x-3)^2+(y-2)^2} \leqslant 2$. So the point (x,y) satisfies the inequality $x^2-6x+y^2-4y \leqslant -9$ if and only if the distance from (x,y) to $(3,2)$ is at most 2, i.e. if and only if (x,y) is inside (or on) the circle $(x-3)^2+(y-2)^2=4$.
- The point (x,y) obeys $y \geqslant 1$ if and only if (x,y) is a vertical distance at least 1 above the x-axis, i.e. is above (or on) the line $y=1$.

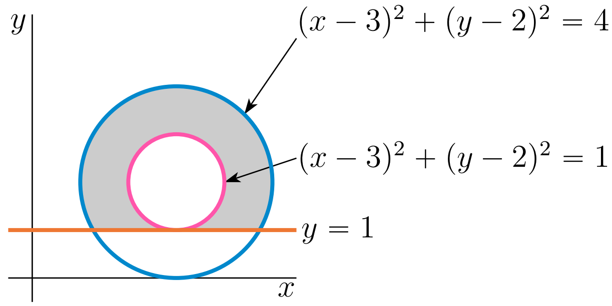
- So the region

$$\left\{ (x,y) \;\middle|\; -12 \leqslant x^2 - 6x + y^2 - 4y \leqslant -9, \;\; y \geqslant 1 \right\}$$

consists of all points (x,y) that

- are inside or on the circle $(x-3)^2 + (y-2)^2 = 4$ and
- are also outside or on the circle $(x-3)^2 + (y-2)^2 = 1$ and
- are also above or on the line $y = 1$.

It is the shaded region in the figure below.



Example 1.1.1

Here are a couple of examples that involve spheres.

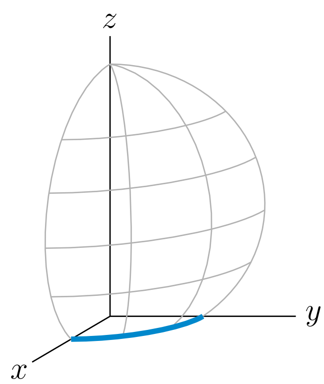
Example 1.1.2

In this example, we are going to find the curve formed by the intersection of the xy-plane and the sphere of radius 5 centred on $(0,0,4)$.

The point (x,y,z) lies on the xy-plane if and only if $z = 0$, and lies on the sphere of radius 5 centred on $(0,0,4)$ if and only if $x^2 + y^2 + (z-4)^2 = 25$. So the point (x,y,z) lies on the curve of intersection if and only if both $z = 0$ and $x^2 + y^2 + (z-4)^2 = 25$, or equivalently

$$z = 0, \quad x^2 + y^2 + (0-4)^2 = 25 \iff z = 0, \quad x^2 + y^2 = 9$$

This is the circle in the xy-plane that is centred on the origin and has radius 3. Here is a sketch that show the parts of the sphere and the circle of intersection that are in the first octant. That is, that have $x \geqslant 0$, $y \geqslant 0$ and $z \geqslant 0$.



Example 1.1.2

Example 1.1.3

In this example, we are going to find all points (x, y, z) for which the distance from (x, y, z) to $(9, -12, 15)$ is twice the distance from (x, y, z) to the origin $(0, 0, 0)$.

The distance from (x, y, z) to $(9, -12, 15)$ is $\sqrt{(x-9)^2 + (y+12)^2 + (z-15)^2}$. The distance from (x, y, z) to $(0, 0, 0)$ is $\sqrt{x^2 + y^2 + z^2}$. So we want to find all points (x, y, z) for which

$$\sqrt{(x-9)^2 + (y+12)^2 + (z-15)^2} = 2\sqrt{x^2 + y^2 + z^2}$$

Squaring both sides of this equation gives

$$x^2 - 18x + 81 + y^2 + 24y + 144 + z^2 - 30z + 225 = 4\big(x^2 + y^2 + z^2\big)$$

Collecting up terms gives

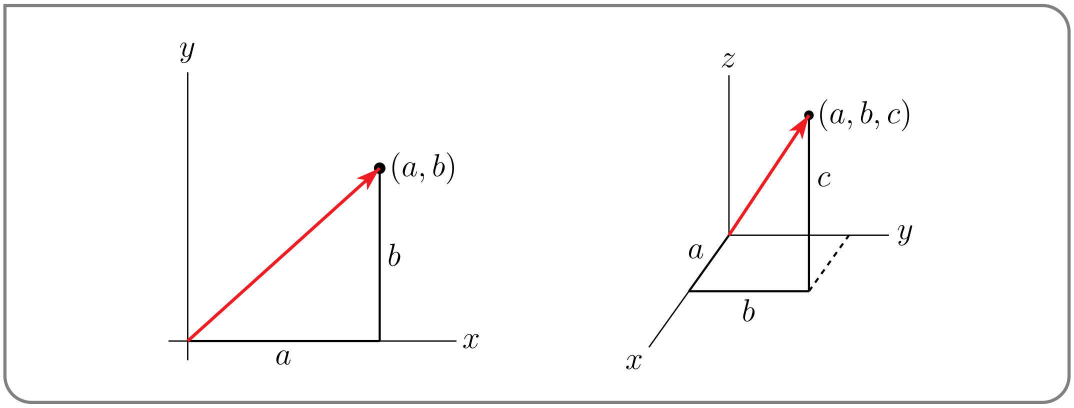
$$\begin{aligned}
3x^2 + 18x + 3y^2 - 24y + 3z^2 + 30z &= 450 && \text{and then, dividing by 3,}\\
x^2 + 6x + y^2 - 8y + z^2 + 10z &= 150 && \text{and then, completing squares,}\\
x^2 + 6x + 9 + y^2 - 8y + 16 + z^2 + 10z + 25 &= 200 && \text{or}\\
(x+3)^2 + (y-4)^2 + (z+5)^2 &= 200
\end{aligned}$$

This is the sphere of radius $10\sqrt{2}$ centred on $(-3, 4, -5)$.

Example 1.1.3

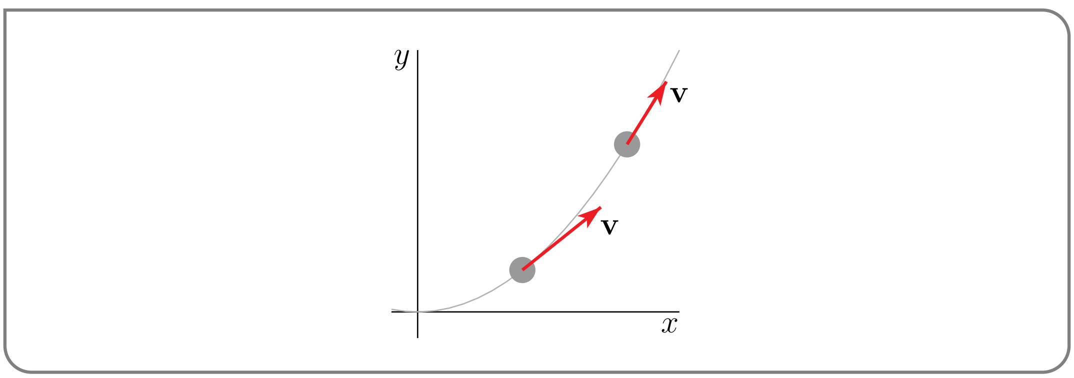
1.2▲ Vectors

In many of our applications in 2d and 3d, we will encounter quantities that have both a magnitude (like a distance) and also a direction. Such quantities are called vectors. That is, a *vector* is a quantity which has both a direction and a magnitude, like a velocity. If you are moving, the magnitude (length) of your velocity vector is your speed (distance travelled per unit time) and the direction of your velocity vector is your direction of motion. To specify a vector in three dimensions you have to give three components, just as for a point. To draw the vector with components $a,\ b,\ c$ you can draw an arrow from the point $(0,0,0)$ to the point (a,b,c). Similarly, to specify a vector in two dimensions you have to

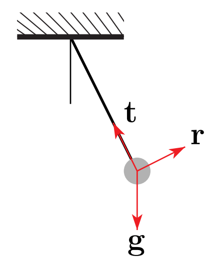


give two components and to draw the vector with components $a,\ b$ you can draw an arrow from the point $(0,0)$ to the point (a,b).

There are many situations in which it is preferable to draw a vector with its tail at some point other than the origin. For example, it is natural to draw the velocity vector of a moving particle with the tail of the velocity vector at the position of the particle, whether or not the particle is at the origin. The sketch below shows a moving particle and its velocity vector at two different times.

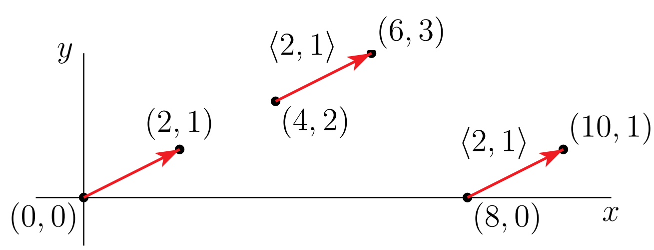


As a second example, suppose that you are analyzing the motion of a pendulum. There are three forces acting on the pendulum bob: gravity **g**, which is pulling the bob straight down, tension **t** in the rod, which is pulling the bob in the direction of the rod, and air resistance **r**, which is pulling the bob in a direction opposite to its direction of motion. All three forces are acting on the bob. So it is natural to draw all three arrows representing the forces with their tails at the bob.



In this text, we will used bold faced letters, like **v**, **t**, **g**, to designate vectors. In handwriting, it is clearer to use a small overhead arrow[4], as in $\vec{v}$, $\vec{t}$, $\vec{g}$, instead. Also, when we want to emphasise that some quantity is a number, rather than a vector, we will call the number a *scalar*.

Both points and vectors in 2d are specified by two numbers. Until you get used to this, it might confuse you sometimes — does a given pair of numbers represent a point or a vector? To distinguish[5] between the components of a vector and the coordinates of the point at its head, when its tail is at some point other than the origin, we shall use angle brackets rather than round brackets around the components of a vector. For example, the figure below shows the two-dimensional vector $\langle 2, 1\rangle$ drawn in three different positions. In each case, when the tail is at the point (u, v) the head is at $(2 + u, 1 + v)$. We warn you that, out in the real world[6], no one uses notation that distinguishes between components of a vector and the coordinates of its head — usually round brackets are used for both. It is up to you to keep straight which is being referred to.



By way of summary,

Notation 1.2.1.

we use

- bold faced letters, like **v**, **t**, **g**, to designate vectors, and
- angle brackets, like $\langle 2, 1\rangle$, around the components of a vector, but use
- round brackets, like $(2, 1)$, around the coordinates of a point, and use
- "scalar" to emphasise that some quantity is a number, rather than a vector.

4 Some people use an underline, as in $\underline{v}$, rather than an arrow.
5 Or, in the Wikipedia jargon, disambiguate.
6 OK. OK. Out in that (admittedly very small) part of the real world that actually knows what a vector is.

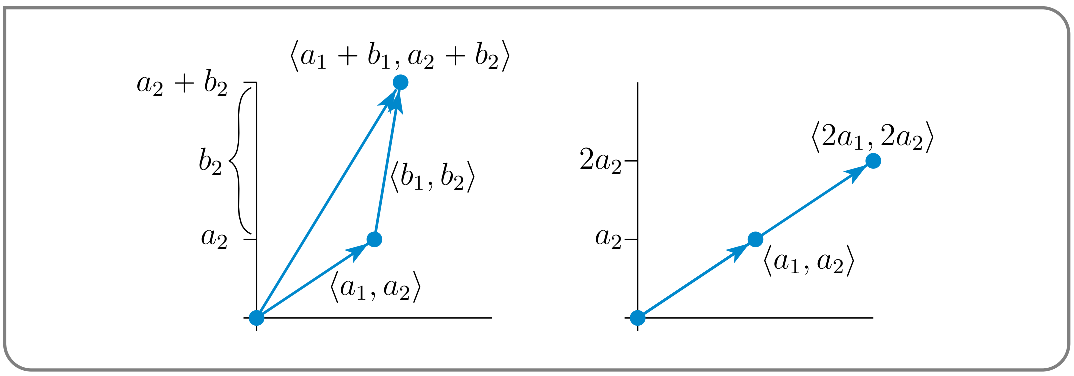
1.2.1 ▸▸ Addition of Vectors and Multiplication of a Vector by a Scalar

Just as we have done many times in the CLP texts, when we define a new type of object, we want to understand how it interacts with the basic operations of addition and multiplication. Vectors are no different, and we shall shortly see a natural way to define addition of vectors. Multiplication will be more subtle, and we shall start with multiplication of a vector by a number (rather than with multiplication of a vector by another vector).

By way of motivation for the definitions of addition and multiplication by a number, imagine that we are out for a walk on the xy-plane.

- Suppose that we take a step and, in doing so, we move a_1 units parallel to the x-axis and a_2 units parallel to the y-axis. Then we say that $\langle a_1, a_2\rangle$ is the displacement vector for the step. Suppose now that we take a second step which moves us an additional b_1 units parallel to the x-axis and an additional b_2 units parallel to the y-axis, as in the figure on the left below. So the displacement vector for the second step is $\langle b_1, b_2\rangle$. All together, we have moved $a_1 + b_1$ units parallel to the x-axis and $a_2 + b_2$ units parallel to the y-axis. The displacement vector for the two steps combined is $\langle a_1 + b_1, a_2 + b_2\rangle$. We shall define the sum of $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$, denoted by $\langle a_1, a_2\rangle + \langle b_1, b_2\rangle$, to be $\langle a_1 + b_1, a_2 + b_2\rangle$.

- Suppose now that, instead, we decide to step in the same direction as the first step above, but to move twice as far, as in the figure on the right below. That is, our step will move us $2a_1$ units in the direction of the x-axis and $2a_2$ units in the direction of the y-axis and the corresponding displacement vector will be $\langle 2a_1, 2a_2\rangle$. We shall define the product of the number 2 and the vector $\langle a_1, a_2\rangle$, denoted by $2\,\langle a_1, a_2\rangle$, to be $\langle 2a_1, 2a_2\rangle$.



Here are the formal definitions.

Definition 1.2.2 (Adding Vectors and Multiplying a Vector by a Number)**.**

These two operations have the obvious definitions

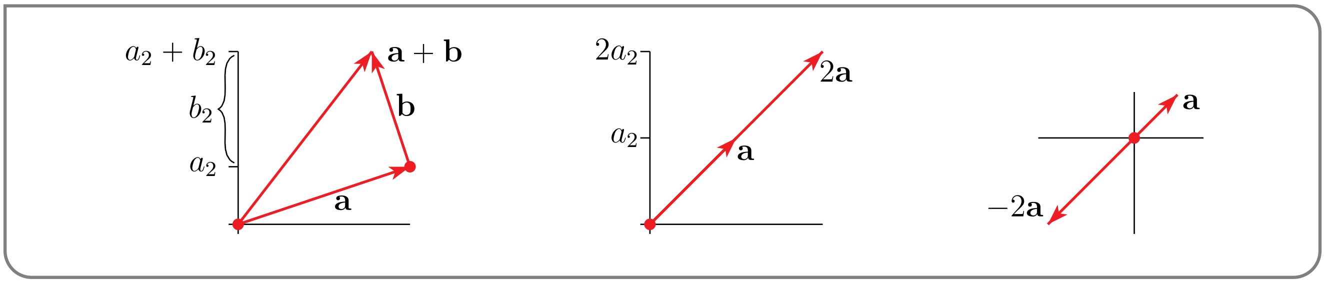
$$\begin{aligned} \mathbf{a} = \langle a_1, a_2\rangle\,,\ \mathbf{b} = \langle b_1, b_2\rangle \qquad &\Longrightarrow \qquad \mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle \\ \mathbf{a} = \langle a_1, a_2\rangle\,,\ s \text{ a number} \qquad &\Longrightarrow \qquad s\mathbf{a} = \langle sa_1, sa_2\rangle \end{aligned}$$

and similarly in three dimensions.

Pictorially, you add the vector **b** to the vector **a** by drawing **b** with its tail at the head of **a** and then drawing a vector from the tail of **a** to the head of **b**, as in the figure on the left below. For a number s, we can draw the vector s**a**, by just

- changing the vector **a**'s length by the factor $|s|$, and,
- if $s < 0$, reversing the arrow's direction,

as in the other two figures below.

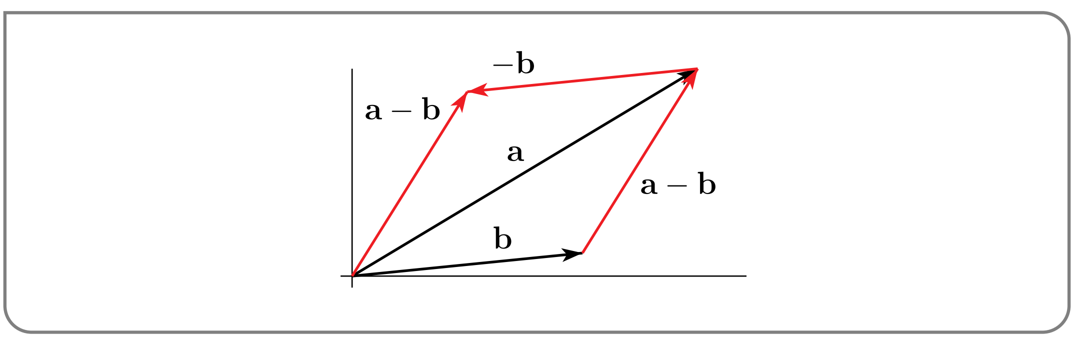


The special case of multiplication by $s = -1$ appears so frequently that $(-1)\mathbf{a}$ is given the shorter notation $-\mathbf{a}$. That is,

$$-\langle a_1, a_2 \rangle = \langle -a_1, -a_2 \rangle$$

Of course $\mathbf{a} + (-\mathbf{a})$ is $\mathbf{0}$, the vector all of whose components are zero.

To subtract **b** from **a** pictorially, you may add $-\mathbf{b}$ (which is drawn by reversing the direction of **b**) to **a**. Alternatively, if you draw **a** and **b** with their tails at a common point, then $\mathbf{a} - \mathbf{b}$ is the vector from the head of **b** to the head of **a**. That is, $\mathbf{a} - \mathbf{b}$ is the vector you must add to **b** in order to get **a**.



The operations of addition and multiplication by a scalar that we have just defined are quite natural and rarely cause any problems, because they inherit from the real numbers the properties of addition and multiplication that you are used to.

Theorem 1.2.3 (Properties of Addition and Scalar Multiplication).

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors and s and t be scalars. Then

(1)	$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$	(2)	$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$
(3)	$\mathbf{a}+\mathbf{0}=\mathbf{a}$	(4)	$\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
(5)	$s(\mathbf{a}+\mathbf{b})=s\mathbf{a}+s\mathbf{b}$	(6)	$(s+t)\mathbf{a}=s\mathbf{a}+t\mathbf{a}$
(7)	$(st)\mathbf{a}=s(t\mathbf{a})$	(8)	$1\mathbf{a}=\mathbf{a}$

We have just been introduced to many definitions. Let's see some of them in action.

Example 1.2.4

For example, if

$$\mathbf{a}=\langle 1,2,3\rangle \qquad \mathbf{b}=\langle 3,2,1\rangle \qquad \mathbf{c}=\langle 1,0,1\rangle$$

then

$$\begin{aligned} 2\mathbf{a} &= 2\,\langle 1,2,3\rangle &&= \langle 2,4,6\rangle \\ -\mathbf{b} &= -\,\langle 3,2,1\rangle &&= \langle -3,-2,-1\rangle \\ 3\mathbf{c} &= 3\,\langle 1,0,1\rangle &&= \langle 3,0,3\rangle \end{aligned}$$

and

$$\begin{aligned} 2\mathbf{a}-\mathbf{b}+3\mathbf{c} &= \langle 2,4,6\rangle + \langle -3,-2,-1\rangle + \langle 3,0,3\rangle \\ &= \langle 2-3+3\,,\,4-2+0\,,\,6-1+3\rangle \\ &= \langle 2,2,8\rangle \end{aligned}$$

Example 1.2.4

Definition 1.2.5.

Two vectors $\mathbf{a}$ and $\mathbf{b}$

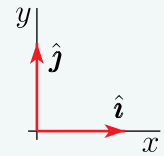
- are said to be parallel if $\mathbf{a}=s\,\mathbf{b}$ for some nonzero real number s and
- are said to have the same direction if $\mathbf{a}=s\,\mathbf{b}$ for some number $s>0$.

There are some vectors that occur sufficiently commonly that they are given special names. One is the vector $\mathbf{0}$. Some others are the "standard basis vectors".

Definition 1.2.6.

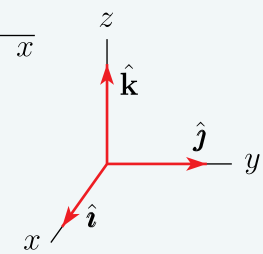
(a) The standard basis vectors in two dimensions are

$$\hat{\imath} = \langle 1, 0 \rangle \qquad \hat{\jmath} = \langle 0, 1 \rangle$$



(b) The standard basis vectors in three dimensions are

$$\hat{\imath} = \langle 1, 0, 0 \rangle \qquad \hat{\jmath} = \langle 0, 1, 0 \rangle \qquad \hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$$



We'll explain the little hats in the notation $\hat{\imath}$, $\hat{\jmath}$, $\hat{\mathbf{k}}$ shortly. Some people rename $\hat{\imath}$, $\hat{\jmath}$ and $\hat{\mathbf{k}}$ to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ respectively. Using the above properties we have, for all vectors,

$$\langle a_1, a_2 \rangle = a_1 \hat{\imath} + a_2 \hat{\jmath} \qquad \langle a_1, a_2, a_3 \rangle = a_1 \hat{\imath} + a_2 \hat{\jmath} + a_3 \hat{\mathbf{k}}$$

A sum of numbers times vectors, like $a_1\hat{\imath} + a_2\hat{\jmath}$ is called a linear combination of the vectors. Thus all vectors can be expressed as linear combinations of the standard basis vectors. This makes basis vectors very helpful in computations. The standard basis vectors are unit vectors, meaning that they are of length one, where the length of a vector $\mathbf{a}$ is denoted[7] $|\mathbf{a}|$ and is defined by

Definition 1.2.7 (Length of a Vector).

$$\mathbf{a} = \langle a_1, a_2 \rangle \implies |\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$
$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \implies |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

A unit vector is a vector of length one. We'll sometimes use the accent ˆ to emphasise that the vector $\hat{\mathbf{a}}$ is a unit vector. That is, $|\hat{\mathbf{a}}| = 1$.

Example 1.2.8

Recall that multiplying a vector $\mathbf{a}$ by a positive number s, changes the length of the vector by a factor s without changing the direction of the vector. So (assuming that $|\mathbf{a}| \neq 0$) $\frac{\mathbf{a}}{|\mathbf{a}|}$ is a unit vector that has the same direction as $\mathbf{a}$. For example, $\frac{\langle 1,1,1 \rangle}{\sqrt{3}}$ is a unit vector that points in the same direction as $\langle 1, 1, 1 \rangle$.

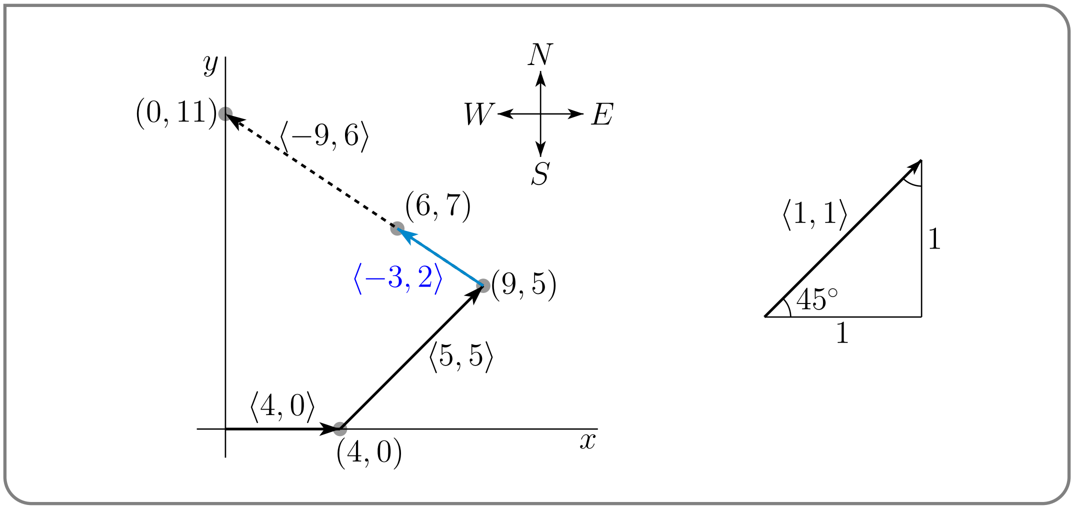
Example 1.2.8

7 The notation $\|\mathbf{a}\|$ is also used for the length of $\mathbf{a}$.

Example 1.2.9

We go for a walk on a flat Earth. We use a coordinate system with the positive x-axis pointing due east and the positive y-axis pointing due north. We

- start at the origin and
- walk due east for 4 units and then
- walk northeast for $5\sqrt{2}$ units and then
- head towards the point $(0, 11)$, but we only go
- one third of the way.



We will now use vectors to figure out our final location.

- On the first leg of our walk, we go 4 units in the positive x-direction. So our displacement vector — the vector whose tail is at our starting point and whose head is at the end point of the first leg — is $\langle 4, 0\rangle$. As we started at $(0, 0)$ we finish the first leg of the walk at $(4, 0)$.
- On the second leg of our walk, our direction of motion is northeast, i.e. is $45°$ above the direction of the positive x-axis. Looking at the figure on the right above, we see that our displacement vector, for the second leg of the walk, has to be in the same direction as the vector $\langle 1, 1\rangle$. So our displacement vector is the vector of length $5\sqrt{2}$ with the same direction as $\langle 1, 1\rangle$. The vector $\langle 1, 1\rangle$ has length $\sqrt{1^2 + 1^2} = \sqrt{2}$ and so $\frac{\langle 1,1\rangle}{\sqrt{2}}$ has length one and our displacement vector is

$$5\sqrt{2}\,\frac{\langle 1,1\rangle}{\sqrt{2}} = 5\,\langle 1,1\rangle = \langle 5,5\rangle$$

  If we draw this displacement vector, $\langle 5, 5\rangle$ with its tail at $(4, 0)$, the starting point of the second leg of the walk, then its head will be at $(4 + 5, 0 + 5) = (9, 5)$ and that is the end point of the second leg of the walk.
- On the final leg of our walk, we start at $(9, 5)$ and walk towards $(0, 11)$. The vector from $(9, 5)$ to $(0, 11)$ is $\langle 0 - 9\,,\, 11 - 5\rangle = \langle -9, 6\rangle$. As we go only one third of the way, our final displacement vector is

$$\frac{1}{3}\,\langle -9,6\rangle = \langle -3,2\rangle$$

If we draw this displacement vector with its tail at $(9, 5)$, the starting point of the final leg, then its head will be at $(9 - 3, 5 + 2) = (6, 7)$ and that is the end point of the final leg of the walk, and our final location.

Example 1.2.9

1.2.2 ▸▸ The Dot Product

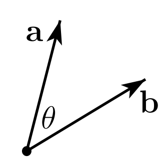
Let's get back to the arithmetic operations of addition and multiplication. We will be using both scalars and vectors. So, for each operation there are three possibilities that we need to explore:

- "scalar plus scalar", "scalar plus vector" and "vector plus vector"
- "scalar times scalar", "scalar times vector" and "vector times vector"

We have been using "scalar plus scalar" and "scalar times scalar" since childhood. "vector plus vector" and "scalar times vector" were just defined above. There is no sensible way to define "scalar plus vector", so we won't. This leaves "vector times vector". There are actually two widely used such products. The first is the *dot product*, which is the topic of this section, and which is used to easily determine the angle θ (or more precisely, $\cos\theta$) between two vectors. We'll get to the second, the cross product, later.

Here is preview of what we will do in this dot product subsection §1.2.2. We are going to give two formulae for the dot product, $\mathbf{a} \cdot \mathbf{b}$, of the pair of vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

- The first formula is $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. We will take it as our official definition of $\mathbf{a} \cdot \mathbf{b}$. This formula provides us with an easy way to compute dot products.
- The second formula is $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.



We will show, in Theorem 1.2.11 below, that this second formula always gives the same answer as the first formula. The second formula provides us with an easy way to determine the angle between two vectors. In particular, it provides us with an easy way to test whether or not two vectors are perpendicular to each other. For example, the vectors $\langle 1, 2, 3 \rangle$ and $\langle -1, -1, 1 \rangle$ have dot product

$$\langle 1, 2, 3 \rangle \cdot \langle -1, -1, 1 \rangle = 1 \times (-1) + 2 \times (-1) + 3 \times 1 = 0$$

This tell us as the angle θ between the two vectors obeys $\cos\theta = 0$, so that $\theta = \frac{\pi}{2}$. That is, the two vectors are perpendicular to each other.

After we give our official definition of the dot product in Definition 1.2.10, and give the important properties of the dot product, including the formula $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, in Theorem 1.2.11, we'll give some examples. Finally, to see the dot product in action, we'll define what it means to project one vector on another vector and give an example.

Definition 1.2.10 (Dot Product)**.**

The dot product of the vectors **a** and **b** is denoted $\mathbf{a}\cdot\mathbf{b}$ and is defined by

$$\begin{aligned}
&\mathbf{a} = \langle a_1, a_2\rangle, & \mathbf{b} &= \langle b_1, b_2\rangle & \implies \quad &\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2\\
&\mathbf{a} = \langle a_1, a_2, a_3\rangle, & \mathbf{b} &= \langle b_1, b_2, b_3\rangle & \implies \quad &\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}$$

in two and three dimensions respectively.

The properties of the dot product are as follows:

Theorem 1.2.11 (Properties of the Dot Product)**.**

Let **a**, **b** and **c** be vectors and let s be a scalar. Then

(0) $\mathbf{a},\mathbf{b}$ are vectors and $\mathbf{a}\cdot\mathbf{b}$ is a scalar

(1) $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

(2) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

(3) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}, \quad (\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}$

(4) $(s\mathbf{a})\cdot\mathbf{b} = s(\mathbf{a}\cdot\mathbf{b})$

(5) $\mathbf{0}\cdot\mathbf{a} = 0$

(6) $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$ where θ is the angle between **a** and **b**

(7) $\mathbf{a}\cdot\mathbf{b} = 0 \iff \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}\perp\mathbf{b}$

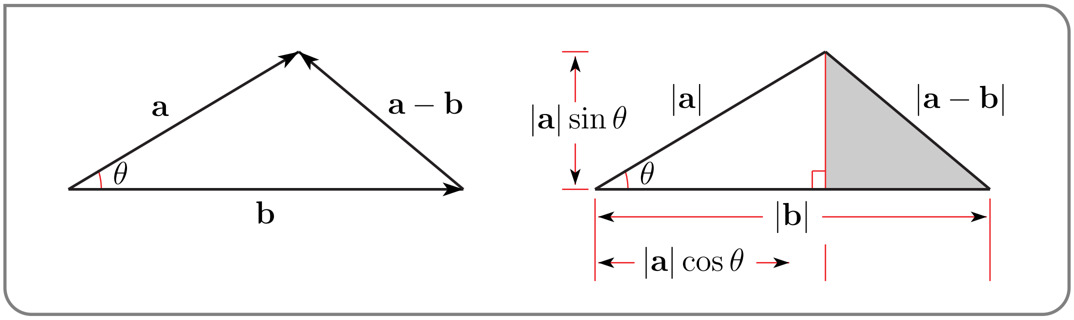
Proof. Properties 0 through 5 are almost immediate consequences of the definition. For example, for property 3 (which is called the distributive law) in dimension 2,

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \langle a_1, a_2\rangle\cdot\langle b_1+c_1, b_2+c_2\rangle\\
&= a_1(b_1+c_1) + a_2(b_2+c_2) = a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2\\
\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} &= \langle a_1, a_2\rangle\cdot\langle b_1, b_2\rangle + \langle a_1, a_2\rangle\cdot\langle c_1, c_2\rangle\\
&= a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2
\end{aligned}$$

Property 6 is sufficiently important that it is often used as the definition of dot product. It is not at all an obvious consequence of the definition. To verify it, we just write $|\mathbf{a}-\mathbf{b}|^2$ in two different ways. The first expresses $|\mathbf{a}-\mathbf{b}|^2$ in terms of $\mathbf{a}\cdot\mathbf{b}$. It is

$$\begin{aligned}
|\mathbf{a}-\mathbf{b}|^2 &\overset{1}{=} (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\
&\overset{3}{=} \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\\
&\overset{1,2}{=} |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}
\end{aligned}$$

Here, $\stackrel{1}{=}$, for example, means that the equality is a consequence of property 1. The second way we write $|\mathbf{a}-\mathbf{b}|^2$ involves $\cos\theta$ and follows from the cosine law for triangles. Just in case you don't remember the cosine law, we'll derive it right now! Start by applying Pythagoras to the shaded triangle in the right hand figure of



That triangle is a right triangle whose hypotenuse has length $|\mathbf{a}-\mathbf{b}|$ and whose other two sides have lengths $\big(|\mathbf{b}|-|\mathbf{a}|\cos\theta\big)$ and $|\mathbf{a}|\sin\theta$. So Pythagoras gives

$$\begin{aligned}|\mathbf{a}-\mathbf{b}|^2 &= \big(|\mathbf{b}|-|\mathbf{a}|\cos\theta\big)^2+\big(|\mathbf{a}|\sin\theta\big)^2\\ &= |\mathbf{b}|^2-2|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta+|\mathbf{a}|^2\cos^2\theta+|\mathbf{a}|^2\sin^2\theta\\ &= |\mathbf{b}|^2-2|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta+|\mathbf{a}|^2\end{aligned}$$

This is precisely the cosine law[8]. Observe that, when $\theta=\frac{\pi}{2}$, this reduces to, (surprise!) Pythagoras' theorem.

Setting our two expressions for $|\mathbf{a}-\mathbf{b}|^2$ equal to each other,

$$|\mathbf{a}-\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}=|\mathbf{b}|^2-2|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta+|\mathbf{a}|^2$$

cancelling the $|\mathbf{a}|^2$ and $|\mathbf{b}|^2$ common to both sides

$$-2\mathbf{a}\cdot\mathbf{b}=-2|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$$

and dividing by -2 gives

$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$$

which is exactly property 6.

Property 7 follows directly from property 6. First note that the dot product $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$ is zero if and only if at least one of the three factors $|\mathbf{a}|$, $|\mathbf{b}|$, $\cos\theta$ is zero. The first factor is zero if and only if $\mathbf{a}=\mathbf{0}$. The second factor is zero if and only if $\mathbf{b}=\mathbf{0}$. The third factor is zero if and only if $\theta=\pm\frac{\pi}{2}+2k\pi$, for some integer k, which in turn is true if and only if $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular. $\square$

Because of Property 7 of Theorem 1.2.11, the dot product can be used to test whether or not two vectors are perpendicular to each other. That is, whether or not the angle between

8 You may be used to seeing it written as $c^2=a^2+b^2-2ab\cos C$, where a, b and c are the lengths of the three sides of the triangle and C is the angle opposite the side of length c

the two vectors is 90°. Another name[9] for "perpendicular" is "orthogonal". Testing for orthogonality is one of the main uses of the dot product.

Example 1.2.12

Consider the three vectors

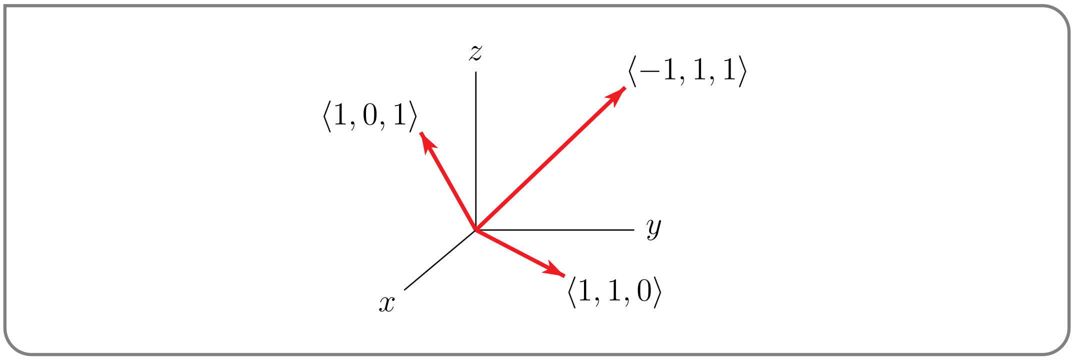
$$\mathbf{a} = \langle 1,1,0\rangle \qquad \mathbf{b} = \langle 1,0,1\rangle \qquad \mathbf{c} = \langle -1,1,1\rangle$$

Their dot products

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \langle 1,1,0\rangle\cdot\langle 1,0,1\rangle &&= 1\times 1 + 1\times 0 + 0\times 1 &&= 1\\
\mathbf{a}\cdot\mathbf{c} &= \langle 1,1,0\rangle\cdot\langle -1,1,1\rangle &&= 1\times(-1) + 1\times 1 + 0\times 1 &&= 0\\
\mathbf{b}\cdot\mathbf{c} &= \langle 1,0,1\rangle\cdot\langle -1,1,1\rangle &&= 1\times(-1) + 0\times 1 + 1\times 1 &&= 0
\end{aligned}$$

tell us that $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Since both $|\mathbf{a}| = |\mathbf{b}| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$ the first dot product tells us that the angle, θ, between $\mathbf{a}$ and $\mathbf{b}$ obeys

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$



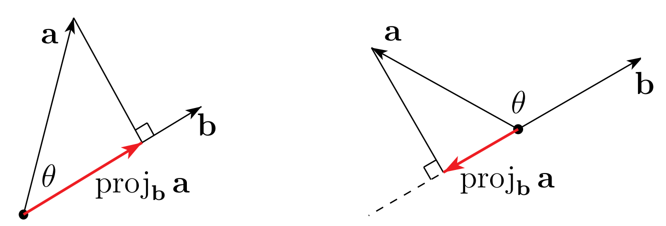
Example 1.2.12

Dot products are also used to compute projections. First, here's the definition.

9 The concepts of the dot product and perpendicularity have been generalized a lot in mathematics (for example, from 2d and 3d vectors to functions). The generalization of the dot product is called the "inner product" and the generalization of perpendicularity is called "orthogonality".

Definition 1.2.13 (Projection).

Draw two vectors, **a** and **b**, with their tails at a common point and drop a perpendicular from the head of **a** to the line that passes through both the head and tail of **b**. By definition, the projection of the vector **a** on the vector **b** is the vector from the tail of **b** to the point on the line where the perpendicular hits.



Think of the projection of **a** on **b** as the part of **a** that is in the direction of **b**.

Now let's develop a formula for the projection of **a** on **b**. Denote by θ the angle between **a** and **b**. If $|\theta|$ is no more than $90°$, as in the figure on the left above, the length of the projection of **a** on **b** is $|\mathbf{a}|\cos\theta$. By Property 6 of Theorem 1.2.11, $|\mathbf{a}|\cos\theta = \mathbf{a}\cdot\mathbf{b}/|\mathbf{b}|$, so the projection is a vector whose length is $\mathbf{a}\cdot\mathbf{b}/|\mathbf{b}|$ and whose direction is given by the unit vector $\mathbf{b}/|\mathbf{b}|$. Hence

$$\text{projection of } \mathbf{a} \text{ on } \mathbf{b} = \mathrm{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}$$

If $|\theta|$ is larger than $90°$, as in the figure on the right above, the projection has length $|\mathbf{a}|\,\cos(\pi-\theta) = -|\mathbf{a}|\cos\theta = -\mathbf{a}\cdot\mathbf{b}/|\mathbf{b}|$ and direction $-\mathbf{b}/|\mathbf{b}|$. In this case

$$\mathrm{proj}_{\mathbf{b}}\,\mathbf{a} = -\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\frac{-\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}$$

So the formula

Equation 1.2.14.

$$\mathrm{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b}$$

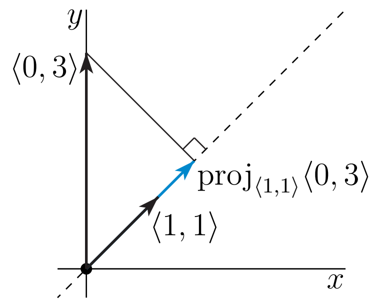
is applicable whenever $\mathbf{b}\neq\mathbf{0}$. We may rewrite $\mathrm{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\frac{\mathbf{b}}{|\mathbf{b}|}$. The coefficient, $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$, of the unit vector $\frac{\mathbf{b}}{|\mathbf{b}|}$, is called the component of **a** in the direction **b**. As a special case, if **b** happens to be a unit vector, which, for emphasis, we'll now write has $\hat{\mathbf{b}}$, the projection formula simplifies to

Equation 1.2.15.

$$\mathrm{proj}_{\hat{\mathbf{b}}}\,\mathbf{a} = (\mathbf{a}\cdot\hat{\mathbf{b}})\,\hat{\mathbf{b}}$$

Example 1.2.16

In this example, we will find the projection of the vector $\langle 0,3\rangle$ on the vector $\langle 1,1\rangle$, as in the figure



By Equation 1.2.14 with $\mathbf{a} = \langle 0,3\rangle$ and $\mathbf{b} = \langle 1,1\rangle$, that projection is

$$\begin{aligned}\text{proj}_{\langle 1,1\rangle}\ \langle 0,3\rangle &= \frac{\langle 0,3\rangle\cdot\langle 1,1\rangle}{|\,\langle 1,1\rangle\,|^2}\ \langle 1,1\rangle \\ &= \frac{0\times 1+3\times 1}{1^2+1^2}\ \langle 1,1\rangle = \left\langle \frac{3}{2},\frac{3}{2}\right\rangle\end{aligned}$$

Example 1.2.16

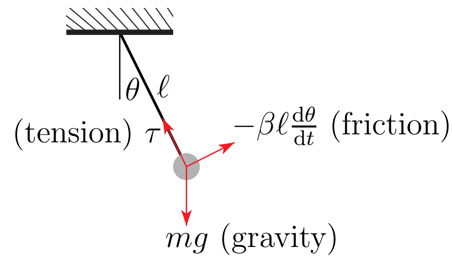
One use of projections is to "resolve forces". There is an example in the next (optional) section.

1.2.3 ▸▸ (Optional) Using Dot Products to Resolve Forces — The Pendulum

Model a pendulum by a mass m that is connected to a hinge by an idealized rod that is massless and of fixed length ℓ. Denote by θ the angle between the rod and vertical. The forces acting on the mass are

- gravity, which has magnitude mg and direction $\langle 0,-1\rangle$,
- tension in the rod, whose magnitude $\tau(t)$ automatically adjusts itself so that the distance between the mass and the hinge is fixed at ℓ (so that the rod does not stretch or contract) and whose direction is always parallel to the rod,
- and possibly some frictional forces, like friction in the hinge and air resistance. Assume that the total frictional force has magnitude proportional[10] to the speed of the mass and has direction opposite to the direction of motion of the mass. We'll call the constant of proportionality β.

10 The behaviour of air resistance (sometimes called drag) is pretty complicated. We're using a reasonable low speed approximation. At high speeds drag is typically proportional to the square of the speed.



If we use a coordinate system centered on the hinge, the (x, y) coordinates of the mass at time t are

$$\begin{aligned} x(t) &= \ell \sin\theta(t) \\ y(t) &= -\ell\cos\theta(t) \end{aligned}$$

where $\theta(t)$ is the angle between the rod and vertical at time t. We are now going to use Newton's law of motion

$$\text{mass} \times \text{acceleration} = \text{total applied force}$$

to determine now θ evolves in time. By definition, the velocity and acceleration vectors[11] for the position vector $\langle x(t), y(t)\rangle$ are

$$\begin{aligned} \frac{d}{dt}\langle x(t), y(t)\rangle &= \left\langle \frac{dx}{dt}(t), \frac{dy}{dt}(t)\right\rangle \\ \frac{d^2}{dt^2}\langle x(t), y(t)\rangle &= \left\langle \frac{d^2x}{dt^2}(t), \frac{d^2y}{dt^2}(t)\right\rangle \end{aligned}$$

So, the velocity and acceleration vectors of our mass are

$$\begin{aligned} \mathbf{v}(t) = \frac{d}{dt}\langle x(t), y(t)\rangle \ &= \left\langle \ell\frac{d}{dt}\sin\theta(t), -\ell\frac{d}{dt}\cos\theta(t)\right\rangle \\ &= \left\langle \ell\cos\theta(t)\,\frac{d\theta}{dt}(t)\,,\ \ell\sin\theta(t)\,\frac{d\theta}{dt}(t)\right\rangle \\ &= \ell\,\frac{d\theta}{dt}(t)\ \langle\cos\theta(t), \sin\theta(t)\rangle \end{aligned}$$

$$\begin{aligned} \mathbf{a}(t) = \frac{d^2}{dt^2}\langle x(t), y(t)\rangle &= \frac{d}{dt}\left\{\ell\,\frac{d\theta}{dt}(t)\ \langle\cos\theta(t), \sin\theta(t)\rangle\right\} \\ &= \ell\,\frac{d^2\theta}{dt^2}(t)\ \langle\cos\theta(t), \sin\theta(t)\rangle + \ell\,\frac{d\theta}{dt}(t)\left\langle\frac{d}{dt}\cos\theta(t), \frac{d}{dt}\sin\theta(t)\right\rangle \\ &= \ell\,\frac{d^2\theta}{dt^2}(t)\,\langle\cos\theta(t), \sin\theta(t)\rangle + \ell\Big(\frac{d\theta}{dt}(t)\Big)^2\langle -\sin\theta(t), \cos\theta(t)\rangle \end{aligned}$$

11 For a more comprehensive treatment of derivatives of vector valued functions $\mathbf{r}(t)$, and in particular of velocity and acceleration, see Section 1.6 in this text and Section 1.1 in the CLP-4 text.

The negative of the velocity vector is $-\ell\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\langle\cos\theta,\sin\theta\rangle$, so the total frictional force is

$$-\beta\ell\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\langle\cos\theta,\sin\theta\rangle$$

with β our constant of proportionality.

The vector

$$\tau(t)\,\langle-\sin\theta(t),\cos\theta(t)\rangle$$

has magnitude $\tau(t)$ and direction parallel to the rod pointing from the mass towards the hinge and so is the force due to tension in the rod.

Hence, for this physical system, Newton's law of motion is

$$\begin{aligned}&\overbrace{m\ell\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\,\langle\cos\theta,\sin\theta\rangle+m\ell\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\langle-\sin\theta,\cos\theta\rangle}^{\text{mass}\times\text{acceleration}}\\ &\qquad=\overbrace{mg\,\langle0,-1\rangle}^{\text{gravity}}+\overbrace{\tau\,\langle-\sin\theta,\cos\theta\rangle}^{\text{tension}}-\overbrace{\beta\ell\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\langle\cos\theta,\sin\theta\rangle}^{\text{friction}}\end{aligned}\tag{$*$}$$

This is a rather complicated looking equation. Writing out its x- and y-components doesn't help. They also look complicated. Instead, the equation can be considerably simplified (and consequently better understood) by "taking its components parallel to and perpendicular to the direction of motion". From the velocity vector $\mathbf{v}(t)$, we see that $\langle\cos\theta(t),\sin\theta(t)\rangle$ is a unit vector parallel to the direction of motion at time t. Recall, from (1.2.15), that the projection of any vector $\mathbf{b}$ on any unit vector $\hat{\mathbf{d}}$ (with the "hat" on $\hat{\mathbf{d}}$ reminding ourselves that the vector is a unit vector) is

$$\big(\mathbf{b}\cdot\hat{\mathbf{d}}\big)\,\hat{\mathbf{d}}$$

The coefficient $\mathbf{b}\cdot\hat{\mathbf{d}}$ is, by definition, the component of $\mathbf{b}$ in the direction $\hat{\mathbf{d}}$. So, by dotting both sides of the equation of motion $(*)$ with $\hat{\mathbf{d}}=\langle\cos\theta(t),\sin\theta(t)\rangle$, we extract the component parallel to the direction of motion. Since

$$\begin{aligned}\langle\cos\theta,\sin\theta\rangle\cdot\langle\cos\theta,\sin\theta\rangle&=1\\ \langle\cos\theta,\sin\theta\rangle\cdot\langle-\sin\theta,\cos\theta\rangle&=0\\ \langle\cos\theta,\sin\theta\rangle\cdot\langle0,-1\rangle&=-\sin\theta\end{aligned}$$

this gives

$$m\ell\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=-mg\sin\theta-\beta\ell\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

which is *much* cleaner than $(*)$! When θ is small, we can approximate $\sin\theta\approx\theta$ and get the equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\frac{\beta}{m}\frac{\mathrm{d}\theta}{\mathrm{d}t}+\frac{g}{\ell}\theta=0$$

which is easily solved. There are systematic procedures for finding the solution, but we'll just guess.

When there is no friction (so that $\beta = 0$), we would expect the pendulum to just oscillate. So it is natural to guess

$$\theta(t) = A\sin(\omega t - \delta)$$

which is an oscillation with (unknown) amplitude A, frequency ω (radians per unit time) and phase δ. Substituting this guess into the left hand side, $\theta'' + \frac{g}{\ell}\theta$, yields

$$-A\omega^2\sin(\omega t - \delta) + A\tfrac{g}{\ell}\sin(\omega t - \delta)$$

which is zero if $\omega = \sqrt{g/\ell}$. So $\theta(t) = A\sin(\omega t - \delta)$ is a solution for any amplitude A and phase δ, provided the frequency $\omega = \sqrt{g/\ell}$.

When there is some, but not too much, friction, so that $\beta > 0$ is relatively small, we would expect "oscillation with decaying amplitude". So we guess

$$\theta(t) = Ae^{-\gamma t}\sin(\omega t - \delta)$$

for some constant decay rate γ, to be determined. With this guess,

$$\begin{aligned}
\theta(t) &= \qquad\qquad Ae^{-\gamma t}\sin(\omega t - \delta)\\
\theta'(t) &= \qquad -\gamma Ae^{-\gamma t}\sin(\omega t - \delta) + \quad \omega Ae^{-\gamma t}\cos(\omega t - \delta)\\
\theta''(t) &= (\gamma^2 - \omega^2)Ae^{-\gamma t}\sin(\omega t - \delta) - 2\gamma\omega Ae^{-\gamma t}\cos(\omega t - \delta)
\end{aligned}$$

and the left hand side

$$\begin{aligned}
&\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\beta}{m}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{g}{\ell}\theta\\
&\qquad = \left[\gamma^2 - \omega^2 - \frac{\beta}{m}\gamma + \frac{g}{\ell}\right]Ae^{-\gamma t}\sin(\omega t - \delta) + \left[-2\gamma\omega + \frac{\beta}{m}\omega\right]Ae^{-\gamma t}\cos(\omega t - \delta)
\end{aligned}$$

vanishes if $\gamma^2 - \omega^2 - \frac{\beta}{m}\gamma + \frac{g}{\ell} = 0$ and $-2\gamma\omega + \frac{\beta}{m}\omega = 0$. The second equation tells us the decay rate $\gamma = \frac{\beta}{2m}$ and then the first tells us the frequency

$$\omega = \sqrt{\gamma^2 - \tfrac{\beta}{m}\gamma + \tfrac{g}{\ell}} = \sqrt{\tfrac{g}{\ell} - \tfrac{\beta^2}{4m^2}}$$

When there is a lot of friction (namely when $\frac{\beta^2}{4m^2} > \frac{g}{\ell}$, so that the frequency ω is not a real number), we would expect damping without oscillation and so would guess $\theta(t) = Ae^{-\gamma t}$. You can determine the allowed values of γ by substituting this guess in.

To extract the components perpendicular to the direction of motion, we dot with $\langle -\sin\theta, \cos\theta\rangle$ rather than $\langle \cos\theta, \sin\theta\rangle$. Note that, because $\langle -\sin\theta, \cos\theta\rangle \cdot \langle \cos\theta, \sin\theta\rangle = 0$, the vector $\langle -\sin\theta, \cos\theta\rangle$ really is perpendicular to the direction of motion. Since

$$\begin{aligned}
\langle -\sin\theta, \cos\theta\rangle \cdot \langle \cos\theta, \sin\theta\rangle &= 0\\
\langle -\sin\theta, \cos\theta\rangle \cdot \langle -\sin\theta, cos\theta\rangle &= 1\\
\langle -\sin\theta, \cos\theta\rangle \cdot \langle 0, -1\rangle &= -\cos\theta
\end{aligned}$$

dotting both sides of the equation of motion $(*)$ with $\langle -\sin\theta, \cos\theta\rangle$ gives

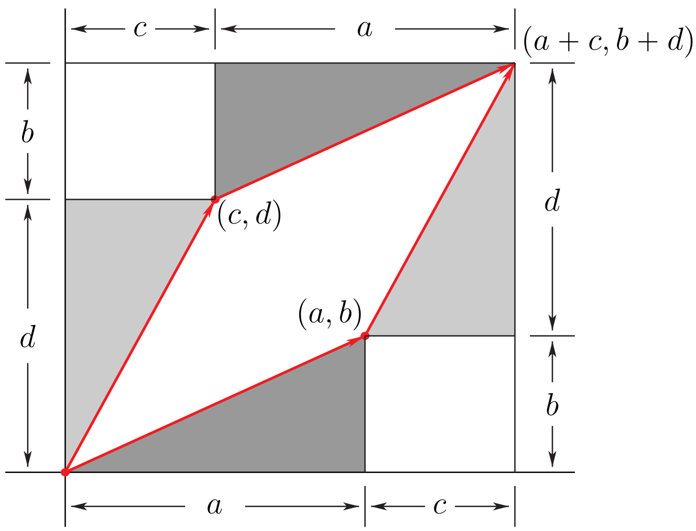
$$m\ell\Big(\frac{\mathrm{d}\theta}{\mathrm{d}t}\Big)^2 = -mg\cos\theta + \tau$$

This equation just determines the tension $\tau = m\ell\big(\frac{\mathrm{d}\theta}{\mathrm{d}t}\big)^2 + mg\cos\theta$ in the rod, once you know $\theta(t)$.

1.2.4 ▸▸ (Optional) Areas of Parallelograms

A parallelogram is naturally determined by the two vectors that define its sides. We'll now develop a formula for the area of a parallelogram in terms of these two vectors.

Construct a parallelogram as follows. Pick two vectors $\langle a, b\rangle$ and $\langle c, d\rangle$. Draw them with their tails at a common point. Then draw $\langle a, b\rangle$ a second time with its tail at the head of $\langle c, d\rangle$ and draw $\langle c, d\rangle$ a second time with its tail at the head of $\langle a, b\rangle$. If the common point is the origin, you get a picture like the figure below. Any parallelogram can



be constructed like this if you pick the common point and two vectors appropriately. Let's compute the area of the parallelogram. The area of the large rectangle with vertices $(0, 0)$, $(0, b+d)$, $(a+c, 0)$ and $(a+c, b+d)$ is $(a+c)(b+d)$. The parallelogram we want can be extracted from the large rectangle by deleting the two small rectangles (each of area bc), and the two lightly shaded triangles (each of area $\frac{1}{2}cd$), and the two darkly shaded triangles (each of area $\frac{1}{2}ab$). So the desired

$$\text{area} = (a+c)(b+d) - (2 \times bc) - \left(2 \times \tfrac{1}{2}cd\right) - \left(2 \times \tfrac{1}{2}ab\right) = ad - bc$$

In the above figure, we have implicitly assumed that $a,\ b,\ c,\ d \geqslant 0$ and $d/c \geqslant b/a$. In words, we have assumed that both vectors $\langle a, b\rangle$, $\langle c, d\rangle$ lie in the first quadrant and that $\langle c, d\rangle$ lies above $\langle a, b\rangle$. By simply interchanging $a \leftrightarrow c$ and $b \leftrightarrow d$ in the picture and throughout the argument, we see that when $a,\ b,\ c,\ d \geqslant 0$ and $b/a \geqslant d/c$, so that the vector $\langle c, d\rangle$ lies below $\langle a, b\rangle$, the area of the parallelogram is $bc - ad$. In fact, all cases are covered by the formula

Equation 1.2.17.

$$\text{area of parallelogram with sides } \langle a, b\rangle \text{ and } \langle c, d\rangle = |ad - bc|$$

Given two vectors $\langle a, b\rangle$ and $\langle c, d\rangle$, the expression $ad - bc$ is generally written

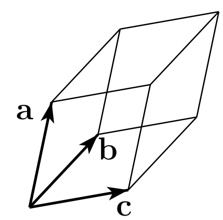
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

and is called the *determinant* of the matrix[12]

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with rows $\langle a, b\rangle$ and $\langle c, d\rangle$. The determinant of a 2×2 matrix is the product of the diagonal entries minus the product of the off-diagonal entries.

There is a similar formula in three dimensions. Any three vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ in three dimensions determine a parallelepiped (three di-



mensional parallelogram). Its volume is given by the formula

Equation 1.2.18.

$$\text{volume of parallelepiped with edges } \mathbf{a},\ \mathbf{b},\ \mathbf{c}\ = \left|\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right|$$

The determinant of a 3×3 matrix can be defined in terms of some 2×2 determinants by

$$\begin{aligned}\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} &= a_1 \det\begin{bmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ \cancel{b_1} & b_2 & b_3 \\ \cancel{c_1} & c_2 & c_3 \end{bmatrix} - a_2 \det\begin{bmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ b_1 & \cancel{b_2} & b_3 \\ c_1 & \cancel{c_2} & c_3 \end{bmatrix} + a_3 \det\begin{bmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ b_1 & b_2 & \cancel{b_3} \\ c_1 & c_2 & \cancel{c_3} \end{bmatrix} \\ &= a_1\,(b_2c_3 - b_3c_2) \quad - a_2\,(b_1c_3 - b_3c_1) \quad + a_3\,(b_1c_2 - b_2c_1)\end{aligned}$$

This formula is called “expansion along the top row”. There is one term in the formula for each entry in the top row of the 3×3 matrix. The term is a sign times the entry itself times the determinant of the 2×2 matrix gotten by deleting the row and column that contains the entry. The sign alternates, starting with a $+$.

We shall not prove this formula completely here[13]. It gets a little tedious. But, there is one case in which we can easily verify that the volume of the parallelepiped is really

12 The topics of matrices and determinants appear prominently in linear algebra courses. We are only going to use them as notation, and we will explicitly explain that notation. A linear algebra course is *not* a prerequisite for this text.

13 For a full derivation, see Example 1.2.25

given by the absolute value of the claimed determinant. If the vectors **b** and **c** happen to lie in the xy plane, so that $b_3 = c_3 = 0$, then

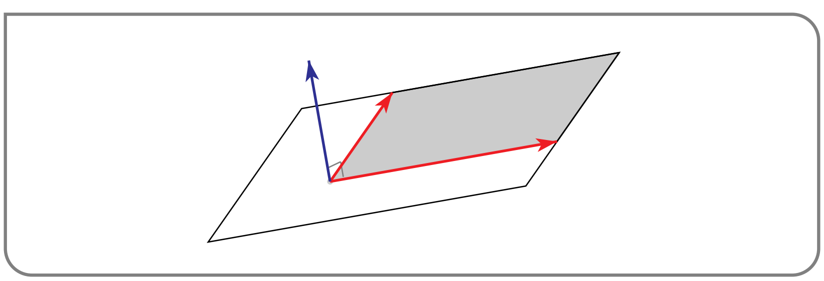
$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \end{bmatrix} = a_1(b_2 0 - 0c_2) - a_2(b_1 0 - 0c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_3(b_1c_2 - b_2c_1)$$

The first factor, a_3, is the z-coordinate of the one vector not contained in the xy-plane. It is (up to a sign) the height of the parallelepiped. The second factor is, up to a sign, the area of the parallelogram determined by **b** and **c**. This parallelogram forms the base of the parallelepiped. The product is indeed, up to a sign, the volume of the parallelepiped. That the formula is true in general is a consequence of the fact (that we will not prove) that the value of a determinant does not change when one rotates the coordinate system and that one can always rotate our coordinate axes around so that **b** and **c** both lie in the xy-plane.

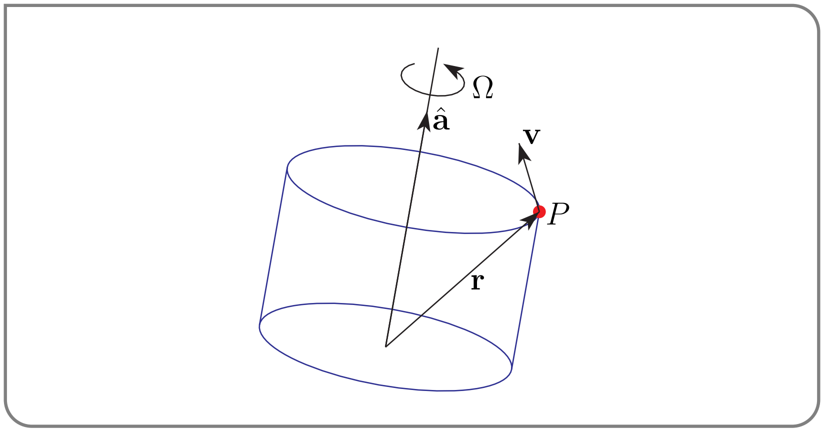
1.2.5 ▸▸ The Cross Product

We have already seen two different products involving vectors — the multiplication of a vector by a scalar and the dot product of two vectors. The dot product of two vectors yields a scalar. We now introduce another product of two vectors, called the *cross product*. The cross product of two vectors will give a vector. There are applications which have two vectors as inputs and produce one vector as an output, and which are related to the cross product. Here is a very brief mention of two such applications. We will look at them in much more detail later.

- Consider a parallelogram in three dimensions. A parallelogram is naturally determined by the two vectors that define its sides. One measure of the size of a parallelogram is its area. One way to specify the orientation of the parallelogram is to give a vector that is perpendicular to it. A very compact way to encode both the area and the orientation of the parallelogram is to give a vector whose direction is perpendicular to the plane in which it lies and whose magnitude is its area. We shall see that such a vector can be easily constructed by taking the cross product (definition coming shortly) of the two vectors that give the sides of the parallelogram.



- Imagine a rigid body which is rotating at a rate Ω radians per second about an axis whose direction is given by the unit vector $\hat{\mathbf{a}}$. Let P be any point on the body. We shall see, in the (optional) §1.2.7, that the velocity, **v**, of the point P is the cross product (again, definition coming shortly) of the vector $\Omega\hat{\mathbf{a}}$ with the vector **r** from any point on the axis of rotation to P.



Finally, here is the definition of the cross product. Note that it applies only to vectors in three dimensions.

Definition 1.2.19 (Cross Product).

The cross product of the vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is denoted $\mathbf{a} \times \mathbf{b}$ and is defined by

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2 \,,\, a_3 b_1 - a_1 b_3 \,,\, a_1 b_2 - a_2 b_1 \rangle$$

Note that each component has the form $a_i b_j - a_j b_i$. The index i of the first a in component number k of $\mathbf{a} \times \mathbf{b}$ is just after k in the list $1, 2, 3, 1, 2, 3, 1, 2, 3, \cdots$. The index j of the first b is just before k in the list.

$$(\mathbf{a} \times \mathbf{b})_k = a_{\text{just after } k}\, b_{\text{just before } k} - a_{\text{just before } k}\, b_{\text{just after } k}$$

For example, for component number $k = 3$,

$$\left.\begin{matrix} \text{“just after 3” is 1} \\ \text{“just before 3” is 2} \end{matrix}\right\} \implies (\mathbf{a} \times \mathbf{b})_3 = a_1 b_2 - a_2 b_1$$

There is a much better way to remember this definition. Recall that a 2×2 matrix is an array of numbers having two rows and two columns and that the determinant of a 2×2 matrix is defined by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

It is the product of the entries on the diagonal minus the product of the entries not on the diagonal.

A 3×3 matrix is an array of numbers having three rows and three columns.

$$\begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

You will shortly see why the entries in the top row have been given the rather peculiar names i, j and k. The determinant of a 3×3 matrix can be defined in terms of some 2×2 determinants by

$$\det\begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = i\det\begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} - j\det\begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} + k\det\begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$
$$= i\,(a_2b_3 - a_3b_2) - j\,(a_1b_3 - a_3b_1) + k\,(a_1b_2 - a_2b_1)$$

This formula is called "expansion of the determinant along the top row". There is one term in the formula for each entry in the top row. The term is a sign times the entry itself times the determinant of the 2×2 matrix gotten by deleting the row and column that contains the entry. The sign alternates, starting with a $+$. If we now replace i by $\hat{\boldsymbol{\imath}}$, j by $\hat{\boldsymbol{\jmath}}$ and k by $\hat{\mathbf{k}}$, we get exactly the formula for $\mathbf{a} \times \mathbf{b}$ of Definition 1.2.19. That is the reason for the peculiar choice of names for the matrix entries. So

$$\mathbf{a} \times \mathbf{b} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$
$$= \hat{\boldsymbol{\imath}}\big(a_2b_3 - a_3b_2\big) - \hat{\boldsymbol{\jmath}}\big(a_1b_3 - a_3b_1\big) + \hat{\mathbf{k}}\big(a_1b_2 - a_2b_1\big)$$

is a mnemonic device for remembering the definition of $\mathbf{a} \times \mathbf{b}$. It is also good from the point of view of evaluating $\mathbf{a} \times \mathbf{b}$. Here are several examples in which we use the determinant mnemonic device to evaluate cross products.

Example 1.2.20

In this example, we'll use the mnemonic device to compute two very simple cross products. First

$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \hat{\boldsymbol{\imath}}\det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \hat{\boldsymbol{\jmath}}\det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$= \hat{\boldsymbol{\imath}}\,(0 \times 0 - 0 \times 1) - \hat{\boldsymbol{\jmath}}\,(1 \times 0 - 0 \times 0) + \hat{\mathbf{k}}\,(1 \times 1 - 0 \times 0) = \hat{\mathbf{k}}$$

Second

$$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \hat{\boldsymbol{\imath}}\det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \hat{\boldsymbol{\jmath}}\det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$= \hat{\boldsymbol{\imath}}\,(1 \times 0 - 0 \times 0) - \hat{\boldsymbol{\jmath}}\,(0 \times 0 - 0 \times 1) + \hat{\mathbf{k}}\,(0 \times 0 - 1 \times 1) = -\hat{\mathbf{k}}$$

Note that, unlike most (or maybe even all) products that you have seen before, $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}$ is *not* the same as $\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}}$!

Example 1.2.20

Example 1.2.21

In this example, we'll use the mnemonic device to compute two more complicated cross products. Let $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 1, -1, 2\rangle$. First

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & 2 & 3\\ 1 & -1 & 2\end{bmatrix} = \hat{\boldsymbol{\imath}}\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & 2 & 3\\ 1 & -1 & 2\end{bmatrix} - \hat{\boldsymbol{\jmath}}\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & 2 & 3\\ 1 & -1 & 2\end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & 2 & 3\\ 1 & -1 & 2\end{bmatrix}\\
&= \hat{\boldsymbol{\imath}}\{2\times 2 - 3\times(-1)\} - \hat{\boldsymbol{\jmath}}\{1\times 2 - 3\times 1\} + \hat{\mathbf{k}}\{1\times(-1) - 2\times 1\}\\
&= 7\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 3\hat{\mathbf{k}}\end{aligned}$$

Second

$$\begin{aligned}\mathbf{b}\times\mathbf{a} &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & -1 & 2\\ 1 & 2 & 3\end{bmatrix} = \hat{\boldsymbol{\imath}}\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & -1 & 2\\ 1 & 2 & 3\end{bmatrix} - \hat{\boldsymbol{\jmath}}\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & -1 & 2\\ 1 & 2 & 3\end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & -1 & 2\\ 1 & 2 & 3\end{bmatrix}\\
&= \hat{\boldsymbol{\imath}}\{(-1)\times 3 - 2\times 2\} - \hat{\boldsymbol{\jmath}}\{1\times 3 - 2\times 1\} + \hat{\mathbf{k}}\{1\times 2 - (-1)\times 1\}\\
&= -7\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + 3\hat{\mathbf{k}}\end{aligned}$$

Here are some important observations.

- The vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ are not the same! In fact $\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$. We shall see in Theorem 1.2.23 below that this was not a fluke.
- The vector $\mathbf{a}\times\mathbf{b}$ has dot product zero with both $\mathbf{a}$ and $\mathbf{b}$. So the vector $\mathbf{a}\times\mathbf{b}$ is prependicular to both $\mathbf{a}$ and $\mathbf{b}$. We shall see in Theorem 1.2.23 below that this was also not a fluke.

Example 1.2.21

Example 1.2.22

Yet again we use the mnemonic device to compute a more complicated cross product. This time let $\mathbf{a} = \langle 3, 2, 1\rangle$ and $\mathbf{b} = \langle 6, 4, 2\rangle$. Then

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 3 & 2 & 1\\ 6 & 4 & 2\end{bmatrix} &= \hat{\boldsymbol{\imath}}\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 3 & 2 & 1\\ 6 & 4 & 2\end{bmatrix} - \hat{\boldsymbol{\jmath}}\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 3 & 2 & 1\\ 6 & 4 & 2\end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 3 & 2 & 1\\ 6 & 4 & 2\end{bmatrix}\\
&= \hat{\boldsymbol{\imath}}(2\times 2 - 1\times 4) - \hat{\boldsymbol{\jmath}}(3\times 2 - 1\times 6) + \hat{\mathbf{k}}(3\times 4 - 2\times 6) = \mathbf{0}\end{aligned}$$

We shall see in Theorem 1.2.23 below that it is not a fluke that the cross product is $\mathbf{0}$. It is a consequence of the fact that $\mathbf{a}$ and $\mathbf{b} = 2\mathbf{a}$ are parallel.

Example 1.2.22

We now move on to learning about the properties of the cross product. Our first properties lead up to a more intuitive geometric definition of $\mathbf{a}\times\mathbf{b}$, which is better for interpreting $\mathbf{a}\times\mathbf{b}$. These properties of the cross product, which state that $\mathbf{a}\times\mathbf{b}$ is a vector and then determine its direction and length, are as follows. We will collect these properties, and a few others, into a theorem shortly.

(0) $\mathbf{a}, \mathbf{b}$ are vectors in three dimensions and $\mathbf{a}\times\mathbf{b}$ is a vector in three dimensions.

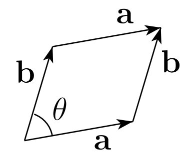
(1) $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Proof: To check that $\mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ are perpendicular, one just has to check that the dot product $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. The six terms in

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$$

cancel pairwise. The computation showing that $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ is similar.

(2) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ where $0 \leqslant \theta \leqslant \pi$ is the angle between $\mathbf{a}$ and $\mathbf{b}$
$= $ the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$



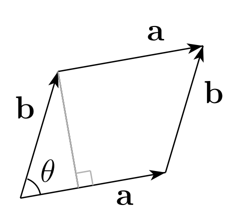
Proof: The formula $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ is gotten by verifying that

$$\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= \big(\mathbf{a} \times \mathbf{b}\big) \cdot \big(\mathbf{a} \times \mathbf{b}\big) \\
&= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2b_3a_3b_2 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_3b_1a_1b_3 + a_1^2b_3^2 \\
&\qquad + a_1^2b_2^2 - 2a_1b_2a_2b_1 + a_2^2b_1^2
\end{aligned}$$

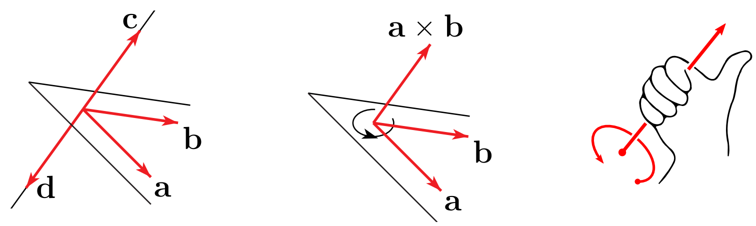
is equal to

$$\begin{aligned}
|\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2\theta &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= \big(a_1^2 + a_2^2 + a_3^2\big)\big(b_1^2 + b_2^2 + b_3^2\big) - \big(a_1b_1 + a_2b_2 + a_3b_3\big)^2 \\
&= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 \\
&\qquad - \big(2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3\big)
\end{aligned}$$

To see that $|\mathbf{a}|\,|\mathbf{b}| \sin\theta$ is the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$, just recall that the area of any parallelogram is given by the length of its base times its height. Think of $\mathbf{a}$ as the base of the parallelogram. Then $|\mathbf{a}|$ is the length of the base and $|\mathbf{b}| \sin\theta$ is the height.



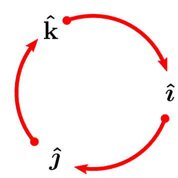
These properties almost determine $\mathbf{a} \times \mathbf{b}$. Property 1 forces the vector $\mathbf{a} \times \mathbf{b}$ to lie on the line perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. There are precisely two vectors on this line that have the length given by property 2. In the left hand figure of



the two vectors are labeled **c** and **d**. Which of these two candidates is correct is determined by the right hand rule[14], which says that if you form your right hand into a fist with your fingers curling from **a** to **b**, then when you stick your thumb straight out from the fist, it points in the direction of $\mathbf{a} \times \mathbf{b}$. This is illustrated in the figure on the right above [15]. The important special cases

(3) $$\begin{aligned} \hat{\imath} \times \hat{\jmath} &= \hat{\mathbf{k}}, & \hat{\jmath} \times \hat{\mathbf{k}} &= \hat{\imath}, & \hat{\mathbf{k}} \times \hat{\imath} &= \hat{\jmath} \\ \hat{\jmath} \times \hat{\imath} &= -\hat{\mathbf{k}}, & \hat{\mathbf{k}} \times \hat{\jmath} &= -\hat{\imath}, & \hat{\imath} \times \hat{\mathbf{k}} &= -\hat{\jmath} \end{aligned}$$

all follow directly from the definition of the cross product (see, for example, Example 1.2.20) and all obey the right hand rule. Combining properties 1, 2 and the right hand rule give the geometric definition of $\mathbf{a} \times \mathbf{b}$. To remember these three special cases, just remember this figure.



The product of any two standard basis vectors, taken in the order of the arrows in the figure, is the third standard basis vector. Going against the arrows introduces a minus sign.

(4) $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta \ \hat{\mathbf{n}}$ where θ is the angle between **a** and **b**, $|\hat{\mathbf{n}}| = 1$, $\hat{\mathbf{n}} \perp \mathbf{a}, \mathbf{b}$ and $(\mathbf{a}, \mathbf{b}, \hat{\mathbf{n}})$ obey the right hand rule.

Outline of Proof: We have already seen that the right hand side has the correct length and, except possibly for a sign, direction. To check that the right hand rule holds in general, rotate your coordinate system around[16] so that **a** points along the positive x axis and **b** lies in the xy-plane with positive y component. That is $\mathbf{a} = \alpha\hat{\imath}$ and $\mathbf{b} = \beta\hat{\imath} + \gamma\hat{\jmath}$ with $\alpha, \gamma \geqslant 0$. Then $\mathbf{a} \times \mathbf{b} = \alpha\hat{\imath} \times (\beta\hat{\imath} + \gamma\hat{\jmath}) = \alpha\beta\,\hat{\imath} \times \hat{\imath} + \alpha\gamma\,\hat{\imath} \times \hat{\jmath}$. The first term vanishes by property 2, because the angle θ between $\hat{\imath}$ and $\hat{\imath}$ is zero. So, by property 3, $\mathbf{a} \times \mathbf{b} = \alpha\gamma\hat{\mathbf{k}}$ points along the positive z axis, which is consistent with the right hand rule.

14 That the cross product uses the right hand rule, rather than the left hand rule, is an example of the tyranny of the masses — only roughly 10% of humans are left-handed.
15 This figure is a variant of https://commons.wikimedia.org/wiki/File:Right_hand_rule_simple.png
16 Note that as you translate or rotate the coordinate system, the right hand rule is preserved. If $(\mathbf{a}, \mathbf{b}, \hat{\mathbf{n}})$ obey the right hand rule so do their rotated and translated versions.

The analog of property 7 of the dot product (which says that $\mathbf{a} \cdot \mathbf{b}$ is zero if and only if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a} \perp \mathbf{b}$) follows immediately from property 2.

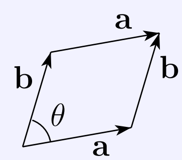
(5) $\mathbf{a} \times \mathbf{b} = \mathbf{0} \iff \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a} \parallel \mathbf{b}$

The remaining properties are all tools for helping do computations with cross products. Here is a theorem which summarizes the properties of the cross product. We have already seen the first five. The other properties are all tools for helping do computations with cross products.

Theorem 1.2.23 (Properties of the Cross Product)**.**

(0) $\mathbf{a}, \mathbf{b}$ are vectors in three dimensions and $\mathbf{a} \times \mathbf{b}$ is a vector in three dimensions.

(1) $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.



(2) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$
$\qquad\qquad =$ the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$

(3) $\hat{\imath} \times \hat{\jmath} = \hat{\mathbf{k}}$, $\hat{\jmath} \times \hat{\mathbf{k}} = \hat{\imath}$, $\hat{\mathbf{k}} \times \hat{\imath} = \hat{\jmath}$

(4) $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\, \mathbf{n}$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $|\mathbf{n}| = 1$, $\mathbf{n} \perp \mathbf{a}, \mathbf{b}$ and $(\mathbf{a}, \mathbf{b}, \mathbf{n})$ obey the right hand rule.

(5) $\mathbf{a} \times \mathbf{b} = \mathbf{0} \iff \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a} \parallel \mathbf{b}$

(6) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

(7) $(s\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (s\mathbf{b}) = s(\mathbf{a} \times \mathbf{b})$ for any scalar (i.e. number) s.

(8) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(9) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

(10) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

Proof. We have already seen the proofs up to number 5. Numbers 6, 7 and 8 follow immediately from the definition, using a little algebra. To prove numbers 9 and 10 we just write out the definitions of the left hand sides and the right hand sides and observe that they are equal.

(9) The left hand side is

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_2c_3 - b_3c_2\,,\, b_3c_1 - b_1c_3\,,\, b_1c_2 - b_2c_1 \rangle \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \end{aligned}$$

The right hand side is

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \langle a_2b_3 - a_3b_2\,,\, a_3b_1 - a_1b_3\,,\, a_1b_2 - a_2b_1 \rangle \cdot \langle c_1, c_2, c_3 \rangle \\ &= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3 \end{aligned}$$

The left and right hand sides are the same.

(10) We will give the straightforward, but slightly tedious, computations in (the optional) §1.2.6. □

Warning 1.2.24.

Take particular care with properties 6 and 10. They are counterintuitive and are a frequent source of errors. In particular, for general vectors **a**, **b**, **c**, the cross product is neither commutative nor associative, meaning that

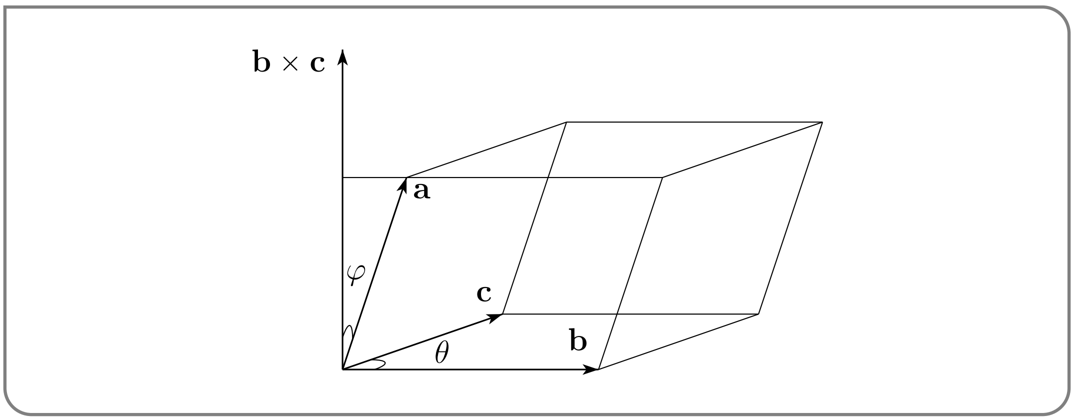
$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

For example

$$\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \hat{\imath} \times \hat{\mathbf{k}} = -\hat{\mathbf{k}} \times \hat{\imath} = -\hat{\jmath}$$
$$(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \mathbf{0} \times \hat{\jmath} = \mathbf{0}$$

Example 1.2.25

As an illustration of the properties of the dot and cross product, we now derive the formula for the volume of the parallelepiped with edges $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ that was mentioned in §1.2.4. The volume of the parallelepiped is the area



of its base times its height[17]. The base is the parallelogram with sides **b** and **c**. Its area is the length of its base, which is $|\mathbf{b}|$, times its height, which is $|\mathbf{c}| \sin\theta$. (Drop a perpendicular from the head of **c** to the line containing **b**). Here θ is the angle between **b** and **c**. So the area of the base is $|\mathbf{b}|\,|\mathbf{c}| \sin\theta = |\mathbf{b} \times \mathbf{c}|$, by property 2 of the cross product. To get the height of the parallelepiped, we drop a perpendicular from the head of **a** to the line that passes through the tail of **a** and is perpendicular to the base of the parallelepiped. In other words, from the head of **a** to the line that contains both the head and the tail of $\mathbf{b} \times \mathbf{c}$. So

17 This is a simple integral calculus exercise.

the height of the parallelepiped is $|\mathbf{a}|\,|\cos\varphi|$. (The absolute values have been included because if the angle between $\mathbf{b}\times\mathbf{c}$ and $\mathbf{a}$ happens to be greater than $90°$, the $\cos\varphi$ produced by taking the dot product of $\mathbf{a}$ and $(\mathbf{b}\times\mathbf{c})$ will be negative.) All together

$$\begin{aligned}
\text{volume of parallelepiped} &= (\text{area of base})\,(\text{height})\\
&= |\mathbf{b}\times\mathbf{c}|\;|\mathbf{a}|\;|\cos\varphi|\\
&= \big|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\big|\\
&= |a_1(\mathbf{b}\times\mathbf{c})_1 + a_2(\mathbf{b}\times\mathbf{c})_2 + a_3(\mathbf{b}\times\mathbf{c})_3|\\
&= \left|a_1\det\begin{bmatrix} b_2 & b_3\\ c_2 & c_3\end{bmatrix} - a_2\det\begin{bmatrix} b_1 & b_3\\ c_1 & c_3\end{bmatrix} + a_3\det\begin{bmatrix} b_1 & b_2\\ c_1 & c_2\end{bmatrix}\right|\\
&= \left|\det\begin{bmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\right|
\end{aligned}$$

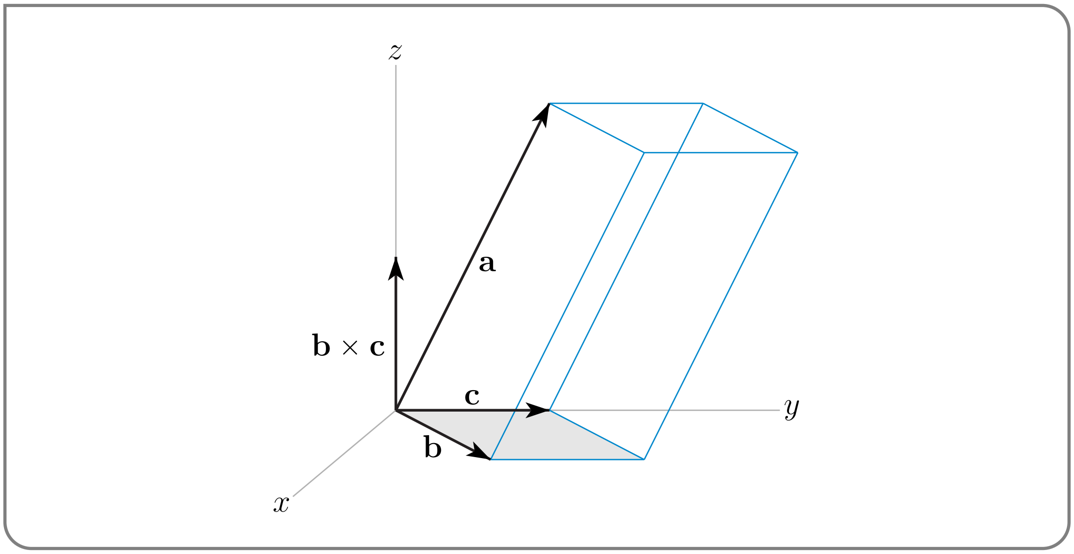
Example 1.2.25

Example 1.2.26

As a concrete example of the computation of the volume of a parallelepiped, we consider the parallelepiped with edges

$$\begin{aligned}
\mathbf{a} &= \langle 0,1,2\rangle\\
\mathbf{b} &= \langle 1,1,0\rangle\\
\mathbf{c} &= \langle 0,1,0\rangle
\end{aligned}$$

Here is a sketch. The base of the parallelepiped is the parallelogram with sides $\mathbf{b}$ and $\mathbf{c}$. It



is the shaded parallelogram in the sketch above. As

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \hat{\boldsymbol{\imath}} \det \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \hat{\boldsymbol{\jmath}} \det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \hat{\mathbf{k}} \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \hat{\boldsymbol{\imath}}(1 \times 0 - 0 \times 1) - \hat{\boldsymbol{\jmath}}(1 \times 0 - 0 \times 0) + \hat{\mathbf{k}}(1 \times 1 - 1 \times 0) \\
&= \hat{\mathbf{k}}
\end{aligned}$$

We should not be surprised that $\mathbf{b} \times \mathbf{c}$ has direction $\hat{\mathbf{k}}$.

- $\mathbf{b} \times \mathbf{c}$ has to be perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ and
- both $\mathbf{b}$ and $\mathbf{c}$ lie in the xy-plane,
- so that $\mathbf{b} \times \mathbf{c}$ has to the perpendicular to the xy-plane,
- so that $\mathbf{b} \times \mathbf{c}$ has to the parallel to the z-axis.

The area of the base, i.e. of the shaded parallelogram in the figure above, is

$$|\mathbf{b} \times \mathbf{c}| = |\hat{\mathbf{k}}| = 1$$

and the volume of the parallelepiped is

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\, \langle 0, 1, 2 \rangle \cdot \langle 0, 0, 1 \rangle \,| = 2$$

Example 1.2.26

1.2.6 ▸▸ (Optional) Some Vector Identities

Here are a few identities involving dot and cross products.

Lemma 1.2.27.

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$

(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

Proof. (a) We proved this in Theorem 1.2.23, by evaluating the left and right hand sides, and observing that they are the same. Here is a second proof, in which we again write out both sides, but this time we express them in terms of determinants.

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} &= (a_1,a_2,a_3)\cdot\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\\
&= a_1\det\begin{bmatrix}b_2 & b_3\\ c_2 & c_3\end{bmatrix} - a_2\det\begin{bmatrix}b_1 & b_3\\ c_1 & c_3\end{bmatrix} + a_3\det\begin{bmatrix}b_1 & b_2\\ c_1 & c_2\end{bmatrix}\\
&= \det\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}\times\mathbf{b}\cdot\mathbf{c} &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}\cdot(c_1,c_2,c_3)\\
&= c_1\det\begin{bmatrix}a_2 & a_3\\ b_2 & b_3\end{bmatrix} - c_2\det\begin{bmatrix}a_1 & a_3\\ b_1 & b_3\end{bmatrix} + c_3\det\begin{bmatrix}a_1 & a_2\\ b_1 & b_2\end{bmatrix}\\
&= \det\begin{bmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{bmatrix}
\end{aligned}$$

Exchanging two rows in a determinant changes the sign of the determinant. Moving the top row of a 3×3 determinant to the bottom row requires two exchanges of rows. So the two 3×3 determinants are equal.

(b) The proof is not exceptionally difficult — just write out both sides and grind. Substituting in

$$\mathbf{b}\times\mathbf{c} \;=\; (b_2c_3-b_3c_2)\hat{\boldsymbol{\imath}} - (b_1c_3-b_3c_1)\hat{\boldsymbol{\jmath}} + (b_1c_2-b_2c_1)\hat{\mathbf{k}}$$

gives, for the left hand side,

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) =\;& \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ a_1 & a_2 & a_3\\ b_2c_3-b_3c_2 & -b_1c_3+b_3c_1 & b_1c_2-b_2c_1\end{bmatrix}\\
=\;& \hat{\boldsymbol{\imath}}\big[a_2(b_1c_2-b_2c_1)-a_3(-b_1c_3+b_3c_1)\big]\\
&-\hat{\boldsymbol{\jmath}}\big[a_1(b_1c_2-b_2c_1)-a_3(b_2c_3-b_3c_2)\big]\\
&+\hat{\mathbf{k}}\big[a_1(-b_1c_3+b_3c_1)-a_2(b_2c_3-b_3c_2)\big]
\end{aligned}$$

On the other hand, the right hand side

$$\begin{aligned}
(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} &= (a_1c_1+a_2c_2+a_3c_3)(b_1\hat{\boldsymbol{\imath}}+b_2\hat{\boldsymbol{\jmath}}+b_3\hat{\mathbf{k}})\\
&\qquad -(a_1b_1+a_2b_2+a_3b_3)(c_1\hat{\boldsymbol{\imath}}+c_2\hat{\boldsymbol{\jmath}}+c_3\hat{\mathbf{k}})\\
&= \hat{\boldsymbol{\imath}}\,\big[a_1b_1c_1+a_2b_1c_2+a_3b_1c_3-a_1b_1c_1-a_2b_2c_1-a_3b_3c_1\big]\\
&\quad+\hat{\boldsymbol{\jmath}}\,\big[a_1b_2c_1+a_2b_2c_2+a_3b_2c_3-a_1b_1c_2-a_2b_2c_2-a_3b_3c_2\big]\\
&\quad+\hat{\mathbf{k}}\,\big[a_1b_3c_1+a_2b_3c_2+a_3b_3c_3-a_1b_1c_3-a_2b_2c_3-a_3b_3c_3\big]\\
&= \hat{\boldsymbol{\imath}}\,\big[a_2b_1c_2+a_3b_1c_3-a_2b_2c_1-a_3b_3c_1\big]\\
&\quad+\hat{\boldsymbol{\jmath}}\,\big[a_1b_2c_1+a_3b_2c_3-a_1b_1c_2-a_3b_3c_2\big]\\
&\quad+\hat{\mathbf{k}}\,\big[a_1b_3c_1+a_2b_3c_2-a_1b_1c_3-a_2b_2c_3\big]
\end{aligned}$$

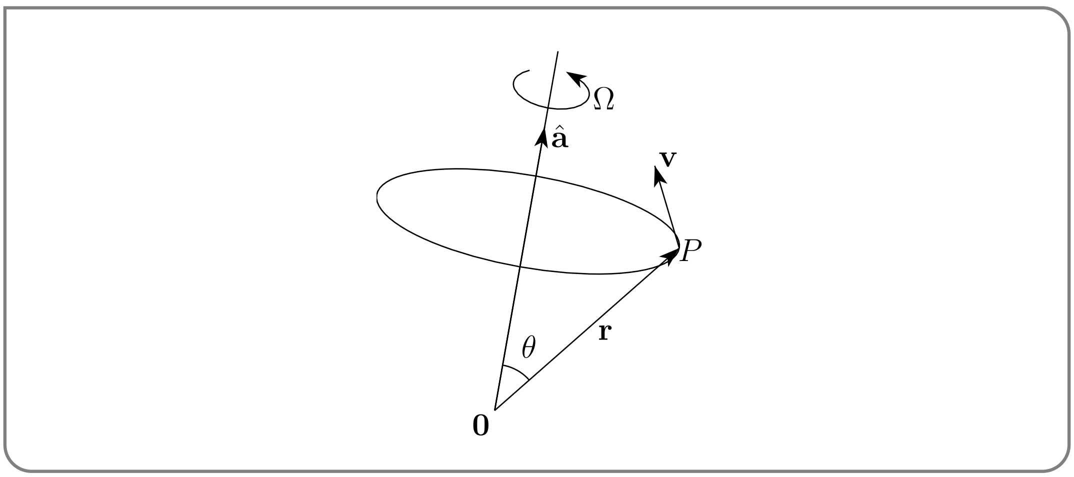
The last formula that we had for the left hand side is the same as the last formula we had for the right hand side. Oof! This is a little tedious to do by hand. But any computer algebra system will do it for you in a flash.
(c) We just apply part (b) three times

$$\begin{aligned}
&\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})\\
&\qquad=(\mathbf{c}\cdot\mathbf{a})\mathbf{b}-(\mathbf{b}\cdot\mathbf{a})\mathbf{c}+(\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{c}\cdot\mathbf{b})\mathbf{a}+(\mathbf{b}\cdot\mathbf{c})\mathbf{a}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}\\
&\qquad=\mathbf{0}
\end{aligned}$$

□

1.2.7 ▸▸ (Optional) Application of Cross Products to Rotational Motion

In most computations involving rotational motion, the cross product shows up in one form or another. This is one of the main applications of the cross product. Consider, for example, a rigid body which is rotating at a constant rate of Ω radians per second about an axis whose direction is given by the unit vector $\hat{\mathbf{a}}$. Let P be any point on the body. Let's figure out its velocity. Pick any point on the axis of rotation and designate it as the origin of our coordinate system. Denote by $\mathbf{r}$ the vector from the origin to the point P. Let θ denote the angle between $\hat{\mathbf{a}}$ and $\mathbf{r}$. As time progresses the point P sweeps out a circle of radius $R=|\mathbf{r}|\sin\theta$.



In one second P travels along an arc that subtends an angle of Ω radians, which is the fraction $\frac{\Omega}{2\pi}$ of a full circle. The length of this arc is $\frac{\Omega}{2\pi}\times 2\pi R=\Omega R=\Omega|\mathbf{r}|\sin\theta$ so P travels the distance $\Omega|\mathbf{r}|\sin\theta$ in one second and its speed, which is also the length of its velocity vector, is $\Omega|\mathbf{r}|\sin\theta$.

Now we just need to figure out the direction of the velocity vector. That is, the direction of motion of the point P. Imagine that both $\hat{\mathbf{a}}$ and $\mathbf{r}$ lie in the plane of a piece of paper, as in the figure above. Then $\mathbf{v}$ points either straight into or straight out of the page and consequently is perpendicular to both $\hat{\mathbf{a}}$ and $\mathbf{r}$. To distinguish between the "into the page" and "out of the page" cases, let's impose the conventions that $\Omega>0$ and the axis of rotation $\hat{\mathbf{a}}$ is chosen to obey the right hand rule, meaning that if the thumb of your right hand is pointing in the direction $\hat{\mathbf{a}}$, then your fingers are pointing in the direction of motion of the rigid body. Under these conventions, the velocity vector $\mathbf{v}$ obeys

- $|\mathbf{v}| = \Omega|\mathbf{r}||\hat{\mathbf{a}}|\sin\theta$
- $\mathbf{v} \perp \hat{\mathbf{a}}, \mathbf{r}$
- $(\hat{\mathbf{a}}, \mathbf{r}, \mathbf{v})$ obey the right hand rule

That is, $\mathbf{v}$ is exactly $\Omega\hat{\mathbf{a}} \times \mathbf{r}$. It is conventional to define the "angular velocity" of a rigid body to be vector $\boldsymbol{\Omega} = \Omega\hat{\mathbf{a}}$. That is, the vector with length given by the rate of rotation and direction given by the axis of rotation of the rigid body. In particular, the bigger the rate of rotation, the longer the angular velocity vector. In terms of this angular velocity vector, the velocity of the point P is

$$\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$$

1.2.8 ▸▸ (Optional) Application of Cross Products to Rotating Reference Frames

Imagine a moving particle that is being tracked by two observers.

(a) One observer is fixed (out in space) and measures the position of the particle to be $\big(X(t), Y(t), Z(t)\big)$.

(b) The other observer is tied to a merry-go-round (the Earth) and measures the position of the particle to be $\big(x(t), y(t), z(t)\big)$.

The merry-go-round is sketched in the figure on the left below. It is rotating about the Z-axis at a (constant) rate of Ω radians per second. The vector $\boldsymbol{\Omega} = \Omega\hat{\mathbf{k}}$, whose length is the rate of rotation and whose direction is the axis of rotation, is called the angular velocity. The x- and y-axes of the moving observer are painted in red on the merry-go-round. The

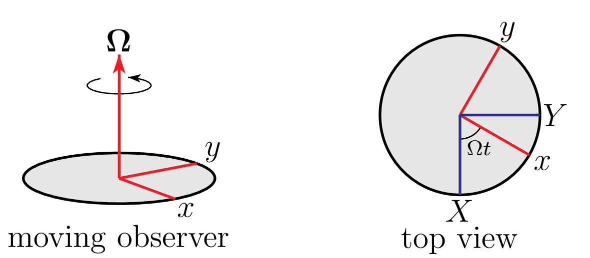


figure on the right above shows a top view of the merry-go-round. The x- and y-axes of the moving observer are again red. The X- and Y-axes of the fixed observer are blue. We are assuming that at time 0, the x-axis of the moving observer and the X-axis of the fixed observer coincide. As the merry-go-round is rotating at Ω radians per second, the angle between the X-axis and x-axis after t seconds is Ωt.

As an example, suppose that the moving particle is tied to the tip of the moving observer's unit x vector. Then

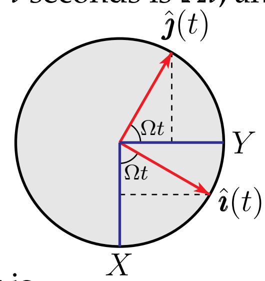
$$\begin{aligned} x(t) &= 1 & y(t) &= 0 & z(t) &= 0 \\ X(t) &= \cos(\Omega t) & Y(t) &= \sin(\Omega t) & Z(t) &= 0 \end{aligned}$$

or, if we write $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$ and $\mathbf{R}(t) = \big(X(t), Y(t), Z(t)\big)$, then

$$\mathbf{r}(t) = (1\,,0\,,0) \qquad \mathbf{R}(t) = \big(\cos(\Omega t)\,,\sin(\Omega t)\,,0\big)$$

In general, denote by $\hat{\imath}(t)$ the coordinates of the unit x-vector of the moving observer at time t, as measured by the fixed observer. Similarly $\hat{\jmath}(t)$ for the unit y-vector, and $\hat{\mathbf{k}}(t)$ for the unit z-vector. As the merry-go-round is rotating about the Z-axis at a rate of Ω radians per second, the angle between the X-axis and x-axis after t seconds is Ωt, and

$$\begin{aligned}\hat{\imath}(t) &= \big(\cos(\Omega t)\,,\sin(\Omega t)\,,0\big)\\ \hat{\jmath}(t) &= \big(-\sin(\Omega t)\,,\cos(\Omega t)\,,0\big)\\ \hat{\mathbf{k}}(t) &= \big(0\,,0\,,1\big)\end{aligned}$$



The position of the moving particle, as seen by the fixed observer is

$$\mathbf{R}(t) = x(t)\,\hat{\imath}(t) + y(t)\,\hat{\jmath}(t) + z(t)\,\hat{\mathbf{k}}(t)$$

Differentiating, the velocity of the moving particle, as measured by the fixed observer is

$$\begin{aligned}\mathbf{V}(t) = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = &\ \frac{\mathrm{d}x}{\mathrm{d}t}(t)\,\hat{\imath}(t) + \frac{\mathrm{d}y}{\mathrm{d}t}(t)\,\hat{\jmath}(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t)\,\hat{\mathbf{k}}(t)\\ &+ x(t)\frac{\mathrm{d}}{\mathrm{d}t}\hat{\imath}(t) + y(t)\frac{\mathrm{d}}{\mathrm{d}t}\hat{\jmath}(t) + z(t)\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{k}}(t)\end{aligned}$$

We saw, in the last (optional) §1.2.7, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\imath}(t) = \boldsymbol{\Omega}\times\hat{\imath}(t) \quad \frac{\mathrm{d}}{\mathrm{d}t}\hat{\jmath}(t) = \boldsymbol{\Omega}\times\hat{\jmath}(t) \quad \frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{k}}(t) = \boldsymbol{\Omega}\times\hat{\mathbf{k}}(t)$$

(You could also verify that these are correct by putting in $\boldsymbol{\Omega} = (0, 0, \Omega)$ and explicitly computing the cross products.) So

$$\mathbf{V}(t) = \Big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,\hat{\imath}(t) + \frac{\mathrm{d}y}{\mathrm{d}t}(t)\,\hat{\jmath}(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t)\,\hat{\mathbf{k}}(t)\Big) + \boldsymbol{\Omega}\times\Big(x(t)\,\hat{\imath}(t) + y(t)\,\hat{\jmath}(t) + z(t)\,\hat{\mathbf{k}}(t)\Big)$$

Differentiating a second time, the acceleration of the moving particle (which is also $\frac{\mathbf{F}}{m}$, where $\mathbf{F}$ is the net force being applied to the particle and m is the mass of the particle) as measured by the fixed observer is

$$\begin{aligned}\frac{\mathbf{F}}{m} = \mathbf{A}(t) = &\Big(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t)\,\hat{\imath}(t) + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}(t)\,\hat{\jmath}(t) + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}(t)\,\hat{\mathbf{k}}(t)\Big)\\ &+ 2\boldsymbol{\Omega}\times\Big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,\hat{\imath}(t) + \frac{\mathrm{d}y}{\mathrm{d}t}(t)\,\hat{\jmath}(t) + \frac{\mathrm{d}z}{\mathrm{d}t}(t)\,\hat{\mathbf{k}}(t)\Big)\\ &+ \boldsymbol{\Omega}\times\Big(\boldsymbol{\Omega}\times\big[x(t)\,\hat{\imath}(t) + y(t)\,\hat{\jmath}(t) + z(t)\,\hat{\mathbf{k}}(t)\big]\Big)\end{aligned}$$

Recall that the angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)$ does not depend on time. The rotating observer sees $\hat{\imath}(t)$ as $\hat{\imath} = (1, 0, 0)$, sees $\hat{\jmath}(t)$ as $\hat{\jmath} = (0, 1, 0)$, and sees $\hat{\mathbf{k}}(t)$ as $\hat{\mathbf{k}} = (0, 0, 1)$ and so sees

$$\frac{\mathbf{F}}{m} = \mathbf{a}(t) + 2\boldsymbol{\Omega}\times\mathbf{v}(t) + \boldsymbol{\Omega}\times\big[\boldsymbol{\Omega}\times\mathbf{r}(t)\big]$$

where, as usual,

$$\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \Big(\frac{\mathrm{d}x}{\mathrm{d}t}(t)\,,\ \frac{\mathrm{d}y}{\mathrm{d}t}(t)\,,\ \frac{\mathrm{d}z}{\mathrm{d}t}(t)\Big)$$
$$\mathbf{a}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}(t) = \Big(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t)\,,\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2}(t)\,,\ \frac{\mathrm{d}^2z}{\mathrm{d}t^2}(t)\Big)$$

So the acceleration of the particle seen by the moving observer is

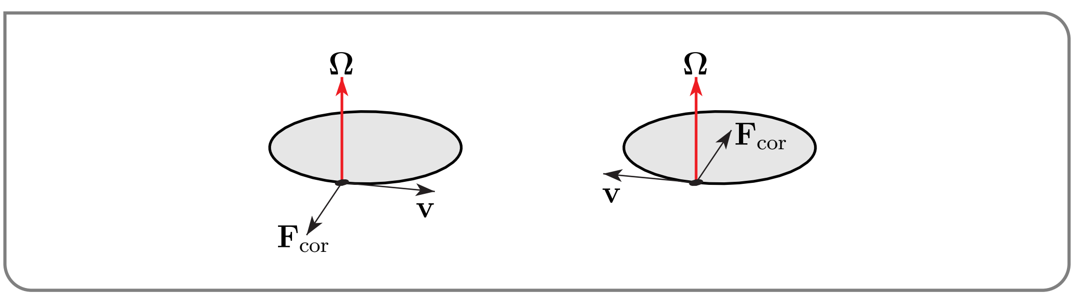
$$\mathbf{a}(t) = \frac{\mathbf{F}}{m} - 2\boldsymbol{\Omega}\times\mathbf{v}(t) - \boldsymbol{\Omega}\times\big[\boldsymbol{\Omega}\times\mathbf{r}(t)\big]$$

Here

- $\mathbf{F}$ is the sum of all external forces acting on the moving particle,
- $\mathbf{F}_{\rm cor} = -2\boldsymbol{\Omega}\times\mathbf{v}(t)$ is called the Coriolis force and
- $-\boldsymbol{\Omega}\times\big[\boldsymbol{\Omega}\times\mathbf{r}(t)\big]$ is called the centrifugal force.

As an example, suppose that you are the moving particle and that you are at the edge of the merry-go-round. Let's say $t = 0$ and you are at $\hat{\boldsymbol{\imath}}$. Then $\mathbf{F}$ is the friction that the surface of the merry-go-round applies to the soles of your shoes. If you are just standing there, $\mathbf{v}(t) = \mathbf{0}$, so that $\mathbf{F}_{\rm cor} = \mathbf{0}$, and the friction $\mathbf{F}$ exactly cancels the centrifugal force $-\boldsymbol{\Omega}\times\big[\boldsymbol{\Omega}\times\mathbf{r}(t)\big]$ so that you remain at $\hat{\boldsymbol{\imath}}(t)$. Assume that $\Omega > 0$. Now suppose that you start walking around the edge of the merry-go-round. Then, at $t = 0$, $\mathbf{r} = \hat{\boldsymbol{\imath}}$ and

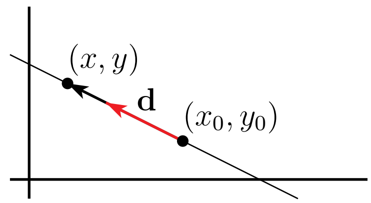
- if you walk in the direction of rotation (with speed one), as in the figure on the left below, $\mathbf{v} = \hat{\boldsymbol{\jmath}}$ and the Coriolis force $\mathbf{F}_{\rm cor} = -2\Omega\hat{\mathbf{k}}\times\hat{\boldsymbol{\jmath}} = 2\Omega\,\hat{\boldsymbol{\imath}}$ tries to push you off of the merry-go-round, while
- if you walk opposite to the direction of rotation (with speed one), as in the figure on the right below, $\mathbf{v} = -\hat{\boldsymbol{\jmath}}$ so that the Coriolis force $\mathbf{F}_{\rm cor} = -2\Omega\hat{\mathbf{k}}\times(-\hat{\boldsymbol{\jmath}}) = -2\Omega\,\hat{\boldsymbol{\imath}}$ tries to pull you into the centre of the merry-go-round.



On a rotating ball, such as the Earth, the Coriolis force deflects wind to the right (counterclockwise) in the northern hemisphere and to the left (clockwise) is the southern hemisphere. In particular, hurricanes/cyclones/typhoons rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. On the other hand, when it comes to water draining out of, for example, a toilet, Coriolis force effects are dominated by other factors like asymmetry of the toilet.

1.3▲ Equations of Lines in 2d

A line in two dimensions can be specified by giving one point (x_0, y_0) on the line and one vector $\mathbf{d} = \langle d_x, d_y \rangle$ whose direction is parallel to the line. If (x, y) is any point on the line



then the vector $\langle x - x_0, y - y_0 \rangle$, whose tail is at (x_0, y_0) and whose head is at (x, y), must be parallel to $\mathbf{d}$ and hence must be a scalar multiple of $\mathbf{d}$. So

Equation 1.3.1 (Parametric Equations)**.**

$$\langle x - x_0, y - y_0 \rangle = t\mathbf{d}$$

or, writing out in components,

$$\begin{aligned} x - x_0 &= td_x \\ y - y_0 &= td_y \end{aligned}$$

These are called the parametric equations of the line, because they contain a free parameter, namely t. As t varies from $-\infty$ to ∞, the point $(x_0 + td_x, y_0 + td_y)$ traverses the entire line.

It is easy to eliminate the parameter t from the equations. Just multiply $x - x_0 = td_x$ by d_y, multiply $y - y_0 = td_y$ by d_x and subtract to give

$$(x - x_0)d_y - (y - y_0)d_x = 0$$

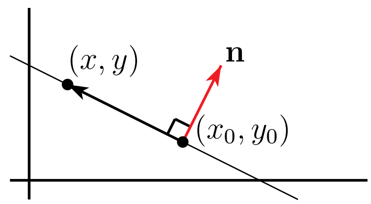
In the event that d_x and d_y are both nonzero, we can rewrite this as

Equation 1.3.2 (Symmetric Equation)**.**

$$\frac{x - x_0}{d_x} = \frac{y - y_0}{d_y}$$

which is called the symmetric equation for the line.

A second way to specify a line in two dimensions is to give one point (x_0, y_0) on the line and one vector $\mathbf{n} = \langle n_x, n_y \rangle$ whose direction is perpendicular to that of the line. If



(x, y) is any point on the line then the vector $\langle x - x_0, y - y_0 \rangle$, whose tail is at (x_0, y_0) and whose head is at (x, y), must be perpendicular to $\mathbf{n}$ so that

Equation 1.3.3.

$$\mathbf{n} \cdot \langle x - x_0, y - y_0 \rangle = 0$$

Writing out in components

$$n_x(x - x_0) + n_y(y - y_0) = 0 \qquad \text{or} \qquad n_x x + n_y y = n_x x_0 + n_y y_0$$

Observe that the coefficients n_x, n_y of x and y in the equation of the line are the components of a vector $\langle n_x, n_y \rangle$ perpendicular to the line. This enables us to read off a vector perpendicular to any given line directly from the equation of the line. Such a vector is called a normal vector for the line.

Example 1.3.4

Consider, for example, the line $y = 3x + 7$. To rewrite this equation in the form

$$n_x x + n_y y = n_x x_0 + n_y y_0$$

we have to move terms around so that x and y are on one side of the equation and 7 is on the other side: $3x - y = -7$. Then n_x is the coefficient of x, namely 3, and n_y is the coefficient of y, namely -1. One normal vector for $y = 3x + 7$ is $\langle 3, -1 \rangle$.

Of course, if $\langle 3, -1 \rangle$ is perpendicular to $y = 3x + 7$, so is $-5 \langle 3, -1 \rangle = \langle -15, 5 \rangle$. In fact, if we first multiply the equation $3x - y = -7$ by -5 to get $-15x + 5y = 35$ and then set n_x and n_y to the coefficients of x and y respectively, we get $\mathbf{n} = \langle -15, 5 \rangle$.

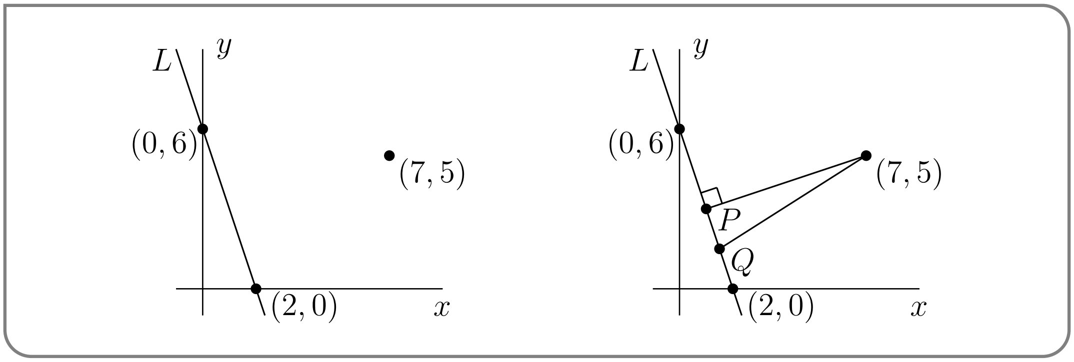
Example 1.3.4

Example 1.3.5

In this example, we find the point on the line $y = 6 - 3x$ (call the line L) that is closest to the point $(7, 5)$.

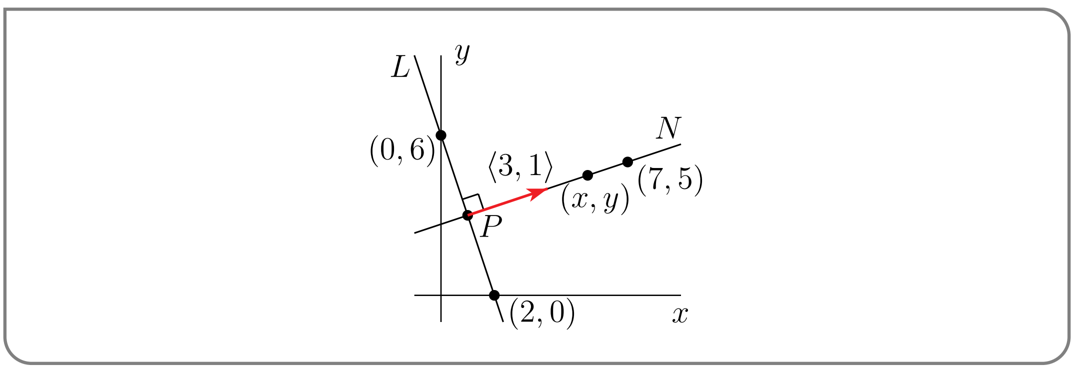
We'll start by sketching the line. To do so, we guess two points on L and then draw the line that passes through the two points.

- If (x, y) is on L and $x = 0$, then $y = 6$. So $(0, 6)$ is on L.
- If (x, y) is on L and $y = 0$, then $x = 2$. So $(2, 0)$ is on L.



Denote by P the point on L that is closest to $(7,5)$. It is characterized by the property that the line from $(7,5)$ to P is perpendicular to L. This is the case just because if Q is any other point on L, then, by Pythagoras, the distance from $(7,5)$ to Q is larger than the distance from $(7,5)$ to P. See the figure on the right above.

Let's use N to denote the line which passes through $(7,5)$ and which is perpendicular to L. Since L has the equation $3x + y = 6$, one vector perpendicular to L, and hence



parallel to N, is $\langle 3,1\rangle$. So if (x,y) is any point on N, the vector $\langle x-7, y-5\rangle$ must be of the form $t\,\langle 3,1\rangle$. So the parametric equations of N are

$$\langle x-7, y-5\rangle = t\,\langle 3,1\rangle \qquad \text{or} \qquad x = 7+3t,\ y = 5+t$$

Now let (x,y) be the coordinates of P. Since P is on N, we have $x = 7+3t$, $y = 5+t$ for some t. Since P is also on L, we also have $3x + y = 6$. So

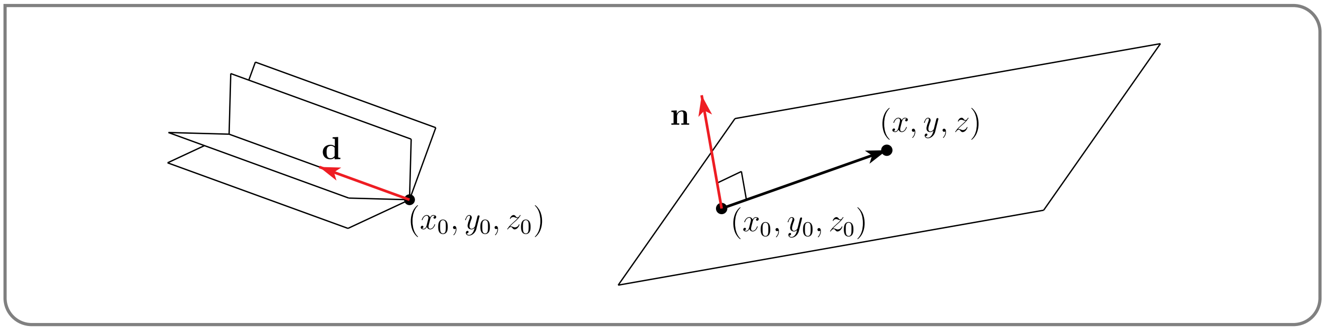
$$\begin{aligned}
& & 3(7+3t) + (5+t) &= 6 \\
&\iff & 10t + 26 &= 6 \\
&\iff & t &= -2 \\
&\implies & x = 7 + 3\times(-2) = 1,\ y &= 5 + (-2) = 3
\end{aligned}$$

and P is $(1,3)$.

Example 1.3.5

1.4▲ Equations of Planes in 3d

Specifying one point (x_0, y_0, z_0) on a plane and a vector $\mathbf{d}$ parallel to the plane does not uniquely determine the plane, because it is free to rotate about $\mathbf{d}$. On the other hand, giving one point on the plane and one vector $\mathbf{n} = \langle n_x, n_y, n_z \rangle$ with direction perpendicular



to that of the plane does uniquely determine the plane. If (x, y, z) is any point on the line then the vector $\langle x - x_0, y - y_0, z - z_0 \rangle$, whose tail is at (x_0, y_0, z_0) and whose head is at (x, y, z), lies entirely inside the plane and so must be perpendicular to $\mathbf{n}$. That is,

Equation 1.4.1 (The Equation of a Plane).

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Writing out in components

$$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0 \quad \text{or} \quad n_x x + n_y y + n_z z = d$$

where $d = n_x x_0 + n_y y_0 + n_z z_0$.

Again, the coefficients n_x, n_y, n_z of x, y and z in the equation of the plane are the components of a vector $\langle n_x, n_y, n_z \rangle$ perpendicular to the plane. The vector $\mathbf{n}$ is often called a normal vector for the plane. Any nonzero multiple of $\mathbf{n}$ will also be perpendicular to the plane and is also called a normal vector.

Example 1.4.2

We have just seen that if we write the equation of a plane in the standard form

$$ax + by + cz = d$$

then it is easy to read off a normal vector for the plane. It is just $\langle a, b, c \rangle$. So for example the planes

$$P: \; x + 2y + 3z = 4 \qquad P': \; 3x + 6y + 9z = 7$$

have normal vectors $\mathbf{n} = \langle 1, 2, 3 \rangle$ and $\mathbf{n}' = \langle 3, 6, 9 \rangle$, respectively. Since $\mathbf{n}' = 3\mathbf{n}$, the two normal vectors $\mathbf{n}$ and $\mathbf{n}'$ are parallel to each other. This tells us that the planes P and P' are parallel to each other.

When the normal vectors of two planes are perpendicular to each other, we say that the planes are perpendicular to each other. For example the planes

$$P:\ x+2y+3z=4 \qquad P'':\ 2x-y=7$$

have normal vectors $\mathbf{n}=\langle 1,2,3\rangle$ and $\mathbf{n}''=\langle 2,-1,0\rangle$, respectively. Since

$$\mathbf{n}\cdot\mathbf{n}''=1\times 2+2\times(-1)+3\times 0=0$$

the normal vectors $\mathbf{n}$ and $\mathbf{n}''$ are mutually perpendicular, so the corresponding planes P and P'' are perpendicular to each other.

Example 1.4.2

Example 1.4.3

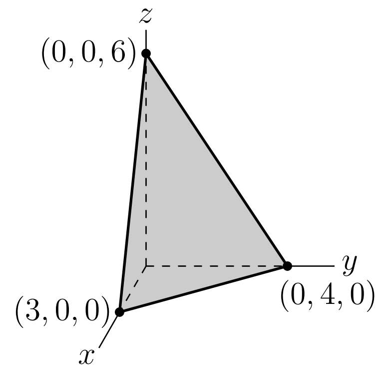
In this example, we'll sketch the plane

$$P:\ 4x+3y+2z=12$$

A good way to prepare for sketching a plane is to find the intersection points of the plane with the x-, y- and z-axes, just as you are used to doing when sketching lines in the xy-plane. For example, any point on the x axis must be of the form $(x,0,0)$. For $(x,0,0)$ to also be on P we need $x={}^{12}/_{4}=3$. So P intersects the x-axis at $(3,0,0)$. Similarly, P intersects the y-axis at $(0,4,0)$ and the z-axis at $(0,0,6)$. Now plot the points $(3,0,0)$, $(0,4,0)$ and $(0,0,6)$. P is the plane containing these three points. Often a visually effective way to sketch a surface in three dimensions is to

- only sketch the part of the surface in the first octant. That is, the part with $x\geqslant 0$, $y\geqslant 0$ and $z\geqslant 0$.
- To do so, sketch the curve of intersection of the surface with the part of the xy-plane in the first octant and,
- similarly, sketch the curve of intersection of the surface with the part of the xz-plane in the first octant and the curve of intersection of the surface with the part of the yz-plane in the first octant.

That's what we'll do. The intersection of the plane P with the xy-plane is the straight line through the two points $(3,0,0)$ and $(0,4,0)$. So the part of that intersection in the first octant is the line segment from $(3,0,0)$ to $(0,4,0)$. Similarly the part of the intersection of P with the xz-plane that is in the first octant is the line segment from $(3,0,0)$ to $(0,0,6)$ and the part of the intersection of P with the yz-plane that is in the first octant is the line segment from $(0,4,0)$ to $(0,0,6)$. So we just have to sketch the three line segments joining the three axis intercepts $(3,0,0)$, $(0,4,0)$ and $(0,0,6)$. That's it.



Example 1.4.3

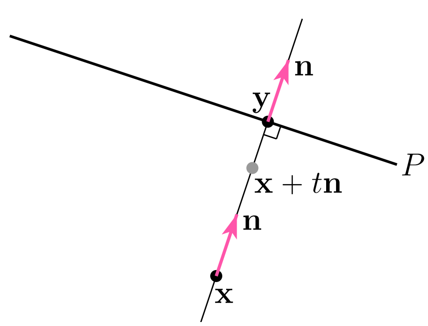
Example 1.4.4

In this example, we'll compute the distance between the point

$$\mathbf{x} = (1, -1, -3) \qquad \text{and the plane} \qquad P:\ x + 2y + 3z = 18$$

By the "distance between **x** and the plane P" we mean the shortest distance between **x** and any point **y** on P. In fact, we'll evaluate the distance in two different ways. In the next Example 1.4.5, we'll use projection. In this example, our strategy for finding the distance will be to

- first observe that the vector $\mathbf{n} = \langle 1, 2, 3 \rangle$ is normal to P and then
- start walking[18] away from **x** in the direction of the normal vector **n** and
- keep walking until we hit P. Call the point on P where we hit, **y**. Then the desired distance is the distance between **x** and **y**. From the figure below it does indeed look like distance between **x** and **y** is the shortest distance between **x** and any point on P. This is in fact true, though we won't prove it.



18 To see why heading in the normal direction gives the shortest walk, revisit Example 1.3.5.

So imagine that we start walking, and that we start at time $t = 0$ at $\mathbf{x}$ and walk in the direction $\mathbf{n}$. Then at time t we might be at

$$\mathbf{x} + t\mathbf{n} = (1, -1, -3) + t\,\langle 1, 2, 3\rangle = (1 + t, -1 + 2t, -3 + 3t)$$

We hit the plane P at exactly the time t for which $(1 + t, -1 + 2t, -3 + 3t)$ satisfies the equation for P, which is $x + 2y + 3z = 18$. So we are on P at the unique time t obeying

$$(1 + t) + 2(-1 + 2t) + 3(-3 + 3t) = 18 \iff 14t = 28 \iff t = 2$$

So the point on P which is closest to $\mathbf{x}$ is

$$\mathbf{y} = \big[\mathbf{x} + t\mathbf{n}\big]_{t=2} = (1 + t, -1 + 2t, -3 + 3t)\big|_{t=2} = (3, 3, 3)$$

and the distance from $\mathbf{x}$ to P is the distance from $\mathbf{x}$ to $\mathbf{y}$, which is

$$|\mathbf{y} - \mathbf{x}| = 2|\mathbf{n}| = 2\sqrt{1^2 + 2^2 + 3^2} = 2\sqrt{14}$$

Example 1.4.4

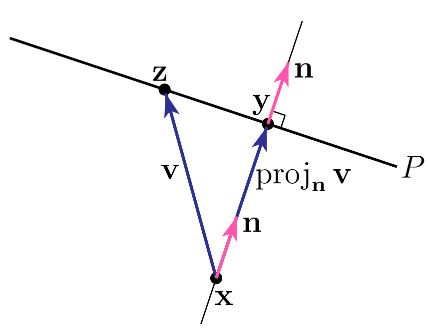
Example 1.4.5 (Example 1.4.4, revisited)

We are again going to find the distance from the point

$$\mathbf{x} = (1, -1, -3) \qquad \text{to the plane} \qquad P:\ x + 2y + 3z = 18$$

But this time we will use the following strategy.

- We'll first find any point $\mathbf{z}$ on P and then
- we'll denote by $\mathbf{y}$ the point on P nearest $\mathbf{x}$, and we'll denote by $\mathbf{v}$ the vector from $\mathbf{x}$ to $\mathbf{z}$ (see the figure below) and then
- we'll realize, by looking at the figure, that the vector from $\mathbf{x}$ to $\mathbf{y}$ is exactly the projection[19] of the vector $\mathbf{v}$ on $\mathbf{n}$ so that
- the distance from $\mathbf{x}$ to P, i.e. the length of the vector from $\mathbf{x}$ to $\mathbf{y}$, is exactly $|\text{proj}_{\mathbf{n}}\mathbf{v}|$.



19 Now might be a good time to review the Definition 1.2.13 of projection.

Now let's find a point on P. The plane P is given by a single equation, namely

$$x + 2y + 3z = 18$$

in the three unknowns, x, y, z. The easiest way to find one solution to this equation is to assign two of the unknowns the value zero and then solve for the third unknown. For example, if we set $x = y = 0$, then the equation reduces to $3z = 18$. So we may take $\mathbf{z} = (0, 0, 6)$.

Then $\mathbf{v}$, the vector from $\mathbf{x} = (1, -1, -3)$ to $\mathbf{z} = (0, 0, 6)$ is $\langle 0 - 1\,,\, 0 - (-1)\,,\, 6 - (-3)\rangle = \langle -1, 1, 9\rangle$ so that, by Equation (1.2.14),

$$\begin{aligned}
\operatorname{proj}_{\mathbf{n}} \mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n} \\
&= \frac{\langle -1, 1, 9\rangle \cdot \langle 1, 2, 3\rangle}{|\,\langle 1, 2, 3\rangle\,|^2}\,\langle 1, 2, 3\rangle \\
&= \frac{28}{14}\,\langle 1, 2, 3\rangle \\
&= 2\,\langle 1, 2, 3\rangle
\end{aligned}$$

and the distance from $\mathbf{x}$ to P is

$$\left|\operatorname{proj}_{\mathbf{n}} \mathbf{v}\right| = \left|2\,\langle 1, 2, 3\rangle\,\right| = 2\sqrt{14}$$

just as we found in Example 1.4.4.

Example 1.4.5

Example 1.4.6

Now we'll increase the degree of difficulty a tiny bit, and compute the distance between the planes

$$P:\ x + 2y + 2z = 1 \qquad \text{and} \qquad P':\ 2x + 4y + 4z = 11$$

By the "distance between the planes P and P'" we mean the shortest distance between any pair of points $\mathbf{x}$ and $\mathbf{x}'$ with $\mathbf{x}$ in P and $\mathbf{x}'$ in P'. First observe that the normal vectors

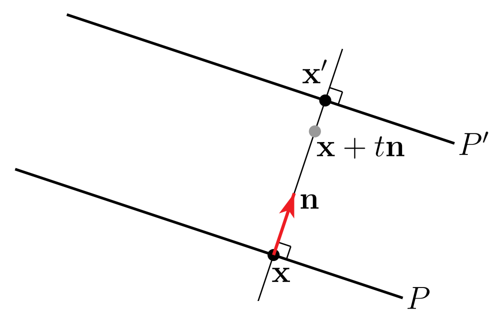
$$\mathbf{n} = \langle 1, 2, 2\rangle \qquad \text{and} \qquad \mathbf{n}' = \langle 2, 4, 4\rangle = 2\mathbf{n}$$

to P and P' are parallel to each other. So the planes P and P' are parallel to each other.

If they had not been parallel, they would have crossed and the distance between them would have been zero.

Our strategy for finding the distance will be to

- first find a point $\mathbf{x}$ on P and then, like we did in Example 1.4.4,
- start walking away from P in the direction of the normal vector $\mathbf{n}$ and
- keep walking until we hit P'. Call the point on P' that we hit $\mathbf{x}'$. Then the desired distance is the distance between $\mathbf{x}$ and $\mathbf{x}'$. From the figure below it does indeed look like distance between $\mathbf{x}$ and $\mathbf{x}'$ is the shortest distance between any pair of points with one point on P and one point on P'. Again, this is in fact true, though we won't prove it.



We can find a point on P just as we did on Example 1.4.5. The plane P is given by the single equation

$$x + 2y + 2z = 1$$

in the three unknowns, x, y, z. We can find one solution to this equation by assigning two of the unknowns the value zero and then solving for the third unknown. For example, if we set $y = z = 0$, then the equation reduces to $x = 1$. So we may take $\mathbf{x} = (1, 0, 0)$.

Now imagine that we start walking, and that we start at time $t = 0$ at $\mathbf{x}$ and walk in the direction $\mathbf{n}$. Then at time t we might be at

$$\mathbf{x} + t\mathbf{n} = (1, 0, 0) + t\ \langle 1, 2, 2\rangle = (1 + t, 2t, 2t)$$

We hit the second plane P' at exactly the time t for which $(1+t, 2t, 2t)$ satisfies the equation for P', which is $2x + 4y + 4z = 11$. So we are on P' at the unique time t obeying

$$2(1 + t) + 4(2t) + 4(2t) = 11 \iff 18t = 9 \iff t = \frac{1}{2}$$

So the point on P' which is closest to $\mathbf{x}$ is

$$\mathbf{x}' = \big[\mathbf{x} + t\mathbf{n}\big]_{t=1/2} = (1 + t, 2t, 2t)\big|_{t=1/2} = (3/2, 1, 1)$$

and the distance from P to P' is the distance from $\mathbf{x}$ to $\mathbf{x}'$ which is

$$\sqrt{(1 - 3/2)^2 + (0 - 1)^2 + (0 - 1)^2} = \sqrt{9/4} = 3/2$$

Example 1.4.6

Example 1.4.7

The orientation (i.e. direction) of a plane is determined by its normal vector. So, by definition, the angle between two planes is the angle between their normal vectors. For example, the normal vectors of the two planes

$$\begin{aligned} P_1 :&\quad 2x + y - z = 3 \\ P_2 :&\quad x + y + z = 4 \end{aligned}$$

are

$$\begin{aligned} \mathbf{n}_1 &= \langle 2, 1, -1 \rangle \\ \mathbf{n}_2 &= \langle 1, 1, 1 \rangle \end{aligned}$$

If we use θ to denote the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, then

$$\begin{aligned} \cos\theta &= \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} \\ &= \frac{\langle 2, 1, -1 \rangle \cdot \langle 1, 1, 1 \rangle}{|\,\langle 2, 1, -1 \rangle\,|\,|\,\langle 1, 1, 1 \rangle\,|} \\ &= \frac{2}{\sqrt{6}\,\sqrt{3}} \end{aligned}$$

so that

$$\theta = \arccos \frac{2}{\sqrt{18}} = 1.0799$$

to four decimal places. That's in radians. In degrees, it is $1.0799\frac{180}{\pi} = 61.87^\circ$ to two decimal places.

Example 1.4.7

1.5 Equations of Lines in 3d

Just as in two dimensions, a line in three dimensions can be specified by giving one point (x_0, y_0, z_0) on the line and one vector $\mathbf{d} = \langle d_x, d_y, d_z \rangle$ whose direction is parallel to that of the line. If (x, y, z) is any point on the line then the vector $\langle x - x_0, y - y_0, z - z_0 \rangle$, whose tail is at (x_0, y_0, z_0) and whose arrow is at (x, y, z), must be parallel to $\mathbf{d}$ and hence a scalar multiple of $\mathbf{d}$. By translating this statement into a vector equation we get

Equation 1.5.1 (Parametric Equations of a Line).

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\mathbf{d}$$

or the three corresponding scalar equations

$$x - x_0 = td_x \qquad y - y_0 = td_y \qquad z - z_0 = td_z$$

These are called the parametric equations of the line. Solving all three equations for the parameter t (assuming that d_x, d_y and d_z are all nonzero)

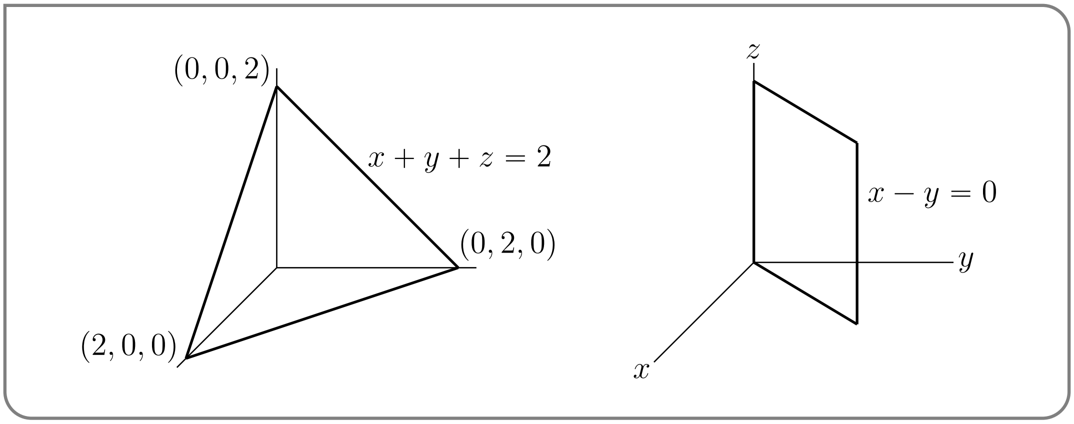
$$t = \frac{x - x_0}{d_x} = \frac{y - y_0}{d_y} = \frac{z - z_0}{d_z}$$

and erasing the "$t =$" again gives the (so called) symmetric equations for the line.

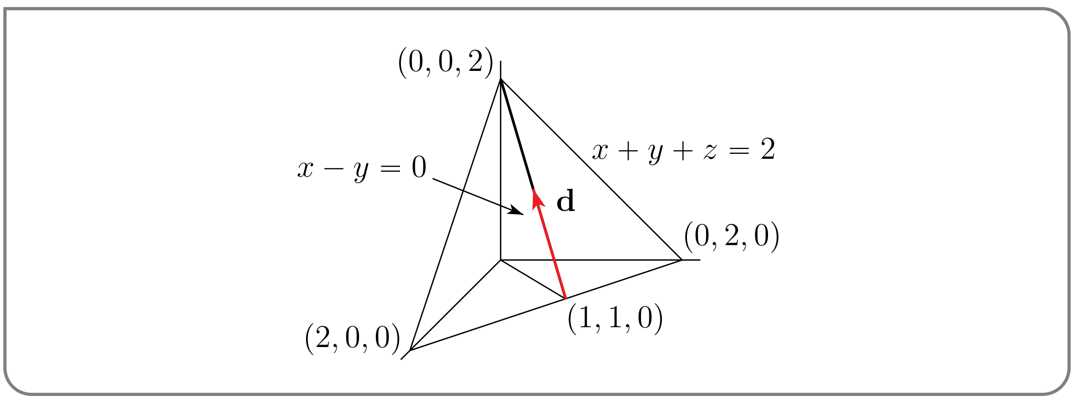
Example 1.5.2

The set of points (x, y, z) that obey $x + y + z = 2$ form a plane. The set of points (x, y, z) that obey $x - y = 0$ form a second plane. The set of points (x, y, z) that obey both $x + y + z = 2$ and $x - y = 0$ lie on the intersection of these two planes and hence form a line. We shall find the parametric equations for that line.

To sketch $x + y + z = 2$ we observe that if any two of x, y, z are zero, then the third is 2. So all of $(0, 0, 2)$, $(0, 2, 0)$ and $(2, 0, 0)$ are on $x + y + z = 2$. The plane $x - y = 0$ contains all of the z-axis, since $(0, 0, z)$ obeys $x - y = 0$ for all z. Here are separate sketches of (parts of) the two planes.



And here is a sketch of their intersection



Method 1. Each point on the line has a different value of z. We'll use z as the parameter. (We could just as well use x or y.) There is no law that requires us to use the parameter name t, but that's what we have done so far, so set $t = z$. If (x, y, z) is on the line then $z = t$ and

$$\begin{aligned} x + y + t &= 2 \\ x - y \qquad &= 0 \end{aligned}$$

The second equation forces $y = x$. Substituting this into the first equation gives

$$2x + t = 2 \implies x = y = 1 - \frac{t}{2}$$

So the parametric equations are

$$x = 1 - \frac{t}{2},\ y = 1 - \frac{t}{2},\ z = t \qquad \text{or} \qquad \langle x-1, y-1, z\rangle = t\left\langle -\frac{1}{2}, -\frac{1}{2}, 1\right\rangle$$

Method 2. We first find one point on the line. There are lots of them. We'll find the point with $z = 0$. (We could just as well use z=123.4, but arguably $z = 0$ is a little easier.) If (x, y, z) is on the line and $z = 0$, then

$$\begin{aligned} x + y &= 2 \\ x - y &= 0 \end{aligned}$$

The second equation again forces $y = x$. Substituting this into the first equation gives

$$2x = 2 \implies x = y = 1$$

So $(1, 1, 0)$ is on the line. Now we'll find a direction vector, $\mathbf{d}$, for the line.

- Since the line is contained in the plane $x + y + z = 2$, any vector lying on the line, like $\mathbf{d}$, is also completely contained in that plane. So $\mathbf{d}$ must be perpendicular to the normal vector of $x + y + z = 2$, which is $\langle 1, 1, 1\rangle$.
- Similarly, since the line is contained in the plane $x - y = 0$, any vector lying on the line, like $\mathbf{d}$, is also completely contained in that plane. So $\mathbf{d}$ must be perpendicular to the normal vector of $x - y = 0$, which is $\langle 1, -1, 0\rangle$.

So we may choose for $\mathbf{d}$ any vector which is perpendicular to both $\langle 1, 1, 1\rangle$ and $\langle 1, -1, 0\rangle$, like, for example,

$$\begin{aligned} \mathbf{d} &= \langle 1, -1, 0\rangle \times \langle 1, 1, 1\rangle \\ &= \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \hat{\imath}\det\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} - \hat{\jmath}\det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \hat{\mathbf{k}}\det\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= -\hat{\imath} - \hat{\jmath} + 2\hat{\mathbf{k}} \end{aligned}$$

We now have both a point on the line (namely $(1, 1, 0)$) and a direction vector for the line (namely $\langle -1, -1, 2\rangle$), so, as usual, the parametric equations for the line are

$$\langle x-1, y-1, z\rangle = t\,\langle -1, -1, 2\rangle \qquad \text{or} \qquad x = 1 - t,\ y = 1 - t,\ z = 2t$$

This looks a little different than the solution from method 1, but we'll see in a moment that they are really the same. Before that, let's do one more method.

Method 3. We'll find two points on the line. We have already found that $(1, 1, 0)$ is on the line. From the picture above, it looks like $(0, 0, 2)$ is also on the line. This is indeed the case since $(0, 0, 2)$ obeys both $x + y + z = 2$ and $x - y = 0$. Notice that we could also have guessed $(0, 0, 2)$ by setting $x = 0$ and then solving $y + z = x + y + z = 2$, $-y = x - y = 0$ for x and y. As both $(1, 1, 0)$ and $(0, 0, 2)$ are on the line, the vector with head at $(1, 1, 0)$ and tail at $(0, 0, 2)$, which is $\langle 1 - 0, 1 - 0, 0 - 2\rangle = \langle 1, 1, -2\rangle$, is a direction vector for the line. As $(0,0,2)$ is a point on the line and $\langle 1,1,-2\rangle$ is a direction vector for the line, the parametric equations for the line are

$$\langle x-0, y-0, z-2\rangle = t\,\langle 1,1,-2\rangle \qquad \text{or} \qquad x=t,\ y=t,\ z=2-2t$$

This also looks similar, but not quite identical, to our previous answers. Time for a comparison.

Comparing the answers. The parametric equations given by the three methods are different. That's just because we have really used different parameters in the three methods, even though we have called the parameter t in each case. To clarify the relation between the three answers, rename the parameter of method 1 to t_1, the parameter of method 2 to t_2 and the parameter of method 3 to t_3. The parametric equations then become

$$\begin{array}{llll} \text{Method 1:} & x = 1-\frac{t_1}{2} & y = 1-\frac{t_1}{2} & z = t_1 \\ \text{Method 2:} & x = 1-t_2 & y = 1-t_2 & z = 2t_2 \\ \text{Method 3:} & x = t_3 & y = t_3 & z = 2-2t_3 \end{array}$$

Substituting $t_1 = 2t_2$ into the Method 1 equations gives the Method 2 equations, and substituting $t_3 = 1-t_2$ into the Method 3 equations gives the Method 2 equations. So all three really give the same line, just parametrized a little differently.

Example 1.5.2

Warning 1.5.3.

A line in three dimensions has infinitely many normal vectors.
For example, the line

$$\langle x-1, y-1, z\rangle = t\,\langle 1,2,-2\rangle$$

has direction vector $\langle 1,2,-2\rangle$. Any vector perpendicular to $\langle 1,2,-2\rangle$ is perpendicular to the line. The vector $\langle n_1,n_2,n_3\rangle$ is perpendicular to $\langle 1,2,-2\rangle$ if and only if

$$0 = \langle 1,2,-2\rangle \cdot \langle n_1,n_2,n_3\rangle = n_1+2n_2-2n_3$$

There is whole plane of $\langle n_1,n_2,n_3\rangle$'s obeying this condition, of which $\langle 2,-1,0\rangle$, $\langle 0,1,1\rangle$ and $\langle 2,0,1\rangle$ are only three examples.

The next two examples illustrate two different methods for finding the distance between a point and a line.

Example 1.5.4

In this example, we find the distance between the point $(2,3,-1)$ and the line

$$L:\ \langle x-1, y-2, z-3\rangle = t\,\langle 1,1,2\rangle \qquad \text{or, equivalently,} \qquad x=1+t,\ y=2+t,\ z=3+2t$$

The vector from $(2,3,-1)$ to the point $(1+t\,,\,2+t\,,\,3+2t)$ on L is $\langle t-1\,,\,t-1\,,\,2t+4\rangle$. The square of the distance between $(2,3,-1)$ and the point $(1+t\,,\,2+t\,,\,3+2t)$ on L is the square of the length of that vector, namely

$$d(t)^2 = (t-1)^2 + (t-1)^2 + (2t+4)^2$$

The point on L that is closest to $(2,3,-1)$ is that whose value of t obeys

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}d(t)^2 = 2(t-1) + 2(t-1) + 2(2)(2t+4) \tag{$*$}$$

Before we solve this equation for t and finish of our computation, observe that this equation (divided by 2) says that

$$\langle 1\,,\,1\,,\,2\rangle \cdot \langle t-1\,,\,t-1\,,\,2t+4\rangle = 0$$

That is, *the vector from* $(2,3,-1)$ *to the point on* L *nearest* $(2,3,-1)$ *is perpendicular to* L*'s direction vector*.

Now back to our computation. The equation $(*)$ simplifies to $12t+12=0$. So the optimal $t=-1$ and the distance is

$$d(-1) = \sqrt{(-1-1)^2 + (-1-1)^2 + (-2+4)^2} = \sqrt{12}$$

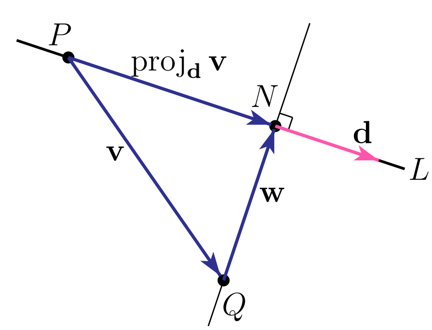
Example 1.5.4

Example 1.5.5 (Example 1.5.4 revisited)

In this example, we again find the distance between the point $(2,3,-1)$ and the line

$$L:\ \langle x-1, y-2, z-3\rangle = t\,\langle 1,1,2\rangle$$

but we use a different method. In the figure below, Q is the point $(2,3,-1)$.



If we drop a perpendicular from Q to the line L, it hits the line L at the point N, which is the point on L that is nearest Q. So the distance from Q to L is exactly the distance from Q to N, which is exactly the length of the vector from Q to N. In the figure above, $\mathbf{w}$ is the vector from Q to N. Now the vector $\mathbf{w}$ has to be perpendicular to the direction vector for L. That is, $\mathbf{w}$ has to be perpendicular to $\mathbf{d} = \langle 1,1,2\rangle$. However, as we saw in Warning 1.5.3, there are a huge number of vectors in different directions that are perpendicular to $\mathbf{d}$. So you might think that it is very hard to even determine the direction of $\mathbf{w}$.

Fortunately, it isn't. Here is the strategy.

- Pick any point on L and call it P.
- It is very easy to find the vector from P to N — it is just the projection of the vector from P to Q (called $\mathbf{v}$ in the figure above) on $\mathbf{d}$.
- Once we know $\mathrm{proj}_{\mathbf{d}}\, \mathbf{v}$, we will be able to compute

$$\mathbf{w} = \mathrm{proj}_{\mathbf{d}}\, \mathbf{v} - \mathbf{v}$$

- and then the distance from Q to the line L is just $|\mathbf{w}|$.

Here is the computation. We'll choose P to be the point on L that has $t = 0$, which is $(1, 2, 3)$. So the vector from $P = (1, 2, 3)$ to $Q = (2, 3, -1)$ is

$$\mathbf{v} = \langle 2-1, 3-2, -1-3\rangle = \langle 1, 1, -4\rangle$$

The projection of $\mathbf{v} = \langle 1, 1, -4\rangle$ on $\mathbf{d} = \langle 1, 1, 2\rangle$ is

$$\mathrm{proj}_{\mathbf{d}}\, \mathbf{v} = \frac{\langle 1, 1, -4\rangle \cdot \langle 1, 1, 2\rangle}{|\,\langle 1, 1, 2\rangle\,|^2} \langle 1, 1, 2\rangle = \frac{-6}{6} \langle 1, 1, 2\rangle = \langle -1, -1, -2\rangle$$

and then

$$\mathbf{w} = \mathrm{proj}_{\mathbf{d}}\, \mathbf{v} - \mathbf{v} = \langle -1, -1, -2\rangle - \langle 1, 1, -4\rangle = \langle -2, -2, 2\rangle$$

and finally the distance from Q to the line L is

$$|\mathbf{w}| = |\,\langle -2, -2, 2\rangle\,| = |2\,\langle -1, -1, 1\rangle\,| = 2\sqrt{3}$$

Example 1.5.5

The next two (optional) examples illustrate two different methods for finding the distance between two lines.

Example 1.5.6 (Optional)

In this example, we find the distance between the lines

$$\begin{aligned} L:&\ \langle x-1, y-2, z-3\rangle = t\,\langle 1, 0, -1\rangle \\ L':&\ \langle x-1, y-2, z-1\rangle = t\,\langle 1, -2, 1\rangle \end{aligned}$$

We can rewrite the equations of the lines as

$$\begin{aligned} L:&\ x = 1+t,\ y = 2,\ z = 3-t \\ L':&\ x = 1+t,\ y = 2-2t,\ z = 1+t \end{aligned}$$

Of course the value of t in the parametric equation for L need not be the same as the value of t in the parametric equation for L'. So let us denote by $\mathbf{x}(s) = (1+s\,, 2\,, 3-s)$ and $\mathbf{y}(t) = (1+t\,, 2-2t\,, 1+t)$ the points on L and L', respectively, that are closest together. Note that the vector from $\mathbf{x}(s)$ to $\mathbf{y}(t)$ is $\langle t-s\,, -2t\,, -2+s+t\rangle$. Then, in particular,

- $\mathbf{x}(s)$ is the point on L that is closest to the point $\mathbf{y}(t)$, and
- $\mathbf{y}(t)$ is the point on L' that is closest to the point $\mathbf{x}(s)$.

So, as we saw in Example 1.5.4, the vector, $\langle t-s\,,\,-2t\,,\,-2+s+t\rangle$, that joins $\mathbf{x}(s)$ and $\mathbf{y}(t)$, must be perpendicular to both the direction vector of L and the direction vector of L'. Consequently

$$\begin{aligned} 0 &= \langle 1,0,-1\rangle\cdot\langle t-s\,,\,-2t\,,\,-2+s+t\rangle = 2-2s\\ 0 &= \langle 1,-2,1\rangle\cdot\langle t-s\,,\,-2t\,,\,-2+s+t\rangle = -2+6t \end{aligned}$$

So $s=1$ and $t={}^1\!/_3$ and the distance between L and L' is

$$\big|\,\langle t-s\,,\,-2t\,,\,-2+s+t\rangle\,\big|_{s=1,\ t=1/3} = \big|\,\langle -2/3\,,\,-2/3\,,\,-2/3\rangle\,\big| = \frac{2}{\sqrt{3}}$$

Example 1.5.6

Example 1.5.7 (Example 1.5.6 revisited, again optional)

In this example, we again find the distance between the lines

$$\begin{aligned} L:&\ \langle x-1,y-2,z-3\rangle = t\,\langle 1,0,-1\rangle\\ L':&\ \langle x-1,y-2,z-1\rangle = t\,\langle 1,-2,1\rangle \end{aligned}$$

this time using a projection, much as in Example 1.4.5. The procedure, which will be justified below, is

- first form a vector $\mathbf{n}$ that is perpendicular to the direction vectors of both lines by taking the cross product of the two direction vectors. In this example,

$$\langle 1,0,-1\rangle\times\langle 1,-2,1\rangle = \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 1 & 0 & -1\\ 1 & -2 & 1\end{bmatrix} = -2\hat{\imath}-2\hat{\jmath}-2\hat{\mathbf{k}}$$

  Since we just want $\hat{\mathbf{n}}$ to be perpendicular to both direction vectors, we may simplify our computations by dividing this vector by -2, and take $\mathbf{n}=\langle 1,1,1\rangle$.

- Next find one point on L and one point on L' and subtract to form a vector $\mathbf{v}$ whose tail is at one point and whose head is at the other point. This vector goes from one line to the other line. In this example, the point $(1,2,3)$ is on L (just set $t=0$ in the equation for L) and the point $(1,2,1)$ is on L' (just set $t=0$ in the equation for L'), so that we may take

$$\mathbf{v} = \langle 1-1\,,\,2-2\,,\,3-1\rangle = \langle 0,0,2\rangle$$

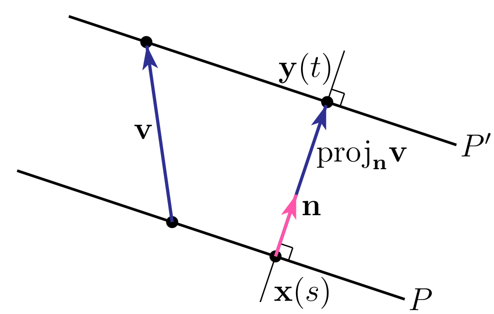
- The distance between the two lines is the length of the projection of $\mathbf{v}$ on $\mathbf{n}$. In this example, by (1.2.14), the distance is

$$\begin{aligned} \big|\mathrm{proj}_{\mathbf{n}}\,\mathbf{v}\big| &= \left|\frac{\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n}\right| = \frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{n}|}\\ &= \frac{|\,\langle 0,0,2\rangle\cdot\langle 1,1,1\rangle\,|}{|\,\langle 1,1,1\rangle\,|}\\ &= \frac{2}{\sqrt{3}} \end{aligned}$$

  just as we found in Example 1.5.6

Now, here is the justification for the procedure.

- As we did in Example 1.5.6, denote by $\mathbf{x}(s)$ and $\mathbf{y}(t)$ the points on L and L', respectively, that are closest together. Note that, as we observed in Example 1.5.6, the vector from $\mathbf{x}(s)$ to $\mathbf{y}(t)$ is perpendicular to the direction vectors of both lines, and so is parallel to $\mathbf{n}$.
- Denote by P the plane through $\mathbf{x}(s)$ that is perpendicular to $\mathbf{n}$. As $\mathbf{x}(s)$ is on L and the direction vector of L is perpendicular to $\mathbf{n}$, the line L is contained in P.
- Denote by P' the plane through $\mathbf{y}(t)$ that is perpendicular to $\mathbf{n}$. As $\mathbf{y}(t)$ is on L' and the direction vector of L' is perpendicular to $\mathbf{n}$, the line L' is contained in P'.
- The planes P and P' are parallel to each other. As $\mathbf{x}(s)$ is on P and $\mathbf{y}(t)$ is on P', and the vector from $\mathbf{x}(s)$ to $\mathbf{y}(t)$ is perpendicular to both P and P', the distance from P to P' is exactly the length of the vector from $\mathbf{x}(s)$ to $\mathbf{y}(t)$. That is also the distance from L to L'.
- The vector $\mathbf{v}$ constructed in the procedure above is a vector between L and L' and so is also a vector between P and P'. Looking at the figure below[20], we see that the vector from $\mathbf{x}(s)$ to $\mathbf{y}(t)$ is (up to a sign) the projection of $\mathbf{v}$ on $\mathbf{n}$.



- So the distance from P to P', and hence the distance from L to L', is exactly the length of $\text{proj}_{\mathbf{n}}\mathbf{v}$.

Example 1.5.7

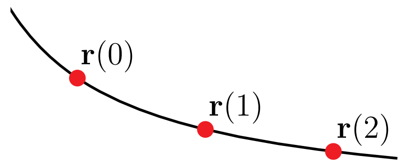
1.6▲ Curves and their Tangent Vectors

The right hand side of the parametric equation $(x,y,z) = (1,1,0) + t\,\langle 1,2,-2\rangle$ that we just saw in Warning 1.5.3 is a vector-valued function of the one real variable t. We are now going to study more general vector-valued functions of one real variable. That is, we are going to study functions that assign to each real number t (typically in some interval) a vector $\mathbf{r}(t)$. For example

$$\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$$

20 and possibly reviewing the Definition 1.2.13 of projection

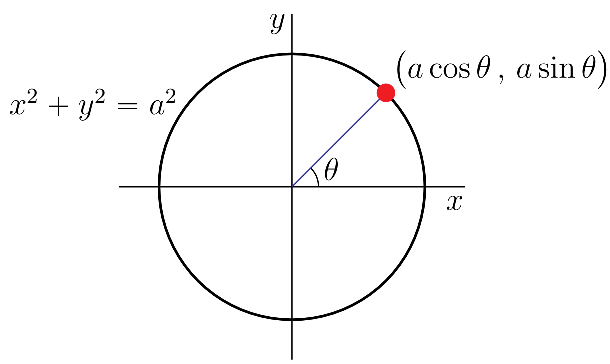
might be the position[21] of a particle at time t. As t varies $\mathbf{r}(t)$ sweeps out a curve.



While in some applications t will indeed be "time", it does not have to be. It can be simply a parameter that is used to label the different points on the curve that $\mathbf{r}(t)$ sweeps out. We then say that $\mathbf{r}(t)$ provides a parametrization of the curve.

Example 1.6.1 (Parametrization of $x^2 + y^2 = a^2$)

While we will often use t as the parameter in a parametrized curve $\mathbf{r}(t)$, there is no need to call it t. Sometimes it is natural to use a different name for the parameter. For example, consider the circle[22] $x^2 + y^2 = a^2$. It is natural to use the angle θ in the sketch below to label the point $\big(a\cos\theta\,,\ a\sin\theta\big)$ on the circle.



That is,

$$\mathbf{r}(\theta) = \big(a\cos\theta\,,\ a\sin\theta\big) \qquad 0 \leqslant \theta < 2\pi$$

is a parametrization of the circle $x^2 + y^2 = a^2$. Just looking at the figure above, it is clear that, as θ runs from 0 to 2π, $\mathbf{r}(\theta)$ traces out the full circle.

However beware that just knowing that $\mathbf{r}(t)$ lies on a specified curve does not guarantee that, as t varies, $\mathbf{r}(t)$ covers the entire curve. For example, as t runs over the whole real line, $\frac{2}{\pi}\arctan(t)$ runs over the interval $(-1, 1)$. For all t,

$$\mathbf{r}(t) = \big(x(t), y(t)\big) = a\left(\frac{2}{\pi}\arctan(t)\,,\ \sqrt{1 - \frac{4}{\pi^2}\arctan^2(t)}\right)$$

is well-defined and obeys $x(t)^2 + y(t)^2 = a^2$. But this $\mathbf{r}(t)$ does not cover the entire circle because $y(t)$ is always positive.

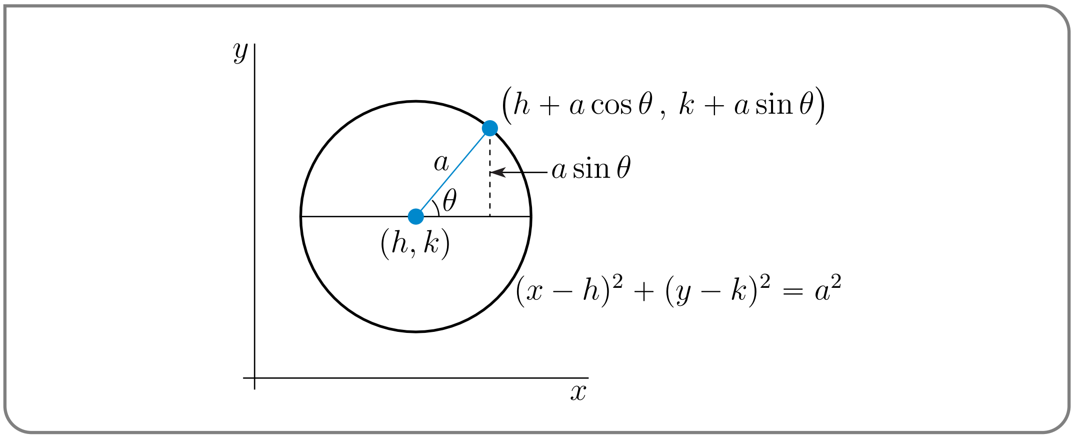
21 When we say $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$, we mean that $\big(x(t), y(t), z(t)\big)$ is the point at the head of the vector $\mathbf{r}(t)$ when its tail is at the origin.

22 We of course assume that the constant $a > 0$.

Example 1.6.1

Example 1.6.2 (Parametrization of $(x-h)^2+(y-k)^2=a^2$)

We can tweak the parametrization of Example 1.6.1 to get a parametrization of the circle of radius a that is centred on (h,k). One way to do so is to redraw the sketch of Example 1.6.1 with the circle translated so that its centre is at (h,k).



We see from the sketch that

$$\mathbf{r}(\theta)=\big(h+a\cos\theta\,,\,k+a\sin\theta\big)\qquad 0\leqslant\theta<2\pi$$

is a parametrization of the circle $(x-h)^2+(y-k)^2=a^2$.

A second way to come up with this parametrization is to observe that we can turn the trig identity $\cos^2 t+\sin^2 t=1$ into the equation $(x-h)^2+(y-k)^2=a^2$ of the circle by

- multiplying the trig identity by a^2 to get $(a\cos t)^2+(a\sin t)^2=a^2$ and then
- setting $a\cos t=x-h$ and $a\sin t=y-k$, which turns $(a\cos t)^2+(a\sin t)^2=a^2$ into $(x-h)^2+(y-k)^2=a^2$.

Example 1.6.2

Example 1.6.3 $\left(\text{Parametrization of } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and of } x^{2/3}+y^{2/3}=a^{2/3}\right)$

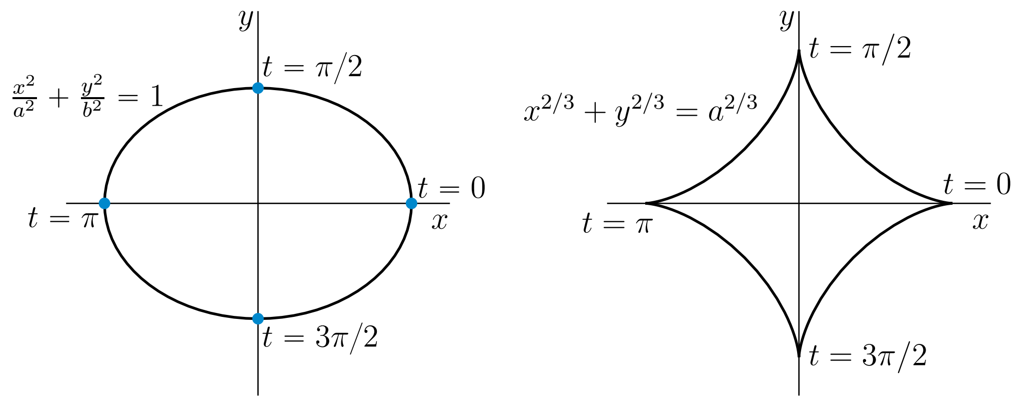
We can build parametrizations of the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $x^{2/3}+y^{2/3}=a^{2/3}$ from the trig identity $\cos^2 t+\sin^2 t=1$, like we did in the second part of the last example.

- Setting $\cos t=\frac{x}{a}$ and $\sin t=\frac{y}{b}$ turns $\cos^2 t+\sin^2 t=1$ into $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
- Setting $\cos t=\left(\frac{x}{a}\right)^{\frac13}$ and $\sin t=\left(\frac{y}{a}\right)^{\frac13}$ turns $\cos^2 t+\sin^2 t=1$ into $\frac{x^{2/3}}{a^{2/3}}+\frac{y^{2/3}}{a^{2/3}}=1$.

So

$$\begin{aligned}\mathbf{r}(t) &= \big(a\cos t\,,\ b\sin t\big) \qquad 0 \leqslant t < 2\pi\\ \mathbf{r}(t) &= \big(a\cos^3 t\,,\ a\sin^3 t\big) \quad 0 \leqslant t < 2\pi\end{aligned}$$

give parametrizations of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^{2/3} + y^{2/3} = a^{2/3}$, respectively. To see that running t from 0 to 2π runs $\mathbf{r}(t)$ once around the curve, look at the figures below.



The curve $x^{2/3} + y^{2/3} = a^{2/3}$ is called an astroid. From its equation, we would expect its sketch to look like a deformed circle. But it is probably not so obvious that it would have the pointy bits of the right hand figure. We will not explain here why they arise. The astroid is studied in some detail in Example 1.1.9 of the CLP-4 text. In particular, the above sketch is carefully developed there.

Example 1.6.3

Example 1.6.4 (Parametrization of $e^y = 1 + x^2$)

A very easy method that can often create parametrizations for a curve is to use x or y as a parameter. Because we can solve $e^y = 1 + x^2$ for y as a function of x, namely $y = \ln\big(1 + x^2\big)$, we can use x as the parameter simply by setting $t = x$. This gives the parametrization

$$\mathbf{r}(t) = \big(t\,,\ \ln(1 + t^2)\big) \qquad -\infty < t < \infty$$

Example 1.6.4

Example 1.6.5 (Parametrization of $x^2 + y^2 = a^2$, again)

It is also quite common that one can use either x or y to parametrize part of, but not all of, a curve. A simple example is the circle $x^2 + y^2 = a^2$. For each $-a < x < a$, there are two points on the circle with that value of x. So one cannot use x to parametrize the whole circle. Similarly, for each $-a < y < a$, there are two points on the circle with that value of y. So one cannot use y to parametrize the whole circle. On the other hand

$$\mathbf{r}(t) = \big(t\,,\ \sqrt{a^2 - t^2}\big) \qquad -a < t < a$$
$$\mathbf{r}(t) = \big(t\,,\ -\sqrt{a^2 - t^2}\big) \qquad -a < t < a$$

provide parametrizations of the top half and bottom half, respectively, of the circle using x as the parameter, and

$$\mathbf{r}(t) = \big(\sqrt{a^2 - t^2}\,,\ t\big) \qquad -a < t < a$$
$$\mathbf{r}(t) = \big(-\sqrt{a^2 - t^2}\,,\ t\big) \qquad -a < t < a$$

provide parametrizations of the right half and left half, respectively, of the circle using y as the parameter.

Example 1.6.5

Example 1.6.6 (Unparametrization of $\mathbf{r}(t) = (\cos t, 7 - t)$)

In this example, we will undo the parametrization $\mathbf{r}(t) = (\cos t, 7 - t)$ and find the Cartesian equation of the curve in question. We may rewrite the parametrization as

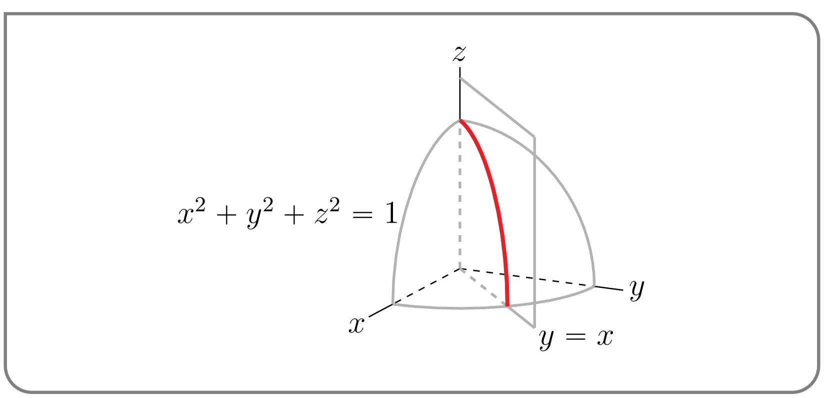
$$x = \cos t$$
$$y = 7 - t$$

Note that we can eliminate the parameter t simply by using the second equation to solve for t as a function of y. Namely $t = 7 - y$. Substituting this into the first equation gives us the Cartesian equation

$$x = \cos(7 - y)$$

Example 1.6.6

Curves often arise as the intersection of two surfaces. For example, the intersection of the sphere $x^2 + y^2 + z^2 = 1$ with the plane $y = x$ is a circle. The part of that circle that is in the first octant is the red curve in the figure below. One way to parametrize such curves



is to choose one of the three coordinates x, y, z as the parameter, and solve the two given equations for the remaining two coordinates, as functions of the parameter. Here are two examples.

Example 1.6.7

The set of all (x, y, z) obeying

$$\begin{aligned} x - y \qquad &= 0 \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

is the circle sketched above. We can choose to use y as the parameter and think of

$$\begin{aligned} x \qquad &= y \\ x^2 + z^2 &= 1 - y^2 \end{aligned}$$

as a system of two equations for the two unknowns x and z, with y being treated as a given constant, rather than as an unknown. We can now (trivially) solve the first equation for x, substitute the result into the second equation, and finally solve for z.

$$x = y,\ x^2 + z^2 = 1 - y^2 \quad \Longrightarrow z^2 = 1 - 2y^2$$

If, for example, we are interested in points (x, y, z) on the curve with $z \geqslant 0$, we have $z = \sqrt{1 - 2y^2}$ and

$$\mathbf{r}(y) = \left(y\,,\, y\,,\, \sqrt{1 - 2y^2}\right), \qquad -\frac{1}{\sqrt{2}} \leqslant y \leqslant \frac{1}{\sqrt{2}}$$

is a parametrization for the part of the circle above the xy-plane. If, on the other hand, we are interested in points (x, y, z) on the curve with $z \leqslant 0$, we have $z = -\sqrt{1 - 2y^2}$ and

$$\mathbf{r}(y) = \left(y\,,\, y\,,\, -\sqrt{1 - 2y^2}\right), \qquad -\frac{1}{\sqrt{2}} \leqslant y \leqslant \frac{1}{\sqrt{2}}$$

is a parametrization for the part of the circle below the xy-plane.

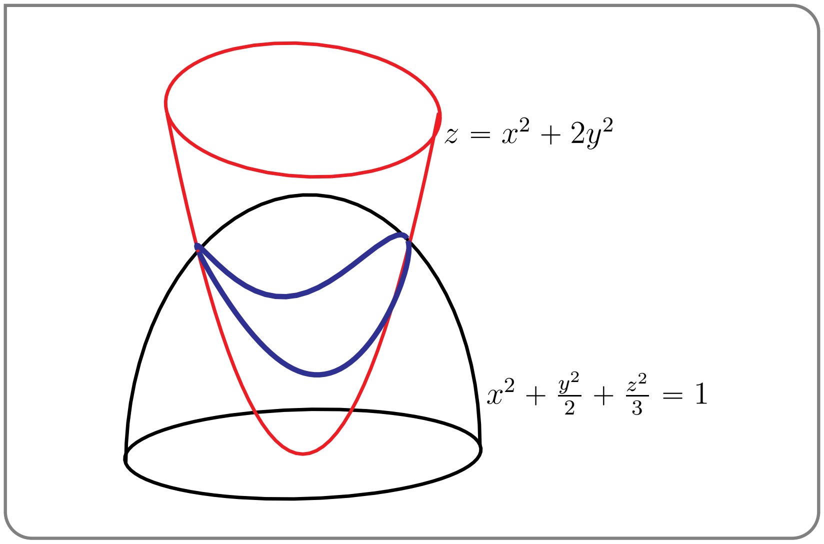
Example 1.6.7

Example 1.6.8

The previous example was rigged so that it was easy to solve for x and z as functions of y. In practice it is not always easy, or even possible, to do so. A more realistic example is the set of all (x, y, z) obeying

$$\begin{aligned} x^2 + \frac{y^2}{2} + \frac{z^2}{3} &= 1 \\ x^2 + 2y^2 &= z \end{aligned}$$

which is the blue curve in the figure



(Don't worry about how we make sketches like this. We'll develop some surface sketching technique in §1.7 below.) Substituting $x^2 = z - 2y^2$ (from the second equation) into the first equation gives

$$-\frac{3}{2}y^2 + z + \frac{z^2}{3} = 1$$

or, completing the square,

$$-\frac{3}{2}y^2 + \frac{1}{3}\left(z + \frac{3}{2}\right)^2 = \frac{7}{4}$$

If, for example, we are interested in points (x, y, z) on the curve with $y \geqslant 0$, this can be solved to give y as a function of z.

$$y = \sqrt{\frac{2}{9}\left(z + \frac{3}{2}\right)^2 - \frac{14}{12}}$$

Then $x^2 = z - 2y^2$ also gives x as a function of z. If $x \geqslant 0$,

$$\begin{aligned} x &= \sqrt{z - \frac{4}{9}\left(z + \frac{3}{2}\right)^2 + \frac{14}{6}} \\ &= \sqrt{\frac{4}{3} - \frac{4}{9}z^2 - \frac{1}{3}z} \end{aligned}$$

The other signs of x and y can be gotten by using the appropriate square roots. In this example, (x, y, z) is on the curve, i.e. satisfies the two original equations, if and only if all of $(\pm x, \pm y, z)$ are also on the curve.

Example 1.6.8

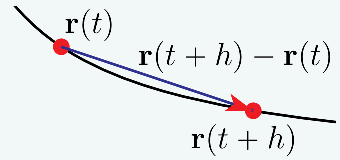
1.6.1 ▸▸ Derivatives and Tangent Vectors

This being a Calculus text, one of our main operations is differentiation. We are now interested in parametrizations $\mathbf{r}(t)$. It is very easy and natural to extend our definition of derivative to $\mathbf{r}(t)$ as follows.

Definition 1.6.9.

The derivative of the vector valued function $\mathbf{r}(t)$ is defined to be

$$\mathbf{r}'(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



when the limit exists. In particular, if $\mathbf{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{\mathbf{k}}$, then

$$\mathbf{r}'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{\mathbf{k}}$$

That is, to differentiate a vector valued function of t, just differentiate each of its components.

And of course differentiation interacts with arithmetic operations, like addition, in the obvious way. Only a little more thought is required to see that differentiation interacts quite nicely with dot and cross products too. Here are some examples.

Example 1.6.10

Let

$$\begin{aligned}
\mathbf{a}(t) &= t^2\,\hat{\imath} + t^4\,\hat{\jmath} + t^6\,\hat{\mathbf{k}} \\
\mathbf{b}(t) &= e^{-t}\,\hat{\imath} + e^{-3t}\,\hat{\jmath} + e^{-5t}\,\hat{\mathbf{k}} \\
\gamma(t) &= t^2 \\
s(t) &= \sin t
\end{aligned}$$

We are about to compute some derivatives. To make it easier to follow what is going on, we'll use some colour. When we apply the product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[f(t)\,g(t)\big] = f'(t)\,g(t) + f(t)\,g'(t)$$

we'll use blue to highlight the factors $f'(t)$ and $g'(t)$. Here we go.

$$\begin{aligned}
\gamma(t)\,\mathbf{b}(t) &= t^2e^{-t}\,\hat{\imath} + t^2e^{-3t}\,\hat{\jmath} + t^2e^{-5t}\,\hat{\mathbf{k}} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\big[\gamma(t)\mathbf{b}(t)\big] &= \big[2te^{-t} - t^2e^{-t}\big]\hat{\imath} + \big[2te^{-3t} - 3t^2e^{-3t}\big]\hat{\jmath} + \big[2te^{-5t} - 5t^2e^{-5t}\big]\hat{\mathbf{k}} \\
&= 2t\big\{e^{-t}\,\hat{\imath} + e^{-3t}\,\hat{\jmath} + e^{-5t}\,\hat{\mathbf{k}}\big\} + t^2\big\{-e^{-t}\,\hat{\imath} - 3e^{-3t}\,\hat{\jmath} - 5e^{-5t}\,\hat{\mathbf{k}}\big\} \\
&= \gamma'(t)\mathbf{b}(t) + \gamma(t)\mathbf{b}'(t)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}(t)\cdot\mathbf{b}(t) &= t^2e^{-t} + t^4e^{-3t} + t^6e^{-5t} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}(t)\cdot\mathbf{b}(t)\big] &= \big[2te^{-t} - t^2e^{-t}\big] + \big[4t^3e^{-3t} - 3t^4e^{-3t}\big] + \big[6t^5e^{-5t} - 5t^6e^{-5t}\big] \\
&= \big[2te^{-t} + 4t^3e^{-3t} + 6t^5e^{-5t}\big] + \big[-t^2e^{-t} - 3t^4e^{-3t} - 5t^6e^{-5t}\big] \\
&= \big\{2t\,\hat{\imath} + 4t^3\,\hat{\jmath} + 6t^5\,\hat{\mathbf{k}}\big\}\cdot\big\{e^{-t}\,\hat{\imath} + e^{-3t}\,\hat{\jmath} + e^{-5t}\,\hat{\mathbf{k}}\big\} \\
&\qquad + \big\{t^2\,\hat{\imath} + t^4\,\hat{\jmath} + t^6\,\hat{\mathbf{k}}\big\}\cdot\big\{ - e^{-t}\,\hat{\imath} - 3e^{-3t}\,\hat{\jmath} - 5e^{-5t}\,\hat{\mathbf{k}}\big\} \\
&= \mathbf{a}'(t)\cdot\mathbf{b}(t) + \mathbf{a}(t)\cdot\mathbf{b}'(t)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}(t)\times\mathbf{b}(t) &= \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ t^2 & t^4 & t^6 \\ e^{-t} & e^{-3t} & e^{-5t}\end{bmatrix} \\
&= \hat{\imath}\big(t^4e^{-5t} - t^6e^{-3t}\big) - \hat{\jmath}\big(t^2e^{-5t} - t^6e^{-t}\big) + \hat{\mathbf{k}}\big(t^2e^{-3t} - t^4e^{-t}\big) \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}(t)\times\mathbf{b}(t)\big] &= \hat{\imath}\big(\,4t^3e^{-5t} - 6t^5e^{-3t}\big) - \hat{\jmath}\big(\,2te^{-5t} - 6t^5e^{-t}\big) + \hat{\mathbf{k}}\big(\,2te^{-3t} - 4t^3e^{-t}\big) \\
&\quad + \hat{\imath}\big(-5t^4e^{-5t} + 3t^6e^{-3t}\big) - \hat{\jmath}\big(-5t^2e^{-5t} + t^6e^{-t}\big) + \hat{\mathbf{k}}\big(-3t^2e^{-3t} + t^4e^{-t}\big) \\
&= \big\{2t\,\hat{\imath} + 4t^3\,\hat{\jmath} + 6t^5\,\hat{\mathbf{k}}\big\}\times\big\{e^{-t}\,\hat{\imath} + e^{-3t}\,\hat{\jmath} + e^{-5t}\,\hat{\mathbf{k}}\big\} \\
&\quad + \big\{t^2\,\hat{\imath} + t^4\,\hat{\jmath} + t^6\,\hat{\mathbf{k}}\big\}\times\big\{ - e^{-t}\,\hat{\imath} - 3e^{-3t}\,\hat{\jmath} - 5e^{-5t}\,\hat{\mathbf{k}}\big\} \\
&= \mathbf{a}'(t)\times\mathbf{b}(t) + \mathbf{a}(t)\times\mathbf{b}'(t)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}\big(s(t)\big) &= (\sin t)^2\,\hat{\imath} + (\sin t)^4\,\hat{\jmath} + (\sin t)^6\,\hat{\mathbf{k}} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}\big(s(t)\big)\big] &= 2(\sin t)\cos t\,\hat{\imath} + 4(\sin t)^3\cos t\,\hat{\jmath} + 6(\sin t)^5\cos t\,\hat{\mathbf{k}} \\
&= \big\{2(\sin t)\,\hat{\imath} + 4(\sin t)^3\hat{\jmath} + 6(\sin t)^5\hat{\mathbf{k}}\big\}\cos t \\
&= \mathbf{a}'\big(s(t)\big)\,s'(t)
\end{aligned}$$

Example 1.6.10

Of course these examples extend to general (differentiable) $\mathbf{a}(t)$, $\mathbf{b}(t)$, $\gamma(t)$ and $s(t)$ and give us (most of) the following theorem.

Theorem 1.6.11 (Arithmetic of differentiation).

Let

- $\mathbf{a}(t), \mathbf{b}(t)$ be vector valued differentiable functions of $t \in \mathbb{R}$ that take values in $\mathbb{R}^n$ and
- $\alpha, \beta \in \mathbb{R}$ be constants and
- $\gamma(t)$ and $s(t)$ be real valued differentiable functions of $t \in \mathbb{R}$

Then

(a) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\alpha\, \mathbf{a}(t) + \beta\, \mathbf{b}(t)\big] = \alpha\, \mathbf{a}'(t) + \beta\, \mathbf{b}'(t)$ (linear combination)

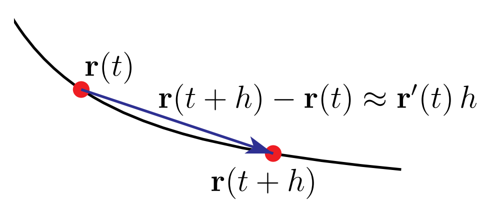
(b) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\gamma(t)\mathbf{b}(t)\big] = \gamma'(t)\mathbf{b}(t) + \gamma(t)\mathbf{b}'(t)$ (multiplication by scalar function)

(c) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}(t) \cdot \mathbf{b}(t)\big] = \mathbf{a}'(t) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot \mathbf{b}'(t)$ (dot product)

(d) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}(t) \times \mathbf{b}(t)\big] = \mathbf{a}'(t) \times \mathbf{b}(t) + \mathbf{a}(t) \times \mathbf{b}'(t)$ (cross product)

(e) $\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{a}\big(s(t)\big)\big] = \mathbf{a}'\big(s(t)\big)\, s'(t)$ (composition)

Let's think about the geometric significance of $\mathbf{r}'(t)$. In particular, let's think about the relationship between $\mathbf{r}'(t)$ and distances along the curve. The derivative $\mathbf{r}'(t)$ is the limit of $\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ as $h \to 0$. The numerator, $\mathbf{r}(t+h) - \mathbf{r}(t)$, is the vector with head at $\mathbf{r}(t+h)$ and tail at $\mathbf{r}(t)$.



When h is very small this vector

- has the essentially the same direction as the tangent vector to the curve at $\mathbf{r}(t)$ and
- has length being essentially the length of the part of the curve between $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$.

Taking the limit as $h \to 0$ yields that

- $\mathbf{r}'(t)$ is a tangent vector to the curve at $\mathbf{r}(t)$ that points in the direction of increasing t and
- if $s(t)$ is the length of the part of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(t)$, then $\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \big|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\big|$.

This is worth stating formally.

Lemma 1.6.12.

Let $\mathbf{r}(t)$ be a parametrized curve.

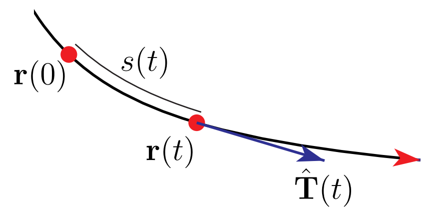
(a) Denote by $\hat{\mathbf{T}}(t)$ the unit tangent vector to the curve at $\mathbf{r}(t)$ pointing in the direction of increasing t. If $\mathbf{r}'(t) \neq 0$ then

$$\hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

(b) Denote by $s(t)$ the length of the part of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(t)$. Then

$$\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right|$$

$$s(T) - s(T_0) = \int_{T_0}^{T} \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t$$



(c) In particular, if the parameter happens to be arc length, i.e. if $t = s$, so that $\frac{\mathrm{d}s}{\mathrm{d}s} = 1$, then

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s)\right| = 1 \qquad \hat{\mathbf{T}}(s) = \mathbf{r}'(s)$$

As an application, we have the

Lemma 1.6.13.

If $\mathbf{r}(t) = \big(x(t)\,,\, y(t)\,,\, z(t)\big)$ is the position of a particle at time t, then

$$\begin{aligned}
\text{velocity at time } t &= \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{\mathbf{k}} = \frac{\mathrm{d}s}{\mathrm{d}t}(t)\,\hat{\mathbf{T}}(t) \\
\text{speed at time } t &= \frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2} \\
\text{acceleration at time } t &= \mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{v}'(t) = x''(t)\hat{\imath} + y''(t)\hat{\jmath} + z''(t)\hat{\mathbf{k}}
\end{aligned}$$

and the distance travelled between times T_0 and T is

$$s(T) - s(T_0) = \int_{T_0}^{T} \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}(t)\right| \mathrm{d}t = \int_{T_0}^{T} \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2}\, \mathrm{d}t$$

Note that the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ is a vector quantity while the speed $\frac{\mathrm{d}s}{\mathrm{d}t}(t) = |\mathbf{r}'(t)|$ is a scalar quantity.

Example 1.6.14 (Circumference of a circle)

In general it can be quite difficult to compute arc lengths. So, as an easy warmup example, we will compute the circumference of the circle[23] $x^2 + y^2 = a^2$. We'll also find a unit tangent to the circle at any point on the circle. We'll use the parametrization

$$\mathbf{r}(\theta) = \big(a\cos\theta\,,\ a\sin\theta\big) \qquad 0 \leqslant \theta \leqslant 2\pi$$

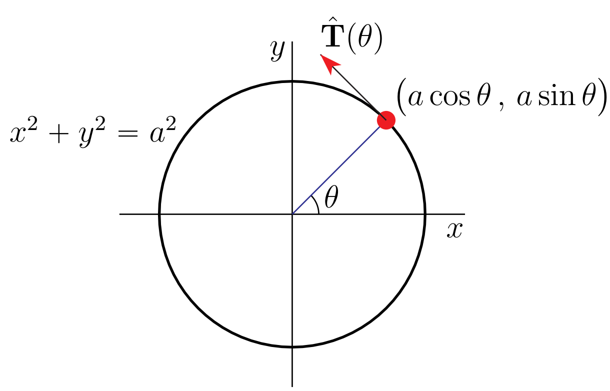
of Example 1.6.1. Using Lemma 1.6.12, but with the parameter t renamed to θ

$$\begin{aligned}
\mathbf{r}'(\theta) &= -a\sin\theta\hat{\boldsymbol{\imath}} + a\cos\theta\hat{\boldsymbol{\jmath}} \\
\hat{\mathbf{T}}(\theta) &= \frac{\mathbf{r}'(\theta)}{|\mathbf{r}'(\theta)|} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}} \\
\frac{\mathrm{d}s}{\mathrm{d}\theta}(\theta) &= \big|\mathbf{r}'(\theta)\big| = a \\
s(\Theta) - s(0) &= \int_0^\Theta \big|\mathbf{r}'(\theta)\big|\,\mathrm{d}\theta = a\Theta
\end{aligned}$$

As[24] $s(\Theta)$ is the arc length of the part of the circle with $0 \leqslant \theta \leqslant \Theta$, the circumference of the whole circle is

$$s(2\pi) = 2\pi a$$

which is reassuring, since this formula has been known[25] for thousands of years. The



formula $s(\Theta) - s(0) = a\Theta$ also makes sense — the part of the circle with $0 \leqslant \theta \leqslant \Theta$ is the fraction $\frac{\Theta}{2\pi}$ of the whole circle, and so should have length $\frac{\Theta}{2\pi} \times 2\pi a$. Also note that

$$\mathbf{r}(\theta)\cdot\hat{\mathbf{T}}(\theta) = \big(a\cos\theta\,,\ a\sin\theta\big)\cdot\big(-\sin\theta\,,\cos\theta\big) = 0$$

23 We of course assume that the constant $a > 0$.
24 You might guess that Θ is a capital Greek theta. You'd be right.
25 The earliest known written approximations of π, in Egypt and Babylon, date from 1900–1600BC. The first recorded algorithm for rigorously evaluating π was developed by Archimedes around 250 BC. The first use of the symbol π, for the ratio between the circumference of a circle and its diameter, in print was in 1706 by William Jones.

so that the tangent to the circle at any point is perpendicular to the radius vector of the circle at that point. This is another geometric fact that has been known[26] for thousands of years.

Example 1.6.14

Example 1.6.15 (Arc length of a helix)

Consider the curve

$$\mathbf{r}(t) = 6\sin(2t)\hat{\imath} + 6\cos(2t)\hat{\jmath} + 5t\hat{\mathbf{k}}$$

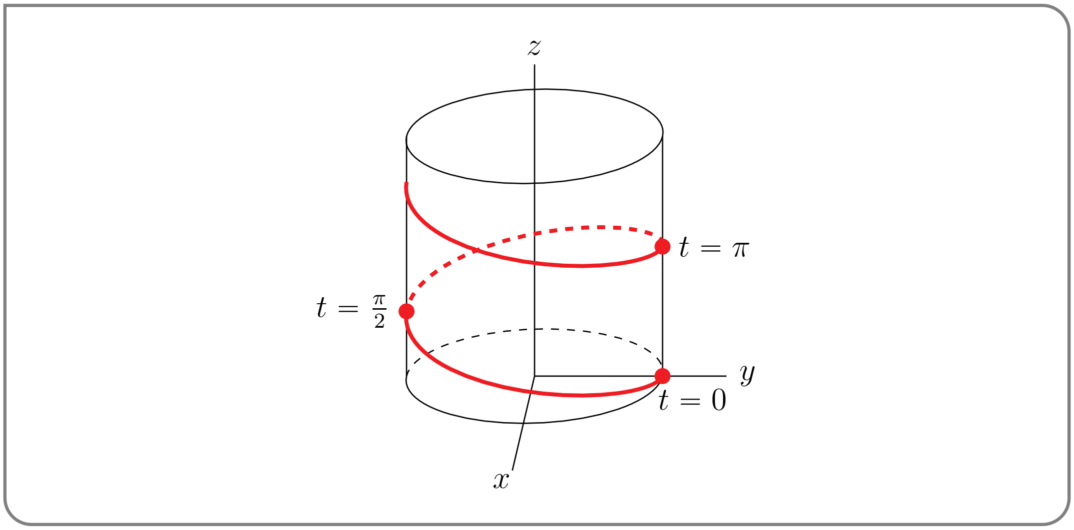
where the standard basis vectors $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$ and $\hat{\mathbf{k}} = (0,0,1)$. We'll first sketch it, by observing that

- $x(t) = 6\sin(2t)$ and $y(t) = 6\cos(2t)$ obey

$$x(t)^2 + y(t)^2 = 36\sin^2(2t) + 36\cos^2(2t) = 36$$

  So all points of the curve lie on the cylinder $x^2 + y^2 = 36$ and
- as t increases, $\big(x(t), y(t)\big)$ runs clockwise around the circle $x^2 + y^2 = 36$ and at the same time $z(t) = 5t$ just increases linearly.

Our curve is the helix



We have marked three points of the curve on the above sketch. The first has $t = 0$ and is $0\hat{\imath} + 6\hat{\jmath} + 0\hat{\mathbf{k}}$. The second has $t = \frac{\pi}{2}$ and is $0\hat{\imath} - 6\hat{\jmath} + \frac{5\pi}{2}\hat{\mathbf{k}}$, and the third has $t = \pi$ and is $0\hat{\imath} + 6\hat{\jmath} + 5\pi\hat{\mathbf{k}}$. We'll now use Lemma 1.6.12 to find a unit tangent $\hat{\mathbf{T}}(t)$ to the curve at $\mathbf{r}(t)$

26 It is Proposition 18 in Book 3 of Euclid's Elements. It was published around 300BC.

and also the arclength of the part of curve between $t = 0$ and $t = \pi$.

$$\begin{aligned}
\mathbf{r}(t) &= 6\sin(2t)\hat{\imath} + 6\cos(2t)\hat{\jmath} + 5t\hat{\mathbf{k}} \\
\mathbf{r}'(t) &= 12\cos(2t)\hat{\imath} - 12\sin(2t)\hat{\jmath} + 5\hat{\mathbf{k}} \\
\frac{ds}{dt}(t) &= |\mathbf{r}'(t)| = \sqrt{12^2\cos^2(2t) + 12^2\sin^2(2t) + 5^2} = \sqrt{12^2 + 5^2} \\
&= 13 \\
\hat{\mathbf{T}}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t))|} = \frac{12}{13}\cos(2t)\hat{\imath} - \frac{12}{13}\sin(2t)\hat{\jmath} + \frac{5}{13}\hat{\mathbf{k}} \\
s(\pi) - s(0) &= \int_0^\pi |\mathbf{r}'(t)|\,dt = 13\pi
\end{aligned}$$

Example 1.6.15

Example 1.6.16 (Velocity and acceleration)

Imagine that, at time t, a particle is at

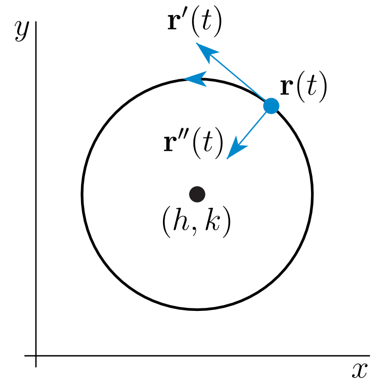
$$\mathbf{r}(t) = \left[h + a\cos\left(2\pi\frac{t}{T}\right)\right]\hat{\imath} + \left[k + a\sin\left(2\pi\frac{t}{T}\right)\right]\hat{\jmath}$$

As $|\mathbf{r}(t) - h\hat{\imath} - k\hat{\jmath}| = a$, the particle is running around the circle of radius a centred on (h, k). When t increases by T, the argument, $2\pi\frac{t}{T}$, of $\cos\left(2\pi\frac{t}{T}\right)$ and $\sin\left(2\pi\frac{t}{T}\right)$ increases by exactly 2π and the particle runs exactly once around the circle. In particular, it travels a distance $2\pi a$. So it is moving at speed $\frac{2\pi a}{T}$. According to Lemma 1.6.13, it has

$$\begin{aligned}
\text{velocity } &= \mathbf{r}'(t) = -\frac{2\pi a}{T}\sin\left(2\pi\frac{t}{T}\right)\hat{\imath} + \frac{2\pi a}{T}\cos\left(2\pi\frac{t}{T}\right)\hat{\jmath} \\
\text{speed} &= \frac{ds}{dt}(t) = |\mathbf{r}'(t)| = \frac{2\pi a}{T} \\
\text{acceleration} &= \mathbf{r}''(t) = -\frac{4\pi^2 a}{T^2}\cos\left(2\pi\frac{t}{T}\right)\hat{\imath} - \frac{4\pi^2 a}{T^2}\sin\left(2\pi\frac{t}{T}\right)\hat{\jmath} = -\frac{4\pi^2}{T^2}\left[\mathbf{r}(t) - h\hat{\imath} - k\hat{\jmath}\right]
\end{aligned}$$

Here are some observations.

- The velocity $\mathbf{r}'(t)$ has dot product zero with $\mathbf{r}(t) - h\hat{\imath} - k\hat{\jmath}$, which is the radius vector from the centre of the circle to the particle. So the velocity is perpendicular to the radius vector, and hence parallel to the tangent vector of the circle at $\mathbf{r}(t)$.
- The speed given by Lemma 1.6.13 is exactly the speed we found above, just before we started applying Lemma 1.6.13.
- The acceleration $\mathbf{r}''(t)$ points in the direction opposite to the radius vector.



Example 1.6.16

1.7 Sketching Surfaces in 3d

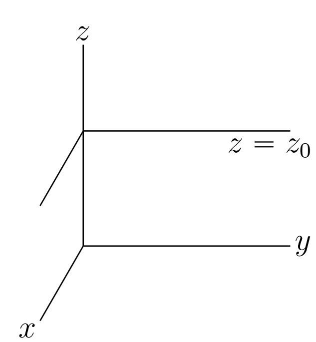
In practice students taking multivariable calculus regularly have great difficulty visualising surfaces in three dimensions, despite the fact that we all live in three dimensions. We'll now develop some technique to help us sketch surfaces in three dimensions[27].

We all have a fair bit of experience drawing curves in two dimensions. Typically the intersection of a surface (in three dimensions) with a plane is a curve lying in the (two dimensional) plane. Such an intersection is usually called a cross-section. In the special case that the plane is one of the coordinate planes, the intersection is sometimes called a trace. One can often get a pretty good idea of what a surface looks like by sketching a bunch of cross-sections. Here are some examples.

Example 1.7.1 $(4x^2 + y^2 - z^2 = 1)$

Sketch the surface that satisfies $4x^2 + y^2 - z^2 = 1$.

Solution. We'll start by fixing any number z_0 and sketching the part of the surface that lies in the horizontal plane $z = z_0$.

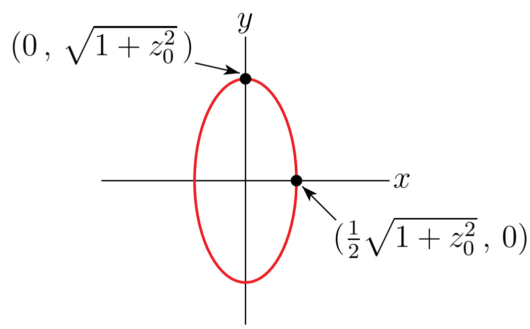


27 Of course you could instead use some fancy graphing software, but part of the point is to build intuition. Not to mention that you can't use fancy graphing software on your exam.

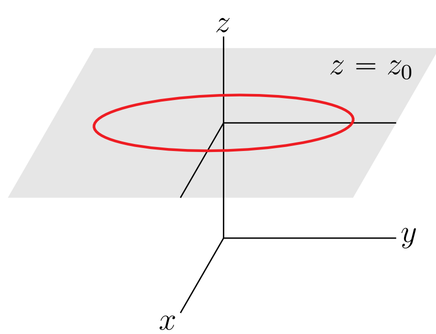
The intersection of our surface with that horizontal plane is a horizontal cross-section. Any point (x, y, z) lying on that horizontal cross-section satisfies both

$$z = z_0 \text{ and } 4x^2 + y^2 - z^2 = 1$$
$$\iff z = z_0 \text{ and } 4x^2 + y^2 = 1 + z_0^2$$

Think of z_0 as a constant. Then $4x^2 + y^2 = 1 + z_0^2$ is a curve in the xy-plane. As $1 + z_0^2$ is a constant, the curve is an ellipse. To determine its semi-axes[28], we observe that when $y = 0$, we have $x = \pm\frac{1}{2}\sqrt{1+z_0^2}$ and when $x = 0$, we have $y = \pm\sqrt{1+z_0^2}$. So the curve is just an ellipse with x semi-axis $\frac{1}{2}\sqrt{1+z_0^2}$ and y semi-axis $\sqrt{1+z_0^2}$. It's easy to sketch.

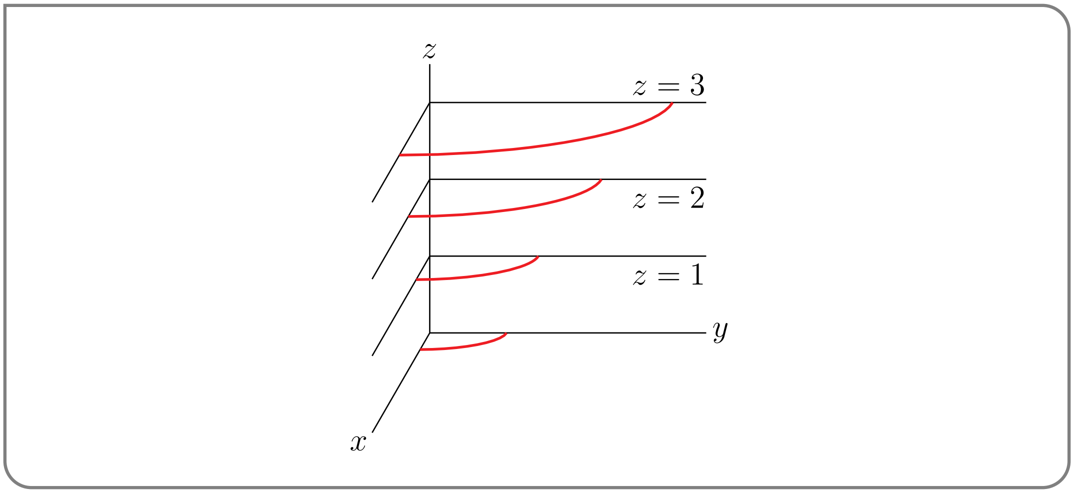


Remember that this ellipse is the part of our surface that lies in the plane $z = z_0$. Imagine that the sketch of the ellipse is on a single sheet of paper. Lift the sheet of paper up, move it around so that the x- and y-axes point in the directions of the three dimensional x- and y-axes and place the sheet of paper into the three dimensional sketch at height z_0. This gives a single horizontal ellipse in 3d, as in the figure below.



We can build up the full surface by stacking many of these horizontal ellipses — one for each possible height z_0. So we now draw a few of them as in the figure below. To reduce the amount of clutter in the sketch, we have only drawn the first octant (i.e. the part of three dimensions that has $x \geqslant 0$, $y \geqslant 0$ and $z \geqslant 0$).

28 The semi-axes of an ellipse are the line segments from the centre of the ellipse to the farthest points on the ellipse and to the nearest points on the ellipse. For a circle the lengths of all of these line segments are just the radius.



Here is why it is OK, in this case, to just sketch the first octant. Replacing x by $-x$ in the equation $4x^2 + y^2 - z^2 = 1$ does not change the equation. That means that a point (x, y, z) is on the surface if and only if the point $(-x, y, z)$ is on the surface. So the surface is invariant under reflection in the yz-plane. Similarly, the equation $4x^2 + y^2 - z^2 = 1$ does not change when y is replaced by $-y$ or z is replaced by $-z$. Our surface is also invariant reflection in the xz- and xy-planes. Once we have the part in the first octant, the remaining octants can be gotten simply by reflecting about the coordinate planes.

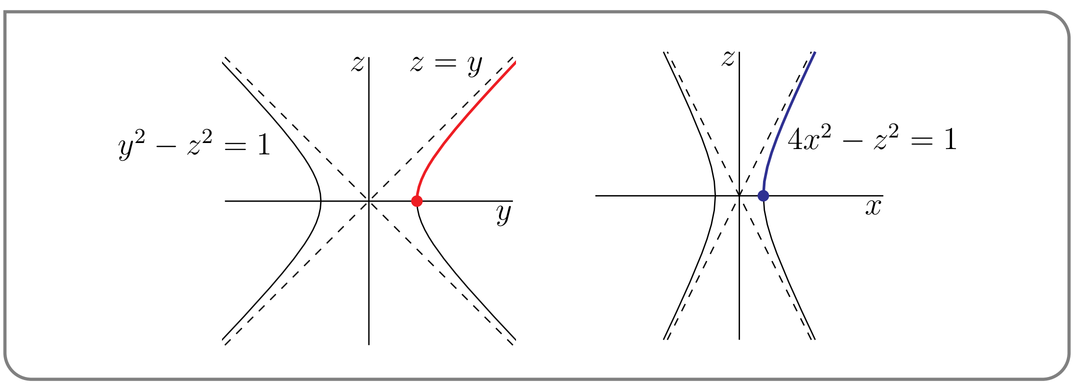
We can get a more visually meaningful sketch by adding in some vertical cross-sections. The $x = 0$ and $y = 0$ cross-sections (also called traces — they are the parts of our surface that are in the yz- and xz-planes, respectively) are

$$x = 0,\ y^2 - z^2 = 1 \qquad \text{and} \qquad y = 0,\ 4x^2 - z^2 = 1$$

These equations describe hyperbolae[29]. If you don't remember how to sketch them, don't worry. We'll do it now. We'll first sketch them in 2d. Since

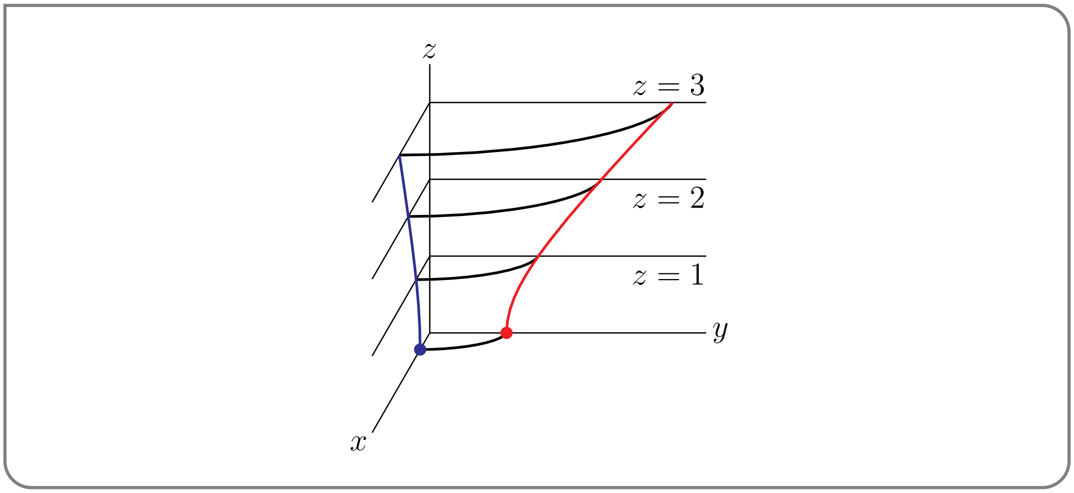
$$\begin{aligned} y^2 = 1 + z^2 &\implies |y| \geqslant 1 \quad \text{and} \quad y = \pm 1 \text{ when } z = 0 \quad \text{and} \quad \text{for large } z,\ y \approx \pm z \\ 4x^2 = 1 + z^2 &\implies |x| \geqslant \tfrac{1}{2} \quad \text{and} \quad x = \pm\tfrac{1}{2} \text{ when } z = 0 \quad \text{and} \quad \text{for large } z,\ x \approx \pm\tfrac{1}{2}z \end{aligned}$$

the sketches are

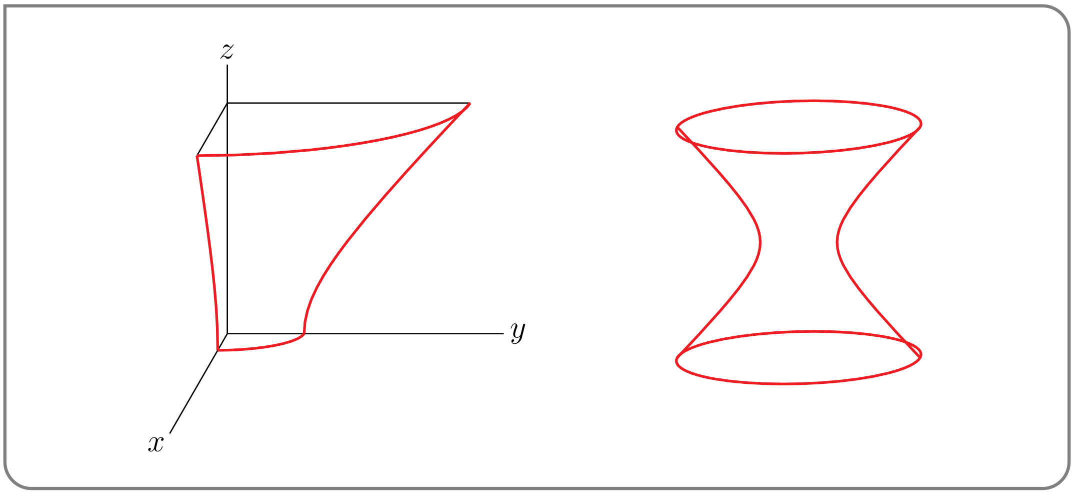


Now we'll incorporate them into the 3d sketch. Once again imagine that each is a single sheet of paper. Pick each up and move it into the 3d sketch, carefully matching up the axes. The red (blue) parts of the hyperbolas above become the red (blue) parts of the 3d sketch below (assuming of course that you are looking at this on a colour screen).

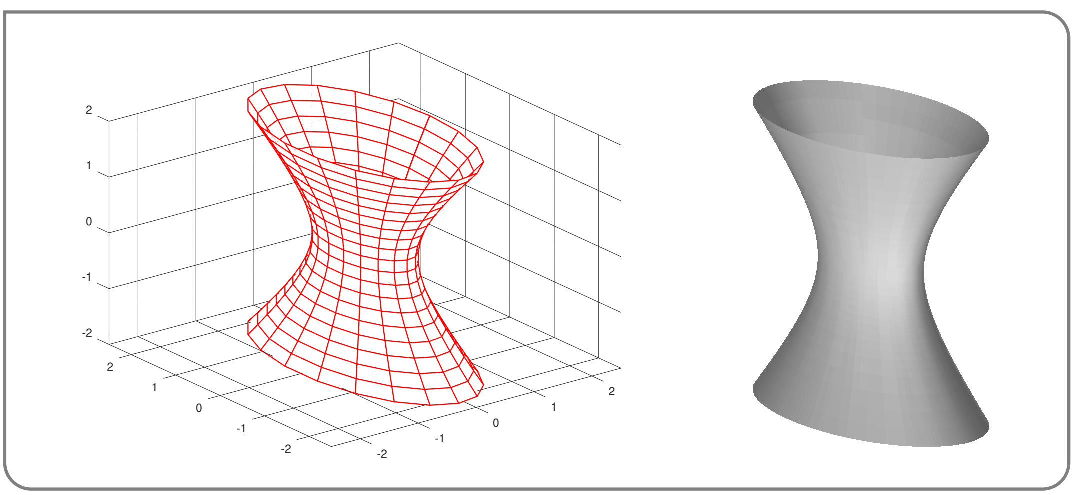
29 It's not just a figure of speech!



Now that we have a pretty good idea of what the surface looks like we can clean up and simplify the sketch. Here are a couple of possibilities.



Here are two figures created by graphing software.



This type of surface is called a hyperboloid of one sheet.

There are also hyperboloids of two sheets. For example, replacing the $+1$ on the right hand side of $x^2 + y^2 - z^2 = 1$ gives $x^2 + y^2 - z^2 = -1$, which is a hyperboloid of two sheets. We'll sketch it quickly in the next example.

Example 1.7.1

Example 1.7.2 $(4x^2 + y^2 - z^2 = -1)$

Sketch the surface that satisfies $4x^2 + y^2 - z^2 = -1$.

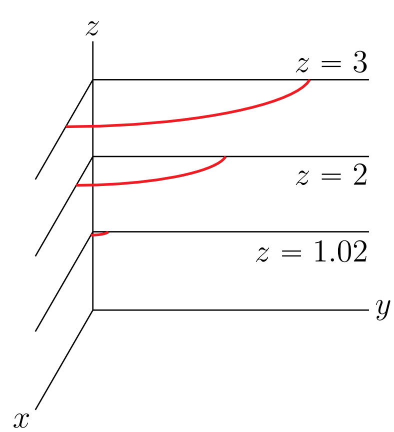
Solution. As in the last example, we'll start by fixing any number z_0 and sketching the part of the surface that lies in the horizontal plane $z = z_0$. The intersection of our surface with that horizontal plane is

$$z = z_0 \text{ and } 4x^2 + y^2 = z_0^2 - 1$$

Think of z_0 as a constant.

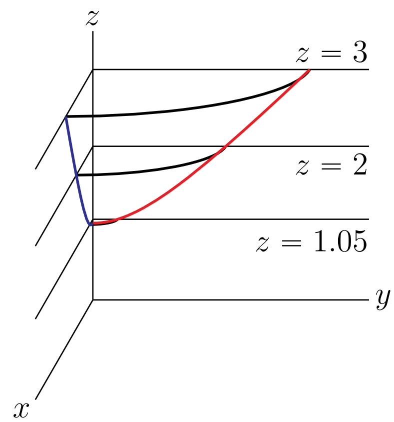
- If $|z_0| < 1$, then $z_0^2 - 1 < 0$ and there are no solutions to $x^2 + y^2 = z_0^2 - 1$.
- If $|z_0| = 1$ there is exactly one solution, namely $x = y = 0$.
- If $|z_0| > 1$ then $4x^2 + y^2 = z_0^2 - 1$ is an ellipse with x semi-axis $\frac{1}{2}\sqrt{z_0^2 - 1}$ and y semi-axis $\sqrt{z_0^2 - 1}$. These semi-axes are small when $|z_0|$ is close to 1 and grow as $|z_0|$ increases.

The first octant parts of a few of these horizontal cross-sections are drawn in the figure below.

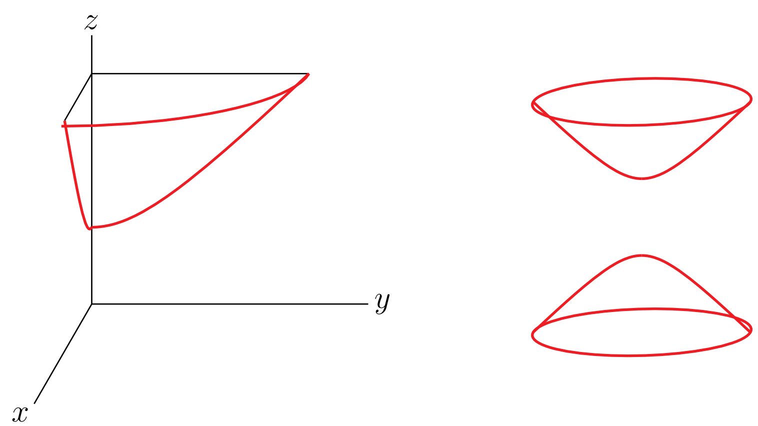


Next we add in the $x = 0$ and $y = 0$ cross-sections (i.e. the parts of our surface that are in the yz- and xz-planes, respectively)

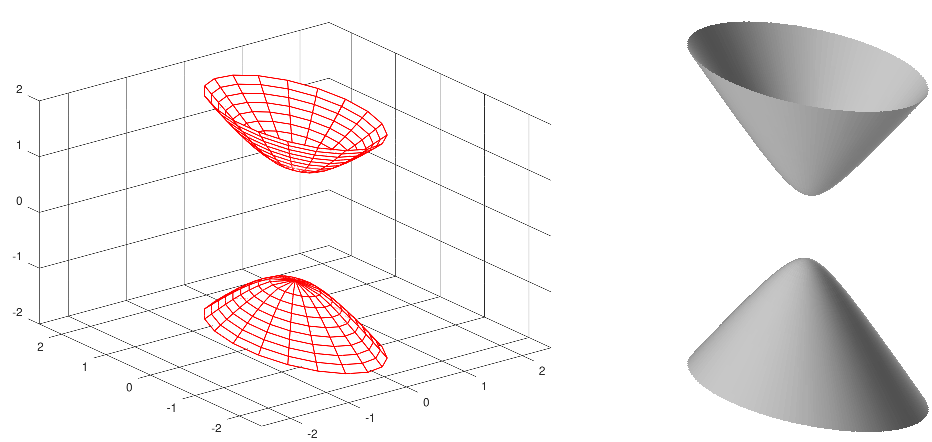
$$x = 0,\ z^2 = 1 + y^2 \qquad \text{and} \qquad y = 0,\ z^2 = 1 + 4x^2$$



Now that we have a pretty good idea of what the surface looks like we clean up and simplify the sketch.



Here are two figures created by graphing software.



This type of surface is called a hyperboloid of two sheets.

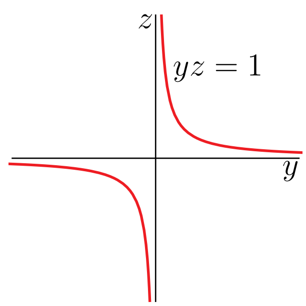
Example 1.7.2

Example 1.7.3 ($yz = 1$)

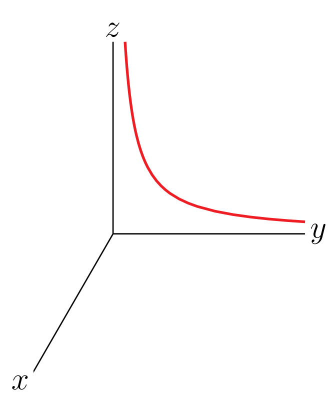
Sketch the surface $yz = 1$.

Solution. This surface has a special property that makes it relatively easy to sketch. There are no x's in the equation $yz = 1$. That means that if some y_0 and z_0 obey $y_0 z_0 = 1$, then the point (x, y_0, z_0) lies on the surface $yz = 1$ for all values of x. As x runs from $-\infty$ to ∞, the point (x, y_0, z_0) sweeps out a straight line parallel to the x-axis. So the surface $yz = 1$ is a union of lines parallel to the x-axis. It is invariant under translations parallel to the x-axis. To sketch $yz = 1$, we just need to sketch its intersection with the yz-plane and then translate the resulting curve parallel to the x-axis to sweep out the surface.

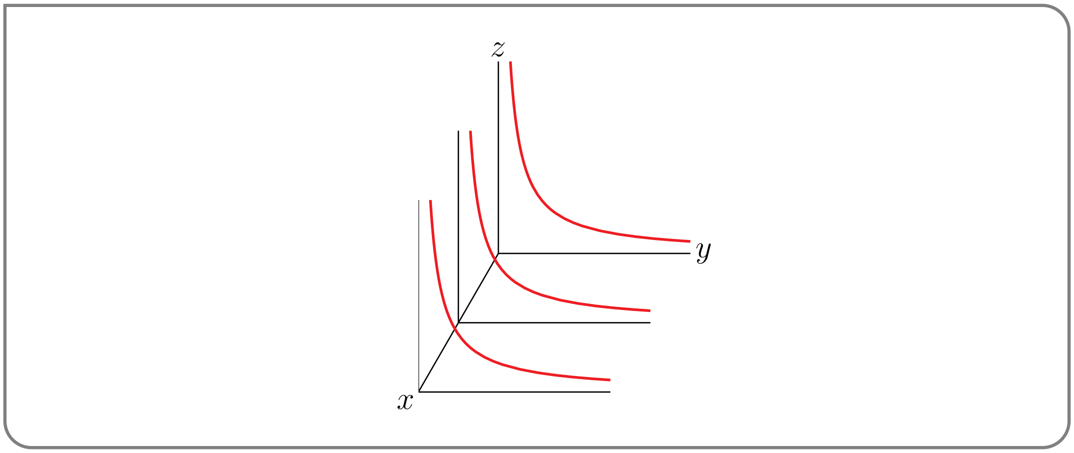
We'll start with a sketch of the hyperbola $yz = 1$ in two dimensions.



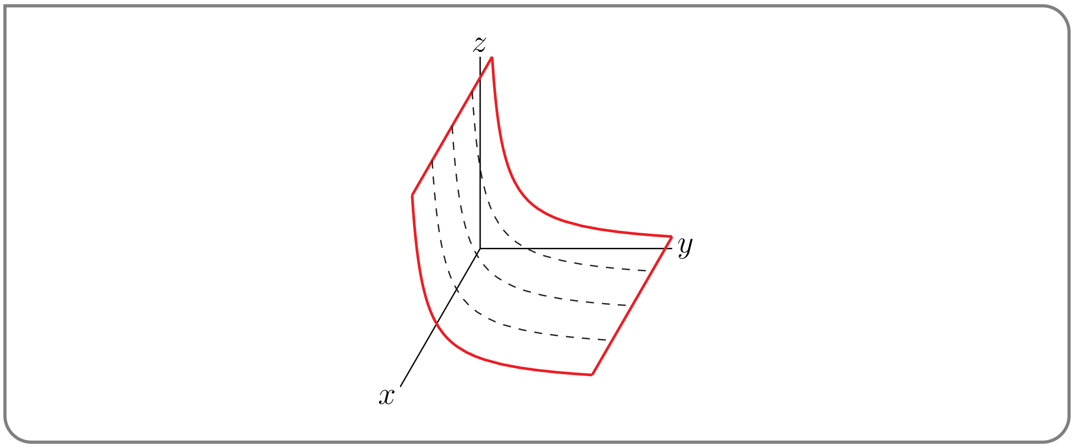
Next we'll move this 2d sketch into the yz-plane, i.e. the plane $x = 0$, in 3d, except that we'll only draw in the part in the first octant.



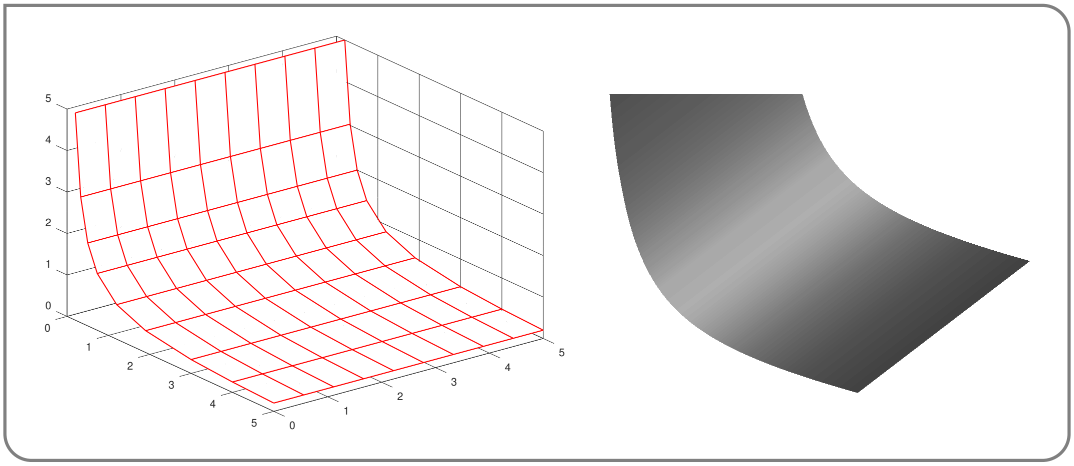
The we'll draw in $x = x_0$ cross-sections for a couple of more values of x_0



and clean up the sketch a bit



Here are two figures created by graphing software.



Example 1.7.3

Example 1.7.4 ($xyz = 4$)

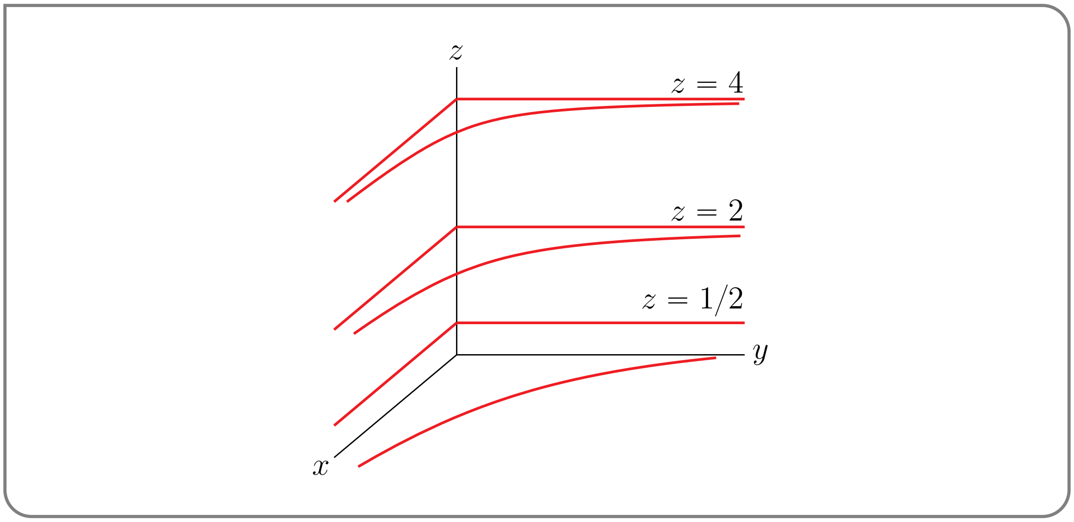
Sketch the surface $xyz = 4$.

Solution. We'll sketch this surface using much the same procedure as we used in Examples 1.7.1 and 1.7.2. We'll only sketch the part of the surface in the first octant. The remaining parts (in the octants with $x, y < 0$, $z \geqslant 0$, with $x, z < 0$, $y \geqslant 0$ and with $y, z < 0$, $x \geqslant 0$) are just reflections of the first octant part.

As usual, we start by fixing any number z_0 and sketching the part of the surface that lies in the horizontal plane $z = z_0$. The intersection of our surface with that horizontal plane is the hyperbola

$$z = z_0 \quad \text{and} \quad xy = \frac{4}{z_0}$$

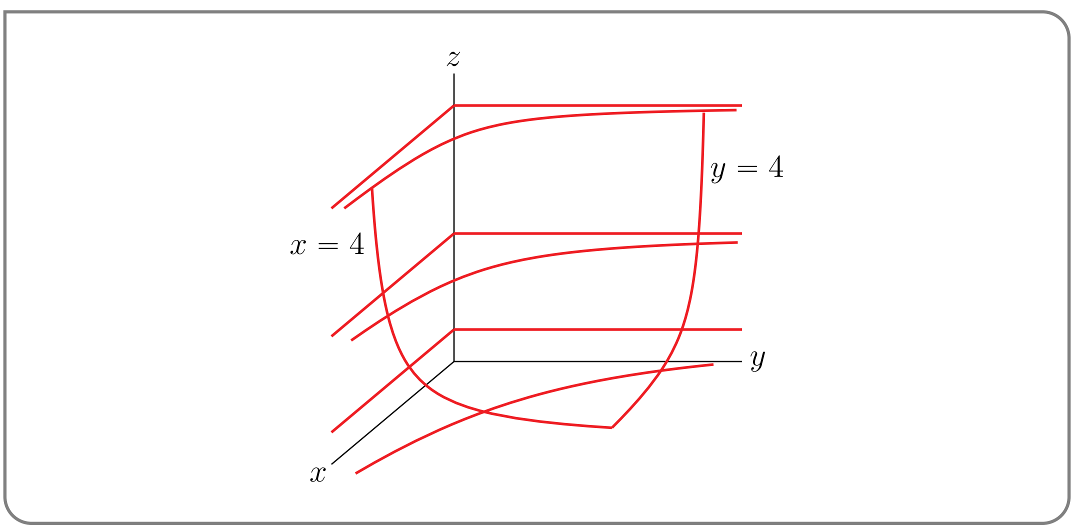
Note that $x \to \infty$ as $y \to 0$ and that $y \to \infty$ as $x \to 0$. So the hyperbola has both the x-axis and the y-axis as asymptotes, when drawn in the xy-plane. The first octant parts of a few of these horizontal cross-sections (namely, $z_0 = 4$, $z_0 = 2$ and $z_0 = \frac{1}{2}$) are drawn in the figure below.



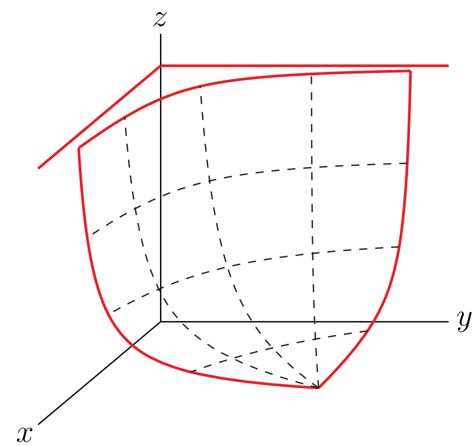
Next we add some vertical cross-sections. We can't use $x = 0$ or $y = 0$ because any point on $xyz = 4$ must have all of x, y, z nonzero. So we use

$$x = 4,\ yz = 1 \qquad \text{and} \qquad y = 4,\ xz = 1$$

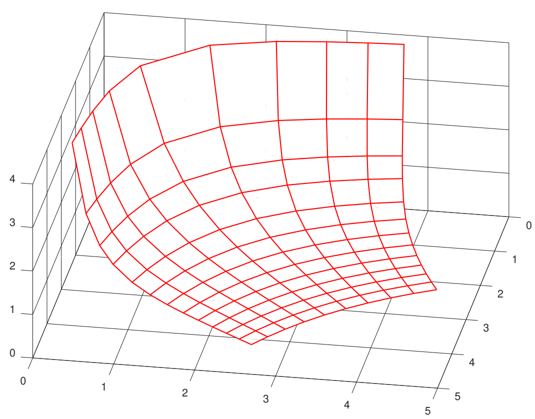
instead. They are again hyperbolae.

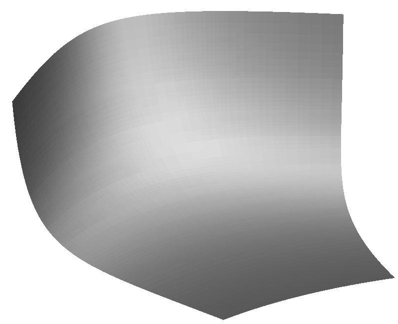


Finally, we clean up and simplify the sketch.



Here are two figures created by graphing software.





Example 1.7.4

1.7.1 ▸▸ Level Curves and Surfaces

Often the reason you are interested in a surface in 3d is that it is the graph $z = f(x,y)$ of a function of two variables $f(x,y)$. Another good way to visualize the behaviour of a function $f(x,y)$ is to sketch what are called its level curves. By definition, a level curve of $f(x,y)$ is a curve whose equation is $f(x,y) = C$, for some constant C. It is the set of points in the xy-plane where f takes the value C. Because it is a curve in 2d, it is usually easier to sketch than the graph of f. Here are a couple of examples.

Example 1.7.5 $\left(f(x,y) = x^2 + 4y^2 - 2x + 2\right)$

Sketch the level curves of $f(x,y) = x^2 + 4y^2 - 2x + 2$.

Solution. Fix any real number C. Then, for the specified function f, the level curve $f(x,y) = C$ is the set of points (x,y) that obey

$$\begin{aligned} x^2 + 4y^2 - 2x + 2 = C &\iff x^2 - 2x + 1 + 4y^2 + 1 = C \\ &\iff (x-1)^2 + 4y^2 = C - 1 \end{aligned}$$

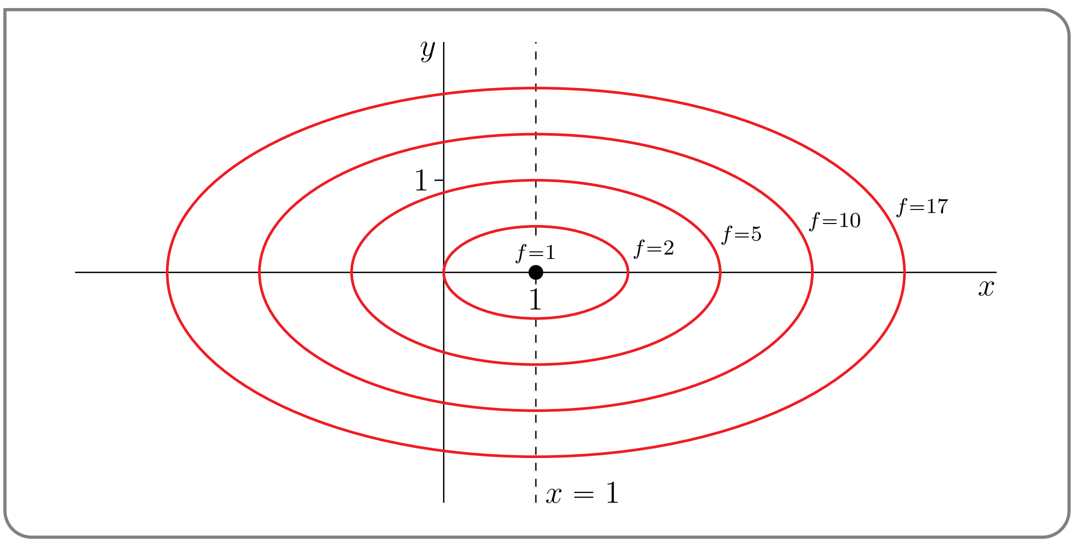
Now $(x-1)^2 + 4y^2$ is the sum of two squares, and so is always at least zero. So if $C - 1 < 0$, i.e. if $C < 1$, there is no curve $f(x,y) = C$. If $C - 1 = 0$, i.e. if $C = 1$, then $f(x,y) = C - 1 = 0$ if and only if both $(x-1)^2 = 0$ and $4y^2 = 0$ and so the level curve consists of the single point $(1,0)$. If $C > 1$, then $f(x,y) = C$ become $(x-1)^2 + 4y^2 = C - 1 > 0$ which describes an ellipse centred on $(1,0)$. It intersects the x-axis when $y = 0$ and

$$(x-1)^2 = C - 1 \iff x - 1 = \pm\sqrt{C-1} \iff x = 1 \pm \sqrt{C-1}$$

and it intersects the line $x = 1$ (i.e. the vertical line through the centre) when

$$4y^2 = C - 1 \iff 2y = \pm\sqrt{C-1} \iff y = \pm\tfrac{1}{2}\sqrt{C-1}$$

So, when $C > 1$, $f(x,y) = C$ is the ellipse centred on $(1,0)$ with x semi-axis $\sqrt{C-1}$ and y semi-axis $\frac{1}{2}\sqrt{C-1}$. Here is a sketch of some representative level curves of $f(x,y) = x^2 + 4y^2 - 2x + 2$.



It is often easier to develop an understanding of the behaviour of a function $f(x,y)$ by looking at a sketch of its level curves, than it is by looking at a sketch of its graph. On the other hand, you can also use a sketch of the level curves of $f(x,y)$ as the first step in building a sketch of the graph $z = f(x,y)$. The next step would be to redraw, for each C, the level curve $f(x,y) = C$, in the plane $z = C$, as we did in Example 1.7.1.

Example 1.7.5

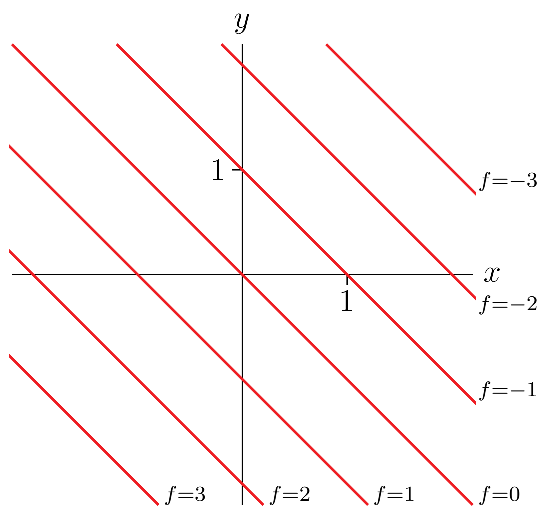
Example 1.7.6 ($e^{x+y+z} = 1$)

The function $f(x,y)$ is given implicitly by the equation $e^{x+y+z} = 1$. Sketch the level curves of f.

Solution. This one is not as nasty as it appears. That "$f(x,y)$ is given implicitly by the equation $e^{x+y+z} = 1$" means that, for each x, y, the solution z of $e^{x+y+z} = 1$ is $f(x,y)$. So, for the specified function f and any fixed real number C, the level curve $f(x,y) = C$ is the set of points (x,y) that obey

$$\begin{aligned} e^{x+y+C} = 1 &\iff x+y+C=0 \qquad \text{(by taking the logarithm of both sides)} \\ &\iff x+y=-C \end{aligned}$$

This is of course a straight line. It intersects the x-axis when $y = 0$ and $x = -C$ and it intersects the y-axis when $x = 0$ and $y = -C$. Here is a sketch of some level curves.

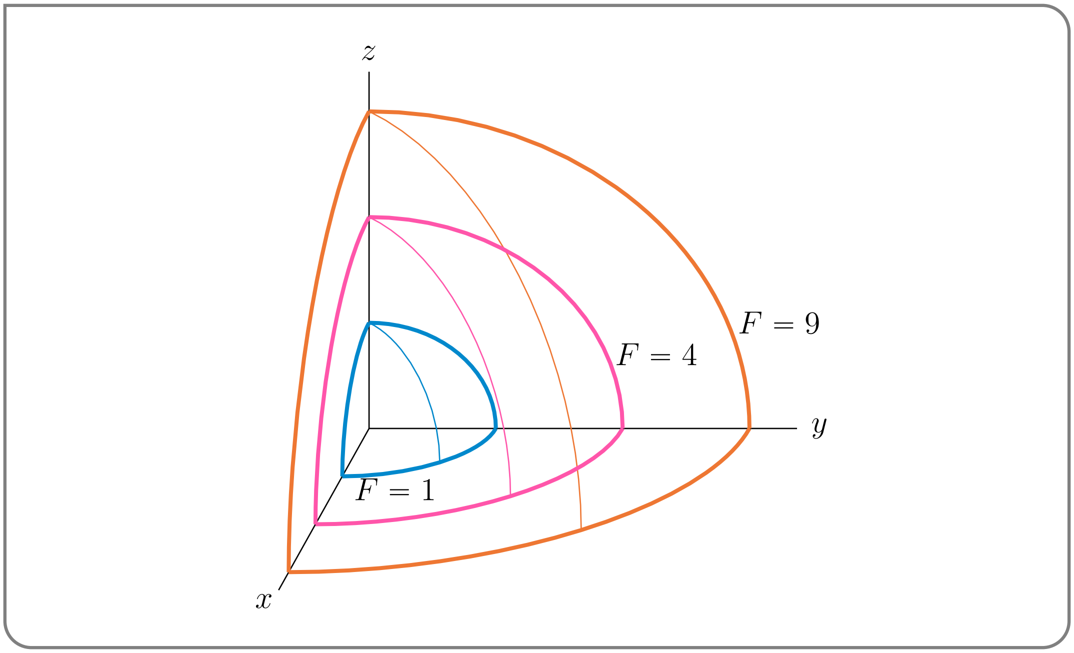


Example 1.7.6

We have just seen that sketching the level curves of a function $f(x,y)$ can help us understand the behaviour of f. We can generalise this to functions $F(x,y,z)$ of three variables. A level surface of $F(x,y,z)$ is a surface whose equation is of the form $F(x,y,z) = C$ for some constant C. It is the set of points (x,y,z) at which F takes the value C.

Example 1.7.7 ($F(x,y,z) = x^2 + y^2 + z^2$)

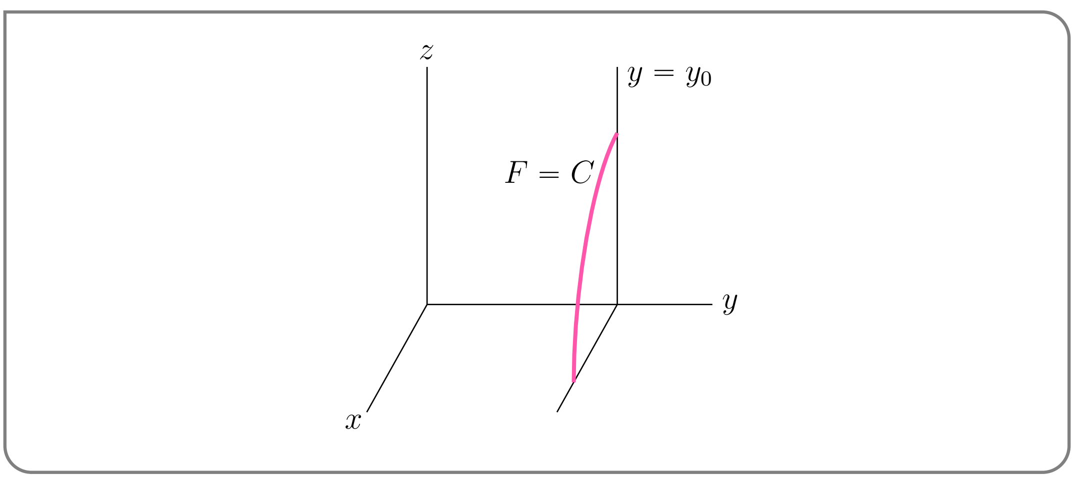
Let $F(x,y,z) = x^2 + y^2 + z^2$. If $C > 0$, then the level surface $F(x,y,z) = C$ is the sphere of radius $\sqrt{C}$ centred on the origin. Here is a sketch of the parts of the level surfaces $F = 1$ (radius 1), $F = 4$ (radius 2) and $F = 9$ (radius 3) that are in the first octant.



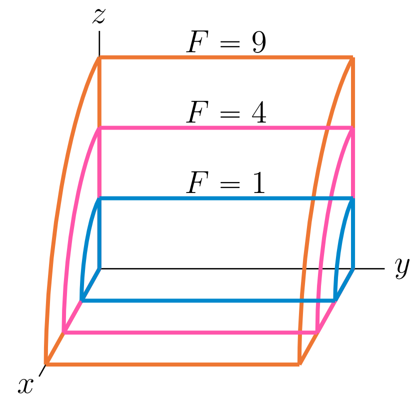
Example 1.7.7

Example 1.7.8 $\left(F(x,y,z) = x^2 + z^2\right)$

Let $F(x,y,z) = x^2 + z^2$ and $C > 0$. Consider the level surface $x^2 + z^2 = C$. The variable y does not appear in this equation. So for any fixed y_0, the intersection of the our surface $x^2 + z^2 = C$ with the plane $y = y_0$ is the circle of radius $\sqrt{C}$ centred on $x = z = 0$. Here is a sketch of the first quadrant part of one such circle.



The full surface is the horizontal stack of all of those circles with y_0 running over $\mathbb{R}$. It is the cylinder of radius $\sqrt{C}$ centred on the y-axis. Here is a sketch of the parts of the level surfaces $F = 1$ (radius 1), $F = 4$ (radius 2) and $F = 9$ (radius 3) that are in the first octant.



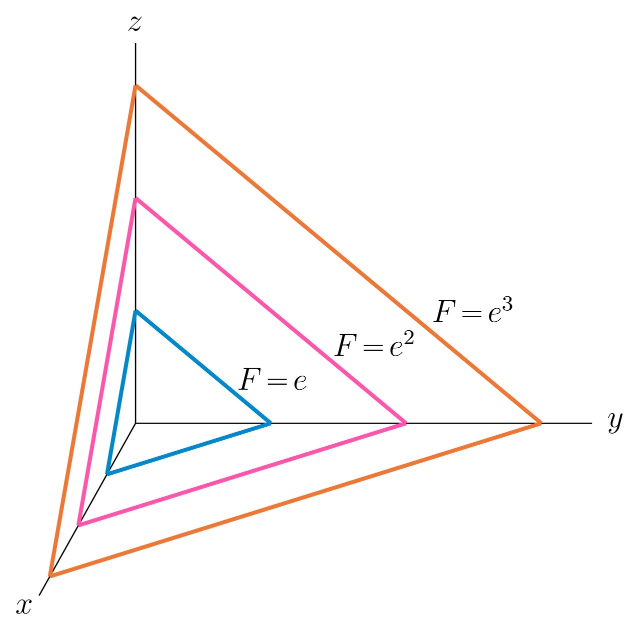
Example 1.7.8

Example 1.7.9 ($F(x, y, z) = e^{x+y+z}$)

Let $F(x, y, z) = e^{x+y+z}$ and $C > 0$. Consider the level surface $e^{x+y+z} = C$, or equivalently, $x + y + z = \ln C$. It is the plane that contains the intercepts $(\ln C, 0, 0)$, $(0, \ln C, 0)$ and $(0, 0, \ln C)$. Here is a sketch of the parts of the level surfaces

- $F = e$ (intercepts $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$),
- $F = e^2$ (intercepts $(2, 0, 0)$, $(0, 2, 0)$, $(0, 0, 2)$) and
- $F = e^3$ (intercepts $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$)

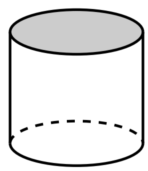
that are in the first octant.



Example 1.7.9

1.8▲ Cylinders

There are some classes of relatively simple, but commonly occurring, surfaces that are given their own names. One such class is cylindrical surfaces. You are probably used to thinking of a cylinder as being something that looks like $x^2 + y^2 = 1$.



In Mathematics, the word "cylinder" is given a more general meaning.

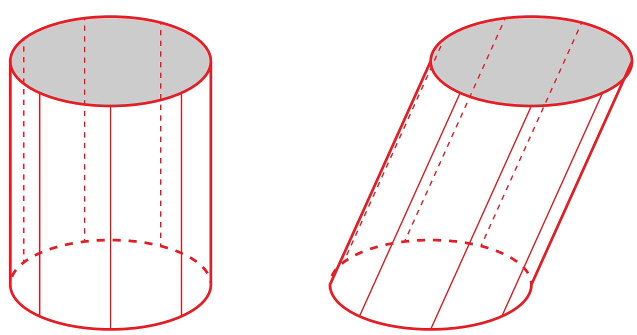
Definition 1.8.1 (Cylinder).

A *cylinder* is a surface that consists of all points that are on all lines that are

- parallel to a given line and
- pass through a given fixed curve, that lies in a fixed plane that is not parallel to the given line.

Example 1.8.2

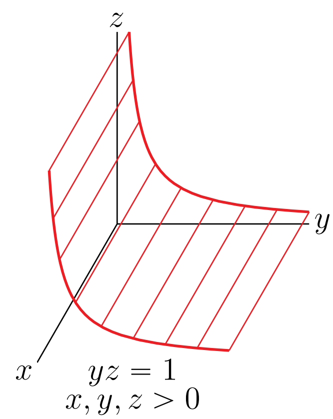
Here are sketches of three cylinders. The familiar cylinder on the left below



is called a right circular cylinder, because the given fixed curve ($x^2 + y^2 = 1$, $z = 0$) is a circle and the given line (the z-axis) is perpendicular (i.e. at right angles) to the fixed curve.

The cylinder on the left above can be thought of as a vertical stack of circles. The cylinder on the right above can also be thought of as a stack of circles, but the centre of the circle at height z has been shifted rightward to $(0, z, z)$. For that cylinder, the given fixed curve is once again the circle $x^2 + y^2 = 1$, $z = 0$, but the given line is $y = z$, $x = 0$.

We have already seen the the third cylinder



in Example 1.7.3. It is called a hyperbolic cylinder. In this example, the given fixed curve is the hyperbola $yz = 1$, $x = 0$ and the given line is the x-axis.

Example 1.8.2

1.9▲ Quadric Surfaces

Another named class of relatively simple, but commonly occurring, surfaces is the quadric surfaces.

Definition 1.9.1 (Quadrics)**.**

A *quadric* surface is surface that consists of all points that obey $Q(x, y, z) = 0$, with Q being a polynomial of degree two[30].

For $Q(x, y, z)$ to be a polynomial of degree two, it must be of the form

$$Q(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J$$

for some constants A, B, $\cdots$, J. Each constant z cross section of a quadric surface has an equation of the form

$$Ax^2 + Dxy + By^2 + gx + hy + j = 0, \quad z = z_0$$

If $A = B = D = 0$ but g and h are not both zero, this is a straight line. If A, B, and D are not all zero, then by rotating and translating our coordinate system the equation of the cross section can be brought into one of the forms[31]

- $\alpha x^2 + \beta y^2 = \gamma$ with $\alpha, \beta > 0$, which, if $\gamma > 0$, is an ellipse (or a circle),

30 Technically, we should also require that the polynomial can't be factored into the product of two polynomials of degree one.

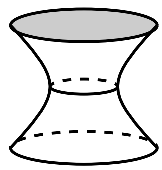
31 This statement can be justified using a linear algebra eigenvalue/eigenvector analysis. It is beyond what we can cover here, but is not too difficult for a standard linear algebra course.

- $\alpha x^2 - \beta y^2 = \gamma$ with $\alpha, \beta > 0$, which, if $\gamma \neq 0$, is a hyperbola, and if $\gamma = 0$ is two lines,
- $x^2 = \delta y$, which, if $\delta \neq 0$ is a parabola, and if $\delta = 0$ is a straight line.

There are similar statements for the constant x cross sections and the constant y cross sections. Hence quadratic surfaces are built by stacking these three types of curves.

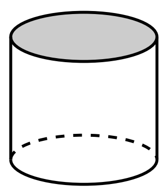
We have already seen a number of quadric surfaces in the last couple of sections.

- We saw the quadric surface $4x^2 + y^2 - z^2 = 1$ in Example 1.7.1.



  Its constant z cross sections are ellipses and its $x = 0$ and $y = 0$ cross sections are hyperbolae. It is called a hyperboloid of one sheet.

- We saw the quadric surface $x^2 + y^2 = 1$ in Example 1.8.2.



  Its constant z cross sections are circles and its $x = 0$ and $y = 0$ cross sections are straight lines. It is called a right circular cylinder.

Appendix H contains other quadric surfaces.

Chapter 2

Partial Derivatives

In this chapter we are going to generalize the definition of "derivative" to functions of more than one variable and then we are going to use those derivatives. We will parallel the development in Chapters 1 and 2 of the CLP-1 text. We shall

- define limits and continuity of functions of more than one variable (Definitions 2.1.2 and 2.1.3) and then
- study the properties of limits in more than one dimension (Theorem 2.1.5) and then
- define derivatives of functions of more than one variable (Definition 2.2.1).

We are going to be able to speed things up considerably by recycling what we have already learned in the CLP-1 text.

We start by generalizing the definition of "limit" to functions of more than one variable.

2.1▲ Limits

Before we really start, let's recall some useful notation.

Notation 2.1.1.

- $\mathbb{N}$ is the set $\{1, 2, 3, \cdots\}$ of all natural numbers.
- $\mathbb{R}$ is the set of all real numbers.
- $\in$ is read "is an element of".
- $\notin$ is read "is not an element of".
- $\left\{\, A \,\middle|\, B \,\right\}$ is read "the set of all A such that B"

Notation 2.1.1 (continued).

- If S is a set and T is a subset of S, then $S\backslash T$ is $\left\{\, x \in S \;\middle|\; x \notin T \,\right\}$, the set S with the elements of T removed. In particular, if S is a set and a is an element of S, then $S\backslash\{a\} = \left\{\, x \in S \;\middle|\; x \neq\ a \,\right\}$ is the set S with the element a removed.
- If n is a natural number, $\mathbb{R}^n$ is used for both the set of n-component vectors $\langle x_1, x_2, \cdots, x_n\rangle$ and the set of points $(x_1, x_2, \cdots, x_n)$ with n coordinates.
- If S and T are sets, then $f : S \to T$ means that f is a function which assigns to each element of S an element of T. The set S is called the domain of f.
- $[a, b] = \left\{\, x \in \mathbb{R} \;\middle|\; a \leqslant x \leqslant b \,\right\}$ $\qquad (a, b] = \left\{\, x \in \mathbb{R} \;\middle|\; a < x \leqslant b \,\right\}$
  $[a, b) = \left\{\, x \in \mathbb{R} \;\middle|\; a \leqslant x < b \,\right\}$ $\qquad (a, b) = \left\{\, x \in \mathbb{R} \;\middle|\; a < x < b \,\right\}$

The definition of the limit of a function of more than one variable looks just like the definition[1] of the limit of a function of one variable. Very roughly speaking

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$$

if $f(\mathbf{x})$ approaches $\mathbf{L}$ whenever $\mathbf{x}$ approaches $\mathbf{a}$. Here is a more careful definition of limit.

Definition 2.1.2 (Limit).

Let

- m and n be natural numbers[2]
- $\mathbf{a} \in \mathbb{R}^m$
- the function $f(\mathbf{x})$ be defined for all $\mathbf{x}$ near[3] $\mathbf{a}$ and take values in $\mathbb{R}^n$
- $\mathbf{L} \in \mathbb{R}^n$

We write

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{L}$$

if[4] the value of the function $f(\mathbf{x})$ is sure to be arbitrarily close to $\mathbf{L}$ whenever the value of $\mathbf{x}$ is close enough to $\mathbf{a}$, without[5] being exactly $\mathbf{a}$.

1 Definition 1.3.3 in the CLP-1 text.
2 In this text, we will interested in $m, n \in \{1, 2, 3\}$, but the definition works for all natural numbers m, n.
3 To be precise, there is a number $r > 0$ such that $f(\mathbf{x})$ is defined for all $\mathbf{x}$ obeying $|\mathbf{x} - \mathbf{a}| < r$.
4 There is a precise, formal version of this definition that looks just like Definition 1.7.1 of the CLP-1 text.
5 You may find the condition "without being exactly $\mathbf{a}$" a little strange, but there is a good reason for it, which we have already seen in Calculus I. In the definition $f'(x) = \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$, the function whose limit is being taken, namely $\frac{f(x)-f(a)}{x-a}$, is not defined at all at $x = a$. This will again happen when we define derivatives of functions of more than one variable.

Now that we have extended the definition of limit, we can extend the definition of continuity.

Definition 2.1.3 (Continuity).

Let

- m and n be natural numbers
- $\mathbf{a} \in \mathbb{R}^m$
- the function $f(\mathbf{x})$ be defined for all $\mathbf{x}$ near $\mathbf{a}$ and take values in $\mathbb{R}^n$

(a) The function f is continuous at a point $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

(b) The function f is continuous on a set D if it is continuous at every point of D.

Here are a few very simple examples. There will be some more substantial examples later — after, as we did in the CLP-1 text, we build some tools that can be used to build complicated limits from simpler ones.

Example 2.1.4

(a) If $f(x,y)$ is the constant function which always takes the value L, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

(b) If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x,y) = (x,y)$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = (a,b)$$

(c) By definition, as (x,y) approaches (a,b), x approaches a and y approaches b, so that if $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = x$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = a$$

Similarly, if $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by $g(x,y) = y$, then

$$\lim_{(x,y)\to(a,b)} g(x,y) = b$$

Example 2.1.4

Limits of multivariable functions have much the same computational properties as limits of functions of one variable. The following theorem summarizes a bunch of them. For simplicity, it concerns primarily real valued functions. That is, functions that output real numbers as opposed to vectors. However it does contain one vector valued function. The function $\mathbf{X}$ in the theorem takes as input an n-component vector and returns an m-component vector. We will not deal with many vector valued functions here in CLP-3, but we will see a lot in CLP-4.

Theorem 2.1.5 (Arithmetic, and Other, Properties of Limits)**.**

Let

- m and n be natural numbers
- $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$
- D be a subset of $\mathbb{R}^m$ that contains all $\mathbf{x} \in \mathbb{R}^m$ that are near $\mathbf{a}$
- $c, F, G \in \mathbb{R}$

and

$$f, g : D\backslash\{\mathbf{a}\} \to \mathbb{R} \qquad \mathbf{X} : \mathbb{R}^n\backslash\{\mathbf{b}\} \to D\backslash\{\mathbf{a}\} \qquad \gamma : \mathbb{R} \to \mathbb{R}$$

Assume that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = F \qquad \lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = G \qquad \lim_{\mathbf{y}\to\mathbf{b}} \mathbf{X}(\mathbf{y}) = \mathbf{a} \qquad \lim_{t\to F} \gamma(t) = \gamma(F)$$

Then

(a) $$\lim_{\mathbf{x}\to\mathbf{a}} \big[f(\mathbf{x}) + g(\mathbf{x})\big] = F + G$$
$$\lim_{\mathbf{x}\to\mathbf{a}} \big[f(\mathbf{x}) - g(\mathbf{x})\big] = F - G$$

(b) $$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})\, g(\mathbf{x}) = FG$$
$$\lim_{\mathbf{x}\to\mathbf{a}} c f(\mathbf{x}) = cF$$

(c) $$\lim_{\mathbf{x}\to\mathbf{a}} \tfrac{f(\mathbf{x})}{g(\mathbf{x})} = \tfrac{F}{G} \text{ if } G \neq 0$$

(d) $$\lim_{\mathbf{y}\to\mathbf{b}} f\big(\mathbf{X}(\mathbf{y})\big) = F$$

(e) $$\lim_{\mathbf{x}\to\mathbf{a}} \gamma\big(f(\mathbf{x})\big) = \gamma(F)$$

This shows that multivariable limits interact very nicely with arithmetic, just as single variable limits did. Also recall, from Theorem 1.6.8 in the CLP-1 text,

Theorem 2.1.6.

The following functions are continuous everywhere in their domains

- polynomials, rational functions
- roots and powers
- trig functions and their inverses
- exponential and the logarithm

Example 2.1.7

In this example we evaluate

$$\lim_{(x,y)\to(2,3)} \frac{x+\sin y}{x^2y^2+1}$$

as a typical application of Theorem 2.1.5. Here " $\stackrel{a}{=}$ " means that part (a) of Theorem 2.1.5 justifies that equality. Start by computing separately the limits of the numerator and denominator.

$$\begin{aligned}
\lim_{(x,y)\to(2,3)} \big(x+\sin y\big) &\stackrel{a}{=} \lim_{(x,y)\to(2,3)} x + \lim_{(x,y)\to(2,3)} \sin y \\
&\stackrel{e}{=} \lim_{(x,y)\to(2,3)} x + \sin\Big(\lim_{(x,y)\to(2,3)} y\Big) \\
&= 2+\sin 3 \\
\lim_{(x,y)\to(2,3)} \big(x^2y^2+1\big) &\stackrel{a}{=} \lim_{(x,y)\to(2,3)} x^2y^2 + \lim_{(x,y)\to(2,3)} 1 \\
&\stackrel{b}{=} \Big(\lim_{(x,y)\to(2,3)} x\Big)\Big(\lim_{(x,y)\to(2,3)} x\Big)\Big(\lim_{(x,y)\to(2,3)} y\Big)\Big(\lim_{(x,y)\to(2,3)} y\Big)+1 \\
&= 2^2 3^2+1
\end{aligned}$$

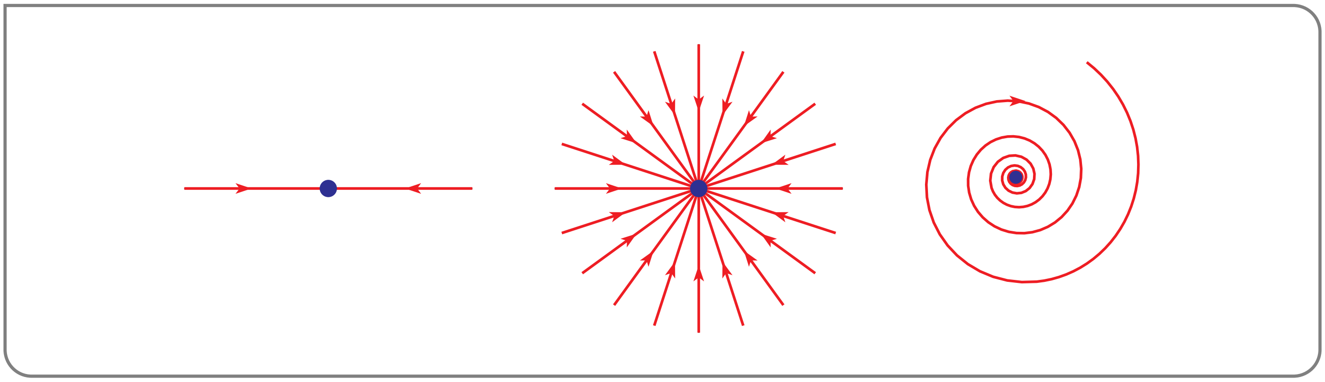
Since the limit of the denominator is nonzero, we can simply divide.

$$\begin{aligned}
\lim_{(x,y)\to(2,3)} \frac{x+\sin y}{x^2y^2+1} &\stackrel{c}{=} \frac{\lim_{(x,y)\to(2,3)} (x+\sin y)}{\lim_{(x,y)\to(2,3)} (x^2y^2+1)} \\
&= \frac{2+\sin 3}{37}
\end{aligned}$$

Here we have used that $\sin x$ is a continuous function.

Example 2.1.7

While the CLP-1 text's Definition 1.3.3 of the limit of a function of one variable, and our Definition 2.1.2 of the limit of a multivariable function look virtually identical, there is a substantial practical difference between the two. In dimension one, you can approach a point from the left or from the right and that's it. There are only two possible directions of approach. In two or more dimensions there is "much more room" and there are infinitely many possible types of approach. One can even spiral in to a point. See the middle and right hand figures below.



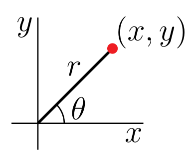
The next few examples illustrate the impact that the"extra room" in dimensions greater than one has on limits.

Example 2.1.8

As a second example, we consider $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$. In this example, both the numerator, x^2y, and the denominator, x^2+y^2, tend to zero as (x,y) approaches $(0,0)$, so we have to be more careful.

A good way to see the behaviour of a function $f(x,y)$ when (x,y) is close to $(0,0)$ is to switch to the polar coordinates, r,θ, that are defined by

$$x = r\cos\theta$$
$$y = r\sin\theta$$



The points (x,y) that are close to $(0,0)$ are those with small r, regardless of what θ is. Recall that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L$ when $f(x,y)$ approaches L as (x,y) approaches $(0,0)$. Substituting $x = r\cos\theta$, $y = r\sin\theta$ into that statement turns it into the statement that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L$ when $f(r\cos\theta, r\sin\theta)$ approaches L as r approaches 0. For our current example

$$\frac{x^2y}{x^2+y^2} = \frac{(r\cos\theta)^2(r\sin\theta)}{r^2} = r\cos^2\theta\sin\theta$$

As $\left|r\cos^2\theta\sin\theta\right| \leqslant r$ tends to 0 as r tends to 0 (regardless of what θ does as r tends to 0) we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$$

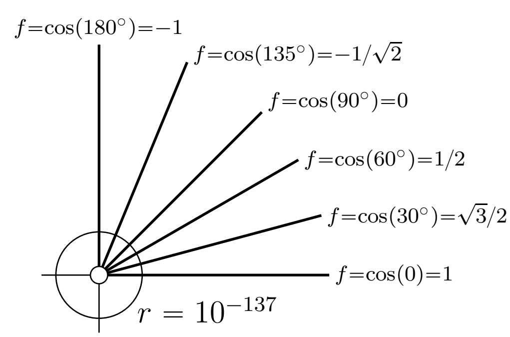
Example 2.1.8

Example 2.1.9

As a third example, we consider $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$. Once again, the best way to see the behaviour of $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ for (x,y) close to $(0,0)$ is to switch to polar coordinates.

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2} = \frac{(r\cos\theta)^2 - (r\sin\theta)^2}{r^2} = \cos^2\theta - \sin^2\theta = \cos(2\theta)$$

Note that, this time, f is independent of r but does depend on θ. Here is a greatly magnified sketch of a number of level curves for $f(x,y)$.



Observe that

- as (x,y) approaches $(0,0)$ along the ray with $2\theta = 30°$, $f(x,y)$ approaches the value $\frac{\sqrt{3}}{2}$ (and in fact $f(x,y)$ takes the value $\cos(30°) = \frac{\sqrt{3}}{2}$ at every point of that ray)
- as (x,y) approaches $(0,0)$ along the ray with $2\theta = 60°$, $f(x,y)$ approaches the value $\frac{1}{2}$ (and in fact $f(x,y)$ takes the value $\cos(60°) = \frac{1}{2}$ at every point of that ray)
- as (x,y) approaches $(0,0)$ along the ray with $2\theta = 90°$, $f(x,y)$ approaches the value 0 (and in fact $f(x,y)$ takes the value $\cos(90°) = 0$ at every point of that ray)
- and so on

So there is not single number L such that $f(x,y)$ approaches L as $r = |(x,y)| \to 0$, no matter what the direction of approach is. The limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Here is another way to come to the same conclusion.

- Pick any really small positive number. We'll use 10^{-137} as an example.
- Pick any real number F between -1 and 1. We'll use $F = \frac{\sqrt{3}}{2}$ as an example.

- Looking at the sketch above, we see that $f(x,y)$ takes the value F along an entire ray $\theta = \text{const}$, $r > 0$. In the case $F = \frac{\sqrt{3}}{2}$, the ray is $2\theta = 30^\circ$, $r > 0$. In particular, because the ray extends all the way to $(0,0)$, f takes the value F for some (x,y) obeying $|(x,y)| < 10^{-137}$.

- That is true regardless of which really small number you picked. So $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ does not approach any single value as $r = |(x,y)|$ approaches 0 and we conclude that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Example 2.1.9

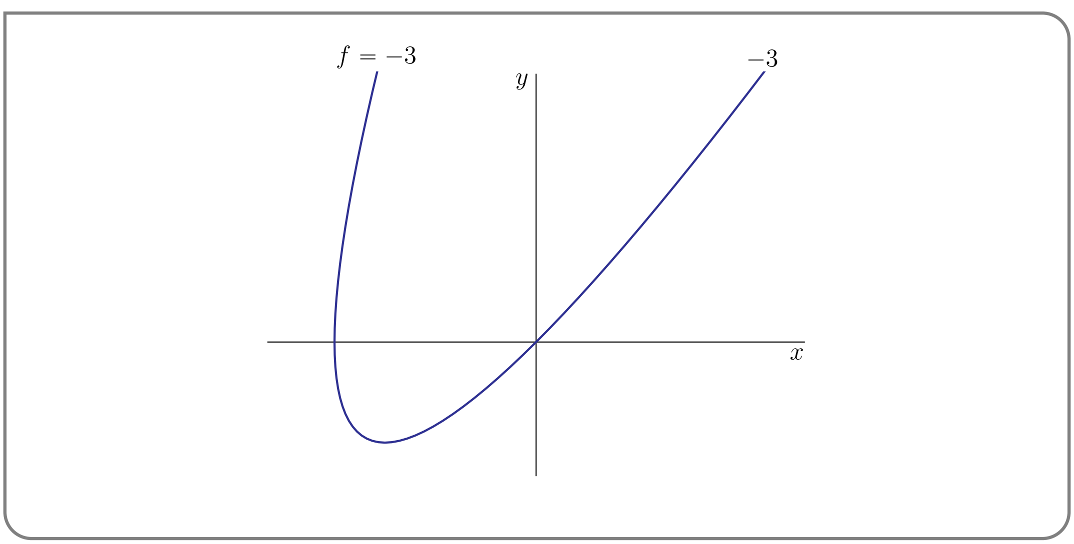
2.1.1 ▸▸ Optional — A Nasty Limit That Doesn't Exist

Example 2.1.10

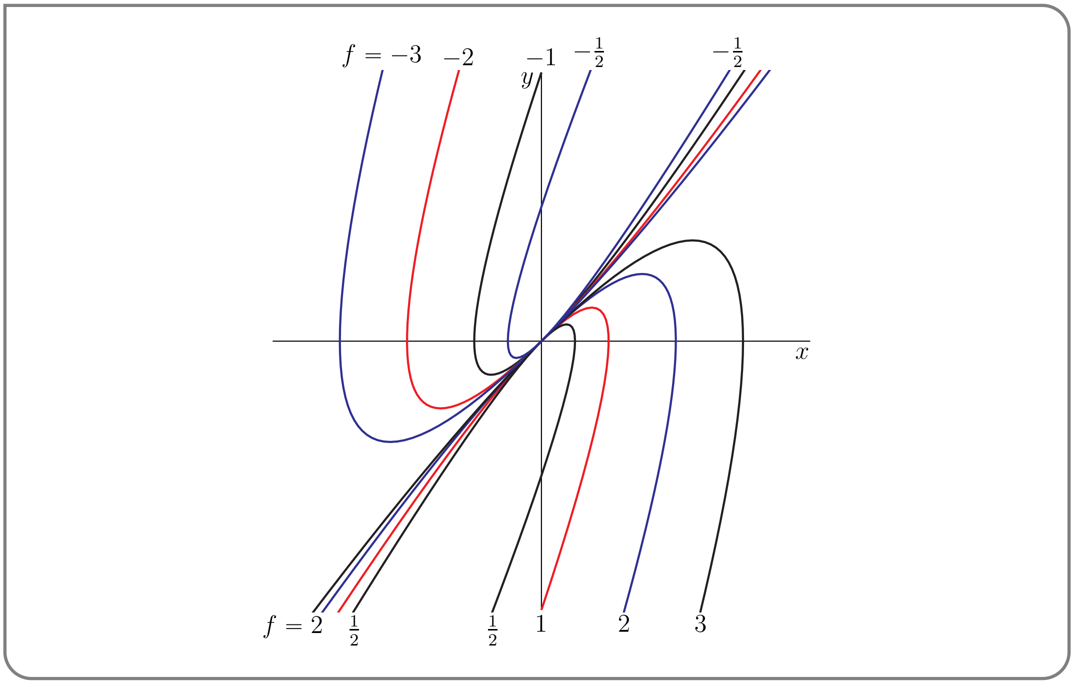
In this example we study the behaviour of the function

$$f(x,y) = \begin{cases} \frac{(2x-y)^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

as $(x,y) \to (0,0)$. Here is a graph of the level curve, $f(x,y) = -3$, for this function.



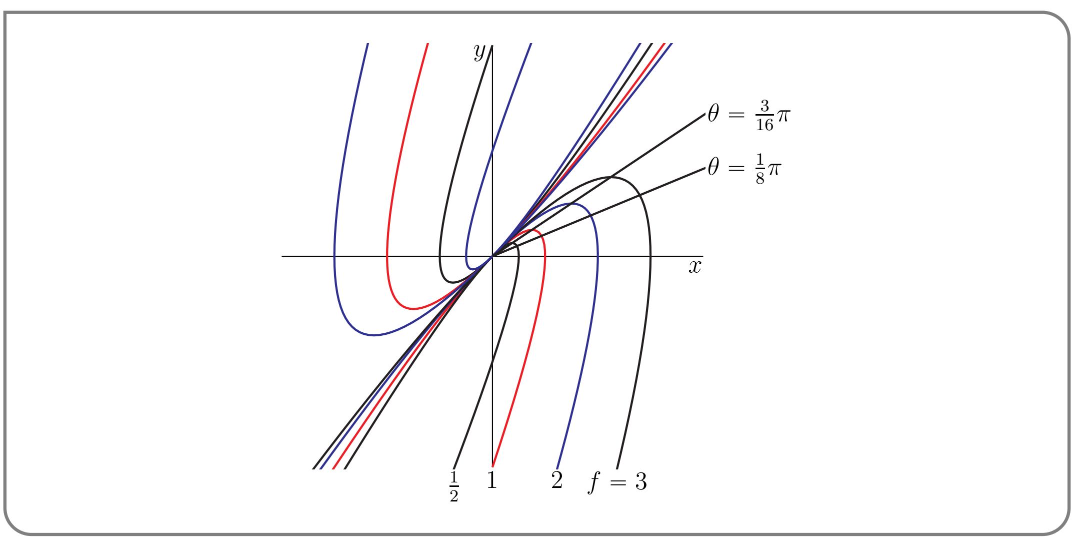
Here is a larger graph of level curves, $f(x,y) = c$, for various values of the constant c.



As before, it helps to convert to polar coordinates — it is a good approach[6]. In polar coordinates

$$f(r\cos\theta, r\sin\theta) = \begin{cases} r\frac{(2\cos\theta-\sin\theta)^2}{\cos\theta-\sin\theta} & \text{if } \cos\theta \neq \sin\theta \\ 0 & \text{if } \cos\theta = \sin\theta \end{cases}$$

If we approach the origin along any fixed ray $\theta = \text{const}$, then $f(r\cos\theta, r\sin\theta)$ is the constant $\frac{(2\cos\theta-\sin\theta)^2}{\cos\theta-\sin\theta}$ (or 0 if $\cos\theta = \sin\theta$) times r and so approaches zero as r approaches zero. You can see this in the figure below, which shows the level curves again, with the rays $\theta = \frac{1}{8}\pi$ and $\theta = \frac{3}{16}\pi$ superimposed. If you move towards the origin on either of

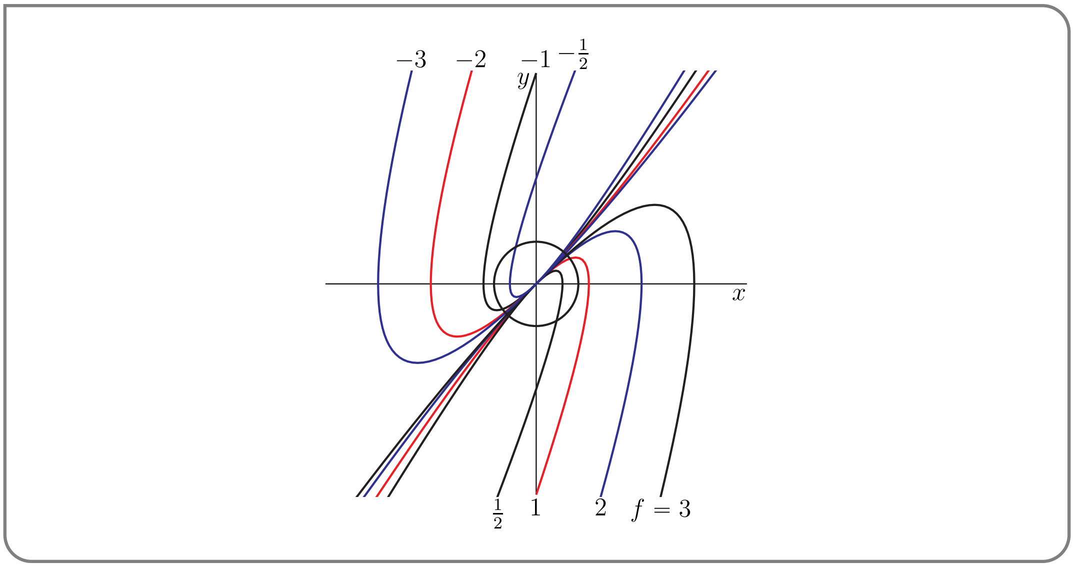


those rays, you first cross the $f = 3$ level curve, then the $f = 2$ level curve, then the $f = 1$ level curve, then the $f = \frac{1}{2}$ level curve, and so on.

6 Not just a pun.

That $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along any fixed ray is suggestive, but *does not imply* that the limit exists and is zero. Recall that to have $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, we need $f(x,y) \to 0$ no matter how $(x,y) \to (0,0)$. It is not sufficient to check only straight line approaches.

In fact, the limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist. A good way to see this is to observe that if you fix any $r > 0$, no matter how small, $f(x,y)$ takes all values from $-\infty$ to $+\infty$ on the circle $x^2 + y^2 = r^2$. You can see this in the figure below, which shows the level curves yet again, with a circle $x^2 + y^2 = r^2$ superimposed. For every single $-\infty < c < \infty$, the level curve $f(x,y) = c$ crosses the circle. Consequently there is no one



number L such that $f(x,y)$ is close to L whenever (x,y) is sufficiently close to $(0,0)$. The limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Another way to see that $f(x,y)$ does not have any limit as $(x,y) \to (0,0)$ is to show that $f(x,y)$ does not have a limit as (x,y) approaches $(0,0)$ along some specific curve. This can be done by picking a curve that makes the denominator, $x - y$, tend to zero very quickly. One such curve is $x - y = x^3$ or, equivalently, $y = x - x^3$. Along this curve, for $x \neq 0$,

$$\begin{aligned} f(x, x - x^3) &= \frac{(2x - x + x^3)^2}{x - x + x^3} = \frac{(x + x^3)^2}{x^3} \\ &= \frac{(1 + x^2)^2}{x} \longrightarrow \begin{cases} +\infty & \text{as } x \to 0 \text{ with } x > 0 \\ -\infty & \text{as } x \to 0 \text{ with } x < 0 \end{cases} \end{aligned}$$

The choice of the specific power x^3 is not important. Any power x^p with $p > 2$ will have the same effect.

If we send (x,y) to $(0,0)$ along the curve $x - y = ax^2$ or, equivalently, $y = x - ax^2$, where a is a nonzero constant,

$$\lim_{x\to 0} f(x, x - ax^2) = \lim_{x\to 0} \frac{(2x - x + ax^2)^2}{x - x + ax^2} = \lim_{x\to 0} \frac{(x + ax^2)^2}{ax^2} = \lim_{x\to 0} \frac{(1 + ax)^2}{a} = \frac{1}{a}$$

This limit depends on the choice of the constant a. Once again, this proves that $f(x,y)$ does not have a limit as $(x,y) \to (0,0)$.

Example 2.1.10

2.2▲ Partial Derivatives

We are now ready to define derivatives of functions of more than one variable. First, recall how we defined the derivative, $f'(a)$, of a function of one variable, $f(x)$. We imagined that we were walking along the x-axis, in the positive direction, measuring, for example, the temperature along the way. We denoted by $f(x)$ the temperature at x. The instantaneous rate of change of temperature that we observed as we passed through $x = a$ was

$$\frac{\mathrm{d}f}{\mathrm{d}x}(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

Next suppose that we are walking in the xy-plane and that the temperature at (x,y) is $f(x,y)$. We can pass through the point $(x,y) = (a,b)$ moving in many different directions, and we cannot expect the measured rate of change of temperature if we walk parallel to the x-axis, in the direction of increasing x, to be the same as the measured rate of change of temperature if we walk parallel to the y-axis in the direction of increasing y. We'll start by considering just those two directions. We'll consider other directions (like walking parallel to the line $y = x$) later.

Suppose that we are passing through the point $(x,y) = (a,b)$ and that we are walking parallel to the x-axis (in the positive direction). Then our y-coordinate will be constant, always taking the value $y = b$. So we can think of the measured temperature as the function of one variable $B(x) = f(x,b)$ and we will observe the rate of change of temperature

$$\frac{\mathrm{d}B}{\mathrm{d}x}(a) = \lim_{h\to 0} \frac{B(a+h) - B(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

This is called the "partial derivative f with respect to x at (a,b)" and is denoted $\frac{\partial f}{\partial x}(a,b)$. Here

- the symbol ∂, which is read "partial", indicates that we are dealing with a function of more than one variable, and
- the x in $\frac{\partial f}{\partial x}$ indicates that we are differentiating with respect to x, while y is being held fixed, i.e. being treated as a constant.
- $\frac{\partial f}{\partial x}$ is read " partial dee f dee x".

Do not write $\frac{\mathrm{d}}{\mathrm{d}x}$ when $\frac{\partial}{\partial x}$ is appropriate. We shall later encounter situations when $\frac{\mathrm{d}}{\mathrm{d}x}f$ and $\frac{\partial}{\partial x}f$ are both defined and have different meanings.

If, instead, we are passing through the point $(x,y) = (a,b)$ and are walking parallel to the y-axis (in the positive direction), then our x-coordinate will be constant, always taking the value $x = a$. So we can think of the measured temperature as the function of one variable $A(y) = f(a,y)$ and we will observe the rate of change of temperature

$$\frac{\mathrm{d}A}{\mathrm{d}y}(b) = \lim_{h\to 0} \frac{A(b+h) - A(b)}{h} = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

This is called the "partial derivative f with respect to y at (a, b)" and is denoted $\frac{\partial f}{\partial y}(a, b)$.

Just as was the case for the ordinary derivative $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$ (see Definition 2.2.6 in the CLP-1 text), it is common to treat the partial derivatives of $f(x, y)$ as functions of (x, y) simply by evaluating the partial derivatives at (x, y) rather than at (a, b).

Definition 2.2.1 (Partial Derivatives)**.**

The x- and y-partial derivatives of the function $f(x, y)$ are

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$
$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

respectively. The partial derivatives of functions of more than two variables are defined analogously.

Partial derivatives are used a lot. And there many notations for them.

Notation 2.2.2.

The partial derivative $\frac{\partial f}{\partial x}(x, y)$ of a function $f(x, y)$ is also denoted

$$\frac{\partial f}{\partial x} \qquad f_x(x,y) \qquad f_x \qquad D_x f(x,y) \qquad D_x f \qquad D_1 f(x,y) \qquad D_1 f$$

The subscript 1 on $D_1 f$ indicates that f is being differentiated with respect to its first variable. The partial derivative $\frac{\partial f}{\partial x}(a, b)$ is also denoted

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$$

with the subscript (a, b) indicating that $\frac{\partial f}{\partial x}$ is being evaluated at $(x, y) = (a, b)$.

The notation $\left(\frac{\partial f}{\partial x}\right)_y$ is used to make explicit that the variable y is being held.[7]

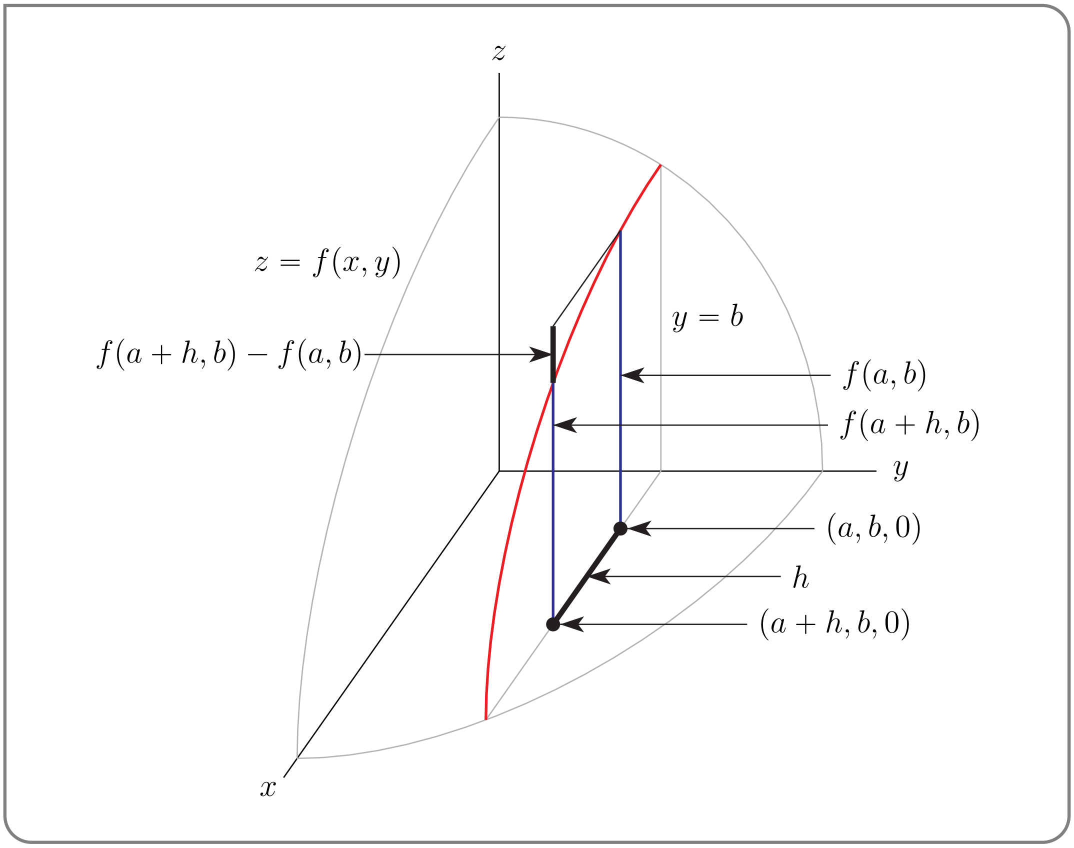
Remark 2.2.3 (The Geometric Interpretation of Partial Derivatives)**.** We'll now develop a geometric interpretation of the partial derivative

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$

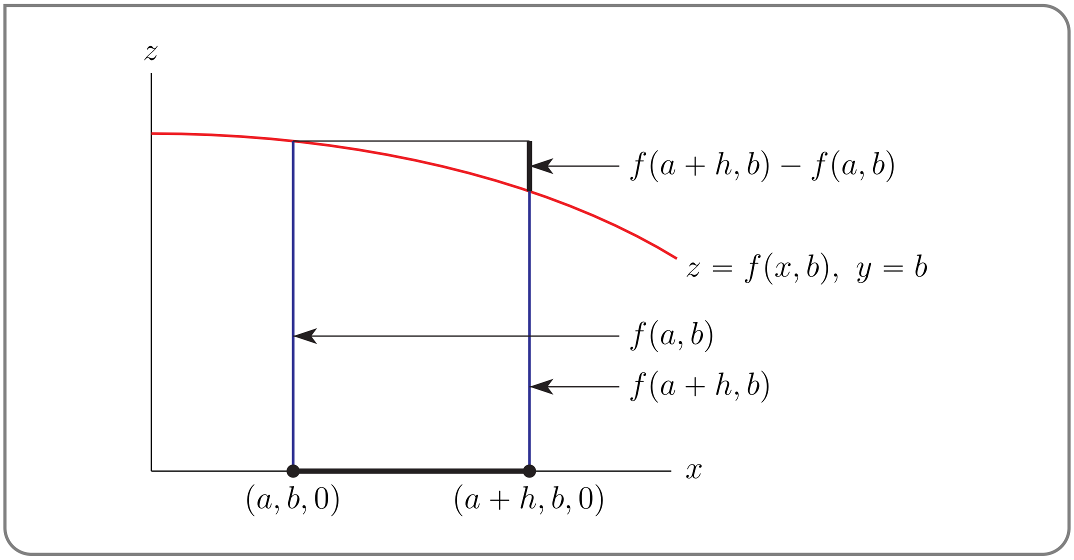
7 There are applications in which there are several variables that cannot be varied independently. For example, the pressure, volume and temperature of an ideal gas are related by the equation of state $PV = (\text{constant})T$. In those applications, it may not be clear from the context which variables are being held fixed.

in terms of the shape of the graph $z = f(x, y)$ of the function $f(x, y)$. That graph appears in the figure below. It looks like the part of a deformed sphere that is in the first octant.

The definition of $\frac{\partial f}{\partial x}(a, b)$ concerns only points on the graph that have $y = b$. In other words, the curve of intersection of the surface $z = f(x, y)$ with the plane $y = b$. That is the red curve in the figure. The two blue vertical line segments in the figure have heights $f(a, b)$ and $f(a + h, b)$, which are the two numbers in the numerator of $\frac{f(a+h,b)-f(a,b)}{h}$.



A side view of the curve (looking from the left side of the y-axis) is sketched in the figure below. Again, the two blue vertical line segments in the figure have heights $f(a, b)$



and $f(a+h,b)$, which are the two numbers in the numerator of $\frac{f(a+h,b)-f(a,b)}{h}$. So the numerator $f(a+h,b)-f(a,b)$ and denominator h are the rise and run, respectively, of the curve $z=f(x,b)$ from $x=a$ to $x=a+h$. Thus $\frac{\partial f}{\partial x}(a,b)$ is exactly *the slope of (the tangent to) the curve of intersection of the surface $z=f(x,y)$ and the plane $y=b$ at the point $\big(a,b,f(a,b)\big)$.* In the same way $\frac{\partial f}{\partial y}(a,b)$ is exactly *the slope of (the tangent to) the curve of intersection of the surface $z=f(x,y)$ and the plane $x=a$ at the point $\big(a,b,f(a,b)\big)$.*

▸▸▸ Evaluation of Partial Derivatives

From the above discussion, we see that we can readily compute partial derivatives $\frac{\partial}{\partial x}$ by using what we already know about ordinary derivatives $\frac{\mathrm{d}}{\mathrm{d}x}$. More precisely,

- to evaluate $\frac{\partial f}{\partial x}(x,y)$, treat the y in $f(x,y)$ as a constant and differentiate the resulting function of x with respect to x.
- To evaluate $\frac{\partial f}{\partial y}(x,y)$, treat the x in $f(x,y)$ as a constant and differentiate the resulting function of y with respect to y.
- To evaluate $\frac{\partial f}{\partial x}(a,b)$, treat the y in $f(x,y)$ as a constant and differentiate the resulting function of x with respect to x. Then evaluate the result at $x=a$, $y=b$.
- To evaluate $\frac{\partial f}{\partial y}(a,b)$, treat the x in $f(x,y)$ as a constant and differentiate the resulting function of y with respect to y. Then evaluate the result at $x=a$, $y=b$.

Now for some examples.

Example 2.2.4

Let

$$f(x,y)=x^3+y^2+4xy^2$$

Then, since $\frac{\partial}{\partial x}$ treats y as a constant,

$$\begin{aligned}\frac{\partial f}{\partial x}&=\frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial x}(y^2)+\frac{\partial}{\partial x}(4xy^2)\\&=3x^2+0+4y^2\frac{\partial}{\partial x}(x)\\&=3x^2+4y^2\end{aligned}$$

and, since $\frac{\partial}{\partial y}$ treats x as a constant,

$$\begin{aligned}\frac{\partial f}{\partial y}&=\frac{\partial}{\partial y}(x^3)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial y}(4xy^2)\\&=0+2y+4x\frac{\partial}{\partial y}(y^2)\\&=2y+8xy\end{aligned}$$

In particular, at $(x, y) = (1, 0)$ these partial derivatives take the values

$$\begin{aligned} \frac{\partial f}{\partial x}(1,0) &= 3(1)^2 + 4(0)^2 \quad = 3 \\ \frac{\partial f}{\partial y}(1,0) &= 2(0) + 8(1)(0) = 0 \end{aligned}$$

Example 2.2.4

Example 2.2.5

Let

$$f(x, y) = y \cos x + x e^{xy}$$

Then, since $\frac{\partial}{\partial x}$ treats y as a constant, $\frac{\partial}{\partial x} e^{yx} = y e^{yx}$ and

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= y \frac{\partial}{\partial x}(\cos x) + e^{xy} \frac{\partial}{\partial x}(x) + x \frac{\partial}{\partial x}\left(e^{xy}\right) \qquad \text{(by the product rule)} \\ &= -y \sin x + e^{xy} + xy e^{xy} \\ \frac{\partial f}{\partial y}(x,y) &= \cos x \frac{\partial}{\partial y}(y) + x \frac{\partial}{\partial y}\left(e^{xy}\right) \\ &= \cos x + x^2 e^{xy} \end{aligned}$$

Example 2.2.5

Let's move up to a function of four variables. Things generalize in a quite straight forward way.

Example 2.2.6

Let

$$f(x, y, z, t) = x \sin(y + 2z) + t^2 e^{3y} \ln z$$

Then

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y,z,t) &= \sin(y + 2z) \\ \frac{\partial f}{\partial y}(x,y,z,t) &= x \cos(y + 2z) + 3t^2 e^{3y} \ln z \\ \frac{\partial f}{\partial z}(x,y,z,t) &= 2x \cos(y + 2z) + t^2 e^{3y}/z \\ \frac{\partial f}{\partial t}(x,y,z,t) &= 2t e^{3y} \ln z \end{aligned}$$

Example 2.2.6

Now here is a more complicated example — our function takes a special value at $(0, 0)$. To compute derivatives there we revert to the definition.

Example 2.2.7

Set

$$f(x,y) = \begin{cases} \frac{\cos x - \cos y}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

If $b \neq a$, then for all (x,y) sufficiently close to (a,b), $f(x,y) = \frac{\cos x - \cos y}{x-y}$ and we can compute the partial derivatives of f at (a,b) using the familiar rules of differentiation. However that is not the case for $(a,b) = (0,0)$. To evaluate $f_x(0,0)$, we need to set $y = 0$ and find the derivative of

$$f(x,0) = \begin{cases} \frac{\cos x - 1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

with respect to x at $x = 0$. As we cannot use the usual differentiation rules, we evaluate the derivative[8] by applying the definition

$$\begin{aligned} f_x(0,0) &= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{\cos h - 1}{h} - 0}{h} && \text{(Recall that } h \neq 0 \text{ in the limit.)} \\ &= \lim_{h \to 0} \frac{\cos h - 1}{h^2} \\ &= \lim_{h \to 0} \frac{-\sin h}{2h} && \text{(By l'Hôpital's rule.)} \\ &= \lim_{h \to 0} \frac{-\cos h}{2} && \text{(By l'Hôpital again.)} \\ &= -\frac{1}{2} \end{aligned}$$

We could also evaluate the limit of $\frac{\cos h - 1}{h^2}$ by substituting in the Taylor expansion

$$\cos h = 1 - \frac{h^2}{2} + \frac{h^4}{4!} - \cdots$$

We can also use Taylor expansions to understand the behaviour of $f(x,y)$ for (x,y) near $(0,0)$. For $x \neq y$,

$$\begin{aligned} \frac{\cos x - \cos y}{x - y} &= \frac{\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right] - \left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right]}{x - y} \\ &= \frac{-\frac{x^2 - y^2}{2!} + \frac{x^4 - y^4}{4!} - \cdots}{x - y} \\ &= -\frac{1}{2!} \frac{x^2 - y^2}{x - y} + \frac{1}{4!} \frac{x^4 - y^4}{x - y} - \cdots \\ &= -\frac{x + y}{2!} + \frac{x^3 + x^2 y + x y^2 + y^3}{4!} - \cdots \end{aligned}$$

8 It is also possible to evaluate the derivative by using the technique of the optional §2.15 in the CLP-1 text.

So for (x, y) near $(0, 0)$,

$$f(x,y) \approx \begin{cases} -\frac{x+y}{2} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

So it sure looks like (and in fact it is true that)

- $f(x, y)$ is continuous at $(0, 0)$ and
- $f(x, y)$ is not continuous at (a, a) for small $a \neq 0$ and
- $f_x(0, 0) = f_y(0, 0) = -\frac{1}{2}$

Example 2.2.7

Example 2.2.8

Again set

$$f(x,y) = \begin{cases} \frac{\cos x - \cos y}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

We'll now compute $f_y(x, y)$ for all (x, y).

The case $y \neq x$: When $y \neq x$,

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y} \frac{\cos x - \cos y}{x - y} \\ &= \frac{(x-y)\frac{\partial}{\partial y}(\cos x - \cos y) - (\cos x - \cos y)\frac{\partial}{\partial y}(x-y)}{(x-y)^2} \qquad \text{by the quotient rule} \\ &= \frac{(x-y)\sin y + \cos x - \cos y}{(x-y)^2} \end{aligned}$$

The case $y = x$: When $y = x$,

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h} = \lim_{h\to 0} \frac{f(x, x+h) - f(x,x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{\cos x - \cos(x+h)}{x-(x+h)} - 0}{h} \qquad \text{(Recall that } h \neq 0 \text{ in the limit.)} \\ &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h^2} \end{aligned}$$

Now we apply L'Hôpital's rule, remembering that, in this limit, x is a constant and h is the variable — so we differentiate with respect to h.

$$f_y(x,y) = \lim_{h\to 0} \frac{-\sin(x+h)}{2h}$$

Note that if x is not an integer multiple of π, then the numerator $-\sin(x+h)$ does *not* tend to zero as h tends to zero, and the limit giving $f_y(x,y)$ does not exist. On the other hand, if x is an integer multiple of π, both the numerator and denominator tend to zero as h tends to zero, and we can apply L'Hôpital's rule a second time. Then

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0} \frac{-\cos(x+h)}{2} \\ &= -\frac{\cos x}{2} \end{aligned}$$

The conclusion:

$$f_y(x,y) = \begin{cases} \frac{(x-y)\sin y+\cos x-\cos y}{(x-y)^2} & \text{if } x \neq y \\ -\frac{\cos x}{2} & \text{if } x = y \text{ with } x \text{ an integer multiple of } \pi \\ DNE & \text{if } x = y \text{ with } x \text{ not an integer multiple of } \pi \end{cases}$$

Example 2.2.8

Example 2.2.9 (Optional — A Little Weirdness)

In this example, we will see that the function

$$f(x,y) = \begin{cases} \frac{x^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is *not* continuous at $(0,0)$ and yet has both partial derivatives $f_x(0,0)$ and $f_y(0,0)$ perfectly well defined. We'll also see how that is possible. First let's compute the partial derivatives. By definition,

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\overbrace{\frac{h^2}{h-0}}^{h} - 0}{h} = \lim_{h\to 0} 1 \\ &= 1 \\ f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0^2}{0-h} - 0}{h} = \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

So the first order partial derivatives $f_x(0,0)$ and $f_y(0,0)$ are perfectly well defined.

To see that, nonetheless, $f(x,y)$ is not continuous at $(0,0)$, we take the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the curve $y = x - x^3$. The limit is

$$\lim_{x\to 0} f\big(x, x-x^3\big) = \lim_{x\to 0} \frac{x^2}{x-(x-x^3)} = \lim_{x\to 0} \frac{1}{x}$$

which does not exist. Indeed as x approaches 0 through positive numbers, $\frac{1}{x}$ approaches $+\infty$, and as x approaches 0 through negative numbers, $\frac{1}{x}$ approaches $-\infty$.

So how is this possible? The answer is that $f_x(0,0)$ only involves values of $f(x,y)$ with $y = 0$. As $f(x,0) = x$, for all values of x, we have that $f(x,0)$ is a continuous, and indeed a differentiable, function. Similarly, $f_y(0,0)$ only involves values of $f(x,y)$ with $x = 0$. As $f(0,y) = 0$, for all values of y, we have that $f(0,y)$ is a continuous, and indeed a differentiable, function. On the other hand, the bad behaviour of $f(x,y)$ for (x,y) near $(0,0)$ only happens for x and y both nonzero.

Example 2.2.9

Our next example uses implicit differentiation.

Example 2.2.10

The equation

$$z^5 + y^2 e^z + e^{2x} = 0$$

implicitly determines z as a function of x and y. That is, the function $z(x,y)$ obeys

$$z(x,y)^5 + y^2 e^{z(x,y)} + e^{2x} = 0$$

For example, when $x = y = 0$, the equation reduces to

$$z(0,0)^5 = -1$$

which forces[9] $z(0,0) = -1$. Let's find the partial derivative $\frac{\partial z}{\partial x}(0,0)$.

We are not going to be able to explicitly solve the equation for $z(x,y)$. All we know is that

$$z(x,y)^5 + y^2 e^{z(x,y)} + e^{2x} = 0$$

for all x and y. We can turn this into an equation for $\frac{\partial z}{\partial x}(0,0)$ by differentiating[10] the whole equation with respect to x, giving

$$5z(x,y)^4 \,\frac{\partial z}{\partial x}(x,y) + y^2 e^{z(x,y)}\,\frac{\partial z}{\partial x}(x,y) + 2e^{2x} = 0$$

and then setting $x = y = 0$, giving

$$5z(0,0)^4 \,\frac{\partial z}{\partial x}(0,0) + 2 = 0$$

As we already know that $z(0,0) = -1$,

$$\frac{\partial z}{\partial x}(0,0) = -\frac{2}{5z(0,0)^4} = -\frac{2}{5}$$

9 The only real number z which obeys $z^5 = -1$ is $z = -1$. However there are four other complex numbers which also obey $z^5 = -1$.

10 You should have already seen this technique, called implicit differentiation, in your first Calculus course. It is covered in Section 2.11 in the CLP-1 text.

Example 2.2.10

Next we have a partial derivative disguised as a limit.

Example 2.2.11

In this example we are going to evaluate the limit

$$\lim_{z\to 0} \frac{(x+y+z)^3-(x+y)^3}{(x+y)z}$$

The critical observation is that, in taking the limit $z \to 0$, x and y are fixed. They do not change as z is getting smaller and smaller. Furthermore this limit is exactly of the form of the limits in the Definition 2.2.1 of partial derivative, disguised by some obfuscating changes of notation.

Set

$$f(x,y,z) = \frac{(x+y+z)^3}{(x+y)}$$

Then

$$\begin{aligned}\lim_{z\to 0} \frac{(x+y+z)^3-(x+y)^3}{(x+y)z} &= \lim_{z\to 0} \frac{f(x,y,z)-f(x,y,0)}{z} = \lim_{h\to 0} \frac{f(x,y,0+h)-f(x,y,0)}{h}\\ &= \frac{\partial f}{\partial z}(x,y,0)\\ &= \left[\frac{\partial}{\partial z}\frac{(x+y+z)^3}{x+y}\right]_{z=0}\end{aligned}$$

Recalling that $\frac{\partial}{\partial z}$ treats x and y as constants, we are evaluating the derivative of a function of the form $\frac{(\text{const}+z)^3}{\text{const}}$. So

$$\begin{aligned}\lim_{z\to 0} \frac{(x+y+z)^3-(x+y)^3}{(x+y)z} &= 3\frac{(x+y+z)^2}{x+y}\bigg|_{z=0}\\ &= 3(x+y)\end{aligned}$$

Example 2.2.11

The next example highlights a potentially dangerous difference between ordinary and partial derivatives.

Example 2.2.12

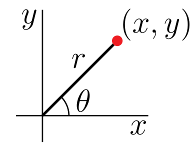
In this example we are going to see that, in contrast to the ordinary derivative case, $\frac{\partial r}{\partial x}$ is not, in general, the same as $\left(\frac{\partial x}{\partial r}\right)^{-1}$.

Recall that Cartesian and polar coordinates[11] (for $(x,y) \neq (0,0)$ and $r > 0$) are related

11 If you are not familiar with polar coordinates, don't worry about it. There will be an introduction to them in §3.2.1.

by

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
r &= \sqrt{x^2+y^2} \\
\tan\theta &= \frac{y}{x}
\end{aligned}
$$



We will use the functions

$$x(r,\theta) = r\cos\theta \qquad \text{and} \qquad r(x,y) = \sqrt{x^2+y^2}$$

Fix any point $(x_0, y_0) \neq (0,0)$ and let (r_0, θ_0), $0 \leqslant \theta_0 < 2\pi$, be the corresponding polar coordinates. Then

$$\frac{\partial x}{\partial r}(r,\theta) = \cos\theta \qquad \frac{\partial r}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$

so that

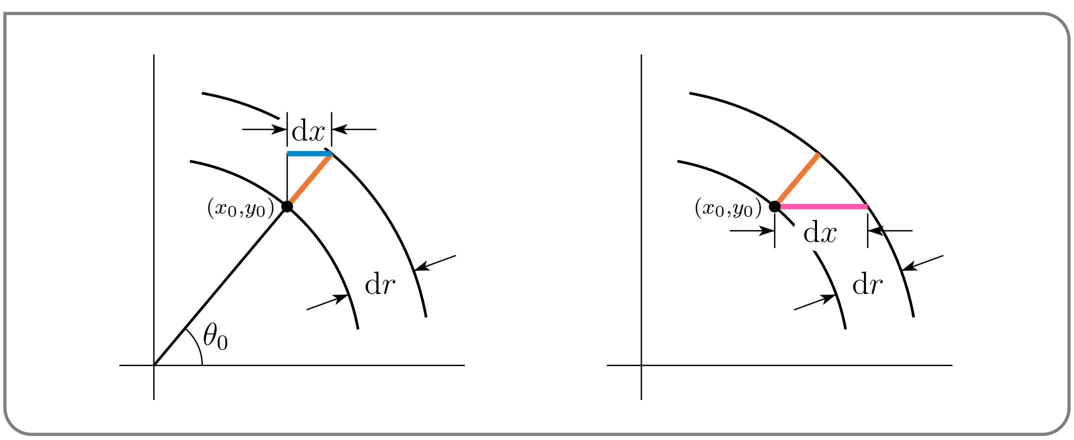
$$
\begin{aligned}
\frac{\partial x}{\partial r}(r_0,\theta_0) = \left(\frac{\partial r}{\partial x}(x_0,y_0)\right)^{-1} &\iff \cos\theta_0 = \left(\frac{x_0}{\sqrt{x_0^2+y_0^2}}\right)^{-1} = (\cos\theta_0)^{-1} \\
&\iff \cos^2\theta_0 = 1 \\
&\iff \theta_0 = 0, \pi
\end{aligned}
$$

We can also see pictorially why this happens. By definition, the partial derivatives

$$
\begin{aligned}
\frac{\partial x}{\partial r}(r_0,\theta_0) &= \lim_{\mathrm{d}r\to 0} \frac{x(r_0+\mathrm{d}r,\theta_0) - x(r_0,\theta_0)}{\mathrm{d}r} \\
\frac{\partial r}{\partial x}(x_0,y_0) &= \lim_{\mathrm{d}x\to 0} \frac{r(x_0+\mathrm{d}x,y_0) - r(x_0,y_0)}{\mathrm{d}x}
\end{aligned}
$$

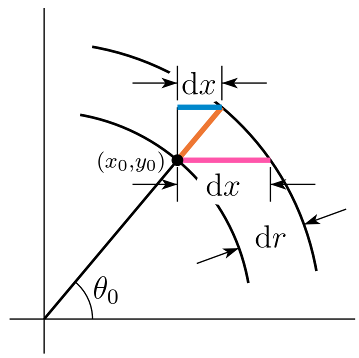
Here we have just renamed the h of Definition 2.2.1 to $\mathrm{d}r$ and to $\mathrm{d}x$ in the two definitions.

In computing $\frac{\partial x}{\partial r}(r_0,\theta_0)$, θ_0 is held fixed, r is changed by a small amount $\mathrm{d}r$ and the resulting $\mathrm{d}x = x(r_0+\mathrm{d}r,\theta_0) - x(r_0,\theta_0)$ is computed. In the figure on the left below, $\mathrm{d}r$ is the length of the orange line segment and $\mathrm{d}x$ is the length of the blue line segment.



On the other hand, in computing $\frac{\partial r}{\partial x}$, y is held fixed, x is changed by a small amount $\mathrm{d}x$ and the resulting $\mathrm{d}r = r(x_0 + \mathrm{d}x, y_0) - r(x_0, y_0)$ is computed. In the figure on the right above, $\mathrm{d}x$ is the length of the pink line segment and $\mathrm{d}r$ is the length of the orange line segment.

Here are the two figures combined together. We have arranged that the same $\mathrm{d}r$ is used in both computations. In order for the $\mathrm{d}r$'s to be the same in both computations, the two $\mathrm{d}x$'s have to be different (unless $\theta_0 = 0, \pi$). So, in general, $\frac{\partial x}{\partial r}(r_0, \theta_0) \neq \big(\frac{\partial r}{\partial x}(x_0, y_0)\big)^{-1}$.



Example 2.2.12

Example 2.2.13 (Optional — Example 2.2.12, continued)

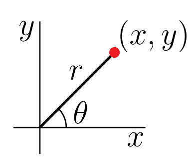
The inverse function theorem, for functions of one variable, says that, if $y(x)$ and $x(y)$ are inverse functions, meaning that $y\big(x(y)\big) = y$ and $x\big(y(x)\big) = x$, and are differentiable with $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$, then

$$\frac{\mathrm{d}x}{\mathrm{d}y}(y) = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}\big(x(y)\big)}$$

To see this, just apply $\frac{\mathrm{d}}{\mathrm{d}y}$ to both sides of $y\big(x(y)\big) = y$ to get $\frac{\mathrm{d}y}{\mathrm{d}x}\big(x(y)\big)\ \frac{\mathrm{d}x}{\mathrm{d}y}(y) = 1$, by the chain rule (see Theorem 2.9.3 in the CLP-1 text). In the CLP-1 text, we used this to compute the derivatives of the logarithm (see Theorem 2.10.1 in the CLP-1 text) and of the inverse trig functions (see Theorem 2.12.7 in the CLP-1 text).

We have just seen, in Example 2.2.12, that we can't be too naive in extending the single variable inverse function theorem to functions of two (or more) variables. On the other hand, there is such an extension, which we will now illustrate, using Cartesian and polar coordinates. For simplicity, we'll restrict our attention to $x > 0$, $y > 0$, or equivalently, $r > 0$, $0 < \theta < \frac{\pi}{2}$. The functions which convert between Cartesian and polar coordinates are

$$x(r, \theta) = r\cos\theta \qquad r(x, y) = \sqrt{x^2 + y^2}$$
$$y(r, \theta) = r\sin\theta \qquad \theta(x, y) = \arctan\left(\frac{y}{x}\right)$$



The two functions on the left convert from polar to Cartesian coordinates and the two functions on the right convert from Cartesian to polar coordinates. The inverse function theorem (for functions of two variables) says that,

- if you form the first order partial derivatives of the left hand functions into the matrix

$$\begin{bmatrix} \frac{\partial x}{\partial r}(r,\theta) & \frac{\partial x}{\partial \theta}(r,\theta) \\ \frac{\partial y}{\partial r}(r,\theta) & \frac{\partial y}{\partial \theta}(r,\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

- and you form the first order partial derivatives of the right hand functions into the matrix

$$\begin{bmatrix} \frac{\partial r}{\partial x}(x,y) & \frac{\partial r}{\partial y}(x,y) \\ \frac{\partial \theta}{\partial x}(x,y) & \frac{\partial \theta}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2} & \frac{\frac{1}{x}}{1+(\frac{y}{x})^2} \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

- and if you evaluate the second matrix at $x = x(r,\theta)$, $y = y(r,\theta)$,

$$\begin{bmatrix} \frac{\partial r}{\partial x}\big(x(r,\theta),y(r,\theta)\big) & \frac{\partial r}{\partial y}\big(x(r,\theta),y(r,\theta)\big) \\ \frac{\partial \theta}{\partial x}\big(x(r,\theta),y(r,\theta)\big) & \frac{\partial \theta}{\partial y}\big(x(r,\theta),y(r,\theta)\big) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$$

- and if you multiply[12] the two matrices together

$$\begin{aligned} &\begin{bmatrix} \frac{\partial r}{\partial x}\big(x(r,\theta),y(r,\theta)\big) & \frac{\partial r}{\partial y}\big(x(r,\theta),y(r,\theta)\big) \\ \frac{\partial \theta}{\partial x}\big(x(r,\theta),y(r,\theta)\big) & \frac{\partial \theta}{\partial y}\big(x(r,\theta),y(r,\theta)\big) \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial r}(r,\theta) & \frac{\partial x}{\partial \theta}(r,\theta) \\ \frac{\partial y}{\partial r}(r,\theta) & \frac{\partial y}{\partial \theta}(r,\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} (\cos\theta)(\cos\theta) + (\sin\theta)(\sin\theta) & (\cos\theta)(-r\sin\theta) + (\sin\theta)(r\cos\theta) \\ (-\frac{\sin\theta}{r})(\cos\theta) + (\frac{\cos\theta}{r})(\sin\theta) & (-\frac{\sin\theta}{r})(-r\sin\theta) + (\frac{\cos\theta}{r})(r\cos\theta) \end{bmatrix} \end{aligned}$$

- then the result is the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and indeed it is!

This two variable version of the inverse function theorem can be derived by applying the derivatives $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ to the equations

$$\begin{aligned} r\big(x(r,\theta),y(r,\theta)\big) &= r \\ \theta\big(x(r,\theta),y(r,\theta)\big) &= \theta \end{aligned}$$

and using the two variable version of the chain rule, which we will see in §2.4.

Example 2.2.13

12 Matrix multiplication is usually covered in courses on linear algebra, which you may or may not have taken. That's why this example is optional.

2.3▲ Higher Order Derivatives

You have already observed, in your first Calculus course, that if $f(x)$ is a function of x, then its derivative, $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$, is also a function of x, and can be differentiated to give the second order derivative $\frac{\mathrm{d}^2f}{\mathrm{d}x^2}(x)$, which can in turn be differentiated yet again to give the third order derivative, $f^{(3)}(x)$, and so on.

We can do the same for functions of more than one variable. If $f(x,y)$ is a function of x and y, then both of its partial derivatives, $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are also functions of x and y. They can both be differentiated with respect to x and they can both be differentiated with respect to y. So there are four possible second order derivatives. Here they are, together with various alternate notations.

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) &= \frac{\partial^2 f}{\partial x^2}(x,y) &&= f_{xx}(x,y)\\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) &= \frac{\partial^2 f}{\partial y\partial x}(x,y) &&= f_{xy}(x,y)\\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) &= \frac{\partial^2 f}{\partial x\partial y}(x,y) &&= f_{yx}(x,y)\\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) &= \frac{\partial^2 f}{\partial y^2}(x,y) &&= f_{yy}(x,y)
\end{aligned}$$

In $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2}{\partial y\,\partial x}f$, the derivative closest to f, in this case $\frac{\partial}{\partial x}$, is applied first.
In f_{xy}, the derivative with respect to the variable closest to f, in this case x, is applied first.

Example 2.3.1

Let $f(x,y) = e^{my}\cos(nx)$. Then

$$\begin{aligned}
f_x &= -ne^{my}\sin(nx) & f_y &= me^{my}\cos(nx)\\
f_{xx} &= -n^2e^{my}\cos(nx) & f_{yx} &= -mne^{my}\sin(nx)\\
f_{xy} &= -mne^{my}\sin(nx) & f_{yy} &= m^2e^{my}\cos(nx)
\end{aligned}$$

Example 2.3.1

Example 2.3.2

Let $f(x,y) = e^{\alpha x+\beta y}$. Then

$$\begin{aligned}
f_x &= \alpha e^{\alpha x+\beta y} & f_y &= \beta e^{\alpha x+\beta y}\\
f_{xx} &= \alpha^2 e^{\alpha x+\beta y} & f_{yx} &= \beta\alpha e^{\alpha x+\beta y}\\
f_{xy} &= \alpha\beta e^{\alpha x+\beta y} & f_{yy} &= \beta^2 e^{\alpha x+\beta y}
\end{aligned}$$

More generally, for any integers $m, n \geqslant 0$,

$$\frac{\partial^{m+n} f}{\partial x^m\, \partial y^n} = \alpha^m \beta^n e^{\alpha x + \beta y}$$

Example 2.3.2

Example 2.3.3

If $f(x_1, x_2, x_3, x_4) = x_1^4\, x_2^3\, x_3^2\, x_4$, then

$$\begin{aligned}
\frac{\partial^4 f}{\partial x_1\, \partial x_2\, \partial x_3\, \partial x_4} &= \frac{\partial^3}{\partial x_1\, \partial x_2\, \partial x_3}\Big(x_1^4\, x_2^3\, x_3^2\Big) \\
&= \frac{\partial^2}{\partial x_1\, \partial x_2}\Big(2\, x_1^4\, x_2^3\, x_3\Big) \\
&= \frac{\partial}{\partial x_1}\Big(6\, x_1^4\, x_2^2\, x_3\Big) \\
&= 24\, x_1^3\, x_2^2\, x_3
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^4 f}{\partial x_4\, \partial x_3\, \partial x_2\, \partial x_1} &= \frac{\partial^3}{\partial x_4\, \partial x_3\, \partial x_2}\Big(4 x_1^3\, x_2^3\, x_3^2\, x_4\Big) \\
&= \frac{\partial^2}{\partial x_4\, \partial x_3}\Big(12\, x_1^3\, x_2^2\, x_3^2\, x_4\Big) \\
&= \frac{\partial}{\partial x_4}\Big(24\, x_1^3\, x_2^2\, x_3\, x_4\Big) \\
&= 24\, x_1^3\, x_2^2\, x_3
\end{aligned}$$

Example 2.3.3

Notice that in Example 2.3.1,

$$f_{xy} = f_{yx} = -mne^{my} \sin(nx)$$

and in Example 2.3.2

$$f_{xy} = f_{yx} = \alpha\beta e^{\alpha x + \beta y}$$

and in Example 2.3.3

$$\frac{\partial^4 f}{\partial x_1\, \partial x_2\, \partial x_3\, \partial x_4} = \frac{\partial^4 f}{\partial x_4\, \partial x_3\, \partial x_2\, \partial x_1} = 24\, x_1^3\, x_2^2\, x_3$$

In all of these examples, it didn't matter what order we took the derivatives in. The following theorem[13] shows that this was no accident.

13 The history of this important theorem is pretty convoluted. See "A note on the history of mixed partial derivatives" by Thomas James Higgins which was published in Scripta Mathematica **7** (1940), 59-62.

Theorem 2.3.4 (Clairaut's Theorem[14] or Schwarz's Theorem[15]).

If the partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are continuous at (x_0, y_0), then

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

2.3.1 ▸▸ Optional — The Proof of Theorem 2.3.4

▸▸▸ Outline

Here is an outline of the proof of Theorem 2.3.4. The (numbered) details are in the subsection below. Fix real numbers x_0 and y_0 and define

$$F(h,k) = \frac{1}{hk}\big[f(x_0+h, y_0+k) - f(x_0, y_0+k) - f(x_0+h, y_0) + f(x_0, y_0)\big]$$

We define $F(h,k)$ in this way because both partial derivatives $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ and $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ are limits of $F(h,k)$ as $h, k \to 0$. Precisely, we show in item (1) in the details below that

$$\begin{aligned}
\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0, y_0) &= \lim_{k\to 0}\lim_{h\to 0} F(h,k) \\
\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0, y_0) &= \lim_{h\to 0}\lim_{k\to 0} F(h,k)
\end{aligned}$$

Note that the two right hand sides here are identical except for the order in which the limits are taken.

Now, by the mean value theorem (four times),

$$\begin{aligned}
F(h,k) &\overset{(2)}{=} \frac{1}{h}\left[\frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) - \frac{\partial f}{\partial y}(x_0, y_0+\theta_1 k)\right] \\
&\overset{(3)}{=} \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0+\theta_2 h, y_0+\theta_1 k) \\
F(h,k) &\overset{(4)}{=} \frac{1}{k}\left[\frac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+k) - \frac{\partial f}{\partial x}(x_0+\theta_3 h, y_0)\right] \\
&\overset{(5)}{=} \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+\theta_4 k)
\end{aligned}$$

for some numbers $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$. All of the numbers $\theta_1, \theta_2, \theta_3, \theta_4$ depend on x_0, y_0, h, k. Hence

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0+\theta_2 h, y_0+\theta_1 k) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+\theta_4 k)$$

14 Alexis Clairaut (1713–1765) was a French mathematician, astronomer, and geophysicist.
15 Hermann Schwarz (1843–1921) was a German mathematician.

for all h and k. Taking the limit $(h,k) \to (0,0)$ and using the assumed continuity of both partial derivatives at (x_0, y_0) gives

$$\lim_{(h,k)\to(0,0)} F(h,k) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0)$$

as desired. To complete the proof we just have to justify the details (1), (2), (3), (4) and (5).

▸▸▸ The Details

(1) By definition,

$$\begin{aligned}
\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0,y_0) &= \lim_{k\to 0}\frac{1}{k}\left[\frac{\partial f}{\partial x}(x_0,y_0+k) - \frac{\partial f}{\partial x}(x_0,y_0)\right] \\
&= \lim_{k\to 0}\frac{1}{k}\left[\lim_{h\to 0}\frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)}{h} - \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}\right] \\
&= \lim_{k\to 0}\lim_{h\to 0}\frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)-f(x_0+h,y_0)+f(x_0,y_0)}{hk} \\
&= \lim_{k\to 0}\lim_{h\to 0} F(h,k)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0,y_0) &= \lim_{h\to 0}\frac{1}{h}\left[\frac{\partial f}{\partial y}(x_0+h,y_0) - \frac{\partial f}{\partial y}(x_0,y_0)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\lim_{k\to 0}\frac{f(x_0+h,y_0+k)-f(x_0+h,y_0)}{k} - \lim_{k\to 0}\frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}\right] \\
&= \lim_{h\to 0}\lim_{k\to 0}\frac{f(x_0+h,y_0+k)-f(x_0+h,y_0)-f(x_0,y_0+k)+f(x_0,y_0)}{hk} \\
&= \lim_{h\to 0}\lim_{k\to 0} F(h,k)
\end{aligned}$$

(2) The mean value theorem (Theorem 2.13.4 in the CLP-1 text) says that, for any differentiable function $\varphi(x)$,

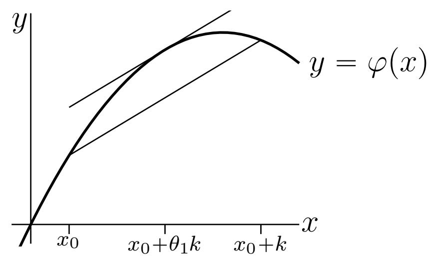
- the slope of the line joining the points $\big(x_0, \varphi(x_0)\big)$ and $\big(x_0+k, \varphi(x_0+k)\big)$ on the graph of φ

is the same as

- the slope of the tangent to the graph at some point between x_0 and x_0+k.

That is, there is some $0 < \theta_1 < 1$ such that

$$\frac{\varphi(x_0+k)-\varphi(x_0)}{k} = \frac{d\varphi}{dx}(x_0+\theta_1 k)$$



Applying this with x replaced by y and φ replaced by $G(y) = f(x_0+h, y) - f(x_0, y)$ gives

$$\begin{aligned} \frac{G(y_0+k) - G(y_0)}{k} &= \frac{\mathrm{d}G}{\mathrm{d}y}(y_0+\theta_1 k) \qquad \text{for some } 0 < \theta_1 < 1 \\ &= \frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) - \frac{\partial f}{\partial y}(x_0, y_0+\theta_1 k) \end{aligned}$$

Hence, for some $0 < \theta_1 < 1$,

$$F(h,k) \;=\; \frac{1}{h}\left[\frac{G(y_0+k) - G(y_0)}{k}\right] = \frac{1}{h}\left[\frac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) - \frac{\partial f}{\partial y}(x_0, y_0+\theta_1 k)\right]$$

(3) Define $H(x) = \frac{\partial f}{\partial y}(x, y_0+\theta_1 k)$. By the mean value theorem,

$$\begin{aligned} F(h,k) &= \frac{1}{h}\left[H(x_0+h) - H(x_0)\right] \\ &= \frac{\mathrm{d}H}{\mathrm{d}x}(x_0+\theta_2 h) \qquad \text{for some } 0 < \theta_2 < 1 \\ &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x_0+\theta_2 h, y_0+\theta_1 k) \end{aligned}$$

(4) Define $A(x) = f(x, y_0+k) - f(x, y_0)$. By the mean value theorem,

$$\begin{aligned} F(h,k) &= \frac{1}{k}\left[\frac{A(x_0+h) - A(x_0)}{h}\right] \\ &= \frac{1}{k}\frac{\mathrm{d}A}{\mathrm{d}x}(x_0+\theta_3 h) \qquad \text{for some } 0 < \theta_3 < 1 \\ &= \frac{1}{k}\left[\frac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+k) - \frac{\partial f}{\partial x}(x_0+\theta_3 h, y_0)\right] \end{aligned}$$

(5) Define $B(y) = \frac{\partial f}{\partial x}(x_0+\theta_3 h, y)$. By the mean value theorem

$$\begin{aligned} F(h,k) &= \frac{1}{k}\left[B(y_0+k) - B(y_0)\right] \\ &= \frac{\mathrm{d}B}{\mathrm{d}y}(y_0+\theta_4 k) \qquad \text{for some } 0 < \theta_4 < 1 \\ &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+\theta_4 k) \end{aligned}$$

This completes the proof of Theorem 2.3.4.

2.3.2 ▸▸ Optional — An Example of $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \neq \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$

In Theorem 2.3.4, we showed that $\frac{\partial^2 f}{\partial x\partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y\partial x}(x_0, y_0)$ **if** the partial derivatives $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$ exist and are continuous at (x_0, y_0). Here is an example which shows that if the partial derivatives $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$ are not continuous at (x_0, y_0), then it is possible that $\frac{\partial^2 f}{\partial x\partial y}(x_0, y_0) \neq \frac{\partial^2 f}{\partial y\partial x}(x_0, y_0)$.

Define

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

This function is continuous everywhere. Note that $f(x,0) = 0$ for all x and $f(0,y) = 0$ for all y. We now compute the first order partial derivatives. For $(x,y) \neq (0,0)$,

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= y\frac{x^2-y^2}{x^2+y^2} + xy\frac{2x}{x^2+y^2} - xy\frac{2x(x^2-y^2)}{(x^2+y^2)^2} = y\frac{x^2-y^2}{x^2+y^2} + xy\frac{4xy^2}{(x^2+y^2)^2} \\ \frac{\partial f}{\partial y}(x,y) &= x\frac{x^2-y^2}{x^2+y^2} - xy\frac{2y}{x^2+y^2} - xy\frac{2y(x^2-y^2)}{(x^2+y^2)^2} = x\frac{x^2-y^2}{x^2+y^2} - xy\frac{4yx^2}{(x^2+y^2)^2} \end{aligned}$$

For $(x,y) = (0,0)$,

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \left[\frac{\mathrm{d}}{\mathrm{d}x} f(x,0)\right]_{x=0} = \left[\frac{\mathrm{d}}{\mathrm{d}x} 0\right]_{x=0} = 0 \\ \frac{\partial f}{\partial y}(0,0) &= \left[\frac{\mathrm{d}}{\mathrm{d}y} f(0,y)\right]_{y=0} = \left[\frac{\mathrm{d}}{\mathrm{d}y} 0\right]_{y=0} = 0 \end{aligned}$$

By way of summary, the two first order partial derivatives are

$$f_x(x,y) = \begin{cases} y\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$f_y(x,y) = \begin{cases} x\frac{x^2-y^2}{x^2+y^2} - \frac{4x^3y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Both $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous. Finally, we compute

$$\begin{aligned} \frac{\partial^2 f}{\partial x\partial y}(0,0) &= \left[\frac{\mathrm{d}}{\mathrm{d}x} f_y(x,0)\right]_{x=0} = \lim_{h\to 0}\frac{1}{h}\left[f_y(h,0) - f_y(0,0)\right] = \lim_{h\to 0}\frac{1}{h}\left[h\frac{h^2-0^2}{h^2+0^2} - 0\right] = 1 \\ \frac{\partial^2 f}{\partial y\partial x}(0,0) &= \left[\frac{\mathrm{d}}{\mathrm{d}y} f_x(0,y)\right]_{y=0} = \lim_{k\to 0}\frac{1}{k}\left[f_x(0,k) - f_x(0,0)\right] = \lim_{k\to 0}\frac{1}{k}\left[k\frac{0^2-k^2}{0^2+k^2} - 0\right] = -1 \end{aligned}$$

2.4▲ The Chain Rule

You already routinely use the one dimensional chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}f\big(x(t)\big) = \frac{\mathrm{d}f}{\mathrm{d}x}\big(x(t)\big)\ \frac{\mathrm{d}x}{\mathrm{d}t}(t)$$

in doing computations like

$$\frac{\mathrm{d}}{\mathrm{d}t}\sin(t^2) = \cos(t^2)\ 2t$$

In this example, $f(x) = \sin(x)$ and $x(t) = t^2$.

We now generalize the chain rule to functions of more than one variable. For concreteness, we concentrate on the case in which all functions are functions of two variables. That is, we find the partial derivatives $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$ of a function $F(s,t)$ that is defined as a composition

$$F(s,t) = f\big(x(s,t)\,,\,y(s,t)\big)$$

We are using the name F for the new function $F(s,t)$ as a reminder that it is closely related to, though not the same as, the function $f(x,y)$. The partial derivative $\frac{\partial F}{\partial s}$ is the rate of change of F when s is varied with t held constant. When s is varied, both the x-argument, $x(s,t)$, and the y-argument, $y(s,t)$, in $f\big(x(s,t)\,,\,y(s,t)\big)$ vary. Consequently, the chain rule for $f\big(x(s,t)\,,\,y(s,t)\big)$ is a sum of two terms — one resulting from the variation of the x-argument and the other resulting from the variation of the y-argument.

Theorem 2.4.1 (The Chain Rule)**.**

Assume that all first order partial derivatives of $f(x,y)$, $x(s,t)$ and $y(s,t)$ exist and are continuous. Then the same is true for $F(s,t) = f\big(x(s,t)\,,\,y(s,t)\big)$ and

$$\begin{aligned}
\frac{\partial F}{\partial s}(s,t) &= \frac{\partial f}{\partial x}\big(x(s,t)\,,\,y(s,t)\big)\ \frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t)\,,\,y(s,t)\big)\ \frac{\partial y}{\partial s}(s,t)\\
\frac{\partial F}{\partial t}(s,t) &= \frac{\partial f}{\partial x}\big(x(s,t)\,,\,y(s,t)\big)\ \frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t)\,,\,y(s,t)\big)\ \frac{\partial y}{\partial t}(s,t)
\end{aligned}$$

We will give the proof of this theorem in §2.4.4, below. It is common to state this chain rule as

$$\begin{aligned}
\frac{\partial F}{\partial s} &= \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s}\\
\frac{\partial F}{\partial t} &= \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t}
\end{aligned}$$

That is, it is common to suppress the function arguments. But you should make sure that you understand what the arguments are before doing so.

Theorem 2.4.1 is given for the case that F is the composition of a function of two variables, $f(x,y)$, with two functions, $x(s,t)$ and $y(s,t)$, of two variables each. There is nothing magical about the number two. There are obvious variants for any numbers of variables. For example,

Equation 2.4.2.

if $F(t) = f\big(x(t), y(t), z(t)\big)$, then

$$\frac{\mathrm{d}F}{\mathrm{d}t}(t) = \frac{\partial f}{\partial x}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\,\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial f}{\partial y}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)$$
$$+\frac{\partial f}{\partial z}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\,\frac{\mathrm{d}z}{\mathrm{d}t}(t)$$

and

Equation 2.4.3.

if $F(s,t) = f\big(x(s,t)\big)$, then

$$\frac{\partial F}{\partial t}(s,t) = \frac{\mathrm{d}f}{\mathrm{d}x}\big(x(s,t)\big)\,\frac{\partial x}{\partial t}(s,t)$$

There will be a large number of examples shortly. First, here is a memory aid.

2.4.1 ▸▸ Memory Aids for the Chain Rule

We recommend strongly that you use the following procedure, without leaving out any steps, the first couple of dozen times that you use the chain rule.

Step 1 List **explicitly** all the functions involved and specify the arguments of each function. Ensure that all different functions have different names. Invent new names for some of the functions if necessary. In the case of the chain rule in Theorem 2.4.1, the list would be

$$f(x,y) \qquad x(s,t) \qquad y(s,t) \qquad F(s,t) = f\big(x(s,t), y(s,t)\big)$$

While the functions f and F are closely related, they are not the same. One is a function of x and y while the other is a function of s and t.

Step 2 Write down the template

$$\frac{\partial F}{\partial s} = \frac{\partial f}{}\frac{}{\partial s}$$

Note that

- The function F appears once in the numerator on the left. The function f, from which F is constructed by a change of variables, appears once in the numerator on the right.
- The variable in the denominator on the left appears once in the denominator on the right.

Step 3 Fill in the blanks with every variable that makes sense. In particular, since f is a function of x and y, it may only be differentiated with respect to x and y. So we add together two copies of our template — one for x and one for y:

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Note that x and y are functions of s so that the derivatives $\frac{\partial x}{\partial s}$ and $\frac{\partial y}{\partial s}$ make sense. The first term, $\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$, arises from the variation of x with respect to s and the second term, $\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$, arises from the variation of y with respect to s.

Step 4 Put in the functional dependence **explicitly**. Fortunately, there is only one functional dependence that makes sense. The left hand side is a function of s and t. Hence the right hand side must also be a function of s and t. As f is a function of x and y, this is achieved by evaluating f at $x = x(s,t)$ and $y = y(s,t)$.

$$\frac{\partial F}{\partial s}(s,t) = \frac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)\frac{\partial y}{\partial s}(s,t)$$

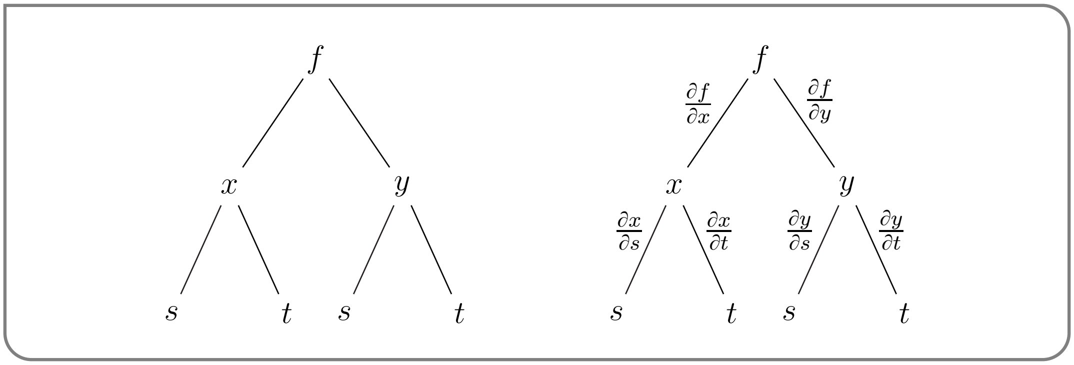
If you fail to put in the arguments, or at least if you fail to remember what the arguments are, you may forget that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ depend on s and t. Then, if you have to compute a second derivative of F, you will probably fail to differentiate the factors $\frac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)$ and $\frac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)$.

To help remember the formulae of Theorem 2.4.1, it is sometimes also useful to pretend that our variables are physical quantities with f, F having units of grams, x, y having units of meters and s, t having units of seconds. Note that

- the left hand side, $\frac{\partial F}{\partial s}$, has units grams per second.
- Each term on the right hand side contains the partial derivative of f with respect to a different independent variable. That independent variable appears once in the denominator and once in the numerator, so that its units (in this case meters) cancel out. Thus both of the terms $\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$ and $\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$ on the right hand side also have the units grams per second.
- Hence both sides of the equation have the same units.

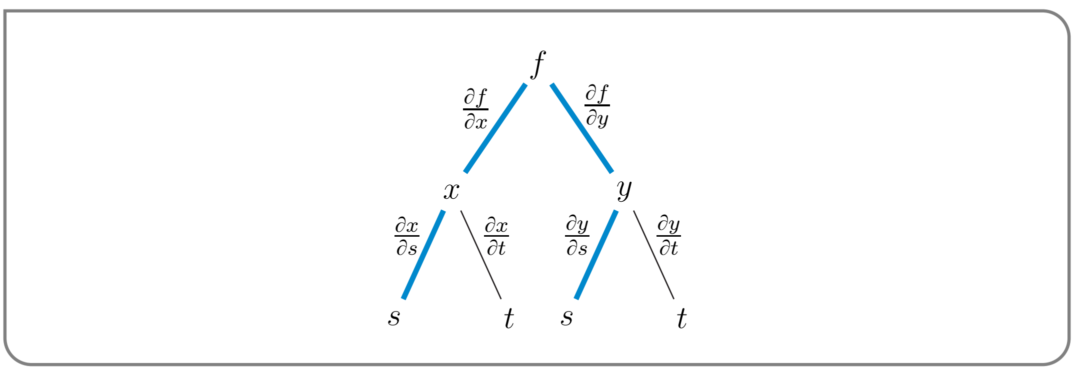
Here is a pictorial procedure that uses a *tree diagram* to help remember the chain rule $\frac{\partial}{\partial s} f\big(x(s,t),y(s,t)\big) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$. As in the figure on the left below,

- ○ write, on the top row, "f".
- ○ Write, on the middle row, each of the variables that the function $f(x,y)$ depends on, namely"x" and "y".
- ○ Write, on the bottom row,
  - below x, each of the variables that the function $x(s,t)$ depends on, namely "s" and "t", and
  - below y, each of the variables that the function $y(s,t)$ depends on, namely "s" and "t".
- ○ Draw a line joining each function with each of the variables that it depends on.
- ○ Then, as in the figure on the right below, write beside each line, the partial derivative of the function at the top of the line with respect to the variable at the bottom of the line.



- Finally
  - observe, from the figure below, that there are two paths from f, on the top, to s, on the bottom. One path goes from f at the top, through x in the middle to s at the bottom. The other path goes from f at the top, through y in the middle to s at the bottom.
  - For each such path, multiply together the partial derivatives beside the lines of the path. In this example, the two products are $\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$, for the first path, and $\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$, for the second path.
  - Then add together those products, giving, in this example, $\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$.
  - Put in the arguments, as in Step 4, above.
- That's it. We have

$$\frac{\partial}{\partial s}f\big(x(s,t),y(s,t)\big) = \frac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)\frac{\partial y}{\partial s}(s,t)$$

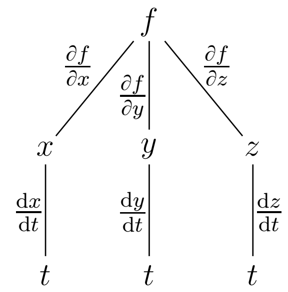


Example 2.4.4

The right hand side of the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}f\big(x(t)\,,\,y(t)\,,\,z(t)\big) = \frac{\partial f}{\partial x}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\,\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial f}{\partial y}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\,\frac{\mathrm{d}y}{\mathrm{d}t}(t)$$
$$+\frac{\partial f}{\partial z}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\,\frac{\mathrm{d}z}{\mathrm{d}t}(t)$$

of Equation (2.4.2), without arguments, is $\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$. The corresponding tree diagram is



Because $x(t)$, $y(t)$ and $z(t)$ are each functions of just one variable, the derivatives beside the lower lines in the tree are ordinary, rather than partial, derivatives.

Example 2.4.4

2.4.2 ▸▸ Chain Rule Examples

Let's do some routine examples first and work our way to some trickier ones.

Example 2.4.5 $\left(\frac{\partial}{\partial s} f\big(x(s,t), y(s,t)\big)\right)$

In this example we find $\frac{\partial}{\partial s} f\big(x(s,t), y(s,t)\big)$ for

$$f(x,y) = e^{xy} \qquad x(s,t) = s \qquad y(s,t) = \cos t$$

Define $F(s,t) = f\big(x(s,t)\,,\,y(s,t)\big)$. The appropriate chain rule for this example is the upper equation of Theorem 2.4.1.

$$\frac{\partial F}{\partial s}(s,t) = \frac{\partial f}{\partial x}\big(x(s,t)\,,\,y(s,t)\big)\,\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t)\,,\,y(s,t)\big)\,\frac{\partial y}{\partial s}(s,t)$$

For the given functions

$$\begin{aligned}
f(x,y) &= e^{xy} \\
\frac{\partial f}{\partial x}(x,y) &= ye^{xy} & \frac{\partial f}{\partial x}\big(x(s,t), y(s,t)\big) &= y(s,t)e^{x(s,t)\,y(s,t)} = \cos t\; e^{s\cos t} \\
\frac{\partial f}{\partial y}(x,y) &= xe^{xy} & \frac{\partial f}{\partial y}\big(x(s,t), y(s,t)\big) &= x(s,t)e^{x(s,t)\,y(s,t)} = s\; e^{s\cos t} \\
\frac{\partial x}{\partial s} &= 1 & \frac{\partial y}{\partial s} &= 0
\end{aligned}$$

so that

$$\frac{\partial F}{\partial s}(s,t) = \overbrace{\left\{\cos t\; e^{s\cos t}\right\}}^{\frac{\partial f}{\partial x}}\;\overbrace{(1)}^{\frac{\partial x}{\partial s}} + \overbrace{\left\{s\; e^{s\cos t}\right\}}^{\frac{\partial f}{\partial y}}\;\overbrace{(0)}^{\frac{\partial y}{\partial s}} = \cos t\; e^{s\cos t}$$

Example 2.4.5

Example 2.4.6 $\left(\frac{\mathrm{d}}{\mathrm{d}t} f\big(x(t), y(t)\big)\right)$

In this example we find $\frac{\mathrm{d}}{\mathrm{d}t} f\big(x(t), y(t)\big)$ for

$$f(x,y) = x^2 - y^2 \qquad x(t) = \cos t \qquad y(t) = \sin t$$

Define $F(t) = f\big(x(t), y(t)\big)$. Since $F(t)$ is a function of one variable its derivative is denoted $\frac{\mathrm{d}F}{\mathrm{d}t}$ rather than $\frac{\partial F}{\partial t}$. The appropriate chain rule for this example (see (2.4.2)) is

$$\frac{\mathrm{d}F}{\mathrm{d}t}(t) = \frac{\partial f}{\partial x}\big(x(t), y(t)\big)\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{\partial f}{\partial y}\big(x(t), y(t)\big)\frac{\mathrm{d}y}{\mathrm{d}t}(t)$$

For the given functions

$$\begin{aligned}
f(x,y) &= x^2 - y^2 \\
\frac{\partial f}{\partial x}(x,y) &= 2x & \frac{\partial f}{\partial x}\big(x(t), y(t)\big) &= 2x(t) = 2\cos t \\
\frac{\partial f}{\partial y}(x,y) &= -2y & \frac{\partial f}{\partial y}\big(x(t), y(t)\big) &= -2y(t) = -2\sin t \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\sin t & \frac{\mathrm{d}y}{\mathrm{d}t} &= \cos t
\end{aligned}$$

so that

$$\frac{\mathrm{d}F}{\mathrm{d}t}(t) = (2\cos t)(-\sin t) + (-2\sin t)(\cos t) = -4\sin t\cos t$$

Of course, in this example we can compute $F(t)$ explicitly

$$F(t) = f\big(x(t), y(t)\big) = x(t)^2 - y(t)^2 = \cos^2 t - \sin^2 t$$

and then differentiate

$$F'(t) = 2(\cos t)(-\sin t) - 2(\sin t)(\cos t) = -4\sin t\cos t$$

Example 2.4.6

Example 2.4.7 $\left(\frac{\partial}{\partial t} f(x+ct)\right)$

Define $u(x,t) = x + ct$ and $w(x,t) = f(x+ct) = f\big(u(x,t)\big)$. Then

$$\frac{\partial}{\partial t} f(x+ct) = \frac{\partial w}{\partial t}(x,t) = \frac{\mathrm{d}f}{\mathrm{d}u}\big(u(x,t)\big)\frac{\partial u}{\partial t}(x,t) = c\, f'(x+ct)$$

Example 2.4.7

Example 2.4.8 $\left(\frac{\partial^2}{\partial t^2} f(x+ct)\right)$

Define $w(x,t) = f(x+ct)$ and $W(x,t) = \frac{\partial w}{\partial t}(x,t) = cf'(x+ct) = F\big(u(x,t)\big)$ where $F(u) = cf'(u)$ and $u(x,t) = x+ct$. Then

$$\frac{\partial^2}{\partial t^2} f(x+ct) = \frac{\partial W}{\partial t}(x,t) = \frac{\mathrm{d}F}{\mathrm{d}u}\big(u(x,t)\big)\frac{\partial u}{\partial t}(x,t) = c\, f''(x+ct)\, c = c^2\, f''(x+ct)$$

Example 2.4.8

Example 2.4.9 (Equation of state)

Suppose that we are told that $F(P,V,T) = 0$ and that we are to find $\frac{\partial P}{\partial T}$.

Before we can find $\frac{\partial P}{\partial T}$, we first have to decide what it means. This happens regularly in applications. In fact, this particular problem comes from thermodynamics. The variables $P,\ V,\ T$ are the pressure, volume and temperature, respectively, of some gas. These three variables are not independent. They are related by an equation of state, here denoted $F(P,V,T) = 0$. Given values for any two of $P,\ V,\ T$, the third can be found by solving $F(P,V,T) = 0$. We are being asked to find $\frac{\partial P}{\partial T}$. This implicitly instructs us to treat P, in this problem, as the dependent variable. So a careful wording of this problem (which you will never encounter in the "real world") would be the following. The function $P(V,T)$ is defined by $F\big(P(V,T),V,T\big) = 0$. Find $\left(\frac{\partial P}{\partial T}\right)_V$. That is, find the rate of change of pressure as the temperature is varied, while holding the volume fixed.

Since we are not told explicitly what F is, we cannot solve explicitly for $P(V,T)$. So, instead we differentiate both sides of

$$F\big(P(V,T),V,T\big) = 0$$

with respect to T, while holding V fixed. Think of the left hand side, $F\big(P(V,T),V,T\big)$, as being $F\big(P(V,T),Q(V,T),R(V,T)\big)$ with $Q(V,T) = V$ and $R(V,T) = T$. By the chain rule,

$$\frac{\partial}{\partial T} F\big(P(V,T),Q(V,T),R(V,T)\big) = F_1\frac{\partial P}{\partial T} + F_2\frac{\partial Q}{\partial T} + F_3\frac{\partial R}{\partial T} = 0$$

with F_j referring to the partial derivative of F with respect to its j^{th} argument. Experienced chain rule users never introduce Q and R. Instead, they just write

$$\frac{\partial F}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial F}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial F}{\partial T}\frac{\partial T}{\partial T} = 0$$

Recalling that V and T are the independent variables and that, in computing $\frac{\partial}{\partial T}$, V is to be treated as a constant,

$$\frac{\partial V}{\partial T} = 0 \qquad\qquad \frac{\partial T}{\partial T} = 1$$

Now putting in the functional dependence

$$\frac{\partial F}{\partial P}\big(P(V,T),V,T\big)\frac{\partial P}{\partial T}(V,T) + \frac{\partial F}{\partial T}\big(P(V,T),V,T\big) = 0$$

and solving

$$\frac{\partial P}{\partial T}(V,T) = -\frac{\frac{\partial F}{\partial T}\big(P(V,T),V,T\big)}{\frac{\partial F}{\partial P}\big(P(V,T),V,T\big)}$$

Example 2.4.9

Example 2.4.10

Suppose that $f(x,y) = 0$ and that we are to find $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

Once again, x and y are not independent variables. Given a value for either x or y, the other is determined by solving $f(x,y) = 0$. Since we are asked to find $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, it is y that is to be viewed as a function of x, rather than the other way around. So $f(x,y) = 0$ really means that, in this problem, $f\big(x,y(x)\big) = 0$ for all x. Differentiating both sides of this equation with respect to x,

$$\begin{aligned} f\big(x,y(x)\big) &= 0 \qquad \text{for all } x \\ \implies \frac{\mathrm{d}}{\mathrm{d}x} f\big(x,y(x)\big) &= 0 \end{aligned}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}x} f\big(x,y(x)\big)$ is not the same as $f_x\big(x,y(x)\big)$. The former is, by definition, the rate of change with respect to x of $g(x) = f\big(x,y(x)\big)$. Precisely,

$$\begin{aligned} \frac{\mathrm{d}g}{\mathrm{d}x} &= \lim_{\Delta x\to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{f\big(x+\Delta x\,,\, y(x+\Delta x)\big) - f\big(x\,,\, y(x)\big)}{\Delta x} \end{aligned} \tag{$*$}$$

On the other hand, by definition,

$$\begin{aligned} f_x(x,y) &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \\ \implies f_x\big(x,y(x)\big) &= \lim_{\Delta x\to 0} \frac{f\big(x+\Delta x\,,\, y(x)\big) - f\big(x\,,\, y(x)\big)}{\Delta x} \end{aligned} \tag{$**$}$$

The right hand sides of $(*)$ and $(**)$ are not the same. In $\frac{\mathrm{d}g}{\mathrm{d}x}$, as Δx varies the value of y that is substituted into the first $f(\cdots)$ on the right hand side, namely $y(x+\Delta x)$, changes as Δx changes. That is, we are computing the rate of change of f along the (curved) path $y = y(x)$. In $(**)$, the corresponding value of y is $y(x)$ and is independent of Δx. That is, we are computing the rate of change of f along a horizontal straight line. As a concrete example, suppose that $f(x,y) = x + y$. Then, $0 = f\big(x\,,\, y(x)\big) = x + y(x)$ gives $y(x) = -x$ so that

$$\frac{\mathrm{d}}{\mathrm{d}x} f\big(x,y(x)\big) = \frac{\mathrm{d}}{\mathrm{d}x} f(x,-x) = \frac{\mathrm{d}}{\mathrm{d}x}[x + (-x)] = \frac{\mathrm{d}}{\mathrm{d}x} 0 = 0$$

But $f(x,y) = x + y$ implies that $f_x(x,y) = 1$ for all x and y so that

$$f_x(x,y(x)) = f_x(x,y)\Big|_{y=-x} = 1\Big|_{y=-x} = 1$$

Now back to

$$\begin{aligned}
& f\big(x,y(x)\big) = 0 \qquad \text{for all } x \\
\implies & \frac{\mathrm{d}}{\mathrm{d}x} f\big(x,y(x)\big) = 0 \\
\implies & f_x\big(x,y(x)\big)\frac{\mathrm{d}x}{\mathrm{d}x} + f_y\big(x,y(x)\big)\frac{\mathrm{d}y}{\mathrm{d}x}(x) = 0 \qquad \text{by the chain rule} \\
\implies & \frac{\mathrm{d}y}{\mathrm{d}x}(x) = -\frac{f_x\big(x,y(x)\big)}{f_y\big(x,y(x)\big)} \\
\implies & \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f_x\big(x,y(x)\big)}{f_y\big(x,y(x)\big)}\right] \\
& \qquad = -\frac{f_y\big(x,y(x)\big)\frac{\mathrm{d}}{\mathrm{d}x}[f_x\big(x,y(x)\big)] - f_x\big(x,y(x)\big)\frac{\mathrm{d}}{\mathrm{d}x}[f_y\big(x,y(x)\big)]}{f_y\big(x,y(x)\big)^2} \qquad (\dagger)
\end{aligned}$$

by the quotient rule. Now it suffices to substitute in $\frac{\mathrm{d}}{\mathrm{d}x}\big[f_x\big(x,y(x)\big)\big]$ and $\frac{\mathrm{d}}{\mathrm{d}x}\big[f_y\big(x,y(x)\big)\big]$. For the former apply the chain rule to $h(x) = u\big(x,y(x)\big)$ with $u(x,y) = f_x\big(x,y\big)$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\big[f_x\big(x,y(x)\big)\big] &= \frac{\mathrm{d}h}{\mathrm{d}x}(x) \\
&= u_x\big(x,y(x)\big)\frac{\mathrm{d}x}{\mathrm{d}x} + u_y\big(x,y(x)\big)\frac{\mathrm{d}y}{\mathrm{d}x}(x) \\
&= f_{xx}\big(x,y(x)\big)\frac{\mathrm{d}x}{\mathrm{d}x} + f_{xy}\big(x,y(x)\big)\frac{\mathrm{d}y}{\mathrm{d}x}(x) \\
&= f_{xx}\big(x,y(x)\big) - f_{xy}\big(x,y(x)\big)\left[\frac{f_x\big(x,y(x)\big)}{f_y\big(x,y(x)\big)}\right]
\end{aligned}$$

Substituting this and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\big[f_y\big(x,y(x)\big)\big] &= f_{yx}\big(x,y(x)\big)\frac{\mathrm{d}x}{\mathrm{d}x} + f_{yy}\big(x,y(x)\big)\frac{\mathrm{d}y}{\mathrm{d}x}(x) \\
&= f_{yx}\big(x,y(x)\big) - f_{yy}\big(x,y(x)\big)\left[\frac{f_x\big(x,y(x)\big)}{f_y\big(x,y(x)\big)}\right]
\end{aligned}$$

into the right hand side of $(\dagger)$ gives the final answer.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(x) = -\frac{f_yf_{xx} - f_yf_{xy}\frac{f_x}{f_y} - f_xf_{yx} + f_xf_{yy}\frac{f_x}{f_y}}{f_y^2} = -\frac{f_y^2f_{xx} - 2f_xf_yf_{xy} + f_x^2f_{yy}}{f_y^3}$$

with all of f_x, f_y, f_{xx}, f_{xy}, f_{yy} having arguments $\big(x\,,\,y(x)\big)$.

Example 2.4.10

We now move on to the proof of Theorem 2.4.1. To give you an idea of how the proof will go, we first review the proof of the familiar one dimensional chain rule.

2.4.3 ▸▸ Review of the Proof of $\frac{\mathrm{d}}{\mathrm{d}t}f\big(x(t)\big) = \frac{\mathrm{d}f}{\mathrm{d}x}\big(x(t)\big)\ \frac{\mathrm{d}x}{\mathrm{d}t}(t)$

As a warm up, let's review the proof of the one dimensional chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}f\big(x(t)\big) = \frac{\mathrm{d}f}{\mathrm{d}x}\big(x(t)\big)\ \frac{\mathrm{d}x}{\mathrm{d}t}(t)$$

assuming that $\frac{\mathrm{d}x}{\mathrm{d}t}$ exists and that $\frac{\mathrm{d}f}{\mathrm{d}x}$ is continuous. We wish to find the derivative of $F(t) = f\big(x(t)\big)$. By definition

$$\begin{aligned} F'(t) &= \lim_{h\to 0}\frac{F(t+h)-F(t)}{h} \\ &= \lim_{h\to 0}\frac{f\big(x(t+h)\big)-f\big(x(t)\big)}{h} \end{aligned}$$

Notice that the numerator is the difference of $f(x)$ evaluated at two nearby values of x, namely $x_1 = x(t+h)$ and $x_0 = x(t)$. The mean value theorem is a good tool for studying the difference in the values of $f(x)$ at two nearby points. Recall that the mean value theorem says that, for any given x_0 and x_1, there exists an (in general unknown) c between them so that

$$f(x_1) - f(x_0) = f'(c)\ (x_1 - x_0)$$

For this proof, we choose $x_0 = x(t)$ and $x_1 = x(t+h)$. The the mean value theorem tells us that there exists a c_h so that

$$f\big(x(t+h)\big) - f\big(x(t)\big) = f(x_1) - f(x_0) = f'(c_h)\ \big[x(t+h) - x(t)\big]$$

We have put the subscript h on c_h to emphasise that c_h, which is between $x_0 = x(t)$ and $x_1 = x(t+h)$, may depend on h. Now since c_h is trapped between $x(t)$ and $x(t+h)$ and since $x(t+h) \to x(t)$ as $h \to 0$, we have that c_h must also tend to $x(t)$ as $h \to 0$. Plugging this into the definition of $F'(t)$,

$$\begin{aligned} F'(t) &= \lim_{h\to 0}\frac{f\big(x(t+h)\big)-f\big(x(t)\big)}{h} \\ &= \lim_{h\to 0}\frac{f'(c_h)\ \big[x(t+h)-x(t)\big]}{h} \\ &= \lim_{h\to 0} f'(c_h)\ \lim_{h\to 0}\frac{x(t+h)-x(t)}{h} \\ &= f'\big(x(t)\big)\ x'(t) \end{aligned}$$

as desired.

2.4.4 ▸▸ Proof of Theorem 2.4.1

We'll now prove the formula for $\frac{\partial}{\partial s}f\big(x(s,t)\,,\,y(s,t)\big)$ that is given in Theorem 2.4.1. The proof uses the same ideas as the proof of the one variable chain rule, that we have just reviewed.

We wish to find the partial derivative with respect to s of $F(s,t) = f\big(x(s,t)\,,\,y(s,t)\big)$. By definition

$$\begin{aligned}\frac{\partial F}{\partial s}(s,t) &= \lim_{h\to 0}\frac{F(s+h,t)-F(s,t)}{h}\\ &= \lim_{h\to 0}\frac{f\big(x(s+h,t)\,,\,y(s+h,t)\big)-f\big(x(s,t)\,,\,y(s,t)\big)}{h}\end{aligned}$$

The numerator is the difference of $f(x,y)$ evaluated at two nearby values of (x,y), namely $(x_1,y_1) = \big(x(s+h,t)\,,\,y(s+h,t)\big)$ and $(x_0,y_0) = \big(x(s,t)\,,\,y(s,t)\big)$. In going from (x_0,y_0) to (x_1,y_1), both the x and y-coordinates change. By adding and subtracting we can separate the change in the x-coordinate from the change in the y-coordinate.

$$f(x_1,y_1)-f(x_0,y_0) = \big\{f(x_1,y_1)-f(x_0,y_1)\big\}+\big\{f(x_0,y_1)-f(x_0,y_0)\big\}$$

The first half, $\big\{f(x_1,y_1)-f(x_0,y_1)\big\}$, has the same y argument in both terms and so is the difference of the function of one variable $g(x) = f(x,y_1)$ (viewing y_1 just as a constant) evaluated at the two nearby values, x_0, x_1, of x. Consequently, we can make use of the mean value theorem as we did in §2.4.3 above. There is a $c_{x,h}$ between $x_0 = x(s,t)$ and $x_1 = x(s+h,t)$ such that

$$\begin{aligned}f(x_1,y_1)-f(x_0,y_1) &= g(x_1)-g(x_0) = g'(c_{x,h})[x_1-x_0] = \frac{\partial f}{\partial x}(c_{x,h}\,,\,y_1)\,[x_1-x_0]\\ &= \frac{\partial f}{\partial x}\big(c_{x,h}\,,\,y(s+h,t)\big)\,\big[x(s+h,t)-x(s,t)\big]\end{aligned}$$

We have introduced the two subscripts in $c_{x,h}$ to remind ourselves that it may depend on h and that it lies between the two x-values x_0 and x_1.

Similarly, the second half, $\big\{f(x_0,y_1)-f(x_0,y_0)\big\}$, is the difference of the function of one variable $h(y) = f(x_0,y)$ (viewing x_0 just as a constant) evaluated at the two nearby values, y_0, y_1, of y. So, by the mean value theorem,

$$\begin{aligned}f(x_0,y_1)-f(x_0,y_0) &= h(y_1)-h(y_0) = h'(c_{y,h})[y_1-y_0] = \frac{\partial f}{\partial y}(x_0,c_{y,h})\,[y_1-y_0]\\ &= \frac{\partial f}{\partial y}\big(x(s,t)\,,\,c_{y,h}\big)\,\big[y(s+h,t)-y(s,t)\big]\end{aligned}$$

for some (unknown) $c_{y,h}$ between $y_0 = y(s,t)$ and $y_1 = y(s+h,t)$. Again, the two subscripts in $c_{y,h}$ remind ourselves that it may depend on h and that it lies between the two y-values y_0 and y_1. So, noting that, as h tends to zero, $c_{x,h}$, which is trapped between $x(s,t)$ and $x(s+h,t)$, must tend to $x(s,t)$, and $c_{y,h}$, which is trapped between $y(s,t)$ and

$y(s+h,t)$, must tend to $y(s,t)$,

$$\begin{aligned}
\frac{\partial F}{\partial s}(s,t)) &= \lim_{h\to 0} \frac{f\big(x(s+h,t)\,,\,y(s+h,t)\big) - f\big(x(s,t)\,,\,y(s,t)\big)}{h} \\
&= \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}\big(c_{x,h}\,,\,y(s+h,t)\big)\,\big[x(s+h,t)-x(s,t)\big]}{h} \\
&\qquad + \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}\big(x(s,t)\,,\,c_{y,h}\big)\,\big[y(s+h,t)-y(s,t)\big]}{h} \\
&= \lim_{h\to 0} \frac{\partial f}{\partial x}\big(c_{x,h}\,,\,y(s+h,t)\big)\ \lim_{h\to 0} \frac{x(s+h,t)-x(s,t)}{h} \\
&\qquad + \lim_{h\to 0} \frac{\partial f}{\partial y}\big(x(s,t)\,,\,c_{y,h}\big)\ \lim_{h\to 0} \frac{y(s+h,t)-y(s,t)}{h} \\
&= \frac{\partial f}{\partial x}\big(x(s,t)\,,\,y(s,t)\big)\,\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t)\,,\,y(s,t)\big)\,\frac{\partial y}{\partial s}(s,t)
\end{aligned}$$

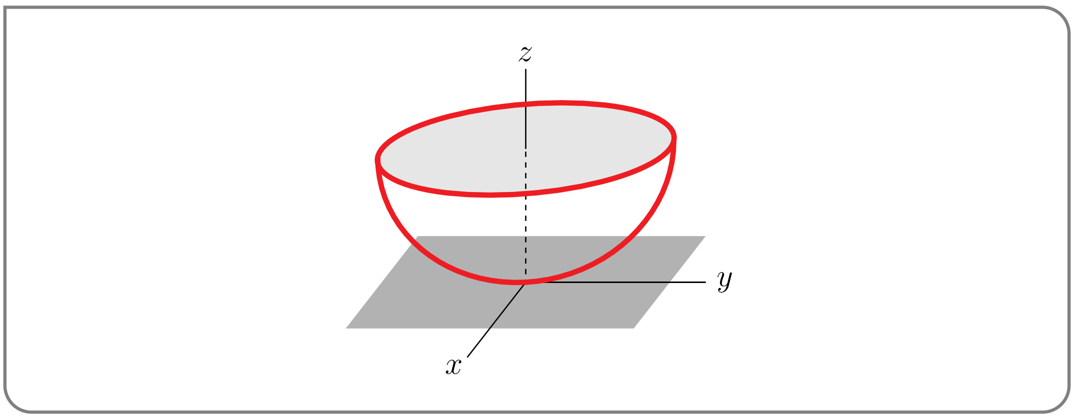
We can of course follow the same procedure to evaluate the partial derivative with respect to t. This concludes the proof of Theorem 2.4.1.

2.5▲ Tangent Planes and Normal Lines

The tangent line to the curve $y = f(x)$ at the point $\big(x_0, f(x_0)\big)$ is the straight line that fits the curve best[16] at that point. Finding tangent lines was probably one of the first applications of derivatives that you saw. See, for example, Theorem 2.3.2 in the CLP-1 text. The analog of the tangent line one dimension up is the tangent plane. The tangent plane to a surface S at a point (x_0, y_0, z_0) is the plane that fits S best at (x_0, y_0, z_0). For example, the tangent plane to the hemisphere

$$S = \big\{\ (x,y,z)\ \big|\ x^2+y^2+(z-1)^2 = 1,\ 0 \leqslant z \leqslant 1\ \big\}$$

at the origin is the xy-plane, $z = 0$.

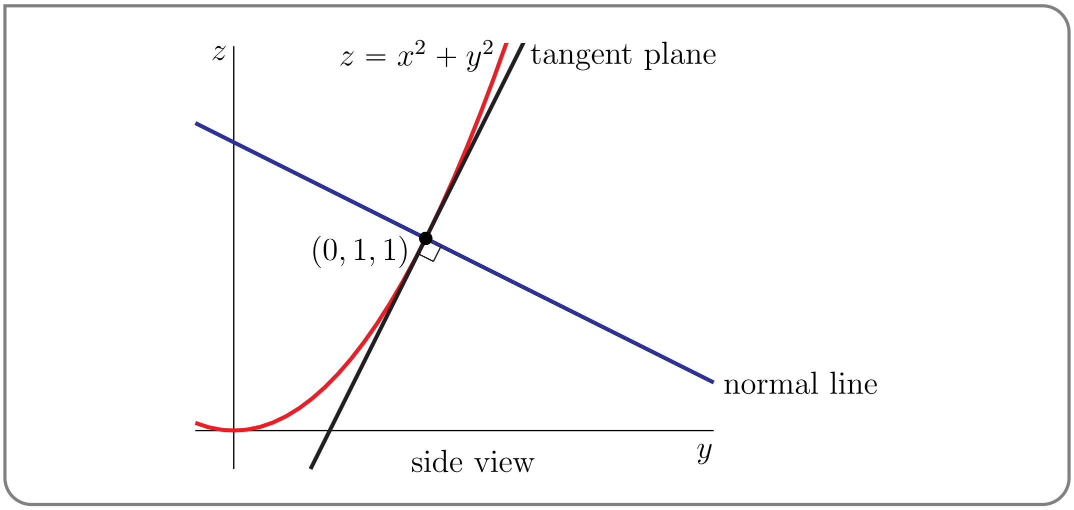


We are now going to determine, as our first application of partial derivatives, the tangent plane to a general surface S at a general point (x_0, y_0, z_0) lying on the surface. We

16 It is possible, but beyond the scope of this text, to give a precise meaning to "fits best".

will also determine the line which passes through (x_0, y_0, z_0) and whose direction is perpendicular to S at (x_0, y_0, z_0). It is called the normal line to S at (x_0, y_0, z_0).

For example, the following figure shows the side view of the tangent plane (in black) and normal line (in blue) to the surface $z = x^2 + y^2$ (in red) at the point $(0, 1, 1)$.



Recall, from (1.4.1), that to specify any plane, we need

- one point on the plane and
- a vector perpendicular to the plane, i.e. a normal vector,

and recall, from (1.5.1), that to specify any line, we need

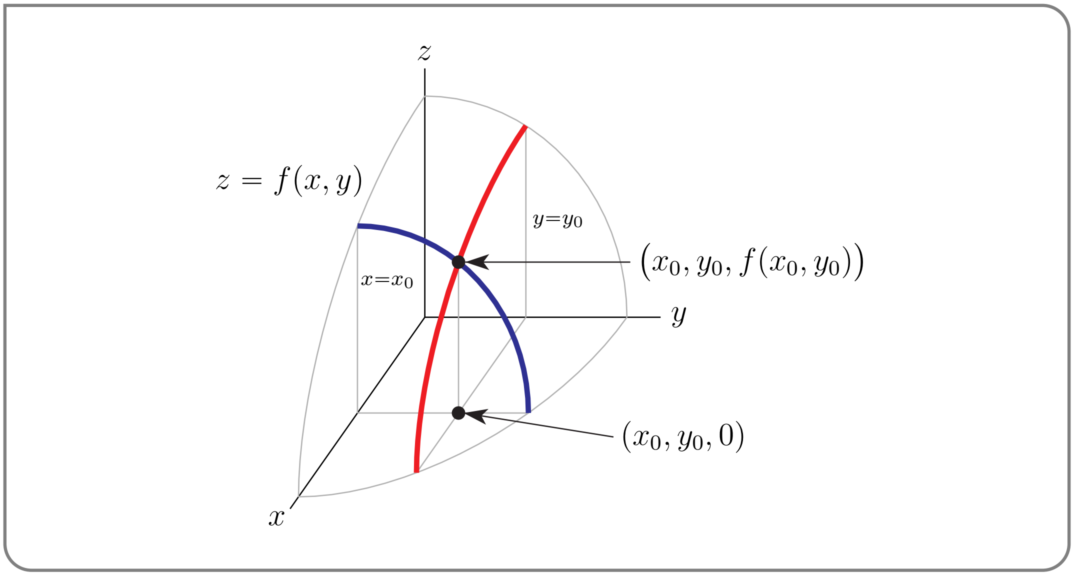
- one point on the line and
- a direction vector for the line.

We already have one point that is on both the tangent plane of interest and the normal line of interest — namely $\big(x_0, y_0, z_0\big)$. Furthermore we can use any (nonzero) vector that is perpendicular to S at (x_0, y_0, z_0) as both the normal vector to the tangent plane and the direction vector of the normal line.

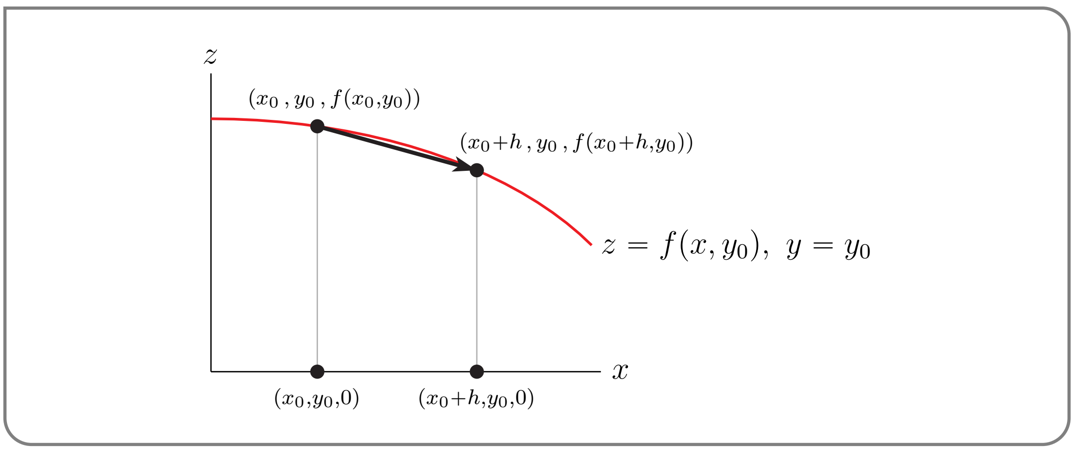
So our main task is to determine a normal vector to the surface S at (x_0, y_0, z_0). That's what we do now, first for surfaces of the form $z = f(x, y)$ and then, more generally, for surfaces of the form $G(x, y, z) = 0$.

2.5.1 ▸▸ Surfaces of the Form $z = f(x, y)$.

We construct a vector perpendicular to the surface $z = f(x, y)$ at $\big(x_0\,,\, y_0\,,\, f(x_0, y_0)\big)$ by, first, constructing two tangent vectors to the specified surface at the specified point, and, second, taking the cross product of those two tangent vectors. Consider the red curve in the figure below. It is the intersection of our surface $z = f(x, y)$ with the plane $y = y_0$.



Here is a side view of the red curve. The vector from the point $\big(x_0\,,\,y_0\,,\,f(x_0,y_0)\big)$, on the



red curve, to the point $\big(x_0+h\,,\,y_0\,,\,f(x_0+h,y_0)\big)$, also on the red curve, is almost tangent to the red curve, if h is very small. As h tends to 0, that vector, which is

$$\langle h\,,\,0\,,\,f(x_0+h,y_0)-f(x_0,y_0)\rangle$$

becomes exactly tangent to the curve. However its length also tends to 0. If we divide by h, and then take the limit $h\to 0$, we get

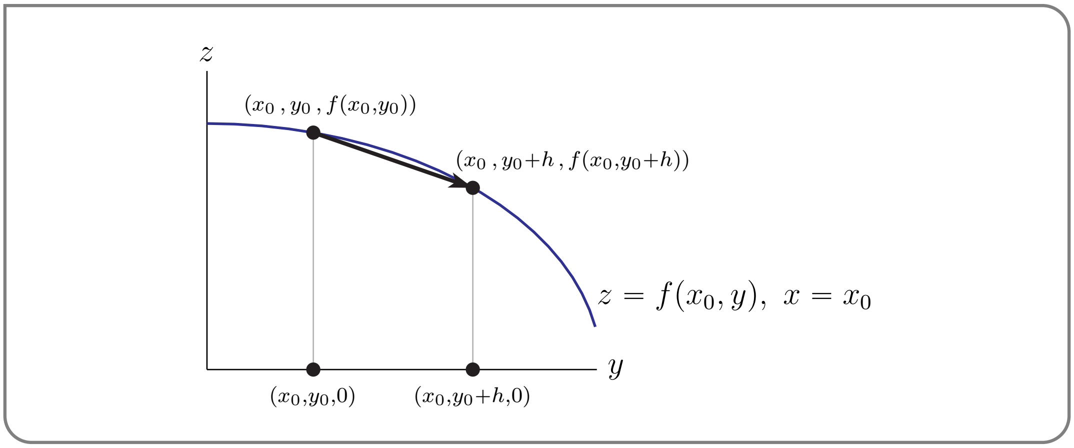
$$\lim_{h\to 0}\frac{1}{h}\,\langle h\,,\,0\,,\,f(x_0+h,y_0)-f(x_0,y_0)\rangle=\lim_{h\to 0}\left\langle 1\,,\,0\,,\,\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}\right\rangle$$

Since the limit $\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$ is the definition of the partial derivative $f_x(x_0,y_0)$, we get that

$$\lim_{h\to 0}\frac{1}{h}\,\langle h\,,\,0\,,\,f(x_0+h,y_0)-f(x_0,y_0)\rangle=\langle 1\,,\,0\,,\,f_x(x_0,y_0)\rangle$$

is a nonzero vector that is exactly tangent to the red curve and hence is also tangent to our surface $z=f(x,y)$ at the point $\big(x_0\,,\,y_0\,,\,f(x_0,y_0)\big)$.

For the second tangent vector, we repeat the process with the blue curve in the figure at the beginning of this subsection. That blue curve is the intersection of our surface $z = f(x, y)$ with the plane $x = x_0$. Here is a front view of the blue curve.



When h is very small, the vector

$$\frac{1}{h}\left\langle 0\,,\, h\,,\, f(x_0, y_0 + h) - f(x_0, y_0)\right\rangle$$

from the point $\big(x_0\,,\, y_0\,,\, f(x_0, y_0)\big)$, on the blue curve, to $\big(x_0\,,\, y_0 + h\,,\, f(x_0, y_0 + h)\big)$, also on the blue curve, (and lengthened by a factor $\frac{1}{h}$) is almost tangent to the blue curve. Taking the limit $h \to 0$ gives the tangent vector

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\left\langle 0\,,\, h\,,\, f(x_0, y_0 + h) - f(x_0, y_0)\right\rangle &= \lim_{h\to 0}\left\langle 0\,,\, 1\,,\, \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}\right\rangle \\ &= \big\langle 0\,,\, 1\,,\, f_y(x_0, y_0)\big\rangle\end{aligned}$$

to the blue curve at the point $\big(a\,,\, b\,,\, f(a, b)\big)$.

Now that we have two vectors in the tangent plane to the surface $z = f(x, y)$ at $\big(x_0\,,\, y_0\,,\, f(x_0, y_0)\big)$, we can find a normal vector to the tangent plane by taking their cross product. Their cross product is

$$\begin{aligned}\left\langle 1\,,\, 0\,,\, f_x(x_0, y_0)\right\rangle \times \big\langle 0\,,\, 1\,,\, f_y(x_0, y_0)\big\rangle &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0)\end{bmatrix} \\ &= -f_x(x_0, y_0)\,\hat{\imath} - f_y(x_0, y_0)\,\hat{\jmath} + \hat{\mathbf{k}}\end{aligned}$$

and we have that the vector

$$-f_x(x_0, y_0)\,\hat{\imath} - f_y(x_0, y_0)\,\hat{\jmath} + \hat{\mathbf{k}}$$

is perpendicular to the surface $z = f(x, y)$ at $\big(x_0\,,\, y_0\,,\, f(x_0, y_0)\big)$.

The tangent plane to the surface $z = f(x, y)$ at $\big(x_0\,,\, y_0\,,\, f(x_0, y_0)\big)$ is the plane through $\big(x_0\,,\, y_0\,,\, f(x_0, y_0)\big)$ with normal vector $-f_x(x_0, y_0)\,\hat{\imath} - f_y(x_0, y_0)\,\hat{\jmath} + \hat{\mathbf{k}}$. This plane has equation

$$-f_x(x_0, y_0)\,(x - x_0) - f_y(x_0, y_0)\,(y - y_0) + \big(z - f(x_0, y_0)\big) = 0$$

or, after a little rearrangement,

$$z = f(x_0,y_0) + f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0)$$

Now that we have the normal vector, finding the equation of the normal line to the surface $z = f(x,y)$ at the point $\big(x_0\,,\,y_0\,,\,f(x_0,y_0)\big)$ is straightforward. Writing it in parametric form,

$$\langle x,y,z\rangle = \langle x_0,y_0,f(x_0,y_0)\rangle + t\,\big\langle -f_x(x_0,y_0)\,,\,-f_y(x_0,y_0)\,,\,1\big\rangle$$

By way of summary

Theorem 2.5.1 (Tangent Plane and Normal Line).

(a) The vector

$$-f_x(x_0,y_0)\,\hat{\imath} - f_y(x_0,y_0)\,\hat{\jmath} + \hat{\mathbf{k}}$$

is normal to the surface $z = f(x,y)$ at $\big(x_0\,,\,y_0\,,\,f(x_0,y_0)\big)$.

(b) The equation of the tangent plane to the surface $z = f(x,y)$ at the point $\big(x_0\,,\,y_0\,,\,f(x_0,y_0)\big)$ may be written as

$$z = f(x_0,y_0) + f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0)$$

(c) The parametric equation of the normal line to the surface $z = f(x,y)$ at the point $\big(x_0\,,\,y_0\,,\,f(x_0,y_0)\big)$ is

$$\langle x,y,z\rangle = \langle x_0,y_0,f(x_0,y_0)\rangle + t\,\big\langle -f_x(x_0,y_0)\,,\,-f_y(x_0,y_0)\,,\,1\big\rangle$$

or, writing it component by component,

$$x = x_0 - t\,f_x(x_0,y_0) \qquad y = y_0 - t\,f_y(x_0,y_0) \qquad z = f(x_0,y_0) + t$$

Example 2.5.2

As a warm-up example, we'll find the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point $(1,0,1)$. To do so, we just apply Theorem 2.5.1 with $x_0 = 1$, $y_0 = 0$ and

$$\begin{aligned} f(x,y) &= x^2 + y^2 & \qquad f(1,0) &= 1\\ f_x(x,y) &= 2x & f_x(1,0) &= 2\\ f_y(x,y) &= 2y & f_y(1,0) &= 0 \end{aligned}$$

So the tangent plane is

$$\begin{aligned} z &= f(x_0,y_0) + f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0) = 1 + 2(x-1) + 0(y-0)\\ &= -1 + 2x \end{aligned}$$

and the normal line is

$$\begin{aligned} \langle x,y,z\rangle &= \langle x_0,y_0,f(x_0,y_0)\rangle + t\,\big\langle -f_x(x_0,y_0)\,,\,-f_y(x_0,y_0)\,,\,1\big\rangle = \langle 1,0,1\rangle + t\,\langle -2\,,\,0\,,\,1\rangle\\ &= \langle 1-2t\,,\,0\,,\,1+t\rangle \end{aligned}$$

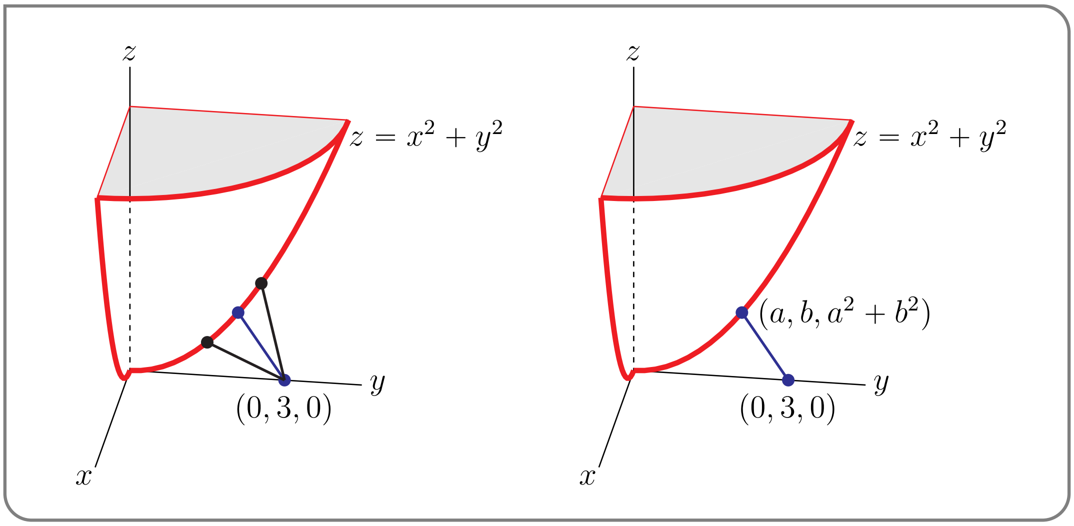
Example 2.5.2

That was pretty simple — find the partial derivatives and substitute in the coordinates. Let's do something a bit more challenging.

Example 2.5.3 (Optional)

Find the distance from $(0,3,0)$ to the surface $z = x^2 + y^2$.

Solution. Write $f(x,y) = x^2 + y^2$. Let's denote by $\big(a,b,f(a,b)\big)$ the point on $z = f(x,y)$ that is nearest $(0,3,0)$. Before we really get into the problem, let's make a simple sketch and think about what the lines from $(0,3,0)$ to the surface look like and, in particular, the angles between these lines and the surface. The line from $(0,3,0)$ to $\big(a,b,f(a,b)\big)$, the



point on $z = f(x,y)$ nearest $(0,3,0)$, is distinguished from the other lines from $(0,3,0)$ to the surface, by being perpendicular to the surface. We will provide a detailed justification for this claim below.

Let's first exploit the fact that the vector from $(0,3,0)$ to $\big(a,b,f(a,b)\big)$ must be perpendicular to the surface to determine $\big(a,b,f(a,b)\big)$, and consequently the distance from $(0,3,0)$ to the surface. By Theorem 2.5.1.a, with $x_0 = a$ and $y_0 = b$, the vector

$$-f_x(a,b)\,\hat{\imath} - f_y(a,b)\,\hat{\jmath} + \hat{\mathbf{k}} = -2a\,\hat{\imath} - 2b\,\hat{\jmath} + \hat{\mathbf{k}} \qquad (*)$$

is normal to the surface $z = f(x,y)$ at $\big(a,b,f(a,b)\big)$. So the vector from $(0,3,0)$ to $\big(a,b,f(a,b)\big)$, namely

$$a\,\hat{\imath} + (b-3)\,\hat{\jmath} + f(a,b)\,\hat{\mathbf{k}} = a\,\hat{\imath} + (b-3)\,\hat{\jmath} + (a^2+b^2)\,\hat{\mathbf{k}} \qquad (**)$$

must be parallel to $(*)$. This does not force the vector $(*)$ to equal $(**)$, but it does force the existence of some number t obeying

$$a\,\hat{\imath} + (b-3)\,\hat{\jmath} + (a^2+b^2)\,\hat{\mathbf{k}} = t\big(-2a\,\hat{\imath} - 2b\,\hat{\jmath} + \hat{\mathbf{k}}\big) \text{ or equivalently } \begin{cases} a = -2a\,t \\ b-3 = -2b\,t \\ a^2+b^2 = t \end{cases}$$

We now have a system of three equations in the three unknowns a, b and t. If we can solve them, we will have found the point on the surface that we want.

- The first equation is $a(1+2t) = 0$ so that either $a = 0$ or $t = -\frac{1}{2}$.
- The third equation forces $t \geqslant 0$, so $a = 0$, and the last equation reduces to $t = b^2$.
- Substituting this into the middle equation gives

$$b - 3 = -2b^3 \qquad \text{or equivalently} \qquad 2b^3 + b - 3 = 0$$

In general, cubic equations are very hard to solve[17]. But, in this case, we can guess one solution[18], namely $b = 1$. So $(b-1)$ must be a factor of $2b^3 + b - 3$ and a little division then gives us

$$0 = 2b^3 + b - 3 = (b-1)(2b^2 + 2b + 3)$$

We can now find the roots of the quadratic factor by using the high school formula

$$\frac{-2 \pm \sqrt{2^2 - 4(2)(3)}}{4}$$

Since $2^2 - 4(2)(3) < 0$, the factor $2b^2 + 2b + 3$ has no real roots. So the only real solution to the cubic equation $2b^3 + b - 3 = 0$ is $b = 1$.

In summary,

- $a = 0$, $b = 1$ and
- the point on $z = x^2 + y^2$ nearest $(0,3,0)$ is $(0,1,1)$ and
- the distance from $(0,3,0)$ to $z = x^2 + y^2$ is the distance from $(0,3,0)$ to $(0,1,1)$, which is $\sqrt{(-2)^2 + 1^2} = \sqrt{5}$.

Finally back to the claim that, because $\big(a, b, f(a,b)\big)$ is the point on $z = f(x,y)$ that is nearest[19] $(0,3,0)$, the vector from $(0,3,0)$ to $\big(a, b, f(a,b)\big)$ must be perpendicular to the surface $z = f(x,y)$ at $\big(a, b, f(a,b)\big)$. Note that the square of the distance from $(0,3,0)$ to a general point $\big(x, y, f(x,y)\big)$ on $z = f(x,y)$ is

$$D(x,y) = x^2 + (y-3)^2 + f(x,y)^2$$

If $x = a$, $y = b$ minimizes $D(x,y)$ then, in particular,

17 The method for solving cubics was developed in the 15th century by del Ferro, Cardano and Ferrari (Cardano's student). Ferrari then went on to discover a formula for the roots of a quartic. Both the cubic and quartic formulae are extremely cumbersome, and no such formula exists for polynomials of degree 5 and higher. This is the famous Abel-Ruffini theorem.

18 See Appendix A.16 in the CLP-2 text. There it is shown that any integer root of a polynomial with integer coefficients must divide the constant term exactly. So in this case only ± 1 and ± 3 could be integer roots. So it is good to check to see if any of these are solutions before moving on to more sophisticated techniques.

19 Note that we are assuming that $\big(a, b, f(a,b)\big)$ is the point on the surface that is nearest $(0,3,0)$. That there exists such a point is intuitively obvious from a sketch of the surface. The mathematical proof that there exists such a point is beyond the scope of this text.

- restricting our attention to the slice $y = b$ of the surface, $x = a$ minimizes $g(x) = D(x,b) = x^2 + (b-3)^2 + f(x,b)^2$ so that

$$\begin{aligned} 0 = g'(a) &= \frac{\partial}{\partial x}\Big[x^2 + (b-3)^2 + f(x,b)^2\Big]\Big|_{x=a} = 2a + 2f(a,b)\, f_x(a,b) \\ &= 2\,\langle a\,,\, b-3\,,\, f(a,b)\rangle \cdot \langle 1\,,\, 0\,,\, f_x(a,b)\rangle \end{aligned}$$

and

- restricting our attention to the slice $x = a$ of the surface, $y = b$ minimizes $h(y) = D(a,y) = a^2 + (y-3)^2 + f(a,y)^2$ so that

$$\begin{aligned} 0 = h'(b) &= \frac{\partial}{\partial y}\Big[a^2 + (y-3)^2 + f(a,y)^2\Big]\Big|_{y=b} = 2(b-3) + 2f(a,b)\, f_y(a,b) \\ &= 2\,\langle a\,,\, b-3\,,\, f(a,b)\rangle \cdot \big\langle 0\,,\, 1\,,\, f_y(a,b)\big\rangle \end{aligned}$$

We have expressed the final right hand sides of both of the above bullets as the dot product of the vector $\langle a\,,\, b-3\,,\, f(a,b)\rangle$ with something because

- $\langle a\,,\, b-3\,,\, f(a,b)\rangle$ is the vector from $(0,3,0)$ to the point $\big(a\,,\, b\,,\, f(a,b)\big)$ on the surface and
- the vanishing of the dot product of two vectors implies that the two vectors are perpendicular.

Thus, that

$$\langle a\,,\, b-3\,,\, f(a,b)\rangle \cdot \langle 1\,,\, 0\,,\, f_x(a,b)\rangle = \langle a\,,\, b-3\,,\, f(a,b)\rangle \cdot \big\langle 0\,,\, 1\,,\, f_y(a,b)\big\rangle = 0$$

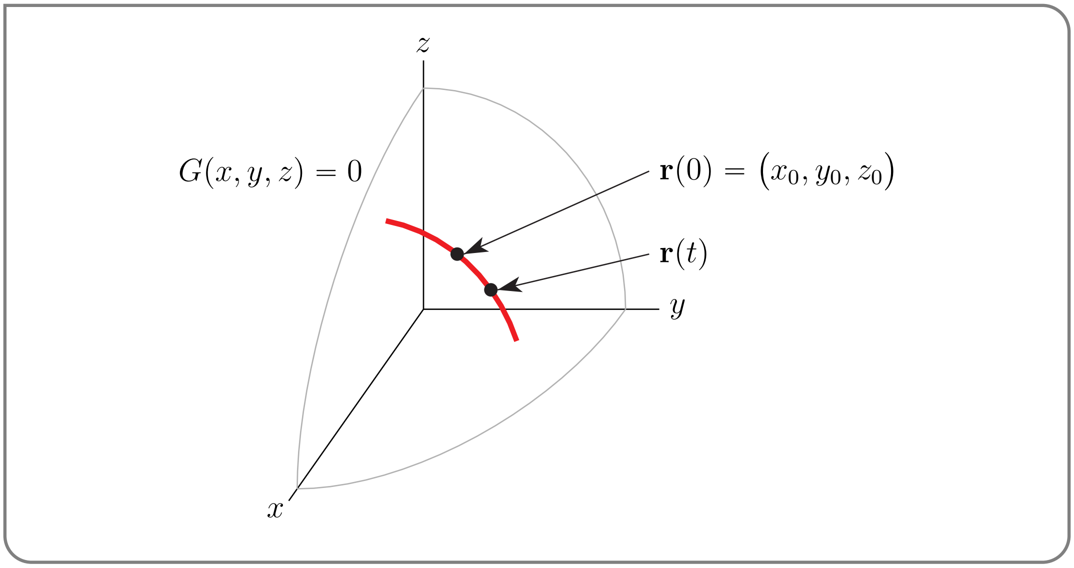
tells us that the vector $\langle a\,,\, b-3\,,\, f(a,b)\rangle$ from $(0,3,0)$ to $\big(a, b, f(a,b)\big)$ is perpendicular to both $\langle 1\,,\, 0\,,\, f_x(a,b)\rangle$ and $\big\langle 0\,,\, 1\,,\, f_y(a,b)\big\rangle$ and hence is parallel to their cross product $\langle 1\,,\, 0\,,\, f_x(a,b)\rangle \times \big\langle 0\,,\, 1\,,\, f_y(a,b)\big\rangle$, which we already know is a normal vector to the surface $z = f(x,y)$ at $\big(a, b, f(a,b)\big)$.

This shows that the point on the surface that minimises the distance to $(0,3,0)$ is joined to $(0,3,0)$ by a line that is parallel to the normal vector — just as we required.

Example 2.5.3

2.5.2 ▸▸ Surfaces of the Form $G(x,y,z) = 0$.

We now use a little trickery to construct a vector perpendicular to the surface $G(x,y,z) = 0$ at the point $\big(x_0\,,\, y_0\,,\, z_0\big)$. Imagine that you are walking on the surface and that at time 0 you are at the point $\big(x_0\,,\, y_0\,,\, z_0\big)$. Let $\mathbf{r}(t) = \big(x(t)\,,\, y(t)\,,\, z(t)\big)$ denote your position at time t. Because you are walking along the surface, we know that $\mathbf{r}(t)$ always lies on the



surface and so

$$G\big(x(t)\,,\,y(t)\,,\,z(t)\big) = 0$$

for all t. Differentiating this equation with respect to t gives, by the chain rule,

$$\frac{\partial G}{\partial x}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\;x'(t) + \frac{\partial G}{\partial y}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\;y'(t) + \frac{\partial G}{\partial z}\big(x(t)\,,\,y(t)\,,\,z(t)\big)\;z'(t) = 0$$

Then setting $t = 0$ gives

$$\frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\;x'(0) + \frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\;y'(0) + \frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\;z'(0) = 0$$

Expressing this as a dot product allows us to turn this into a statement about vectors.

$$\left\langle \frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\right\rangle \cdot \mathbf{r}'(0) = 0 \qquad (*)$$

The first vector in this dot product is sufficiently important that it is given its own name.

Definition 2.5.4 (Gradient).

The gradient[20] of the function $G(x, y, z)$ at the point $(x_0\,,\,y_0\,,\,z_0)$ is

$$\left\langle \frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\,,\,\frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0)\right\rangle$$

It is denoted $\nabla G(x_0, y_0, z_0)$.

So $(*)$ tells us that the gradient $\nabla G(x_0, y_0, z_0)$, is perpendicular to the vector $\mathbf{r}'(0)$.

20 The gradient will also play a big role in Section 2.7.

Now if t is very close to zero, the vector $\mathbf{r}(t) - \mathbf{r}(0)$, from $\mathbf{r}(0)$ to $\mathbf{r}(t)$, is almost tangent to the path that we are walking on. The limit

$$\mathbf{r}'(0) = \lim_{t\to 0} \frac{\mathbf{r}(t) - \mathbf{r}(0)}{t}$$

is thus exactly tangent to our path, and consequently to the surface $G(x, y, z) = 0$ at (x_0, y_0, z_0). This is true for all paths on the surface that pass through (x_0, y_0, z_0) at time $t = 0$, which tells us that $\nabla G(x_0, y_0, z_0)$ is perpendicular to the surface at (x_0, y_0, z_0). We have just found a normal vector!

The above argument goes through unchanged for surfaces of the form[21] $G(x, y, z) = K$, for any constant K. So we have

Theorem 2.5.5 (Tangent Plane and Normal Line)**.**

Let K be a constant and (x_0, y_0, z_0) be a point on the surface $G(x, y, z) = K$. Assume that the gradient

$$\nabla G(x_0, y_0, z_0) = \left\langle \frac{\partial G}{\partial x}(x_0\,,\,y_0\,,\,z_0)\,,\ \frac{\partial G}{\partial y}(x_0\,,\,y_0\,,\,z_0)\,,\ \frac{\partial G}{\partial z}(x_0\,,\,y_0\,,\,z_0) \right\rangle$$

of G at (x_0, y_0, z_0) is nonzero.

(a) The vector $\nabla G(x_0, y_0, z_0)$ is normal to the surface $G(x, y, z) = K$ at (x_0, y_0, z_0).

(b) The equation of the tangent plane to the surface $G(x, y, z) = K$ at (x_0, y_0, z_0) is
$$\nabla G(x_0, y_0, z_0) \cdot \langle x - x_0\,,\, y - y_0\,,\, z - z_0 \rangle = 0$$

(c) The parametric equation of the normal line to the surface $G(x, y, z) = K$ at (x_0, y_0, z_0) is
$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\,\nabla G(x_0, y_0, z_0)$$

Remark 2.5.6. Theorem 2.5.1 about the tangent planes and normal lines to the surface $z = f(x, y)$ is actually a very simple consequence of Theorem 2.5.5 about the tangent planes and normal lines to the surface $G(x, y, z) = 0$. This is just because we can always rewrite the equation $z = f(x, y)$ as $z - f(x, y) = 0$ and apply Theorem 2.5.5 with $G(x, y, z) = z - f(x, y)$. Since
$$\nabla G(x_0, y_0, z_0) = -f_x(x_0, y_0)\,\hat{\boldsymbol{\imath}} - f_y(x_0, y_0)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$$

Theorem 2.5.5 then gives[22] Theorem 2.5.1.

Here are a couple of routine examples.

21 Alternatively, one could rewrite $G = K$ as $G - K = 0$ and replace G by $G - K$ in the above argument.
22 Indeed we could write Theorem 2.5.1 as a corollary of Theorem 2.5.5. But in a textbook one tries to start with the concrete and move to the more general.

Example 2.5.7

Problem: Find the tangent plane and the normal line to the surface

$$z = x^2 + 5xy - 2y^2$$

at the point $(1, 2, 3)$.

Solution. As a preliminary check, note that

$$1^2 + 5 \times 1 \times 2 - 2(2)^2 = 3$$

which verifies that the point $(1, 2, 3)$ is indeed on the surface. This is a good reality check and also increases our confidence that the question is asking what we think that it is asking. Rewrite the equation of the surface as $G(x, y, z) = x^2 + 5xy - 2y^2 - z = 0$. Then the gradient

$$\nabla G(x, y, z) = (2x + 5y)\,\hat{\imath} + (5x - 4y)\,\hat{\jmath} - \hat{\mathbf{k}}$$

so that, by Theorem 2.5.5,

$$\mathbf{n} = \nabla G(1, 2, 3) = 12\hat{\imath} - 3\hat{\jmath} - \hat{\mathbf{k}}$$

is a normal vector to the surface at $(1, 2, 3)$. Equipped[23] with the normal, it is easy to work out an equation for the tangent plane.

$$\mathbf{n} \cdot \langle x-1\,,\, y-2\,,\, z-3 \rangle = \langle 12\,,\, -3\,,\, -1 \rangle \cdot \langle x-1\,,\, y-2\,,\, z-3 \rangle = 0 \quad \text{or} \quad 12x - 3y - z = 3$$

We can quickly check that the point $(1, 2, 3)$ does indeed lie on the plane:

$$12 \times 1 - 3 \times 2 - 3 = 3$$

The normal line is

$$\langle x-1\,,\, y-2\,,\, z-3 \rangle = t\,\mathbf{n} = t\,\langle 12\,,\, -3\,,\, -1 \rangle \quad \text{or} \quad \frac{x-1}{12} = \frac{y-2}{-3} = \frac{z-3}{-1} \quad \big(= t\big)$$

Example 2.5.7

Another warm-up example. This time the surface is a hyperboloid of one sheet.

Example 2.5.8

Problem: Find the tangent plane and the normal line to the surface

$$x^2 + y^2 - z^2 = 4$$

at the point $(2, -3, 3)$.

Solution. As a preliminary check, note that the point $(2, -3, 3)$ is indeed on the surface:

$$2^2 + (-3)^2 - (3)^2 = 4$$

23 The spelling "equipt" is a bit archaic. There must be a joke here about quips.

The equation of the surface is $G(x,y,z) = x^2 + y^2 - z^2 = 4$. Then the gradient of G is

$$\nabla G(x,y,z) = 2x\,\hat{\imath} + 2y\,\hat{\jmath} - 2z\,\hat{\mathbf{k}}$$

so that, at $(2,-3,3)$,

$$\nabla G(2,-3,3) = 4\,\hat{\imath} - 6\,\hat{\jmath} - 6\,\hat{\mathbf{k}}$$

and so, by Theorem 2.5.5,

$$\mathbf{n} = \frac{1}{2}\big(4\,\hat{\imath} - 6\,\hat{\jmath} - 6\,\hat{\mathbf{k}}\big) = 2\,\hat{\imath} - 3\,\hat{\jmath} - 3\,\hat{\mathbf{k}}$$

is a normal vector to the surface at $(2,-3,3)$. The tangent plane is

$$\mathbf{n}\cdot\langle x-2\,,\,y+3\,,\,z-3\rangle = \langle 2\,,\,-3\,,\,-3\rangle\cdot\langle x-2\,,\,y+3\,,\,z-3\rangle = 0 \quad\text{or}\quad 2x-3y-3z=4$$

Again, as a check, we can verify that our point $(2,-3,3)$ is indeed on the plane:

$$2\times 2 - 3\times(-3) - 3\times 3 = 4$$

The normal line is

$$\langle x-2\,,\,y+3\,,\,z-3\rangle = t\,\mathbf{n} = t\,\langle 2\,,\,-3\,,\,-3\rangle \quad\text{or}\quad \frac{x-2}{2} = \frac{y+3}{-3} = \frac{z-3}{-3} \quad \Big(=t\Big)$$

Example 2.5.8

Warning 2.5.9.

The vector $\nabla G(x,y,z)$ *is not a normal vector to the surface* $G(x,y,z)=K$ *at* (x_0,y_0,z_0). *The vector* $\nabla G(x_0,y_0,z_0)$ *is a normal vector to* $G(x,y,z)=K$ *at* (x_0,y_0,z_0) *(provided* $G(x_0,y_0,z_0)=K$*).*

As an example of the consequences of failing to evaluate $\nabla G(x,y,z)$ at the point (x_0,y_0,z_0), consider the problem

Find the tangent plane to the surface $x^2+y^2+z^2=1$ at the point $(0,0,1)$.

In this case, the surface is $G(x,y,z) = x^2+y^2+z^2 = 1$. The gradient of G is $\nabla G(x,y,z) = 2x\,\hat{\imath}+2y\,\hat{\jmath}+2z\,\hat{\mathbf{k}}$. To correctly apply part (b) of Theorem 2.5.5, we evaluate $\nabla G(0,0,1) = 2\,\hat{\mathbf{k}}$ and find that the tangent plane at $(0,0,1)$ is

$$\nabla G(0,0,1)\cdot\langle x-0\,,\,y-0\,,\,z-1\rangle = 0 \qquad \text{or} \qquad 2(z-1)=0 \qquad \text{or} \qquad z=1$$

This is of course correct — the tangent plane to the unit sphere at the north pole is indeed horizontal.

But if we were to incorrectly apply part (b) of Theorem 2.5.5 by failing to evaluate $\nabla G(x,y,z)$ at $(0,0,1)$, we would find that the "tangent plane" is

$$\begin{aligned} &\nabla G(x,y,z)\cdot\langle x-0\,,\,y-0\,,\,z-1\rangle = 0\\ \text{or}\quad &2x(x-0)+2y(y-0)+2z(z-1)=0\\ \text{or}\quad &x^2+y^2+z^2-z=0\end{aligned}$$

This is horribly wrong. It is not even a plane, as any plane has an equation of the form $ax+by+cz=d$, with a, b, c and d constants.

Now we'll move on to some more involved examples.

Example 2.5.10

Suppose that we wish to find the highest and lowest points on the surface $G(x,y,z) = x^2-2x+y^2-4y+z^2-6z=2$. That is, we wish to find the points on the surface with the maximum value of z and with the minimum[24] value of z.

Completing three squares,

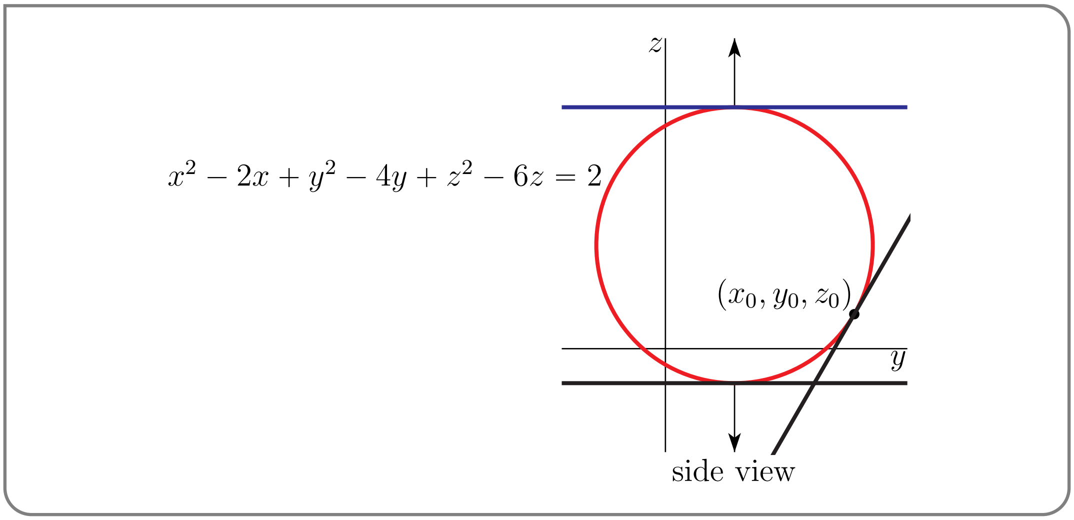
$$G(x,y,z) = x^2-2x+y^2-4y+z^2-6z = (x-1)^2+(y-2)^2+(z-3)^2-14.$$

So the surface $G(x,y,z)=2$ is a sphere, whose highest point is the north pole and whose lowest point is the south pole. But let's pretend that $G(x,y,z)=2$ is some complicated surface that we can't easily picture.

We'll find its highest and lowest points by exploiting the fact that the tangent plane to $G=2$ is horizontal at the highest and lowest points. Equivalently, the normal vector to $G=2$ is vertical at the highest and lowest points. To see that this is the case, look

24 Recall that "minimum" means the most negative, not the closest to zero.

at the figure below. If the tangent plane at (x_0, y_0, z_0) is not horizontal, then the tangent plane contains points near (x_0, y_0, z_0) with z bigger than z_0 and points near (x_0, y_0, z_0) with z smaller than z_0. Near (x_0, y_0, z_0), the tangent plane is a good approximation to the surface. So the surface also contains[25] such points.



The gradient is

$$\nabla G(x,y,z) = (2x-2)\,\hat{\imath} + (2y-4)\,\hat{\jmath} + (2z-6)\,\hat{\mathbf{k}}$$

It is vertical when the $\hat{\imath}$ and $\hat{\jmath}$ components are both zero. This happens when $2x - 2 = 0$ and $2y - 4 = 0$, i.e. when $x = 1$ and $y = 2$. So the normal vector to the surface $G = 2$ at the point (x, y, z) is vertical when $x = 1$, $y = 2$ and (don't forget that (x, y, z) has to be on $G = 2$)

$$\begin{aligned} G(1,2,z) &= 1^2 - 2\times 1 + 2^2 - 4\times 2 + z^2 - 6z = 2 \\ \iff & z^2 - 6z - 7 = 0 \\ \iff & (z-7)(z+1) = 0 \\ \iff & z = 7,\ -1 \end{aligned}$$

The highest point is $(1, 2, 7)$ and the lowest point is $(1, 2, -1)$, as expected.

Example 2.5.10

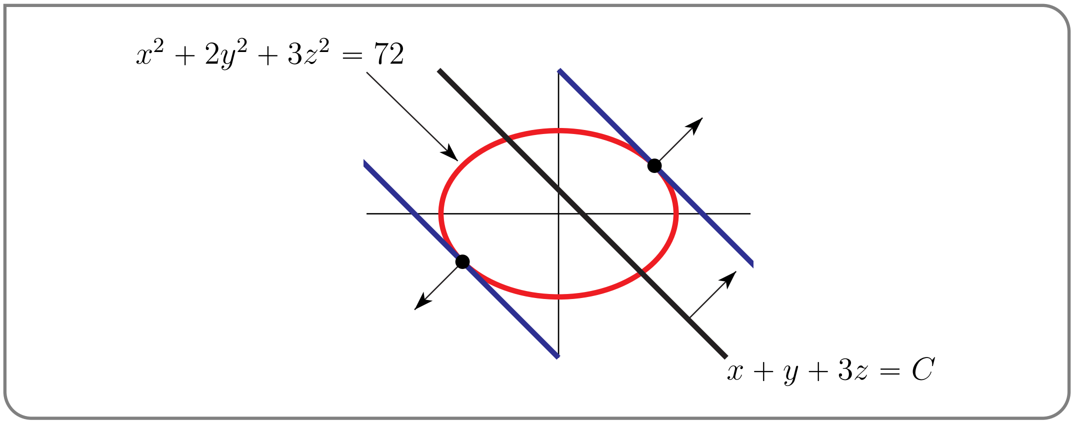
We could have short-cut the last example by using that the surface was a sphere. Here is an example in the same spirit for which we don't have an easy short-cut.

Example 2.5.11

In the last example, we found the points on a specified surface having the largest and smallest values of z. We'll now ramp up the level of difficulty a bit and find the points on the surface $x^2 + 2y^2 + 3z^2 = 72$ that have the largest and smallest values of $x + y + 3z$.

To develop a strategy for tackling this problem, consider the following sketch. The

25 While this is intuitively obvious, proving it is beyond the scope of this text.



red ellipse in the sketch is intended to represent (schematically) our surface

$$x^2 + 2y^2 + 3z^2 = 72$$

which is an ellipsoid. The middle diagonal (black) line is intended to represent (schematically) the plane $x + y + 3z = C$ for some more or less randomly chosen value of the constant C. At each point on that plane, the function, $x + y + 3z$, (that we are trying to maximize and minimize) takes the value C. In particular, for the C chosen in the figure, $x + y + 3z = C$ does intersect our surface, indicating that $x + y + 3z$ does indeed take the value C somewhere on our surface.

To maximize $x + y + 3z$, imagine slowly increasing the value of C. As we do so, the plane $x + y + 3z = C$ moves to the right. We want to stop increasing C at the biggest value of C for which the plane $x + y + 3z = C$ intersects our surface $x^2 + 2y^2 + 3z^2 = 72$. For that value of C the plane $x + y + 3z = C$, which is represented by the right hand blue line in the sketch, is tangent to our surface.

Similarly, to minimize $x + y + 3z$, imagine slowly decreasing the value of C. As we do so, the plane $x + y + 3z = C$ moves to the left. We want to stop decreasing C at the smallest value of C for which the plane $x + y + 3z = C$ intersects our surface $x^2 + 2y^2 + 3z^2 = 72$. For that value of C the plane $x + y + 3z = C$, which is represented by the left hand blue line in the sketch, is again tangent to our surface. The previous Example 2.5.10 was similar, except that the plane was $z = C$.

We are now ready to compute. We need to find the points (a, b, c) (in the sketch, they are the black dot points of tangency) for which

- (a, b, c) is on the surface and
- the normal vector to the surface $x^2 + 2y^2 + 3z^2 = 72$ at (a, b, c) is parallel to $\langle 1, 1, 3\rangle$, which is a normal vector to the plane $x + y + 3z = C$

Since the gradient of $x^2 + 2y^2 + 3z^2$ is $\langle 2x\,,\, 4y\,,\, 6z\rangle = 2\,\langle x\,,\, 2y\,,\, 3z\rangle$, these two conditions are, in equations,

$$\begin{aligned} a^2 + 2b^2 + 3c^2 &= 72 \\ \langle a\,,\, 2b\,,\, 3c\rangle &= t\,\langle 1, 1, 3\rangle \qquad \text{for some number } t \end{aligned}$$

The second equation says that $a = t$, $b = \frac{t}{2}$ and $c = t$. Substituting this into the first equation gives

$$t^2 + \frac{1}{2}t^2 + 3t^2 = 72 \iff \frac{9}{2}t^2 = 72 \iff t^2 = 16 \iff t = \pm 4$$

So

- the point on the surface $x^2 + 2y^2 + 3z^2 = 72$ at which $x + y + 3z$ takes its maximum value is $(a, b, c) = \left(t, \frac{t}{2}, t\right)\Big|_{t=4} = (4, 2, 4)$ and
- $x + y + 3z$ takes the value $4 + 2 + 3 \times 4 = 18$ there.
- The point on the surface $x^2 + 2y^2 + 3z^2 = 72$ at which $x + y + 3z$ takes its minimum value is $(a, b, c) = \left(t, \frac{t}{2}, t\right)\Big|_{t=-4} = (-4, -2, -4)$ and
- $x + y + 3z$ takes the value $-4 - 2 + 3 \times (-4) = -18$ there.

Example 2.5.11

Example 2.5.12

Problem: Find the distance from the point $(1, 1, 1)$ to the plane $x + 2y + 3z = 20$.

Solution 1. First note that the point $(1, 1, 1)$ is not itself on the plane $x + 2y + 3z = 20$ because

$$1 + 2 \times 1 + 3 \times 1 = 6 \neq 20$$

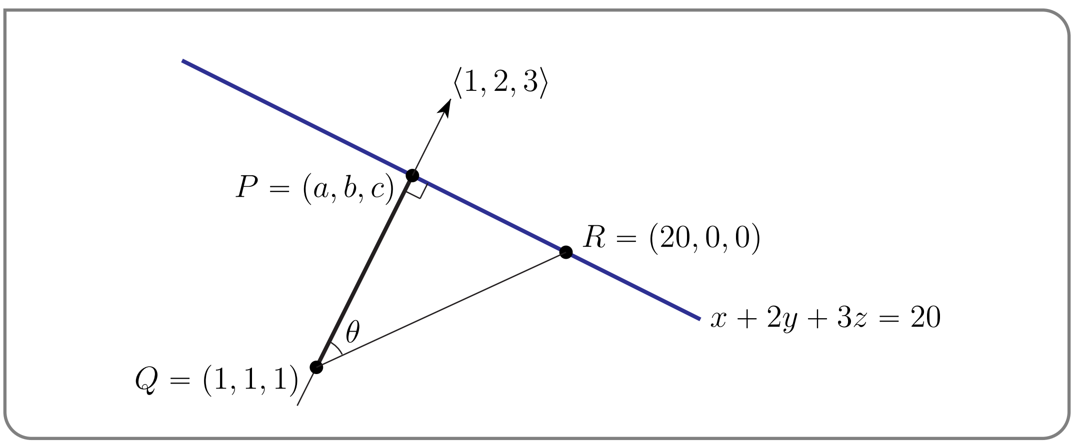
Denote by (a, b, c) the point on the plane $x + 2y + 3z = 20$ that is nearest $(1, 1, 1)$. Then the vector from $(1, 1, 1)$ to (a, b, c), namely $\langle a-1\,,\, b-1\,,\, c-1 \rangle$, must be perpendicular[26] to the plane. As the gradient of $x + 2y + 3z$, namely $\langle 1\,,\, 2\,,\, 3 \rangle$, is a normal vector to the plane, $\langle a-1\,,\, b-1\,,\, c-1 \rangle$ must be parallel to $\langle 1\,,\, 2\,,\, 3 \rangle$. So there must be some number t so that

$$\langle a-1\,,\, b-1\,,\, c-1 \rangle = t\,\langle 1\,,\, 2\,,\, 3 \rangle \qquad \text{or} \qquad a = t+1,\ b = 2t+1,\ c = 3t+1$$

As (a, b, c) must be on the plane, we know that $a + 2b + 3c = 20$ and so

$$(t+1) + 2(2t+1) + 3(3t+1) = 20 \implies 14t = 14 \implies t = 1$$

The distance from $(1, 1, 1)$ to the plane $x + 2y + 3z = 20$ is the length of the vector $\langle a-1\,,\, b-1\,,\, c-1 \rangle = t\,\langle 1\,,\, 2\,,\, 3 \rangle = \langle 1\,,\, 2\,,\, 3 \rangle$ which is $\sqrt{14}$.



26 We saw why this vector must be perpendicular to the plane in Example 2.5.3.

Solution 2. Denote by $P = (a, b, c)$ the point on the plane $x + 2y + 3z = 20$ that is nearest the point $Q = (1, 1, 1)$. Pick any other point on the plane and call it R. For example $(x, y, z) = (20, 0, 0)$ obeys $x + 2y + 3z = 20$ and so $R = (20, 0, 0)$ is a point on the plane.

The triangle PQR is right angled. Denote by θ the angle between the hypotenuse QR and the side QP. The distance from $Q = (1, 1, 1)$ to the plane is the length of the line segment QP, which is

$$\text{distance} = |QP| = |QR| \cos\theta$$

Now, the dot product between the vector from Q to R, which is $\langle 19, -1, -1\rangle$, with the vector $\langle 1, 2, 3\rangle$, which is normal to the plane and hence parallel to the side QP is

$$\begin{aligned}\langle 19, -1, -1\rangle \cdot \langle 1, 2, 3\rangle &= 14 \\ &= |\, \langle 19, -1, -1\rangle \,|\, |\, \langle 1, 2, 3\rangle \,|\ \cos\theta = |QR|\ \sqrt{14}\ \cos\theta\end{aligned}$$

so that, finally,

$$\text{distance} = |QR| \cos\theta = \frac{14}{\sqrt{14}} = \sqrt{14}$$

Example 2.5.12

Example 2.5.13

Problem: Let $F(x, y, z) = 0$ and $G(x, y, z) = 0$ be two surfaces. These two surfaces intersect along a curve. Find a tangent vector to this curve at the point (x_0, y_0, z_0).

Solution. Call the tangent vector $\mathbf{T}$. Then $\mathbf{T}$ has to be

- tangent to the surface $F(x, y, z) = 0$ at (x_0, y_0, z_0) and
- tangent to the surface $G(x, y, z) = 0$ at (x_0, y_0, z_0).

Consequently $\mathbf{T}$ has to be

- perpendicular to the vector $\nabla F(x_0, y_0, z_0)$, which is normal to $F(x, y, z) = 0$ at (x_0, y_0, z_0), and at the same time has to be
- perpendicular to the vector $\nabla G(x_0, y_0, z_0)$, which is normal to $G(x, y, z) = 0$ at (x_0, y_0, z_0).

Recall that an easy way to construct a vector that is perpendicular to two other vectors is to take their cross product. So we take

$$\begin{aligned}\mathbf{T} = \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0) &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{bmatrix} \\ &= \big(F_y G_z - F_z G_y\big)\, \hat{\imath} + \big(F_z G_x - F_x G_z\big)\, \hat{\jmath} + \big(F_x G_y - F_y G_x\big)\, \hat{\mathbf{k}}\end{aligned}$$

where all partial derivatives are evaluated at $(x, y, z) = (x_0, y_0, z_0)$.

Example 2.5.13

Let's put Example 2.5.13 into action.

Example 2.5.14

Problem: Consider the curve that is the intersection of the surfaces

$$x^2 + y^2 + z^2 = 5 \qquad \text{and} \qquad x^2 + y^2 = 4z$$

Find a tangent vector to this curve at the point $\big(\sqrt{3}\,,\,1\,,\,1\big)$.

Solution. As a preliminary check, we verify that the point $\big(\sqrt{3}\,,\,1\,,\,1\big)$ really is on the curve. To do so, we check that $\big(\sqrt{3}\,,\,1\,,\,1\big)$ satisfies both equations:

$$\big(\sqrt{3}\big)^2 + 1^2 + 1^2 = 5 \qquad \big(\sqrt{3}\big)^2 + 1^2 = 4 \times 1$$

We'll find the specified tangent vector by using the strategy of Example 2.5.13.

Write $F(x,y,z) = x^2 + y^2 + z^2$ and $G(x,y,z) = x^2 + y^2 - 4z$. Then

- the vector

$$\boldsymbol{\nabla} F(\sqrt{3},1,1) = \langle 2x\,,\,2y\,,\,2z\rangle\Big|_{(x,y,z)=(\sqrt{3},1,1)} = 2\left\langle \sqrt{3}\,,\,1\,,\,1\right\rangle$$

  is normal to the surface $F(x,y,z) = 5$ at $\big(\sqrt{3}\,,\,1\,,\,1\big)$, and
- the vector

$$\boldsymbol{\nabla} G(\sqrt{3},1,1) = \langle 2x\,,\,2y\,,\,-4\rangle\Big|_{(x,y,z)=(\sqrt{3},1,1)} = 2\left\langle \sqrt{3}\,,\,1\,,\,-2\right\rangle$$

  is normal to the surface $G(x,y,z) = 0$ at $\big(\sqrt{3}\,,\,1\,,\,1\big)$.

So a tangent vector is

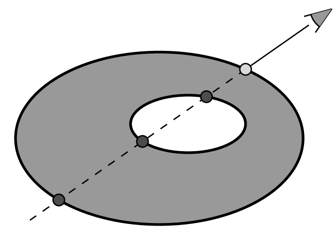
$$\begin{aligned}\left\langle \sqrt{3}\,,\,1\,,\,1\right\rangle \times \left\langle \sqrt{3}\,,\,1\,,\,-2\right\rangle &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ \sqrt{3} & 1 & 1\\ \sqrt{3} & 1 & -2\end{bmatrix}\\ &= \big(-2-1\big)\,\hat{\boldsymbol{\imath}} + \big(\sqrt{3}+2\sqrt{3}\big)\,\hat{\boldsymbol{\jmath}} + \big(\sqrt{3}-\sqrt{3}\big)\,\hat{\mathbf{k}}\\ &= -3\,\hat{\boldsymbol{\imath}} + 3\sqrt{3}\,\hat{\boldsymbol{\jmath}}\end{aligned}$$

There is an easy common factor of 3 in both components. So we can create a slightly neater tangent vector by dividing the length of $-3\,\hat{\boldsymbol{\imath}} + 3\sqrt{3}\,\hat{\boldsymbol{\jmath}}$ by 3, giving $\langle -1\,,\,\sqrt{3}\,,\,0\rangle$.

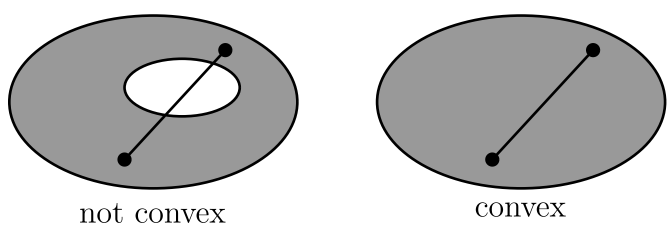
Example 2.5.14

Example 2.5.15 ((Optional) computer graphics hidden-surface elimination)

When you look at a solid three dimensional object, you do not see all of the surface of the object — parts of the surface are hidden from your view by other parts of the object. For example, the following sketch shows, schematically, a ray of light leaving your eye and hitting the surface of the object at the light dot. The object is solid, so the light cannot penetrate any further. But, if it could, it would follow the dotted line, hitting the surface of the object three more times. Your eye can see the light dot, but cannot see the other three dark dots.



Recreating this effect in computer generated graphics is called "hidden-surface elimination". In general, implementing hidden-surface elimination can be quite complicated. Often a technique called "ray tracing" is used[27]. However, it is easy if you know about vectors and gradients, and you are only looking at a single convex body. By definition, a solid is convex if, whenever two points are in the solid, then the line segment joining the two points is also contained in the solid.

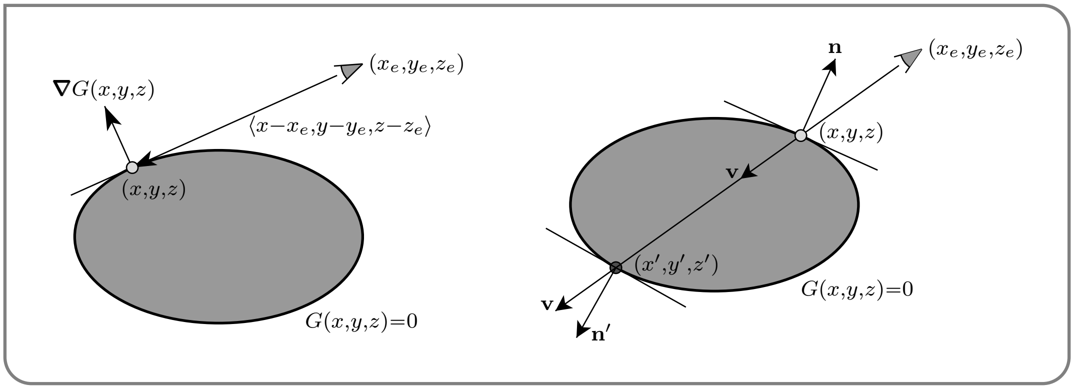


So suppose that we are looking at a convex solid, that the equation of the surface of the solid is $G(x, y, z) = 0$, and that our eye is at (x_e, y_e, z_e).

- First consider a light ray that leaves our eye and then just barely nicks the solid at the point (x, y, z), as in the figure on the left below. The light ray is a tangent line to the surface at (x, y, z). So the direction vector of the light ray, $\langle x - x_e, y - y_e, z - z_e \rangle$, is tangent to the surface at (x, y, z) and consequently is perpendicular to the normal vector, $\mathbf{n} = \nabla G(x, y, z)$, of the surface at (x, y, z). Thus

$$\langle x - x_e, y - y_e, z - z_e \rangle \cdot \nabla G(x, y, z) = 0$$

27 You can find out more about it by plugging "ray tracing" into the search engine of your choice.



- Now consider a light ray that leaves our eye and then passes through the solid, as in the figure on the right above. Call the point at which the light ray first enters the solid (x, y, z) and the point at which the light ray leaves the solid $(x', y,' z')$.

  - Let $\mathbf{v}$ be a vector that has the same direction as, i.e. is a positive multiple of, the vector $\langle x - x_e, y - y_e, z - z_e\rangle$.
  - Let $\mathbf{n}$ be an outward pointing normal to the solid at (x, y, z). It will be either $\nabla G(x, y, z)$ or $-\nabla G(x, y, z)$.
  - Let $\mathbf{n}'$ be an outward pointing normal to the solid at (x', y', z'). It will be either $\nabla G(x', y', z')$ or $-\nabla G(x', y', z')$.

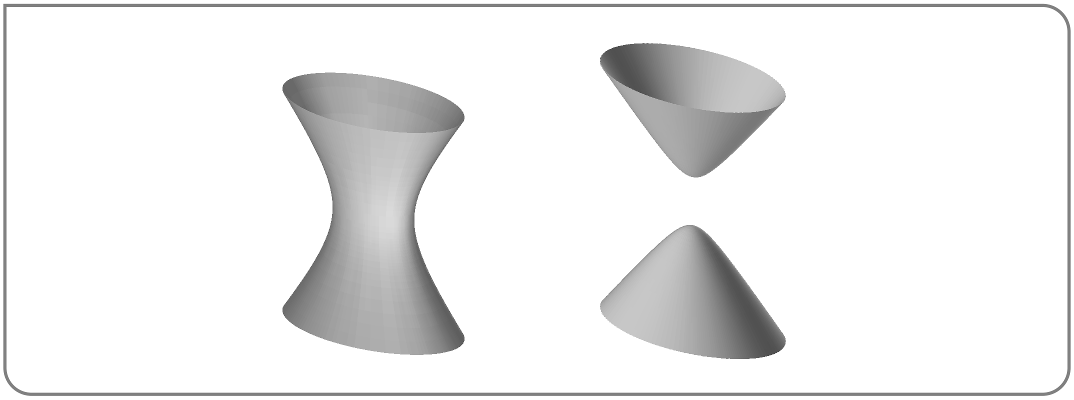
  Then

  - at the point (x, y, z) where the ray enters the solid, which is a visible point, the direction vector $\mathbf{v}$ points into the solid. The angle θ between $\mathbf{v}$ and the outward pointing normal $\mathbf{n}$ is greater than 90°, so that the dot product $\mathbf{v} \cdot \mathbf{n} = |\mathbf{v}|\,|\mathbf{n}|\cos\theta < 0$. But
  - at the point (x', y', z') where the ray leaves the solid, which is a hidden point, the direction vector $\mathbf{v}$ points out of the solid. The angle θ between $\mathbf{v}$ and the outward pointing normal $\mathbf{n}'$ is less than 90°, so that the dot product $\mathbf{v} \cdot \mathbf{n}' = |\mathbf{v}|\,|\mathbf{n}'|\cos\theta > 0$.

Our conclusion is that, if we are looking in the direction $\mathbf{v}$, and if the outward pointing normal[28] to the surface of the solid at (x, y, z) is $\nabla G(x, y, z)$ then the point (x, y, z) is hidden if and only if $\mathbf{v} \cdot \nabla G(x, y, z) > 0$.

This method was used by the computer graphics program that created the shaded figures[29] in Examples 1.7.1 and 1.7.2, which are reproduced here.

28 If $\nabla G(x, y, z)$ is the inward pointing normal, just replace G by $-G$.
29 Those figures are not convex. But it was still possible to use the method discussed above because any light ray from our eye that passes through the figure intersects the figure at most twice. It first enters the figure at a visible point and then exits the figure at a hidden point.



Example 2.5.15

Tangent planes, in addition to being geometric objects, provide a simple but powerful tool for approximating functions of two variables near a specified point. We saw something very similar in the CLP-1 text where we approximated functions of one variable by their tangent lines. This brings us to our next topic — approximating functions.

2.6▲ Linear Approximations and Error

A frequently used, and effective, strategy for building an understanding of the behaviour of a complicated function near a point is to approximate it by a simple function. The following suite of such approximations is standard fare in Calculus I courses. See, for example, §3.4 in the CLP-1 text.

$$\begin{aligned} &g(t_0+\Delta t) \approx g(t_0) && \text{constant approximation}\\ &g(t_0+\Delta t) \approx g(t_0)+g'(t_0)\,\Delta t && \text{linear, or tangent line, approximation}\\ &g(t_0+\Delta t) \approx g(t_0)+g'(t_0)\,\Delta t+\tfrac{1}{2}g''(t_0)\,\Delta t^2 && \text{quadratic approximation}\end{aligned}$$

More generally, for any natural number n, the approximation

$$g(t_0+\Delta t) \approx g(t_0)+g'(t_0)\,\Delta t+\tfrac{1}{2}g''(t_0)\,\Delta t^2+\cdots+\tfrac{1}{n!}g^{(n)}(t_0)\,\Delta t^n$$

is known as the Taylor polynomial of degree n. You may have also found a formula for the error introduced in making this approximation. The error $E_n(\Delta t)$ is defined by

$$g(t_0+\Delta t) = g(t_0)+g'(t_0)\Delta t+\tfrac{1}{2!}g''(t_0)\Delta t^2+\cdots+\tfrac{1}{n!}g^{(n)}(t_0)\Delta t^n+E_n(\Delta t)$$

and obeys[30]

$$E_n(\Delta t) = \tfrac{1}{(n+1)!}g^{(n+1)}(t_0+c\Delta t)\Delta t^{n+1}$$

for some (unknown) $0 \leqslant c \leqslant 1$.

It is a simple matter to use these one dimensional approximations to generate the analogous multidimensional approximations. To introduce the ideas, we'll generate the linear approximation to a function, $f(x,y)$, of two variables, near the point (x_0,y_0). Define

$$g(t) = f\big(x_0+t\,\Delta x\,,\,y_0+t\,\Delta y\big)$$

30 You may have seen it written as $E_n(x) = \frac{1}{(n+1)!}g^{(n+1)}(c)(x-a)^{n+1}$

We have defined $g(t)$ so that

$$g(0) = f(x_0\,,\,y_0) \qquad \text{and} \qquad g(1) = f(x_0+\Delta x\,,\,y_0+\Delta y)$$

Consequently, setting $t_0 = 0$ and $\Delta t = 1$,

$$\begin{aligned} f(x_0+\Delta x\,,\,y_0+\Delta y) &= g(1) = g(t_0+\Delta t) \\ &\approx g(t_0) + g'(t_0)\,\Delta t \\ &= g(0) + g'(0) \end{aligned}$$

We can now compute $g'(0)$ using the multivariable chain rule of (2.4.2):

$$g'(t) = \frac{\partial f}{\partial x}(x_0+t\,\Delta x\,,\,y_0+t\,\Delta y)\,\Delta x + \frac{\partial f}{\partial y}(x_0+t\,\Delta x\,,\,y_0+t\,\Delta y)\,\Delta y$$

so that,

Equation 2.6.1.

$$f(x_0+\Delta x\,,\,y_0+\Delta y) \approx f(x_0\,,\,y_0) + \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\Delta y$$

Of course exactly the same procedure works for functions of three or more variables. In particular

Equation 2.6.2.

$$\begin{aligned} &f(x_0+\Delta x\,,\,y_0+\Delta y\,,\,z_0+\Delta z) \\ &\quad\approx f(x_0\,,\,y_0\,,\,z_0) + \frac{\partial f}{\partial x}(x_0\,,\,y_0\,,\,z_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0\,,\,y_0\,,\,z_0)\,\Delta y + \frac{\partial f}{\partial z}(x_0\,,\,y_0\,,\,z_0)\,\Delta z \end{aligned}$$

While these linear approximations are quite simple, they tend to be pretty decent provided Δx and Δy are small. See the optional §2.6.1 for a more precise statement.

Remark 2.6.3. Applying (2.6.1), with $\Delta x = x - x_0$ and $\Delta y = y - y_0$. gives

$$f(x\,,\,y) \approx f(x_0\,,\,y_0) + \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,(x-x_0) + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,(y-y_0)$$

Looking at part (b) of Theorem 2.5.1, we see that this just says that the tangent plane to the surface $z = f(x,y)$ at the point $\big(x_0\,,\,y_0\,,\,f(x_0,y_0)\big)$ remains close to the surface when (x,y) is close to (x_0,y_0).

Example 2.6.4

Let

$$f(x,y) = \sqrt{x^2+y^2}$$

Then

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} \qquad f_x(x_0,y_0) = \frac{x_0}{\sqrt{x_0^2+y_0^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} \qquad f_y(x_0,y_0) = \frac{y_0}{\sqrt{x_0^2+y_0^2}}$$

so that the linear approximation to $f(x,y)$ at (x_0,y_0) is

$$\begin{aligned} f(x_0+\Delta x\,,\,y_0+\Delta y) &\approx f(x_0\,,\,y_0) + f_x(x_0\,,\,y_0)\,\Delta x + f_y(x_0\,,\,y_0)\,\Delta y \\ &= \sqrt{x_0^2+y_0^2} + \frac{x_0}{\sqrt{x_0^2+y_0^2}}\,\Delta x + \frac{y_0}{\sqrt{x_0^2+y_0^2}}\,\Delta y \end{aligned}$$

Example 2.6.4

Notation 2.6.5.

People often write Δf for the change $f(x_0+\Delta x\,,\,y_0+\Delta y) - f(x_0\,,\,y_0)$ in the value of f. Then the linear approximation (2.6.1) becomes

$$\Delta f \approx \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\Delta y$$

If they want to emphasize that that Δx, Δy and Δf are really small (they may even say "infinitesimal"), they'll write[31] $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}f$ instead. In this notation

$$\mathrm{d}f \approx \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\mathrm{d}x + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\mathrm{d}y$$

People sometimes call $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}f$ "differentials" and sometimes $\mathrm{d}f$ is called the "total differential of f" to indicate that it includes the impact of small changes in both x and y.

Definition 2.6.6.

Suppose that we wish to approximate a quantity Q and that the approximation turns out to be $Q + \Delta Q$. Then

- the absolute error in the approximation is $|\Delta Q|$ and
- the relative error in the approximation is $\left|\frac{\Delta Q}{Q}\right|$ and
- the percentage error in the approximation is $100\left|\frac{\Delta Q}{Q}\right|$

31 Don't take the notation $\mathrm{d}x$ or the terminology "infinitesimal" too seriously. It is just intended to signal "very small".

In Example 3.4.5 of the CLP-1 text we found an approximate value for the number $\sqrt{4.1}$ by using a linear approximation to the single variable function $f(x) = \sqrt{x}$. We can make similar use of linear approximations to multivariable functions.

Example 2.6.7

Problem: Find an approximate value for $\frac{(0.998)^3}{1.003}$.

Solution: Set $f(x,y) = \dfrac{x^3}{y}$. We are to find (approximately) $f(0.998\,,\,1.003)$. We can easily find

$$f(1,1) = \frac{1^3}{1} = 1$$

and since

$$\frac{\partial f}{\partial x} = \frac{3x^2}{y} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = -\frac{x^3}{y^2}$$

we can also easily find

$$\begin{aligned} \frac{\partial f}{\partial x}(1,1) &= 3\frac{1^2}{1} = 3 \\ \frac{\partial f}{\partial y}(1,1) &= 1\frac{1^3}{1^2} = -1 \end{aligned}$$

So, setting $\Delta x = -0.002$ and $\Delta y = 0.003$, we have

$$\begin{aligned} \frac{0.998^3}{1.003} &= f(0.998\,,\,1.003) = f(1+\Delta x\,,\,1+\Delta y) \approx f\big(1,1\big) + \frac{\partial f}{\partial x}\big(1,1\big)\,\Delta x + \frac{\partial f}{\partial y}\big(1,1\big)\,\Delta y \\ &\approx 1 + 3(-0.002) - 1(0.003) = 0.991 \end{aligned}$$

By way of comparison, the exact answer is 0.9910389 to seven decimal places.

Example 2.6.7

Example 2.6.8

Problem: Find an approximate value for $(4.2)^{1/2} + (26.7)^{1/3} + (256.4)^{1/4}$.

Solution: Set $f(x,y,z) = x^{1/2} + y^{1/3} + z^{1/4}$. We are to find (approximately) $f(4.2\,,\,26.7\,,\,256.4)$. We can easily find

$$f(4,27,256) = (4)^{1/2} + (27)^{1/3} + (256)^{1/4} = 2 + 3 + 4 = 9$$

and since

$$\frac{\partial f}{\partial x} = \frac{1}{2x^{1/2}} \qquad \frac{\partial f}{\partial y} = \frac{1}{3y^{2/3}} \qquad \frac{\partial f}{\partial z} = \frac{1}{4z^{3/4}}$$

we can also easily find

$$\begin{aligned}\frac{\partial f}{\partial x}(4,27,256) &= \frac{1}{2(4)^{1/2}} = \frac{1}{2}\times\frac{1}{2}\\ \frac{\partial f}{\partial y}(4,27,256) &= \frac{1}{3(27)^{2/3}} = \frac{1}{3}\times\frac{1}{9}\\ \frac{\partial f}{\partial z}(4,27,256) &= \frac{1}{4(256)^{3/4}} = \frac{1}{4}\times\frac{1}{64}\end{aligned}$$

So, setting $\Delta x = 0.2$, $\Delta y = -0.3$, and $\Delta z = 0.4$, we have

$$\begin{aligned}(4.2)^{1/2} + (26.7)^{1/3} + (256.4)^{1/4} &= f(4.2\,,\,26.7\,,\,256.4) = f(4+\Delta x\,,\,27+\Delta y\,,\,256+\Delta z)\\ &\approx f(4,27,256) + \frac{\partial f}{\partial x}(4,27,256)\,\Delta x + \frac{\partial f}{\partial y}(4,27,256)\,\Delta y + \frac{\partial f}{\partial z}(4,27,256)\,\Delta z\\ &\approx 9 + \frac{0.2}{2\times 2} - \frac{0.3}{3\times 9} + \frac{0.4}{4\times 64} = 9 + \frac{1}{20} - \frac{1}{90} + \frac{1}{640} = 9.0405\end{aligned}$$

to four decimal places. The exact answer is 9.03980 to five decimal places. That's a difference of about

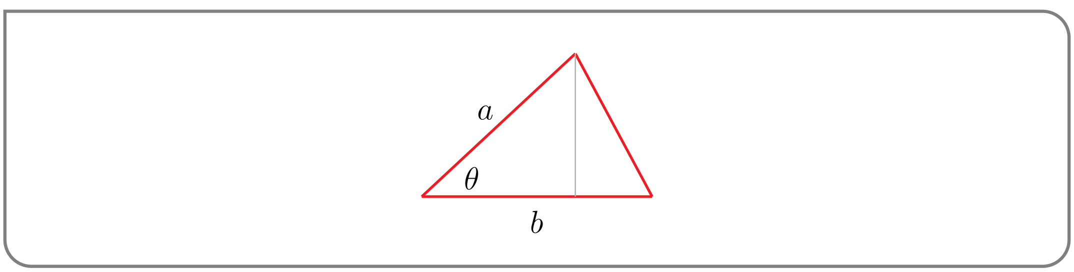
$$100\frac{9.0405 - 9.0398}{9}\% = 0.008\%$$

Note that we could have used the single variable approximation techniques in the CLP-1 text to separately approximate $(4.2)^{1/2}$, $(26.7)^{1/3}$ and $(256.4)^{1/4}$ and then added the results together. Indeed what we have done here is equivalent.

Example 2.6.8

Example 2.6.9

Problem: A triangle has sides $a = 10.1$cm and $b = 19.8$cm which include an angle $35°$. Approximate the area of the triangle.



Solution: The triangle has height $h = a\sin\theta$ and hence has area

$$A(a,b,\theta) = \frac{1}{2}bh = \frac{1}{2}ab\sin\theta$$

The $\sin\theta$ in this formula hides a booby trap built into this problem. In preparing the linear approximation we will need to use the derivative of $\sin\theta$. But the standard derivative

$\frac{d}{d\theta}\sin\theta = \cos\theta$ only applies when θ is expressed in radians — not in degrees. See Warning 3.4.23 in the CLP-1 text.

So we are obliged to convert 35° into

$$35^\circ = (30+5)\frac{\pi}{180} \text{ radians} = \left(\frac{\pi}{6}+\frac{\pi}{36}\right) \text{ radians}$$

We need to compute (approximately) $A(10.1\,,\,19.8\,,\,\pi/6+\pi/36)$. We will, of course[32], choose

$$\begin{aligned} a_0 &= 10 & b_0 &= 20 & \theta_0 &= \pi/6 \\ \Delta a &= 0.1 & \Delta b &= -0.2 & \Delta\theta &= \pi/36 \end{aligned}$$

By way of preparation, we evaluate

$$\begin{aligned} A\big(a_0,b_0,\theta_0\big) &= \frac{1}{2}a_0 b_0 \sin\theta_0 = \frac{1}{2}(10)(20)\frac{1}{2} = 50 \\ \frac{\partial A}{\partial a}\big(a_0,b_0,\theta_0\big) &= \frac{1}{2}b_0\sin\theta_0 = \frac{1}{2}(20)\frac{1}{2} = 5 \\ \frac{\partial A}{\partial b}\big(a_0,b_0,\theta_0\big) &= \frac{1}{2}a_0\sin\theta_0 = \frac{1}{2}(10)\frac{1}{2} = \frac{5}{2} \\ \frac{\partial A}{\partial \theta}\big(a_0,b_0,\theta_0\big) &= \frac{1}{2}a_0 b_0\cos\theta_0 = \frac{1}{2}(10)(20)\frac{\sqrt{3}}{2} = 50\sqrt{3} \end{aligned}$$

So the linear approximation gives

$$\begin{aligned} \text{Area} &= A(10.1\,,\,19.8\,,\,\pi/6+\pi/36) = A(a_0+\Delta a\,,\,b_0+\Delta b\,,\,\theta_0+\Delta\theta) \\ &\approx A\big(a_0,b_0,\theta_0\big) + \frac{\partial A}{\partial a}\big(a_0,b_0,\theta_0\big)\Delta a + \frac{\partial A}{\partial b}\big(a_0,b_0,\theta_0\big)\Delta b + \frac{\partial A}{\partial \theta}\big(a_0,b_0,\theta_0\big)\Delta\theta \\ &= 50 + 5\times 0.1 + \frac{5}{2}\times(-0.2) + 50\sqrt{3}\frac{\pi}{36} \\ &= 50 + \frac{5}{10} - \frac{5}{10} + 50\sqrt{3}\frac{\pi}{36} \\ &= 50\left(1+\sqrt{3}\frac{\pi}{36}\right) \\ &\approx 57.56 \end{aligned}$$

to two decimal places. The exact answer is 57.35 to two decimal places. Our approximation has an error of about

$$100\,\frac{57.56-57.35}{57.35}\% = 0.37\%$$

Example 2.6.9

32 There are other choices possible. For example, we could write $35^\circ = 45^\circ - 10^\circ$. To get a good approximation we try to make $\Delta\theta$ as small as possible, while keeping the arithmetic reasonably simple.

Another practical use of these linear approximations is to quantify how errors made in measured quantities propagate in computations using those measured quantities. Let's explore this idea a little by recycling the last example.

Example 2.6.10 (Example 2.6.9, continued)

Suppose, that, as in Example 2.6.9, we are attempting to determine the area of a triangle by measuring the lengths of two of its sides together with the angle between them and then using the formula

$$A(a, b, \theta) = \frac{1}{2}ab\sin\theta$$

Of course, in the real world[33], we cannot measure lengths and angles exactly. So if we need to know the area to within 1%, the question becomes: "How accurately do we have to measure the side lengths and included angle if we want the area that we compute to have an error of no more than about 1%?"

Let's call the exact side lengths and included angle a_0, b_0 and θ_0, respectively, and the measured side lengths and included angle $a_0 + \Delta a$, $b_0 + \Delta b$ and $\theta_0 + \Delta\theta$. So Δa, Δb and $\Delta\theta$ represent the errors in our measurements. Then, by (2.6.2), the error in our computed area will be approximately

$$\begin{aligned}\Delta A &\approx \frac{\partial A}{\partial a}\big(a_0, b_0, \theta_0\big)\,\Delta a + \frac{\partial A}{\partial b}\big(a_0, b_0, \theta_0\big)\,\Delta b + \frac{\partial A}{\partial \theta}\big(a_0, b_0, \theta_0\big)\,\Delta\theta \\ &= \frac{\Delta a}{2} b_0\sin\theta_0 + \frac{\Delta b}{2} a_0\sin\theta_0 + \frac{\Delta\theta}{2} a_0 b_0\cos\theta_0\end{aligned}$$

and the percentage error in our computed area will be

$$100\frac{|\Delta A|}{A(a_0, b_0, \theta_0)} \approx \left|100\frac{\Delta a}{a_0} + 100\frac{\Delta b}{b_0} + 100\Delta\theta\frac{\cos\theta_0}{\sin\theta_0}\right|$$

By the triangle inequality, $|u + v| \leqslant |u| + |v|$, and the fact that $|uv| = |u|\,|v|$,

$$\left|100\frac{\Delta a}{a_0} + 100\frac{\Delta b}{b_0} + 100\Delta\theta\frac{\cos\theta_0}{\sin\theta_0}\right| \leqslant 100\left|\frac{\Delta a}{a_0}\right| + 100\left|\frac{\Delta b}{b_0}\right| + 100|\Delta\theta|\,\left|\frac{\cos\theta_0}{\sin\theta_0}\right|$$

We want this to be less than 1.

Of course we do not know exactly what a_0, b_0 and θ_0 are. But suppose that we are confident that $a_0 \geqslant 10$, $b_0 \geqslant 10$ and $\frac{\pi}{6} \leqslant \theta_0 \leqslant \frac{\pi}{2}$ so that $\cot\theta_0 \leqslant \cot\frac{\pi}{6} = \sqrt{3} \leqslant 2$. Then

$$\begin{aligned}100\left|\frac{\Delta a}{a_0}\right| &\leqslant 100\left|\frac{\Delta a}{10}\right| = 10\,|\Delta a| \\ 100\left|\frac{\Delta b}{b_0}\right| &\leqslant 100\left|\frac{\Delta b}{10}\right| = 10\,|\Delta b| \\ 100|\Delta\theta|\,\left|\frac{\cos\theta_0}{\sin\theta_0}\right| &\leqslant 100|\Delta\theta|\,2 = 200\,|\Delta\theta|\end{aligned}$$

and

$$100\frac{|\Delta A|}{A(a_0, b_0, \theta_0)} \lesssim 10\,|\Delta a| + 10\,|\Delta b| + 200\,|\Delta\theta|$$

33 Of course in our "real world" everyone uses calculus.

So it will suffice to have measurement errors $|\Delta a|$, $|\Delta b|$ and $|\Delta\theta|$ obey

$$10\,|\Delta a| + 10\,|\Delta b| + 200\,|\Delta\theta| < 1$$

Example 2.6.10

Example 2.6.11

A Question:

Suppose that three variables are measured with percentage error ε_1, ε_2 and ε_3 respectively. In other words, if the exact value of variable number i is x_i and measured value of variable number i is $x_i + \Delta x_i$ then

$$100\left|\frac{\Delta x_i}{x_i}\right| = \varepsilon_i$$

Suppose further that a quantity P is then computed by taking the product of the three variables. So the exact value of P is

$$P(x_1, x_2, x_3) = x_1x_2x_3$$

and the measured value is $P(x_1 + \Delta x_1\,,\ x_2 + \Delta x_2\,,\ x_3 + \Delta x_3)$. What is the percentage error in this measured value of P?

The Answer:

The percentage error in the measured value $P(x_1 + \Delta x_1\,,\ x_2 + \Delta x_2\,,\ x_3 + \Delta x_3)$ is

$$100\left|\frac{P(x_1 + \Delta x_1\,,\ x_2 + \Delta x_2\,,\ x_3 + \Delta x_3) - P(x_1, x_2, x_3)}{P(x_1, x_2, x_3)}\right|$$

We can get a much simpler approximate expression for this percentage error, which is good enough for virtually all applications, by applying

$$\begin{aligned}&P(x_1 + \Delta x_1\,,\ x_2 + \Delta x_2\,,\ x_3 + \Delta x_3)\\ &\qquad\approx P(x_1, x_2, x_3) + P_{x_1}(x_1, x_2, x_3)\,\Delta x_1 + P_{x_2}(x_1, x_2, x_3)\,\Delta x_2 + P_{x_3}(x_1, x_2, x_3)\,\Delta x_3\end{aligned}$$

The three partial derivatives are

$$\begin{aligned}P_{x_1}(x_1, x_2, x_3) &= \frac{\partial}{\partial x_1}\big[x_1x_2x_3\big] = x_2x_3\\ P_{x_2}(x_1, x_2, x_3) &= \frac{\partial}{\partial x_2}\big[x_1x_2x_3\big] = x_1x_3\\ P_{x_3}(x_1, x_2, x_3) &= \frac{\partial}{\partial x_3}\big[x_1x_2x_3\big] = x_1x_2\end{aligned}$$

So

$$P(x_1 + \Delta x_1\,,\ x_2 + \Delta x_2\,,\ x_3 + \Delta x_3) \approx P(x_1, x_2, x_3) + x_2x_3\,\Delta x_1 + x_1x_3\,\Delta x_2 + x_1x_2\,\Delta x_3$$

and the (approximate) percentage error in P is

$$\begin{aligned}100\left|\frac{P(x_1+\Delta x_1, x_2+\Delta x_2, x_3+\Delta x_3)-P(x_1,x_2,x_3)}{P(x_1,x_2,x_3)}\right| \\ &\approx 100\left|\frac{x_2x_3\Delta x_1+x_1x_3\Delta x_2+x_1x_2\Delta x_3}{P(x_1,x_2,x_3)}\right| \\ &= 100\left|\frac{x_2x_3\Delta x_1+x_1x_3\Delta x_2+x_1x_2\Delta x_3}{x_1x_2x_3}\right| \\ &= \left|100\frac{\Delta x_1}{x_1}+100\frac{\Delta x_2}{x_2}+100\frac{\Delta x_3}{x_3}\right| \\ &\leqslant \varepsilon_1+\varepsilon_2+\varepsilon_3\end{aligned}$$

More generally, if we take a product of n, rather than three, variables the percentage error in the product becomes at most (approximately) $\sum\limits_{i=1}^{n}\varepsilon_i$. This is the basis of the experimentalist's rule of thumb that when you take products, percentage errors add.

Still more generally, if we take a "product" $\prod_{i=1}^{n}x_i^{m_i}$, the percentage error in the "product" becomes at most (approximately) $\sum\limits_{i=1}^{n}|m_i|\varepsilon_i$.

Example 2.6.11

2.6.1 ▸▸ Quadratic Approximation and Error Bounds

Recall that, in the CLP-1 text, we started with the constant approximation, then improved it to the linear approximation by adding in degree one terms, then improved that to the quadratic approximation by adding in degree two terms, and so on. We can do the same thing here. Once again, set

$$g(t)=f\big(x_0+t\,\Delta x\,,\,y_0+t\,\Delta y\big)$$

and recall that

$$g(0)=f\big(x_0\,,\,y_0\big)\qquad\text{and}\qquad g(1)=f\big(x_0+\Delta x\,,\,y_0+\Delta y\big)$$

We'll now see what the quadratic approximation

$$g(t_0+\Delta t)\approx g(t_0)+g'(t_0)\,\Delta t+\tfrac{1}{2}g''(t_0)\,\Delta t^2$$

and the corresponding exact formula (see (3.4.32) in the CLP-1 text)

$$g(t_0+\Delta t)=g(t_0)+g'(t_0)\,\Delta t+\tfrac{1}{2}g''(t_0+c\Delta t)\,\Delta t^2\qquad\text{for some } 0\leqslant c\leqslant 1$$

tells us about f. We have already found, using the chain rule, that

$$g'(t)=\frac{\partial f}{\partial x}\big(x_0+t\,\Delta x\,,\,y_0+t\,\Delta y\big)\,\Delta x+\frac{\partial f}{\partial y}\big(x_0+t\,\Delta x\,,\,y_0+t\,\Delta y\big)\,\Delta y$$

We now need to evaluate $g''(t)$. Temporarily write $f_1 = \frac{\partial f}{\partial x}$ and $f_2 = \frac{\partial f}{\partial y}$ so that

$$g'(t) = f_1\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta x + f_2\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta y$$

Then we have, again using the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[f_1\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\big] &= \frac{\partial f_1}{\partial x}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta x + \frac{\partial f_1}{\partial y}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta y \\ &= \frac{\partial^2 f}{\partial x^2}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta x + \frac{\partial^2 f}{\partial y \partial x}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta y \qquad (*)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[f_2\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\big] &= \frac{\partial f_2}{\partial x}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta x + \frac{\partial f_2}{\partial y}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta y \\ &= \frac{\partial^2 f}{\partial x \partial y}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta x + \frac{\partial^2 f}{\partial y^2}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta y \qquad (**)\end{aligned}$$

Adding Δx times $(*)$ to Δy times $(**)$ and recalling that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, gives

$$\begin{aligned}g''(t) = \frac{\partial^2 f}{\partial x^2}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta x \Delta y \\ + \frac{\partial^2 f}{\partial y^2}\big(x_0 + t\,\Delta x\,,\, y_0 + t\,\Delta y\big)\,\Delta y^2\end{aligned}$$

Now setting $t_0 = 0$ and $\Delta t = 1$, the quadratic approximation

$$f\big(x_0 + \Delta x\,,\, y_0 + \Delta y\big) = g(1) \approx g(0) + g'(0) + \tfrac{1}{2}g''(0)$$

is

Equation 2.6.12.

$$\begin{aligned}f\big(x_0 + \Delta x\,,\, y_0 + \Delta y\big) \approx f\big(x_0\,,\, y_0\big) + \frac{\partial f}{\partial x}\big(x_0\,,\, y_0\big)\,\Delta x + \frac{\partial f}{\partial y}\big(x_0\,,\, y_0\big)\,\Delta y \\ + \frac{1}{2}\left\{\frac{\partial^2 f}{\partial x^2}\big(x_0, y_0\big)\,\Delta x^2 + 2\frac{\partial^2 f}{\partial x \partial y}\big(x_0, y_0\big)\,\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}\big(x_0, y_0\big)\,\Delta y^2\right\}\end{aligned}$$

and the corresponding exact formula

$$f\big(x_0 + \Delta x\,,\, y_0 + \Delta y\big) = g(1) = g(0) + g'(0) + \tfrac{1}{2}g''(c)$$

is

Equation 2.6.13.

$$f(x_0+\Delta x\,,\,y_0+\Delta y) = f(x_0\,,\,y_0) + \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\Delta y$$
$$+\frac{1}{2}\left\{\frac{\partial^2 f}{\partial x^2}(\mathbf{r}(c))\,\Delta x^2 + 2\frac{\partial^2 f}{\partial x\partial y}(\mathbf{r}(c))\,\Delta x\Delta y + \frac{\partial^2 f}{\partial y^2}(\mathbf{r}(c))\,\Delta y^2\right\}$$

where $\mathbf{r}(c) = \big(x_0 + c\,\Delta x\,,\,y_0 + c\,\Delta y\big)$ and c is some (unknown) number satisfying $0 \leqslant c \leqslant 1$.

Equation 2.6.14.

If we can bound the second derivatives

$$\left|\frac{\partial^2 f}{\partial x^2}(\mathbf{r}(c))\right|\,,\,\left|\frac{\partial^2 f}{\partial x\partial y}(\mathbf{r}(c))\right|\,,\,\left|\frac{\partial^2 f}{\partial y^2}(\mathbf{r}(c))\right| \leqslant M$$

we can massage (2.6.13) into the form

$$\left|f(x_0+\Delta x\,,\,y_0+\Delta y) - \left\{f(x_0\,,\,y_0) + \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\Delta y\right\}\right|$$
$$\leqslant \frac{M}{2}\left(|\Delta x|^2 + 2|\Delta x|\,|\Delta y| + |\Delta y|^2\right)$$

Why might we want to do this? The left hand side of (2.6.14) is exactly the error in the linear approximation (2.6.1). So the right hand side is a rigorous bound on the error in the linear approximation.

Example 2.6.15 (Example 2.6.7, continued)

Suppose that we approximate $\frac{(0.998)^3}{1.003}$ as in Example 2.6.7 and we want a rigorous bound on the approximation. We can get such a rigorous bound by applying (2.6.13). Set

$$f(x,y) = \frac{x^3}{y}$$

and

$$x_0 = 1 \qquad \Delta x = -0.002 \qquad y_0 = 1 \qquad \Delta y = 0.003$$

Then the exact answer is $f(x_0+\Delta x\,,\,y_0+\Delta y)$ and the approximate answer is $f(x_0\,,\,y_0) + \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\Delta y$, so that, by (2.6.13), the error in the approximation is exactly

$$\frac{1}{2}\left|\frac{\partial^2 f}{\partial x^2}(\mathbf{r}(c))\,\Delta x^2 + 2\frac{\partial^2 f}{\partial x\partial y}(\mathbf{r}(c))\,\Delta x\Delta y + \frac{\partial^2 f}{\partial y^2}(\mathbf{r}(c))\,\Delta y^2\right|$$

with $\mathbf{r}(c) = \big(1 - 0.002c\,,\, 1 + 0.0003c\big)$ for some, unknown, $0 \leqslant c \leqslant 1$. For our function f

$$f(x,y) = \frac{x^3}{y} \qquad \frac{\partial f}{\partial x}(x,y) = \frac{3x^2}{y} \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{x^3}{y^2}$$
$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{6x}{y} \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = -\frac{3x^2}{y^2} \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2x^3}{y^3}$$

We don't know what $\mathbf{r}(c) = \big(1 - 0.002c\,,\, 1 + 0.0003c\big)$ is. But we know that $0 \leqslant c \leqslant 1$, so we definitely know that the x component of $\mathbf{r}(c)$ is smaller that 1 and the y component of $\mathbf{r}(c)$ is bigger than 1. So

$$\left|\frac{\partial^2 f}{\partial x^2}\big(\mathbf{r}(c)\big)\right| \leqslant 6 \qquad \left|\frac{\partial^2 f}{\partial x \partial y}\big(\mathbf{r}(c)\big)\right| \leqslant 3 \qquad \left|\frac{\partial^2 f}{\partial y^2}\big(\mathbf{r}(c)\big)\right| \leqslant 2$$

and

$$\begin{aligned} \text{error} &\leqslant \frac{1}{2}\left[6\Delta x^2 + 2 \times 3|\Delta x\, \Delta y| + 2\Delta y^2\right] \\ &\leqslant 3(0.002)^2 + 3(0.002)(0.003) + (0.003)^2 \\ &= 0.000039 \end{aligned}$$

By way of comparison, the exact error is 0.0000389, to seven decimal places.

Example 2.6.15

Example 2.6.16

In this example, we find the quadratic approximation of $f(x,y) = \sqrt{1 + 4x^2 + y^2}$ at $(x_0, y_0) = (1,2)$ and use it to compute approximately $f(1.1\,,\,2.05)$. We know that we will need all partial derivatives up to order 2, so we first compute them and evaluate them at $(x_0, y_0) = (1,2)$.

$$\begin{aligned} f(x,y) &= \sqrt{1 + 4x^2 + y^2} & f(x_0,y_0) &= 3 \\ f_x(x,y) &= \frac{4x}{\sqrt{1 + 4x^2 + y^2}} & f_x(x_0,y_0) &= \frac{4}{3} \\ f_y(x,y) &= \frac{y}{\sqrt{1 + 4x^2 + y^2}} & f_y(x_0,y_0) &= \frac{2}{3} \\ f_{xx}(x,y) &= \frac{4}{\sqrt{1 + 4x^2 + y^2}} - \frac{16x^2}{[1 + 4x^2 + y^2]^{3/2}} & f_{xx}(x_0,y_0) &= \frac{4}{3} - \frac{16}{27} = \frac{20}{27} \\ f_{xy}(x,y) &= -\frac{4xy}{[1 + 4x^2 + y^2]^{3/2}} & f_{xy}(x_0,y_0) &= -\frac{8}{27} \\ f_{yy}(x,y) &= \frac{1}{\sqrt{1 + 4x^2 + y^2}} - \frac{y^2}{[1 + 4x^2 + y^2]^{3/2}} & f_{yy}(x_0,y_0) &= \frac{1}{3} - \frac{4}{27} = \frac{5}{27} \end{aligned}$$

We now just substitute them into (2.6.12) to get that the quadratic approximation to f about (x_0, y_0) is

$$\begin{aligned} f\big(x_0+\Delta x\,,\, y_0+\Delta y\big) &\approx f(x_0,y_0)+f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y \\ &\qquad + \frac{1}{2}\Big[f_{xx}(x_0,y_0)\Delta x^2+2f_{xy}(x_0,y_0)\Delta x\Delta y+f_{yy}(x_0,y_0)\Delta y^2\Big] \\ &= 3+\frac{4}{3}\Delta x+\frac{2}{3}\Delta y+\frac{10}{27}\Delta x^2-\frac{8}{27}\Delta x\Delta y+\frac{5}{54}\Delta y^2 \end{aligned}$$

In particular, with $\Delta x = 0.1$ and $\Delta y = 0.05$,

$$f(1.1\,,\,2.05)\approx 3+\frac{4}{3}(0.1)+\frac{2}{3}(0.05)+\frac{10}{27}(0.01)-\frac{8}{27}(0.005)+\frac{5}{54}(0.0025)=3.1691$$

The actual value, to four decimal places, is 3.1690. The percentage error is about 0.004%.

Example 2.6.16

Example 2.6.17

In this example, we find the quadratic approximation of $f(x,y) = e^{2x}\sin(3y)$ about $(x_0, y_0) = (0,0)$ in two different ways.

The first way uses the canned formula (2.6.12). We compute all partial derivatives up to order 2 at (x_0, y_0).

$$\begin{aligned} f(x,y) &= e^{2x}\sin(3y) & f(x_0,y_0) &= 0 \\ f_x(x,y) &= 2e^{2x}\sin(3y) & f_x(x_0,y_0) &= 0 \\ f_y(x,y) &= 3e^{2x}\cos(3y) & f_y(x_0,y_0) &= 3 \\ f_{xx}(x,y) &= 4e^{2x}\sin(3y) & f_{xx}(x_0,y_0) &= 0 \\ f_{xy}(x,y) &= 6e^{2x}\cos(3y) & f_{xy}(x_0,y_0) &= 6 \\ f_{yy}(x,y) &= -9e^{2x}\sin(3y) & f_{yy}(x_0,y_0) &= 0 \end{aligned}$$

So the quadratic approximation to f about $(0,0)$ is

$$\begin{aligned} f\big(x\,,\,y\big) &\approx f(x,y)+f_x(x,y)x+f_y(0,0)y+\frac{1}{2}\Big[f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2\Big] \\ &= 3y+6xy \end{aligned}$$

That's pretty simple — just compute a bunch of partial derivatives and substitute into the formula (2.6.12).

But there is also a sneakier, and often computationally more efficient, method to get the same result. It exploits the single variable Taylor expansions

$$\begin{aligned} e^x &= 1+x+\frac{1}{2!}x^2+\cdots \\ \sin y &= y-\frac{1}{3!}y^3+\cdots \end{aligned}$$

Replacing x by $2x$ in the first and y by $3y$ in the second and multiplying the two together, keeping track only of terms of degree at most two, gives

$$\begin{aligned}
f(x,y) &= e^{2x}\sin(3y) \\
&= \Big[1+(2x)+\frac{1}{2!}(2x)^2+\cdots\Big]\Big[(3y)-\frac{1}{3!}(3y)^3+\cdots\Big] \\
&= \Big[1+2x+2x^2+\cdots\Big]\Big[3y-\frac{9}{2}y^3+\cdots\Big] \\
&= 3y+6xy+6x^2y+\cdots-\frac{9}{2}y^3-9xy^3-9x^2y^3+\cdots \\
&= 3y+6xy+\cdots
\end{aligned}$$

just as in the first computation.

Example 2.6.17

2.6.2 ▸▸ Optional — Taylor Polynomials

We have just found linear and quadratic approximations to the function $f(x,y)$, for (x,y) near the point (x_0,y_0). In CLP-1, we found not only linear and quadratic approximations, but in fact a whole hierarchy of approximations. For each integer $n \geqslant 0$, the n^{th} degree Taylor polynomial for $f(x)$ about $x=a$ was defined, in Definition 3.4.11 of the CLP-1 text, to be

$$\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)\cdot(x-a)^k$$

We'll now define, and find, the Taylor polynomial of degree n for the function $f(x,y)$ about $(x,y)=(x_0,y_0)$. It is going to be a polynomial of degree n in Δx and Δy. The most general such polynomial is

$$T_n(\Delta x,\Delta y) = \sum_{\substack{\ell,m\geqslant 0\\ \ell+m\leqslant n}} a_{\ell,m}\,(\Delta x)^\ell(\Delta y)^m$$

with all of the coefficients $a_{\ell,m}$ being constants. The specific coefficients for the Taylor polynomial are determined by the requirement that all partial derivatives of $T_n(\Delta x,\Delta y)$ at $\Delta x=\Delta y=0$ are the same as the corresponding partial derivatives of $f\big(x_0+\Delta x\,,\,y_0+\Delta y\big)$ at $\Delta x=\Delta y=0$.

By way of preparation for our computation of the derivatives of $T_n(\Delta x,\Delta y)$, consider

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}t^4 = 4t^3 && \frac{\mathrm{d}^2}{\mathrm{d}t^2}t^4=(4)(3)t^2 && \frac{\mathrm{d}^3}{\mathrm{d}t^3}t^4=(4)(3)(2)t \\
&\frac{\mathrm{d}^4}{\mathrm{d}t^4}t^4=(4)(3)(2)(1)=4! && \frac{\mathrm{d}^5}{\mathrm{d}t^5}t^4=0 && \frac{\mathrm{d}^6}{\mathrm{d}t^6}t^4=0
\end{aligned}$$

and

$$\frac{d}{dt}t^4\Big|_{t=0} = 0 \qquad \frac{d^2}{dt^2}t^4\Big|_{t=0} = 0 \qquad \frac{d^3}{dt^3}t^4\Big|_{t=0} = 0$$
$$\frac{d^4}{dt^4}t^4\Big|_{t=0} = 4! \qquad \frac{d^5}{dt^5}t^4\Big|_{t=0} = 0 \qquad \frac{d^6}{dt^6}t^4\Big|_{t=0} = 0$$

More generally, for any natural numbers p, m,

$$\frac{d^p}{dt^p}t^m = \begin{cases} m(m-1)\cdots(m-p+1)t^{m-p} & \text{if } p \leqslant m \\ 0 & \text{if } p > m \end{cases}$$

so that

$$\frac{d^p}{dt^p}t^m\Big|_{t=0} = \begin{cases} m! & \text{if } p = m \\ 0 & \text{if } p \neq m \end{cases}$$

Consequently

$$\frac{\partial^p}{\partial(\Delta x)^p}\frac{\partial^q}{\partial(\Delta y)^q}(\Delta x)^\ell(\Delta y)^m\Big|_{\Delta x=\Delta y=0} = \begin{cases} \ell!\,m! & \text{if } p = \ell \text{ and } q = m \\ 0 & \text{if } p \neq \ell \text{ or } q \neq m \end{cases}$$

and

$$\begin{aligned} \frac{\partial^{p+q}\ T_n}{\partial(\Delta x)^p\,\partial(\Delta y)^q}(0,0) &= \sum_{\substack{\ell,m\geqslant 0\\ \ell+m\leqslant n}} a_{\ell,m}\ \frac{\partial^p}{\partial(\Delta x)^p}\frac{\partial^q}{\partial(\Delta y)^q}(\Delta x)^\ell(\Delta y)^m\Big|_{\Delta x=\Delta y=0} \\ &= \begin{cases} p!\,q!\,a_{p,q} & \text{if } p+q \leqslant n \\ 0 & \text{if } p+q > n \end{cases} \end{aligned}$$

Our requirement that the derivatives of f and T_n match is the requirement that, for all $p + q \leqslant n$,

$$\frac{\partial^{p+q}\ T_n}{\partial(\Delta x)^p\,\partial(\Delta y)^q}(0,0) = \frac{\partial^{p+q}}{\partial(\Delta x)^p\,\partial(\Delta y)^q}f\big(x_0+\Delta x\,,\,y_0+\Delta y\big)\Big|_{\Delta x=\Delta y=0} = \frac{\partial^{p+q}\,f}{\partial x^p\,\partial y^q}(x_0,y_0)$$

This requirement gives

$$p!\,q!\,a_{p,q} = \frac{\partial^{p+q}\,f}{\partial x^p\,\partial y^q}(x_0,y_0)$$

So the Taylor polynomial of degree n for the function $f(x,y)$ about $(x,y) = (x_0,y_0)$ is the right hand side of

Equation 2.6.18.

$$f\big(x_0+\Delta x\,,\,y_0+\Delta y\big) \approx \sum_{\substack{\ell,m\geqslant 0\\ \ell+m\leqslant n}} \frac{1}{\ell!\ m!}\,\frac{\partial^{\ell+m}\,f}{\partial x^\ell\,\partial y^m}(x_0,y_0)\ (\Delta x)^\ell(\Delta y)^m$$

This is for functions, $f(x,y)$, of two variables. There are natural extensions of this for functions of any (finite) number of variables. For example, the Taylor polynomial of degree n for a function, $f(x,y,z)$, of three variables is the right hand side of

$$f\big(x_0+\Delta x\,,\,y_0+\Delta y\,,\,z_0+\Delta z\big) \approx \sum_{\substack{k,\ell,m\geqslant 0\\ k+\ell+m\leqslant n}} \frac{1}{k!\,\ell!\,m!}\,\frac{\partial^{k+\ell+m} f}{\partial x^k\,\partial y^\ell\,\partial z^m}(x_0,y_0,z_0)\,(\Delta x)^k(\Delta y)^\ell(\Delta z)^m$$

2.7▲ Directional Derivatives and the Gradient

The principal interpretation of $\frac{\mathrm{d}f}{\mathrm{d}x}(a)$ is the rate of change of $f(x)$, per unit change of x, at $x = a$. The natural analog of this interpretation for multivariable functions is the directional derivative, which we now introduce through a question.

▸▸▸ A Question

Suppose that you are standing at (a,b) near a campfire. The temperature you feel at (x,y) is $f(x,y)$. You start to move with velocity $\mathbf{v} = \langle v_1, v_2\rangle$. What rate of change of temperature do you feel?

▸▸▸ The Answer

Let's set the beginning of time, $t = 0$, to the time at which you leave (a,b). Then

- at time 0 you are at (a,b) and feel the temperature $f(a,b)$ and
- at time t you are at $(a+v_1t\,,\,b+v_2t)$ and feel the temperature $f(a+v_1t\,,\,b+v_2t)$. So
- the change in temperature between time 0 and time t is $f(a+v_1t\,,\,b+v_2t)-f(a,b)$,
- the average rate of change of temperature, per unit time, between time 0 and time t is $\frac{f(a+v_1t\,,\,b+v_2t)-f(a,b)}{t}$ and the
- instantaneous rate of change of temperature per unit time as you leave (a,b) is $\lim\limits_{t\to 0}\frac{f(a+v_1t\,,\,b+v_2t)-f(a,b)}{t}$.

Concentrate on the t dependence in this limit by writing $f(a+v_1t\,,\,b+v_2t) = g(t)$. Then

$$\begin{aligned}\lim_{t\to 0}\frac{f(a+v_1t\,,\,b+v_2t)-f(a,b)}{t} &= \lim_{t\to 0}\frac{g(t)-g(0)}{t}\\ &= \frac{\mathrm{d}g}{\mathrm{d}t}(0)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big[f(a+v_1t\,,\,b+v_2t)\big]\Big|_{t=0}\end{aligned}$$

By the chain rule, we can write the right hand side in terms of partial derivatives of f.

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[f(a+v_1t\,,\,b+v_2t)\big] = f_x(a+v_1t\,,\,b+v_2t)\,v_1 + f_y(a+v_1t\,,\,b+v_2t)\,v_2$$

So, the instantaneous rate of change per unit time as you leave (a,b) is

$$\begin{aligned}\lim_{t\to 0}\frac{f(a+v_1t\,,\,b+v_2t)-f(a,b)}{t} &= \Big[f_x(a+v_1t\,,\,b+v_2t)\,v_1+f_y(a+v_1t\,,\,b+v_2t)\,v_2\Big]\Big|_{t=0}\\ &= f_x(a,b)\,v_1+f_y(a,b)\,v_2\\ &= \big\langle f_x(a,b)\,,\,f_y(a,b)\big\rangle\cdot\langle v_1,v_2\rangle\end{aligned}$$

Notice that we have expressed the rate of change as the dot product of the velocity vector with a vector of partial derivatives of f. We have seen such a vector of partial derivatives of f before; in Definition 2.5.4, we defined the gradient of the three variable function $G(x,y,z)$ at the point $(x_0\,,\,y_0\,,\,z_0)$ to be $\big\langle G_x(x_0\,,\,y_0\,,\,z_0)\,,\,G_y(x_0\,,\,y_0\,,\,z_0)\,,\,G_z(x_0\,,\,y_0\,,\,z_0)\big\rangle$. Here we see the natural two dimensional analog.

Definition 2.7.1.

The vector $\big\langle f_x(a,b)\,,\,f_y(a,b)\big\rangle$ is denoted $\boldsymbol{\nabla} f(a,b)$ and is called "the **gradient** of the function f at the point (a,b)".

In general, the gradient of f is a vector with one component for each variable of f. The j^{th} component is the partial derivative of f with respect to the j^{th} variable.

Now because the dot product $\boldsymbol{\nabla} f(a,b)\cdot\mathbf{v}$ appears frequently, we introduce some handy notation.

Notation 2.7.2.

Given any vector $\mathbf{v}=\langle v_1,v_2\rangle$, the expression

$$\big\langle f_x(a,b),f_y(a,b)\big\rangle\cdot\langle v_1,v_2\rangle=\boldsymbol{\nabla} f(a,b)\cdot\mathbf{v}$$

is denoted $D_{\mathbf{v}}f(a,b)$.

Armed with this useful notation we can answer our question very succinctly.

Equation 2.7.3.

The rate of change of f per unit time as you leave (a,b) moving with velocity $\mathbf{v}$ is

$$D_{\mathbf{v}}f(a,b)=\boldsymbol{\nabla} f(a,b)\cdot\mathbf{v}$$

We can compute the rate of change of temperature per unit distance (as opposed to per unit time) in a similar way. The change in temperature between time 0 and time t is $f(a+v_1t,b+v_2t)-f(a,b)$. Between time 0 and time t, you have travelled a distance $|\mathbf{v}|t$. So the instantaneous rate of change of temperature per unit distance as you leave (a,b) is

$$\lim_{t\to 0}\frac{f(a+v_1t,b+v_2t)-f(a,b)}{t|\mathbf{v}|}$$

This is exactly $\frac{1}{|\mathbf{v}|}$ times $\lim\limits_{t\to 0}\frac{f(a+v_1t,b+v_2t)-f(a,b)}{t}$ which we computed above to be $D_{\mathbf{v}}f(a,b)$. So

Equation 2.7.4.

Given any nonzero vector $\mathbf{v}$, the rate of change of f per unit distance as you leave (a, b) moving in direction $\mathbf{v}$ is

$$\nabla f(a,b) \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = D_{\mathbf{v}/|\mathbf{v}|} f(a,b)$$

Definition 2.7.5.

$D_{\mathbf{v}/|\mathbf{v}|} f(a,b)$ is called the *directional derivative* of the function $f(x, y)$ at the point (a, b) in the direction[34] $\mathbf{v}$.

▸▸▸ The Implications

We have just seen that the instantaneous rate of change of f per unit distance as we leave (a, b) moving in direction $\mathbf{v}$ is a dot product, which we can write as

$$\nabla f(a,b) \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = |\nabla f(a,b)| \cos\theta$$

where θ is the angle between the gradient vector $\nabla f(a, b)$ and the direction vector $\mathbf{v}$. Writing it in this way allows us to make some useful observations. Since $\cos\theta$ is always between -1 and $+1$

- the direction of maximum rate of increase is that having $\theta = 0$. So to get maximum rate of increase per unit distance, as you leave (a, b), you should move in the same direction as the gradient $\nabla f(a, b)$. Then the rate of increase per unit distance is $|\nabla f(a, b)|$.

- The direction of minimum (i.e. most negative) rate of increase is that having $\theta = 180°$. To get minimum rate of increase per unit distance you should move in the direction opposite $\nabla f(a, b)$. Then the rate of increase per unit distance is $-|\nabla f(a, b)|$.

- The directions giving zero rate of increase are those perpendicular to $\nabla f(a, b)$. If you move in a direction perpendicular to $\nabla f(a, b)$, then $f(x, y)$ remains constant as you leave (a, b). At that instant, you are moving so that $f(x, y)$ remains constant and consequently you are moving along the level curve $f(x, y) = f(a, b)$. So $\nabla f(a, b)$ is perpendicular to the level curve $f(x, y) = f(a, b)$ at (a, b). The corresponding statement in three dimensions is that $\nabla F(a, b, c)$ is perpendicular to the level surface $F(x, y, z) = F(a, b, c)$ at (a, b, c). Hence a good way to find a vector normal to the surface $F(x, y, z) = F(a, b, c)$ at the point (a, b, c) is to compute the gradient $\nabla F(a, b, c)$. This is precisely what we saw back in Theorem 2.5.5.

34 Some people require direction vectors to have unit length. We don't.

Now that we have defined the directional derivative, here are some examples.

Example 2.7.6

Problem: Find the directional derivative of the function $f(x,y) = e^{x+y^2}$ at the point $(0,1)$ in the direction $-\hat{\imath} + \hat{\jmath}$.

Solution. To compute the directional derivative, we need the gradient. To compute the gradient, we need some partial derivatives. So we start with the partial derivatives of f at $(0,1)$:

$$\begin{aligned} f_x(0,1) &= e^{x+y^2}\Big|_{\substack{x=0\\y=1}} &&= e \\ f_y(0,1) &= 2ye^{x+y^2}\Big|_{\substack{x=0\\y=1}} &&= 2e \end{aligned}$$

So the gradient of f at $(0,1)$ is

$$\nabla f(0,1) = f_x(0,1)\,\hat{\imath} + f_y(0,1)\,\hat{\jmath} = e\,\hat{\imath} + 2e\,\hat{\jmath}$$

and the direction derivative in the direction $-\hat{\imath} + \hat{\jmath}$ is

$$D_{\frac{-\hat{\imath}+\hat{\jmath}}{|-\hat{\imath}+\hat{\jmath}|}} f(0,1) = \nabla f(0,1) \cdot \frac{-\hat{\imath}+\hat{\jmath}}{|-\hat{\imath}+\hat{\jmath}|} = \big(e\,\hat{\imath} + 2e\,\hat{\jmath}\big) \cdot \frac{-\hat{\imath}+\hat{\jmath}}{\sqrt{2}} = \frac{e}{\sqrt{2}}$$

Example 2.7.6

Example 2.7.7

Problem: Find the directional derivative of the function $w(x,y,z) = xyz + \ln(xz)$ at the point $(1,3,1)$ in the direction $\langle 1\,,\,0\,,\,1\rangle$. In what directions is the directional derivative zero?

Solution. First, the partial derivatives of w at $(1,3,1)$ are

$$\begin{aligned} w_x(1,3,1) &= \left[yz + \frac{1}{x}\right]\Big|_{(1,3,1)} &&= 3\times 1 + \frac{1}{1} = 4 \\ w_y(1,3,1) &= xz\Big|_{(1,3,1)} &&= 1\times 1 \qquad = 1 \\ w_z(1,3,1) &= \left[xy + \frac{1}{z}\right]\Big|_{(1,3,1)} &&= 1\times 3 + \frac{1}{1} = 4 \end{aligned}$$

so the gradient of w at $(1,3,1)$ is

$$\nabla w(1,3,1) = \big\langle w_x(1,3,1)\,,\, w_y(1,3,1)\,,\, w_z(1,3,1)\big\rangle = \langle 4\,,\,1\,,\,4\rangle$$

and the direction derivative in the direction $\langle 1\,,\,0\,,\,1\rangle$ is

$$D_{\frac{\langle 1,0,1\rangle}{|\langle 1,0,1\rangle|}} w(1,3,1) = \nabla w(1,3,1) \cdot \frac{\langle 1\,,\,0\,,\,1\rangle}{|\,\langle 1\,,\,0\,,\,1\rangle\,|} = \langle 4\,,\,1\,,\,4\rangle \cdot \frac{\langle 1\,,\,0\,,\,1\rangle}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

The directional derivative of w at $(1,3,1)$ in the direction $\mathbf{t} \neq \mathbf{0}$ is zero if and only if

$$0 = D_{\frac{\mathbf{t}}{|\mathbf{t}|}} w(1,3,1) = \nabla w(1,3,1) \cdot \frac{\mathbf{t}}{|\mathbf{t}|} = \langle 4\,,\,1\,,\,4\rangle \cdot \frac{\mathbf{t}}{|\mathbf{t}|}$$

which is the case if and only if $\mathbf{t}$ is perpendicular to $\langle 4\,,\,1\,,\,4\rangle$. So if we walk in the direction of any vector in the plane, $4x + y + 4z = 0$ (which has normal vector $\langle 4\,,\,1\,,\,4\rangle$) then the directional derivative is zero.

Example 2.7.7

Example 2.7.8

Let

$$f(x,y) = 5 - x^2 - 2y^2 \qquad (a,b) = \big(-1,-1\big)$$

In this example, we'll explore the behaviour of the function $f(x,y)$ near the point (a,b).

Note that for any fixed $f_0 < 5$, $f(x,y) = f_0$ is the ellipse $x^2 + 2y^2 = 5 - f_0$. So the graph $z = f(x,y)$ consists of a bunch of horizontal ellipses stacked one on top of each other.

- Since the ellipse $x^2 + 2y^2 = 5 - f_0$ has x-semi-axis $\sqrt{5 - f_0}$ and y-semi-axis $\sqrt{\frac{5-f_0}{2}}$,
  - the ellipses start with a point on the z axis when $f_0 = 5$ and
  - increase in size as f_0 decreases.
- The part of the graph $z = f(x,y)$ in the first octant is sketched in the top figure below.
- Several level curves, $f(x,y) = f_0$, are sketched in the bottom figure below.
- The gradient vector

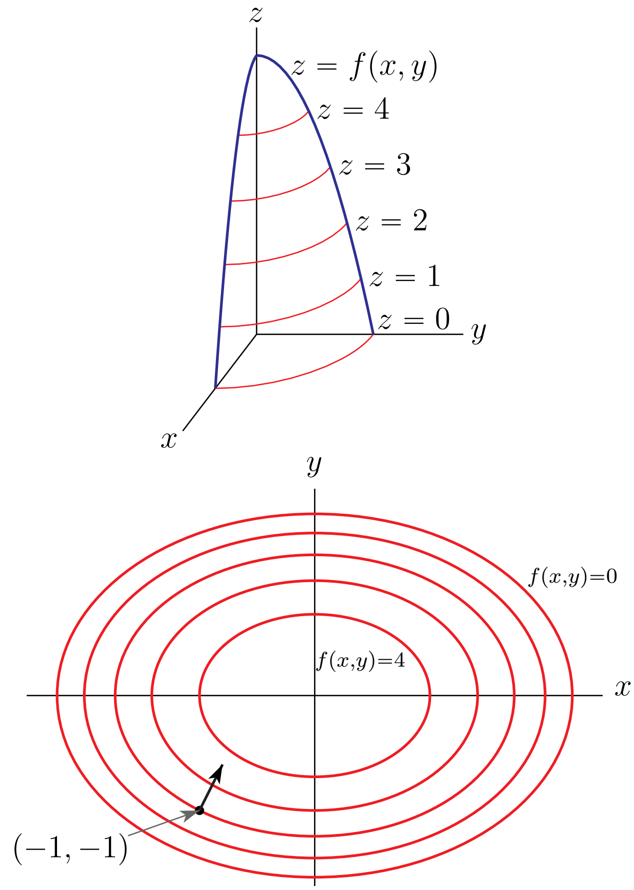
  $$\nabla f(a,b) = \langle -2x, -4y\rangle \big|_{(-1,-1)} = \langle 2,4\rangle = 2\,\langle 1,2\rangle$$

  at $(-1,-1)$ is also illustrated in the bottom sketch.

We have that, at $(a,b) = (-1,-1)$,

- the unit vector giving the direction of maximum rate of increase is the unit vector in the direction of the gradient vector $2\,\langle 1,2\rangle$, which is $\frac{1}{\sqrt{5}}\,\langle 1,2\rangle$. The maximum rate of increase is $|\,\langle 2,4\rangle\,| = 2\sqrt{5}$.
- The unit vector giving the direction of minimum rate of increase is $-\frac{1}{\sqrt{5}}\,\langle 1,2\rangle$ and that minimum rate is $-|\,\langle 2,4\rangle\,| = -2\sqrt{5}$.
- The directions giving zero rate of increase are perpendicular to $\nabla f(a,b)$. One vector perpendicular[35] to $\langle 1,2\rangle$ is $\langle 2,-1\rangle$. So the unit vectors giving the direction of zero rate of increase are the $\pm\frac{1}{\sqrt{5}}(2,-1)$. These are the directions of the tangent vector at (a,b) to the level curve of f through (a,b), which is the curve $f(x,y) = f(a,b)$.

35 Check this by taking the dot product of $\langle 1,2\rangle$ and $\langle 2,-1\rangle$.



Example 2.7.8

Example 2.7.9

Problem: What is the rate of change of $f(x,y,z) = x^2 + y^2 + z^2$ at $(3,5,4)$ moving in the positive x-direction along the curve of intersection of the surfaces $G(x,y,z) = 25$ and $H(x,y,z) = 0$ where

$$G(x,y,z) = 2x^2 - y^2 + 2z^2 \qquad \text{and} \qquad H(x,y,z) = x^2 - y^2 + z^2$$

Solution. As a first check note that $(3,5,4)$ really does lie on both surfaces because

$$\begin{aligned} G(3,5,4) &= 2\big(3^2\big) - 5^2 + 2\big(4^2\big) = 18 - 25 + 32 = 25 \\ H(3,5,4) &= \quad 3^2 - 5^2 + \quad 4^2 \quad = \ 9 - 25 + 16 = 0 \end{aligned}$$

We compute gradients to get the normal vectors to the surfaces $G(x,y,z) = 25$ and $H(x,y,z) = 0$ at $(3,5,4)$.

$$\begin{aligned} \nabla G(3,5,4) &= \Big[4x\,\hat{\boldsymbol{\imath}} - 2y\,\hat{\boldsymbol{\jmath}} + 4z\,\hat{\mathbf{k}}\Big]_{(3,5,4)} = 12\,\hat{\boldsymbol{\imath}} - 10\,\hat{\boldsymbol{\jmath}} + 16\,\hat{\mathbf{k}} = 2\big(6\,\hat{\boldsymbol{\imath}} - 5\,\hat{\boldsymbol{\jmath}} + 8\,\hat{\mathbf{k}}\big) \\ \nabla H(3,5,4) &= \Big[2x\,\hat{\boldsymbol{\imath}} - 2y\,\hat{\boldsymbol{\jmath}} + 2z\,\hat{\mathbf{k}}\Big]_{(3,5,4)} = 6\,\hat{\boldsymbol{\imath}} - 10\,\hat{\boldsymbol{\jmath}} + 8\,\hat{\mathbf{k}} \quad = 2\big(3\,\hat{\boldsymbol{\imath}} - 5\,\hat{\boldsymbol{\jmath}} + 4\,\hat{\mathbf{k}}\big) \end{aligned}$$

The direction of interest is tangent to the curve of intersection. So the direction of interest is tangent to both surfaces and hence is perpendicular to both gradients. Consequently one tangent vector to the curve at $(3,5,4)$ is

$$\begin{aligned}\nabla G(3,5,4)\ \times \nabla H(3,5,4) &= 4\big(6\,\hat{\imath} - 5\,\hat{\jmath} + 8\,\hat{\mathbf{k}}\big) \times \big(3\,\hat{\imath} - 5\,\hat{\jmath} + 4\,\hat{\mathbf{k}}\big) \\ &= 4\ \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ 6 & -5 & 8 \\ 3 & -5 & 4\end{bmatrix} \\ &= 4\ \big(20\,\hat{\imath} - 15\,\hat{\mathbf{k}}\big) = 20\ \big(4\,\hat{\imath} - 3\,\hat{\mathbf{k}}\big)\end{aligned}$$

and the unit tangent vector to the curve at $(3,5,4)$ that has positive x component is

$$\frac{4\,\hat{\imath} - 3\,\hat{\mathbf{k}}}{|4\,\hat{\imath} - 3\,\hat{\mathbf{k}}|} = \frac{4}{5}\,\hat{\imath} - \frac{3}{5}\,\hat{\mathbf{k}}$$

The desired rate of change is

$$\begin{aligned}D_{\frac{4}{5}\hat{\imath}-\frac{3}{5}\hat{\mathbf{k}}}f(3,5,4) &= \nabla f(3,5,4)\cdot\left(\frac{4}{5}\,\hat{\imath} - \frac{3}{5}\,\hat{\mathbf{k}}\right) = \overbrace{\big(6\,\hat{\imath} + 10\,\hat{\jmath} + 8\,\hat{\mathbf{k}}\big)}^{[2x\,\hat{\imath}+2y\,\hat{\jmath}+2z\,\hat{\mathbf{k}}]_{(x,y,z)=(3,5,4)}}\cdot\left(\frac{4}{5}\,\hat{\imath} - \frac{3}{5}\,\hat{\mathbf{k}}\right) \\ &= 0\end{aligned}$$

Actually, we could have known that the rate of change would be zero.

- Any point (x,y,z) on the curve obeys both $y^2 = x^2 + z^2$ and $2x^2 - y^2 + 2z^2 = 25$.
- Substituting $y^2 = x^2 + z^2$ into $2x^2 - y^2 + 2z^2 = 25$ gives $x^2 + z^2 = 25$.
- So, at any point on the curve, $x^2 + z^2 = 25$ and $y^2 = x^2 + z^2 = 25$ so that $x^2 + y^2 + z^2 = 50$.
- That is, $f(x,y,z) = x^2 + y^2 + z^2$ takes the value 50 at every point of the curve.
- So of course the rate of change of f along the curve is 0.

Example 2.7.9

Let's change things up a little. In the next example, we are told the rates of change in two different directions. From this we are to determine the rate of change in a third direction.

Example 2.7.10

Problem: The rate of change of a given function $f(x,y)$ at the point $P_0 = (1,2)$ in the direction towards $P_1 = (2,3)$ is $2\sqrt{2}$ and in the direction towards $P_2 = (1,0)$ is -3. What is the rate of change of f at P_0 towards the origin $P_3 = (0,0)$?

Solution. We can easily determine the rate of change of f at the point P_0 in any direction once we know the gradient $\nabla f(1,2) = a\,\hat{\imath} + b\,\hat{\jmath}$. So we will first use the two given rates of change to determine a and b, and then we determine the rate of change towards $(0,0)$.

The two rates of change that we are given are those in the directions of the vectors

$$\overrightarrow{P_0P_1} = \langle 1, 1\rangle \qquad \overrightarrow{P_0P_2} = \langle 0, -2\rangle$$

As you might guess, the notation $\overrightarrow{PQ}$ means the vector whose tail is at P and whose head is at Q. So the given rates of change tell us that

$$\begin{aligned} 2\sqrt{2} = D_{\frac{\langle 1,1\rangle}{|\langle 1,1\rangle|}} f(1,2) &= \nabla f(1,2)\cdot\frac{\langle 1,1\rangle}{|\langle 1,1\rangle|} = \langle a,b\rangle\cdot\frac{\langle 1,1\rangle}{\sqrt{2}} = \frac{a}{\sqrt{2}}+\frac{b}{\sqrt{2}} \\ -3 = D_{\frac{\langle 0,-2\rangle}{|\langle 0,-2\rangle|}} f(1,2) &= \nabla f(1,2)\cdot\frac{\langle 0,-2\rangle}{|\langle 0,-2\rangle|} = \langle a,b\rangle\cdot\frac{\langle 0,-2\rangle}{2} = -b \end{aligned}$$

These two lines give us two linear equations in the two unknowns a and b. The second equation directly gives us $b = 3$. Substituting $b = 3$ into the first equation gives

$$\frac{a}{\sqrt{2}}+\frac{3}{\sqrt{2}} = 2\sqrt{2} \implies a+3=4 \implies a=1$$

A direction vector from $P_0 = (1,2)$ towards $P_3 = (0,0)$ is

$$\overrightarrow{P_0P_3} = \langle -1, -2\rangle$$

and the rate of change (per unit distance) in that direction is

$$D_{\frac{\langle -1,-2\rangle}{|\langle -1,-2\rangle|}} f(1,2) = \nabla f(1,2)\cdot\frac{\langle -1,-2\rangle}{|\langle -1,-2\rangle|} = \langle a,b\rangle\cdot\frac{\langle -1,-2\rangle}{\sqrt{5}} = \langle 1,3\rangle\cdot\frac{\langle -1,-2\rangle}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$$

Example 2.7.10

Example 2.7.11 (Optional)

Problem: Find all points (a, b, c) for which the spheres $(x-a)^2+(y-b)^2+(z-c)^2 = 1$ and $x^2+y^2+z^2 = 1$ intersect orthogonally. That is, the tangent planes to the two spheres are to be perpendicular at each point of intersection.

Solution. Let (x_0, y_0, z_0) be a point of intersection. That is

$$\begin{aligned} (x_0-a)^2+(y_0-b)^2+(z_0-c)^2 &= 1 \\ x_0^2+y_0^2+z_0^2 &= 1 \end{aligned}$$

A normal vector to $G(x, y, z) = (x-a)^2+(y-b)^2+(z-c)^2 = 1$ at (x_0, y_0, z_0) is

$$\mathbf{N} = \nabla G(x_0, y_0, z_0) = \langle 2(x_0-a)\,,\, 2(y_0-b)\,,\, 2(z_0-c)\rangle$$

A normal vector to $g(x, y, z) = x^2+y^2+z^2 = 1$ at (x_0, y_0, z_0) is

$$\mathbf{n} = \nabla g(x_0, y_0, z_0) = \langle 2x_0\,,\, 2y_0\,,\, 2z_0\rangle$$

The two tangent planes are perpendicular if and only if $\hat{\mathbf{N}}$ and $\hat{\mathbf{n}}$ are perpendicular, which is the case if and only if

$$0 = \hat{\mathbf{N}} \cdot \hat{\mathbf{n}} = 4x_0(x_0 - a) + 4y_0(y_0 - b) + 4z_0(z_0 - c)$$

or, dividing the equation by 4,

$$x_0(x_0 - a) + y_0(y_0 - b) + z_0(z_0 - c) = 0$$

Let's pause to take stock. We need to find all (a, b, c)'s such that the statement

$$(x_0, y_0, z_0) \text{ is a point of intersection of the two spheres} \tag{S1}$$

implies the statement

$$\text{the normal vectors } \hat{\mathbf{N}} \text{ and } \hat{\mathbf{n}} \text{ are perpendicular} \tag{S2}$$

In equations, we need to find all (a, b, c)'s such that the statement

$$(x_0, y_0, z_0) \quad \text{obeys} \quad x_0^2 + y_0^2 + z_0^2 = 1 \text{ and } (x_0 - a)^2 + (y_0 - b)^2 + (z_0 - c)^2 = 1 \tag{S1}$$

implies the statement

$$(x_0, y_0, z_0) \quad \text{obeys} \quad x_0(x_0 - a) + y_0(y_0 - b) + z_0(z_0 - c) = 0 \tag{S2}$$

Now if we expand (S2) then we can, with a little care, massage it into something that looks more like (S1).

$$\begin{aligned} x_0(x_0 - a) + y_0(y_0 - b) + z_0(z_0 - c) &= x_0^2 + y_0^2 + z_0^2 - ax_0 - by_0 - cz_0 \\ &= \frac{1}{2}\Big\{\big[x_0^2 + y_0^2 + z_0^2\big] + \big[(x_0 - a)^2 + (y_0 - b)^2 + (z_0 - c)^2\big] - a^2 - b^2 - c^2\Big\} \end{aligned}$$

If (S1) is true, then $\big[x_0^2 + y_0^2 + z_0^2\big] = 1$ and $\big[(x_0 - a)^2 + (y_0 - b)^2 + (z_0 - c)^2\big] = 1$ so that

$$x_0(x_0 - a) + y_0(y_0 - b) + z_0(z_0 - c) = \frac{1}{2}\Big\{1 \ + \ 1 \ - a^2 - b^2 - c^2\Big\}$$

and statement (S2) is true if and only if

$$a^2 + b^2 + c^2 = 2$$

Our conclusion is that the set of allowed points (a, b, c) is the sphere of radius $\sqrt{2}$ centred on the origin.

Example 2.7.11

Example 2.7.12 (Optional — The gradient in polar coordinates)

Question: What is the gradient of a function in polar coordinates?

Answer. As was the case in Examples 2.4.9 and 2.4.10, figuring out what the question is asking is half the battle. By Definition 2.5.4, the gradient of a function $g(x, y)$ is the vector

$\big\langle g_x(x,y), g_y(x,y)\big\rangle$. In this question we are told that we are given some function $f(r,\theta)$ of the polar coordinates[36] r and θ. We are supposed to convert this function to Cartesian coordinates.

This means that we are to consider the function

$$g(x,y) = f\big(r(x,y), \theta(x,y)\big)$$

with

$$\begin{aligned} r(x,y) &= \sqrt{x^2+y^2} \\ \theta(x,y) &= \arctan\frac{y}{x} \end{aligned}$$

Then we are to compute the gradient of $g(x,y)$ and express the answer in terms of r and θ. By the chain rule,

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial f}{\partial r}\,\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\,\frac{\partial \theta}{\partial x} \\ &= \frac{\partial f}{\partial r}\,\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \theta}\,\frac{-y/x^2}{1+(y/x)^2} \\ &= \frac{\partial f}{\partial r}\,\frac{x}{\sqrt{x^2+y^2}} - \frac{\partial f}{\partial \theta}\,\frac{y}{x^2+y^2} \\ &= \frac{\partial f}{\partial r}\,\frac{r\cos\theta}{r} - \frac{\partial f}{\partial \theta}\,\frac{r\sin\theta}{r^2} \end{aligned}$$

since $x = r\cos\theta$ and $y = r\sin\theta$

$$= \frac{\partial f}{\partial r}\,\cos\theta - \frac{\partial f}{\partial \theta}\,\frac{\sin\theta}{r}$$

Similarly

$$\begin{aligned} \frac{\partial g}{\partial y} &= \frac{\partial f}{\partial r}\,\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\,\frac{\partial \theta}{\partial y} \\ &= \frac{\partial f}{\partial r}\,\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \theta}\,\frac{1/x}{1+(y/x)^2} \\ &= \frac{\partial f}{\partial r}\,\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial \theta}\,\frac{x}{x^2+y^2} \\ &= \frac{\partial f}{\partial r}\,\sin\theta + \frac{\partial f}{\partial \theta}\,\frac{\cos\theta}{r} \end{aligned}$$

So

$$\big\langle g_x, g_y\big\rangle = f_r\ \langle\cos\theta, \sin\theta\rangle + \frac{f_\theta}{r}\,\langle -\sin\theta, \cos\theta\rangle$$

or, with all the arguments written explicitly,

$$\begin{aligned} \big\langle g_x(x,y), g_y(x,y)\big\rangle &= f_r\big(r(x,y),\theta(x,y)\big)\ \langle\cos\theta(x,y)\,,\,\sin\theta(x,y)\rangle \\ &\qquad + \frac{1}{r(x,y)} f_\theta\big(r(x,y),\theta(x,y)\big)\ \langle -\sin\theta(x,y)\,,\,\cos\theta(x,y)\rangle \end{aligned}$$

36 Polar coordinates were defined in Example 2.1.8.

Example 2.7.12

2.8▲ A First Look at Partial Differential Equations

Many phenomena are modelled by equations that relate the rates of change of various quantities. As rates of change are given by derivatives, the resulting equations contain derivatives and so are called differential equations. We saw a number of such differential equations in §2.4 of the CLP-2 text. In particular, a partial differential equation is an equation for an unknown function of two or more variables that involves the partial derivatives of the unknown function. The standard acronym for partial differential equation is PDE. PDEs[37] play a central role in modelling a huge number of different phenomena. Here is a table giving a bunch of named PDEs and what they are used for. It is far from complete.

Maxwell's equations	describes electromagnetic radiation
Navier–Stokes equations	describes fluid motion
Heat equation	describes heat flow
Wave equation	describes wave motion
Schrödinger equation	describes atoms, molecules and crystals
Black–Scholes equation	used for pricing stock options
Einstein's equations	connects gravity and geometry
Laplace's equation	used in many applications, including electrostatics

We are just going to scratch the surface of the study of partial of differential equations. Many of you will take a separate course on the subject. Some very important PDEs are very hard. One of the millon U.S. dollar prizes[38] announced in 2000 by the Clay Institute concerns the Navier-Stokes equations. On the other hand, we already know enough to accomplish some PDE tasks. In particular, we can check if a given function really does satisfy a given PDE. Here are some examples.

Example 2.8.1 $\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0\right)$

Show that the function $z(x,y) = \frac{x+y}{x-y}$ obeys

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

37 There is a divided community on what the plural of PDE should be. Most people use PDEs as the plural. But some people use PDE as its own plural.
38 See https://www.claymath.org/millennium-problems or https://en.wikipedia.org/wiki/Millennium_Prize_Problems

Solution. We simply evaluate the two terms on the left hand side when $z = z(x,y) = \frac{x+y}{x-y}$.

$$x\frac{\partial z}{\partial x} = x\frac{\partial}{\partial x}\left(\frac{x+y}{x-y}\right) = x\frac{(1)(x-y)-(x+y)(1)}{(x-y)^2} = \frac{-2xy}{(x-y)^2}$$
$$y\frac{\partial z}{\partial y} = y\frac{\partial}{\partial y}\left(\frac{x+y}{x-y}\right) = y\frac{(1)(x-y)-(x+y)(-1)}{(x-y)^2} = \frac{2xy}{(x-y)^2}$$

So

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{-2xy}{(x-y)^2} + \frac{2xy}{(x-y)^2} = 0$$

and $z(x,y) = \frac{x+y}{x-y}$ really does solve the PDE $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

Beware however, that while we have found one solution to the given PDE, we have not found all solutions. There are many others. Trivially, if $z(x,y) = 7$, or any other constant, then we certainly have $x\frac{\partial z}{\partial x} = 0$ and $y\frac{\partial z}{\partial y} = 0$ so that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$. Less trivially, in the next example, we'll find a ton[39][40] of solutions.

Example 2.8.1

Example 2.8.2 $\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0, \text{ again}\right)$

Let $G(u)$ be any differentiable function. Show that the function $z(x,y) = G\left(\frac{y}{x}\right)$ obeys

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

for all $x \neq 0$.

Solution. We again simply evaluate the two terms on the left hand side when $z = z(x,y) = G\left(\frac{y}{x}\right)$. By the chain rule

$$x\frac{\partial z}{\partial x} = x\frac{\partial}{\partial x}\left(G\left(\frac{y}{x}\right)\right) = xG'\left(\frac{y}{x}\right)\left(\frac{\partial}{\partial x}\frac{y}{x}\right) = xG'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = -G'\left(\frac{y}{x}\right)\frac{y}{x}$$
$$y\frac{\partial z}{\partial y} = y\frac{\partial}{\partial y}\left(G\left(\frac{y}{x}\right)\right) = yG'\left(\frac{y}{x}\right)\left(\frac{\partial}{\partial y}\frac{y}{x}\right) = yG'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = G'\left(\frac{y}{x}\right)\frac{y}{x}$$

So

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -G'\left(\frac{y}{x}\right)\frac{y}{x} + G'\left(\frac{y}{x}\right)\frac{y}{x} = 0$$

39 Or, if you prefer, we will find 1.01605 tonnes of solutions. Although the authors of this text believe strongly in the supremacy of the modern metric system over the archaic chaos of imperial units, they are less certain of the appropriateness of revising well established colloquialisms. It is not at all clear that rewriting "I have a ton of work to do" as "I have a tonne of work to do" achieves very much except to give the impression that the author is wasting time adding two letters when they are expressing the sheer quantity of tasks that require their attention. Speaking of which, the authors should end this footnote, and get on with the next example.

40 In the previous footnote, the authors, writing from Canada, are using imperial tons rather than U.S. tons. The interested reader is invited to proceed to their favourite search engine to discover just how much time they can waste investigating the history, similarities and differences of these systems.

and $z(x,y) = G\big(\frac{y}{x}\big)$ really does solve the PDE $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$. Note that we can rewrite the solution $\frac{x+y}{x-y}$ of Example 2.8.1 as $\frac{1+y/x}{1-y/x}$, which is of the form $G\big(\frac{y}{x}\big)$.

Example 2.8.2

Example 2.8.3 (Harmonic)

A function $u(x,y)$ is said to be harmonic if it satisfies Laplace's equation

$$u_{xx} + u_{yy} = 0$$

We will now find all harmonic polynomials (in the variables x and y) of degree at most two. Any polynomial of degree at most two is of the form

$$u(x,y) = a + bx + cy + \alpha x^2 + \beta xy + \gamma y^2$$

for some constants a, b, c, α, β, γ. We will need u_{xx} and u_{yy}, so we compute them now.

$$\begin{aligned} u(x,y) &= a + bx + cy + \alpha x^2 + \beta xy + \gamma y^2 \\ u_x(x,y) &= b + 2\alpha x + \beta y \\ u_{xx}(x,y) &= 2\alpha \\ u_y(x,y) &= c + \beta x + 2\gamma y \\ u_{yy}(x,y) &= 2\gamma \end{aligned}$$

The polynomial $u(x,y)$ is harmonic if and only if

$$0 = u_{xx}(x,y) + u_{yy}(x,y) = 2\alpha + 2\gamma$$

So the polynomial $u(x,y)$ is harmonic if and only if $\alpha + \gamma = 0$, i.e. if and only if the polynomial is of the form

$$u(x,y) = a + \overbrace{bx + cy}^{\text{degree 1}} + \overbrace{\alpha(x^2 - y^2)\ +\ \beta xy}^{\text{degree 2}}$$

with a, b, c, α, and β all constants. Notice that since both terms in the equation involve a second derivative, we would not expect there to be any conditions on the constant and linear terms. There aren't. Beware that, while we have found all harmonic degree-two polynomials, there are many other *harmonic functions*, like, for example $e^x \cos y$.

Example 2.8.3

2.8.1 ▸▸ Optional — Solving the Advection and Wave Equations

In this section we consider

$$\frac{\partial^2 w}{\partial x^2}(x,t) - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}(x,t) = 0$$

This is an extremely important[41]partial differential equation called the "wave equation" (in one spatial dimension) that is used in modelling water waves, sound waves, seismic waves, light waves and so on. The reason that we are looking at it here is that we can use what we have just learned to see that its solutions are waves travelling with speed c.

To start, we'll use gradients and the chain rule to find the solution of the slightly simpler equation

$$\frac{\partial w}{\partial x}(x,t) - \frac{1}{c}\frac{\partial w}{\partial t}(x,t) = 0$$

which is called an advection equation. By way of motivation for what will follow, note that

- we can rewrite the above equation as

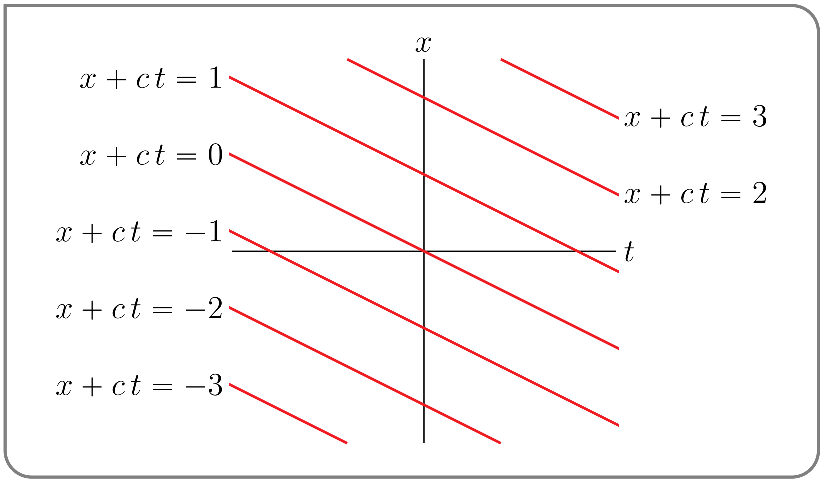
$$\left\langle 1\,,\,-\frac{1}{c}\right\rangle \cdot \nabla w(x,t) = 0$$

- This equation tells that the gradient of any solution $w(x,t)$ must always be perpendicular to the constant vector $\langle 1\,,\,-\frac{1}{c}\rangle$.
- A vector $\langle a,b\rangle$ is perpendicular to $\langle 1\,,\,-\frac{1}{c}\rangle$ if and only if

$$\langle a,b\rangle \cdot \left\langle 1\,,\,-\frac{1}{c}\right\rangle = 0 \iff a - \frac{b}{c} = 0 \iff b = ac \iff \langle a,b\rangle = a\,\langle 1,c\rangle$$

  That is, a vector is perpendicular to $\langle 1\,,\,-\frac{1}{c}\rangle$ if and only if it is parallel to $\langle 1,c\rangle$.
- Thus the gradient of any solution $w(x,t)$ must always be parallel to the constant vector $\langle 1\,,\,c\rangle$.
- Recall that one of our implications following Definition 2.7.5 is that the gradient of $w(x,t)$ must always be perpendicular to the level curves of w.
- So the level curves of $w(x,t)$ are always perpendicular to the constant vector $\langle 1\,,\,c\rangle$. They must be straight lines with equations of the form

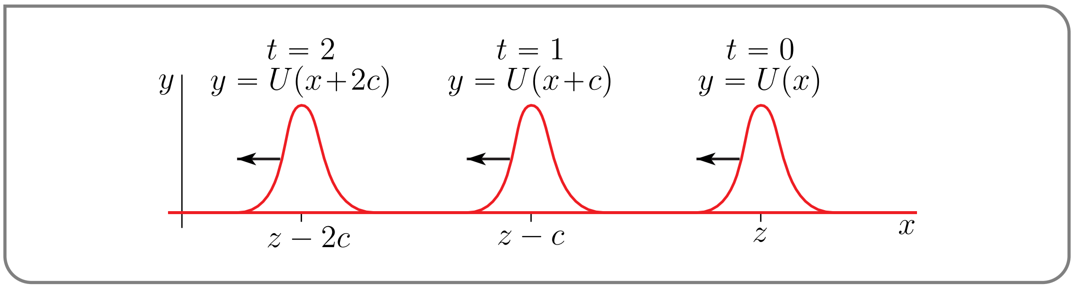
$$\langle 1\,,\,c\rangle \cdot \langle x - x_0\,,\,t - t_0\rangle = 0 \qquad \text{or} \qquad x + ct = u \quad \text{with } u \text{ a constant}$$



41 If you plug "wave equation" into your favourite search engine you will get more than a million hits.

- That is, for each constant u, $w(x,t)$ takes the same value at each point of the straight line $x+ct=u$. Call that value $U(u)$. So $w(x,t)=U(u)=U(x+ct)$ for some function U.

This solution represents a wave packet moving to the left with speed c. You can see this by observing that all points (x,t) in space-time for which $x+ct$ takes the same fixed value, say z, have the same value of $U(x+ct)$, namely $U(z)$. So if you move so that your position at time t is $x=z-ct$ (i.e. move the left with speed c) you always see the same value of w. The figure below illustrates this. It contains the graphs of $U(x)$, $U(x+c)=U(x+ct)\big|_{t=1}$ and $U(x+2c)=U(x+ct)\big|_{t=2}$ for a bump shaped $U(x)$. In the figure the location of the tick z on the x-axis was chosen so that so that $U(z)=\max_x U(x)$.



The above argument that lead to the solution $w(x,t)=U(x+ct)$ was somewhat hand-wavy. But we can easily turn it into a much tighter argument by simply changing variables from (x,y) to (u,v) with $u=x+ct$. It doesn't much matter what we choose (within reason) for the new variable v. Let's take $v=x-ct$. Then $x=\frac{u+v}{2}$ and $t=\frac{u-v}{2c}$ and it is easy to translate back and forth between x,t and u,v.

Now define the function $W(u,v)$ by

$$w(x,t)=W(x+ct\,,\,x-ct)$$

By the chain rule

$$\begin{aligned}
\frac{\partial w}{\partial x}(x,t) &= \frac{\partial}{\partial x}\big[W(x+ct\,,\,x-ct)\big]\\
&= \frac{\partial W}{\partial u}(x+ct\,,\,x-ct)\frac{\partial}{\partial x}(x+ct)+\frac{\partial W}{\partial v}(x+ct\,,\,x-ct)\frac{\partial}{\partial x}(x-ct)\\
&= \frac{\partial W}{\partial u}(x+ct\,,\,x-ct)+\frac{\partial W}{\partial v}(x+ct\,,\,x-ct)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial w}{\partial t}(x,t) &= \frac{\partial}{\partial t}\big[W(x+ct\,,\,x-ct)\big]\\
&= \frac{\partial W}{\partial u}(x+ct\,,\,x-ct)\frac{\partial}{\partial t}(x+ct)+\frac{\partial W}{\partial v}(x+ct\,,\,x-ct)\frac{\partial}{\partial t}(x-ct)\\
&= \frac{\partial W}{\partial u}(x+ct\,,\,x-ct)\times c+\frac{\partial W}{\partial v}(x+ct\,,\,x-ct)\times(-c)
\end{aligned}$$

Subtracting $\frac{1}{c}$ times the second equation from the first equation gives

$$\frac{\partial w}{\partial x}(x,t)-\frac{1}{c}\frac{\partial w}{\partial t}(x,t)=2\frac{\partial W}{\partial v}(x+ct\,,\,x-ct)$$

So

$$w(x,t) \text{ obeys the equation } \frac{\partial w}{\partial x}(x,t) - \frac{1}{c}\frac{\partial w}{\partial t}(x,t) = 0 \text{ for all } x \text{ and } t$$

if and only if

$$W(u,v) \text{ obeys the equation } \frac{\partial W}{\partial v}(x+ct\,,\,x-ct) = 0 \text{ for all } x \text{ and } t,$$

which, substituting in $x = \frac{u+v}{2}$ and $t = \frac{u-v}{2c}$, is the case if and only if

$$W(u,v) \text{ obeys the equation } \frac{\partial W}{\partial v}(u\,,\,v) = 0 \text{ for all } u \text{ and } v$$

The equation $\frac{\partial W}{\partial v}(u\,,\,v) = 0$ means that $W(u,v)$ is independent of v, so that $W(u,v)$ is of the form $W(u,v) = U(u)$, for some function U, and, so finally,

$$w(x,t) = W(x+ct\,,\,x-ct) = U(x+ct)$$

Now that we have solved our toy equation, let's move on to the 1d wave equation.

Example 2.8.4 (Wave Equation)

We'll now expand the above argument to find the general solution to

$$\frac{\partial^2 w}{\partial x^2}(x,t) - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}(x,t) = 0$$

We'll again make the change of variables from (x,y) to (u,v) with $u = x+ct$ and $v = x-ct$ and again define the function $W(u,v)$ by

$$w(x,t) = W(x+ct\,,\,x-ct)$$

By the chain rule, we still have

$$\begin{aligned}
\frac{\partial w}{\partial x}(x,t) &= \frac{\partial}{\partial x}\big[W(x+ct\,,\,x-ct)\big] = \frac{\partial W}{\partial u}(x+ct\,,\,x-ct) + \frac{\partial W}{\partial v}(x+ct\,,\,x-ct)\\
\frac{\partial w}{\partial t}(x,t) &= \frac{\partial}{\partial t}\big[W(x+ct\,,\,x-ct)\big] = \frac{\partial W}{\partial u}(x+ct\,,\,x-ct)\times c + \frac{\partial W}{\partial v}(x+ct\,,\,x-ct)\times(-c)
\end{aligned}$$

We now need to differentiate a second time. Write $W_1(u,v) = \frac{\partial W}{\partial u}(u,v)$ and $W_2(u,v) = \frac{\partial W}{\partial v}(u,v)$ so that

$$\begin{aligned}
\frac{\partial w}{\partial x}(x,t) &= W_1(x+ct\,,\,x-ct) + W_2(x+ct\,,\,x-ct)\\
\frac{\partial w}{\partial t}(x,t) &= c\,W_1(x+ct\,,\,x-ct) - c\,W_2(x+ct\,,\,x-ct)
\end{aligned}$$

Using the chain rule again

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2}(x,t) &= \frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x}(x,t)\right] \\
&= \frac{\partial}{\partial x}\left[W_1(x+ct\,,\,x-ct)\right] + \frac{\partial}{\partial x}\left[W_2(x+ct\,,\,x-ct)\right] \\
&= \frac{\partial W_1}{\partial u} + \frac{\partial W_1}{\partial v} \qquad + \frac{\partial W_2}{\partial u} + \frac{\partial W_2}{\partial v} \\
&= \frac{\partial^2 W}{\partial u^2} + \frac{\partial^2\, W}{\partial v\, \partial u} \qquad + \frac{\partial^2\, W}{\partial u\, \partial v} + \frac{\partial^2 W}{\partial v^2} \\
\frac{\partial^2 w}{\partial t^2}(x,t) &= \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}(x,t)\right] \\
&= c\frac{\partial}{\partial t}\left[W_1(x+ct\,,\,x-ct)\right] - c\frac{\partial}{\partial t}\left[W_2(x+ct\,,\,x-ct)\right] \\
&= c^2\frac{\partial W_1}{\partial u} - c^2\frac{\partial W_1}{\partial v} \qquad - c^2\frac{\partial W_2}{\partial u} + c^2\frac{\partial W_2}{\partial v} \\
&= c^2\frac{\partial^2 W}{\partial u^2} - c^2\frac{\partial^2 W}{\partial v \partial u} \qquad - c^2\frac{\partial^2 W}{\partial u \partial v} + c^2\frac{\partial^2 W}{\partial v^2}
\end{aligned}$$

with all of the functions on the right hand sides having arguments $(x+ct\,,\,x-ct)$. So, subtracting $\frac{1}{c^2}$ times the second from the first, we get

$$\frac{\partial^2 w}{\partial x^2}(x,t) - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}(x,t) = 4\frac{\partial^2 W}{\partial u \partial v}(x+ct\,,\,x-ct)$$

and $w(x,t)$ obeys $\frac{\partial^2 w}{\partial x^2}(x,t) - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}(x,t) = 0$ for all x and t if and only if

$$\frac{\partial^2 W}{\partial u \partial v}(u\,,\,v) = 0$$

for all u and v.

- This tells us that the u-derivative of $\frac{\partial W}{\partial v}$ is zero, so that $\frac{\partial W}{\partial v}$ is independent of u. That is $\frac{\partial W}{\partial v}(u,v) = \tilde{V}(v)$ for some function $\tilde{V}$. The reason that we have called it $\tilde{V}$ instead of V with become evident shortly.
- Recall that to apply $\frac{\partial}{\partial v}$, you treat u as a constant and differentiate with respect to v.
- So $\frac{\partial W}{\partial v}(u,v) = \tilde{V}(v)$ says that, when u is thought of as a constant, W is an antiderivative of $\tilde{V}$.
- That is, $W(u,v) = \int \tilde{V}(v)\,\mathrm{d}v + U$, with U being an arbitrary constant. As u is being thought of as a constant, U is allowed to depend on u.

So, denoting by V any antiderivative of $\tilde{V}$, we can write our solution in a very neat form.

$$W(u,v) = U(u) + V(v)$$

and the function we want is[42]

$$w(x,t) = W(x+ct\,,\ x-ct) = U(x+ct) + V(x-ct)$$

As we saw above $U(x+ct)$ represents a wave packet moving to the left with speed c. Similarly, $V(x-ct)$ represents a wave packet moving to the right with speed c.

Notice that $w(x,t) = U(x+ct) + V(x-ct)$ is a solution regardless of what U and V are. The differential equation cannot tell us what U and V are. To determine them, we need more information about the system — usually in the form of initial conditions, like $w(x,0) = \cdots$ and $\frac{\partial w}{\partial t}(x,0) = \cdots$. General techniques for solving partial differential equations lie beyond this text — but definitely require a good understanding of multivariable calculus. A good reason to keep on reading!

Example 2.8.4

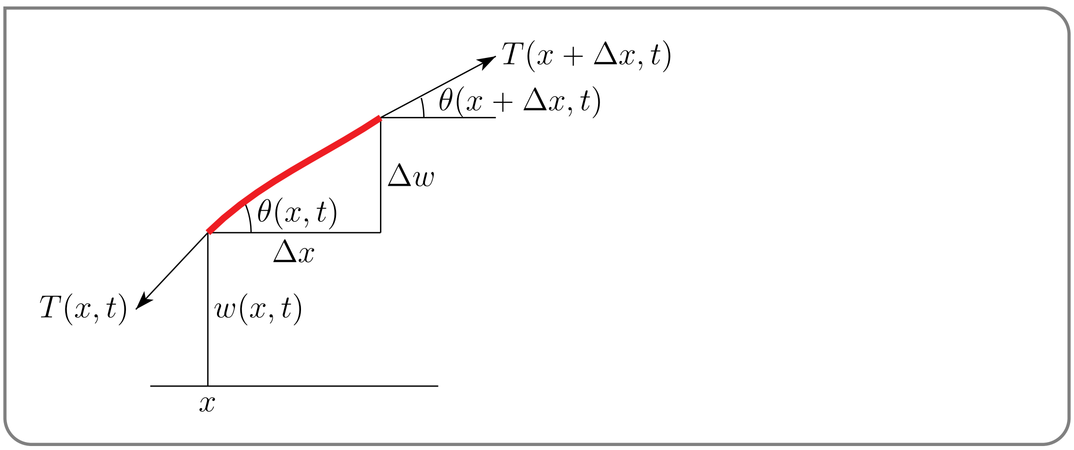
2.8.2 ▸▸ Really Optional — Derivation of the Wave Equation

In this section we derive the wave equation

$$\frac{\partial^2 w}{\partial x^2}(x,t) - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}(x,t) = 0$$

in one application. To be precise, we apply Newton's law to an elastic string, and conclude that small amplitude transverse vibrations of the string obey the wave equation.

Here is a sketch of a tiny element of the string.



The basic notation that we will use (most of which appears in the sketch) is

$$\begin{aligned} w(x,t) &= \text{vertical displacement of the string from the } x \text{ axis at position } x \text{ and time } t \\ \theta(x,t) &= \text{angle between the string and a horizontal line at position } x \text{ and time } t \\ T(x,t) &= \text{tension in the string at position } x \text{ and time } t \\ \rho(x) &= \text{mass density (per unit length) of the string at position } x \end{aligned}$$

The forces acting on the tiny element of string at time t are

42 This is known as d'Alembert's form of the solution. It is named after Jean le Rond d'Alembert, 1717–1783, who was a French mathematician, physicist, philosopher and music theorist.

(a) tension pulling to the right, which has magnitude $T(x+\Delta x, t)$ and acts at an angle $\theta(x+\Delta x, t)$ above horizontal

(b) tension pulling to the left, which has magnitude $T(x,t)$ and acts at an angle $\theta(x,t)$ below horizontal and, possibly,

(c) various external forces, like gravity. We shall assume that all of the external forces act vertically and we shall denote by $F(x,t)\Delta x$ the net magnitude of the external force acting on the element of string.

The length of the element of string is essentially $\sqrt{\Delta x^2 + \Delta w^2}$ so that the mass of the element of string is essentially $\rho(x)\sqrt{\Delta x^2 + \Delta w^2}$ and the vertical component of Newton's law $\mathbf{F} = m\mathbf{a}$ says that

$$\rho(x)\sqrt{\Delta x^2 + \Delta w^2}\,\frac{\partial^2 w}{\partial t^2}(x,t) = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) + F(x,t)\Delta x$$

Dividing by Δx and taking the limit as $\Delta x \to 0$ gives

$$\begin{aligned}\rho(x)\sqrt{1+\left(\frac{\partial w}{\partial x}\right)^2}\,\frac{\partial^2 w}{\partial t^2}(x,t) &= \frac{\partial}{\partial x}\big[T(x,t)\sin\theta(x,t)\big] + F(x,t)\\ &= \frac{\partial T}{\partial x}(x,t)\sin\theta(x,t) + T(x,t)\cos\theta(x,t)\,\frac{\partial\theta}{\partial x}(x,t) + F(x,t)\end{aligned} \tag{E1}$$

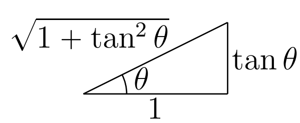
We can dispose of all the θ's by observing from the figure above that

$$\tan\theta(x,t) = \lim_{\Delta x\to 0}\frac{\Delta w}{\Delta x} = \frac{\partial w}{\partial x}(x,t)$$

which implies, using the figure on the right below, that

$$\sin\theta(x,t) = \frac{\frac{\partial w}{\partial x}(x,t)}{\sqrt{1+\big(\frac{\partial w}{\partial x}(x,t)\big)^2}} \qquad \cos\theta(x,t) = \frac{1}{\sqrt{1+\big(\frac{\partial w}{\partial x}(x,t)\big)^2}}$$

$$\theta(x,t) = \arctan\frac{\partial w}{\partial x}(x,t) \qquad \frac{\partial\theta}{\partial x}(x,t) = \frac{\frac{\partial^2 w}{\partial x^2}(x,t)}{1+\big(\frac{\partial w}{\partial x}(x,t)\big)^2}$$



Substituting these formulae into (E1) give a horrendous mess. However, we can get considerable simplification by looking only at small vibrations. By a small vibration, we mean that $|\theta(x,t)| \ll 1$ for all x and t. This implies that $|\tan\theta(x,t)| \ll 1$, hence that $\big|\frac{\partial w}{\partial x}(x,t)\big| \ll 1$ and hence that

$$\sqrt{1+\left(\frac{\partial w}{\partial x}\right)^2} \approx 1 \qquad \sin\theta(x,t) \approx \frac{\partial w}{\partial x}(x,t) \qquad \cos\theta(x,t)\approx 1 \qquad \frac{\partial\theta}{\partial x}(x,t) \approx \frac{\partial^2 w}{\partial x^2}(x,t) \tag{E2}$$

Substituting these into equation (E1) give

$$\rho(x)\frac{\partial^2 w}{\partial t^2}(x,t) = \frac{\partial T}{\partial x}(x,t)\frac{\partial w}{\partial x}(x,t) + T(x,t)\,\frac{\partial^2 w}{\partial x^2}(x,t) + F(x,t) \tag{E3}$$

which is indeed relatively simple, but still exhibits a problem. This is one equation in the two unknowns w and T.

Fortunately there is a second equation lurking in the background, that we haven't used yet. Namely, the horizontal component of Newton's law of motion. As a second simplification, we assume that there are only transverse vibrations. That is, our tiny string element moves only vertically. Then the net horizontal force on it must be zero. That is,

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t)-T(x,t)\cos\theta(x,t)=0$$

Dividing by Δx and taking the limit as Δx tends to zero gives

$$\frac{\partial}{\partial x}\big[T(x,t)\cos\theta(x,t)\big]=0$$

Thus $T(x,t)\cos\theta(x,t)$ is independent of x. For small amplitude vibrations, $\cos\theta$ is very close to one, for all x. So T is a function of t only, which is determined by how hard you are pulling on the ends of the string at time t. So for small, transverse vibrations, (E3) simplifies further to

$$\rho(x)\frac{\partial^2 w}{\partial t^2}(x,t)=T(t)\,\frac{\partial^2 w}{\partial x^2}(x,t)+F(x,t) \tag{E4}$$

In the event that the string density ρ is a constant, independent of x, the string tension $T(t)$ is a constant independent of t (in other words you are not continually playing with the tuning pegs) and there are no external forces F we end up with the wave equation

$$\frac{\partial^2 w}{\partial t^2}(x,t)=c^2\,\frac{\partial^2 w}{\partial x^2}(x,t)\qquad\text{where}\qquad c=\sqrt{\frac{T}{\rho}}$$

as desired.

The equation that is called the wave equation has built into it a lot of approximations. By going through the derivation, we have seen what those approximations are, and we can get some idea as to when they are applicable.

2.9▲ Maximum and Minimum Values

One of the core topics in single variable calculus courses is finding the maxima and minima of functions of one variable. We'll now extend that discussion to functions of more than one variable[43]. Rather than leaping into the deep end, we'll not be too ambitious and concentrate on functions of two variables. That being said, many of the techniques work more generally. To start, we have the following natural extensions to some familiar definitions.

43 Life is not (always) one-dimensional and sometimes we have to embrace it.

Definition 2.9.1.

Let the function $f(x,y)$ be defined for all (x,y) in some subset R of $\mathbb{R}^2$. Let (a,b) be a point in R.

- (a,b) is a *local maximum* of $f(x,y)$ if $f(x,y) \leqslant f(a,b)$ for all (x,y) close to (a,b). More precisely, (a,b) is a local maximum of $f(x,y)$ if there is an $r > 0$ such that $f(x,y) \leqslant f(a,b)$ for all points (x,y) within a distance r of (a,b).
- (a,b) is a *local minimum* of $f(x,y)$ if $f(x,y) \geqslant f(a,b)$ for all (x,y) close to (a,b).
- Local maximum and minimum values are also called extremal values.
- (a,b) is an *absolute maximum* or *global maximum* of $f(x,y)$ if $f(x,y) \leqslant f(a,b)$ for all (x,y) in R.
- (a,b) is an *absolute minimum* or *global minimum* of $f(x,y)$ if $f(x,y) \geqslant f(a,b)$ for all (x,y) in R.

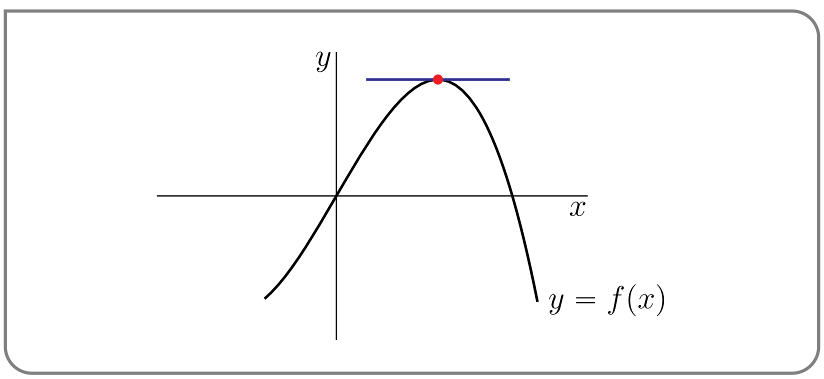
▸▸ Local Maxima and Minima

One of the first things you did when you were developing the techniques used to find the maximum and minimum values of $f(x)$ was ask yourself[44]

> Suppose that the largest value of $f(x)$ is $f(a)$. What does that tell us about a?

After a little thought you answered

> If the largest value of $f(x)$ is $f(a)$ and f is differentiable at a, then $f'(a) = 0$.



Let's recall why that's true. Suppose that the largest value of $f(x)$ is $f(a)$. Then for all $h > 0$,

$$f(a+h) \leqslant f(a) \implies f(a+h) - f(a) \leqslant 0 \implies \frac{f(a+h) - f(a)}{h} \leqslant 0 \quad \text{if } h > 0$$

44 Or perhaps your instructor asked you.

Taking the limit $h \to 0$ tells us that $f'(a) \leqslant 0$. Similarly[45], for all $h < 0$,

$$f(a+h) \leqslant f(a) \implies f(a+h) - f(a) \leqslant 0 \implies \frac{f(a+h) - f(a)}{h} \geqslant 0 \quad \text{if } h < 0$$

Taking the limit $h \to 0$ now tells us that $f'(a) \geqslant 0$. So we have both $f'(a) \geqslant 0$ and $f'(a) \leqslant 0$ which forces $f'(a) = 0$.

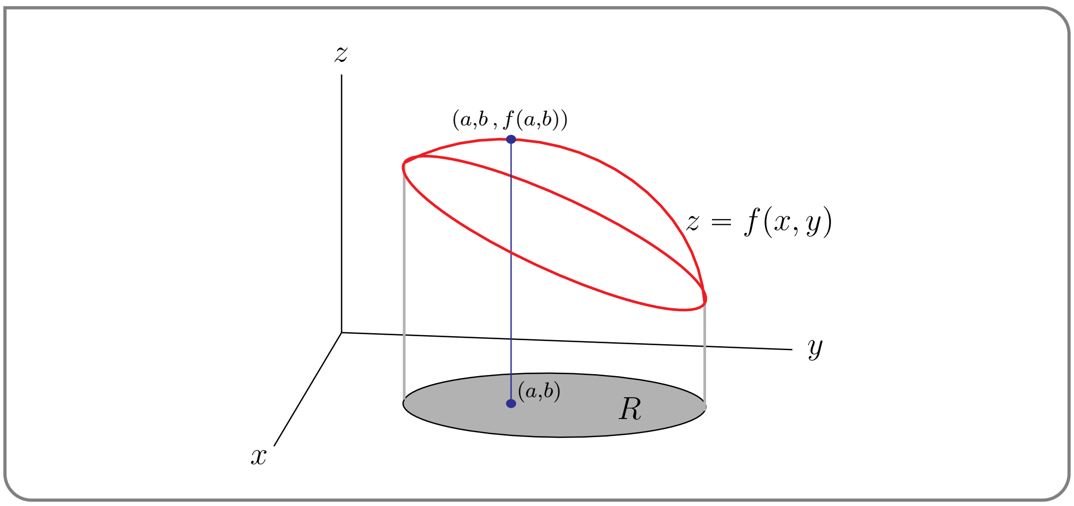
You also observed at the time that for this argument to work, you only need $f(x) \leqslant f(a)$ for all x's close to a, not necessarily for all x's in the whole world. (In the above inequalities, we only used $f(a+h)$ with h small.) Since we care only about $f(x)$ for x near a, we can refine the above statement.

If $f(a)$ is a local maximum for $f(x)$ and f is differentiable at a, then $f'(a) = 0$.

Precisely the same reasoning applies to minima.

If $f(a)$ is a local minimum for $f(x)$ and f is differentiable at a, then $f'(a) = 0$.

Let's use the ideas of the above discourse to extend the study of local maxima and local minima to functions of more than one variable. Suppose that the function $f(x, y)$ is defined for all (x, y) in some subset R of $\mathbb{R}^2$, that (a, b) is point of R that is not on the boundary of R, and that f has a local maximum at (a, b). See the figure below.



Then the function $f(x, y)$ must decrease in value as (x, y) moves away from (a, b) in *any* direction. No matter which direction $\mathbf{d}$ we choose, the directional derivative of f at (a, b) in direction $\mathbf{d}$ must be zero or smaller. Writing this in mathematical symbols, we get

$$D_{\mathbf{d}} f(a, b) = \nabla f(a, b) \cdot \frac{\mathbf{d}}{|\mathbf{d}|} \leqslant 0$$

And the directional derivative of f at (a, b) in the direction $-\mathbf{d}$ also must be zero or negative.

$$D_{-\mathbf{d}} f(a, b) = \nabla f(a, b) \cdot \frac{-\mathbf{d}}{|\mathbf{d}|} \leqslant 0 \quad \text{which implies that} \quad \nabla f(a, b) \cdot \frac{\mathbf{d}}{|\mathbf{d}|} \geqslant 0$$

45 Recall that if $h < 0$ and $A \leqslant B$, then $hA \geqslant hB$. This is because the product of any two negative numbers is positive, so that $h < 0,\ A \leqslant B \implies A - B \leqslant 0 \implies h(A - B) \geqslant 0 \implies hA \geqslant hB$.

As $\nabla f(a,b) \cdot \frac{\mathbf{d}}{|\mathbf{d}|}$ must be both positive (or zero) and negative (or zero) at the same time, it must be zero. In particular, choosing $\mathbf{d} = \hat{\imath}$ forces the x component of $\nabla f(a,b)$ to be zero, and choosing $\mathbf{d} = \hat{\jmath}$ forces the y component of $\nabla f(a,b)$ to be zero. We have thus shown that $\nabla f(a,b) = \mathbf{0}$. The same argument shows that $\nabla f(a,b) = \mathbf{0}$ when (a,b) is a local minimum too. This is an important and useful result, so let's theoremise it.

Theorem 2.9.2.

Let the function $f(x,y)$ be defined for all (x,y) in some subset R of $\mathbb{R}^2$. Assume that

- (a,b) is a point of R that is not on the boundary of R and
- (a,b) is a local maximum or local minimum of f and that
- the partial derivatives of f exist at (a,b).

Then

$$\nabla f(a,b) = \mathbf{0}.$$

Definition 2.9.3.

Let $f(x,y)$ be a function and let (a,b) be a point in its domain. Then

- if $\nabla f(a,b)$ exists and is zero we call (a,b) a critical point (or a stationary point) of the function, and
- if $\nabla f(a,b)$ does not exist then we call (a,b) a singular point of the function.

Warning 2.9.4.

Note that some people (and texts) combine both of these cases and call (a,b) a critical point when either the gradient is zero or does not exist.

Warning 2.9.5.

Theorem 2.9.2 tells us that every local maximum or minimum (in the interior of the domain of a function whose partial derivatives exist) is a critical point. Beware that it does *not*[46] tell us that every critical point is either a local maximum or a local minimum.

46 A very common error of logic that people make is "Affirming the consequent". "If P then Q" is true, does not imply that "If Q then P" is true . The statement "If he is Shakespeare then he is dead" is true. But concluding from "That man is dead" that "He must be Shakespeare" is just silly.

In fact, we shall see later[47], in Examples 2.9.13 and 2.9.15, critical points that are neither local maxima nor a local minima. None-the-less, Theorem 2.9.2 is very useful because often functions have only a small number of critical points. To find local maxima and minima of such functions, we only need to consider its critical and singular points. We'll return later to the question of how to tell if a critical point is a local maximum, local minimum or neither. For now, we'll just practice finding critical points.

Example 2.9.6 $\big(f(x,y) = x^2 - 2xy + 2y^2 + 2x - 6y + 12\big)$

Find all critical points of $f(x,y) = x^2 - 2xy + 2y^2 + 2x - 6y + 12$.

Solution. To find the critical points, we need to find the gradient. To find the gradient we need to find the first order partial derivatives. So, as a preliminary calculation, we find the two first order partial derivatives of $f(x,y)$.

$$\begin{aligned} f_x(x,y) &= 2x - 2y + 2 \\ f_y(x,y) &= -2x + 4y - 6 \end{aligned}$$

So the critical points are the solutions of the pair of equations

$$2x - 2y + 2 = 0 \qquad -2x + 4y - 6 = 0$$

or equivalently (dividing by two and moving the constants to the right hand side)

$$x - y = -1 \tag{E1}$$

$$-x + 2y = 3 \tag{E2}$$

This is a system of two equations in two unknowns (x and y). One strategy for solving a system like this is to

- First use one of the equations to solve for one of the unknowns in terms of the other unknown. For example, (E1) tells us that $y = x + 1$. This expresses y in terms of x. We say that we have solved for y in terms of x.
- Then substitute the result, $y = x + 1$ in our case, into the other equation, (E2). In our case, this gives

$$-x + 2(x+1) = 3 \iff x + 2 = 3 \iff x = 1$$

- We have now found that $x = 1, y = x + 1 = 2$ is the only solution. So the only critical point is $(1,2)$. Of course it only takes a moment to verify that $\nabla f(1,2) = \langle 0, 0\rangle$. It is a good idea to do this as a simple check of our work.

An alternative strategy for solving a system of two equations in two unknowns, like (E1) and (E2), is to

47 And you also saw, for example in Example 3.6.4 of the CLP-1 text, that critical points that are also inflection points are neither local maxima nor local minima.

- add equations (E1) and (E2) together. This gives

$$(E1) + (E2) : \quad (1-1)x + (-1+2)y = -1+3 \iff y = 2$$

  The point here is that adding equations (E1) and (E2) together eliminates the unknown x, leaving us with one equation in the unknown y, which is easily solved. For other systems of equations you might have to multiply the equations by some numbers before adding them together.

- We now know that $y = 2$. Substituting it into (E1) gives us

$$x - 2 = -1 \implies x = 1$$

- Once again (thankfully) we have found that the only critical point is $(1, 2)$.

Example 2.9.6

This was pretty easy because we only had to solve linear equations, which in turn was a consequence of the fact that $f(x, y)$ was a polynomial of degree two. Here is an example with some slightly more challenging algebra.

Example 2.9.7 $\left(f(x,y) = 2x^3 - 6xy + y^2 + 4y\right)$

Find all critical points of $f(x, y) = 2x^3 - 6xy + y^2 + 4y$.

Solution. As in the last example, we need to find where the gradient is zero, and to find the gradient we need the first order partial derivatives.

$$f_x = 6x^2 - 6y \qquad f_y = -6x + 2y + 4$$

So the critical points are the solutions of

$$6x^2 - 6y = 0 \qquad -6x + 2y + 4 = 0$$

We can rewrite the first equation as $y = x^2$, which expresses y as a function of x. We can then substitute $y = x^2$ into the second equation, giving

$$\begin{aligned} -6x + 2y + 4 = 0 &\iff -6x + 2x^2 + 4 = 0 \iff x^2 - 3x + 2 = 0 \iff (x-1)(x-2) = 0 \\ &\iff x = 1 \text{ or } 2 \end{aligned}$$

When $x = 1$, $y = 1^2 = 1$ and when $x = 2$, $y = 2^2 = 4$. So, there are two critical points: $(1, 1)$, $(2, 4)$.

Alternatively, we could have also used the second equation to write $y = 3x - 2$, and then substituted that into the first equation to get

$$6x^2 - 6(3x-2) = 0 \iff x^2 - 3x + 2 = 0$$

just as above.

Example 2.9.7

And here is an example for which the algebra requires a bit more thought.

Example 2.9.8 ($f(x,y) = xy(5x+y-15)$)

Find all critical points of $f(x,y) = xy(5x+y-15)$.

Solution. The first order partial derivatives of $f(x,y) = xy(5x+y-15)$ are

$$\begin{aligned} f_x(x,y) &= y(5x+y-15) + xy(5) = y(5x+y-15) + y(5x) = y(10x+y-15) \\ f_y(x,y) &= x(5x+y-15) + xy(1) = x(5x+y-15) + x(y) = x(5x+2y-15) \end{aligned}$$

The critical points are the solutions of $f_x(x,y) = f_y(x,y) = 0$. That is, we need to find all x, y that satisfy the pair of equations

$$y(10x+y-15) = 0 \tag{E1}$$
$$x(5x+2y-15) = 0 \tag{E2}$$

The first equation, $y(10x+y-15) = 0$, is satisfied if at least one of the two factors y, $(10x+y-15)$ is zero. So the first equation is satisfied if at least one of the two equations

$$y = 0 \tag{E1a}$$
$$10x + y = 15 \tag{E1b}$$

is satisfied. The second equation, $x(5x+2y-15) = 0$, is satisfied if at least one of the two factors x, $(5x+2y-15)$ is zero. So the second equation is satisfied if at least one of the two equations

$$x = 0 \tag{E2a}$$
$$5x + 2y = 15 \tag{E2b}$$

is satisfied.

So both critical point equations (E1) and (E2) are satisfied if and only if at least one of (E1a), (E1b) is satisfied and in addition at least one of (E2a), (E2b) is satisfied. So both critical point equations (E1) and (E2) are satisfied if and only if at least one of the following four possibilities hold.

- (E1a) and (E2a) are satisfied if and only if $x = y = 0$
- (E1a) and (E2b) are satisfied if and only if $y = 0,\ 5x + 2y = 15 \iff y = 0,\ 5x = 15$
- (E1b) and (E2a) are satisfied if and only if $10x + y = 15,\ x = 0 \iff y = 15,\ x = 0$
- (E1b) and (E2b) are satisfied if and only if $10x+y = 15$, $5x+2y = 15$. We can use, for example, the second of these equations to solve for x in terms of y: $x = \frac{1}{5}(15-2y)$. When we substitute this into the first equation we get $2(15-2y)+y = 15$, which we can solve for y. This gives $-3y = 15-30$ or $y = 5$ and then $x = \frac{1}{5}(15-2\times 5) = 1$.

In conclusion, the critical points are $(0,0)$, $(3,0)$, $(0,15)$ and $(1,5)$.

A more compact way to write what we have just done is

$$\begin{array}{rccc} & f_x(x,y)=0 & \text{and} & f_y(x,y)=0 \\ \iff & y(10x+y-15)=0 & \text{and} & x(5x+2y-15)=0 \\ \iff & \{y=0 \ \text{ or } \ 10x+y=15\} & \text{and} & \{x=0 \ \text{ or } \ 5x+2y=15\} \\ \iff & \multicolumn{3}{l}{\{y=0,\ x=0\} \text{ or } \{y=0,\ 5x+2y=15\} \text{ or } \{10x+y=15,\ x=0\} \text{ or }} \\ & \multicolumn{3}{c}{\{10x+y=15,\ 5x+2y=15\}} \\ \iff & \multicolumn{3}{l}{\{x=y=0\} \text{ or } \{y=0,\ x=3\} \text{ or } \{x=0,\ y=15\} \text{ or } \{x=1,\ y=5\}} \end{array}$$

Example 2.9.8

Let's try a more practical example — something from the real world. Well, a mathematician's "real world". The interested reader should search-engine their way to a discussion of "idealisation", "game theory" "Cournot models" and "Bertrand models". But don't spend too long there. A discussion of breweries is about to take place.

Example 2.9.9

In a certain community, there are two breweries in competition[48], so that sales of each negatively affect the profits of the other. If brewery A produces x litres of beer per month and brewery B produces y litres per month, then the profits of the two breweries are given by

$$P = 2x - \frac{2x^2+y^2}{10^6} \qquad Q = 2y - \frac{4y^2+x^2}{2\times 10^6}$$

respectively. Find the sum of the two profits if each brewery independently sets its own production level to maximize its own profit and assumes that its competitor does likewise. Then, assuming cartel behaviour, find the sum of the two profits if the two breweries cooperate so as to maximize that sum[49].

Solution. If A adjusts x to maximize P (for y held fixed) and B adjusts y to maximize Q (for x held fixed) then x and y are determined by the equations

$$P_x = 2 - \tfrac{4x}{10^6} = 0 \tag{E1}$$

$$Q_y = 2 - \tfrac{8y}{2\times 10^6} = 0 \tag{E2}$$

Equation (E1) yields $x = \frac{1}{2}10^6$ and equation (E2) yields $y = \frac{1}{2}10^6$. Knowing x and y we can determine P, Q and the total profit

$$\begin{aligned} P+Q &= 2(x+y) - \tfrac{1}{10^6}\big(\tfrac{5}{2}x^2 + 3y^2\big) \\ &= 10^6\big(1+1-\tfrac{5}{8}-\tfrac{3}{4}\big) = \tfrac{5}{8}10^6 \end{aligned}$$

48 We have both types of music here — country and western.
49 This sort of thing is generally illegal.

On the other hand if (A, B) adjust (x, y) to maximize $P + Q = 2(x + y) - \frac{1}{10^6}\left(\frac{5}{2}x^2 + 3y^2\right)$, then x and y are determined by

$$(P+Q)_x = 2 - \frac{5x}{10^6} = 0 \tag{E1}$$
$$(P+Q)_y = 2 - \frac{6y}{10^6} = 0 \tag{E2}$$

Equation (E1) yields $x = \frac{2}{5}10^6$ and equation (E2) yields $y = \frac{1}{3}10^6$. Again knowing x and y we can determine the total profit

$$\begin{aligned} P + Q &= 2(x+y) - \frac{1}{10^6}\left(\frac{5}{2}x^2 + 3y^2\right) \\ &= 10^6\left(\frac{4}{5} + \frac{2}{3} - \frac{2}{5} - \frac{1}{3}\right) = \frac{11}{15}10^6 \end{aligned}$$

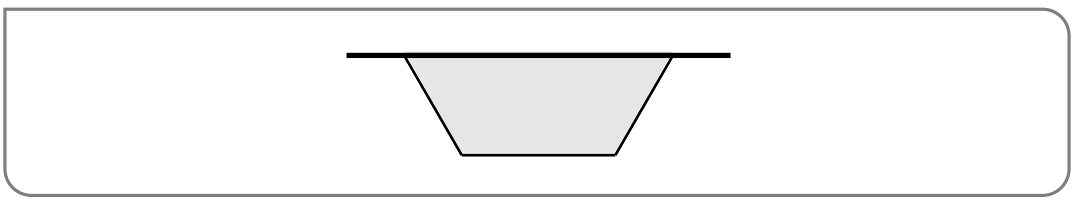
So cooperating really does help their profits. Unfortunately, like a very small tea-pot, consumers will be a little poorer[50].

Example 2.9.9

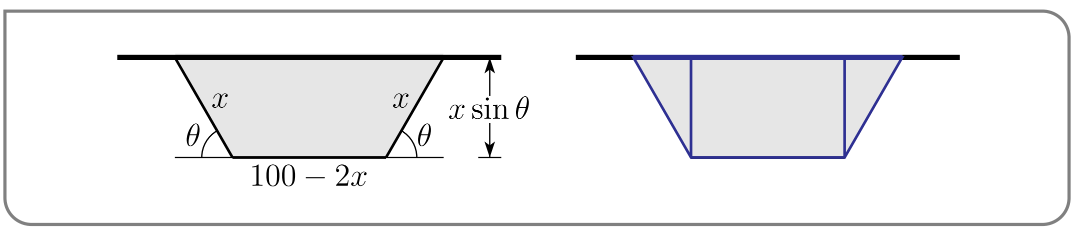
Moving swiftly away from the last pun, let's do something a little more geometric.

Example 2.9.10

Equal angle bends are made at equal distances from the two ends of a 100 metre long fence so the resulting three segment fence can be placed along an existing wall to make an enclosure of trapezoidal shape. What is the largest possible area for such an enclosure?



Solution. This is a very geometric problem (fenced off from pun opportunities), and as such we should start by drawing a sketch and introducing some variable names.



The area enclosed by the fence is the area inside the blue rectangle (in the figure on the right above) plus the area inside the two blue triangles.

$$\begin{aligned} A(x,\theta) &= (100-2x)x\sin\theta + 2\cdot\frac{1}{2}\cdot x\sin\theta\cdot x\cos\theta \\ &= (100x - 2x^2)\sin\theta + x^2\sin\theta\ \cos\theta \end{aligned}$$

50 Sorry about the pun.

To maximize the area, we need to solve

$$0 = \frac{\partial A}{\partial x} = (100 - 4x)\sin\theta + 2x\sin\theta\cos\theta$$
$$0 = \frac{\partial A}{\partial \theta} = (100x - 2x^2)\cos\theta + x^2\big\{\cos^2\theta - \sin^2\theta\big\}$$

Note that both terms in the first equation contain the factor $\sin\theta$ and all terms in the second equation contain the factor x. If either $\sin\theta$ or x are zero the area $A(x,\theta)$ will also be zero, and so will certainly not be maximal. So we may divide the first equation by $\sin\theta$ and the second equation by x, giving

$$(100 - 4x) + 2x\cos\theta = 0 \tag{E1}$$
$$(100 - 2x)\cos\theta + x\big\{\cos^2\theta - \sin^2\theta\big\} = 0 \tag{E2}$$

These equations might look a little scary. But there is no need to panic. They are not as bad as they look because θ enters only through $\cos\theta$ and $\sin^2\theta$, which we can easily write in terms of $\cos\theta$. Furthermore we can eliminate $\cos\theta$ by observing that the first equation forces $\cos\theta = -\frac{100-4x}{2x}$ and hence $\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{(100-4x)^2}{4x^2}$. Substituting these into the second equation gives

$$\begin{aligned}
& -(100-2x)\frac{100-4x}{2x} + x\left[\frac{(100-4x)^2}{2x^2} - 1\right] = 0 \\
\implies \quad & -(100-2x)(100-4x) + (100-4x)^2 - 2x^2 = 0 \\
\implies \quad & 6x^2 - 200x = 0 \\
\implies \quad & x = \frac{100}{3} \quad \cos\theta = -\frac{-100/3}{200/3} = \frac{1}{2} \quad \theta = 60^\circ
\end{aligned}$$

and the maximum area enclosed is

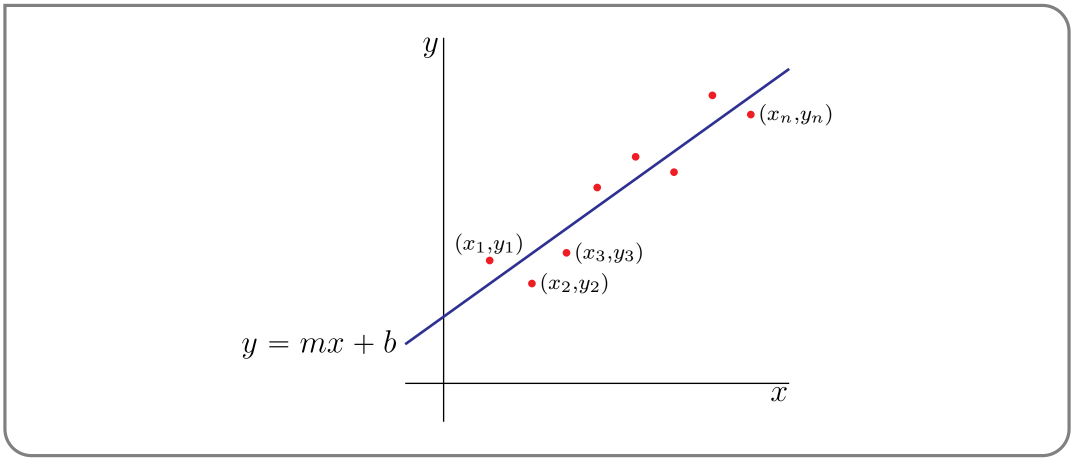
$$A = \Big(100\frac{100}{3} - 2\frac{100^2}{3^2}\Big)\frac{\sqrt{3}}{2} \;+\; \frac{1}{2}\frac{100^2}{3^2}\frac{\sqrt{3}}{2} \;=\; \frac{2500}{\sqrt{3}}$$

Example 2.9.10

Now here is a very useful (even practical!) statistical example — finding the line that best fits a given collection of points.

Example 2.9.11 (Linear regression)

An experiment yields n data points (x_i, y_i), $i = 1, 2, \cdots, n$. We wish to find the straight line $y = mx + b$ which "best" fits the data. The definition of "best" is "minimizes the



root mean square error", i.e. minimizes

$$E(m,b) = \sum_{i=1}^{n}(mx_i + b - y_i)^2$$

Note that

- term number i in $E(m,b)$ is the square of the difference between y_i, which is the i^{th} measured value of y, and $\Big[mx+b\Big]_{x=x_i}$, which is the approximation to y_i given by the line $y = mx + b$.
- All terms in the sum are positive, regardless of whether the points (x_i, y_i) are above or below the line.

Our problem is to find the m and b that minimizes $E(m,b)$. This technique for drawing a line through a bunch of data points is called "linear regression". It is used *a lot*[51] [52]. Even in the real world — and not just the real world that you find in mathematics problems. The actual real world that involves jobs.

Solution. We wish to choose m and b so as to minimize $E(m,b)$. So we need to determine where the partial derivatives of E are zero.

$$0 = \frac{\partial E}{\partial m} = \sum_{i=1}^{n} 2(mx_i + b - y_i)x_i = m\Big[\sum_{i=1}^{n} 2x_i^2\Big] + b\Big[\sum_{i=1}^{n} 2x_i\Big] - \Big[\sum_{i=1}^{n} 2x_iy_i\Big]$$
$$0 = \frac{\partial E}{\partial b} = \sum_{i=1}^{n} 2(mx_i + b - y_i) \quad = m\Big[\sum_{i=1}^{n} 2x_i\Big] + b\Big[\sum_{i=1}^{n} 2\Big] - \Big[\sum_{i=1}^{n} 2y_i\Big]$$

There are a lot of symbols here. But remember that all of the x_i's and y_i's are given constants. They come from, for example, experimental data. The only unknowns are m and b. To emphasize this, and to save some writing, define the constants

$$S_x = \sum_{i=1}^{n} x_i \qquad S_y = \sum_{i=1}^{n} y_i \qquad S_{x^2} = \sum_{i=1}^{n} x_i^2 \qquad S_{xy} = \sum_{i=1}^{n} x_iy_i$$

51 Proof by search engine.
52 And has been used for a long time. It was introduced by the French mathematician Adrien-Marie Legendre, 1752–1833, in 1805, and by the German mathematician and physicist Carl Friedrich Gauss, 1777–1855, in 1809.

The equations which determine the critical points are (after dividing by two)

$$S_{x^2}\, m + S_x\, b = S_{xy} \tag{E1}$$
$$S_x\, m + n\, b = S_y \tag{E2}$$

These are two linear equations on the unknowns m and b. They may be solved in any of the usual ways. One is to use (E2) to solve for b in terms of m

$$b = \frac{1}{n}\big(S_y - S_x m\big) \tag{E3}$$

and then substitute this into (E1) to get the equation

$$S_{x^2}\, m + \frac{1}{n} S_x \left(S_y - S_x m\right) = S_{xy} \qquad \Longrightarrow \qquad \big(nS_{x^2} - S_x^2\big)m = nS_{xy} - S_x S_y$$

for m. We can then solve this equation for m and substitute back into (E3) to get b. This gives

$$m = \frac{nS_{xy} - S_x S_y}{nS_{x^2} - S_x^2}$$
$$b = \frac{S_y}{n}\,\frac{nS_{x^2} - S_x^2}{nS_{x^2} - S_x^2} - \frac{S_x}{n}\,\frac{nS_{xy} - S_x S_y}{nS_{x^2} - S_x^2} = \frac{nS_y S_{x^2} - nS_x S_{xy}}{n(nS_{x^2} - S_x^2)} = -\frac{S_x S_{xy} - S_y S_{x^2}}{nS_{x^2} - S_x^2}$$

Another way to solve the system of equations is

$$\begin{aligned} n(\text{E1}) - S_x(\text{E2}) :&\quad \Big[nS_{x^2} - S_x^2\Big]m = nS_{xy} - S_x S_y \\ -S_x(\text{E1}) + S_{x^2}(\text{E2}) :&\quad \Big[nS_{x^2} - S_x^2\Big]b \ = -S_x S_{xy} + S_y S_{x^2} \end{aligned}$$

which gives the same solution.

So given a bunch of data points, it only takes a quick bit of arithmetic — no calculus required — to apply the above formulae and so to find the best fitting line. Of course while you don't need any calculus to apply the formulae, you do need calculus to understand where they came from. The same technique can be extended to other types of curve fitting problems. For example, polynomial regression.

Example 2.9.11

▸▸ The Second Derivative Test

Now let's start thinking about how to tell if a critical point is a local minimum or maximum. Remember what happens for functions of one variable. Suppose that $x = a$ is a critical point of the function $f(x)$. Any (sufficiently smooth) function is well approximated, when x is close to a, by the first few terms of its Taylor expansion

$$f(x) = f(a) + f'(a)\,(x-a) + \tfrac{1}{2} f''(a)\,(x-a)^2 + \tfrac{1}{3!} f^{(3)}(a)\,(x-a)^3 + \cdots$$

As a is a critical point, we know that $f'(a) = 0$ and

$$f(x) = f(a) + \tfrac{1}{2}f''(a)\,(x-a)^2 + \tfrac{1}{3!}f^{(3)}(a)\,(x-a)^3 + \cdots$$

If $f''(a) \neq 0$, $f(x)$ is going to look a lot like $f(a) + \frac{1}{2}f''(a)\,(x-a)^2$ when x is really close to a. In particular

- if $f''(a) > 0$, then we will have $f(x) > f(a)$ when x is close to (but not equal to) a, so that a will be a local minimum and
- if $f''(a) < 0$, then we will have $f(x) < f(a)$ when x is close to (but not equal to) a, so that a will be a local maximum, but
- if $f''(a) = 0$, then we cannot draw any conclusions without more work.

A similar, but messier, analysis is possible for functions of two variables. Here are some simple quadratic examples that provide a warmup for that messier analysis.

Example 2.9.12 $\big(f(x,y) = x^2 + 3xy + 3y^2 - 6x - 3y - 6\big)$

Consider $f(x,y) = x^2 + 3xy + 3y^2 - 6x - 3y - 6$. The gradient of f is

$$\nabla f(x,y) = \big(2x + 3y - 6\big)\,\hat{\imath} + \big(3x + 6y - 3\big)\,\hat{\jmath}$$

So (x,y) is a critical point of f if and only if

$$2x + 3y = 6 \tag{E1}$$
$$3x + 6y = 3 \tag{E2}$$

Multiplying the first equation by 2 and subtracting the second equation gives

$$x = 9 \tag{2(E1) - (E2)}$$

Then substituting $x = 9$ back into the first equation gives

$$2 \times 9 + 3y = 6 \implies y = -4$$

So $f(x,y)$ has precisely one critical point, namely $(9\,,\,-4)$.

Now let's try to determine if $f(x,y)$ has a local minimum, or a local maximum, or neither, at $(9,-4)$. A good way to determine the behaviour of $f(x,y)$ for (x,y) near $(9,-4)$ is to make the change of variables[53]

$$x = 9 + \Delta x \qquad y = -4 + \Delta y$$

and study the behaviour of f for Δx and Δy near zero.

$$\begin{aligned} f\big(9+\Delta x\,,\,-4+\Delta y\big) &= (9+\Delta x)^2 + 3(9+\Delta x)(-4+\Delta y) + 3(-4+\Delta y)^2 \\ &\qquad - 6(9+\Delta x) - 3(-4+\Delta y) - 6 \\ &= (\Delta x)^2 + 3\Delta x\,\Delta y + 3(\Delta y)^2 - 27 \end{aligned}$$

53 This is equivalent to translating the graph so that the critical point lies at $(0,0)$.

And a good way to study the sign of quadratic expressions like $(\Delta x)^2 + 3\Delta x\,\Delta y + 3(\Delta y)^2$ is to complete the square. So far you have probably just completed the square for quadratic expressions that involve only a single variable. For example

$$x^2 + 3x + 3 = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + 3$$

When there are two variables around, like Δx and Δy, you can just pretend that one of them is a constant and complete the square as before. For example, if you pretend that Δy is a constant,

$$\begin{aligned}(\Delta x)^2 + 3\Delta x\,\Delta y + 3(\Delta y)^2 &= \left(\Delta x + \frac{3}{2}\Delta y\right)^2 + \left(3 - \frac{9}{4}\right)(\Delta y)^2 \\ &= \left(\Delta x + \frac{3}{2}\Delta y\right)^2 + \frac{3}{4}(\Delta y)^2\end{aligned}$$

To this point, we have expressed

$$f\big(9 + \Delta x\,,\ -4 + \Delta y\big) = \left(\Delta x + \frac{3}{2}\Delta y\right)^2 + \frac{3}{4}(\Delta y)^2 - 27$$

As the smallest values of $\left(\Delta x + \frac{3}{2}\Delta y\right)^2$ and $\frac{3}{4}(\Delta y)^2$ are both zero, we have that

$$f(x,y) = f\big(9 + \Delta x\,,\ -4 + \Delta y\big) \geqslant -27 = f(9,-4)$$

for all (x,y) so that $(9,-4)$ is both a local minimum and a global minimum for f.

Example 2.9.12

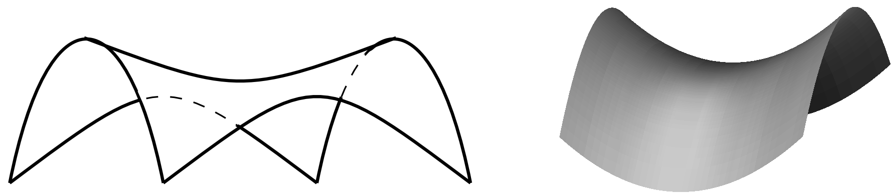
You have already encountered single variable functions that have a critical point which is neither a local max nor a local min. See Example 3.5.9 in the CLP-1 text. Here are a couple of examples which show that this can also happen for functions of two variables. We'll start with the simplest possible such example.

Example 2.9.13 $\left(f(x,y) = x^2 - y^2\right)$

The first partial derivatives of $f(x,y) = x^2 - y^2$ are $f_x(x,y) = 2x$ and $f_y(x,y) = -2y$. So the only critical point of this function is $(0,0)$. Is this a local minimum or maximum? Well let's start with (x,y) at $(0,0)$ and then move (x,y) away from $(0,0)$ and see if $f(x,y)$ gets bigger or smaller. At the origin $f(0,0) = 0$. Of course we can move (x,y) away from $(0,0)$ in many different directions.

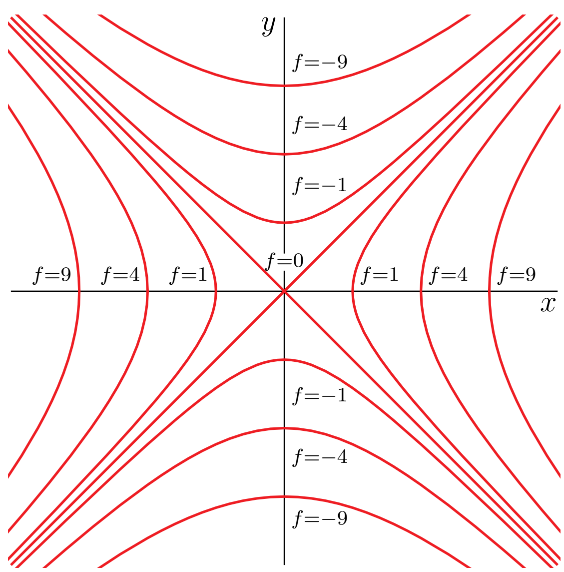
- First consider moving (x,y) along the x-axis. Then $(x,y) = (x,0)$ and $f(x,y) = f(x,0) = x^2$. So when we start with $x = 0$ and then increase x, the value of the function f increases — which means that $(0,0)$ cannot be a local maximum for f.
- Next let's move (x,y) away from $(0,0)$ along the y-axis. Then $(x,y) = (0,y)$ and $f(x,y) = f(0,y) = -y^2$. So when we start with $y = 0$ and then increase y, the value of the function f decreases — which means that $(0,0)$ cannot be a local minimum for f.

So moving away from $(0,0)$ in one direction causes the value of f to increase, while moving away from $(0,0)$ in a second direction causes the value of f to decrease. Consequently $(0,0)$ is neither a local minimum or maximum for f. It is called a saddle point, because the graph of f looks like a saddle. (The full definition of "saddle point" is given immediately after this example.) Here are some figures showing the graph of f.



The figure below show some level curves of f. Observe from the level curves that

- f increases as you leave $(0,0)$ walking along the x axis
- f decreases as you leave $(0,0)$ walking along the y axis



Example 2.9.13

Approximately speaking, if a critical point (a,b) is neither a local minimum nor a local maximum, then it is a saddle point. For (a,b) to not be a local minimum, f has to take values bigger than $f(a,b)$ at some points nearby (a,b). For (a,b) to not be a local maximum, f has to take values smaller than $f(a,b)$ at some points nearby (a,b). Writing this more mathematically we get the following definition.

Definition 2.9.14.

The critical point (a, b) is called a saddle point for the function $f(x, y)$ if, for each $r > 0$,

- there is at least one point (x, y), within a distance r of (a, b), for which $f(x, y) > f(a, b)$ and
- there is at least one point (x, y), within a distance r of (a, b), for which $f(x, y) < f(a, b)$.

Here is another example of a saddle point. This time we have to work a bit to see it.

Example 2.9.15 $\big(f(x,y) = x^2 - 2xy - y^2 + 4y - 2\big)$

Consider $f(x, y) = x^2 - 2xy - y^2 + 4y - 2$. The gradient of f is

$$\nabla f(x,y) = \big(2x - 2y\big)\,\hat{\imath} + \big(-2x - 2y + 4\big)\,\hat{\jmath}$$

So (x, y) is a critical point of f if and only if

$$\begin{aligned} 2x - 2y &= 0 \\ -2x - 2y &= -4 \end{aligned}$$

The first equation gives that $x = y$. Substituting $y = x$ into the second equation gives

$$-2y - 2y = -4 \implies x = y = 1$$

So $f(x, y)$ has precisely one critical point, namely $(1, 1)$.

To determine if $f(x, y)$ has a local minimum, or a local maximum, or neither, at $(1, 1)$, we proceed as in Example 2.9.12. We make the change of variables

$$x = 1 + \Delta x \qquad y = 1 + \Delta y$$

to give

$$\begin{aligned} f\big(1 + \Delta x\,,\,1 + \Delta y\big) &= (1 + \Delta x)^2 - 2(1 + \Delta x)(1 + \Delta y) - (1 + \Delta y)^2 + 4(1 + \Delta y) - 2 \\ &= (\Delta x)^2 - 2\Delta x\,\Delta y - (\Delta y)^2 \end{aligned}$$

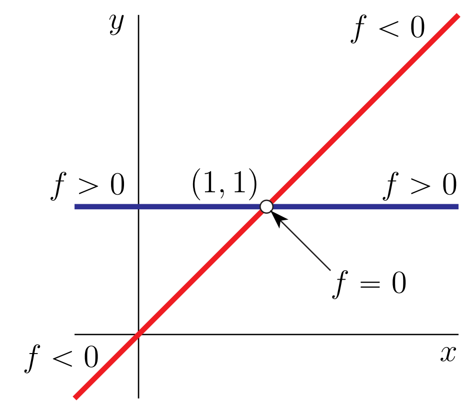
Completing the square,

$$f\big(1 + \Delta x\,,\,1 + \Delta y\big) = (\Delta x)^2 - 2\Delta x\,\Delta y - (\Delta y)^2 = \big(\Delta x - \Delta y\big)^2 - 2(\Delta y)^2$$

Notice that f has now been written as the difference of two squares, much like the f in the saddle point Example 2.9.13.

- If Δx and Δy are such that the first square $\big(\Delta x - \Delta y\big)^2$ is nonzero, but the second square $(\Delta y)^2$ is zero, then $f\big(1 + \Delta x\,,\,1 + \Delta y\big) = \big(\Delta x - \Delta y\big)^2 > 0 = f(1, 1)$. That is, whenever $\Delta y = 0$ and $\Delta x \neq \Delta y$, then $f\big(1 + \Delta x\,,\,1 + \Delta y\big) = \big(\Delta x - \Delta y\big)^2 > 0 = f(1, 1)$.

- On the other hand, if Δx and Δy are such that the first square $(\Delta x - \Delta y)^2$ is zero but the second square $(\Delta y)^2$ is nonzero, then $f\big(1+\Delta x\,,\,1+\Delta y\big) = -2(\Delta y)^2 < 0 = f(1,1)$. That is, whenever $\Delta x = \Delta y \neq 0$, then $f\big(1+\Delta x\,,\,1+\Delta y\big) = -2(\Delta y)^2 < 0 = f(1,1)$.



So

- $f(x,y) > f(1,1)$ at all points on the blue line in the figure above, and
- $f(x,y) < f(1,1)$ at all point on the red line.

We conclude that $(1,1)$ is the only critical point for $f(x,y)$, and furthermore that it is a saddle point.

Example 2.9.15

The above three examples show that we can find all critical points of quadratic functions of two variables. We can also classify each critical point as either a minimum, a maximum or a saddle point.

Of course not every function is quadratic. But by using the quadratic approximation (2.6.12) we can apply the same ideas much more generally. Suppose that (a,b) is a critical point of some function $f(x,y)$. For Δx and Δy small, the quadratic approximation (2.6.12) gives

$$\begin{aligned} f\big(a+\Delta x\,,\,b+\Delta y\big) &\approx f\big(a\,,\,b\big) + f_x\big(a\,,\,b\big)\,\Delta x + f_y\big(a\,,\,b\big)\,\Delta y \\ &\qquad + \frac{1}{2}\Big\{f_{xx}\big(a,b\big)\,\Delta x^2 + 2f_{xy}\big(a,b\big)\,\Delta x\Delta y + f_{yy}\big(a,b\big)\,\Delta y^2\Big\} \qquad (*) \\ &= f\big(a\,,\,b\big) + \frac{1}{2}\Big\{f_{xx}\big(a,b\big)\,\Delta x^2 + 2f_{xy}\big(a,b\big)\,\Delta x\Delta y + f_{yy}\big(a,b\big)\,\Delta y^2\Big\} \end{aligned}$$

since (a,b) is a critical point so that $f_x(a,b) = f_y(a,b) = 0$. Then using the technique of Examples 2.9.12 and 2.9.15, we get[54] (details below)

54 There are analogous results in higher dimensions that are accessible to people who have learned some linear algebra. They are derived by diagonalizing the matrix of second derivatives, which is called the Hessian matrix.

Theorem 2.9.16 (Second Derivative Test)**.**

Let $r > 0$ and assume that all second order derivatives of the function $f(x, y)$ are continuous at all points (x, y) that are within a distance r of (a, b). Assume that $f_x(a, b) = f_y(a, b) = 0$. Define

$$D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - f_{xy}(x, y)^2$$

It is called the discriminant of f. Then

- if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(x, y)$ has a local minimum at (a, b),
- if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(x, y)$ has a local maximum at (a, b),
- if $D(a, b) < 0$, then $f(x, y)$ has a saddle point at (a, b), but
- if $D(a, b) = 0$, then we cannot draw any conclusions without more work.

"Proof". We are putting quotation marks around the word "Proof", because we are not going to justify the fact that it suffices to analyse the quadratic approximation in equation $(*)$. Let's temporarily suppress the arguments (a, b). If $f_{xx}(a, b) \neq 0$, then by completing the square we can write

$$\begin{aligned} f_{xx}\,\Delta x^2 + 2f_{xy}\,\Delta x\Delta y + f_{yy}\,\Delta y^2 &= f_{xx}\left(\Delta x + \frac{f_{xy}}{f_{xx}}\Delta y\right)^2 + \left(f_{yy} - \frac{f_{xy}^2}{f_{xx}}\right)\Delta y^2 \\ &= \frac{1}{f_{xx}}\left\{\left(f_{xx}\,\Delta x + f_{xy}\,\Delta y\right)^2 + \left(f_{xx}f_{yy} - f_{xy}^2\right)\Delta y^2\right\} \end{aligned}$$

Similarly, if $f_{yy}(a, b) \neq 0$,

$$f_{xx}\,\Delta x^2 + 2f_{xy}\,\Delta x\Delta y + f_{yy}\,\Delta y^2 = \frac{1}{f_{yy}}\left\{\left(f_{xy}\,\Delta x + f_{yy}\,\Delta y\right)^2 + \left(f_{xx}f_{yy} - f_{xy}^2\right)\Delta x^2\right\}$$

Note that this algebra breaks down if $f_{xx}(a, b) = f_{yy}(a, b) = 0$. We'll deal with that case shortly. More importantly, note that

- if $\left(f_{xx}f_{yy} - f_{xy}^2\right) > 0$ then both f_{xx} and f_{yy} must be nonzero and of the same sign and furthermore, whenever Δx or Δy are nonzero,

  $$\begin{aligned} &\left\{\left(f_{xx}\,\Delta x + f_{xy}\,\Delta y\right)^2 + \left(f_{xx}f_{yy} - f_{xy}^2\right)\Delta y^2\right\} > 0 \quad \text{and} \\ &\left\{\left(f_{xy}\,\Delta x + f_{yy}\,\Delta y\right)^2 + \left(f_{xx}f_{yy} - f_{xy}^2\right)\Delta x^2\right\} > 0 \end{aligned}$$

  so that, recalling $(*)$,

  - if $f_{xx}(a, b) > 0$, then (a, b) is a local minimum and
  - if $f_{xx}(a, b) < 0$, then (a, b) is a local maximum.

- If $\left(f_{xx}f_{yy} - f_{xy}^2\right) < 0$ and f_{xx} is nonzero then

$$\left\{\left(f_{xx}\,\Delta x + f_{xy}\,\Delta y\right)^2 + \left(f_{xx}f_{yy} - f_{xy}^2\right)\Delta y^2\right\}$$

  is strictly positive whenever $\Delta x \neq 0$, $\Delta y = 0$ and is strictly negative whenever $f_{xx}\,\Delta x + f_{xy}\,\Delta y = 0$, $\Delta y \neq 0$, so that (a,b) is a saddle point. Similarly, (a,b) is also a saddle point if $\left(f_{xx}f_{yy} - f_{xy}^2\right) < 0$ and f_{yy} is nonzero.

- Finally, if $f_{xy} \neq 0$ and $f_{xx} = f_{yy} = 0$, then

$$f_{xx}\,\Delta x^2 + 2f_{xy}\,\Delta x\,\Delta y + f_{yy}\,\Delta y^2 = 2f_{xy}\,\Delta x\,\Delta y$$

  is strictly positive for one sign of $\Delta x\,\Delta y$ and is strictly negative for the other sign of $\Delta x\,\Delta y$. So (a,b) is again a saddle point.

$\square$

You might wonder why, in the local maximum/local minimum cases of Theorem 2.9.16, $f_{xx}(a,b)$ appears rather than $f_{yy}(a,b)$. The answer is only that x is before y in the alphabet[55]. You can use $f_{yy}(a,b)$ just as well as $f_{xx}(a,b)$. The reason is that if $D(a,b) > 0$ (as in the first two bullets of the theorem), then because $D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}(a,b)^2 > 0$, we necessarily have $f_{xx}(a,b)\,f_{yy}(a,b) > 0$ so that $f_{xx}(a,b)$ and $f_{yy}(a,b)$ must have the same sign — either both are positive or both are negative.

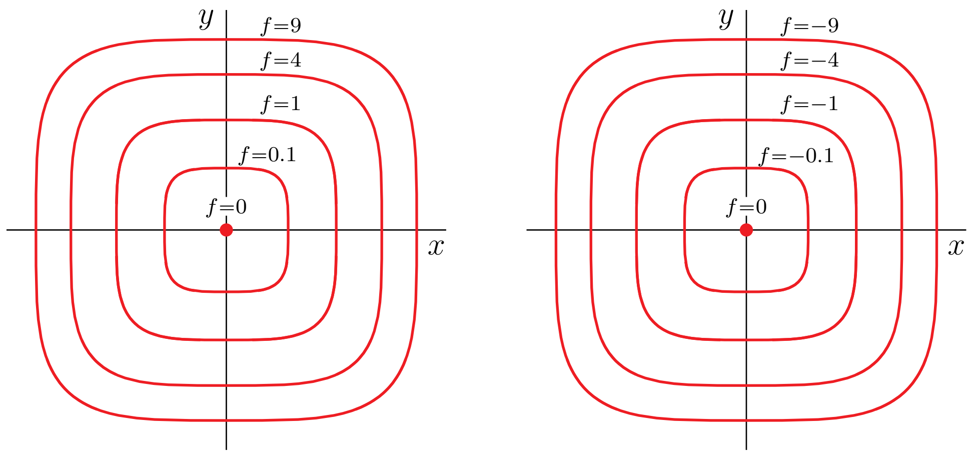
You might also wonder why we cannot draw any conclusions when $D(a,b) = 0$ and what happens then. The second derivative test for functions of two variables was derived in precisely the same way as the second derivative test for functions of one variable is derived — you approximate the function by a polynomial that is of degree two in $(x-a)$, $(y-b)$ and then you analyze the behaviour of the quadratic polynomial near (a,b). For this to work, the contributions to $f(x,y)$ from terms that are of degree two in $(x-a)$, $(y-b)$ had better be bigger than the contributions to $f(x,y)$ from terms that are of degree three and higher in $(x-a)$, $(y-b)$ when $(x-a)$, $(y-b)$ are really small. If this is not the case, for example when the terms in $f(x,y)$ that are of degree two in $(x-a)$, $(y-b)$ all have coefficients that are exactly zero, the analysis will certainly break down. That's exactly what happens when $D(a,b) = 0$. Here are some examples. The functions

$$f_1(x,y) = x^4 + y^4 \qquad f_2(x,y) = -x^4 - y^4 \qquad f_3(x,y) = x^3 + y^3 \qquad f_4(x,y) = x^4 - y^4$$

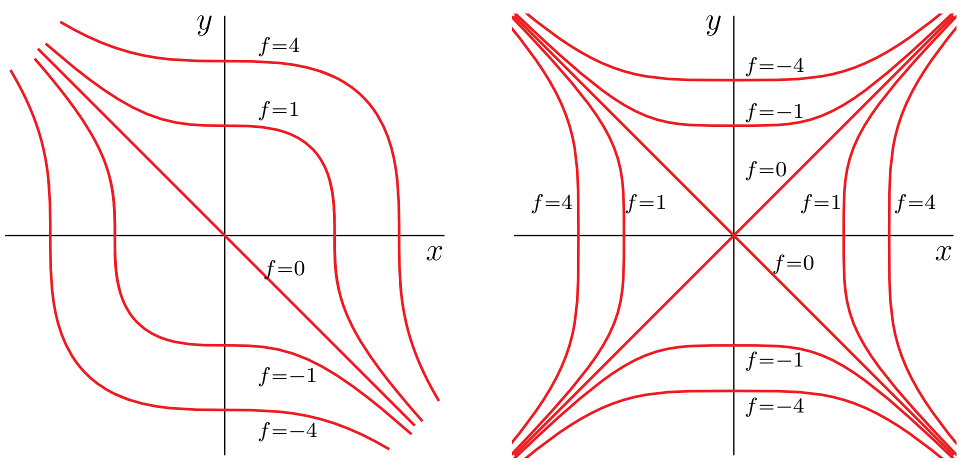
all have $(0,0)$ as the only critical point and all have $D(0,0) = 0$. The first, f_1 has its minimum there. The second, f_2, has its maximum there. The third and fourth have a saddle point there.

Here are sketches of some level curves for each of these four functions (with all renamed to simply f).

55 The shackles of convention are not limited to mathematics. Election ballots often have the candidates listed in alphabetic order.



level curves of $f(x,y) = x^4 + y^4$ level curves of $f(x,y) = -x^4 - y^4$



level curves of $f(x,y) = x^3 + y^3$ level curves of $f(x,y) = x^4 - y^4$

Example 2.9.17 $\big(f(x,y) = 2x^3 - 6xy + y^2 + 4y\big)$

Find and classify all critical points of $f(x,y) = 2x^3 - 6xy + y^2 + 4y$.

Solution. Thinking a little way ahead, to find the critical points we will need the gradient and to apply the second derivative test of Theorem 2.9.16 we will need all second order partial derivatives. So we need all partial derivatives of order up to two. Here they are.

$$\begin{array}{lll} f = 2x^3 - 6xy + y^2 + 4y & & \\ f_x = 6x^2 - 6y & f_{xx} = 12x & f_{xy} = -6 \\ f_y = -6x + 2y + 4 & f_{yy} = 2 & f_{yx} = -6 \end{array}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

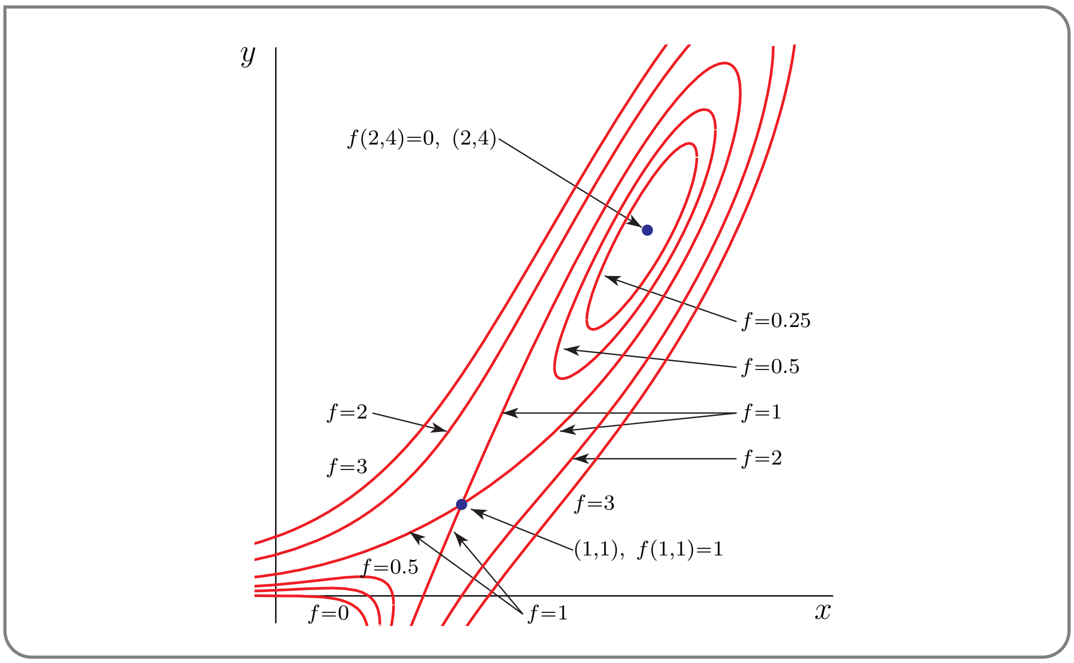
We have already found, in Example 2.9.7, that the critical points are $(1,1)$, $(2,4)$. The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(1,1)$	$12 \times 2 - (-6)^2 < 0$		saddle point
$(2,4)$	$24 \times 2 - (-6)^2 > 0$	24	local min

We were able to leave the f_{xx} entry in the top row blank, because

- we knew that $f_{xx}(1,1)f_{yy}(1,1) - f_{xy}^2(1,1) < 0$, and
- we knew, from Theorem 2.9.16, that $f_{xx}(1,1)f_{yy}(1,1) - f_{xy}^2(1,1) < 0$, by itself, was enough to ensure that $(1,1)$ was a saddle point.

Here is a sketch of some level curves of our $f(x,y)$. They are not needed to answer this question, but can give you some idea as to what the graph of f looks like.



Example 2.9.17

Example 2.9.18 ($f(x,y) = xy(5x+y-15)$)

Find and classify all critical points of $f(x,y) = xy(5x+y-15)$.

Solution. We have already computed the first order partial derivatives

$$f_x(x,y) = y(10x+y-15) \qquad f_y(x,y) = x(5x+2y-15)$$

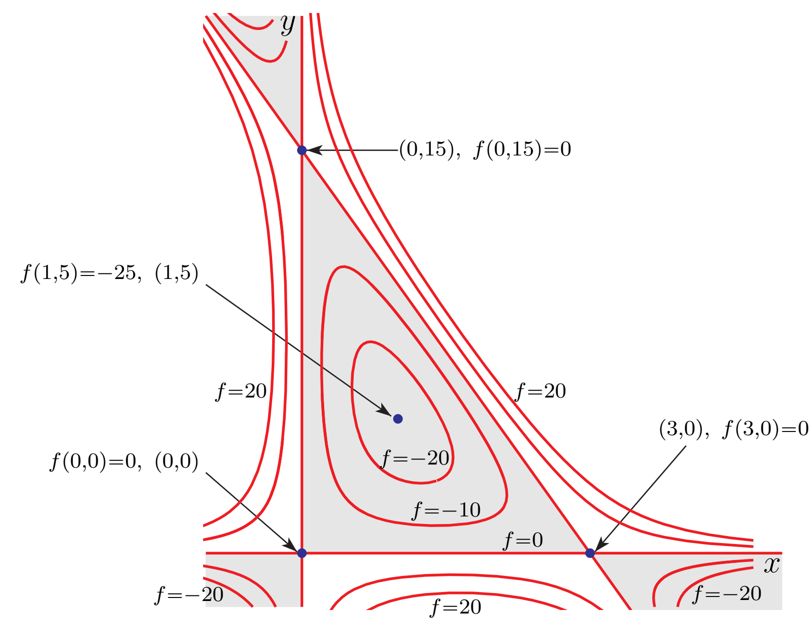
of $f(x,y)$ in Example 2.9.8. Again, to classify the critical points we need the second order partial derivatives. They are

$$\begin{aligned}
f_{xx}(x,y) &= 10y \\
f_{yy}(x,y) &= 2x \\
f_{xy}(x,y) &= (1)(10x+y-15) + y(1) = 10x+2y-15 \\
f_{yx}(x,y) &= (1)(5x+2y-15) + x(5) = 10x+2y-15
\end{aligned}$$

(Once again, we have computed both f_{xy} and f_{yx} to guard against mechanical errors.) We have already found, in Example 2.9.8, that the critical points are $(0,0)$, $(0,15)$, $(3,0)$ and $(1,5)$. The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-15)^2 < 0$		saddle point
$(0,15)$	$150 \times 0 - 15^2 < 0$		saddle point
$(3,0)$	$0 \times 6 - 15^2 < 0$		saddle point
$(1,5)$	$50 \times 2 - 5^2 > 0$	75	local min

Here is a sketch of some level curves of our $f(x,y)$. f is negative in the shaded regions and f is positive in the unshaded regions. Again this is not needed to answer this



question, but can give you some idea as to what the graph of f looks like.

Example 2.9.18

Example 2.9.19

Find and classify all of the critical points of $f(x,y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$.

Solution. We know the drill now. We start by computing all of the partial derivatives of f up to order 2.

$$
\begin{aligned}
f &= x^3 + xy^2 - 3x^2 - 4y^2 + 4 \\
f_x &= 3x^2 + y^2 - 6x & f_{xx} &= 6x - 6 & f_{xy} &= 2y \\
f_y &= 2xy - 8y & f_{yy} &= 2x - 8 & f_{yx} &= 2y
\end{aligned}
$$

The critical points are then the solutions of $f_x = 0$, $f_y = 0$. That is

$$f_x = 3x^2 + y^2 - 6x = 0 \tag{E1}$$
$$f_y = 2y(x-4) = 0 \tag{E2}$$

The second equation, $2y(x-4) = 0$, is satisfied if and only if at least one of the two equations $y = 0$ and $x = 4$ is satisfied.

- When $y = 0$, equation (E1) forces x to obey

  $$0 = 3x^2 + 0^2 - 6x = 3x(x-2)$$

  so that $x = 0$ or $x = 2$.

- When $x = 4$, equation (E1) forces y to obey

  $$0 = 3 \times 4^2 + y^2 - 6 \times 4 = 24 + y^2$$

  which is impossible.

So, there are two critical points: $(0,0)$, $(2,0)$. Here is a table that classifies the critical points.

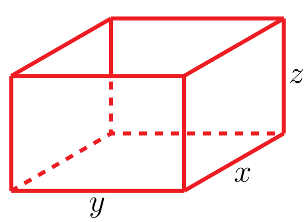
critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6) \times (-8) - 0^2 > 0$	$-6 < 0$	local max
$(2,0)$	$6 \times (-4) - 0^2 < 0$		saddle point

Example 2.9.19

Example 2.9.20

A manufacturer wishes to make an open rectangular box of given volume V using the least possible material. Find the design specifications.

Solution. Denote by x, y and z, the length, width and height, respectively, of the box.



The box has two sides of area xz, two sides of area yz and a bottom of area xy. So the total surface area of material used is

$$S = 2xz + 2yz + xy$$

However the three dimensions x, y and z are not independent. The requirement that the box have volume V imposes the constraint

$$xyz = V$$

We can use this constraint to eliminate one variable. Since z is at the end of the alphabet (poor z), we eliminate z by substituting $z = \frac{V}{xy}$. So we have find the values of x and y that minimize the function

$$S(x,y) = \frac{2V}{y} + \frac{2V}{x} + xy$$

Let's start by finding the critical points of S. Since

$$S_x(x,y) = -\frac{2V}{x^2} + y$$

$$S_y(x,y) = -\frac{2V}{y^2} + x$$

(x,y) is a critical point if and only if

$$x^2y = 2V \tag{E1}$$

$$xy^2 = 2V \tag{E2}$$

Solving (E1) for y gives $y = \frac{2V}{x^2}$. Substituting this into (E2) gives

$$x\frac{4V^2}{x^4} = 2V \implies x^3 = 2V \implies x = \sqrt[3]{2V} \quad \text{and} \quad y = \frac{2V}{(2V)^{2/3}} = \sqrt[3]{2V}$$

As there is only one critical point, we would expect it to give the minimum[56]. But let's use the second derivative test to verify that at least the critical point is a local minimum. The various second partial derivatives are

$$S_{xx}(x,y) = \frac{4V}{x^3} \qquad\qquad S_{xx}\big(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\big) = 2$$

$$S_{xy}(x,y) = 1 \qquad\qquad S_{xy}\big(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\big) = 1$$

$$S_{yy}(x,y) = \frac{4V}{y^3} \qquad\qquad S_{yy}\big(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\big) = 2$$

So

$$S_{xx}\big(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\big)\ S_{yy}\big(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\big) - S_{xy}\big(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\big)^2 = 3 > 0 \qquad S_{xx}\big(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\big) = 2 > 0$$

and, by Theorem 2.9.16.b, $\big(\sqrt[3]{2V}\,,\,\sqrt[3]{2V}\big)$ is a local minimum and the desired dimensions are

$$x = y = \sqrt[3]{2V} \qquad z = \sqrt[3]{\frac{V}{4}}$$

Note that our solution has $x = y$. That's a good thing — the function $S(x,y)$ is symmetric in x and y. Because the box has no top, the symmetry does not extend to z.

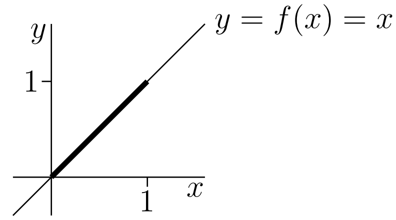
Example 2.9.20

56 Indeed one can use the facts that $0 < x < \infty$, that $0 < y < \infty$, and that $S \to \infty$ as $x \to 0$ and as $y \to 0$ and as $x \to \infty$ and as $y \to \infty$ to prove that the single critical point gives the global minimum.

2.9.1 ▸▸ Absolute Minima and Maxima

Of course a local maximum or minimum of a function need not be the absolute maximum of minimum. We'll now consider how to find the absolute maximum and minimum. Let's start by reviewing how one finds the absolute maximum and minimum of a function of one variable on an interval.

For concreteness, let's suppose that we want to find the extremal[57] values of a function $f(x)$ on the interval $0 \leqslant x \leqslant 1$. If an extremal value is attained at some $x = a$ which is in the interior of the interval, i.e. if $0 < a < 1$, then a is also a local maximum or minimum and so has to be a critical point of f. But if an extremal value is attained at a boundary point a of the interval, i.e. if $a = 0$ or $a = 1$, then a need not be a critical point of f. This happens, for example, when $f(x) = x$. The largest value of $f(x)$ on the interval $0 \leqslant x \leqslant 1$ is 1 and is attained at $x = 1$, but $f'(x) = 1$ is never zero, so that f has no critical points.



So to find the maximum and minimum of the function $f(x)$ on the interval $[0, 1]$, you

1. build up a list of all candidate points $0 \leqslant a \leqslant 1$ at which the maximum or minimum could be attained, by finding all a's for which either
   (a) $0 < a < 1$ and $f'(a) = 0$ or
   (b) $0 < a < 1$ and $f'(a)$ does not exist[58] or
   (c) a is a boundary point, i.e. $a = 0$ or $a = 1$,
2. and then you evaluate $f(a)$ at each a on the list of candidates. The biggest of these candidate values of $f(a)$ is the absolute maximum and the smallest of these candidate values is the absolute minimum.

The procedure for finding the maximum and minimum of a function of two variables, $f(x, y)$ in a set like, for example, the unit disk $x^2 + y^2 \leqslant 1$, is similar. You again

1. build up a list of all candidate points (a, b) in the set at which the maximum or minimum could be attained, by finding all (a, b)'s for which either[59]
   (a) (a, b) is in the interior of the set (for our example, $a^2 + b^2 < 1$) and $f_x(a, b) = f_y(a, b) = 0$ or
   (b) (a, b) is in the interior of the set and $f_x(a, b)$ or $f_y(a, b)$ does not exist or

57 Recall that "extremal value" means "either maximum value or minimum value".
58 Recall that if $f'(a)$ does not exist, then a is called a singular point of f.
59 This is probably a good time to review the statement of Theorem 2.9.2.

(c) (a,b) is a boundary[60] point, (for our example, $a^2+b^2=1$), and could give the maximum or minimum on the boundary — more about this shortly —

2. and then you evaluate $f(a,b)$ at each (a,b) on the list of candidates. The biggest of these candidate values of $f(a,b)$ is the absolute maximum and the smallest of these candidate values is the absolute minimum.

The boundary of a set, like $x^2+y^2\leqslant 1$, in $\mathbb{R}^2$ is a curve, like $x^2+y^2=1$. This curve is a one dimensional set, meaning that it is like a deformed x-axis. We can find the maximum and minimum of $f(x,y)$ on this curve by converting $f(x,y)$ into a function of one variable (on the curve) and using the standard function of one variable techniques. This is best explained by some examples.

Example 2.9.21

Find the maximum and minimum of $T(x,y)=(x+y)e^{-x^2-y^2}$ on the region defined by $x^2+y^2\leqslant 1$ (i.e. on the unit disk).

Solution. Let's follow our checklist. First critical points, then points where the partial derivatives don't exist, and finally the boundary.
Interior Critical Points: If T takes its maximum or minimum value at a point in the interior, $x^2+y^2<1$, then that point must be either a critical point of T or a singular point of T. To find the critical points we compute the first order derivatives.

$$T_x(x,y)=(1-2x^2-2xy)e^{-x^2-y^2}\qquad T_y(x,y)=(1-2xy-2y^2)e^{-x^2-y^2}$$

Because the exponential $e^{-x^2-y^2}$ is never zero, the critical points are the solutions of

$$\begin{aligned} T_x=0 &\quad\iff\quad 2x(x+y)=1\\ T_y=0 &\quad\iff\quad 2y(x+y)=1 \end{aligned}$$

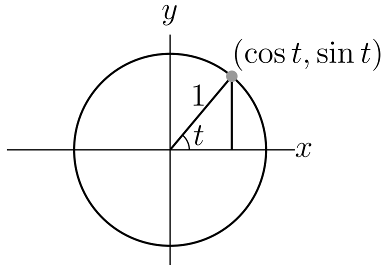
- As both $2x(x+y)$ and $2y(x+y)$ are nonzero, we may divide the two equations, which gives $\frac{x}{y}=1$, forcing $x=y$.
- Substituting this into either equation gives $2x(2x)=1$ so that $x=y=\pm 1/2$.

So the only critical points are $(1/2,1/2)$ and $(-1/2,-1/2)$. Both are in $x^2+y^2<1$.
Singular points: In this problem, there are no singular points.
Boundary: Points on the boundary satisfy $x^2+y^2=1$. That is they lie on a circle. We may use the figure below to express $x=\cos t$ and $y=\sin t$, in terms of the angle t. This will make the formula for T on the boundary quite a bit easier to deal with. On the boundary,

$$T=(\cos t+\sin t)e^{-\cos^2 t-\sin^2 t}=(\cos t+\sin t)e^{-1}$$

As all t's are allowed, this function takes its max and min at zeroes of

60 It should intuitively obvious from a sketch that the boundary of the disk $x^2+y^2\leqslant 1$ is the circle $x^2+y^2=1$. But if you really need a formal definition, here it is. A point (a,b) is on the boundary of a set S if there is a sequence of points in S that converges to (a,b) and there is also a sequence of points in the complement of S that converges to (a,b).



$$\frac{dT}{dt} = \big(-\sin t + \cos t\big)e^{-1}$$

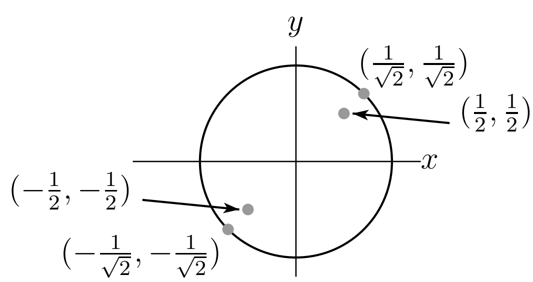
That is, $(\cos t + \sin t)e^{-1}$ takes its max and min

- when $\sin t = \cos t$,
- that is, when $x = y$ and $x^2 + y^2 = 1$,
- which forces $x^2 + x^2 = 1$ and hence $x = y = \pm\frac{1}{\sqrt{2}}$.

All together, we have the following candidates for max and min, with the max and min indicated.

point	$(\frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
value of T	$\frac{1}{\sqrt{e}} \approx 0.61$	$-\frac{1}{\sqrt{e}}$	$\frac{\sqrt{2}}{e} \approx 0.52$	$-\frac{\sqrt{2}}{e}$
	max	min		

The following sketch shows all of the critical points. It is a good idea to make such a sketch so that you don't accidentally include a critical point that is outside of the allowed region.



Example 2.9.21

In the last example, we analyzed the behaviour of f on the boundary of the region of interest by using the parametrization $x = \cos t$, $y = \sin t$ of the circle $x^2 + y^2 = 1$. Sometimes using this parametrization is not so clean. And worse, some curves don't have such a simple parametrization. In the next problem we'll look at the boundary a little differently.

Example 2.9.22

Find the maximum and minimum values of $f(x, y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$ on the disk $x^2 + y^2 \leqslant 1$.

Solution. Again, we first find all critical points, then find all singular points and, finally, analyze the boundary.

Interior Critical Points: If f takes its maximum or minimum value at a point in the interior, $x^2 + y^2 < 1$, then that point must be either a critical point of f or a singular point of f. To find the critical points[61] we compute the first order derivatives.

$$f_x = 3x^2 + y^2 - 6x \qquad f_y = 2xy - 8y$$

The critical points are the solutions of

$$f_x = 3x^2 + y^2 - 6x = 0 \tag{E1}$$
$$f_y = 2y(x - 4) = 0 \tag{E2}$$

The second equation, $2y(x - 4) = 0$, is satisfied if and only if at least one of the two equations $y = 0$ and $x = 4$ is satisfied.

- When $y = 0$, equation (E1) forces x to obey

  $$0 = 3x^2 + 0^2 - 6x = 3x(x - 2)$$

  so that $x = 0$ or $x = 2$.

- When $x = 4$, equation (E1) forces y to obey

  $$0 = 3 \times 4^2 + y^2 - 6 \times 4 = 24 + y^2$$

  which is impossible.

So, there are only two critical points: $(0, 0)$, $(2, 0)$.

Singular points: In this problem, there are no singular points.

Boundary: On the boundary, $x^2 + y^2 = 1$, we could again take advantage of having a circle and write $x = \cos t$ and $y = \sin t$. But, for practice, we'll use another method[62]. We know that (x, y) satisfies $x^2 + y^2 = 1$, and hence $y^2 = 1 - x^2$. Examining the formula for $f(x, y)$,

61 We actually found the critical points in Example 2.9.19. But, for the convenience of the reader, we'll repeat that here.

62 Even if you don't believe that "you can't have too many tools", it is pretty dangerous to have to rely on just one tool.

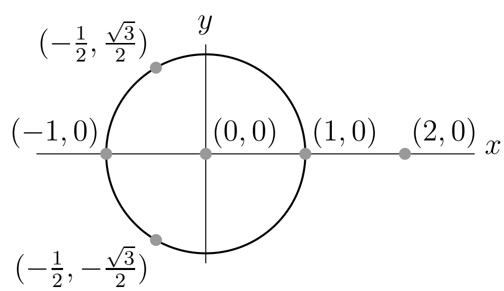
we see that it contains only even[63] powers of y, so we can eliminate y by substituting $y^2 = 1 - x^2$ into the formula.

$$f = x^3 + x(1-x^2) - 3x^2 - 4(1-x^2) + 4 = x + x^2$$

The max and min of $x + x^2$ for $-1 \leqslant x \leqslant 1$ must occur either

- when $x = -1$ ($\Rightarrow y = f = 0$) or
- when $x = +1$ ($\Rightarrow y = 0, f = 2$) or
- when $0 = \frac{d}{dx}(x + x^2) = 1 + 2x$ ($\Rightarrow x = -\frac{1}{2}, y = \pm\sqrt{\frac{3}{4}}, f = -\frac{1}{4}$).

Here is a sketch showing all of the points that we have identified.



Note that the point $(2, 0)$ is outside the allowed region[64]. So all together, we have the following candidates for max and min, with the max and min indicated.

point	$(0,0)$	$(-1,0)$	$(1,0)$	$(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$
value of f	4	0	2	$-\frac{1}{4}$
	max			min

Example 2.9.22

Example 2.9.23

Find the maximum and minimum values of $f(x, y) = xy - x^3y^2$ when (x, y) runs over the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

63 If it contained odd powers too, we could consider the cases $y \geqslant 0$ and $y \leqslant 0$ separately and substitute $y = \sqrt{1-x^2}$ in the former case and $y = -\sqrt{1-x^2}$ in the latter case.
64 We found $(2, 0)$ as a solution to the critical point equations (E1), (E2). That's because, in the course of solving those equations, we ignored the constraint that $x^2 + y^2 \leqslant 1$.

Solution. As usual, let's examine the critical points, singular points and boundary in turn.
Interior Critical Points: If f takes its maximum or minimum value at a point in the interior, $0 < x < 1$, $0 < y < 1$, then that point must be either a critical point of f or a singular point of f. To find the critical points we compute the first order derivatives.

$$f_x(x, y) = y - 3x^2y^2 \qquad f_y(x, y) = x - 2x^3y$$

The critical points are the solutions of

$$\begin{aligned} f_x = 0 &\iff y(1 - 3x^2y) = 0 &&\iff y = 0 \text{ or } 3x^2y = 1 \\ f_y = 0 &\iff x(1 - 2x^2y) = 0 &&\iff x = 0 \text{ or } 2x^2y = 1 \end{aligned}$$

- If $y = 0$, we cannot have $2x^2y = 1$, so we must have $x = 0$.
- If $3x^2y = 1$, we cannot have $x = 0$, so we must have $2x^2y = 1$. Dividing gives $1 = \frac{3x^2y}{2x^2y} = \frac{3}{2}$ which is impossible.

So the only critical point in the square is $(0, 0)$. There $f = 0$.
Singular points: Yet again there are no singular points in this problem.
Boundary: The region is a square, so its boundary consists of its four sides.

- First, we look at the part of the boundary with $x = 0$. On that entire side $f = 0$.
- Next, we look at the part of the boundary with $y = 0$. On that entire side $f = 0$.
- Next, we look at the part of the boundary with $y = 1$. There $f = f(x, 1) = x - x^3$. To find the maximum and minimum of $f(x, y)$ on the part of the boundary with $y = 1$, we must find the maximum and minimum of $x - x^3$ when $0 \leqslant x \leqslant 1$.

  Recall that, in general, the maximum and minimum of a function $h(x)$ on the interval $a \leqslant x \leqslant b$, must occur either at $x = a$ or at $x = b$ or at an x for which either $h'(x) = 0$ or $h'(x)$ does not exist. In this case, $\frac{d}{dx}(x - x^3) = 1 - 3x^2$, so the max and min of $x - x^3$ for $0 \leqslant x \leqslant 1$ must occur

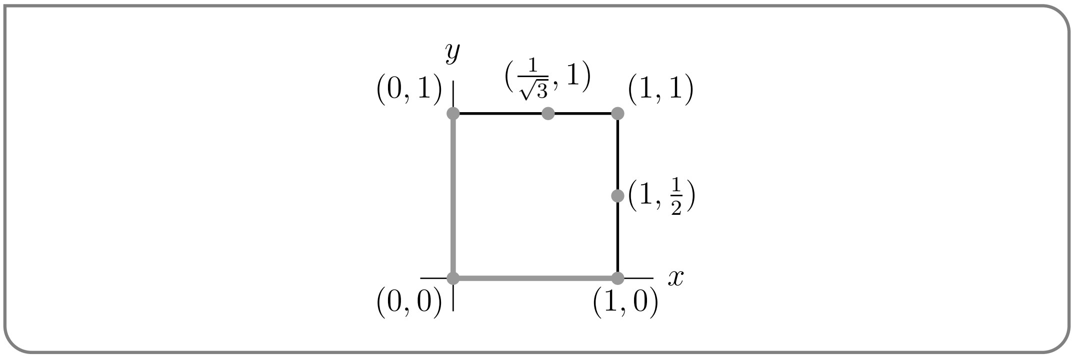
  - either at $x = 0$, where $f = 0$,
  - or at $x = \frac{1}{\sqrt{3}}$, where $f = \frac{2}{3\sqrt{3}}$,
  - or at $x = 1$, where $f = 0$.

- Finally, we look at the part of the boundary with $x = 1$. There $f = f(1, y) = y - y^2$. As $\frac{d}{dy}(y - y^2) = 1 - 2y$, the only critical point of $y - y^2$ is at $y = \frac{1}{2}$. So the the max and min of $y - y^2$ for $0 \leqslant y \leqslant 1$ must occur

  - either at $y = 0$, where $f = 0$,
  - or at $y = \frac{1}{2}$, where $f = \frac{1}{4}$,
  - or at $y = 1$, where $f = 0$.

All together, we have the following candidates for max and min, with the max and min indicated.

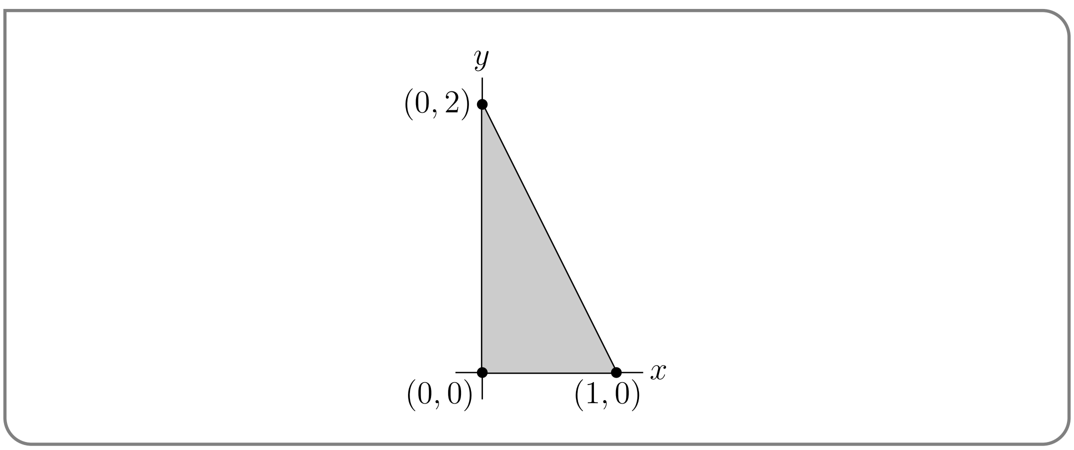
point	$(0,0)$	$(0, 0\leqslant y\leqslant 1)$	$(0\leqslant x\leqslant 1, 0)$	$(1,0)$	$(1,\frac{1}{2})$	$(1,1)$	$(0,1)$	$(\frac{1}{\sqrt{3}},1)$
value of f	0	0	0	0	$\frac{1}{4}$	0	0	$\frac{2}{3\sqrt{3}}\approx 0.385$
	min	min	min	min		min	min	max



Example 2.9.23

Example 2.9.24

Find the maximum and minimum values of $f(x,y) = xy + 2x + y$ when (x,y) runs over the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,2)$. The triangular region is sketched in



Solution. As usual, let's examine the critical points, singular points and boundary in turn.
Interior Critical Points: If f takes its maximum or minimum value at a point in the interior, then that point must be either a critical point of f or a singular point of f. The critical points are the solutions of

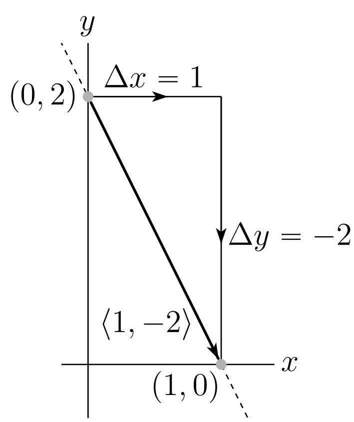
$$f_x(x,y) = y + 2 = 0 \qquad f_y(x,y) = x + 1 = 0$$

So there is exactly one critical point, namely $(-1,-2)$. This is well outside the triangle and so is not a candidate for the location of the max and min.

Singular points: Yet again there are no singular points for this f.

Boundary: The region is a triangle, so its boundary consists of its three sides.

- First, we look at the side that runs from $(0,0)$ to $(0,2)$. On that entire side $x=0$, so that $f(0,y)=y$. The smallest value of f on that side is $f=0$ at $(0,0)$ and the largest value of f on that side is $f=2$ at $(0,2)$.
- Next, we look at the side that runs from $(0,0)$ to $(1,0)$. On that entire side $y=0$, so that $f(x,0)=2x$. The smallest value of f on that side is $f=0$ at $(0,0)$ and the largest value of f on that side is $f=2$ at $(1,0)$.
- Finally, we look at the side that runs from $(0,2)$ to $(1,0)$. Or first job is to find the equation of the line that contains $(0,2)$ and $(1,0)$. By way of review, we'll find the equation using three different methods.
  - *Method 1:* You (probably) learned in high school that any line in the xy-plane[65] has equation $y=mx+b$ where b is the y intercept and m is the slope. In this case, the line crosses the y axis at $y=2$ and so has y intercept $b=2$. The line passes through $(0,2)$ and $(1,0)$ and so, as we see in the figure below, has slope $m=\frac{\Delta y}{\Delta x}=\frac{0-2}{1-0}=-2$. Thus the side of the triangle that runs from $(0,2)$ to $(1,0)$ is $y=2-2x$ with $0\leqslant x\leqslant 1$.



  - *Method 2:* Every line in the xy-plane has an equation of the form $ax+by=c$. In this case $(0,0)$ is *not* on the line so that $c\neq 0$ and we can divide the equation by c, giving $\frac{a}{c}x+\frac{b}{c}y=1$. Rename $\frac{a}{c}=A$ and $\frac{b}{c}=B$. Thus, because the line does not pass through the origin, it has an equation of the form $Ax+By=1$, for some constants A and B. In order for $(0,2)$ to lie on the line, $x=0$, $y=2$ has to be a solution of $Ax+By=1$. That is, $Ax\big|_{x=0}+By\big|_{y=2}=1$, so that $B=1/2$. In order for $(1,0)$ to lie on the line, $x=1$, $y=0$ has to be a solution of $Ax+By=1$. That is $Ax\big|_{x=1}+By\big|_{y=0}=1$, so that $A=1$. Thus the line has equation $x+\frac{1}{2}y=1$, or equivalently, $y=2-2x$.
  - *Method 3:* The vector from $(0,2)$ to $(1,0)$ is $\langle 1-0\,,\,0-2\rangle=\langle 1,-2\rangle$. As we see from the figure above, it is a direction vector for the line. One point on the line is $(0,2)$. So a parametric equation for the line (see Equation 1.3.1) is

$$\langle x-0\,,\,y-2\rangle=t\,\langle 1,-2\rangle \qquad \text{or} \qquad x=t,\ y=2-2t$$

65 To be picky, any line the xy-plane that is not parallel to the y axis.

By any of these three methods[66], we have that the side of the triangle that runs from $(0,2)$ to $(1,0)$ is $y = 2-2x$ with $0 \leqslant x \leqslant 1$. On that side of the triangle

$$f(x, 2-2x) = x(2-2x) + 2x + (2-2x) = -2x^2 + 2x + 2$$

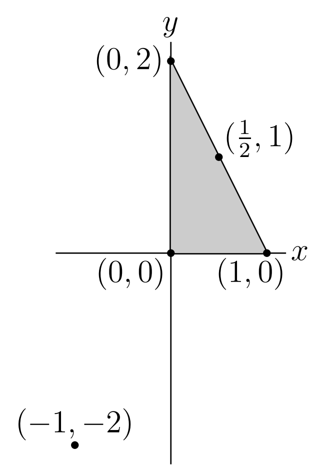
Write $g(x) = -2x^2 + 2x + 2$. The maximum and minimum of $g(x)$ for $0 \leqslant x \leqslant 1$, and hence the maximum and minimum values of f on the hypotenuse of the triangle, must be achieved either at

- $x = 0$, where $f(0,2) = g(0) = 2$, or at
- $x = 1$, where $f(1,0) = g(1) = 2$, or when
- $0 = g'(x) = -4x + 2$ so that $x = \frac{1}{2}$, $y = 2 - \frac{2}{2} = 1$ and

$$f(1/2, 1) = g(1/2) = -\frac{2}{4} + \frac{2}{2} + 2 = \frac{5}{2}$$

All together, we have the following candidates for max and min, with the max and min indicated.

point	$(0,0)$	$(0,2)$	$(1,0)$	$(1/2,1)$
value of f	0	2	2	$5/2$
	min			max



Example 2.9.24

Example 2.9.25

Find the high and low points of the surface $z = \sqrt{x^2+y^2}$ with (x,y) varying over the square $|x| \leqslant 1$, $|y| \leqslant 1$.

Solution. The function $f(x,y) = \sqrt{x^2+y^2}$ has a particularly simple geometric interpretation — it is the distance from the point (x,y) to the origin. So

66 In the third method, x has just be renamed to t.

- the minimum of $f(x,y)$ is achieved at the point in the square that is nearest the origin — namely the origin itself. So $(0,0,0)$ is the lowest point on the surface and is at height 0.

- The maximum of $f(x,y)$ is achieved at the points in the square that are farthest from the origin — namely the four corners of the square $(\pm 1, \pm 1)$. At those four points $z = \sqrt{2}$. So the highest points on the surface are $(\pm 1, \pm 1, \sqrt{2})$.

Even though we have already answered this question, it will be instructive to see what we would have found if we had followed our usual protocol. The partial derivatives of $f(x,y) = \sqrt{x^2+y^2}$ are defined for $(x,y) \neq (0,0)$ and are

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

- There are no critical points because
  - $f_x = 0$ only for $x = 0$, and
  - $f_y = 0$ only for $y = 0$, but
  - $(0,0)$ is not a critical point because f_x and f_y are not defined there.

- There is one singular point — namely $(0,0)$. The minimum value of f is achieved at the singular point.

- The boundary of the square consists of its four sides. One side is

$$\big\{\, (x,y) \,\big|\, x = 1,\ -1 \leqslant y \leqslant 1 \,\big\}$$

  On this side $f = \sqrt{1+y^2}$. As $\sqrt{1+y^2}$ increases with $|y|$, the smallest value of f on that side is 1 (when $y = 0$) and the largest value of f is $\sqrt{2}$ (when $y = \pm 1$). The same thing happens on the other three sides. The maximum value of f is achieved at the four corners. Note that f_x and f_y are both nonzero at all four corners.

Example 2.9.25

2.10▲ Lagrange Multipliers

In the last section we had to solve a number of problems of the form "What is the maximum value of the function f on the curve C?" In those examples, the curve C was simple enough that we could reduce the problem to finding the maximum of a function of one variable. For more complicated problems this reduction might not be possible. In this section, we introduce another method for solving such problems. First some nomenclature.

Definition 2.10.1.

A problem of the form

> "Find the maximum and minimum values of the function $f(x, y)$ for (x, y) on the curve $g(x, y) = 0$."

is one type of *constrained optimization* problem. The function being maximized or minimized, $f(x, y)$, is called the *objective function*. The function, $g(x, y)$, whose zero set is the curve of interest, is called the *constraint function*.

Such problems are quite common. As we said above, we have already encountered them in the last section on absolute maxima and minima, when we were looking for the extreme values of a function on the boundary of a region. In economics "utility functions" are used to model the relative "usefulness" or "desirability" or "preference" of various economic choices. For example, a utility function $U(w, \kappa)$ might specify the relative level of satisfaction a consumer would get from purchasing a quantity w of wine and κ of coffee. If the consumer wants to spend \$100 and wine costs \$20 per unit and coffee costs \$5 per unit, then the consumer would like to maximize $U(w, \kappa)$ subject to the constraint that $20w + 5\kappa = 100$.

To this point we have always solved such constrained optimization problems either by

- solving $g(x, y) = 0$ for y as a function of x (or for x as a function of y) or by
- parametrizing the curve $g(x, y) = 0$. This means writing all points of the curve in the form $\big(x(t), y(t)\big)$ for some functions $x(t)$ and $y(t)$. For example we used $x(t) = \cos t$, $y(t) = \sin t$ as a parametrization of the circle $x^2 + y^2 = 1$ in Example 2.9.21.

However quite often the function $g(x, y)$ is so complicated that one cannot explicitly solve $g(x, y) = 0$ for y as a function of x or for x as a function of y and one also cannot explicitly parametrize $g(x, y) = 0$. Or sometimes you can, for example, solve $g(x, y) = 0$ for y as a function of x, but the resulting solution is so complicated that it is really hard, or even virtually impossible, to work with. Direct attacks become even harder in higher dimensions when, for example, we wish to optimize a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$.

There is another procedure called the method of "Lagrange[67] multipliers" that comes to our rescue in these scenarios. Here is the three dimensional version of the method. There are obvious analogs is other dimensions.

67 Joseph-Louis Lagrange was actually born Giuseppe Lodovico Lagrangia in Turin, Italy in 1736. He moved to Berlin in 1766 and then to Paris in 1786. He eventually acquired French citizenship and then the French claimed he was a French mathematician, while the Italians continued to claim that he was an Italian mathematician.

Theorem 2.10.2 (Lagrange Multipliers)**.**

Let $f(x,y,z)$ and $g(x,y,z)$ have continuous first partial derivatives in a region of $\mathbb{R}^3$ that contains the surface S given by the equation $g(x,y,z) = 0$. Further assume that $\nabla g(x,y,z) \neq \mathbf{0}$ on S.

If f, restricted to the surface S, has a local extreme value at the point (a,b,c) on S, then there is a real number λ such that

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c)$$

that is

$$\begin{aligned} f_x(a,b,c) &= \lambda\, g_x(a,b,c) \\ f_y(a,b,c) &= \lambda\, g_y(a,b,c) \\ f_z(a,b,c) &= \lambda\, g_z(a,b,c) \end{aligned}$$

The number λ is called a *Lagrange multiplier*.

Proof. Suppose that (a,b,c) is a point of S and that $f(x,y,z) \geqslant f(a,b,c)$ for all points (x,y,z) on S that are close to (a,b,c). That is (a,b,c) is a local minimum for f on S. Of course the argument for a local maximum is virtually identical.

Imagine that we go for a walk on S, with the time t running, say, from $t = -1$ to $t = +1$ and that at time $t = 0$ we happen to be exactly at (a,b,c). Let's say that our position is $\big(x(t), y(t), z(t)\big)$ at time t. Write

$$F(t) = f\big(x(t), y(t), z(t)\big)$$

So $F(t)$ is the value of f that we see on our walk at time t. Then for all t close to 0, $\big(x(t), y(t), z(t)\big)$ is close to $\big(x(0), y(0), z(0)\big) = (a,b,c)$ so that

$$F(0) = f\big(x(0), y(0), z(0)\big) = f(a,b,c) \leqslant f\big(x(t), y(t), z(t)\big) = F(t)$$

for all t close to zero. So $F(t)$ has a local minimum at $t = 0$ and consequently $F'(0) = 0$.

By the chain rule, Theorem 2.4.1,

$$\begin{aligned} F'(0) &= \frac{\mathrm{d}}{\mathrm{d}t} f\big(x(t), y(t), z(t)\big)\Big|_{t=0} \\ &= f_x\big(a,b,c\big)x'(0) + f_y\big(a,b,c\big)y'(0) + f_z\big(a,b,c\big)z'(0) = 0 \end{aligned} \tag{$*$}$$

We may rewrite this as a dot product:

$$\begin{aligned} 0 = F'(0) &= \nabla f(a,b,c) \cdot \langle x'(0)\,,\, y'(0)\,,\, z'(0)\rangle \\ &\implies \nabla f(a,b,c) \perp \langle x'(0)\,,\, y'(0)\,,\, z'(0)\rangle \end{aligned}$$

This is true for all paths on S that pass through (a,b,c) at time 0. In particular it is true for all vectors $\langle x'(0)\,,\, y'(0)\,,\, z'(0)\rangle$ that are tangent to S at (a,b,c). So $\nabla f(a,b,c)$ is perpendicular to S at (a,b,c).

But we already know, by Theorem 2.5.5.a, that $\nabla g(a,b,c)$ is also perpendicular to S at (a,b,c). So $\nabla f(a,b,c)$ and $\nabla g(a,b,c)$ have to be parallel vectors. That is,

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c)$$

for some number λ. That's the Lagrange multiplier rule of our theorem. $\square$

So to find the maximum and minimum values of $f(x,y,z)$ on a surface $g(x,y,z) = 0$, assuming that both the objective function $f(x,y,z)$ and constraint function $g(x,y,z)$ have continuous first partial derivatives and that $\nabla g(x,y,z) \neq \mathbf{0}$, you

1. build up a list of candidate points (x,y,z) by finding all solutions to the equations

$$\begin{aligned} f_x(x,y,z) &= \lambda\, g_x(x,y,z) \\ f_y(x,y,z) &= \lambda\, g_y(x,y,z) \\ f_z(x,y,z) &= \lambda\, g_z(x,y,z) \\ g(x,y,z) &= 0 \end{aligned}$$

   Note that there are four equations and four unknowns, namely x, y, z and λ.

2. Then you evaluate $f(x,y,z)$ at each (x,y,z) on the list of candidates. The biggest of these candidate values is the absolute maximum and the smallest of these candidate values is the absolute minimum.

Another way to write the system of equations in the first step is

$$L_x(a,b,c,\lambda) = L_y(a,b,c,\lambda) = L_z(a,b,c,\lambda) = L_\lambda(a,b,c,\lambda) = 0$$

where $L(x,y,z,\lambda)$ is the auxiliary function[68] [69]

$$L(x,y,z,\lambda) = f(x,y,z) - \lambda\, g(x,y,z)$$

Now for a bunch of examples.

Example 2.10.3

Find the maximum and minimum of the function $x^2 - 10x - y^2$ on the ellipse whose equation is $x^2 + 4y^2 = 16$.

Solution. For this problem the objective function is $f(x,y) = x^2 - 10x - y^2$ and the constraint function is $g(x,y) = x^2 + 4y^2 - 16$. To apply the method of Lagrange multipliers we need ∇f and ∇g. So we start by computing the first order derivatives of these functions.

$$f_x = 2x - 10 \qquad f_y = -2y \qquad g_x = 2x \qquad g_y = 8y$$

68 We call L an auxiliary function because, while we use it to help solve the problem, it doesn't actually appear in either the statement of the question or in the answer itself.

69 Some people use $L(x,y,z,\lambda) = f(x,y,z) + \lambda\, g(x,y,z)$ instead. This amounts to renaming λ to $-\lambda$. While we care that λ has a value, we don't care what it is.

So, according to the method of Lagrange multipliers, we need to find all solutions to

$$\begin{aligned} 2x - 10 &= \lambda(2x) \\ -2y &= \lambda(8y) \\ x^2 + 4y^2 - 16 &= 0 \end{aligned}$$

Rearranging these equations gives

$$(\lambda - 1)x = -5 \tag{E1}$$
$$(4\lambda + 1)y = 0 \tag{E2}$$
$$x^2 + 4y^2 - 16 = 0 \tag{E3}$$

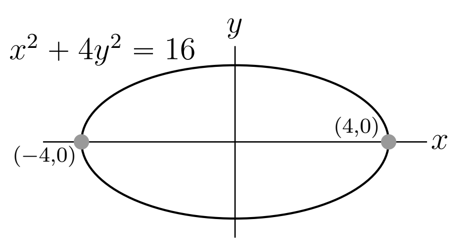
From (E2), we see that we must have either $\lambda = -1/4$ or $y = 0$.

- If $\lambda = -1/4$, (E1) gives $-\frac{5}{4}x = -5$, i.e. $x = 4$, and then (E3) gives $y = 0$.
- If $y = 0$, then (E3) gives $x = \pm 4$ (and while we could easily use (E1) to solve for λ, we don't actually need λ).

So the method of Lagrange multipliers, Theorem 2.10.2 (actually the dimension two version of Theorem 2.10.2), gives that the only possible locations of the maximum and minimum of the function f are $(4, 0)$ and $(-4, 0)$. To complete the problem, we only have to compute f at those points.

point	$(4, 0)$	$(-4, 0)$
value of f	-24	56
	min	max

Hence the maximum value of $x^2 - 10x - y^2$ on the ellipse is 56 and the minimum value is -24.



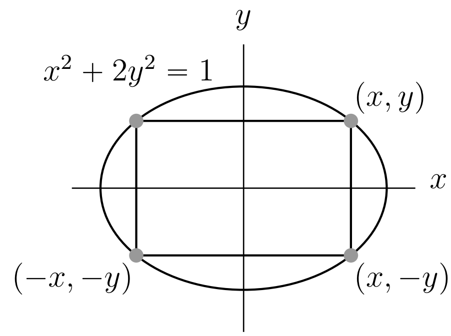
Example 2.10.3

In the previous example, the objective function and the constraint were specified explicitly. That will not always be the case. In the next example, we have to do a little geometry to extract them.

Example 2.10.4

Find the rectangle of largest area (with sides parallel to the coordinates axes) that can be inscribed in the ellipse $x^2 + 2y^2 = 1$.

Solution. Since this question is so geometric, it is best to start by drawing a picture.



Call the coordinates of the upper right corner of the rectangle (x, y), as in the figure above. The four corners of the rectangle are $(\pm x, \pm y)$ so the rectangle has width $2x$ and height $2y$ and the objective function is $f(x, y) = 4xy$. The constraint function for this problem is $g(x, y) = x^2 + 2y^2 - 1$. Again, to use Lagrange multipliers we need the first order partial derivatives.

$$f_x = 4y \qquad f_y = 4x \qquad g_x = 2x \qquad g_y = 4y$$

So, according to the method of Lagrange multipliers, we need to find all solutions to

$$4y = \lambda(2x) \tag{E1}$$
$$4x = \lambda(4y) \tag{E2}$$
$$x^2 + 2y^2 - 1 = 0 \tag{E3}$$

Equation (E1) gives $y = \frac{1}{2}\lambda x$. Substituting this into equation (E2) gives

$$4x = 2\lambda^2 x \qquad \text{or} \qquad 2x\big(2 - \lambda^2\big) = 0$$

So (E2) is satisfied if either $x = 0$ or $\lambda = \sqrt{2}$ or $\lambda = -\sqrt{2}$.

- If $x = 0$, then (E1) gives $y = 0$ too. But $(0, 0)$ violates the constraint equation (E3). Note that, to have a solution, *all* of the equations (E1), (E2) and (E3) must be satisfied.
- If $\lambda = \sqrt{2}$, then
  - (E2) gives $x = \sqrt{2}y$ and then
  - (E3) gives $2y^2 + 2y^2 = 1$ or $y^2 = \frac{1}{4}$ so that
  - $y = \pm 1/2$ and $x = \sqrt{2}y = \pm 1/\sqrt{2}$.
- If $\lambda = -\sqrt{2}$, then
  - (E2) gives $x = -\sqrt{2}y$ and then
  - (E3) gives $2y^2 + 2y^2 = 1$ or $y^2 = \frac{1}{4}$ so that
  - $y = \pm 1/2$ and $x = -\sqrt{2}y = \mp 1/\sqrt{2}$.

We now have four possible values of (x, y), namely $\big(1/\sqrt{2}\,,\,1/2\big)$, $\big(-1/\sqrt{2}\,,\,-1/2\big)$, $\big(1/\sqrt{2}\,,\,-1/2\big)$ and $\big(-1/\sqrt{2}\,,\,1/2\big)$. They are the four corners of a single rectangle. We said that we wanted (x, y) to be the upper right corner, i.e. the corner in the first quadrant. It is $\big(1/\sqrt{2}\,,\,1/2\big)$.

Example 2.10.4

Example 2.10.5

Find the ends of the major and minor axes of the ellipse $3x^2 - 2xy + 3y^2 = 4$. They are the points on the ellipse that are farthest from and nearest to the origin.

Solution. Let (x, y) be a point on $3x^2 - 2xy + 3y^2 = 4$. This point is at the end of a major axis when it maximizes its distance from the centre, $(0, 0)$ of the ellipse. It is at the end of a minor axis when it minimizes its distance from $(0, 0)$. So we wish to maximize and minimize the distance $\sqrt{x^2 + y^2}$ subject to the constraint

$$g(x, y) = 3x^2 - 2xy + 3y^2 - 4 = 0$$

Now maximizing/minimizing $\sqrt{x^2 + y^2}$ is equivalent[70] to maximizing/minimizing its square $\left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$. So we are free to choose the objective function

$$f(x, y) = x^2 + y^2$$

which we will do, because it makes the derivatives cleaner. Again, we use Lagrange multipliers to solve this problem, so we start by finding the partial derivatives.

$$f_x(x, y) = 2x \qquad f_y(x, y) = 2y \qquad g_x(x, y) = 6x - 2y \qquad g_y(x, y) = -2x + 6y$$

We need to find all solutions to

$$\begin{aligned} 2x &= \lambda(6x - 2y) \\ 2y &= \lambda(-2x + 6y) \\ 3x^2 - 2xy + 3y^2 - 4 &= 0 \end{aligned}$$

Dividing the first two equations by 2, and then collecting together the x's and the y's gives

$$(1 - 3\lambda)x + \lambda y = 0 \tag{E1}$$

$$\lambda x + (1 - 3\lambda)y = 0 \tag{E2}$$

$$3x^2 - 2xy + 3y^2 - 4 = 0 \tag{E3}$$

To start, let's concentrate on the first two equations. Pretend, for a couple of minutes, that we already know the value of λ and are trying to find x and y. Note that λ cannot be zero because if it is, (E1) forces $x = 0$ and (E2) forces $y = 0$ and $(0, 0)$ is not on the ellipse, i.e. violates (E3). So we may divide by λ and (E1) gives

$$y = -\frac{1 - 3\lambda}{\lambda} x$$

Subbing this into (E2) gives

$$\lambda x - \frac{(1 - 3\lambda)^2}{\lambda} x = 0$$

70 The function $S(z) = z^2$ is a strictly increasing function for $z \geqslant 0$. So, for $a, b \geqslant 0$, the statement "$a < b$" is equivalent to the statement "$S(a) < S(b)$".

Again, x cannot be zero, since then $y = -\frac{1-3\lambda}{\lambda}x$ would give $y = 0$ and $(0,0)$ is still not on the ellipse.

So we may divide $\lambda x - \frac{(1-3\lambda)^2}{\lambda}x = 0$ by x, giving

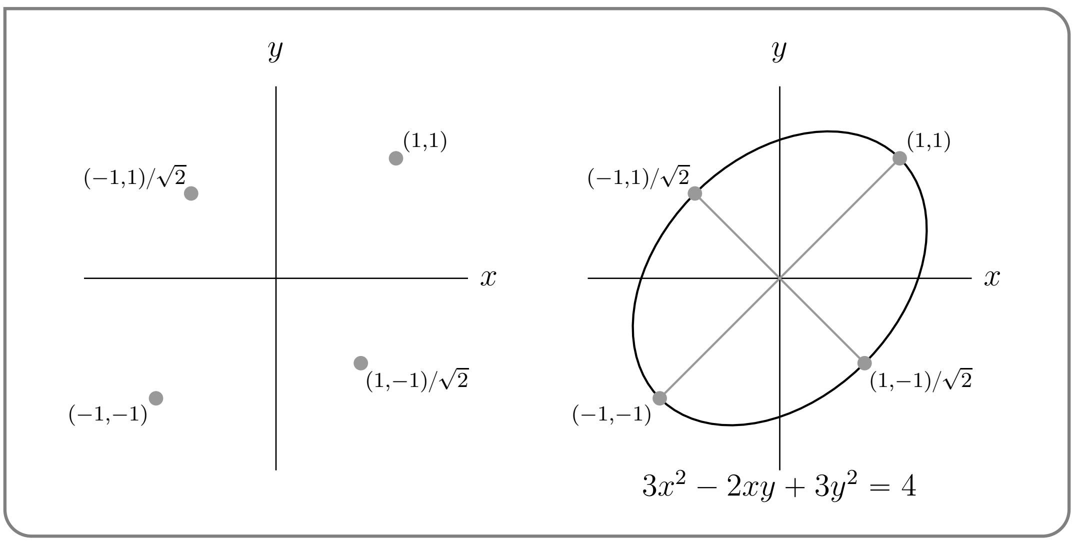
$$\begin{aligned} \lambda - \frac{(1-3\lambda)^2}{\lambda} = 0 &\iff (1-3\lambda)^2 - \lambda^2 = 0 \\ &\iff 8\lambda^2 - 6\lambda + 1 = (2\lambda - 1)(4\lambda - 1) = 0 \end{aligned}$$

We now know that λ must be either $\frac{1}{2}$ or $\frac{1}{4}$. Subbing these into either (E1) or (E2) gives

$$\lambda = \frac{1}{2} \implies -\frac{1}{2}x + \frac{1}{2}y = 0 \implies x = y \overset{(E3)}{\implies} 3x^2 - 2x^2 + 3x^2 = 4 \implies x = \pm 1$$

$$\lambda = \frac{1}{4} \implies \frac{1}{4}x + \frac{1}{4}y = 0 \implies x = -y \overset{(E3)}{\implies} 3x^2 + 2x^2 + 3x^2 = 4 \implies x = \pm\frac{1}{\sqrt{2}}$$

Here " $\overset{(E3)}{\implies}$ " indicates that we have just used (E3). We now have $(x,y) = \pm(1,1)$, from $\lambda = \frac{1}{2}$, and $(x,y) = \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ from $\lambda = \frac{1}{4}$. The distance from $(0,0)$ to $\pm(1,1)$, namely $\sqrt{2}$, is larger than the distance from $(0,0)$ to $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, namely 1. So the ends of the minor axes are $\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and the ends of the major axes are $\pm(1,1)$. Those ends are sketched in the figure on the left below. Once we have the ends, it is an easy matter[71] to sketch the ellipse as in the figure on the right below.



Example 2.10.5

Example 2.10.6

Find the values of $w \geqslant 0$ and $\kappa \geqslant 0$ that maximize the utility function

$$U(w,\kappa) = 6w^{2/3}\kappa^{1/3} \qquad \text{subject to the constraint} \qquad 4w + 2\kappa = 12$$

71 if you tilt your head so that the line through $(1,1)$ and $(-1,-1)$ appears horizontal

Solution. The constraint $4w + 2\kappa = 12$ is simple enough that we can easily use it to express κ in terms of w, then substitute $\kappa = 6 - 2w$ into $U(w,\kappa)$, and then maximize $U(w, 6-2w) = 6w^{2/3}(6-2w)^{1/3}$ using the techniques of §3.5 in the CLP-1 textbook.

However, for practice purposes, we'll use Lagrange multipliers with the objective function $U(w,\kappa) = 6w^{2/3}\kappa^{1/3}$ and the constraint function $g(w,\kappa) = 4w + 2\kappa - 12$. The first order derivatives of these functions are

$$U_w = 4w^{-1/3}\kappa^{1/3} \qquad U_\kappa = 2w^{2/3}\kappa^{-2/3} \qquad g_w = 4 \qquad g_\kappa = 2$$

The boundary values $w = 0$ and $\kappa = 0$ give utility 0, which is obviously not going to be the maximum utility. So it suffices to consider only local maxima. According to the method of Lagrange multipliers, we need to find all solutions to

$$4w^{-1/3}\kappa^{1/3} = 4\lambda \tag{E1}$$
$$2w^{2/3}\kappa^{-2/3} = 2\lambda \tag{E2}$$
$$4w + 2\kappa - 12 = 0 \tag{E3}$$

Then

- equation (E1) gives $\lambda = w^{-1/3}\kappa^{1/3}$.
- Substituting this into (E2) gives $w^{2/3}\kappa^{-2/3} = \lambda = w^{-1/3}\kappa^{1/3}$ and hence $w = \kappa$.
- Then substituting $w = \kappa$ into (E3) gives $6\kappa = 12$.

So $w = \kappa = 2$ and the maximum utility is $U(2,2) = 12$.

Example 2.10.6

Example 2.10.7

Find the point on the sphere $x^2 + y^2 + z^2 = 1$ that is farthest from $(1,2,3)$.

Solution. As before, we simplify the algebra by maximizing the square of the distance rather than the distance itself. So we are to maximize

$$f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2$$

subject to the constraint

$$g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$$

Since

$$f_x(x,y,z) = 2(x-1) \qquad f_y(x,y,z) = 2(y-2) \qquad f_z(x,y,z) = 2(z-3)$$
$$g_x(x,y,z) = 2x \qquad g_y(x,y,z) = 2y \qquad g_z(x,y,z) = 2z$$

we need to find all solutions to

$$2(x-1) = \lambda(2x) \qquad \iff \qquad x = \frac{1}{1-\lambda} \tag{E1}$$
$$2(y-2) = \lambda(2y) \qquad \iff \qquad y = \frac{2}{1-\lambda} \tag{E2}$$
$$2(z-3) = \lambda(2z) \qquad \iff \qquad z = \frac{3}{1-\lambda} \tag{E3}$$
$$0 = x^2 + y^2 + z^2 - 1 \tag{E4}$$

Substituting (E1), (E2) and (E3) into (E4) gives

$$\frac{1+4+9}{(1-\lambda)^2} - 1 = 0 \implies (1-\lambda)^2 = 14 \implies 1-\lambda = \pm\sqrt{14}$$

We can then substitute these two values of λ back into the expressions for x, y, z in terms of λ to get the two points $\frac{1}{\sqrt{14}}(1,2,3)$ and $-\frac{1}{\sqrt{14}}(1,2,3)$.

The vector from $\frac{1}{\sqrt{14}}(1,2,3)$ to $(1,2,3)$, namely $\left\{1-\frac{1}{\sqrt{14}}\right\}(1,2,3)$, is obviously shorter than the vector from $-\frac{1}{\sqrt{14}}(1,2,3)$ to $(1,2,3)$, which is $\left\{1+\frac{1}{\sqrt{14}}\right\}(1,2,3)$. So the nearest point is $\frac{1}{\sqrt{14}}(1,2,3)$ and the farthest point is $-\frac{1}{\sqrt{14}}(1,2,3)$.

Example 2.10.7

2.10.1 ▸▸ (Optional) An Example with Two Lagrange Multipliers

In this optional section, we consider an example of a problem of the form "maximize (or minimize) $f(x,y,z)$ subject to the two constraints $g(x,y,z) = 0$ and $h(x,y,z) = 0$". We use the following variant of Theorem 2.10.2.

Theorem 2.10.8 (Two Lagrange Multipliers)**.**

Let $f(x,y,z)$, $g(x,y,z)$ and $h(x,y,z)$ have continuous first partial derivatives in a region of $\mathbb{R}^3$ that contains the curve C given by the equations

$$g(x,y,z) = h(x,y,z) = 0$$

Assume[72] that $\nabla g(x,y,z) \times \nabla h(x,y,z) \neq 0$ on C. If f, restricted to the curve C, has a local extreme value at the point (a,b,c) on C, then there are real numbers λ and μ such that

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c) + \mu \nabla h(a,b,c)$$

that is

$$\begin{aligned} f_x(a,b,c) &= \lambda\, g_x(a,b,c) + \mu\, h_x(a,b,c) \\ f_y(a,b,c) &= \lambda\, g_y(a,b,c) + \mu\, h_y(a,b,c) \\ f_z(a,b,c) &= \lambda\, g_z(a,b,c) + \mu\, h_z(a,b,c) \end{aligned}$$

We can reformulate this theorem in terms of the auxiliary function

$$L(x,y,z,\lambda,\mu) = f(x,y,z) - \lambda\, g(x,y,z) - \mu\, h(x,y,z)$$

It is a function of five variables — the original variables x, y and z, and two auxiliary variables λ and μ. If there is a local extreme value at (a,b,c) then (a,b,c) must obey

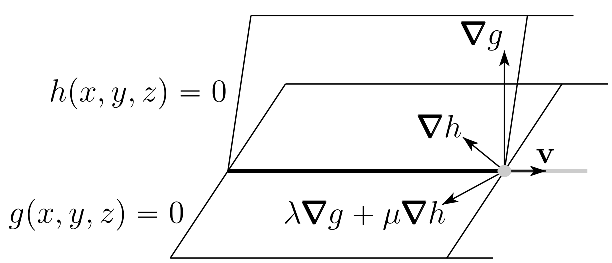
72 This condition says that the normal vectors to $g = 0$ and $h = 0$ at (x,y,z) are not parallel. This ensures that the surfaces $g = 0$ and $h = 0$ are not tangent to each other at (x,y,z). They intersect in a curve.

Equation 2.10.9.

$$\begin{aligned}
0 = L_x(a,b,c,\lambda,\mu) &= f_x(a,b,c) - \lambda g_x(a,b,c) - \mu h_x(a,b,c) \\
0 = L_y(a,b,c,\lambda,\mu) &= f_y(a,b,c) - \lambda g_y(a,b,c) - \mu h_y(a,b,c) \\
0 = L_z(a,b,c,\lambda,\mu) &= f_z(a,b,c) - \lambda g_z(a,b,c) - \mu h_z(a,b,c) \\
0 = L_\lambda(a,b,c,\lambda,\mu) &= g(a,b,c) \\
0 = L_\mu(a,b,c,\lambda,\mu) &= h(a,b,c)
\end{aligned}$$

for some λ and μ. So solving this system of five equations in five unknowns gives all possible candidates for the locations of local maxima and minima. We'll go through an example shortly.

Proof of Theorem 2.10.8. Before we get to the example itself, here is why the above approach works. Assume that a local minimum occurs at (a,b,c), which is the grey point in the schematic figure below. Imagine that you start walking away from (a,b,c) along the curve $g = h = 0$. Your path is the grey line in the schematic figure below. Call your



velocity vector $\mathbf{v}$. It is tangent to the curve $g(x,y,z) = h(x,y,z) = 0$. Because f has a local minimum at (a,b,c), f must be increasing (or constant) as we leave (a,b,c). So the directional derivative

$$D_{\mathbf{v}} f(a,b,c) = \nabla f(a,b,c) \cdot \mathbf{v} \geqslant 0$$

Now start over. Again walk away from (a,b,c) along the curve $g = h = 0$, but this time moving in the opposite direction, with velocity vector $-\mathbf{v}$. Again f must be increasing (or constant) as we leave (a,b,c), so the directional derivative

$$D_{-\mathbf{v}} f(a,b,c) = \nabla f(a,b,c) \cdot (-\mathbf{v}) \geqslant 0$$

As both $\nabla f(a,b,c) \cdot \mathbf{v}$ and $-\nabla f(a,b,c) \cdot \mathbf{v}$ are at least zero, we now have that

$$\nabla f(a,b,c) \cdot \mathbf{v} = 0 \tag{$*$}$$

for all vectors $\mathbf{v}$ that are tangent to the curve $g = h = 0$ at (a,b,c). Let's denote by $\mathcal{T}$ the set of all vectors $\mathbf{v}$ that are tangent to the curve $g = h = 0$ at (a,b,c) and let's denote by $\mathcal{T}^\perp$ the set of all vectors that are perpendicular to all vectors in $\mathcal{T}$. So $(*)$ says that $\nabla f(a,b,c)$ must in $\mathcal{T}^\perp$.

We now find all vectors in $\mathcal{T}^\perp$. We can easily guess two such vectors. Since the curve $g = h = 0$ lies inside the surface $g = 0$ and $\nabla g(a,b,c)$ is normal to $g = 0$ at (a,b,c), we have

$$\nabla g(a,b,c) \cdot \mathbf{v} = 0 \tag{E1}$$

Similarly, since the the curve $g = h = 0$ lies inside the surface $h = 0$ and $\nabla h(a,b,c)$ is normal to $h = 0$ at (a,b,c), we have

$$\nabla h(a,b,c) \cdot \mathbf{v} = 0 \tag{E2}$$

Picking any two constants λ and μ, multiplying (E1) by λ, multiplying (E2) by μ and adding gives that

$$\big(\lambda \nabla g(a,b,c) + \mu \nabla h(a,b,c)\big) \cdot \mathbf{v} = 0$$

for all vectors $\mathbf{v}$ in $\mathcal{T}$. Thus, for all λ and μ, the vector $\lambda \nabla g(a,b,c) + \mu \nabla h(a,b,c)$ is in $\mathcal{T}^\perp$.

Now the vectors in $\mathcal{T}$ form a line. (They are all tangent to the same curve at the same point.) So, $\mathcal{T}^\perp$, the set of all vectors perpendicular to $\mathcal{T}$, forms a plane. As λ and μ run over all real numbers, the vectors $\lambda \nabla g(a,b,c) + \mu \nabla h(a,b,c)$ form a plane. Thus we have found all vector in $\mathcal{T}^\perp$ and we conclude that $\nabla f(a,b,c)$ must be of the form $\lambda \nabla g(a,b,c) + \mu \nabla h(a,b,c)$ for some real numbers λ and μ. The three components of the equation

$$\nabla f(a,b,c) = \lambda \nabla g(a,b,c) + \mu \nabla h(a,b,c)$$

are exactly the first three equations of (2.10.9). This completes the explanation of why Lagrange multipliers work in this setting. $\square$

Example 2.10.10

Find the distance from the origin to the curve that is the intersection of the two surfaces

$$z^2 = x^2 + y^2 \qquad x - 2z = 3$$

Solution. Yet again, we simplify the algebra by maximizing the square of the distance rather than the distance itself. So we are to maximize

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to the constraints

$$0 = g(x,y,z) = x^2 + y^2 - z^2 \qquad 0 = h(x,y,z) = x - 2z - 3$$

Since

$$\begin{array}{lll} f_x = 2x & f_y = 2y & f_z = 2z \\ g_x = 2x & g_y = 2y & g_z = -2z \\ h_x = 1 & h_y = 0 & h_z = -2 \end{array}$$

the method of Lagrange multipliers requires us to find all solutions to

$$2x = \lambda(2x) + \mu(1) \tag{E1}$$
$$2y = \lambda(2y) + \mu(0) \qquad \iff \qquad (1-\lambda)y = 0 \tag{E2}$$
$$2z = \lambda(-2z) + \mu(-2) \tag{E3}$$
$$z^2 = x^2 + y^2 \tag{E4}$$
$$x - 2z = 3 \tag{E5}$$

Since equation (E2) factors so nicely we start there. It tells us that either $y = 0$ or $\lambda = 1$.

Case $\lambda = 1$: When $\lambda = 1$ the remaining equations reduce to

$$0 = \mu \tag{E1}$$
$$0 = 4z + 2\mu \tag{E3}$$
$$z^2 = x^2 + y^2 \tag{E4}$$
$$x - 2z = 3 \tag{E5}$$

So

- equation (E1) gives $\mu = 0$.
- Then substituting $\mu = 0$ into (E3) gives $z = 0$.
- Then substituting $z = 0$ into (E5) gives $x = 3$.
- Then substituting $z = 0$ and $x = 3$ into (E4) gives $0 = 9 + y^2$, which is impossible, since $9 + y^2 \geqslant 9 > 0$ for all y.

So we can't have $\lambda = 1$.

Case $y = 0$: When $y = 0$ the remaining equations reduce to

$$2(1-\lambda)x = \mu \tag{E1}$$
$$(1+\lambda)z = -\mu \tag{E3}$$
$$z^2 = x^2 \tag{E4}$$
$$x - 2z = 3 \tag{E5}$$

These don't clean up quite so nicely as in the $\lambda = 1$ case. But at least equation (E4) tells us that $z = \pm x$. So we have to consider those two possibilities.

Subcase $y = 0$, $z = x$: When $y = 0$ and $z = x$, the remaining equations reduce to

$$2(1-\lambda)x = \mu \tag{E1}$$
$$(1+\lambda)x = -\mu \tag{E3}$$
$$-x = 3 \tag{E5}$$

So equation (E5) now tells us that $x = -3$ so that $(x, y, z) = (-3, 0, -3)$. (We don't really care what λ and μ are. But as they obey $-6(1-\lambda) = \mu$, $-3(1+\lambda) = -\mu$ we have, adding the two equations together

$$-9 + 3\lambda = 0 \implies \lambda = 3$$

and then, subbing into either equation, $\mu = 12$.)

Subcase $y = 0, z = -x$: When $y = 0$ and $z = -x$, the remaining equations reduce to

$$2(1 - \lambda)x = \mu \tag{E1}$$
$$(1 + \lambda)x = \mu \tag{E3}$$
$$3x = 3 \tag{E5}$$

So equation (E5) now tells us that $x = 1$ so that $(x, y, z) = (1, 0, -1)$. (Again, we don't really care what λ and μ are. But as they obey $2(1 - \lambda) = \mu$, $(1 + \lambda) = \mu$ we have, subtracting the second equation from the first,

$$1 - 3\lambda = 0 \implies \lambda = \frac{1}{3}$$

and then, subbing into either equation, $\mu = \frac{4}{3}$.)

Conclusion: We have two candidates for the location of the max and min, namely $(-3, 0, -3)$ and $(1, 0, -1)$. The first is a distance $3\sqrt{2}$ from the origin, giving the maximum, and the second is a distance $\sqrt{2}$ from the origin, giving the minimum. In particular, the distance is $\sqrt{2}$.

Example 2.10.10

Chapter 3

MULTIPLE INTEGRALS

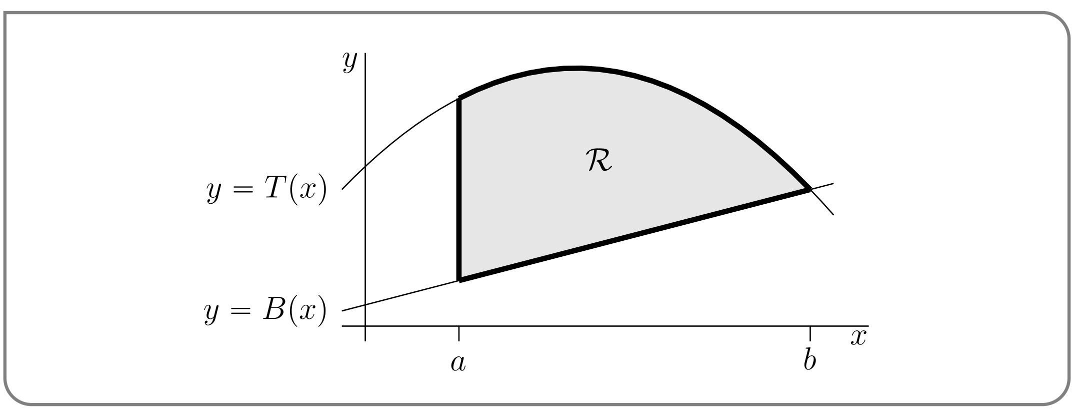
In your previous calculus courses you defined and worked with single variable integrals, like $\int_a^b f(x)\,\mathrm{d}x$. In this chapter, we define and work with multivariable integrals, like $\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ and $\iiint_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. We start with two variable integrals.

3.1▲ Double Integrals

3.1.1 ▸▸ Vertical Slices

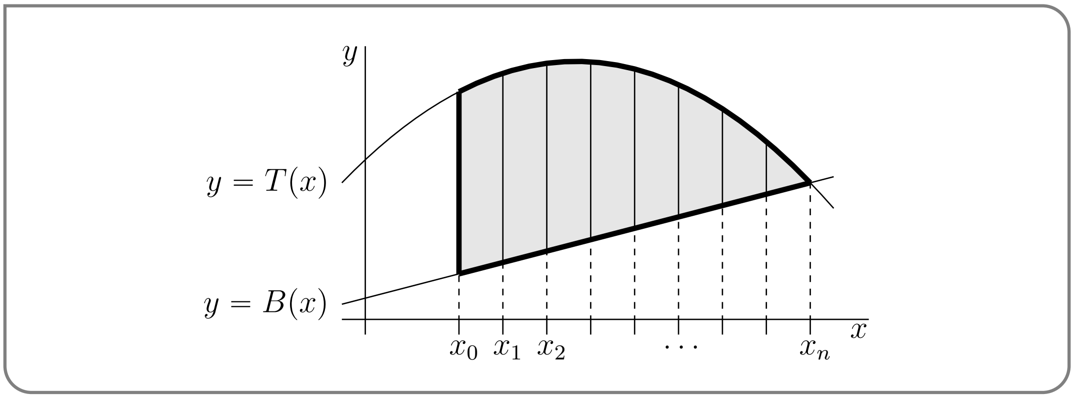
Suppose that you want to compute the mass of a plate that fills the region $\mathcal{R}$ in the xy-plane. Suppose further that the density of the plate, say in kilograms per square meter, depends on position. Call the density $f(x,y)$. For simplicity we'll assume that $\mathcal{R}$ is the region between the bottom curve $y = B(x)$ and the top curve $y = T(x)$ with x running from a to b. That is,

$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ a \leqslant x \leqslant b,\ B(x) \leqslant y \leqslant T(x)\ \right\}$$

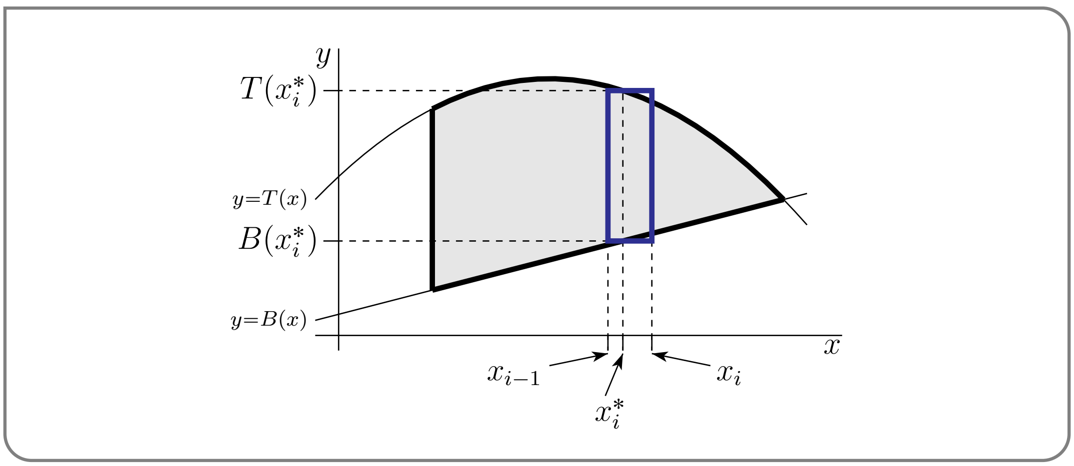


We'll shortly express that mass as a two dimensional integral. As a warmup, recall the procedure that we used to set up a (one dimensional) integral representing the area of $\mathcal{R}$ in Example 1.5.1 of the CLP-2 text.

- Pick a natural number n (that we will later send to infinity), and then
- subdivide $\mathcal{R}$ into n narrow vertical slices, each of width $\Delta x = \frac{b-a}{n}$. Denote by $x_i = a + i\,\Delta x$ the x-coordinate of the right hand edge of slice number i.



- For each $i = 1, 2, \ldots, n$, slice number i has x running from x_{i-1} to x_i. We approximate its area by the area of a rectangle. We pick a number x_i^* between x_{i-1} and x_i and approximate the slice by a rectangle whose top is at $y = T(x_i^*)$ and whose bottom is at $y = B(x_i^*)$. The rectangle is outlined in blue in the figure below.



- Thus the area of slice i is approximately $\big[T(x_i^*) - B(x_i^*)\big]\Delta x$.
- So the Riemann sum approximation of the area of $\mathcal{R}$ is

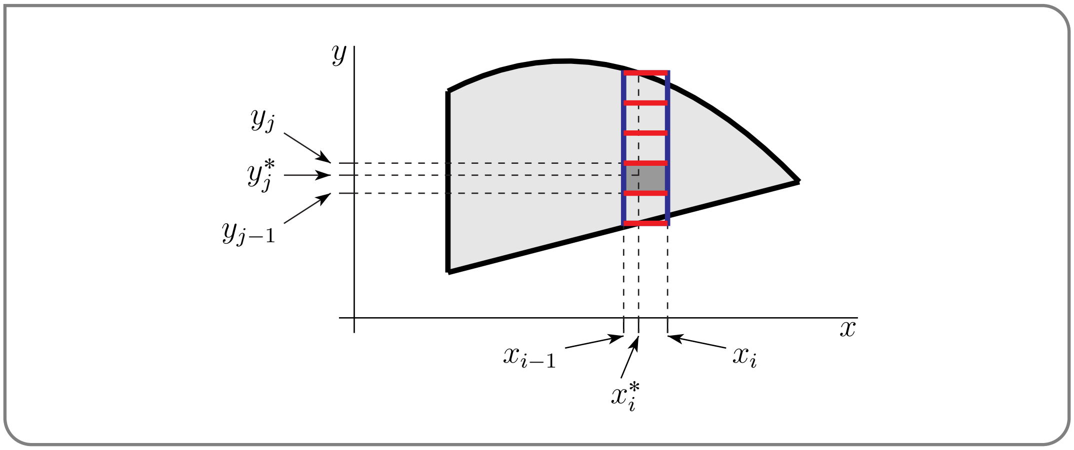
$$\text{Area} \approx \sum_{i=1}^{n} \big[T(x_i^*) - B(x_i^*)\big]\Delta x$$

- By taking the limit as $n \to \infty$ (i.e. taking the limit as the width of the rectangles goes to zero), we convert the Riemann sum into a definite integral (see Definition 1.1.9 in the CLP-2 text) and at the same time our approximation of the area becomes the exact area:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \big[T(x_i^*) - B(x_i^*)\big]\Delta x = \int_a^b \big[T(x) - B(x)\big]\,\mathrm{d}x$$

Now we can expand that procedure to yield the mass of $\mathcal{R}$ rather than the area of $\mathcal{R}$. We just have to replace our approximation $\big[T(x_i^*) - B(x_i^*)\big]\Delta x$ of the area of slice i by an approximation to the mass of slice i. To do so, we

- Pick a natural number m (that we will later send to infinity), and then
- subdivide slice number i into m tiny rectangles, each of width Δx and of height $\Delta y = \frac{1}{m}\big[T(x_i^*) - B(x_i^*)\big]$. Denote by $y_j = B(x_i^*) + j\,\Delta y$ the y-coordinate of the top of rectangle number j.



- At this point we approximate the density inside each rectangle by a constant. For each $j = 1, 2, \ldots, m$, rectangle number j has y running from y_{j-1} to y_j. We pick a number y_j^* between y_{j-1} and y_j and approximate the density on rectangle number j in slice number i by the constant $f\big(x_i^*, y_j^*\big)$.
- Thus the mass of rectangle number j in slice number i is approximately $f\big(x_i^*, y_j^*\big)\,\Delta x\,\Delta y$.
- So the Riemann sum approximation of the mass of slice number i is

$$\text{Mass of slice } i \approx \sum_{j=1}^{m} f\big(x_i^*, y_j^*\big)\,\Delta x\,\Delta y$$

  Note that the y_j^*'s depend on i and m.

- By taking the limit as $m \to \infty$ (i.e. taking the limit as the height of the rectangles goes to zero), we convert the Riemann sum into a definite integral:

$$\text{Mass of slice } i \approx \Delta x \int_{B(x_i^*)}^{T(x_i^*)} f\big(x_i^*, y\big)\,\mathrm{d}y = F(x_i^*)\,\Delta x$$

  where

$$F(x) = \int_{B(x)}^{T(x)} f\big(x, y\big)\,\mathrm{d}y$$

  Notice that, while we started with the density $f(x, y)$ being a function of both x and y, by taking the limit of this Riemann sum, we have "integrated out" the dependence on y. As a result, $F(x)$ is a function of x only, not of x and y.

- Finally taking the limit as $n \to \infty$ (i.e. taking the limit as the slice width goes to zero), we get

$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \int_{B(x_i^*)}^{T(x_i^*)} f\big(x_i^*, y\big)\,\mathrm{d}y = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\,\Delta x$$

  Now we are back in familiar 1-variable territory. The sum $\sum\limits_{i=1}^{n} F(x_i^*)\,\Delta x$ is a Riemann sum approximation to the integral $\int_a^b F(x)\,\mathrm{d}x$. So

$$\text{Mass} = \int_a^b F(x)\,\mathrm{d}x = \int_a^b \left[\int_{B(x)}^{T(x)} f\big(x,y\big)\,\mathrm{d}y\right]\mathrm{d}x$$

This is our first double integral. There are a couple of different standard notations for this integral.

Notation 3.1.1.

$$\begin{aligned}\iint\limits_{\mathcal{R}} f\big(x,y\big)\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \left[\int_{B(x)}^{T(x)} f\big(x,y\big)\,\mathrm{d}y\right]\mathrm{d}x \\ &= \int_a^b \int_{B(x)}^{T(x)} f\big(x,y\big)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\, f\big(x,y\big)\end{aligned}$$

The last three integrals here are called iterated integrals, for obvious reasons.

Note that

- to evaluate the integral $\displaystyle\int_a^b \int_{B(x)}^{T(x)} f\big(x,y\big)\,\mathrm{d}y\,\mathrm{d}x$,
  - first evaluate the inside integral $\int_{B(x)}^{T(x)} f\big(x,y\big)\,\mathrm{d}y$ using the inside limits of integration, and by treating x as a constant and using standard single variable integration techniques, such as those in the CLP-2 text. The result of the inside integral is a function of x only. Call it $F(x)$.
  - Then evaluate the outside integral $\int_a^b F(x)\,\mathrm{d}x$, whose integrand is the answer to the inside integral. Again, this integral is evaluated using standard single variable integration techniques.
- To evaluate the integral $\displaystyle\int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\, f\big(x,y\big)$,
  - first evaluate the inside integral $\int_{B(x)}^{T(x)} \mathrm{d}y\, f\big(x,y\big)$ using the limits of integration that are directly beside the $\mathrm{d}y$. Indeed the $\mathrm{d}y$ is written directly beside $\int_{B(x)}^{T(x)}$ to

make it clear that the limits of integration $B(x)$ and $T(x)$ are for the y-integral. In the past you probably wrote this integral as $\int_{B(x)}^{T(x)} f(x,y)\,\mathrm{d}y$. The result of the inside integral is again a function of x only. Call it $F(x)$.

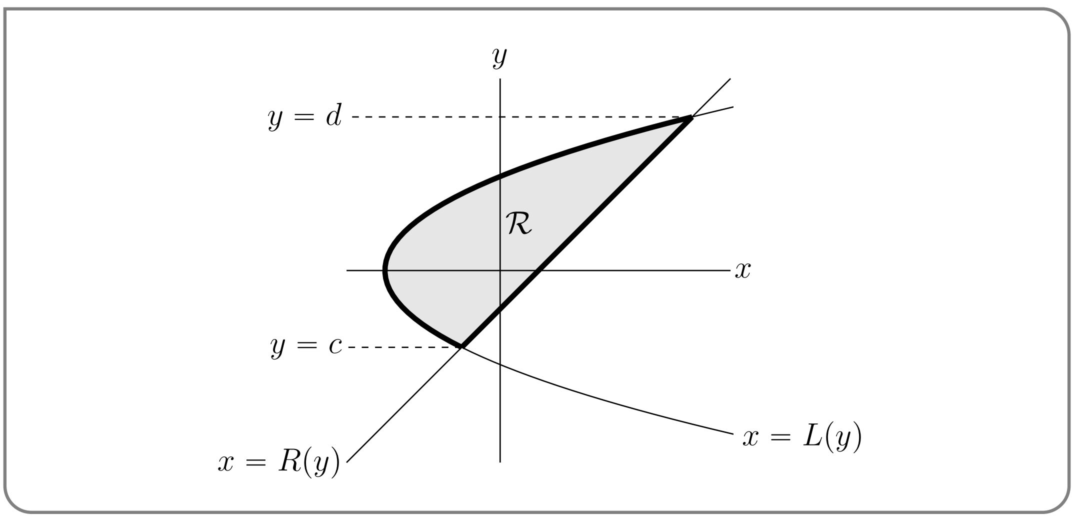
– Then evaluate the outside integral $\int_a^b \mathrm{d}x\, F(x)$, whose integrand is the answer to the inside integral and whose limits of integration are directly beside the $\mathrm{d}x$.

At this point you may be wondering "Do we always have to use vertical slices?" and "Do we always have to integrate with respect to y first?" The answer is "no". This brings us to consider "horizontal slices".

3.1.2 ▸▸ Horizontal Slices

We found, when computing areas of regions in the xy-plane, that it is often advantageous to use horizontal slices, rather than vertical slices. See, for example, Example 1.5.4 in the CLP-2 text. The same is true when setting up multidimensional integrals. So we now repeat the setup procedure of the last section, but starting with horizontal slices, rather than vertical slices. This procedure will be useful when dealing with regions of the form

$$\mathcal{R} = \big\{\ (x,y)\ \big|\ c \leqslant y \leqslant d,\ L(y) \leqslant x \leqslant R(y)\ \big\}$$

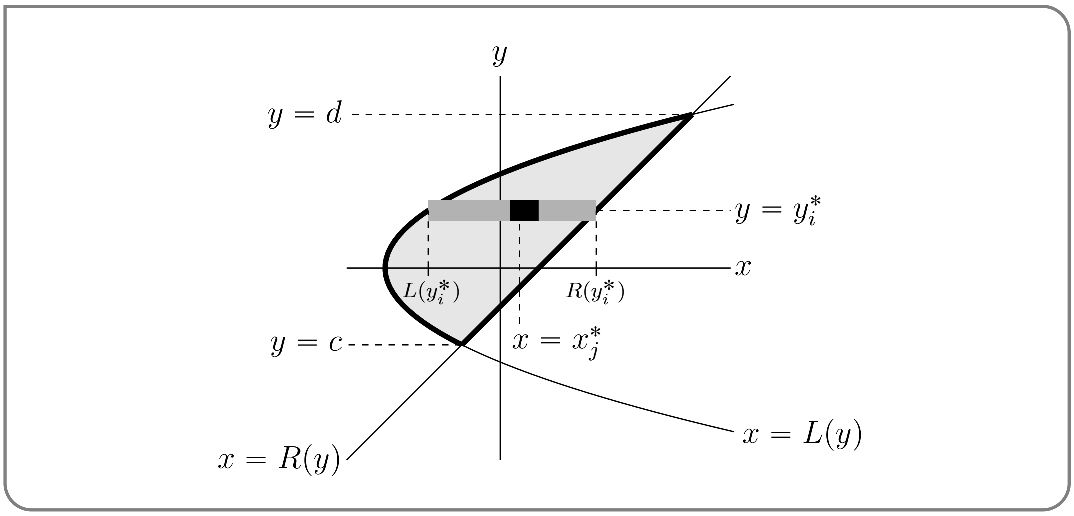


Here $L(y)$ ("L" stands for "left") is the smallest[1] allowed value of x, when the y-coordinate is y, and $R(y)$ ("R" stands for "right") is the largest allowed value of x, when the y-coordinate is y. Suppose that we wish to evaluate the mass of a plate that fills the region $\mathcal{R}$, and that the density of the plate is $f(x,y)$. We follow essentially the same the procedure as we used with vertical slices, but with the roles of x and y swapped.

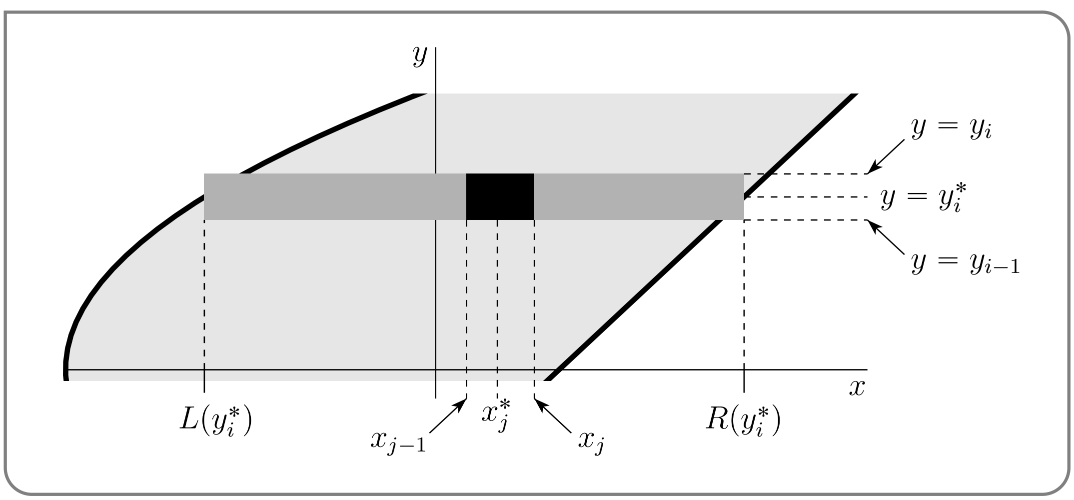
- Pick a natural number n (that we will later send to infinity). Then
- subdivide the interval $c \leqslant y \leqslant d$ into n narrow subintervals, each of width $\Delta y = \frac{d-c}{n}$. Each subinterval cuts a thin horizontal slice from the region (see the figure below).

1 By the "smallest " x we mean the x farthest to the left along the number line, not the x closest to 0.

- We approximate slice number i by a thin horizontal rectangle (indicated by the long darker gray rectangle in the figure below). On this slice, the y-coordinate runs over a very narrow range. We pick a number y_i^*, somewhere in that range. We approximate slice i by a rectangle whose left side is at $x = L(y_i^*)$ and whose right side is at $x = R(y_i^*)$.
- If we were computing the area of $\mathcal{R}$, we would now approximate the area of slice i by $\big[R(x_i^*) - L(x_i^*)\big]\Delta y$, which is the area of the rectangle with width $\big[R(x_i^*) - L(x_i^*)\big]$ and height Δy.
- To get the mass, just as we did above with vertical slices, we
  - pick another natural number m (that we will later send to infinity), and then
  - subdivide slice number i into m tiny rectangles, each of height Δy and of width $\Delta x = \frac{1}{m}\big[R(y_i^*) - L(y_i^*)\big]$.
  - For each $j = 1, 2, \dots, m$, rectangle number j has x running over a very narrow range. We pick a number x_j^* somewhere in that range. See the small black rectangle in the figure below.



Here is a magnified sketch of slice number i



- On rectangle number j in slice number i, we approximate the density by $f(x_j^*, y_i^*)$, giving us that the mass of rectangle number j in slice number i is approximately $f(x_j^*, y_i^*)\,\Delta x\,\Delta y$.
- So the Riemann sum approximation of the mass of (horizontal) slice number i is

$$\text{Mass of slice } i \approx \sum_{j=1}^{m} f(x_j^*, y_i^*)\,\Delta x\,\Delta y$$

- By taking the limit as $m \to \infty$ (i.e. taking the limit as the width of the rectangles goes to zero), we convert the Riemann sum into a definite integral:

$$\text{Mass of slice } i \approx \Delta y \int_{L(y_i^*)}^{R(y_i^*)} f(x, y_i^*)\,\mathrm{d}x = F(y_i^*)\,\Delta y$$

where

$$F(y) = \int_{L(y)}^{R(y)} f(x, y)\,\mathrm{d}x$$

Observe that, as x has been integrated out, $F(y)$ is a function of y only, not of x and y.

- Finally taking the limit as $n \to \infty$ (i.e. taking the limit as the slice width goes to zero), we get

$$\text{Mass} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta y \int_{L(y_i^*)}^{R(y_i^*)} f(x, y_i^*)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} F(y_i^*)\,\Delta y$$

Now $\sum\limits_{i=1}^{n} F(y_i^*)\,\Delta y$ is a Riemann sum approximation to the integral $\int_c^d F(y)\,\mathrm{d}y$. So

$$\text{Mass} = \int_c^d F(y)\,\mathrm{d}y = \int_c^d \left[\int_{L(y)}^{R(y)} f(x, y)\,\mathrm{d}x\right]\mathrm{d}y$$

The standard notations of Notation 3.1.1 also apply to this integral.

Notation 3.1.2.

$$\iint_{\mathcal{R}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left[\int_{L(y)}^{R(y)} f(x, y)\,\mathrm{d}x\right]\mathrm{d}y$$
$$= \int_c^d \int_{L(y)}^{R(y)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\, f(x, y)$$

Note that

- to evaluate the integral $\displaystyle\int_c^d \int_{L(y)}^{R(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$,
  - first evaluate the inside integral $\int_{L(y)}^{R(y)} f(x,y)\,\mathrm{d}x$ using the inside limits of integration. The result of the inside integral is a function of y only. Call it $F(y)$.
  - Then evaluate the outside integral $\int_c^d F(y)\,\mathrm{d}y$, whose integrand is the answer to the inside integral.
- To evaluate the integral $\displaystyle\int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\, f(x,y)$,
  - first evaluate the inside integral $\int_{L(y)}^{R(y)} \mathrm{d}x\, f(x,y)$ using the limits of integration that are directly beside the $\mathrm{d}x$. Again, the $\mathrm{d}x$ is written directly beside $\int_{L(y)}^{R(y)}$ to make it clear that the limits of integration $L(y)$ and $R(y)$ are for the x-integral. In the past you probably wrote this integral as $\int_{L(y)}^{R(y)} f(x,y)\,\mathrm{d}x$. The result of the inside integral is again a function of y only. Call it $F(y)$.
  - Then evaluate the outside integral $\int_c^d \mathrm{d}y\, F(y)$, whose integrand is the answer to the inside integral and whose limits of integration are directly beside the $\mathrm{d}y$.

By way of summary, we now have two integral representations for the mass of regions in the xy-plane.

Theorem 3.1.3.

Let $\mathcal{R}$ be a region in the xy-plane and let the function $f(x,y)$ be defined and continuous on $\mathcal{R}$.

(a) If

$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ a \leqslant x \leqslant b,\ B(x) \leqslant y \leqslant T(x)\ \right\}$$

with $B(x)$ and $T(x)$ being continuous, and if the mass density in $\mathcal{R}$ is $f(x,y)$, then the mass of $\mathcal{R}$ is

$$\int_a^b \left[\int_{B(x)}^{T(x)} f(x,y)\,\mathrm{d}y\right]\mathrm{d}x = \int_a^b \int_{B(x)}^{T(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\, f(x,y)$$

(b) If

$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ c \leqslant y \leqslant d,\ L(y) \leqslant x \leqslant R(y)\ \right\}$$

with $L(y)$ and $R(y)$ being continuous, and if the mass density in $\mathcal{R}$ is $f(x,y)$, then the mass of $\mathcal{R}$ is

$$\int_c^d \left[\int_{L(y)}^{R(y)} f(x,y)\,\mathrm{d}x\right]\mathrm{d}y = \int_c^d \int_{L(y)}^{R(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\, f(x,y)$$

Implicit in Theorem 3.1.3 is the statement that, if

$$\begin{aligned} &\{\ (x,y)\ \big|\ a \leqslant x \leqslant b,\ B(x) \leqslant y \leqslant T(x)\ \} \\ =&\{\ (x,y)\ \big|\ c \leqslant y \leqslant d,\ L(y) \leqslant x \leqslant R(y)\ \} \end{aligned}$$

and if $f(x,y)$ is continuous, then

$$\int_a^b \int_{B(x)}^{T(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_{L(y)}^{R(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

This is called Fubini's theorem[2]. It will be discussed more in the optional §3.1.5.

Notation 3.1.4.

The integrals of Theorem 3.1.3 are often denoted

$$\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y \qquad \text{or} \qquad \iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}A$$

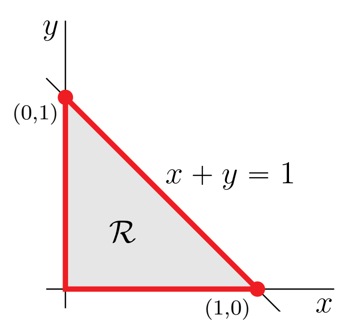
The symbol $\mathrm{d}A$ represents the area of an "infinitesimal" piece of $\mathcal{R}$.

Here is a simple example. We'll do some more complicated examples in §3.1.4.

Example 3.1.5

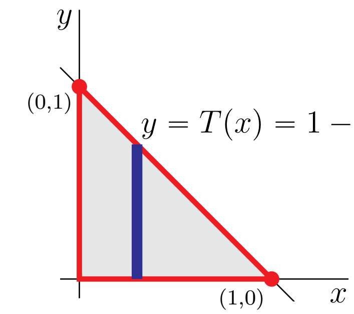
Let $\mathcal{R}$ be the triangular region above the x-axis, to the right of the y-axis and to the left of the line $x + y = 1$. Find the mass of $\mathcal{R}$ if it has density $f(x,y) = y$.

Solution. We'll do this problem twice — once using vertical strips and once using horizontal strips. First, here is a sketch of $\mathcal{R}$.



Solution using vertical strips. We'll now set up a double integral for the mass using vertical strips. Note, from the figure

2 This theorem is named after the Italian mathematician Guido Fubini (1879–1943).



that

- the leftmost points in $\mathcal{R}$ have $x = 0$ and the rightmost point in $\mathcal{R}$ has $x = 1$ and
- for each fixed x between 0 and 1, the point (x, y) in $\mathcal{R}$ with the smallest y has $y = 0$ and the point (x, y) in $\mathcal{R}$ with the largest y has $y = 1 - x$.

Thus
$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ 0 = a \leqslant x \leqslant b = 1,\ 0 = B(x) \leqslant y \leqslant T(x) = 1 - x\ \right\}$$
and, by part (a) of Theorem 3.1.3

$$\text{Mass} = \int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\, f(x,y) = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, y$$

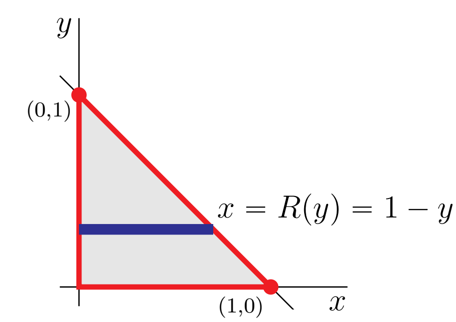
Now the inside integral is

$$\int_0^{1-x} y\ \mathrm{d}y = \left[\frac{y^2}{2}\right]_0^{1-x} = \frac{1}{2}(1-x)^2$$

so that the

$$\text{Mass} = \int_0^1 \mathrm{d}x\ \frac{(1-x)^2}{2} = \left[-\frac{(1-x)^3}{6}\right]_0^1 = \frac{1}{6}$$

Solution using horizontal strips. This time we'll set up a double integral for the mass using horizontal strips. Note, from the figure



that

- the lowest points in $\mathcal{R}$ have $y = 0$ and the topmost point in $\mathcal{R}$ has $y = 1$ and
- for each fixed y between 0 and 1, the point (x, y) in $\mathcal{R}$ with the smallest x has $x = 0$ and the point (x, y) in $\mathcal{R}$ with the largest x has $x = 1 - y$.

Thus

$$\mathcal{R} = \{ (x,y) \,|\, 0 = c \leqslant y \leqslant d = 1,\ 0 = L(y) \leqslant x \leqslant R(y) = 1 - y \}$$

and, by part (b) of Theorem 3.1.3

$$\text{Mass} = \int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\, f(x,y) = \int_0^1 \mathrm{d}y \int_0^{1-y} \mathrm{d}x\, y$$

Now the inside integral is

$$\int_0^{1-y} y\, \mathrm{d}x = [xy]_0^{1-y} = y - y^2$$

since the y integral treats x as a constant. So the

$$\text{Mass} = \int_0^1 \mathrm{d}y\, \left[y - y^2\right] = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Example 3.1.5

Double integrals share the usual basic properties that we are used to from integrals of functions of one variable. See, for example, Theorem 1.2.1 and Theorem 1.2.12 in the CLP-2 text. Indeed the following theorems follow from them.

Theorem 3.1.6 (Arithmetic of Integration)**.**

Let A, B, C be real numbers. Under the hypotheses of Theorem 3.1.3,

(a) $$\iint_{\mathcal{R}} \big(f(x,y) + g(x,y)\big)\, \mathrm{d}x\mathrm{d}y = \iint_{\mathcal{R}} f(x,y)\, \mathrm{d}x\mathrm{d}y + \iint_{\mathcal{R}} g(x,y)\, \mathrm{d}x\mathrm{d}y$$

(b) $$\iint_{\mathcal{R}} \big(f(x,y) - g(x,y)\big)\, \mathrm{d}x\mathrm{d}y = \iint_{\mathcal{R}} f(x,y)\, \mathrm{d}x\mathrm{d}y - \iint_{\mathcal{R}} g(x,y)\, \mathrm{d}x\mathrm{d}y$$

(c) $$\iint_{\mathcal{R}} C f(x,y)\, \mathrm{d}x\mathrm{d}y = C \iint_{\mathcal{R}} f(x,y)\, \mathrm{d}x\mathrm{d}y$$

Theorem 3.1.6 (continued).

Combining these three rules we have

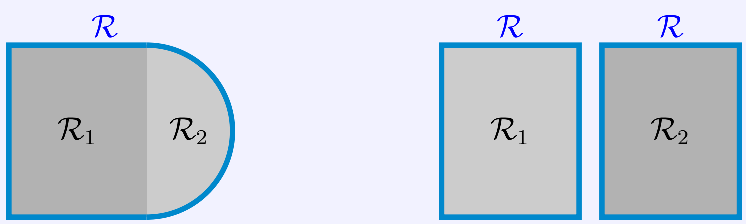
(d) $$\iint_{\mathcal{R}} \big(Af(x,y) + Bg(x,y)\big)\,\mathrm{d}x\mathrm{d}y = A\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y + B\iint_{\mathcal{R}} g(x,y)\,\mathrm{d}x\mathrm{d}y$$

That is, integrals depend linearly on the integrand.

(e) $$\iint_{\mathcal{R}} \mathrm{d}x\mathrm{d}y = \mathrm{Area}(\mathcal{R})$$

If the region $\mathcal{R}$ in the xy-plane is the union of regions $\mathcal{R}_1$ and $\mathcal{R}_2$ that do not overlap (except possibly on their boundaries), then

(f) $$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{R}_1} f(x,y)\,\mathrm{d}x\mathrm{d}y + \iint_{\mathcal{R}_2} f(x,y)\,\mathrm{d}x\mathrm{d}y$$



In the very special (but not that uncommon) case that $\mathcal{R}$ is the rectangle

$$\mathcal{R} = \big\{\ (x,y)\ \big|\ a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\ \big\}$$

and the integrand is the product $f(x,y) = g(x)h(y)$,

$$\begin{aligned}
\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y &= \int_a^b \mathrm{d}x \int_c^d \mathrm{d}y\ g(x)h(y) \\
&= \int_a^b \mathrm{d}x\ g(x) \int_c^d \mathrm{d}y\ h(y) \\
&\qquad \text{since } g(x) \text{ is a constant as far as the } y\text{-integral is concerned} \\
&= \left[\int_a^b \mathrm{d}x\ g(x)\right] \left[\int_c^d \mathrm{d}y\ h(y)\right] \\
&\qquad \text{since } \int_c^d \mathrm{d}y\ h(y) \text{ is a constant as far as the } x\text{-integral is concerned}
\end{aligned}$$

This is worth stating as a theorem

Theorem 3.1.7.

If the domain of integration

$$\mathcal{R} = \{\ (x, y)\ \big|\ a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\ \}$$

is a rectangle and the integrand is the product $f(x, y) = g(x)h(y)$, then

$$\iint\limits_{\mathcal{R}} f(x, y)\,\mathrm{d}x\mathrm{d}y = \left[\int_a^b \mathrm{d}x\ g(x)\right]\ \left[\int_c^d \mathrm{d}y\ h(y)\right]$$

Just as was the case for single variable integrals, sometimes we don't actually need to know the value of a double integral exactly. We are instead interested in bounds on its value. The following theorem provides some simple tools for generating such bounds. They are the multivariable analogs of the single variable tools in Theorem 1.2.12 of the CLP-2 text.

Theorem 3.1.8 (Inequalities for Integrals)**.**

Under the hypotheses of Theorem 3.1.3,

(a) If $f(x, y) \geqslant 0$ for all (x, y) in $\mathcal{R}$, then

$$\iint\limits_{\mathcal{R}} f(x, y)\,\mathrm{d}x\mathrm{d}y \geqslant 0$$

(b) If there are constants m and M such that $m \leqslant f(x, y) \leqslant M$ for all (x, y) in $\mathcal{R}$, then

$$m\,\mathrm{Area}(\mathcal{R}) \leqslant \iint\limits_{\mathcal{R}} f(x, y)\,\mathrm{d}x\mathrm{d}y \leqslant M\,\mathrm{Area}(\mathcal{R})$$

(c) If $f(x, y) \leqslant g(x, y)$ for all (x, y) in $\mathcal{R}$, then

$$\iint\limits_{\mathcal{R}} f(x, y)\,\mathrm{d}x\mathrm{d}y \leqslant \iint\limits_{\mathcal{R}} g(x, y)\,\mathrm{d}x\mathrm{d}y$$

(d) We have

$$\left|\iint\limits_{\mathcal{R}} f(x, y)\,\mathrm{d}x\mathrm{d}y\right| \leqslant \iint\limits_{\mathcal{R}} |f(x, y)|\,\mathrm{d}x\mathrm{d}y$$

3.1.3 ▸▸ Volumes

Now that we have defined double integrals, we should start putting them to use. One of the most immediate applications arises from interpreting $f(x,y)$, not as a density, but rather as the height of the part of a solid above the point (x,y) in the xy-plane. Then Theorem 3.1.3 gives the volume between the xy-plane and the surface $z = f(x,y)$.

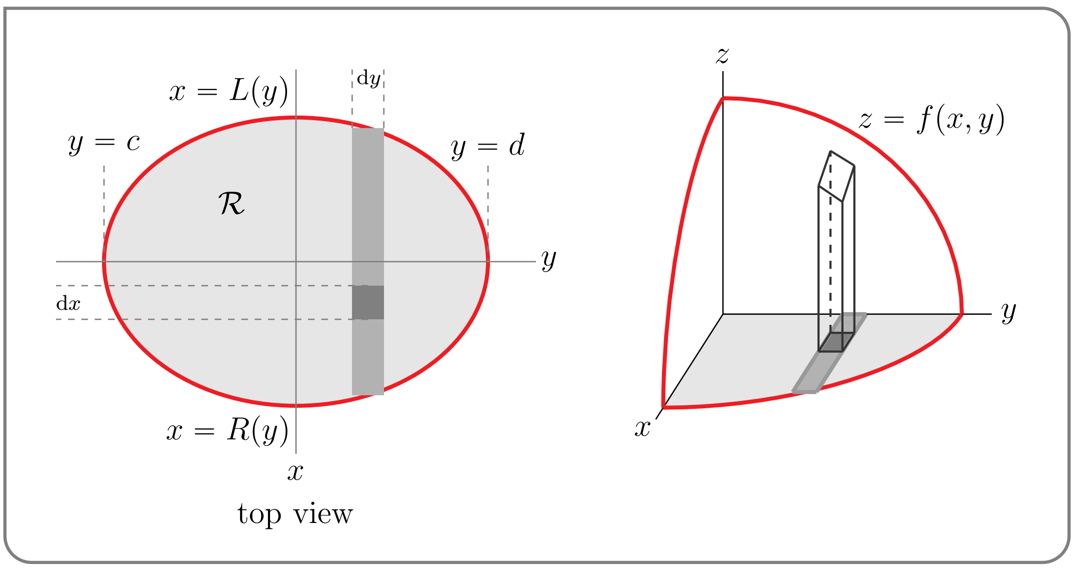
We'll now see how this goes in the case of part (b) of Theorem 3.1.3. The case of part (a) works in the same way. So we assume that the solid $\mathcal{V}$ lies above the base region

$$\mathcal{R} = \left\{ (x,y) \mid c \leqslant y \leqslant d,\ L(y) \leqslant x \leqslant R(y) \right\}$$

and that

$$\mathcal{V} = \left\{ (x,y,z) \mid (x,y) \in \mathcal{R},\ 0 \leqslant z \leqslant f(x,y) \right\}$$

The base region $\mathcal{R}$ (which is also the top view of $\mathcal{V}$) is sketched in the figure on the left below and the part of $\mathcal{V}$ in the first octant is sketched in the figure on the right below.



To find the volume of $\mathcal{V}$ we shall

- Pick a natural number n and slice $\mathcal{R}$ into strips of width $\Delta y = \frac{d-c}{n}$.
- Subdivide slice number i into m tiny rectangles, each of height Δy and of width $\Delta x = \frac{1}{m} \cdots$.
- Compute, approximately, the volume of the part of $\mathcal{V}$ that is above each rectangle.
- Take the limit $m \to \infty$ and then the limit $n \to \infty$.

We have just been through this type of argument twice. So we'll abbreviate the argument and just say

- slice the base region $\mathcal{R}$ into long "infinitesimally" thin strips of width $\mathrm{d}y$.
- Subdivide each strip into "infinitesimal" rectangles each of height $\mathrm{d}y$ and of width $\mathrm{d}x$. See the figure on the left above.

- The volume of the part of $\mathcal{V}$ that is above the rectangle centred on (x,y) is essentially $f(x,y)\,\mathrm{d}x\,\mathrm{d}y$. See the figure on the right above.
- So the volume of the part of $\mathcal{V}$ that is above the strip centred on y is essentially[3] $\mathrm{d}y\int_{L(y)}^{R(y)}\mathrm{d}x\ f(x,y)$ and
- we arrive at the following conclusion.

Equation 3.1.9.

If

$$\mathcal{V}=\big\{\ (x,y,z)\ \big|\ (x,y)\in\mathcal{R},\ 0\leqslant z\leqslant f(x,y)\ \big\}$$

where

$$\mathcal{R}=\big\{\ (x,y)\ \big|\ c\leqslant y\leqslant d,\ L(y)\leqslant x\leqslant R(y)\ \big\}$$

then

$$\text{Volume}(\mathcal{V})=\int_c^d\mathrm{d}y\int_{L(y)}^{R(y)}\mathrm{d}x\ f(x,y)$$

Similarly

Equation 3.1.10.

If

$$\mathcal{V}=\big\{\ (x,y,z)\ \big|\ (x,y)\in\mathcal{R},\ 0\leqslant z\leqslant f(x,y)\ \big\}$$

where

$$\mathcal{R}=\big\{\ (x,y)\ \big|\ a\leqslant x\leqslant b,\ B(x)\leqslant y\leqslant T(x)\ \big\}$$

then

$$\text{Volume}(\mathcal{V})=\int_a^b\mathrm{d}x\int_{B(x)}^{T(x)}\mathrm{d}y\ f(x,y)$$

3.1.4 ▸▸ Examples

Oof — we have had lots of equations and theory. It's time to put all of this to work. Let's start with a mass example and then move on to a volume example. You will notice that the mathematics is really very similar. Just the interpretation changes.

Example 3.1.11 (Mass)

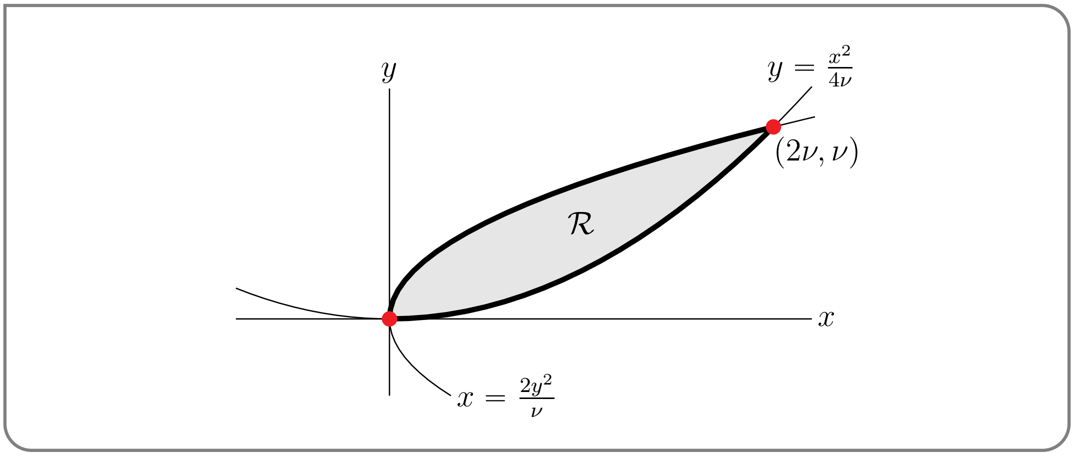
Let $\nu>0$ be a constant and let $\mathcal{R}$ be the region above the curve $x^2=4\nu y$ and to the right of the curve $y^2=\frac{1}{2}\nu x$. Find the mass of $\mathcal{R}$ if it has density $f(x,y)=xy$.

3 Think of the part of $\mathcal{V}$ that is above the strip as being a thin slice of bread. Then the factor $\mathrm{d}y$ in $\mathrm{d}y\int_{L(y)}^{R(y)}\mathrm{d}x\ f(x,y)$ is the thickness of the slice of bread. The factor $\int_{L(y)}^{R(y)}\mathrm{d}x\ f(x,y)$ is the surface area of the constant y cross-section $\big\{\ (x,z)\ \big|\ L(y)\leqslant x\leqslant R(y),\ 0\leqslant z\leqslant f(x,y)\ \big\}$, i.e. the surface area of the slice of bread.

Solution. For practice, we'll do this problem twice — once using vertical strips and once using horizontal strips. We'll start by sketching $\mathcal{R}$. First note that, since $y \geqslant \frac{x^2}{4\nu}$ and $x \geqslant \frac{2y^2}{\nu}$, both x and y are positive throughout $\mathcal{R}$. The two curves intersect at points (x, y) that satisfy both

$$x = \frac{2y^2}{\nu} \text{ and } y = \frac{x^2}{4\nu} \quad \Longrightarrow \quad x = \frac{2y^2}{\nu} = \frac{2}{\nu}\left(\frac{x^2}{4\nu}\right)^2 = \frac{x^4}{8\nu^3} \quad \Longleftrightarrow \quad \left(\frac{x^3}{8\nu^3} - 1\right) x = 0$$

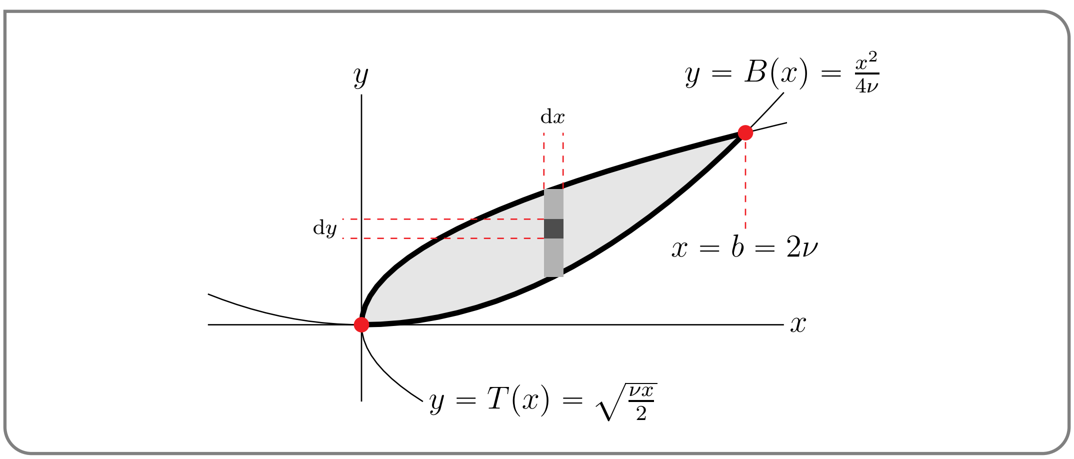
This equation has only two real[4] solutions — $x = 0$ and $x = 2\nu$. So the upward opening parabola $y = \frac{x^2}{4\nu}$ and the rightward opening parabola $x = \frac{2y^2}{\nu}$ intersect at $(0, 0)$ and $(2\nu, \nu)$.



Solution using vertical strips. We'll now set up a double integral for the mass using vertical strips and using the abbreviated argument of the end of the last section (on volumes). Note, from the figure above, that

$$\mathcal{R} = \left\{ (x, y) \;\middle|\; 0 = a \leqslant x \leqslant b = 2\nu,\ \frac{x^2}{4\nu} = B(x) \leqslant y \leqslant T(x) = \sqrt{\frac{\nu x}{2}} \right\}$$

- Slice $\mathcal{R}$ into long "infinitesimally" thin vertical strips of width $\mathrm{d}x$.
- Subdivide each strip into "infinitesimal" rectangles each of height $\mathrm{d}y$ and of width $\mathrm{d}x$. See the figure below.



4 It also has two complex solutions that play no role here.

- The mass of the rectangle centred on (x, y) is essentially $f(x, y)\,\mathrm{d}x\,\mathrm{d}y = xy\,\mathrm{d}x\,\mathrm{d}y$.

- So the mass of the strip centred on x is essentially $\mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\ f(x, y)$ (the integral over y adds up the masses of all of the different rectangles on the single vertical strip in question) and

- we conclude that the

$$\text{Mass}(\mathcal{R}) = \int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\ f(x,y) = \int_0^{2\nu} \mathrm{d}x \int_{x^2/(4\nu)}^{\sqrt{\nu x/2}} \mathrm{d}y\ xy$$

Here the integral over x adds up the masses of all of the different strips.

Recall that, when integrating y, x is held constant, so we may factor the constant x out of the inner y integral.

$$\begin{aligned}\int_{x^2/(4\nu)}^{\sqrt{\nu x/2}} \mathrm{d}y\ xy &= x \int_{x^2/(4\nu)}^{\sqrt{\nu x/2}} \mathrm{d}y\ y \\ &= x \left[\frac{y^2}{2}\right]_{x^2/(4\nu)}^{\sqrt{\nu x/2}} \\ &= \frac{\nu x^2}{4} - \frac{x^5}{32\nu^2}\end{aligned}$$

and the

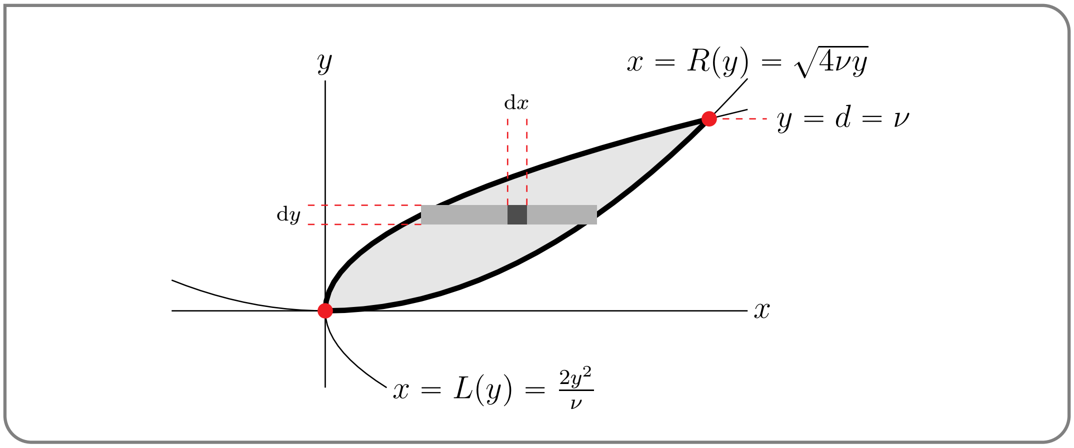
$$\begin{aligned}\text{Mass}(\mathcal{R}) &= \int_0^{2\nu} \mathrm{d}x\ \left[\frac{\nu x^2}{4} - \frac{x^5}{32\nu^2}\right] \\ &= \frac{\nu(2\nu)^3}{3 \times 4} - \frac{(2\nu)^6}{6 \times 32\nu^2} = \frac{\nu^4}{3}\end{aligned}$$

Solution using horizontal strips. We'll now set up a double integral for the mass using horizontal strips, again using the abbreviated argument of the end of the last section (on volumes). Note, from the figure at the beginning of this example, that

$$\mathcal{R} = \left\{ (x,y) \;\middle|\; 0 = c \leqslant y \leqslant d = \nu,\ \frac{2y^2}{\nu} = L(y) \leqslant x \leqslant R(y) = \sqrt{4\nu y} \right\}$$

- Slice $\mathcal{R}$ into long “infinitesimally” thin horizontal strips of width $\mathrm{d}y$.

- Subdivide each strip into “infinitesimal” rectangles each of height $\mathrm{d}y$ and of width $\mathrm{d}x$. See the figure below.



- The mass of the rectangle centred on (x, y) is essentially $f(x, y)\,\mathrm{d}x\,\mathrm{d}y = xy\,\mathrm{d}x\,\mathrm{d}y$.
- So the mass of the strip centred on y is essentially $\mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\ f(x, y)$ (the integral over x adds up the masses of all of the different rectangles on the single horizontal strip in question) and
- we conclude that the

$$\text{Mass}(\mathcal{R}) = \int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\ f(x, y) = \int_0^\nu \mathrm{d}y \int_{2y^2/\nu}^{\sqrt{4\nu y}} \mathrm{d}x\ xy$$

Here the integral over y adds up the masses of all of the different strips. Recalling that, when integrating x, y is held constant

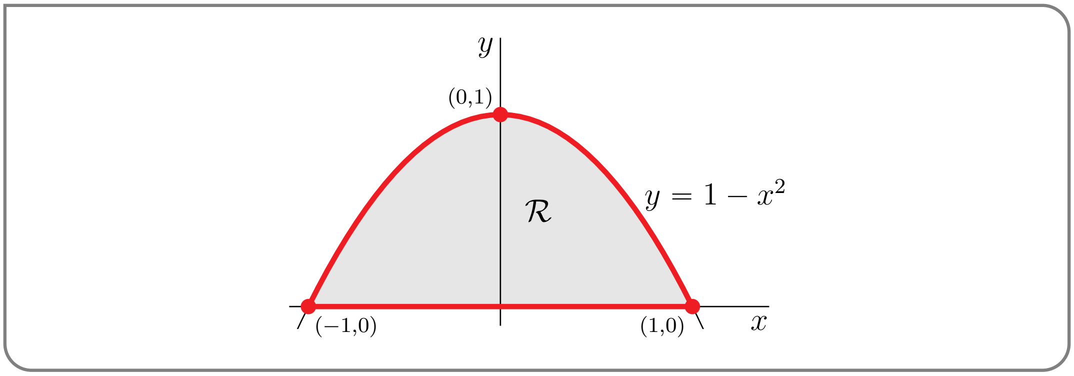
$$\begin{aligned}\text{Mass}(\mathcal{R}) &= \int_0^\nu \mathrm{d}y\ y \left[\int_{2y^2/\nu}^{\sqrt{4\nu y}} \mathrm{d}x\ x\right] \\ &= \int_0^\nu \mathrm{d}y\ y \left[\frac{x^2}{2}\right]_{2y^2/\nu}^{\sqrt{4\nu y}} \\ &= \int_0^\nu \mathrm{d}y\ \left[2\nu y^2 - \frac{2y^5}{\nu^2}\right] \\ &= \frac{2\nu(\nu)^3}{3} - \frac{2\nu^6}{6\nu^2} = \frac{\nu^4}{3}\end{aligned}$$

Example 3.1.11

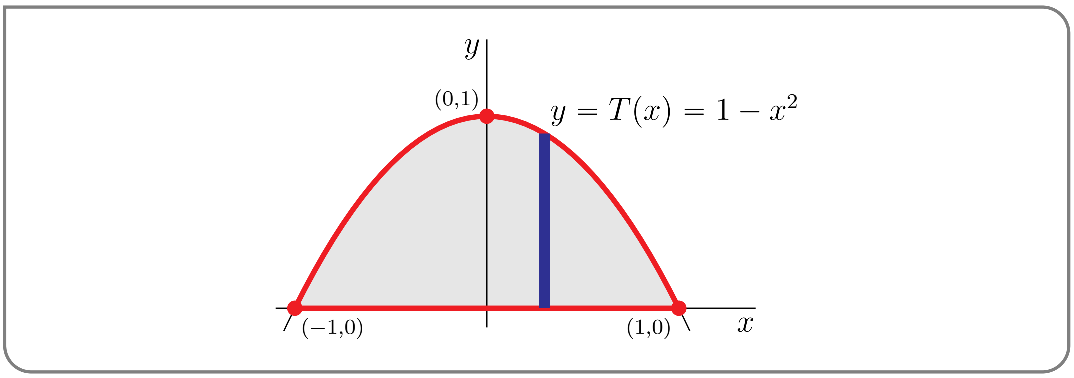
Example 3.1.12 (Volume)

Let $\mathcal{R}$ be the part of the xy-plane above the x-axis and below the parabola $y = 1 - x^2$. Find the volume between $\mathcal{R}$ and the surface $z = x^2\sqrt{1 - y}$.

Solution. Yet again, for practice, we'll do this problem twice — once using vertical strips and once using horizontal strips. First, here is a sketch of $\mathcal{R}$.



Solution using vertical strips. We'll now set up a double integral for the volume using vertical strips. Note, from the figure



that

- the leftmost point in $\mathcal{R}$ has $x = -1$ and the rightmost point in $\mathcal{R}$ has $x = 1$ and
- for each fixed x between -1 and 1, the point (x, y) in $\mathcal{R}$ with the smallest y has $y = 0$ and the point (x, y) in $\mathcal{R}$ with the largest y has $y = 1 - x^2$.

Thus

$$\mathcal{R} = \left\{ \ (x, y) \ \middle| \ -1 = a \leqslant x \leqslant b = 1, \ 0 = B(x) \leqslant y \leqslant T(x) = 1 - x^2 \ \right\}$$

and, by (3.1.10)

$$\begin{aligned} \text{Volume} &= \int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y \, f(x, y) = \int_{-1}^{1} \mathrm{d}x \int_0^{1-x^2} \mathrm{d}y \, x^2 \sqrt{1-y} \\ &= 2 \int_0^1 \mathrm{d}x \int_0^{1-x^2} \mathrm{d}y \, x^2 \sqrt{1-y} \end{aligned}$$

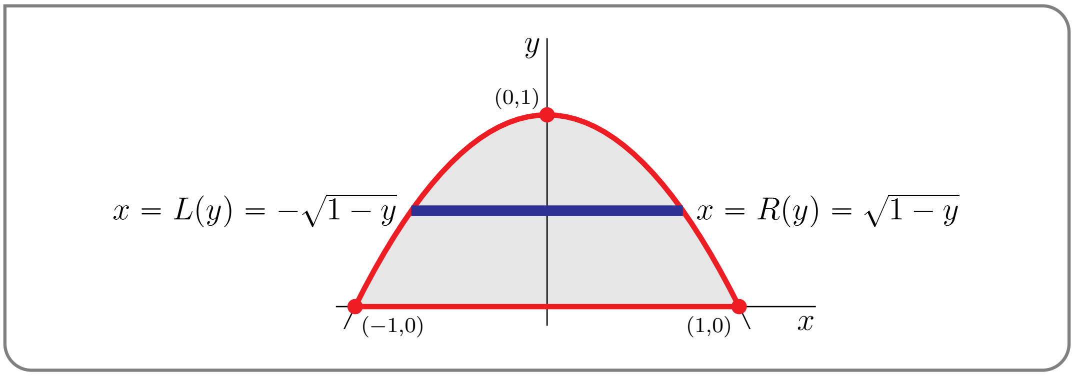
since the inside integral $F(x) = \int_0^{1-x^2} \mathrm{d}y \, x^2 \sqrt{1-y}$ is an even function of x. Now, for $x \geqslant 0$, the inside integral is

$$\int_0^{1-x^2} x^2 \sqrt{1-y} \, \mathrm{d}y = x^2 \int_0^{1-x^2} \sqrt{1-y} \, \mathrm{d}y = x^2 \left[-\frac{2}{3}(1-y)^{3/2} \right]_0^{1-x^2} = \frac{2}{3} x^2 (1 - x^3)$$

so that the

$$\text{Volume} = 2\int_0^1 \mathrm{d}x\ \frac{2}{3}x^2(1-x^3) = \frac{4}{3}\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = \frac{2}{9}$$

Solution using horizontal strips. This time we'll set up a double integral for the volume using horizontal strips. Note, from the figure



that

- the lowest points in $\mathcal{R}$ have $y = 0$ and the topmost point in $\mathcal{R}$ has $y = 1$ and
- for each fixed y between 0 and 1, the point (x, y) in $\mathcal{R}$ with the leftmost x has $x = -\sqrt{1-y}$ and the point (x, y) in $\mathcal{R}$ with the rightmost x has $x = \sqrt{1-y}$.

Thus

$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ 0 = c \leqslant y \leqslant d = 1,\ -\sqrt{1-y} = L(y) \leqslant x \leqslant R(y) = \sqrt{1-y}\ \right\}$$

and, by (3.1.9)

$$\text{Volume} = \int_c^d \mathrm{d}y \int_{L(y)}^{R(y)} \mathrm{d}x\ f(x,y) = \int_0^1 \mathrm{d}y \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \mathrm{d}x\ x^2\sqrt{1-y}$$

Now the inside integral has an even integrand (in x) and so is

$$\int_{-\sqrt{1-y}}^{\sqrt{1-y}} \mathrm{d}x\ x^2\sqrt{1-y} = 2\sqrt{1-y}\int_0^{\sqrt{1-y}} x^2\ \mathrm{d}x = 2\sqrt{1-y}\left[\frac{x^3}{3}\right]_0^{\sqrt{1-y}} = \frac{2}{3}(1-y)^2$$

So the

$$\text{Volume} = \frac{2}{3}\int_0^1 \mathrm{d}y\ (1-y)^2 = \frac{2}{3}\left[-\frac{(1-y)^3}{3}\right]_0^1 = \frac{2}{9}$$

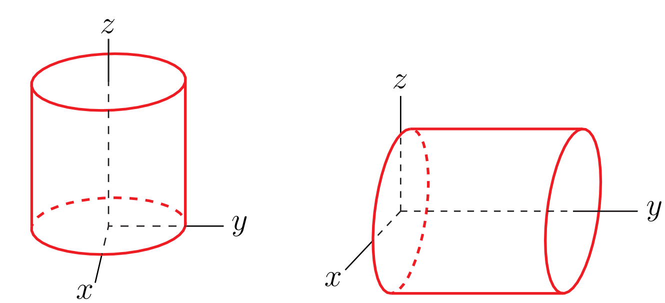
Example 3.1.12

Example 3.1.13 (Volume)

Find the volume common to the two cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution. Our first job is figure out what the specified solid looks like. Note that

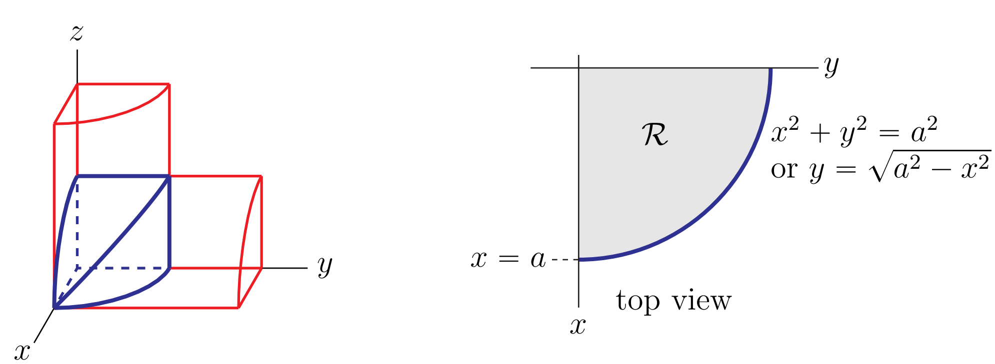
- The variable z does not appear in the equation $x^2 + y^2 = a^2$. So, for every value of the constant z_0, the part of the cylinder $x^2 + y^2 = a^2$ in the plane $z = z_0$, is the circle $x^2 + y^2 = a^2$, $z = z_0$. So the cylinder $x^2 + y^2 = a^2$ consists of many circles stacked vertically, one on top of the other. The part of the cylinder $x^2 + y^2 = a^2$ that lies above the xy-plane is sketched in the figure on the left below.
- The variable y does not appear in the equation $x^2 + z^2 = a^2$. So, for every value of the constant y_0, the part of the cylinder $x^2 + z^2 = a^2$ in the plane $y = y_0$, is the circle $x^2 + z^2 = a^2$, $y = y_0$. So the cylinder $x^2 + z^2 = a^2$ consists of many circles stacked horizontally, one beside the other. The part of the cylinder $x^2 + z^2 = a^2$ that lies to the right of the xz-plane is sketched in the figure on the right below.



We have to compute the volume common to these two intersecting cylinders.

- The equations $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ do not change at all if x is replaced by $-x$. Consequently both cylinders, and hence our solid, is symmetric about the yz-plane. In particular the volume of the part of the solid in the octant $x \leqslant 0$, $y \geqslant 0$, $z \geqslant 0$ is the same as the volume in the first octant $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$. Similarly, the equations do not change at all if y is replaced by $-y$ or if z is replaced by $-z$. Our solid is also symmetric about both the xz-plane and the xy-plane. Hence the volume of the part of our solid in each of the eight octants is the same.
- So we will compute the volume of the part of the solid in the first octant, i.e. with $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$. The total volume of the solid is eight times that.

The part of the solid in the first octant is sketched in the figure on the left below. A point (x, y, z) lies in the first cylinder if and only if $x^2 + y^2 \leqslant a^2$. It lies in the second cylinder if



and only if $x^2 + z^2 \leqslant a^2$. So the part of the solid in the first octant is

$$\mathcal{V}_1 = \left\{ (x,y,z) \,\middle|\, x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0,\ x^2+y^2 \leqslant a^2,\ x^2+z^2 \leqslant a^2 \right\}$$

Notice that, in $\mathcal{V}_1$, $z^2 \leqslant a^2 - x^2$ so that $z \leqslant \sqrt{a^2-x^2}$ and

$$\mathcal{V}_1 = \left\{ (x,y,z) \,\middle|\, x \geqslant 0,\ y \geqslant 0,\ x^2+y^2 \leqslant a^2,\ 0 \leqslant z \leqslant \sqrt{a^2-x^2} \right\}$$

The top view of the part of the solid in the first octant is sketched in the figure on the right above. In that top view, x runs from 0 to a. For each fixed x, y runs from 0 to $\sqrt{a^2-x^2}$. So we may rewrite

$$\mathcal{V}_1 = \left\{ (x,y,z) \,\middle|\, (x,y) \in \mathcal{R},\ 0 \leqslant z \leqslant f(x,y) \right\}$$

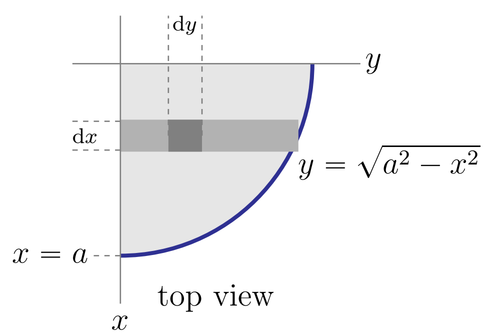
where

$$\mathcal{R} = \left\{ (x,y) \,\middle|\, 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant \sqrt{a^2-x^2} \right\} \qquad \text{and} \qquad f(x,y) = \sqrt{a^2-x^2}$$

and "$(x,y) \in \mathcal{R}$" is read "(x,y) is an element of $\mathcal{R}$.". Note that $f(x,y)$ is actually independent of y. This will make things a bit easier below.

We can now compute the volume of $\mathcal{V}_1$ using our usual abbreviated protocol.

- Slice $\mathcal{R}$ into long "infinitesimally" thin horizontal strips of height $\mathrm{d}x$.
- Subdivide each strip into "infinitesimal" rectangles each of width $\mathrm{d}y$ and of height $\mathrm{d}x$. See the figure below.



- The volume of the part of $\mathcal{V}_1$ above rectangle centred on (x,y) is essentially

$$f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sqrt{a^2-x^2}\,\mathrm{d}x\,\mathrm{d}y$$

- So the volume of the part of $\mathcal{V}_1$ above the strip centred on x is essentially

$$\mathrm{d}x \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,\mathrm{d}y$$

  (the integral over y adds up the volumes over all of the different rectangles on the single horizontal strip in question) and

- we conclude that the

$$\text{Volume}(\mathcal{V}_1) = \int_0^a \mathrm{d}x \int_0^{\sqrt{a^2-x^2}} \mathrm{d}y\ \sqrt{a^2-x^2}$$

Here the integral over x adds up the volumes over all of the different strips. Recalling that, when integrating y, x is held constant

$$\begin{aligned}\text{Volume}(\mathcal{V}_1) &= \int_0^a \mathrm{d}x\ \sqrt{a^2-x^2}\left[\int_0^{\sqrt{a^2-x^2}} \mathrm{d}y\right]\\ &= \int_0^a \mathrm{d}x\ \left(a^2-x^2\right)\\ &= \left[a^2x - \frac{x^3}{3}\right]_0^a\\ &= \frac{2a^3}{3}\end{aligned}$$

and the total volume of the solid in question is

$$\text{Volume}(\mathcal{V}) = 8\,\text{Volume}(\mathcal{V}_1) = \frac{16a^3}{3}$$

Example 3.1.13

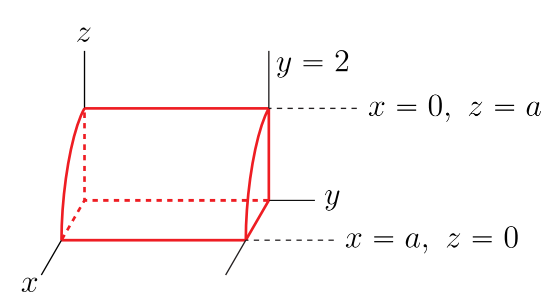
Example 3.1.14 (Geometric Interpretation)

Evaluate $\displaystyle\int_0^2 \int_0^a \sqrt{a^2-x^2}\,\mathrm{d}x\,\mathrm{d}y$.

Solution. This integral represents the volume of a simple geometric figure and so can be evaluated without using any calculus at all. The domain of integration is

$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ 0 \leqslant y \leqslant 2,\ 0 \leqslant x \leqslant a\ \right\}$$

and the integrand is $f(x,y) = \sqrt{a^2-x^2}$, so the integral represents the volume between the xy-plane and the surface $z = \sqrt{a^2-x^2}$, with (x,y) running over $\mathcal{R}$. We can rewrite the equation of the surface as $x^2+z^2 = a^2$, which, as in Example 3.1.13, we recognize as the equation of a cylinder of radius a centred on the y-axis. We want the volume of the part of this cylinder that lies above $\mathcal{R}$. It is sketched in the figure below.



The constant y cross-sections of this volume are quarter circles of radius a and hence of area $\frac{1}{4}\pi a^2$. The inside integral, $\int_0^a \sqrt{a^2-x^2}\,\mathrm{d}x$, is exactly this area. So, as y runs from 0 to 2,

$$\int_0^2 \int_0^a \sqrt{a^2-x^2}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{4}\pi a^2 \times 2 = \frac{\pi a^2}{2}$$

Example 3.1.14

Example 3.1.15 (Example 3.1.14, the hard way)

It is possible, but very tedious, to evaluate the integral $\int_0^2\int_0^a \sqrt{a^2-x^2}\,\mathrm{d}x\,\mathrm{d}y$ of Example 3.1.14, using single variable calculus techniques. We do so now as a review of a couple of those techniques.

The inside integral is $\int_0^a \sqrt{a^2-x^2}\,\mathrm{d}x$. The standard procedure for eliminating square roots like $\sqrt{a^2-x^2}$ from integrands is the method of trigonometric substitution, that was covered in §1.9 of the CLP-2 text. In this case, the appropriate substitution is

$$x = a\sin\theta \qquad \mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$$

The lower limit of integration $x=0$, i.e. $a\sin\theta=0$, corresponds to $\theta=0$, and the upper limit $x=a$, i.e. $a\sin\theta=a$, corresponds to $\theta=\frac{\pi}{2}$, so that

$$\int_0^a \sqrt{a^2-x^2}\,\mathrm{d}x = \int_0^{\pi/2} \sqrt{\underbrace{a^2-a^2\sin^2\theta}_{a^2\cos^2\theta}}\;a\cos\theta\,\mathrm{d}\theta = a^2\int_0^{\pi/2}\cos^2\theta\,\mathrm{d}\theta$$

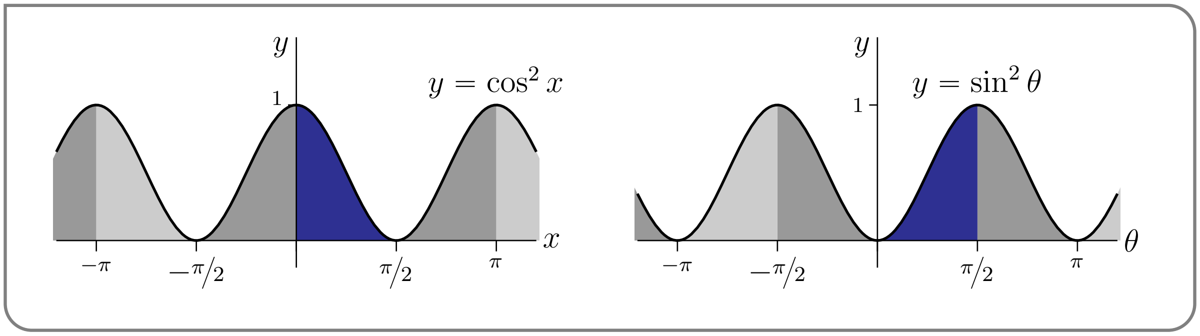
The orthodox procedure for evaluating the resulting trigonometric integral $\int_0^{\pi/2}\cos^2\theta\,\mathrm{d}\theta$, covered in §1.8 of the CLP-2 text, uses the trigonometric double angle formula

$$\cos(2\theta) = 2\cos^2\theta - 1 \qquad \text{to write} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

and then

$$\begin{aligned}\int_0^a \sqrt{a^2-x^2}\,\mathrm{d}x &= a^2\int_0^{\pi/2}\cos^2\theta\,\mathrm{d}\theta = \frac{a^2}{2}\int_0^{\pi/2}\big[1+\cos(2\theta)\big]\,\mathrm{d}\theta\\ &= \frac{a^2}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/2}\\ &= \frac{\pi a^2}{4}\end{aligned}$$

However we remark that there is also an efficient, sneaky, way to evaluate definite integrals like $\int_0^{\pi/2}\cos^2\theta\,\mathrm{d}\theta$. Looking at the figures



we see that

$$\int_0^{\pi/2} \cos^2\theta \, \mathrm{d}\theta = \int_0^{\pi/2} \sin^2\theta \, \mathrm{d}\theta$$

Thus

$$\int_0^{\pi/2} \cos^2\theta \, \mathrm{d}\theta = \int_0^{\pi/2} \sin^2\theta \, \mathrm{d}\theta = \int_0^{\pi/2} \frac{1}{2}\big[\sin^2\theta + \cos^2\theta\big] \, \mathrm{d}\theta = \frac{1}{2}\int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi}{4}$$

In any event, the inside integral

$$\int_0^a \sqrt{a^2 - x^2} \, \mathrm{d}x = \frac{\pi a^2}{4}$$

and the full integral

$$\int_0^2 \int_0^a \sqrt{a^2 - x^2} \, \mathrm{d}x \, \mathrm{d}y = \frac{\pi a^2}{4} \int_0^2 \mathrm{d}y = \frac{\pi a^2}{2}$$

just as we saw in Example 3.1.14.

Example 3.1.15

Example 3.1.16 (Order of Integration)

The integral $\displaystyle\int_{-1}^2 \int_{x^2}^{x+2} \mathrm{d}y \, \mathrm{d}x$ represents the area of a region in the xy-plane. Express the same area as a double integral with the order of integration reversed.

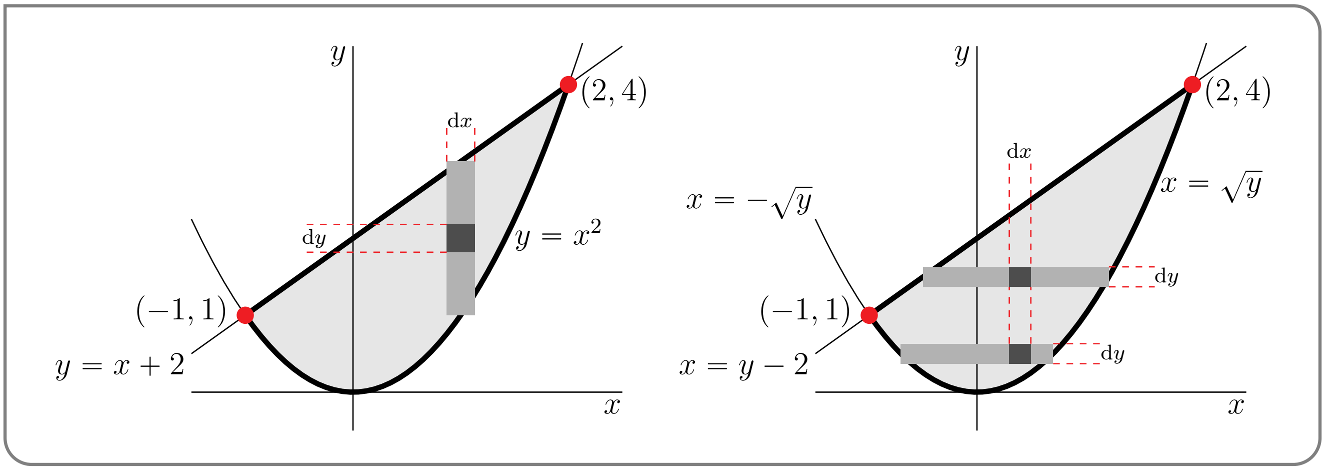
Solution. The critical step in reversing the order of integration is to sketch the region in the xy-plane. Rewrite the given integral as

$$\int_{-1}^2 \int_{x^2}^{x+2} \mathrm{d}y \, \mathrm{d}x = \int_{-1}^2 \left[\int_{x^2}^{x+2} \mathrm{d}y\right] \mathrm{d}x$$

From this we see that, on the domain of integration,

- x runs from -1 to 2 and
- for each fixed x, y runs from the parabola $y = x^2$ to the straight line $y = x + 2$.

The given iterated integral corresponds to the (vertical) slicing in the figure on the left below.



To reverse the order of integration we have to switch to horizontal slices as in the figure on the right above.

There we see a new wrinkle: the formula giving the value of x at the left hand end of a slice depends on whether the y coordinate of the slice is bigger than, or smaller than $y = 1$. Looking at the figure on the right, we see that, on the domain of integration,

- y runs from 0 to 4 and
- for each fixed $0 \leqslant y \leqslant 1$, x runs from $x = -\sqrt{y}$ to $x = +\sqrt{y}$.
- for each fixed $1 \leqslant y \leqslant 4$, x runs from $x = y - 2$ to $x = +\sqrt{y}$.

So

$$\int_{-1}^{2} \mathrm{d}x \int_{x^2}^{x+2} \mathrm{d}y = \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{\sqrt{y}} \mathrm{d}x + \int_1^4 \mathrm{d}y \int_{y-2}^{\sqrt{y}} \mathrm{d}x$$

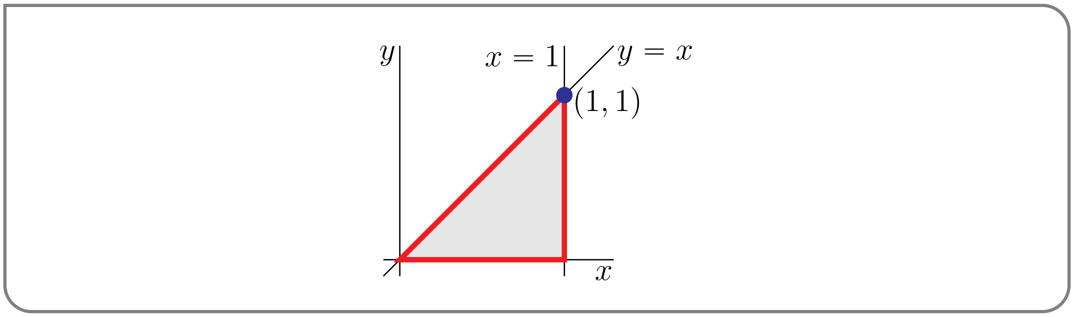
Example 3.1.16

There was a moral to the last example. Just because both orders of integration have to give the same answer doesn't mean that they are equally easy to evaluate. Here is an extreme example illustrating that moral.

Example 3.1.17

Evaluate the integral of $\frac{\sin x}{x}$ over the region in the xy-plane that is above the x-axis, to the right of the line $y = x$ and to the left of the line $x = 1$.

Solution. Here is a sketch of the specified domain.



We'll try to evaluate the specified integral twice — once using horizontal strips (the impossibly hard way) and once using vertical strips (the easy way).

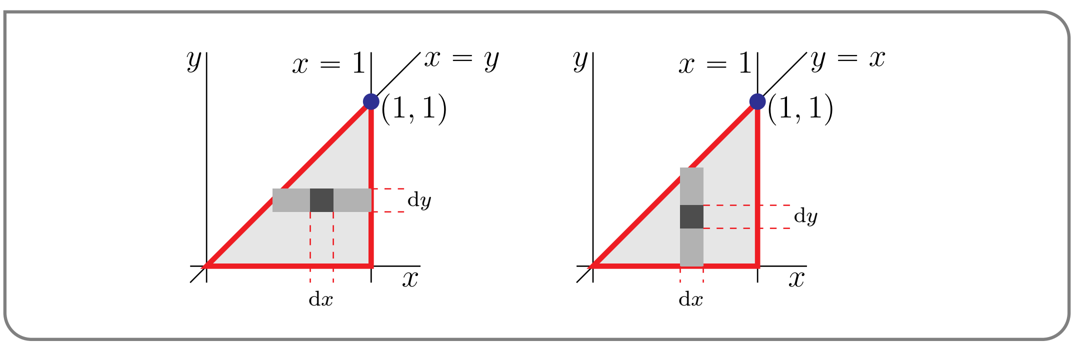
Solution using horizontal strips. To set up the integral using horizontal strips, as in the figure on the left below, we observe that, on the domain of integration,

- y runs from 0 to 1 and
- for each fixed y, x runs from $x = y$ to 1.

So the iterated integral is

$$\int_0^1 \mathrm{d}y \int_y^1 \mathrm{d}x \, \frac{\sin x}{x}$$

And we have a problem. The integrand $\frac{\sin x}{x}$ does not have an antiderivative that can be expressed in terms of elementary functions[5]. It is impossible to evaluate $\int_y^1 \mathrm{d}x\, \frac{\sin x}{x}$ without resorting to, for example, numerical methods or infinite series[6].



Solution using vertical strips. To set up the integral using vertical strips, as in the figure on the right above, we observe that, on the domain of integration,

- x runs from 0 to 1 and
- for each fixed x, y runs from 0 to $y = x$.

5 Perhaps the best known function whose antiderivative cannot be expressed in terms of elementary functions is e^{-x^2}. It is the integrand of the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$ that is used in computing "bell curve" probabilities. See Example 3.6.14 in the CLP-2 text.

6 See, for example, Example 3.6.14 in the CLP-2 text.

So the iterated integral is

$$\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\ \frac{\sin x}{x}$$

This time, because x is treated as a constant in the inner integral, it is trivial to evaluate the iterated integral.

$$\int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\ \frac{\sin x}{x} = \int_0^1 \mathrm{d}x\ \frac{\sin x}{x} \int_0^x \mathrm{d}y = \int_0^1 \mathrm{d}x\ \sin x = 1 - \cos 1$$

Example 3.1.17

Here is an example which is included as an excuse to review some integration technique from CLP-2.

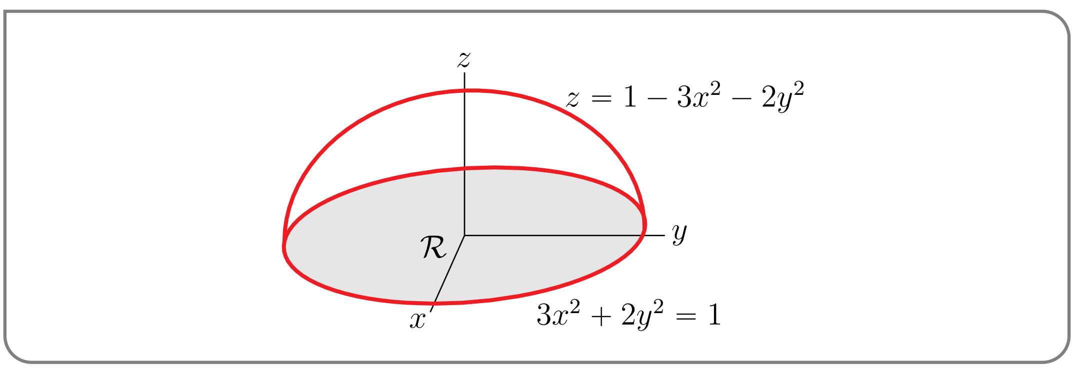
Example 3.1.18

Find the volume under the surface $z = 1 - 3x^2 - 2y^2$ and above the xy-plane.

Solution. Before leaping into integration, we should try to understand what the surface and volume look like. For each constant z_0, the part of the surface $z = 1 - 3x^2 - 2y^2$ that lies in the horizontal plane $z = z_0$ is the ellipse $3x^2 + 2y^2 = 1 - z_0$. The biggest of these ellipses is that in the xy-plane, where $z_0 = 0$. It is the ellipse $3x^2 + 2y^2 = 1$. As z_0 increases the ellipse shrinks, degenerating to a single point, namely $(0, 0, 1)$, when $z_0 = 1$. So the surface consists of a stack of ellipses and our solid is

$$\mathcal{V} = \left\{ (x, y, z) \mid 3x^2 + 2y^2 \leqslant 1,\ 0 \leqslant z \leqslant 1 - 3x^2 - 2y^2 \right\}$$

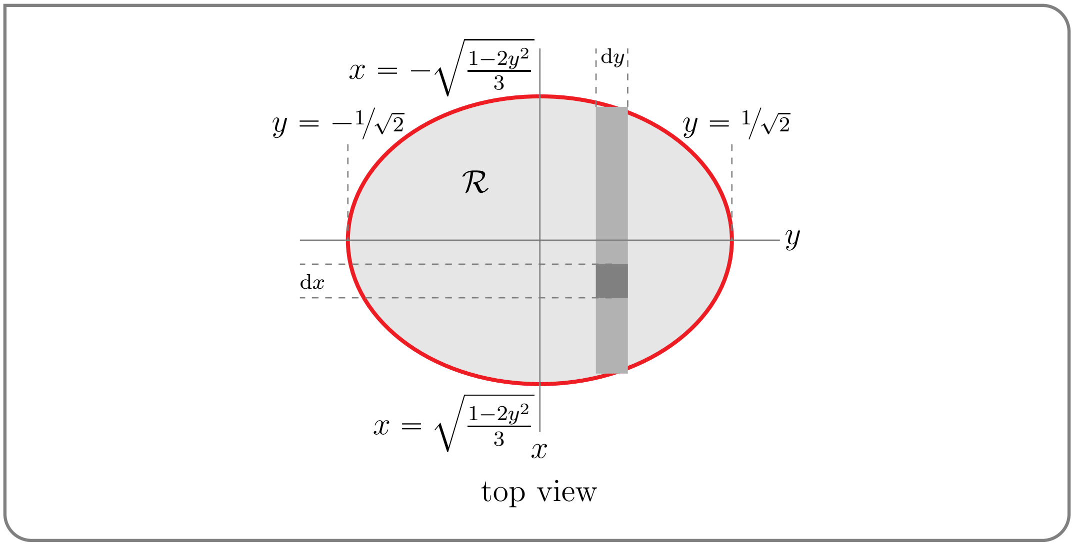
This is sketched in the figure below



The top view of the base region

$$\mathcal{R} = \left\{ (x, y) \mid 3x^2 + 2y^2 \leqslant 1 \right\}$$

is sketched in the figure below.



Considering that the x-dependence in $z = 1 - 3x^2 - 2y^2$ is almost identical to the y-dependence in $z = 1 - 3x^2 - 2y^2$ (only the coefficients 2 and 3 are interchanged), using vertical slices is likely to lead to exactly the same level of difficulty as using horizontal slices. So we'll just pick one — say vertical slices.

The fattest part of $\mathcal{R}$ is on the y-axis. The intersection points of the ellipse with the y-axis have $x = 0$ and y obeying $3(0)^2 + 2y^2 = 1$ or $y = \pm 1/\sqrt{2}$. So in $\mathcal{R}$, $-1/\sqrt{2} \leqslant y \leqslant 1/\sqrt{2}$ and, for each such y, $3x^2 \leqslant 1 - 2y^2$ or $-\sqrt{\frac{1-2y^2}{3}} \leqslant x \leqslant \sqrt{\frac{1-2y^2}{3}}$. So using vertical strips as in the figure above

$$\begin{aligned}
\text{Volume}(\mathcal{V}) &= \iint_{\mathcal{R}} \left(1 - 3x^2 - 2y^2\right) \mathrm{d}x\, \mathrm{d}y \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \mathrm{d}y \int_{-\sqrt{\frac{1-2y^2}{3}}}^{\sqrt{\frac{1-2y^2}{3}}} \mathrm{d}x\ \left(1 - 3x^2 - 2y^2\right) \\
&= 4\int_0^{1/\sqrt{2}} \mathrm{d}y \int_0^{\sqrt{\frac{1-2y^2}{3}}} \mathrm{d}x\ \left(1 - 3x^2 - 2y^2\right) \\
&= 4\int_0^{1/\sqrt{2}} \mathrm{d}y\ \Big[(1-2y^2)x - x^3\Big]_0^{\sqrt{\frac{1-2y^2}{3}}} \\
&= 4\int_0^{1/\sqrt{2}} \mathrm{d}y\ \sqrt{\frac{1-2y^2}{3}}\ \left[(1-2y^2) - \frac{1-2y^2}{3}\right] \\
&= 8\int_0^{1/\sqrt{2}} \mathrm{d}y\ \left[\frac{1-2y^2}{3}\right]^{3/2}
\end{aligned}$$

To evaluate this integral, we use the trig substitution[7] $2y^2 = \sin^2\theta$, or

$$y = \frac{\sin\theta}{\sqrt{2}} \qquad \mathrm{d}y = \frac{\cos\theta}{\sqrt{2}}\,\mathrm{d}\theta$$

7 See §1.9 in the CLP-2 text for a general discussion of trigonometric substitution.

to give

$$\text{Volume}(\mathcal{V}) = 8\int_0^{\pi/2} \overbrace{\mathrm{d}\theta\ \frac{\cos\theta}{\sqrt{2}}}^{\mathrm{d}y} \left[\frac{\cos^2\theta}{3}\right]^{3/2} = \frac{8}{\sqrt{54}}\int_0^{\pi/2} \mathrm{d}\theta\ \cos^4\theta$$

Then to integrate $\cos^4\theta$, we use the double angle formula[8]

$$\begin{aligned} \cos^2\theta &= \frac{\cos(2\theta)+1}{2} \\ \implies \cos^4\theta &= \frac{\big(\cos(2\theta)+1\big)^2}{4} = \frac{\cos^2(2\theta)+2\cos(2\theta)+1}{4} = \frac{\frac{\cos(4\theta)+1}{2}+2\cos(2\theta)+1}{4} \\ &= \frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta) \end{aligned}$$

Finally, since $\int_0^{\pi/2}\cos(4\theta)\,\mathrm{d}\theta = \int_0^{\pi/2}\cos(2\theta)\,\mathrm{d}\theta = 0$,

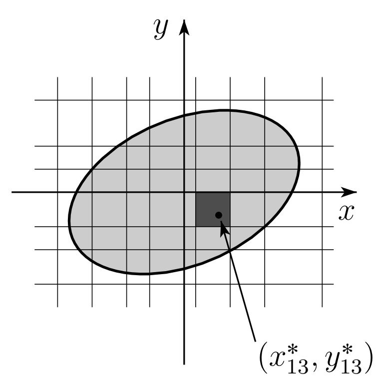
$$\text{Volume}(\mathcal{V}) = \frac{8}{\sqrt{54}}\,\frac{3}{8}\,\frac{\pi}{2} = \frac{\pi}{2\sqrt{6}}$$

Example 3.1.18

3.1.5 ▸▸ Optional — More about the Definition of $\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y$

Technically, the integral $\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, where $\mathcal{R}$ is a bounded region in $\mathbb{R}^2$, is defined as follows.

- Subdivide $\mathcal{R}$ by drawing lines parallel to the x and y axes.



- Number the resulting rectangles contained in $\mathcal{R}$, 1 through n. Notice that we are numbering all of the rectangles in $\mathcal{R}$, not just those in one particular row or column.

8 We weren't joking about his being a good review of single variable integration techniques. See Example 1.8.8 in the CLP-2 text.

- Denote by ΔA_i the area of rectangle #i.
- Select an arbitrary point (x_i^*, y_i^*) in rectangle #i.
- Form the sum $\sum\limits_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i$. Again note that the sum runs over all of the rectangles in $\mathcal{R}$, not just those in one particular row or column.

Now repeat this construction over and over again, using finer and finer grids. If, as the size[9] of the rectangles approaches zero, this sum approaches a unique limit (independent of the choice of parallel lines and of points (x_i^*, y_i^*)), then we define

$$\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta A_i$$

Theorem 3.1.19.

If $f(x,y)$ is continuous in a region $\mathcal{R}$ described by

$$\begin{aligned} a \leqslant x \leqslant b \\ B(x) \leqslant y \leqslant T(x) \end{aligned}$$

for continuous functions $B(x)$, $T(x)$, then

$$\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{and} \qquad \int_a^b \mathrm{d}x \left[\int_{B(x)}^{T(x)} \mathrm{d}y\; f(x,y)\right]$$

both exist and are equal. Similarly, if $\mathcal{R}$ is described by

$$\begin{aligned} c \leqslant y \leqslant d \\ L(y) \leqslant x \leqslant R(y) \end{aligned}$$

for continuous functions $L(y)$, $R(y)$, then

$$\iint\limits_{R} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{and} \qquad \int_c^d \mathrm{d}y \left[\int_{L(y)}^{R(y)} \mathrm{d}x\; f(x,y)\right]$$

both exist and are equal.

The proof of this theorem is not particularly difficult, but is still beyond the scope of this text. The main ideas in the proof can already be seen in section 1.1.6 of the CLP-2 text. An important consequence of this theorem is

9 For example, let p_i be the perimeter of rectangle number i and require that $\max_{1\leqslant i\leqslant n} p_i$ tends to zero. This way both the heights and widths of all rectangles also tend to zero.

Theorem 3.1.20 (Fubini).

If $f(x,y)$ is continuous in a region $\mathcal{R}$ described by both

$$\left\{\begin{matrix} a \leqslant x \leqslant b \\ B(x) \leqslant y \leqslant T(x) \end{matrix}\right\} \qquad \text{and} \qquad \left\{\begin{matrix} c \leqslant y \leqslant d \\ L(y) \leqslant x \leqslant R(y) \end{matrix}\right\}$$

for continuous functions $B(x)$, $T(x)$, $L(y)$, $R(y)$, then both

$$\int_a^b \mathrm{d}x \left[\int_{B(x)}^{T(x)} \mathrm{d}y\, f(x,y)\right] \qquad \text{and} \qquad \int_c^d \mathrm{d}y \left[\int_{L(y)}^{R(y)} \mathrm{d}x\, f(x,y)\right]$$

exist and are equal.

The hypotheses of both of these theorems can be relaxed a bit, but not too much. For example, if

$$\mathcal{R} = \{\, (x,y) \bigm| 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \,\} \qquad f(x,y) = \begin{cases} 1 & \text{if } x, y \text{ are both rational numbers} \\ 0 & \text{otherwise} \end{cases}$$

then the integral $\iint_{\mathcal{R}} f(x,y)\, \mathrm{d}x\, \mathrm{d}y$ does not exist. This is easy to see. If all of the x_i^*'s and y_i^*'s are chosen to be rational numbers, then

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta A_i = \sum_{i=1}^{n} \Delta A_i = \text{Area}(\mathcal{R})$$

But if we choose all the x_i^*'s and y_i^*'s to be irrational numbers, then

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta A_i = \sum_{i=1}^{n} 0\, \Delta A_i = 0$$

So the limit of $\sum\limits_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta A_i$, as the maximum diagonal of the rectangles approaches zero, depends on the choice of points (x_i^*, y_i^*). So the integral $\iint_{\mathcal{R}} f(x,y)\, \mathrm{d}x\, \mathrm{d}y$ does not exist.

Here is an even more pathological[10] example.

Example 3.1.21

In this example, we relax exactly one of the hypotheses of Fubini's Theorem, namely the continuity of f, and construct an example in which both of the integrals in Fubini's Theorem exist, but are **not equal**. In fact, we choose $\mathcal{R} = \{\, (x,y) \bigm| 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 \,\}$ and we use a function $f(x,y)$ that is continuous on $\mathcal{R}$, except at exactly one point — the origin.

10 For mathematicians, "pathological" is a synonym for "cool".

First, let $\delta_1, \delta_2, \delta_3, \cdots$ be any sequence of real numbers obeying

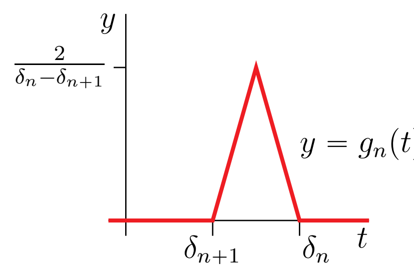
$$1 = \delta_1 > \delta_2 > \delta_3 > \cdots > \delta_n \to 0$$

For example $\delta_n = \frac{1}{n}$ or $\delta_n = \frac{1}{2^{n-1}}$ are both acceptable. For each positive integer n, let $I_n = (\delta_{n+1}, \delta_n] = \{ t \mid \delta_{n+1} < t \leqslant \delta_n \}$ and let $g_n(t)$ be any nonnegative continuous function obeying

- $g_n(t) = 0$ if t is not in I_n and
- $\int_{I_n} g(t)\, dt = 1$

There are many such functions. For example

$$g_n(t) = \left(\frac{2}{\delta_n - \delta_{n+1}}\right)^2 \begin{cases} \delta_n - t & \text{if } \frac{1}{2}(\delta_{n+1} + \delta_n) \leqslant t \leqslant \delta_n \\ t - \delta_{n+1} & \text{if } \delta_{n+1} \leqslant t \leqslant \frac{1}{2}(\delta_{n+1} + \delta_n) \\ 0 & \text{otherwise} \end{cases}$$



Here is a summary of what we have done so far.

- We subdivided the interval $0 < x \leqslant 1$ into infinitely many subintervals I_n. As n increases, the subinterval I_n gets smaller and smaller and also gets closer and closer to zero.
- We defined, for each n, a nonnegative continuous function g_n that is zero everywhere outside of I_n and whose integral over I_n is one.

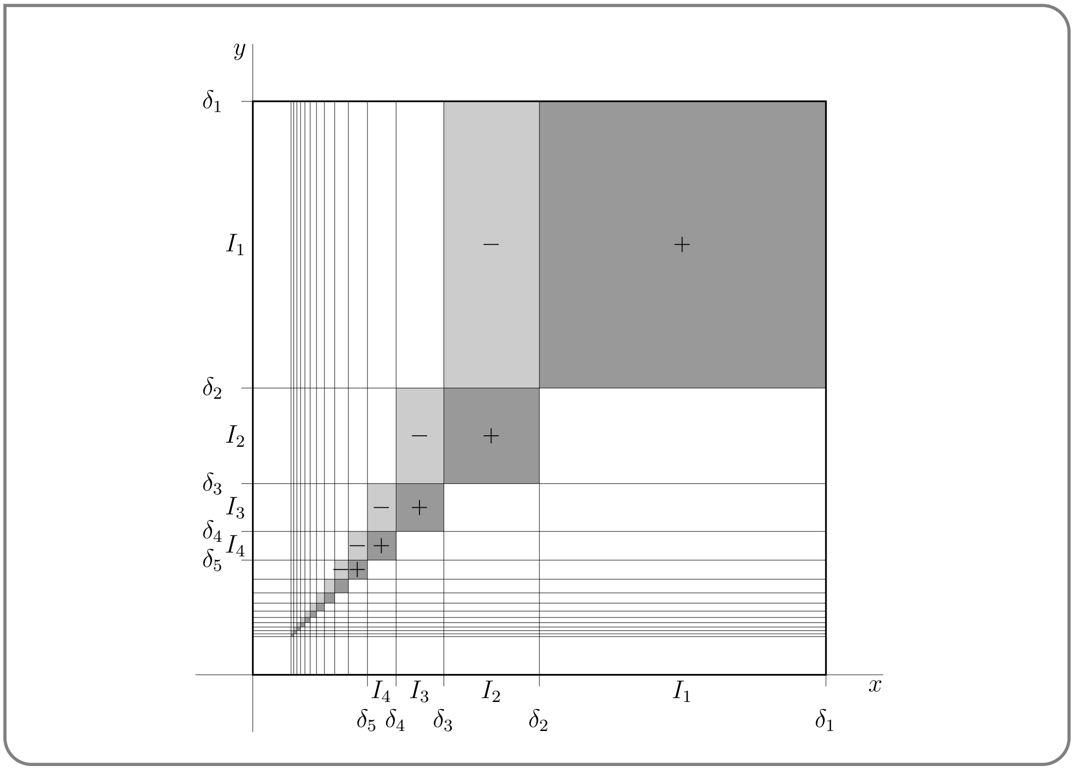
Now we define the integrand $f(x, y)$ in terms of these subintervals I_n and functions g_n.

$$f(x, y) = \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } y = 0 \\ g_m(x) g_n(y) & \text{if } x \in I_m,\ y \in I_n \text{ with } m = n \\ -g_m(x) g_n(y) & \text{if } x \in I_m,\ y \in I_n \text{ with } m = n + 1 \\ 0 & \text{otherwise} \end{cases}$$

You should think of $(0, 1] \times (0, 1]$ as a union of a bunch of small rectangles $I_m \times I_n$, as in the figure below. On most of these rectangles, $f(x, y)$ is just zero. The exceptions are the darkly shaded rectangles $I_n \times I_n$ on the "diagonal" of the figure and the lightly shaded rectangles $I_{n+1} \times I_n$ just to the left of the "diagonal".

On each darkly shaded rectangle, $f(x, y) \geqslant 0$ and the graph of $f(x, y)$ is the graph of $g_n(x) g_n(y)$ which looks like a pyramid. On each lightly shaded rectangle, $f(x, y) \leqslant 0$ and the graph of $f(x, y)$ is the graph of $-g_{n+1}(x) g_n(y)$ which looks like a pyramidal hole in the ground.

Now fix any $0 \leqslant y \leqslant 1$ and let's compute $\int_0^1 f(x, y)\, \mathrm{d}x$. That is, we are integrating f along a line that is parallel to the x-axis. If $y = 0$, then $f(x, y) = 0$ for all x, so



$\int_0^1 f(x,y)\,\mathrm{d}x = 0$. If $0 < y \leqslant 1$, then there is exactly one positive integer n with $y \in I_n$ and $f(x,y)$ is zero, except for x in I_n or I_{n+1}. So for $y \in I_n$

$$\begin{aligned}\int_0^1 f(x,y)\,\mathrm{d}x &= \sum_{m=n,n+1} \int_{I_m} f(x,y)\,\mathrm{d}x = \int_{I_n} g_n(x)g_n(y)\,\mathrm{d}x - \int_{I_{n+1}} g_{n+1}(x)g_n(y)\,\mathrm{d}x \\ &= g_n(y)\int_{I_n} g_n(x)\,\mathrm{d}x - g_n(y)\int_{I_{n+1}} g_{n+1}(x)\,\mathrm{d}x \\ &= g_n(y) - g_n(y) = 0\end{aligned}$$

Here we have twice used that $\int_{I_m} g(t)\,dt = 1$ for all m. Thus $\int_0^1 f(x,y)\,\mathrm{d}x = 0$ for all y and hence $\int_0^1 \mathrm{d}y\left[\int_0^1 \mathrm{d}x\, f(x,y)\right] = 0$.

Finally, fix any $0 \leqslant x \leqslant 1$ and let's compute $\int_0^1 f(x,y)\,\mathrm{d}y$. That is, we are integrating f along a line that is parallel to the y-axis. If $x = 0$, then $f(x,y) = 0$ for all y, so $\int_0^1 f(x,y)\,\mathrm{d}y = 0$. If $0 < x \leqslant 1$, then there is exactly one positive integer m with $x \in I_m$. If $m \geqslant 2$, then $f(x,y)$ is zero, except for y in I_m and I_{m-1}. But, if $m = 1$, then $f(x,y)$ is zero, except for y in I_1. (Take another look at the figure above.) So for $x \in I_m$, with $m \geqslant 2$,

$$\begin{aligned}\int_0^1 f(x,y)\,\mathrm{d}y &= \sum_{n=m,m-1} \int_{I_n} f(x,y)\,\mathrm{d}y = \int_{I_m} g_m(x)g_m(y)\,\mathrm{d}y - \int_{I_{m-1}} g_m(x)g_{m-1}(y)\,\mathrm{d}y \\ &= g_m(x)\int_{I_m} g_m(y)\,\mathrm{d}y - g_m(x)\int_{I_{m-1}} g_{m-1}(y)\,\mathrm{d}y \\ &= g_m(x) - g_m(x) = 0\end{aligned}$$

But for $x \in I_1$,

$$\int_0^1 f(x,y)\,\mathrm{d}y = \int_{I_1} f(x,y)\,\mathrm{d}y = \int_{I_1} g_1(x)g_1(y)\,\mathrm{d}y = g_1(x)\int_{I_1} g_1(y)\,\mathrm{d}y = g_1(x)$$

Thus

$$\int_0^1 f(x,y)\,\mathrm{d}y = \begin{cases} 0 & \text{if } x \leqslant \delta_2 \\ g_1(x) & \text{if } x \in I_1 \end{cases}$$

and hence

$$\int_0^1 \mathrm{d}x \left[\int_0^1 \mathrm{d}y\, f(x,y)\right] = \int_{I_1} g_1(x)\,\mathrm{d}x = 1$$

The conclusion is that for the $f(x,y)$ above, which is defined for all $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ and is continuous except at $(0,0)$,

$$\int_0^1 \mathrm{d}y \left[\int_0^1 \mathrm{d}x\, f(x,y)\right] = 0 \qquad \int_0^1 \mathrm{d}x \left[\int_0^1 \mathrm{d}y\, f(x,y)\right] = 1$$

Example 3.1.21

3.1.6 ▸▸ Even and Odd Functions

During the course of our study of integrals of functions of one variable, we found that the evaluation of certain integrals could be substantially simplified by exploiting symmetry properties of the integrand. Concretely, in §1.2.1 of the CLP-2 text, we gave the

Definition 3.1.22 (Definition 1.2.8 in the CLP-2 text)**.**

Let $f(x)$ be a function of one variable. Then,

- we say that $f(x)$ is even when $f(-x) = f(x)$ for all x, and
- we say that $f(x)$ is odd when $f(-x) = -f(x)$ for all x

and we saw that

- $f(x) = |x|$, $f(x) = \cos x$ and $f(x) = x^2$ are even functions and
- $f(x) = \sin x$, $f(x) = \tan x$ and $f(x) = x^3$ are odd functions.
- In fact, if $f(x)$ is any even power of x, then $f(x)$ is an even function and if $f(x)$ is any odd power of x, then $f(x)$ is an odd function.

We also learned how to exploit evenness and oddness to simplify integration.

Theorem 3.1.23 (Theorem 1.2.11 in the CLP-2 text)**.**

Let $a > 0$.

(a) If $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$$

(b) If $f(x)$ is an odd function, then

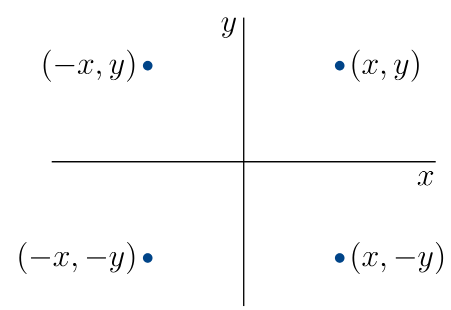
$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$$

We will now see that we can similarly exploit evenness and oddness of functions of more than one variable. But for functions of more than one variable there is also more than one kind of oddness and evenness. In the Definition 3.1.22 (Definition 1.2.8 in the CLP-2 text) of evenness and oddness of the function $f(x)$, we compared the value of f at x with the value of f at $-x$. The points x and $-x$ are the same distance from the origin, 0, and are on opposite sides of 0. The point $-x$ is called the reflection of x across the origin. To prepare for our definitions of evenness and oddness of functions of two variables, we now define three different reflections in the two dimensional world of the xy-plane.

Definition 3.1.24.

Let x and y be two real numbers.

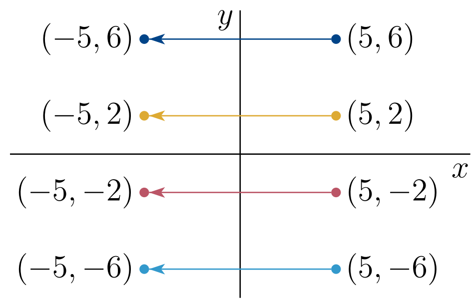
- The reflection of (x, y) across the y-axis is $(-x, y)$.
- The reflection of (x, y) across the x-axis is $(x, -y)$.
- The reflection of (x, y) across the origin is $(-x, -y)$.



- To get from the point (x, y) to its image reflected across the y-axis, you
  - start from (x, y), and

- walk horizontally straight to the y-axis, and
- cross the y-axis, and
- continue horizontally the same distance as you have already travelled to $(-x, y)$.

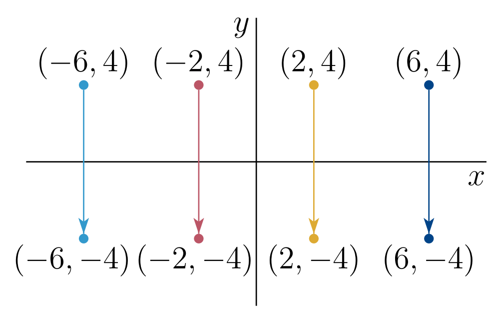
Here are four examples.



reflection across the y-axis

- To get from the point (x, y) to its image reflected across the x-axis, you
  - start from (x, y), and
  - walk vertically straight to the x-axis, and
  - cross the x-axis, and
  - continue vertically the same distance as you have already travelled to the reflected image $(x, -y)$.

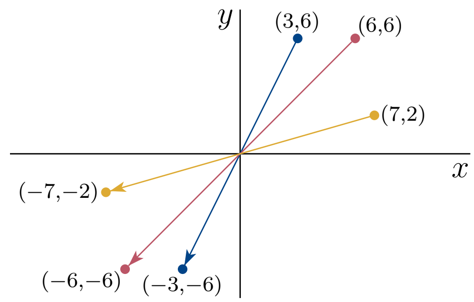
Here are four examples.



reflection across the x-axis

- To get from the point (x, y) to its image reflected across the origin, you
  - start from (x, y), and
  - walk radially straight to the origin, and
  - cross the origin, and
  - continue radially in the same direction the same distance as you have already travelled to the reflected image $(-x, -y)$.

Here are three examples.



reflection across the origin

For each of these three types of reflection, there is a corresponding kind of oddness and evenness.

Definition 3.1.25.

Let $f(x,y)$ be a function of two variables. Then,

- we say that $f(x,y)$ is even (under reflection across the origin) when $f(-x,-y) = f(x,y)$ for all x and y, and
- we say that $f(x,y)$ is odd (under reflection across the origin) when $f(-x,-y) = -f(x,y)$ for all x and y

and

- we say that $f(x,y)$ is even under $x \to -x$ (i.e. under reflection across the y-axis) when $f(-x,y) = f(x,y)$ for all x and y, and
- we say that $f(x,y)$ is odd under $x \to -x$ (i.e. under reflection across the y-axis) when $f(-x,y) = -f(x,y)$ for all x and y

and

- we say that $f(x,y)$ is even under $y \to -y$ (i.e. under reflection across the x-axis) when $f(x,-y) = f(x,y)$ for all x and y, and
- we say that $f(x,y)$ is odd under $y \to -y$ (i.e. under reflection across the x-axis) when $f(x,-y) = -f(x,y)$ for all x and y.

Example 3.1.26

Let m and n be two integers and set $f(x,y) = x^m y^n$. Then

$$\begin{aligned} f(-x,y) &= (-x)^m y^n = (-1)^m x^m y^n = (-1)^m f(x,y) \\ f(x,-y) &= x^m (-y)^n = (-1)^n x^m y^n = (-1)^n f(x,y) \\ f(-x,-y) &= (-x)^m (-y)^n = (-1)^{m+n} x^m y^n = (-1)^{m+n} f(x,y) \end{aligned}$$

Consequently

- if m is even, then $f(x, y)$ is even under $x \to -x$ and
- if m is odd, then $f(x, y)$ is odd under $x \to -x$ and
- if n is even, then $f(x, y)$ is even under $y \to -y$ and
- if n is odd, then $f(x, y)$ is odd under $y \to -y$ and
- if $m + n$ is even, then $f(x, y)$ is even (under reflection across the origin) and
- if $m + n$ is odd, then $f(x, y)$ is odd (under reflection across the origin).

Example 3.1.26

Recall from Theorem 3.1.23 (or Theorem 1.2.11 in the CLP-2 text) that we can exploit the evenness or oddness of the integrand, $f(x)$, of the integral $\int_b^a f(x)\,\mathrm{d}x$ to simplify the evaluation of the integral when $b = -a$, i.e. when the domain of integration is invariant under reflection across the origin. Similarly, we will be able to simplify the evaluation of the double integral $\iint_{\mathcal{R}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ when the integrand is even or odd and the domain of integration $\mathcal{R}$ is invariant under the corresponding reflection — meaning that the reflected $\mathcal{R}$ is identical to the original $\mathcal{R}$. Here are some details for "reflection across the y-axis". The details for the other reflections are similar.

- If $\mathcal{R}$ is any subset of the xy-plane,

  $$\text{the reflection of } \mathcal{R} \text{ across the } y\text{-axis} = \big\{\ (-x, y)\ \big|\ (x, y) \in \mathcal{R}\ \big\}$$

  The set notation on the right hand side means "the set of all points $(-x, y)$ with (x, y) a point of $\mathcal{R}$".

- In the special case[11] that

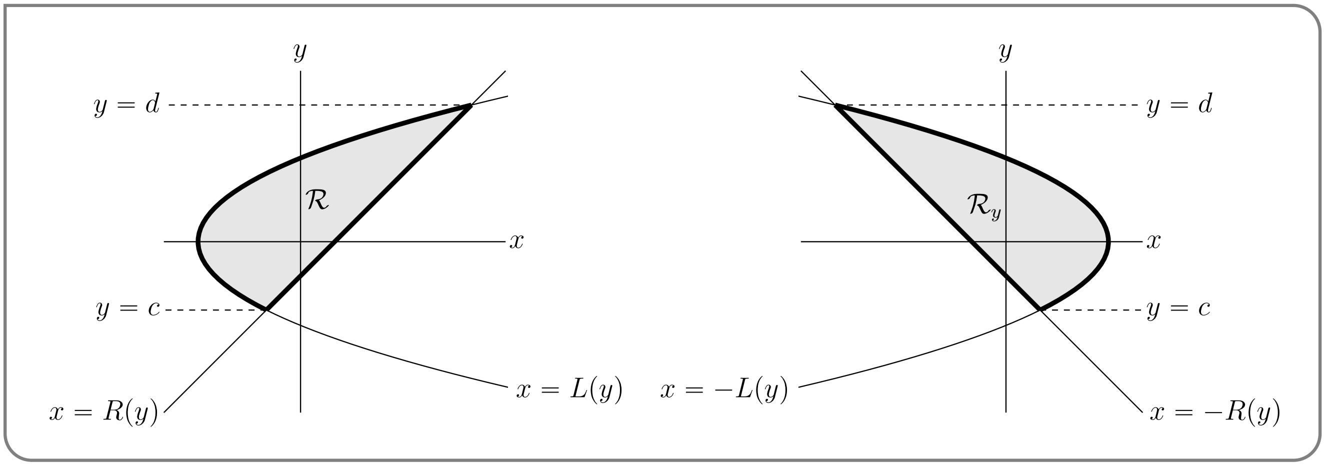
  $$\mathcal{R} = \big\{\ (x, y)\ \big|\ c \leqslant y \leqslant d, L(y) \leqslant x \leqslant R(y)\ \big\}$$

  (see §3.1.2 on horizontal slices) then

  $$\text{the reflection of } \mathcal{R} \text{ across the } y\text{-axis} = \big\{\ (x, y)\ \big|\ c \leqslant y \leqslant d, -R(y) \leqslant x \leqslant -L(y)\ \big\}$$

  In the sketch below $\mathcal{R}_y$ is the reflection of $\mathcal{R}$ across the y-axis.

11 Here $L(y)$ ("L" stands for "left") is the leftmost allowed value of x when the y-coordinate is y, and $R(y)$ ("R" stands for "right") is the rightmost allowed value of x, when the y-coordinate is y.



- A subset $\mathcal{R}$ of the xy-plane is invariant under reflection across the y-axis (or is also known as "symmetric about the y-axis") when

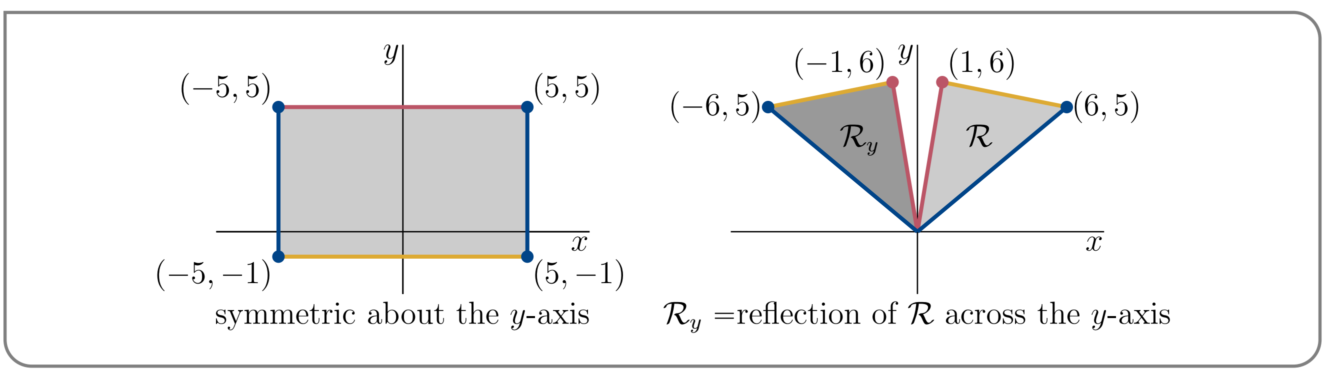
$$(-x, y) \text{ is in } \mathcal{R} \iff (x, y) \text{ is in } \mathcal{R}$$

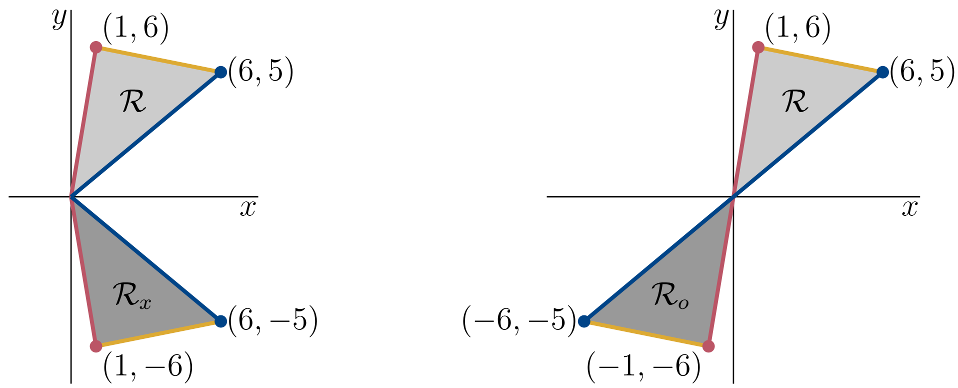
Recall that the symbol $\iff$ is read "if and only if". In the special case that

$$\mathcal{R} = \big\{ \ (x, y) \ \big| \ c \leqslant y \leqslant d, L(y) \leqslant x \leqslant R(y) \ \big\}$$

$\mathcal{R}$ is is invariant under reflection across the y-axis when $L(y) = -R(y)$.

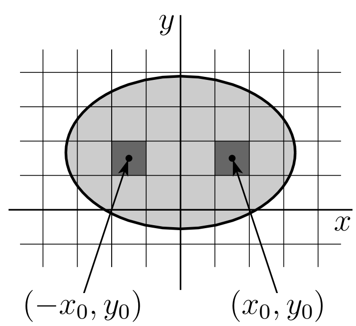
Here are some more sketches. The first sketch is of a rectangle that is invariant under reflection across the y-axis, but is not invariant under reflection across the x-axis. The remaining three sketches show a triangle and its reflections across the y-axis, across the x-axis and across the origin.





$\mathcal{R}_x$ =reflection of $\mathcal{R}$ across the x-axis $\mathcal{R}_o$ =reflection of $\mathcal{R}$ across the origin

We are finally ready for the analog of Theorem 3.1.23 (Theorem 1.2.11 in the CLP-2 text) for functions of two variables. By way of motivation for that theorem, consider the integral $\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y$, with the integrand, $f(x,y)$, odd under $x \to -x$, and the domain of integration, $\mathcal{R}$, symmetric about the y-axis. Slice up $\mathcal{R}$ into tiny (think "infinitesmal") squares, either by subdividing vertical slices into tiny squares, as in §3.1.1, or by subdividing horizontal slices into tiny squares, as in §3.1.2. Concentrate on any point (x_0, y_0) in $\mathcal{R}$. The contribution to the integral coming from the square that contains (x_0, y_0) is



(essentially[12]) $f(x_0, y_0)\,\Delta x\,\Delta y$. That contribution is cancelled by the contribution coming from the square containing (the reflected point) $(-x_0, y_0)$, which is

$$f(-x_0, y_0)\,\Delta x\,\Delta y = -f(x_0, y_0)\,\Delta x\,\Delta y$$

This is the case for all points (x_0, y_0) in $\mathcal{R}$. Consequently

$$\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = 0$$

Here is the analog of Theorem 3.1.23 for functions of two variables.

12 In this motivation, we suppress the $\Delta x \to 0$ and $\Delta y \to 0$ limits.

Theorem 3.1.27 (2d Even and Odd).

(a) Let $\mathcal{R}$ be a subset of the xy-plane that is symmetric about the y-axis. If $f(x,y)$ is odd under $x \to -x$, then

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = 0$$

Denote by $\mathcal{R}_+$ the set of all points in $\mathcal{R}$ that have $x \geqslant 0$. If $f(x,y)$ is even under $x \to -x$, then

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = 2\iint_{\mathcal{R}_+} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

(b) Let $\mathcal{R}$ be a subset of the xy-plane that is symmetric about the x-axis. If $f(x,y)$ is odd under $y \to -y$, then

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = 0$$

Denote by $\mathcal{R}_+$ the set of all points in $\mathcal{R}$ that have $y \geqslant 0$. If $f(x,y)$ is even under $y \to -y$, then

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = 2\iint_{\mathcal{R}_+} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

(c) Let $\mathcal{R}$ be a subset of the xy-plane that is invariant under reflection across the origin. If $f(x,y)$ is odd (under reflection across the origin), then

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = 0$$

Denote by $\mathcal{R}_+$ either the set of all points in $\mathcal{R}$ that have $x \geqslant 0$ or the set of all points in $\mathcal{R}$ that have $y \geqslant 0$. If $f(x,y)$ is even (under reflection across the origin), then

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = 2\iint_{\mathcal{R}_+} f(x,y)\,\mathrm{d}x\mathrm{d}y$$

Proof. We will give only the proof for part (a) in the special case that

$$\mathcal{R} = \left\{\ (x,y)\ \middle|\ c \leqslant y \leqslant d,\ L(y) \leqslant x \leqslant R(y)\ \right\}$$

In part (a), we are assuming that $\mathcal{R}$ is symmetric about the y-axis, so that $L(y) = -R(y)$.

So, using horizontal strips, as described in §3.1.2,

$$\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_c^d \mathrm{d}y \int_{-R(y)}^{R(y)} \mathrm{d}x\ f(x,y)$$

Fix any $c \leqslant y \leqslant d$.

- If $f(x,y)$ is odd under $x \to -x$, then $f(-x,y) = -f(x,y)$ for all $-R(y) \leqslant x \leqslant R(y)$ and
$$\int_{-R(y)}^{R(y)} \mathrm{d}x\ f(x,y) = 0$$
by part (b) of Theorem 3.1.23 (Theorem 1.2.11 in the CLP-2 text).
- If $f(x,y)$ is even under $x \to -x$, then $f(-x,y) = f(x,y)$ for all $-R(y) \leqslant x \leqslant R(y)$ and
$$\int_{-R(y)}^{R(y)} \mathrm{d}x\ f(x,y) = 2\int_0^{R(y)} \mathrm{d}x\ f(x,y)$$
by part (a) of Theorem 3.1.23.

As the statements of the two bullets are true for each fixed $c \leqslant y \leqslant d$, we have that

- if $f(x,y)$ is odd under $x \to -x$, then
$$\begin{aligned}\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y &= \int_c^d \mathrm{d}y \int_{-R(y)}^{R(y)} \mathrm{d}x\ f(x,y) = \int_c^d \mathrm{d}y\ 0\\ &= 0\end{aligned}$$
- and if $f(x,y)$ is even under $x \to -x$, then
$$\begin{aligned}\iint\limits_{\mathcal{R}} f(x,y)\,\mathrm{d}x\mathrm{d}y &= \int_c^d \mathrm{d}y \int_{-R(y)}^{R(y)} \mathrm{d}x\ f(x,y) = \int_c^d \mathrm{d}y\ 2\int_0^{R(y)} \mathrm{d}x\ f(x,y)\\ &= 2\iint\limits_{\mathcal{R}_+} f(x,y)\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

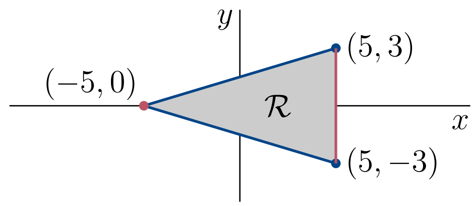
The proof of part (a) when $\mathcal{R}$ is not of the form $\mathcal{R} = \{\ (x,y)\ |\ c \leqslant y \leqslant d, L(y) \leqslant x \leqslant R(y)\ \}$ (for example if $\mathcal{R}$ has holes in it) is most easily done using the change of variables $x = -u$, $y = v$ in Theorem 3.8.3, which is part of the optional §3.8.

The proof of part (b) is similar to the proof of part (a).

The proof of part (c) is most easily done using the change of variables $x = -u$, $y = -v$ in Theorem 3.8.3, which is part of the optional §3.8. □

Example 3.1.28

Evaluate the integral $\iint_{\mathcal{R}} e^x \sin(y + y^3)\,\mathrm{d}x\mathrm{d}y$ over the triangular region $\mathcal{R}$ in the sketch



Solution. Start by checking the evenness and oddness properties of the integrand $f(x,y) = e^x \sin(y+y^3)$. Since

$$\begin{aligned} f(-x,y) &= e^{-x}\sin(y+y^3) \\ f(x,-y) &= e^x \sin\big(-y+(-y)^3\big) = e^x\sin(-y-y^3) = -e^x\sin(y+y^3) \\ &= -f(x,y) \\ f(-x,-y) &= -e^{-x}\sin(y+y^3) \end{aligned}$$

the integrand is odd under $y \to -y$ but is neither even nor odd under $x \to -x$ and $(x,y) \to -(x,y)$. Fortunately (or by rigging), the domain of integration $\mathcal{R}$ is invariant under $y \to -y$ (i.e. is symmetric about the x-axis) and so

$$\iint_{\mathcal{R}} e^x \sin(y+y^3)\,\mathrm{d}x\mathrm{d}y = 0$$

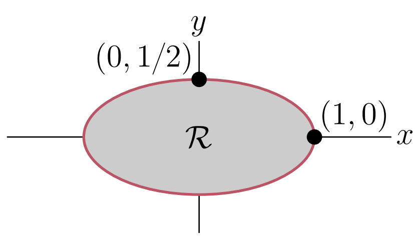
by part (b) of Theorem 3.1.27 (Theorem 1.2.11 in the CLP-2 text).

Example 3.1.28

Example 3.1.29

Evaluate the integral $\iint_{\mathcal{R}}(xe^y + ye^x + xe^{xy} + 7)\,\mathrm{d}x\mathrm{d}y$ over the region $\mathcal{R}$ whose outer boundary is the ellipse $x^2+4y^2=1$.

Solution. First, let's sketch the ellipse $x^2+4y^2=1$. Notice that its x intercepts are the points $(x,0)$ that obey $x^2+4(0)^2=1$. So the x-intercepts are $(\pm1,0)$. Similarly its y intercepts are the points $(0,y)$ that obey $0^2+4y^2=1$. So the y-intercepts are $(0,\pm1/2)$. Here is a sketch of $\mathcal{R}$.



From the sketch, it looks like $\mathcal{R}$ is invariant under $x \to -x$ (i.e. is symmetric about the y-axis) and is also invariant under $y \to -y$ (i.e. is symmetric about the x-axis) and is also invariant under $(x,y) \to -(x,y)$. It is easy to check analytically that this is indeed the case. The point (x,y) is in $\mathcal{R}$ if and only if it is inside $x^2+4y^2=1$. That is the case if and only if $x^2+4y^2 \leqslant 1$. Since

$$(-x)^2+4y^2 = x^2+(-4y)^2 = (-x)^2+4(-y)^2 = x^2+4y^2$$

we have

$$\begin{aligned}(x,y) \text{ is in } \mathcal{R} &\iff (-x,y) \text{ is in } \mathcal{R}\\ &\iff (x,-y) \text{ is in } \mathcal{R}\\ &\iff (-x,-y) \text{ is in } \mathcal{R}\end{aligned}$$

Now let's check the evenness and oddness properties of the integrand.

$$\begin{aligned} f(x,y) &= xe^y \quad + ye^x \quad + xe^{xy} \quad +7\\ f(-x,y) &= -xe^y \quad + ye^{-x} - xe^{-xy} + 7\\ f(x,-y) &= xe^{-y} \quad - ye^x \quad + xe^{-xy} + 7\\ f(-x,-y) &= -xe^{-y} - ye^{-x} - xe^{xy} \quad + 7\end{aligned}$$

So $f(x,y)$ is neither even nor odd under any of $x \to -x$, $y \to -y$, and $(x,y) \to -(x,y)$. BUT, look at the four terms of $f(x,y)$ separately.

- The first term of $f(x,y)$, namely xe^y, is odd under $x \to -x$.
- The second term of $f(x,y)$, namely ye^x, is odd under $y \to -y$.
- The third term of $f(x,y)$, namely xe^{xy}, is odd under $(x,y) \to -(x,y)$.
- The fourth term of $f(x,y)$, namely 7, is even under all of $x \to -x$, $y \to -y$, and $(x,y) \to -(x,y)$.

So, by parts (a), (b) and (c) of Theorem 3.1.27, in order,

$$\begin{aligned}\iint_{\mathcal{R}} (xe^y + ye^x + xe^{xy} + 7)\, \mathrm{d}x\mathrm{d}y &= \iint_{\mathcal{R}} xe^y\, \mathrm{d}x\mathrm{d}y + \iint_{\mathcal{R}} ye^x\, \mathrm{d}x\mathrm{d}y + \iint_{\mathcal{R}} xe^{xy}\, \mathrm{d}x\mathrm{d}y + 7\iint_{\mathcal{R}} \mathrm{d}x\mathrm{d}y\\ &= 0+0+0+7\,\mathrm{Area}(\mathcal{R})\end{aligned}$$

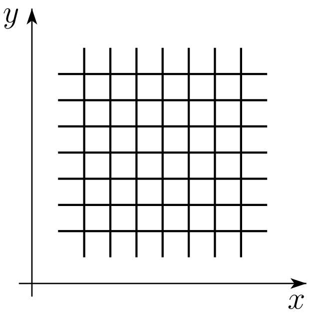
Since $\mathcal{R}$ is an ellipse with semi-major axis $a=1$ and semi-minor axis $b=\frac{1}{2}$, it has area $\pi ab = \frac{1}{2}\pi$ and

$$\iint_{\mathcal{R}} (xe^y + ye^x + xe^{xy} + 7)\, \mathrm{d}x\mathrm{d}y = \frac{7}{2}\pi$$

Example 3.1.29

3.2▲ Double Integrals in Polar Coordinates

So far, in setting up integrals, we have always cut up the domain of integration into tiny rectangles by drawing in many lines of constant x and many lines of constant y.



There is no law that says that we must cut up our domains of integration into tiny pieces in that way. Indeed, when the objects of interest are sort of round and centered on the origin, it is often advantageous[13] to use polar coordinates, rather than Cartesian coordinates.

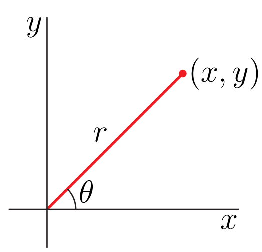
3.2.1 ▸▸ Polar Coordinates

It may have been a while since you did anything in polar coordinates. So let's review before we resume integrating.

Definition 3.2.1.

The polar coordinates[14] of any point (x, y) in the xy-plane are

$r =$ the distance from $(0, 0)$ to (x, y)
$\theta =$ the (counter-clockwise) angle between the x axis and the line joining (x, y) to $(0, 0)$



13 The "golden hammer" (also known as Maslow's hammer and as the law of the instrument) refers to a tendency to always use the same tool, even when it isn't the best tool for the job. It is just as bad in mathematics as it is in carpentry.

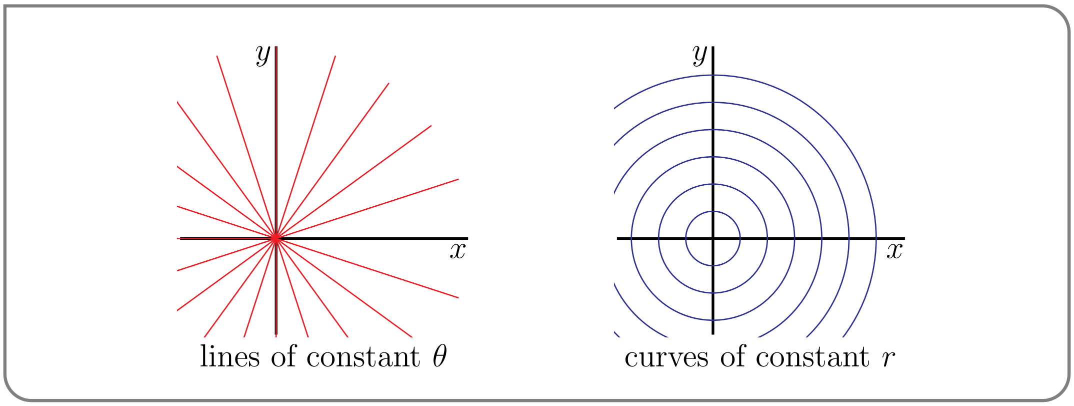
14 In the mathematical literature, the angular coordinate is usually denoted θ, as we do here. The symbol ϕ is also often used for the angular coordinate. In fact there is an ISO standard (#80000-2) which specifies that ϕ should be used in the natural sciences and in technology. See Appendix G.

Cartesian and polar coordinates are related, via a quick bit of trigonometry, by

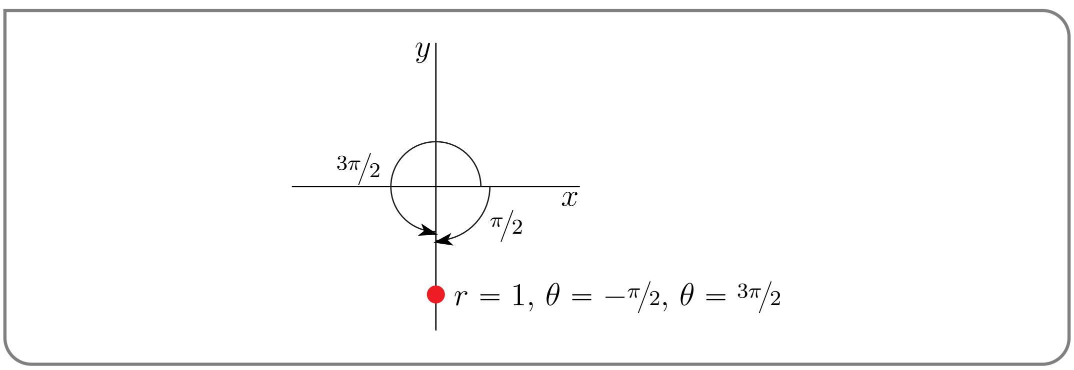
Equation 3.2.2.

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r = \sqrt{x^2+y^2} \qquad \theta = \arctan\frac{y}{x}$$

The following two figures show a number of lines of constant θ, on the left, and curves of constant r, on the right.

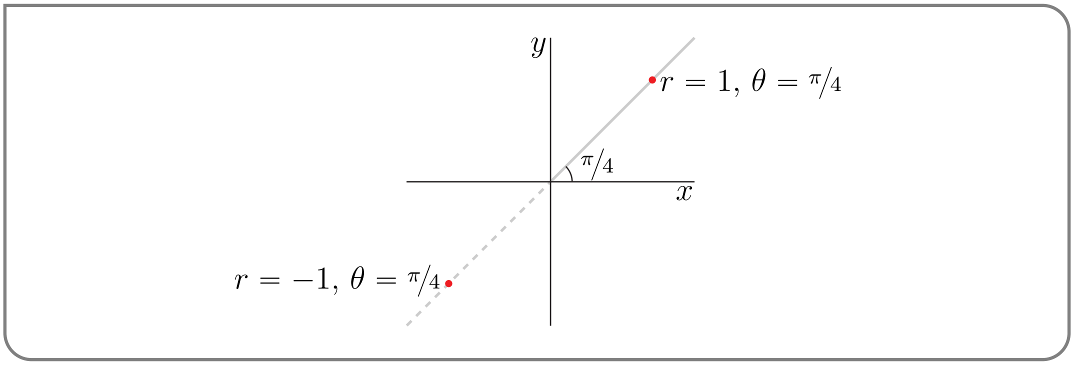


Note that the polar angle θ is only defined up to integer multiples of 2π. For example, the point $(1,0)$ on the x-axis could have $\theta = 0$, but could also have $\theta = 2\pi$ or $\theta = 4\pi$. It is sometimes convenient to assign θ negative values. When $\theta < 0$, the counter-clockwise[15] angle θ refers to the clockwise angle $|\theta|$. For example, the point $(0,-1)$ on the negative y-axis can have $\theta = -\frac{\pi}{2}$ and can also have $\theta = \frac{3\pi}{2}$.



It is also sometimes convenient to extend the above definitions by saying that $x = r\cos\theta$ and $y = r\sin\theta$ even when r is negative. For example, the following figure shows (x,y) for $r = 1$, $\theta = \pi/4$ and for $r = -1$, $\theta = \pi/4$. Both points lie on the line through

15 or anti-clockwise or widdershins. Yes, widdershins is a real word, though the Oxford English Dictionary lists its frequency of usage as between 0.01 and 0.1 times per million words. Of course both "counter-clockwise" and "anti-clockwise" assume that your clock is not a sundial in the southern hemisphere.



the origin that makes an angle of 45° with the x-axis and both are a distance one from the origin. But they are on opposite sides of the the origin.

3.2.2 ▸▸ Polar Curves

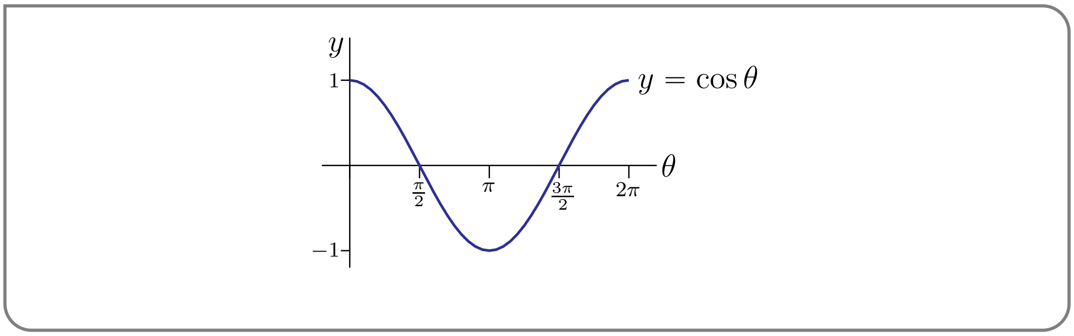
Here are a couple of examples in which we sketch curves specified by equations in terms of polar coordinates.

Example 3.2.3 (The Cardioid)

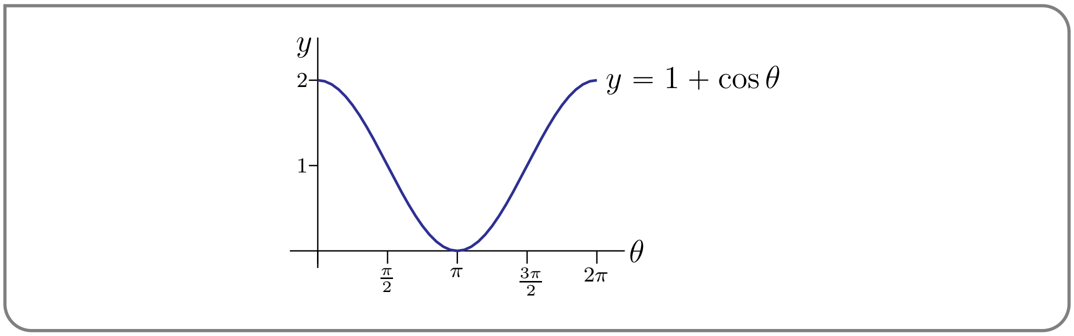
Let's sketch the curve

$$r = 1 + \cos\theta$$

Our starting point will be to understand how $1 + \cos\theta$ varies with θ. So it will be helpful to remember what the graph of $\cos\theta$ looks like for $0 \leqslant \theta \leqslant 2\pi$.



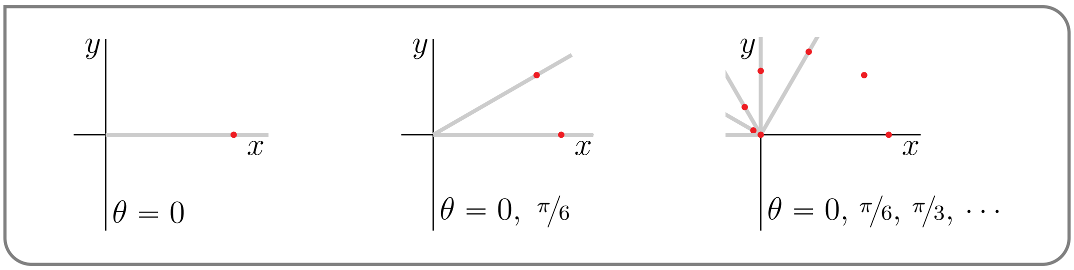
From this we see that the graph of $y = 1 + \cos\theta$ is



Now let's pick some easy θ values, find the corresponding r's and sketch them.

- When $\theta = 0$, we have $r = 1 + \cos 0 = 1 + 1 = 2$. To sketch the point with $\theta = 0$ and $r = 2$, we first draw in the half-line consisting of all points with $\theta = 0$, $r > 0$. That's the positive x-axis, sketched in gray in the leftmost figure below. Then we put in a dot on that line a distance 2 from the origin. That's the red dot in the leftmost figure below.

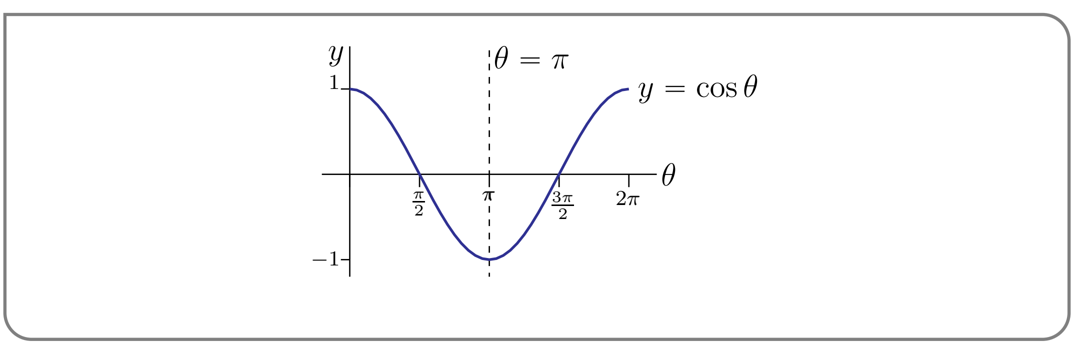
- Now increase θ a bit (to another easy place to evaluate), say to $\theta = \frac{\pi}{6}$. As we do so $r = 1 + \cos\theta$ decreases to $r = 1 + \cos\frac{\pi}{6} = 1 + \frac{\sqrt{3}}{2} \approx 1.87$. To sketch the point with $\theta = \frac{\pi}{6}$ and $r \approx 1.87$, we first draw in the half-line consisting of all points with $\theta = \frac{\pi}{6}$, $r > 0$. That's the upper gray line in the second figure below. Then we put in a dot on that line a distance 1.87 from the origin. That's the upper red dot in the second figure below.



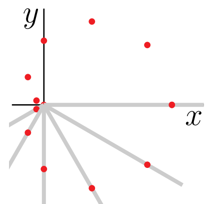
- Now increase θ still more, say to
  - $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$,
  - followed by $\theta = \frac{3\pi}{6} = \frac{\pi}{2}$,
  - followed by $\theta = \frac{4\pi}{6} = \frac{2\pi}{3}$,
  - followed by $\theta = \frac{5\pi}{6}$,
  - followed by $\theta = \frac{6\pi}{6} = \pi$.

  As θ increases, $r = 1 + \cos\theta$ decreases, hitting $r = 1$ when $\theta = \frac{\pi}{2}$ and ending at $r = 0$ when $\theta = \pi$. For each of these θ's, we first draw in the half-line consisting of all points with that θ and $r \geqslant 0$. Those are the five gray lines in the figure on the right above. Then we put in a dot on each θ-line a distance $r = 1 + \cos\theta$ from the origin. Those are the red dots on the gray lines in the figure on the right above.

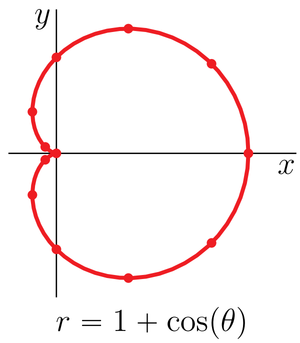
- We could continue the above procedure for $\pi \leqslant \theta \leqslant 2\pi$. Or we can look at the graph of $\cos\theta$ above and notice that the graph of $\cos\theta$ for $\pi \leqslant \theta \leqslant 2\pi$ is exactly the mirror image, about $\theta = \pi$, of the graph of $\cos\theta$ for $0 \leqslant \theta \leqslant \pi$. That is,



$\cos(\pi + \theta) = \cos(\pi - \theta)$ so that $r(\pi + \theta) = r(\pi - \theta)$. So we get the figure.



- Finally, we fill in a smooth curve through the dots and we get the graph below. This curve is called a *cardioid* because it looks like a heart[16].



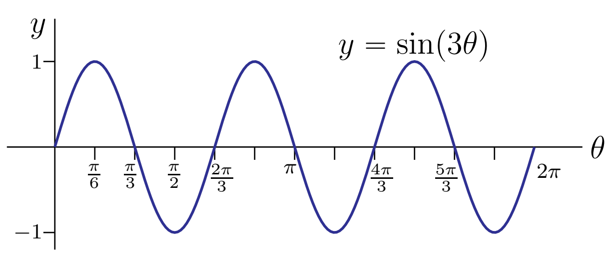
Example 3.2.3

Example 3.2.4 (The Three Petal Rose)

Now we'll use the same procedure as in the last example to sketch the graph of

$$r = \sin(3\theta)$$

Again it will be useful to remember what the graph of $\sin(3\theta)$ looks like for $0 \leqslant \theta \leqslant 2\pi$.



- We'll first consider $0 \leqslant \theta \leqslant \frac{\pi}{3}$, so that $0 \leqslant 3\theta \leqslant \pi$. On this interval $r(\theta) = \sin(3\theta)$

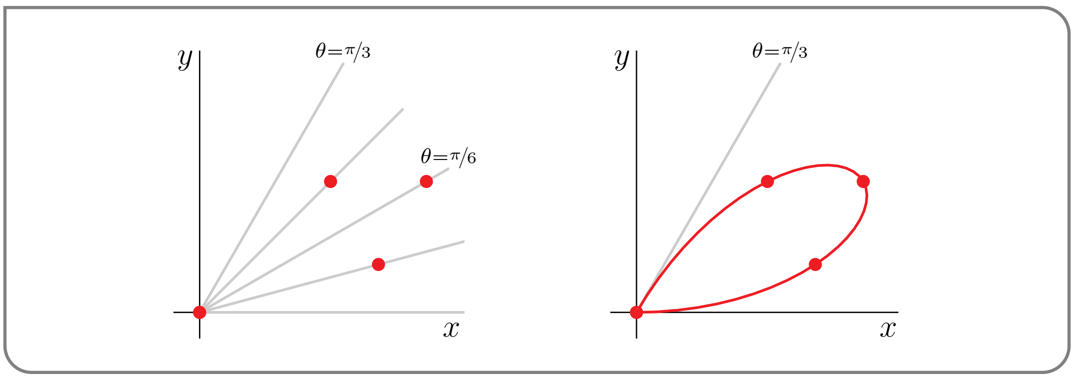
16 Well, a mathematician's heart. The name "cardioid" comes from the Greek word $\kappa\alpha\rho\delta\iota\alpha$ (which anglicizes to kardia) for heart.

- starts with $r(0) = 0$, and then
- increases as θ increases until
- $3\theta = \frac{\pi}{2}$, i.e. $\theta = \frac{\pi}{6}$, where $r\left(\pi/6\right) = 1$, and then
- decreases as θ increases until
- $3\theta = \pi$, i.e. $\theta = \frac{\pi}{3}$, where $r\left(\pi/3\right) = 0$, again.

Here is a table giving a few values of $r(\theta)$ for $0 \leqslant \theta \leqslant \frac{\pi}{3}$. Notice that we have chosen values of θ for which $\sin(3\theta)$ is easy to compute.

θ	3θ	$r(\theta)$
0	0	0
$\pi/12$	$\pi/4$	$1/\sqrt{2} \approx 0.71$
$2\pi/12$	$\pi/2$	1
$3\pi/12$	$3\pi/4$	$1/\sqrt{2} \approx 0.71$
$4\pi/12$	π	0

and here is a sketch exhibiting those values and another sketch of the part of the curve with $0 \leqslant \theta \leqslant \frac{\pi}{3}$.

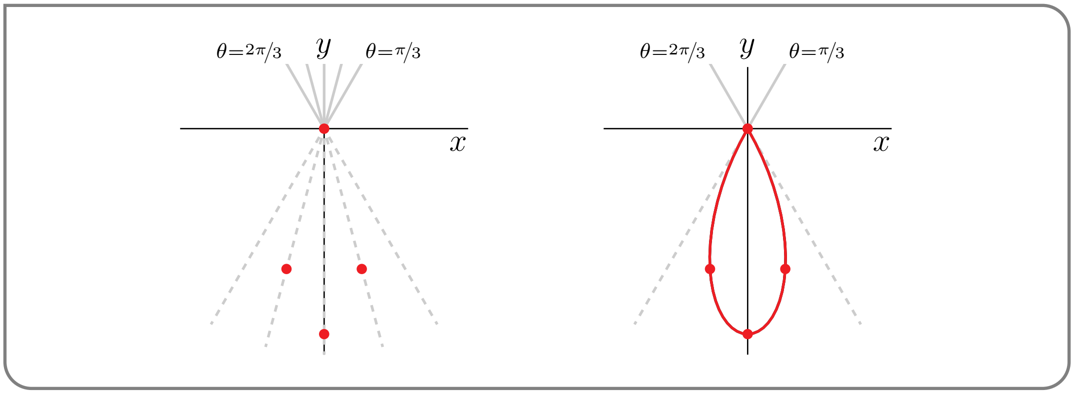


- Next consider $\frac{\pi}{3} \leqslant \theta \leqslant \frac{2\pi}{3}$, so that $\pi \leqslant 3\theta \leqslant 2\pi$. On this interval $r(\theta) = \sin(3\theta)$
  - starts with $r\left(\pi/3\right) = 0$, and then
  - decreases as θ increases until
  - $3\theta = \frac{3\pi}{2}$, i.e. $\theta = \frac{\pi}{2}$, where $r\left(\pi/2\right) = -1$, and then
  - increases as θ increases until
  - $3\theta = 2\pi$, i.e. $\theta = \frac{2\pi}{3}$, where $r\left(2\pi/3\right) = 0$, again.

  We are now encountering, for the first time, $r(\theta)$'s that are negative. The figure on the left below contains, for each of $\theta = \frac{4\pi}{12} = \frac{\pi}{3}$, $\frac{5\pi}{12}$, $\frac{6\pi}{12} = \frac{\pi}{2}$, $\frac{7\pi}{12}$ and $\frac{8\pi}{12} = \frac{2\pi}{3}$

  - the (dashed) half-line consisting of all points with that θ and $r < 0$ and
  - the dot with that θ and $r(\theta) = \sin(3\theta)$.

  The figure on the right below provides a sketch of the part of the curve $r = \sin(3\theta)$ with $\frac{\pi}{3} \leqslant \theta \leqslant \frac{2\pi}{3}$.



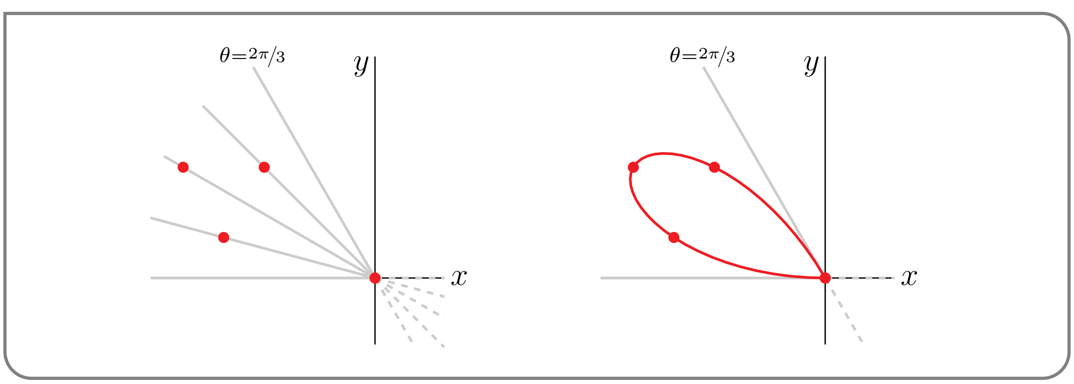
- Finally consider $\frac{2\pi}{3} \leqslant \theta \leqslant \pi$ (because $r(\theta+\pi) = \sin(3\theta+3\pi) = -\sin(3\theta) = -r(\theta)$, the part of the curve with $\pi \leqslant \theta \leqslant 2\pi$ just retraces the part with $0 \leqslant \theta \leqslant \pi$), so that $2\pi \leqslant 3\theta \leqslant 3\pi$. On this interval $r(\theta) = \sin(3\theta)$

  - starts with $r\big({}^{2\pi}/{}_{3}\big) = 0$, and then
  - increases as θ increases until
  - $3\theta = \frac{5\pi}{2}$, i.e. $\theta = \frac{10\pi}{12}$, where $r\big({}^{5\pi}/{}_{2}\big) = 1$, and then
  - decreases as θ increases until
  - $3\theta = 3\pi$, i.e. $\theta = \frac{12\pi}{12} = \pi$, where $r\big(\pi\big) = 0$, again.

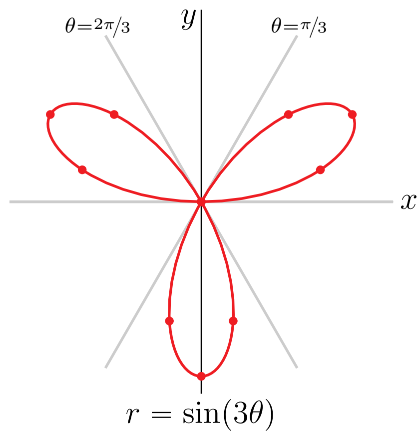
  The figure on the left below contains, for each of $\theta = \frac{8\pi}{12} = \frac{2\pi}{3}, \frac{9\pi}{12}, \frac{10\pi}{12}, \frac{11\pi}{12}$ and $\frac{12\pi}{12} = \pi$

  - the (solid) half-line consisting of all points with that θ and $r \geqslant 0$ and
  - the dot with that θ and $r(\theta) = \sin(3\theta)$.

  The figure on the right below provides a sketch of the part of the curve $r = \sin(3\theta)$ with $\frac{2\pi}{3} \leqslant \theta \leqslant \pi$.



Putting the three lobes together gives the full curve, which is called the “three petal rose”.

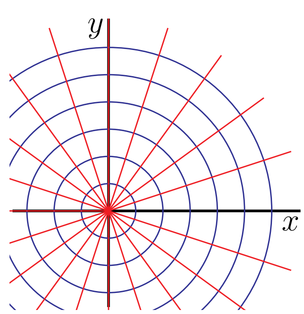


There is an infinite family of similar rose curves (also called rhodonea[17] curves).

Example 3.2.4

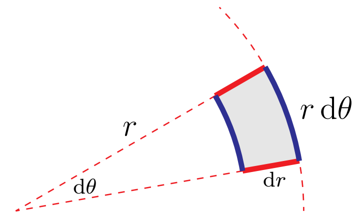
3.2.3 ▸▸ Integrals in Polar Coordinates

We now return to the problem of using polar coordinates to set up double integrals. So far, we have used Cartesian coordinates, in the sense that we have cut up our domains of integration into tiny rectangles (on which the integrand is essentially constant) by drawing in many lines of constant x and many lines of constant y. To use polar coordinates, we instead draw in both lines of constant θ and curves of constant r. This cuts the xy-plane up into approximate rectangles.



Here is an enlarged sketch of one such approximate rectangle.

17 The name rhodenea first appeared in the 1728 publication *Flores geometrici* of the Italian monk, theologian, mathematician and engineer, Guido Grandi (1671–1742).



One side has length $\mathrm{d}r$, the spacing between the curves of constant r. The other side is a portion of a circle of radius r that subtends, at the origin, an angle $\mathrm{d}\theta$, the angle between the lines of constant θ. As the circumference of the full circle is $2\pi r$ and as $\mathrm{d}\theta$ is the fraction $\frac{\mathrm{d}\theta}{2\pi}$ of a full circle[18], the other side of the approximate rectangle has length $\frac{\mathrm{d}\theta}{2\pi}2\pi r = r\mathrm{d}\theta$. So the shaded region has area approximately

Equation 3.2.5.

$$\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$$

By way of comparison, using Cartesian coordinates we had $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$.

This intuitive computation has been somewhat handwavy[19]. But using it in the usual integral setup procedure, in which we choose $\mathrm{d}r$ and $\mathrm{d}\theta$ to be constants times $\frac{1}{n}$ and then take the limit $n \to 0$, gives, in the limit, error exactly zero. A sample argument, in which we see the error going to zero in the limit $n \to \infty$, is provided in the (optional) section §3.2.4.

Example 3.2.6 (Mass)

Let $0 \leqslant a < b \leqslant 2\pi$ be constants and let $\mathcal{R}$ be the region

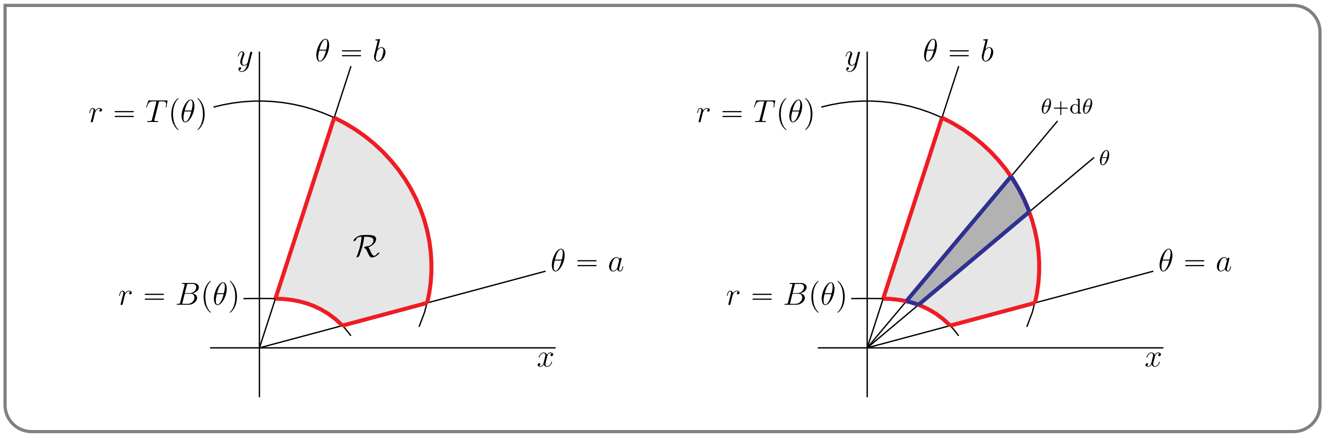
$$\mathcal{R} = \left\{ (r\cos\theta, r\sin\theta) \,\middle|\, a \leqslant \theta \leqslant b, B(\theta) \leqslant r \leqslant T(\theta) \right\}$$

where the functions $T(\theta)$ and $B(\theta)$ are continuous and obey $B(\theta) \leqslant T(\theta)$ for all $a \leqslant \theta \leqslant b$. Find the mass of $\mathcal{R}$ if it has density $f(x,y)$.

Solution. The figure on the left below is a sketch of $\mathcal{R}$. Notice that $r = T(\theta)$ is the outer curve while $r = B(\theta)$ is the inner curve. Divide $\mathcal{R}$ into wedges (as in wedges of pie[20]

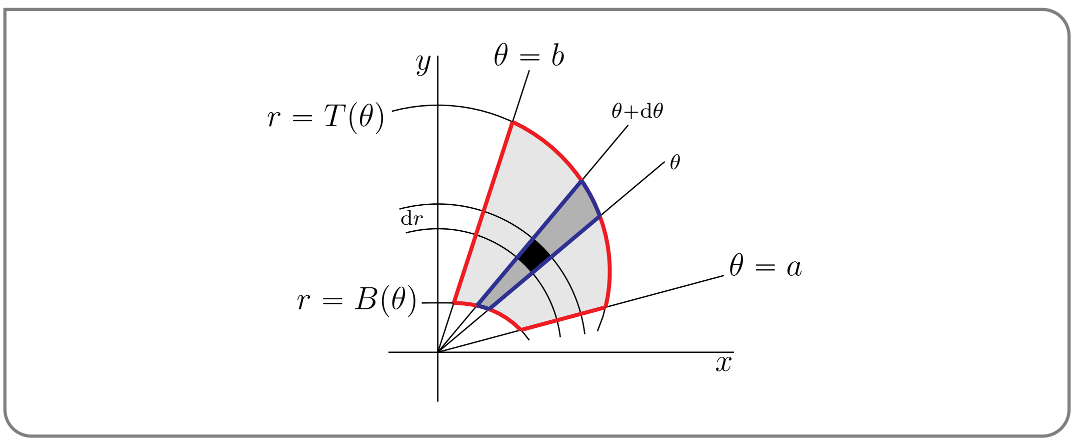
18 Recall that θ has to be measured in radians for this to be true.

19 "Handwaving" is sometimes used as a pejorative to refer to an argument that lacks substance. Here we are just using it to indicate that we have left out a bunch of technical details. In mathematics, "nose-following" is sometimes used as the polar opposite of handwaving. It refers to a very narrow, mechanical, line of reasoning.



or wedges of cheese) by drawing in many lines of constant θ, with the various values of θ differing by a tiny amount $\mathrm{d}\theta$. The figure on the right above shows one such wedge, outlined in blue.

Concentrate on any one wedge. Subdivide the wedge further into approximate rectangles by drawing in many circles of constant r, with the various values of r differing by a tiny amount $\mathrm{d}r$. The figure below shows one such approximate rectangle, in black.



Now concentrate on one such rectangle. Let's say that it contains the point with polar coordinates r and θ. As we saw in (3.2.5) above,

- the area of that rectangle is essentially $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$.
- As the mass density on the rectangle is essentially $f\big(r\cos\theta\,,\,r\sin\theta\big)$, the mass of the rectangle is essentially $f\big(r\cos\theta\,,\,r\sin\theta\big)\,r\,\mathrm{d}r\,\mathrm{d}\theta$.
- To get the mass of any one wedge, say the wedge whose polar angle runs from θ to $\theta + \mathrm{d}\theta$, we just add up the masses of the approximate rectangles in that wedge, by integrating r from its smallest value on the wedge, namely $B(\theta)$, to its largest value on the wedge, namely $T(\theta)$. The mass of the wedge is thus

$$\mathrm{d}\theta \int_{B(\theta)}^{T(\theta)} \mathrm{d}r\, r\, f\big(r\cos\theta\,,\,r\sin\theta\big)$$

20 There is a pie/pi/pye pun in there somewhere.

- Finally, to get the mass of $\mathcal{R}$, we just add up the masses of all of the different wedges, by integrating θ from its smallest value on $\mathcal{R}$, namely a, to its largest value on $\mathcal{R}$, namely b.

In conclusion,

$$\text{Mass}(\mathcal{R}) = \int_a^b \mathrm{d}\theta \int_{B(\theta)}^{T(\theta)} \mathrm{d}r\, r\, f\big(r\cos\theta\,,\, r\sin\theta\big)$$

We have repeatedly used the word "essentially" above to avoid getting into the nitty-gritty details required to prove things rigorously. The mathematically correct proof of (3.2.7) follows the same intuition, but requires some more careful error bounds, as in the optional §3.2.4 below.

Example 3.2.6

In the last example, we derived the important formula that the mass of the region

$$\mathcal{R} = \big\{\ (r\cos\theta, r\sin\theta)\ \big|\ a \leqslant \theta \leqslant b, B(\theta) \leqslant r \leqslant T(\theta)\ \big\}$$

with mass density $f(x,y)$ is

Equation 3.2.7.

$$\text{Mass}(\mathcal{R}) = \int_a^b \mathrm{d}\theta \int_{B(\theta)}^{T(\theta)} \mathrm{d}r\, r\, f\big(r\cos\theta\,,\, r\sin\theta\big)$$

We can immediately adapt that example to calculate areas and derive the formula that the area of the region

$$\mathcal{R} = \big\{\ (r\cos\theta, r\sin\theta)\ \big|\ a \leqslant \theta \leqslant b, 0 \leqslant r \leqslant R(\theta)\ \big\}$$

is

Equation 3.2.8.

$$\text{Area}(\mathcal{R}) = \frac{1}{2}\int_a^b R(\theta)^2\, \mathrm{d}\theta$$

We just have to set the density to 1. We do so in the next example.

Example 3.2.9 (Polar Area)

Let $0 \leqslant a < b \leqslant 2\pi$ be constants. Find the area of the region

$$\mathcal{R} = \big\{\ (r\cos\theta, r\sin\theta)\ \big|\ a \leqslant \theta \leqslant b, 0 \leqslant r \leqslant R(\theta)\ \big\}$$

where the function $R(\theta) \geqslant 0$ is continuous.

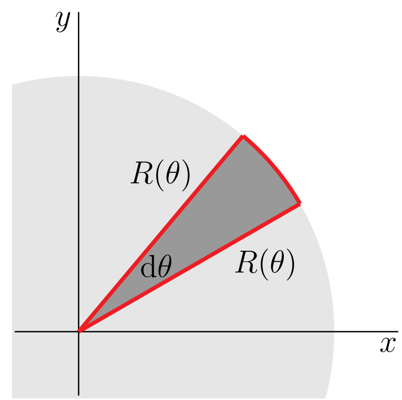
Solution. To get the area of $\mathcal{R}$ we just need to assign it a density one and find the resulting mass. So, by (3.2.7), with $f(x,y) = 1$, $B(\theta) = 0$ and $T(\theta) = R(\theta)$,

$$\text{Area}(\mathcal{R}) = \int_a^b \mathrm{d}\theta \int_0^{R(\theta)} \mathrm{d}r\, r$$

In this case we can easily do the inner r integral, giving

$$\text{Area}(\mathcal{R}) = \frac{1}{2}\int_a^b R(\theta)^2\,\mathrm{d}\theta$$

The expression $\frac{1}{2}R(\theta)^2\,\mathrm{d}\theta$ in (3.2.8) has a geometric interpretation. It is just the area of a wedge of a circular disk of radius $R(\theta)$ (with $R(\theta)$ treated as a constant) that subtends the angle $\mathrm{d}\theta$. To see this, note that area of the wedge is the fraction $\frac{\mathrm{d}\theta}{2\pi}$ of the area of the entire



disk, which is $\pi R(\theta)^2$. So (3.2.8) just says that the area of $\mathcal{R}$ can be computed by cutting $\mathcal{R}$ up into tiny wedges and adding up the areas of all of the tiny wedges.

Example 3.2.9

Example 3.2.10 (Polar Area)

Find the area of one petal of the three petal rose $r = \sin(3\theta)$.

Solution. Looking at the last figure in Example 3.2.4, we see that we want the area of

$$\mathcal{R} = \left\{ (r\cos\theta, r\sin\theta) \,\middle|\, 0 \leqslant \theta \leqslant \pi/3,\ 0 \leqslant r \leqslant \sin(3\theta) \right\}$$

So, by (3.2.8) with $a = 0$, $b = \pi/3$, and $R(\theta) = \sin(3\theta)$,

$$\begin{aligned}
\text{area}(\mathcal{R}) &= \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,\mathrm{d}\theta \\
&= \frac{1}{4}\int_0^{\pi/3} \big(1-\cos(6\theta)\big)\,\mathrm{d}\theta \\
&= \frac{1}{4}\left[\theta - \frac{1}{6}\sin(6\theta)\right]_0^{\pi/3} \\
&= \frac{\pi}{12}
\end{aligned}$$

In the first step we used the double angle formula $\cos(2\phi) = 1 - 2\sin^2(\phi)$. Unsurprisingly, trig identities show up a lot when polar coordinates are used.

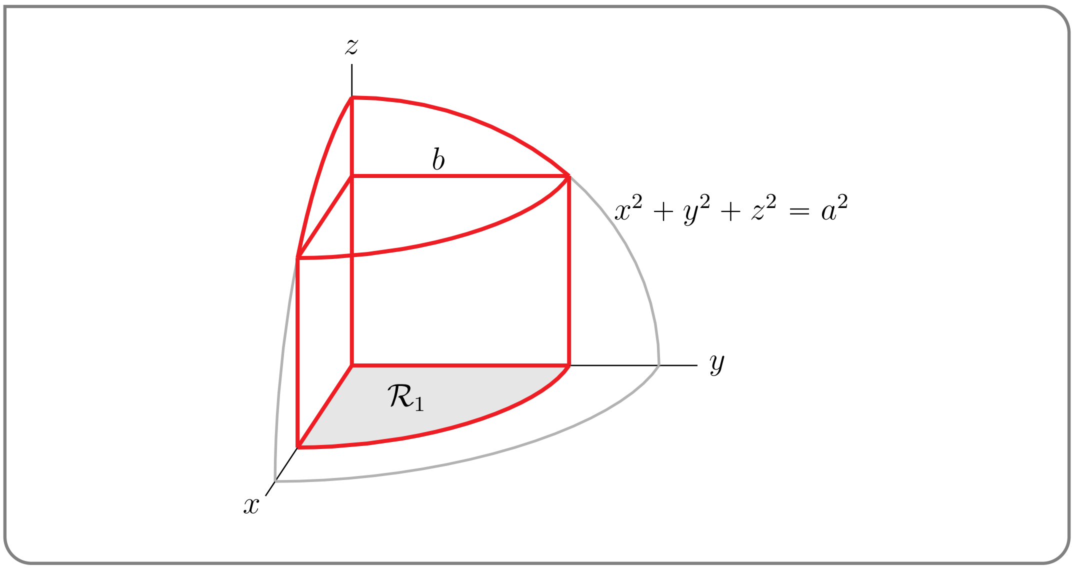
Example 3.2.10

Example 3.2.11 (Volumes Using Polar Coordinates)

A cylindrical hole of radius b is drilled symmetrically (i.e. along a diameter) through a metal sphere of radius $a \geqslant b$. Find the volume of metal removed.

Solution. Let's use a coordinate system with the sphere centred on $(0, 0, 0)$ and with the centre of the drill hole following the z-axis. In particular, the sphere is $x^2 + y^2 + z^2 \leqslant a^2$.

Here is a sketch of the part of the sphere in the first octant. The hole in the sphere made by the drill is outlined in red. By symmetry the total amount of metal removed will be eight times the amount from the first octant. That is, the volume of metal removed will



be eight times the volume of the solid

$$\mathcal{V}_1 = \left\{ (x, y, z) \mid (x, y) \in \mathcal{R}_1,\ 0 \leqslant z \leqslant \sqrt{a^2 - x^2 - y^2} \right\}$$

where the base region

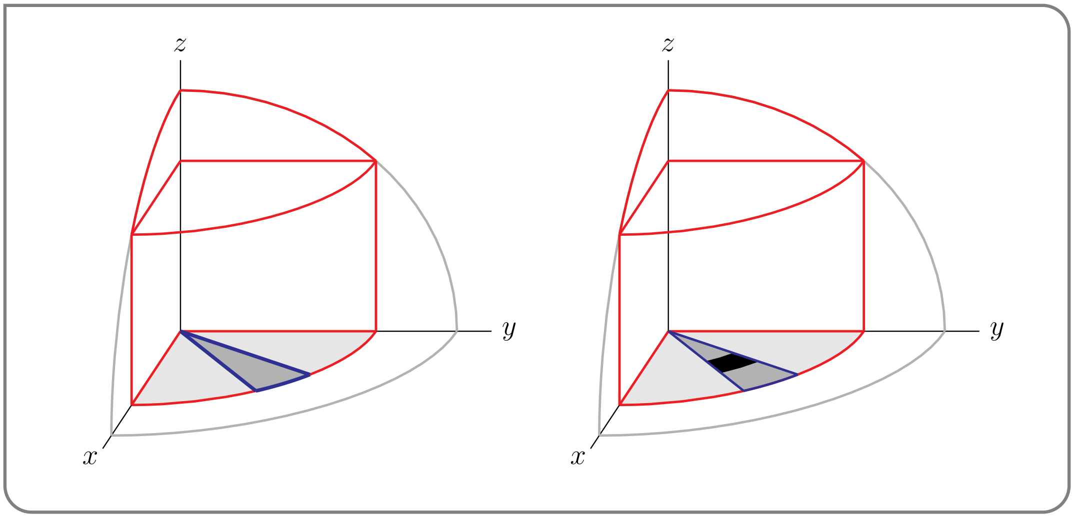
$$\mathcal{R}_1 = \left\{ (x, y) \mid x^2 + y^2 \leqslant b^2,\ x \geqslant 0,\ y \geqslant 0 \right\}$$

In polar coordinates

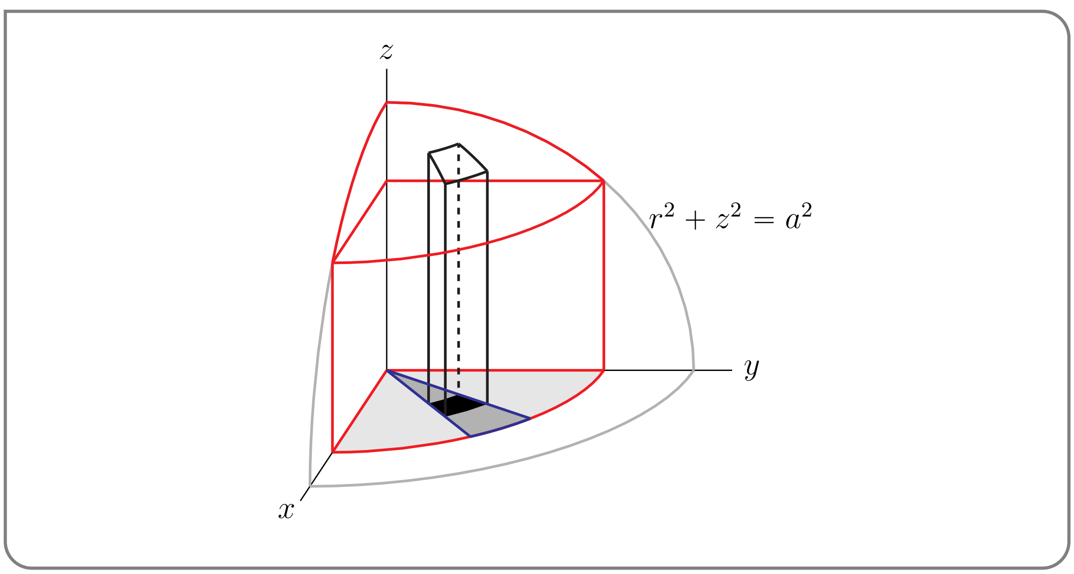
$$\mathcal{V}_1 = \left\{ (r\cos\theta, r\sin\theta, z) \mid (r\cos\theta, r\sin\theta) \in \mathcal{R}_1,\ 0 \leqslant z \leqslant \sqrt{a^2 - r^2} \right\}$$
$$\mathcal{R}_1 = \left\{ (r\cos\theta, r\sin\theta) \mid 0 \leqslant r \leqslant b,\ 0 \leqslant \theta \leqslant \pi/2 \right\}$$

We follow our standard divide and sum up strategy. We will cut the base region $\mathcal{R}_1$ into small pieces and sum up the volumes that lie above each small piece.

- Divide $\mathcal{R}_1$ into wedges by drawing in many lines of constant θ, with the various values of θ differing by a tiny amount $d\theta$. The figure on the left below shows one such wedge, outlined in blue.



- Concentrate on any one wedge. Subdivide the wedge further into approximate rectangles by drawing in many circles of constant r, with the various values of r differing by a tiny amount $\mathrm{d}r$. The figure on the right above shows one such approximate rectangle, in black.
- Concentrate on one such rectangle. Let's say that it contains the point with polar coordinates r and θ. As we saw in (3.2.5) above,
  - the area of that rectangle is essentially $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$.
  - The part of $\mathcal{V}_1$ that is above that rectangle is like an office tower whose height is essentially $\sqrt{a^2 - r^2}$, and whose base has area $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. It is outlined in black in the figure below. So the volume of the part of $\mathcal{V}_1$ that is above the rectangle is essentially $\sqrt{a^2 - r^2}\, r\,\mathrm{d}r\,\mathrm{d}\theta$.

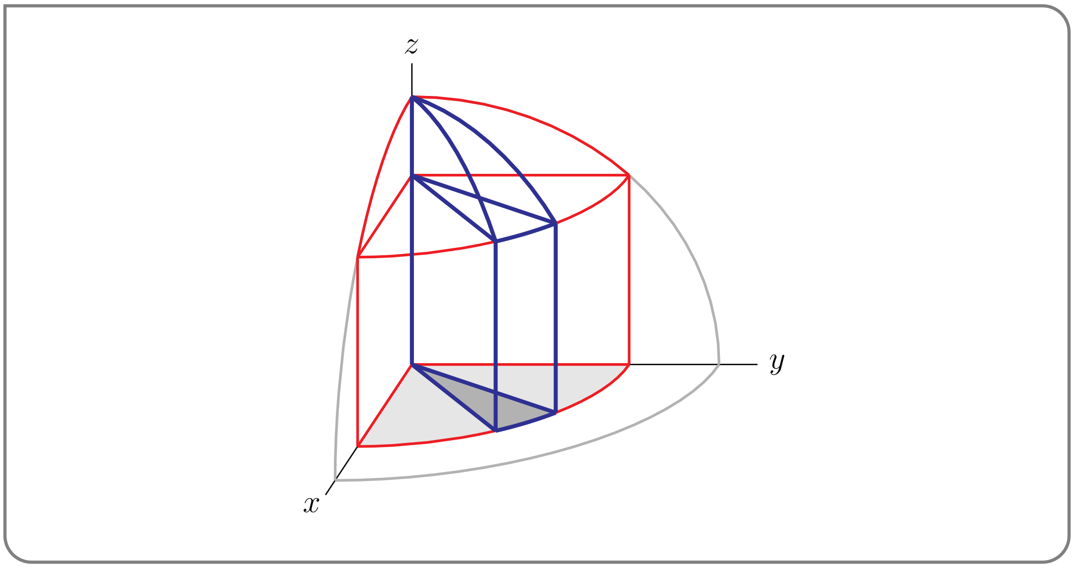


- To get the volume of the part of $\mathcal{V}_1$ above any one wedge (outlined in blue in the figure below), say the wedge whose polar angle runs from θ to $\theta + \mathrm{d}\theta$, we just add up the volumes above the approximate rectangles in that wedge, by integrating r

from its smallest value on the wedge, namely 0, to its largest value on the wedge, namely b. The volume above the wedge is thus

$$\begin{aligned} \mathrm{d}\theta \int_0^b \mathrm{d}r\, r\sqrt{a^2-r^2} &= \mathrm{d}\theta \int_{a^2}^{a^2-b^2} \frac{\mathrm{d}u}{-2}\sqrt{u} \qquad \text{where } u=a^2-r^2,\ \mathrm{d}u=-2r\,\mathrm{d}r \\ &= \mathrm{d}\theta \left[\frac{u^{3/2}}{-3}\right]_{a^2}^{a^2-b^2} \\ &= \frac{1}{3}\mathrm{d}\theta\left[a^3-\left(a^2-b^2\right)^{3/2}\right] \end{aligned}$$

Notice that this quantity is independent of θ. If you think about this for a moment, you can see that this is a consequence of the fact that our solid is invariant under rotations about the z-axis.



- Finally, to get the volume of $\mathcal{V}_1$, we just add up the volumes over all of the different wedges, by integrating θ from its smallest value on $\mathcal{R}_1$, namely 0, to its largest value on $\mathcal{R}_1$, namely $\pi/2$.

$$\begin{aligned} \text{Volume}(\mathcal{V}_1) &= \frac{1}{3}\int_0^{\pi/2} \mathrm{d}\theta\left[a^3-\left(a^2-b^2\right)^{3/2}\right] \\ &= \frac{\pi}{6}\left[a^3-\left(a^2-b^2\right)^{3/2}\right] \end{aligned}$$

- In conclusion, the total volume of metal removed is

$$\begin{aligned} \text{Volume}(\mathcal{V}) &= 8\,\text{Volume}(\mathcal{V}_1) \\ &= \frac{4\pi}{3}\left[a^3-\left(a^2-b^2\right)^{3/2}\right] \end{aligned}$$

Note that we can easily apply a couple of sanity checks to our answer.

- If the radius of the drill bit $b = 0$, no metal is removed at all. So the total volume removed should be zero. Our answer does indeed give 0 in this case.
- If the radius of the drill bit $b = a$, the radius of the sphere, then the entire sphere disappears. So the total volume removed should be the volume of a sphere of radius a. Our answer does indeed give $\frac{4}{3}\pi a^3$ in this case.
- If the radius, a, of the sphere and the radius, b, of the drill bit are measured in units of meters, then the remaining volume $\frac{4\pi}{3}\left[a^3 - \left(a^2 - b^2\right)^{3/2}\right]$, has units meters^3, as it should.

Example 3.2.11

The previous two problems were given to us (or nearly given to us) in polar coordinates. We'll now get a little practice converting integrals into polar coordinates, and recognising when it is helpful to do so.

Example 3.2.12 (Changing to Polar Coordinates)

Convert the integral $\int_0^1 \int_0^x y\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$ to polar coordinates and evaluate the result.

Solution. First recall that in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$ so that the integrand (and $\mathrm{d}A$)

$$y\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x = (r\sin\theta)\,r\,r\,\mathrm{d}r\,\mathrm{d}\theta = r^3\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta$$

is very simple. So whether or not this integral will be easy to evaluate using polar coordinates will be largely determined by the domain of integration.

So our main task is to sketch the domain of integration. To prepare for the sketch, note that in the integral

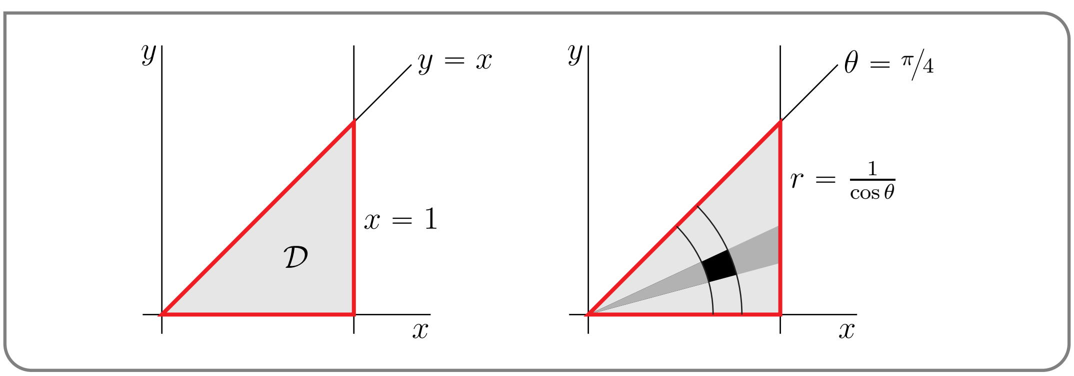
$$\int_0^1 \int_0^x y\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \mathrm{d}x\left[\int_0^x \mathrm{d}y\; y\sqrt{x^2+y^2}\right]$$

- the variable x runs from 0 to 1 and
- for each fixed $0 \leqslant x \leqslant 1$, y runs from 0 to x.

So the domain of integration is

$$\mathcal{D} = \left\{\ (x,y)\ \middle|\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x\ \right\}$$

which is sketched in the figure on the left below. It is a right angled triangle.



Next we express the domain of integration in terms of polar coordinates, by expressing the equations of each of the boundary lines in terms of polar coordinates.

- The x-axis, i.e. $y = r\sin\theta = 0$, is $\theta = 0$.
- The line $x = 1$ is $r\cos\theta = 1$ or $r = \frac{1}{\cos\theta}$.
- Finally, (in the first quadrant) the line

$$y = x \iff r\sin\theta = r\cos\theta \iff \tan\theta = \frac{\sin\theta}{\cos\theta} = 1 \iff \theta = \frac{\pi}{4}$$

So, in polar coordinates, we can write the domain of integration as

$$\mathcal{R} = \left\{(r,\theta)\ \middle|\ 0 \leqslant \theta \leqslant \frac{\pi}{4},\ 0 \leqslant r \leqslant \frac{1}{\cos\theta}\right\}$$

We can now slice up $\mathcal{R}$ using polar coordinates.

- Divide $\mathcal{R}$ into wedges by drawing in many lines of constant θ, with the various values of θ differing by a tiny amount $\mathrm{d}\theta$. The figure on the right above shows one such wedge.
  - The first wedge has $\theta = 0$.
  - The last wedge has $\theta = \frac{\pi}{4}$.
- Concentrate on any one wedge. Subdivide the wedge further into approximate rectangles by drawing in many circles of constant r, with the various values of r differing by a tiny amount $\mathrm{d}r$. The figure on the right above shows one such approximate rectangle, in black.
  - The rectangle that contains the point with polar coordinates r and θ has area (essentially) $r\,\mathrm{d}r\,\mathrm{d}\theta$.
  - The first rectangle has $r = 0$.
  - The last rectangle has $r = \frac{1}{\cos\theta}$.

So our integral is

$$\int_0^1 \int_0^x y\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\pi/4} \mathrm{d}\theta \int_0^{\frac{1}{\cos\theta}} \mathrm{d}r\ r\,\overbrace{(r^2\sin\theta)}^{y\sqrt{x^2+y^2}}$$

Because the r-integral treats θ as a constant, we can pull the $\sin\theta$ out of the inner r-integral.

$$\begin{aligned}\int_0^1 \int_0^x y\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{\pi/4} \mathrm{d}\theta\ \sin\theta \int_0^{\frac{1}{\cos\theta}} \mathrm{d}r\ r^3 \\ &= \frac{1}{4}\int_0^{\pi/4} \mathrm{d}\theta\ \sin\theta \frac{1}{\cos^4\theta}\end{aligned}$$

Make the substitution

$$u = \cos\theta,\ \mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$$

When $\theta = 0$, $u = \cos\theta = 1$ and when $\theta = \frac{\pi}{4}$, $u = \cos\theta = \frac{1}{\sqrt{2}}$. So

$$\begin{aligned}\int_0^1 \int_0^x y\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x &= \frac{1}{4}\int_1^{1/\sqrt{2}} (-\mathrm{d}u)\frac{1}{u^4} \\ &= -\frac{1}{4}\left[\frac{u^{-3}}{-3}\right]_1^{1/\sqrt{2}} = \frac{1}{12}\left[2\sqrt{2}-1\right]\end{aligned}$$

Example 3.2.12

Example 3.2.13 (Changing to Polar Coordinates)

Evaluate $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x$.

Solution. This is actually a trick question. In fact it is a famous trick question[21]. The integrand e^{-x^2} does not have an antiderivative that can be expressed in terms of elementary functions[22]. So we cannot evaluate this integral using the usual Calculus II methods. However we can evaluate it's square

$$\left[\int_0^\infty e^{-x^2}\,\mathrm{d}x\right]^2 = \int_0^\infty e^{-x^2}\,\mathrm{d}x \int_0^\infty e^{-y^2}\,\mathrm{d}y = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\; e^{-x^2-y^2}$$

precisely because this double integral can be easily evaluated just by changing to polar coordinates! The domain of integration is the first quadrant $\left\{\ (x,y)\ \middle|\ x \geqslant 0,\ y \geqslant 0\ \right\}$. In polar coordinates, $\mathrm{d}x\mathrm{d}y = r\,\mathrm{d}r\mathrm{d}\theta$ and the first quadrant is

$$\left\{\ (r\cos\theta\,,\ r\sin\theta)\ \middle|\ r \geqslant 0,\ 0 \leqslant \theta \leqslant \pi/2\ \right\}$$

So

$$\left[\int_0^\infty e^{-x^2}\,\mathrm{d}x\right]^2 = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\; e^{-x^2-y^2} = \int_0^{\pi/2} \mathrm{d}\theta \int_0^\infty \mathrm{d}r\; r\, e^{-r^2}$$

As r runs all the way to $+\infty$, this is an improper integral, so we should be a little bit

21 The solution is attributed to the French Mathematician Siméon Denis Poisson (1781–840) and was published in the textbook *Cours d'Analyse de l'école polytechnique* by Jacob Karl Franz Sturm (1803–1855).

22 On the other hand it is the core of the function $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,\mathrm{d}t$, which gives Gaussian (i.e. bell curve) probabilities. "erf" stands for "error function".

careful.

$$\begin{aligned}
\left[\int_0^\infty e^{-x^2}\,\mathrm{d}x\right]^2 &= \lim_{R\to\infty}\int_0^{\pi/2}\mathrm{d}\theta\int_0^R \mathrm{d}r\, r\, e^{-r^2}\\
&= \lim_{R\to\infty}\int_0^{\pi/2}\mathrm{d}\theta\int_0^{R^2}\frac{\mathrm{d}u}{2}\,e^{-u} \qquad \text{where } u=r^2,\ \mathrm{d}u=2r\,\mathrm{d}r\\
&= \lim_{R\to\infty}\int_0^{\pi/2}\mathrm{d}\theta\,\left[-\frac{e^{-u}}{2}\right]_0^{R^2}\\
&= \lim_{R\to\infty}\frac{\pi}{2}\left[\frac{1}{2}-\frac{e^{-R^2}}{2}\right]\\
&= \frac{\pi}{4}
\end{aligned}$$

and so we get the famous result

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

Example 3.2.13

Example 3.2.14

Find the area of the region that is inside the circle $r = 4\cos\theta$ and to the left of the line $x = 1$.

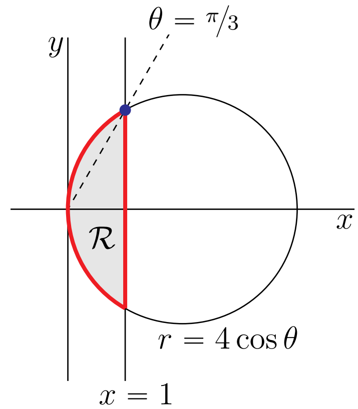
Solution. First, let's check that $r = 4\cos\theta$ really is a circle and figure out what circle it is. To do so, we'll convert the equation $r = 4\cos\theta$ into Cartesian coordinates. Multiplying both sides by r gives

$$r^2 = 4r\cos\theta \iff x^2+y^2 = 4x \iff (x-2)^2+y^2 = 4$$

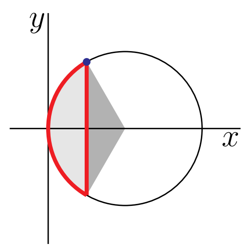
So $r = 4\cos\theta$ is the circle of radius 2 centred on $(2,0)$. We'll also need the intersection point(s) of $x = r\cos\theta = 1$ and $r = 4\cos\theta$. At such an intersection point

$$\begin{aligned}
r\cos\theta = 1,\ r = 4\cos\theta &\implies \frac{1}{\cos\theta} = 4\cos\theta\\
&\implies \cos^2\theta = \frac{1}{4}\\
&\implies \cos\theta = \frac{1}{2} \qquad \text{since } r\cos\theta = 1 > 0\\
&\implies \theta = \pm\frac{\pi}{3}
\end{aligned}$$

Here is a sketch of the region of interest, which we'll call $\mathcal{R}$.

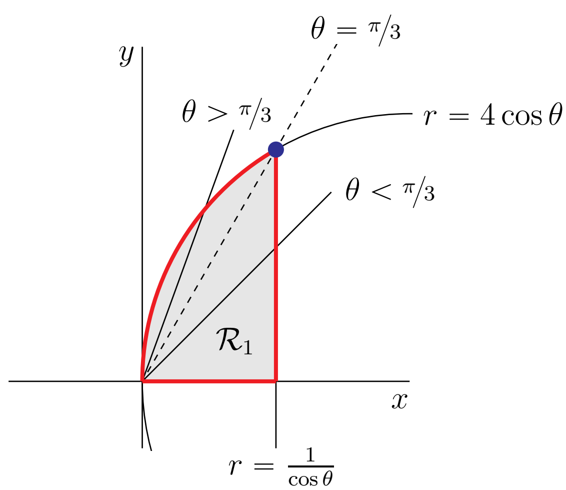


We could figure out the area of $\mathcal{R}$ by using some high school geometry, because $\mathcal{R}$ is a circular wedge with a triangle removed. (See Example 3.2.15, below.) Instead, we'll treat



its computation as an exercise in integration using polar coordinates.

As $\mathcal{R}$ is symmetric about the x-axis, the area of $\mathcal{R}$ is twice the area of the part that is above the x-axis. We'll denote by $\mathcal{R}_1$ the upper half of $\mathcal{R}$. Note that we can write the equation $x = 1$ in polar coordinates as $r = \frac{1}{\cos\theta}$. Here is a sketch of $\mathcal{R}_1$.



Observe that, on $\mathcal{R}_1$, for any fixed θ between 0 and $\pi/2$,

- if $\theta < \pi/3$, then r runs from 0 to $\frac{1}{\cos\theta}$, while

- if $\theta > \pi/3$, then r runs from 0 to $4\cos\theta$.

This naturally leads us to split the domain of integration at $\theta = \frac{\pi}{3}$:

$$\text{Area}(\mathcal{R}_1) = \int_0^{\pi/3} \mathrm{d}\theta \int_0^{1/\cos\theta} \mathrm{d}r\ r + \int_{\pi/3}^{\pi/2} \mathrm{d}\theta \int_0^{4\cos\theta} \mathrm{d}r\ r$$

As $\int r\,\mathrm{d}r = \frac{r^2}{2} + C$,

$$\begin{aligned}\text{Area}(\mathcal{R}_1) &= \int_0^{\pi/3} \mathrm{d}\theta\ \frac{\sec^2\theta}{2} + \int_{\pi/3}^{\pi/2} \mathrm{d}\theta\ 8\cos^2\theta \\ &= \frac{1}{2}\tan\theta\Big|_0^{\pi/3} + 4\int_{\pi/3}^{\pi/2} \mathrm{d}\theta\ \big[1+\cos(2\theta)\big] \\ &= \frac{\sqrt{3}}{2} + 4\left[\theta + \frac{\sin(2\theta)}{2}\right]_{\pi/3}^{\pi/2} \\ &= \frac{\sqrt{3}}{2} + 4\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] \\ &= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\end{aligned}$$

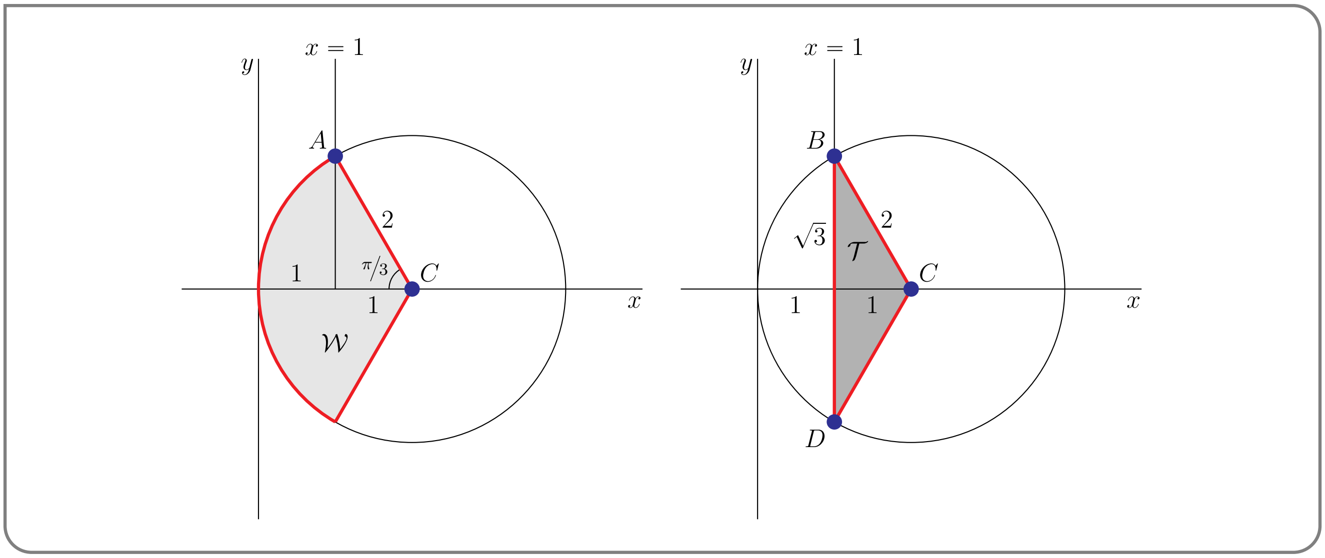
and

$$\text{Area}(\mathcal{R}) = 2\text{Area}(\mathcal{R}_1) = \frac{4\pi}{3} - \sqrt{3}$$

Example 3.2.14

Example 3.2.15 (Optional — Example 3.2.14 by high school geometry)

We'll now again compute the area of the region $\mathcal{R}$ that is inside the circle $r = 4\cos\theta$ and to the left of the line $x = 1$. That was the region of interest in Example 3.2.14. This time we'll just use some geometry. Think of $\mathcal{R}$ as being the wedge $\mathcal{W}$, of the figure on the left below, with the triangle $\mathcal{T}$, of the figure on the right below, removed.



- First we'll get the area of $\mathcal{W}$. The cosine of the angle between the x axis and the radius vector from C to A is $\frac{1}{2}$. So that angle is $\frac{\pi}{3}$ and $\mathcal{W}$ subtends an angle of $\frac{2\pi}{3}$. The entire circle has area $\pi 2^2$, so that $\mathcal{W}$, which is the fraction $\frac{2\pi/3}{2\pi} = \frac{1}{3}$ of the full circle, has area $\frac{4\pi}{3}$.
- Now we'll get the area of the triangle $\mathcal{T}$. Think of $\mathcal{T}$ as having base BD. Then the length of the base of $\mathcal{T}$ is $2\sqrt{3}$ and the height of $\mathcal{T}$ is 1. So $\mathcal{T}$ has area $\frac{1}{2}(2\sqrt{3})(1) = \sqrt{3}$.

All together

$$\text{Area}(\mathcal{R}) = \text{Area}(\mathcal{W}) - \text{Area}(\mathcal{T}) = \frac{4\pi}{3} - \sqrt{3}$$

Example 3.2.15

We used some hand waving in deriving the area formula (3.2.8): the word "essentially" appeared quite a few times. Here is how do that derivation more rigorously.

3.2.4 ▸▸ Optional— Error Control for the Polar Area Formula

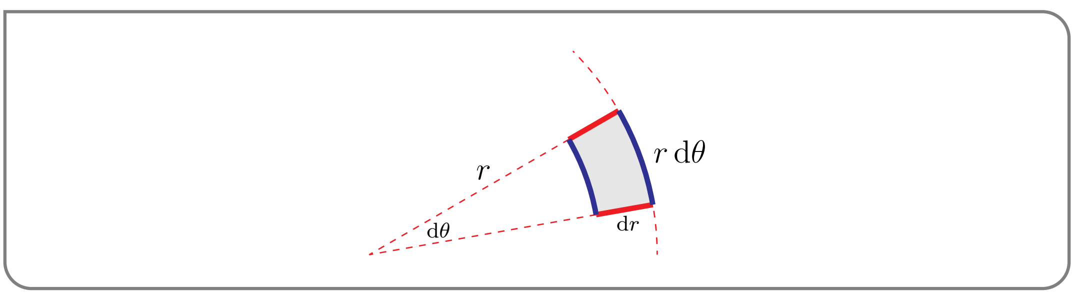
Let $0 \leqslant a < b \leqslant 2\pi$. In Examples 3.2.6 and 3.2.9 we derived the formula

$$A = \frac{1}{2}\int_a^b R(\theta)^2\, d\theta$$

for the area of the region

$$\mathcal{R} = \left\{ \left(r\cos\theta, r\sin\theta\right) \bigm| a \leqslant \theta \leqslant b,\ 0 \leqslant r \leqslant R(\theta) \right\}$$

In the course of that derivation we approximated the area of the shaded region in



by $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$.

We will now justify that approximation, under the assumption that

$$0 \leqslant R(\theta) \leqslant M \qquad |R'(\theta)| \leqslant L$$

for all $a \leqslant \theta \leqslant b$. That is, $R(\theta)$ is bounded and its derivative exists and is bounded too.

Divide the interval $a \leqslant \theta \leqslant b$ into n equal subintervals, each of length $\Delta\theta = \frac{b-a}{n}$. Let θ_i^* be the midpoint of the i^{th} interval. On the i^{th} interval, θ runs from $\theta_i^* - \frac{1}{2}\Delta\theta$ to $\theta_i^* + \frac{1}{2}\Delta\theta$.

By the mean value theorem

$$R(\theta) - R(\theta_i^*) = R'(c)(\theta - \theta_i^*)$$

for some c between θ and θ_i^*. Because $|R'(\theta)| \leqslant L$

$$\big|R(\theta) - R(\theta_i^*)\big| \leqslant L\big|\theta - \theta_i^*\big| \tag{$*$}$$

This tells us that the difference between $R(\theta)$ and $R(\theta_i^*)$ can't be too big compared to $|\theta - \theta_i^*|$.

On the i^{th} interval, the radius $r = R(\theta)$ runs over all values of $R(\theta)$ with θ satisfying $\big|\theta - \theta_i^*\big| \leqslant \frac{1}{2}\Delta\theta$. By $(*)$, all of these values of $R(\theta)$ lie between $r_i = R(\theta_i^*) - \frac{1}{2}L\Delta\theta$ and $R_i = R(\theta_i^*) + \frac{1}{2}L\Delta\theta$. Consequently the part of $\mathcal{R}$ having θ in the i^{th} subinterval, namely,

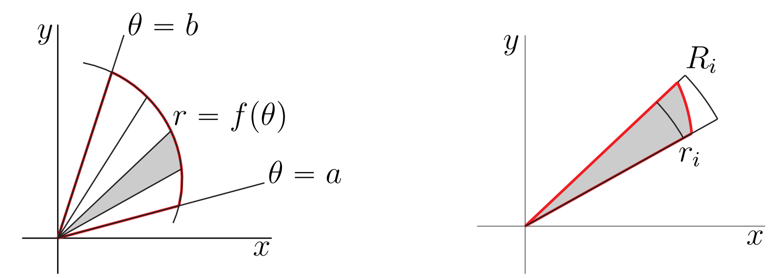
$$\mathcal{R}_i = \big\{\ \big(r\cos\theta, r\sin\theta\big)\ \big|\ \theta_i^* - \tfrac{1}{2}\Delta\theta \leqslant \theta \leqslant \theta_i^* + \tfrac{1}{2}\Delta\theta,\ 0 \leqslant r \leqslant R(\theta)\ \big\}$$

must contain all of the circular sector

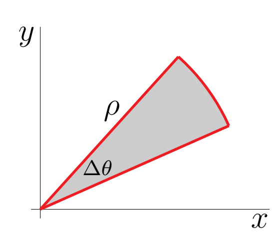
$$\big\{\ \big(r\cos\theta, r\sin\theta\big)\ \big|\ \theta_i^* - \tfrac{1}{2}\Delta\theta \leqslant \theta \leqslant \theta_i^* + \tfrac{1}{2}\Delta\theta,\ 0 \leqslant r \leqslant r_i\ \big\}$$

and must be completely contained inside the circular sector

$$\big\{\ \big(r\cos\theta, r\sin\theta\big)\ \big|\ \theta_i^* - \tfrac{1}{2}\Delta\theta \leqslant \theta \leqslant \theta_i^* + \tfrac{1}{2}\Delta\theta,\ 0 \leqslant r \leqslant R_i\ \big\}$$



That is, we have found one circular sector that is bigger than the one we are approximating, and one circular sector that is smaller. The area of a circular disk of radius ρ is $\pi\rho^2$. A circular sector of radius ρ that subtends an angle $\Delta\theta$ is the fraction $\frac{\Delta\theta}{2\pi}$ of the full disk and so has the area $\frac{\Delta\theta}{2\pi}\pi\rho^2 = \frac{\Delta\theta}{2}\rho^2$.



So the area of $\mathcal{R}_i$ must lie between

$$\frac{1}{2}\Delta\theta\, r_i^2 = \frac{1}{2}\Delta\theta\left[R(\theta_i^*) - \frac{1}{2}L\Delta\theta\right]^2 \quad \text{and} \quad \frac{1}{2}\Delta\theta\, R_i^2 = \frac{1}{2}\Delta\theta\left[R(\theta_i^*) + \frac{1}{2}L\Delta\theta\right]^2$$

Observe that

$$\left[R(\theta_i^*) \pm \frac{1}{2}L\Delta\theta\right]^2 = R(\theta_i^*)^2 \pm LR(\theta_i^*)\Delta\theta + \frac{1}{4}L^2\Delta\theta^2$$

implies that, since $0 \leqslant R(\theta) \leqslant M$,

$$R(\theta_i^*)^2 - LM\Delta\theta + \frac{1}{4}L^2\Delta\theta^2 \leqslant \left[R(\theta_i^*) \pm \frac{1}{2}L\Delta\theta\right]^2 \leqslant R(\theta_i^*)^2 + LM\Delta\theta + \frac{1}{4}L^2\Delta\theta^2$$

Hence (multiplying by $\frac{\Delta\theta}{2}$ to turn them into areas)

$$\frac{1}{2}R(\theta_i^*)^2\Delta\theta - \frac{1}{2}LM\Delta\theta^2 + \frac{1}{8}L^2\Delta\theta^3 \leqslant \text{Area}(\mathcal{R}_i) \leqslant \frac{1}{2}R(\theta_i^*)^2\Delta\theta + \frac{1}{2}LM\Delta\theta^2 + \frac{1}{8}L^2\Delta\theta^3$$

and the total area A obeys the bounds

$$\sum_{i=1}^{n}\left[\frac{1}{2}R(\theta_i^*)^2\Delta\theta - \frac{1}{2}LM\Delta\theta^2 + \frac{1}{8}L^2\Delta\theta^3\right] \leqslant A \leqslant \sum_{i=1}^{n}\left[\frac{1}{2}R(\theta_i^*)^2\Delta\theta + \frac{1}{2}LM\Delta\theta^2 + \frac{1}{8}L^2\Delta\theta^3\right]$$
$$\frac{1}{2}\sum_{i=1}^{n}R(\theta_i^*)^2\Delta\theta - \frac{1}{2}nLM\Delta\theta^2 + \frac{1}{8}nL^2\Delta\theta^3 \leqslant A \leqslant \sum_{i=1}^{n}\frac{1}{2}R(\theta_i^*)^2\Delta\theta + \frac{1}{2}nLM\Delta\theta^2 + \frac{1}{8}nL^2\Delta\theta^3$$

Since $\Delta\theta = \frac{b-a}{n}$,

$$\frac{1}{2}\sum_{i=1}^{n}R(\theta_i^*)^2\Delta\theta - \frac{LM}{2}\frac{(b-a)^2}{n} + \frac{L^2}{8}\frac{(b-a)^3}{n^2} \leqslant A \leqslant \frac{1}{2}\sum_{i=1}^{n}R(\theta_i^*)^2\Delta\theta + \frac{LM}{2}\frac{(b-a)^2}{n} + \frac{L^2}{8}\frac{(b-a)^3}{n^2}$$

Now take the limit as $n \to \infty$. Since

$$\begin{aligned}
\lim_{n\to\infty}&\left[\frac{1}{2}\sum_{i=1}^{n}R(\theta_i^*)^2\Delta\theta \pm \frac{LM}{2}\frac{(b-a)^2}{n} + \frac{L^2}{8}\frac{(b-a)^3}{n^2}\right] \\
&= \frac{1}{2}\int_a^b R(\theta)^2\,d\theta \pm \lim_{n\to\infty}\frac{LM}{2}\frac{(b-a)^2}{n} + \lim_{n\to\infty}\frac{L^2}{8}\frac{(b-a)^3}{n^2} \\
&= \frac{1}{2}\int_a^b R(\theta)^2\,d\theta \qquad \text{(since } L, M, a \text{ and } b \text{ are all constants)}
\end{aligned}$$

we have that

$$A = \frac{1}{2}\int_a^b R(\theta)^2\,d\theta$$

exactly, as desired.

3.3▲ Applications of Double Integrals

Double integrals are useful for more than just computing areas and volumes. Here are a few other applications that lead to double integrals.

▸▸▸ Averages

In Section 2.2 of the CLP-2 text, we defined the average value of a function of one variable. We'll now extend that discussion to functions of two variables. First, we recall the definition of the average of a finite set of numbers.

Definition 3.3.1.

The average (mean) of a set of n numbers f_1, f_2, $\cdots$, f_n is

$$\bar{f} = \langle f \rangle = \frac{f_1 + f_2 + \cdots + f_n}{n}$$

The notations $\bar{f}$ and $\langle f \rangle$ are both commonly used to represent the average.

Now suppose that we want to take the average of a function $f(x,y)$ with (x,y) running continuously over some region $\mathcal{R}$ in the xy-plane. A natural approach to defining what we mean by the average value of f over $\mathcal{R}$ is to

- First fix any natural number n.
- Subdivide the region $\mathcal{R}$ into tiny (approximate) squares each of width $\Delta x = \frac{1}{n}$ and height $\Delta y = \frac{1}{n}$. This can be done by, for example, subdividing vertical strips into tiny squares, like in Example 3.1.11.
- Name the squares (in any fixed order) R_1, R_2, $\cdots$, R_N, where N is the total number of squares.
- Select, for each $1 \leqslant i \leqslant N$, one point in square number i and call it (x_i^*, y_i^*). So $(x_i^*, y_i^*) \in R_i$.
- The average value of f at the selected points is

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i^*, y_i^*) = \frac{\sum_{i=1}^{N} f(x_i^*, y_i^*)}{\sum_{i=1}^{N} 1} = \frac{\sum_{i=1}^{N} f(x_i^*, y_i^*)\,\Delta x\,\Delta y}{\sum_{i=1}^{N} \Delta x\,\Delta y}$$

  We have transformed the average into a ratio of Riemann sums.

Once we have the Riemann sums it is clear what to do next. Taking the limit $n \to \infty$, we get exactly $\frac{\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\iint_{\mathcal{R}} \mathrm{d}x\,\mathrm{d}y}$. That's why we define

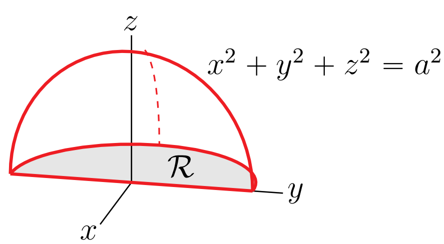
Definition 3.3.2.

Let $f(x, y)$ be an integrable function defined on region $\mathcal{R}$ in the xy-plane. The average value of f on $\mathcal{R}$ is

$$\bar{f} = \langle f \rangle = \frac{\displaystyle\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\iint_{\mathcal{R}} \mathrm{d}x\,\mathrm{d}y}$$

Example 3.3.3 (Average)

Let $a > 0$. A mountain, call it Half Dome[23], has height $z(x, y) = \sqrt{a^2 - x^2 - y^2}$ above each point (x, y) in the base region $\mathcal{R} = \{\ (x, y) \mid x^2 + y^2 \leqslant a^2, x \leqslant 0\ \}$. Find its average height.



Solution. By Definition 3.3.2 the average height is

$$\bar{z} = \frac{\iint_{\mathcal{R}} z(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\iint_{\mathcal{R}} \mathrm{d}x\,\mathrm{d}y} = \frac{\iint_{\mathcal{R}} \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y}{\iint_{\mathcal{R}} \mathrm{d}x\,\mathrm{d}y}$$

The integrals in both the numerator and denominator are easily evaluated by interpreting them geometrically.

- The numerator $\iint_{\mathcal{R}} z(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{R}} \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y$ can be interpreted as the volume of

$$\left\{\ (x,y,z)\ \middle|\ x^2 + y^2 \leqslant a^2,\ x \leqslant 0,\ 0 \leqslant z \leqslant \sqrt{a^2 - x^2 - y^2}\ \right\}$$
$$= \{\ (x,y,z) \mid x^2 + y^2 + z^2 \leqslant a^2,\ x \leqslant 0,\ z \geqslant 0\ \}$$

  which is one quarter of the interior of a sphere of radius a. So the numerator is $\frac{1}{3}\pi a^3$.

- The denominator $\iint_{\mathcal{R}} \mathrm{d}x\,\mathrm{d}y$ is the area of one half of a circular disk of radius a. So the denominator is $\frac{1}{2}\pi a^2$.

23 There is a real Half-Dome mountain in Yosemite National Park. It has $a = 1445\,\mathrm{m}$.

All together, the average height is

$$\bar{z} = \frac{\frac{1}{3}\pi a^3}{\frac{1}{2}\pi a^2} = \frac{2}{3}\,a$$

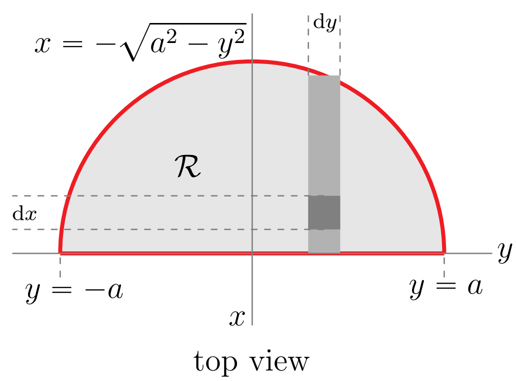
Notice this this number is bigger than zero and less than the maximum height, which is a. That makes sense.

Example 3.3.3

Example 3.3.4 (Example 3.3.3, the hard way)

This last example was relatively easy because we could reinterpret the integrals as geometric quantities. For practice, let's go back and evaluate the numerator $\iint_{\mathcal{R}} \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y$ of Example 3.3.3 as an iterated integral.

Here is a sketch of the top view of the base region $\mathcal{R}$.



Using the slicing in the figure

$$\iint_{\mathcal{R}} \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_{-a}^{a} \mathrm{d}y \int_{-\sqrt{a^2-y^2}}^{0} \mathrm{d}x\, \sqrt{a^2 - x^2 - y^2}$$

Note that, in the inside integral $\displaystyle\int_{-\sqrt{a^2-y^2}}^{0} \mathrm{d}x\, \sqrt{a^2 - x^2 - y^2}$, the variable y is treated as a constant, so that the integrand $\sqrt{a^2 - y^2 - x^2} = \sqrt{C^2 - x^2}$ with C being the constant $\sqrt{a^2 - y^2}$. The standard protocol for evaluating this integral uses the trigonometric substitution

$$\begin{aligned} x &= C\sin\theta \qquad \text{with } -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \\ \mathrm{d}x &= C\cos\theta\,\mathrm{d}\theta \end{aligned}$$

Trigonometric substitution was discussed in detail in Section 1.9 in the CLP-2 text. Since

$$\begin{aligned} x &= 0 & &\implies C\sin\theta = 0 & &\implies \theta = 0 \\ x &= -\sqrt{a^2 - y^2} = -C & &\implies C\sin\theta = -C & &\implies \theta = -\frac{\pi}{2} \end{aligned}$$

and

$$\sqrt{a^2 - x^2 - y^2} = \sqrt{C^2 - C^2 \sin^2\theta} = C\cos\theta$$

the inner integral

$$\begin{aligned}
\int_{-\sqrt{a^2-y^2}}^{0} \mathrm{d}x\, \sqrt{a^2 - x^2 - y^2} &= \int_{-\pi/2}^{0} C^2\cos^2\theta\, \mathrm{d}\theta \\
&= C^2 \int_{-\pi/2}^{0} \frac{1+\cos(2\theta)}{2}\, \mathrm{d}\theta = C^2 \left[\frac{\theta + \frac{\sin(2\theta)}{2}}{2}\right]_{-\pi/2}^{0} \\
&= \frac{\pi C^2}{4} = \frac{\pi}{4}(a^2 - y^2)
\end{aligned}$$

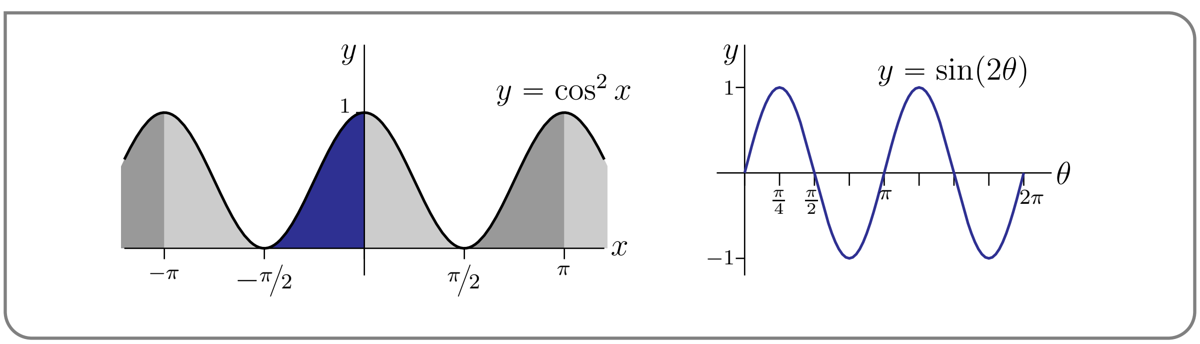
and the full integral

$$\begin{aligned}
\iint_{\mathcal{R}} \sqrt{a^2 - x^2 - y^2}\, \mathrm{d}x\, \mathrm{d}y &= \frac{\pi}{4}\int_{-a}^{a} \left(a^2 - y^2\right)\, \mathrm{d}y = \frac{\pi}{2}\int_{0}^{a} \left(a^2 - y^2\right)\, \mathrm{d}y = \frac{\pi}{2}\left[a^3 - \frac{a^3}{3}\right] \\
&= \frac{1}{3}\pi a^3
\end{aligned}$$

just as we saw in Example 3.3.3.

Example 3.3.4

Remark 3.3.5. We remark that there is an efficient, sneaky, way to evaluate definite integrals like $\int_{-\pi/2}^{0} \cos^2\theta\, \mathrm{d}\theta$. Looking at the figures



we see that

$$\int_{-\pi/2}^{0} \cos^2\theta\, \mathrm{d}\theta = \int_{-\pi/2}^{0} \sin^2\theta\, \mathrm{d}\theta$$

Thus

$$\int_{-\pi/2}^{0} \cos^2\theta\, \mathrm{d}\theta = \int_{-\pi/2}^{0} \sin^2\theta\, \mathrm{d}\theta = \int_{-\pi/2}^{0} \frac{1}{2}\left[\sin^2\theta + \cos^2\theta\right] \mathrm{d}\theta = \frac{1}{2}\int_{-\pi/2}^{0} \mathrm{d}\theta = \frac{\pi}{4}$$

It is not at all unusual to want to find the average value of some function $f(x,y)$ with (x,y) running over some region $\mathcal{R}$, but to also want some (x,y)'s to play a greater role in determining the average than other (x,y)'s. One common way to do so is to create a "weight function" $w(x,y) > 0$ with $\frac{w(x_1,y_1)}{w(x_2,y_2)}$ giving the relative importance of (x_1,y_1) and (x_2,y_2). That is, (x_1,y_1) is $\frac{w(x_1,y_1)}{w(x_2,y_2)}$ times as important as (x_2,y_2). This leads to the definition

Definition 3.3.6.

$$\frac{\iint_{\mathcal{R}} f(x,y)\, w(x,y)\, \mathrm{d}x\, \mathrm{d}y}{\iint_{\mathcal{R}} w(x,y)\, \mathrm{d}x\, \mathrm{d}y}$$

is called the weighted average of f over $\mathcal{R}$ with weight $w(x,y)$.

Note that if $f(x,y) = F$, a constant, then the weighted average of f is just F, just as you would want.

▸▸▸ Centre of Mass

One important example of a weighted average is the centre of mass. If you support a body at its centre of mass (in a uniform gravitational field) it balances perfectly. That's the definition of the centre of mass of the body. In Section 2.3 of the CLP-2 text, we found that the centre of mass of a body that consists of mass distributed continuously along a straight line, with mass density $\rho(x)\mathrm{kg/m}$ and with x running from a to b, is at

$$\bar{x} = \frac{\int_a^b x\, \rho(x)\, \mathrm{d}x}{\int_a^b \rho(x)\, \mathrm{d}x}$$

That is, the centre of mass is at the average of the x-coordinate weighted by the mass density.

In two dimensions, the centre of mass of a plate that covers the region $\mathcal{R}$ in the xy-plane and that has mass density $\rho(x,y)$ is the point $(\bar{x},\bar{y})$ where

Equation 3.3.7 (Centre of Mass)**.**

$$\begin{aligned} \bar{x} &= \text{the weighted average of } x \text{ over } \mathcal{R} \\ &= \frac{\iint_{\mathcal{R}} x\, \rho(x,y)\, \mathrm{d}x\, \mathrm{d}y}{\iint_{\mathcal{R}} \rho(x,y)\, \mathrm{d}x\, \mathrm{d}y} = \frac{\iint_{\mathcal{R}} x\, \rho(x,y)\, \mathrm{d}x\, \mathrm{d}y}{\mathrm{Mass}(\mathcal{R})} \\ \bar{y} &= \text{the weighted average of } y \text{ over } \mathcal{R} \\ &= \frac{\iint_{\mathcal{R}} y\, \rho(x,y)\, \mathrm{d}x\, \mathrm{d}y}{\iint_{\mathcal{R}} \rho(x,y)\, \mathrm{d}x\, \mathrm{d}y} = \frac{\iint_{\mathcal{R}} y\, \rho(x,y)\, \mathrm{d}x\, \mathrm{d}y}{\mathrm{Mass}(\mathcal{R})} \end{aligned}$$

If the mass density is a constant, the centre of mass is also called the centroid, and is the geometric centre of $\mathcal{R}$. In this case

Equation 3.3.8 (Centroid).

$$\bar{x} = \frac{\iint_{\mathcal{R}} x \,\mathrm{d}x\,\mathrm{d}y}{\iint_{\mathcal{R}} \mathrm{d}x\,\mathrm{d}y} = \frac{\iint_{\mathcal{R}} x \,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Area}(\mathcal{R})}$$

$$\bar{y} = \frac{\iint_{\mathcal{R}} y \,\mathrm{d}x\,\mathrm{d}y}{\iint_{\mathcal{R}} \mathrm{d}x\,\mathrm{d}y} = \frac{\iint_{\mathcal{R}} y \,\mathrm{d}x\,\mathrm{d}y}{\mathrm{Area}(\mathcal{R})}$$

Example 3.3.9 (Centre of Mass)

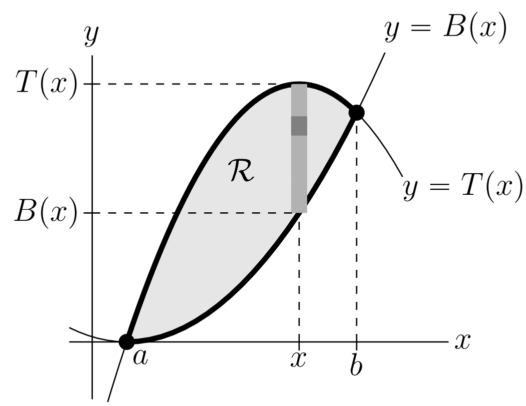
In Section 2.3 of the CLP-2 text, we did not have access to multivariable integrals, so we used some physical intuition to derive that the centroid of a body that fills the region

$$\mathcal{R} = \big\{\ (x,y)\ \big|\ a \leqslant x \leqslant b,\ B(x) \leqslant y \leqslant T(x)\ \big\}$$

in the xy-plane is $(\bar{x},\bar{y})$ where

$$\bar{x} = \frac{\int_a^b x[T(x) - B(x)]\,\mathrm{d}x}{A}$$
$$\bar{y} = \frac{\int_a^b [T(x)^2 - B(x)^2]\,\mathrm{d}x}{2A}$$

and $A = \int_a^b [T(x) - B(x)]\,\mathrm{d}x$ is the area of $\mathcal{R}$. Now that we do have access to multivariable integrals, we can derive these formulae directly from (3.3.8). Using vertical slices, as in this figure,



we see that the area of $\mathcal{R}$ is

$$A = \iint\limits_{\mathcal{R}} \mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y = \int_a^b \mathrm{d}x\ \big[T(x) - B(x)\big]$$

and that (3.3.8) gives

$$\bar{x} = \frac{1}{A}\iint_{\mathcal{R}} x \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{A}\int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\; x = \frac{1}{A}\int_a^b \mathrm{d}x\; x\big[T(x) - B(x)\big]$$

$$\bar{y} = \frac{1}{A}\iint_{\mathcal{R}} y \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{A}\int_a^b \mathrm{d}x \int_{B(x)}^{T(x)} \mathrm{d}y\; y = \frac{1}{A}\int_a^b \mathrm{d}x\; \left[\frac{T(x)^2}{2} - \frac{B(x)^2}{2}\right]$$

just as desired.

Example 3.3.9

We'll start with a simple mechanical example.

Example 3.3.10 (Quarter Circle)

In Example 2.3.4 of the CLP-2 text, we found the centroid of the quarter circular disk

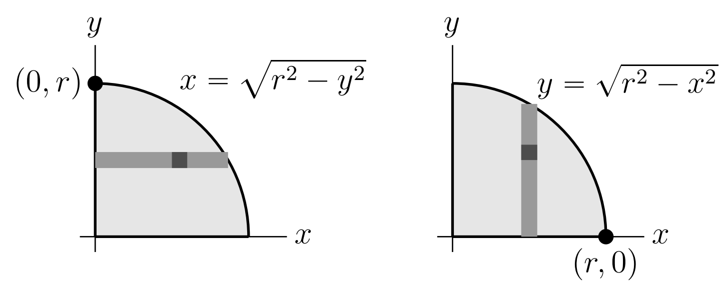
$$D = \left\{\ (x,y)\ \middle|\ x \geqslant 0,\ y \geqslant 0,\ x^2 + y^2 \leqslant r^2\ \right\}$$

by using the formulae of the last example. We'll now find it again using (3.3.8).

Since the area of D is $\frac{1}{4}\pi r^2$, we have

$$\bar{x} = \frac{\iint_D x \,\mathrm{d}x\,\mathrm{d}y}{\frac{1}{4}\pi r^2} \qquad \bar{y} = \frac{\iint_D y \,\mathrm{d}x\,\mathrm{d}y}{\frac{1}{4}\pi r^2}$$

We'll evaluate $\iint_D x \,\mathrm{d}x\,\mathrm{d}y$ by using horizontal slices, as in the figure on the left below.



Looking at that figure, we see that

- y runs from 0 to r and
- for each y in that range, x runs from 0 to $\sqrt{r^2 - y^2}$.

So

$$\iint_D x \,\mathrm{d}x\,\mathrm{d}y = \int_0^r \mathrm{d}y \int_0^{\sqrt{r^2-y^2}} \mathrm{d}x\; x = \int_0^r \mathrm{d}y\; \left[\frac{x^2}{2}\right]_0^{\sqrt{r^2-y^2}} = \frac{1}{2}\int_0^r \mathrm{d}y\; \big[r^2 - y^2\big]$$

$$= \frac{1}{2}\left[r^3 - \frac{r^3}{3}\right] = \frac{r^3}{3}$$

and

$$\bar{x} = \frac{4}{\pi r^2}\left[\frac{r^3}{3}\right] = \frac{4r}{3\pi}$$

This is the same answer as we got in Example 2.3.4 of the CLP-2 text. But because we were able to use horizontal slices, the integral in this example was a little easier to evaluate than the integral in CLP-2. Had we used vertical slices, we would have ended up with exactly the integral of CLP-2.

By symmetry, we should have $\bar{y} = \bar{x}$. We'll check that by evaluating $\iint_D y\,\mathrm{d}x\,\mathrm{d}y$ by using vertical slices slices, as in the figure on the right above. From that figure, we see that

- x runs from 0 to r and
- for each x in that range, y runs from 0 to $\sqrt{r^2 - x^2}$.

So

$$\iint\limits_D y\,\mathrm{d}x\,\mathrm{d}y = \int_0^r \mathrm{d}x \int_0^{\sqrt{r^2-x^2}} \mathrm{d}y\; y = \frac{1}{2}\int_0^r \mathrm{d}x\,\left[r^2 - x^2\right]$$

This is exactly the integral $\frac{1}{2}\int_0^r \mathrm{d}y\,\left[r^2 - y^2\right]$ that we evaluated above, with y renamed to x. So $\iint_D y\,\mathrm{d}x\,\mathrm{d}y = \frac{r^3}{3}$ too and

$$\bar{y} = \frac{4}{\pi r^2}\left[\frac{r^3}{3}\right] = \frac{4r}{3\pi} = \bar{x}$$

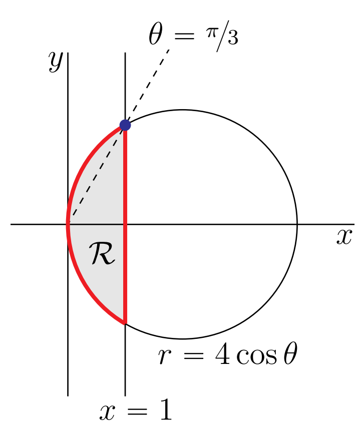
as expected.

Example 3.3.10

Example 3.3.11 (Example 3.2.14, continued)

Find the centroid of the region that is inside the circle $r = 4\cos\theta$ and to the left of the line $x = 1$.

Solution. Recall that we saw in Example 3.2.14 that $r = 4\cos\theta$ was indeed a circle, and in fact is the circle $(x-2)^2 + y^2 = 4$. Here is a sketch of that circle and of the region of interest, $\mathcal{R}$.



From the sketch, we see that $\mathcal{R}$ is symmetric about the x-axis. So we expect that its centroid, $(\bar{x}, \bar{y})$, has $\bar{y} = 0$. To see this from the integral definition, note that the integral $\iint_{\mathcal{R}} y \, \mathrm{d}x \, \mathrm{d}y$

- has domain of integration, namely $\mathcal{R}$, invariant under $y \to -y$ (i.e. under reflection across the x-axis), and
- has integrand, namely y, that is odd under $y \to -y$.

So $\iint_{\mathcal{R}} y \, \mathrm{d}x \, \mathrm{d}y = 0$ and consequently $\bar{y} = 0$.

We now just have to find $\bar{x}$:

$$\bar{x} = \frac{\iint_{\mathcal{R}} x \, \mathrm{d}x \, \mathrm{d}y}{\iint_{\mathcal{R}} \mathrm{d}x \, \mathrm{d}y}$$

We have already found, in Example 3.2.14, that

$$\iint_{\mathcal{R}} \mathrm{d}x \, \mathrm{d}y = \frac{4\pi}{3} - \sqrt{3}$$

So we just have to compute $\iint_{\mathcal{R}} x \, \mathrm{d}x \, \mathrm{d}y$. Using $\mathcal{R}_1$ to denote the top half of $\mathcal{R}$, and using polar coordinates, like we did in Example 3.2.14,

$$\begin{aligned}
\iint_{\mathcal{R}_1} x \, \mathrm{d}x \, \mathrm{d}y &= \int_0^{\pi/3} \mathrm{d}\theta \int_0^{1/\cos\theta} \mathrm{d}r \; r \, \overbrace{(r\cos\theta)}^{x} + \int_{\pi/3}^{\pi/2} \mathrm{d}\theta \int_0^{4\cos\theta} \mathrm{d}r \; r \, \overbrace{(r\cos\theta)}^{x} \\
&= \int_0^{\pi/3} \mathrm{d}\theta \; \cos\theta \int_0^{1/\cos\theta} \mathrm{d}r \; r^2 + \int_{\pi/3}^{\pi/2} \mathrm{d}\theta \; \cos\theta \int_0^{4\cos\theta} \mathrm{d}r \; r^2 \\
&= \int_0^{\pi/3} \mathrm{d}\theta \; \frac{\sec^2\theta}{3} + \int_{\pi/3}^{\pi/2} \mathrm{d}\theta \; \frac{64}{3} \cos^4\theta
\end{aligned}$$

The first integral is easy, provided we remember that $\tan\theta$ is an antiderivative for $\sec^2\theta$. For the second integral, we'll need the double angle formula $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$:

$$\begin{aligned}
\cos^4\theta = \big(\cos^2\theta\big)^2 = \left[\frac{1+\cos(2\theta)}{2}\right]^2 &= \frac{1}{4}\Big[1 + 2\cos(2\theta) + \cos^2(2\theta)\Big] \\
&= \frac{1}{4}\left[1 + 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2}\right] \\
&= \frac{3}{8} + \frac{\cos(2\theta)}{2} + \frac{\cos(4\theta)}{8}
\end{aligned}$$

so

$$\begin{aligned}
\iint_{\mathcal{R}_1} x \, \mathrm{d}x \, \mathrm{d}y &= \frac{1}{3}\tan\theta\Big|_0^{\pi/3} + \frac{64}{3}\left[\frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32}\right]_{\pi/3}^{\pi/2} \\
&= \frac{1}{3} \times \sqrt{3} + \frac{64}{3}\left[\frac{3}{8} \times \frac{\pi}{6} - \frac{\sqrt{3}}{4 \times 2} + \frac{\sqrt{3}}{32 \times 2}\right] \\
&= \frac{4\pi}{3} - 2\sqrt{3}
\end{aligned}$$

The integral we want, namely $\iint_{\mathcal{R}} x \, \mathrm{d}x \, \mathrm{d}y$,

- has domain of integration, namely $\mathcal{R}$, invariant under $y \to -y$ (i.e. under reflection across the x-axis), and
- has integrand, namely x, that is even under $y \to -y$.

So $\iint_{\mathcal{R}} x\, \mathrm{d}x\, \mathrm{d}y = 2\iint_{\mathcal{R}_1} x\, \mathrm{d}x\, \mathrm{d}y$ and, all together,

$$\bar{x} = \frac{2\left(\frac{4\pi}{3} - 2\sqrt{3}\right)}{\frac{4\pi}{3} - \sqrt{3}} = \frac{\frac{8\pi}{3} - 4\sqrt{3}}{\frac{4\pi}{3} - \sqrt{3}} \approx 0.59$$

As a check, note that $0 \leqslant x \leqslant 1$ on $\mathcal{R}$ and more of $\mathcal{R}$ is closer to $x = 1$ than to $x = 0$. So it makes sense that $\bar{x}$ is between $\frac{1}{2}$ and 1.

Example 3.3.11

Example 3.3.12 (Reverse Centre of Mass)

Evaluate $\displaystyle\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (2x + 3y)\,\mathrm{d}y\,\mathrm{d}x$.

Solution. This is another integral that can be evaluated without using any calculus at all. This time by relating it to a centre of mass. By (3.3.8),

$$\iint_{\mathcal{R}} x\, \mathrm{d}x\, \mathrm{d}y = \bar{x}\ \mathrm{Area}(\mathcal{R})$$
$$\iint_{\mathcal{R}} y\, \mathrm{d}x\, \mathrm{d}y = \bar{y}\ \mathrm{Area}(\mathcal{R})$$

so that we can easily evaluate $\iint_{\mathcal{R}} x\, \mathrm{d}x\, \mathrm{d}y$ and $\iint_{\mathcal{R}} y\, \mathrm{d}x\, \mathrm{d}y$ provided $\mathcal{R}$ is sufficiently simple and symmetric that we can easily determine its area and its centroid.

That is the case for the integral in this example. Rewrite

$$\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (2x + 3y)\,\mathrm{d}y\,\mathrm{d}x = 2\int_0^2 \mathrm{d}x \left[\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \mathrm{d}y\ x\right] + 3\int_0^2 \mathrm{d}x \left[\int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \mathrm{d}y\ y\right]$$

On the domain of integration

- x runs from 0 to 2 and
- for each fixed $0 \leqslant x \leqslant 2$, y runs from $-\sqrt{2x - x^2}$ to $+\sqrt{2x - x^2}$

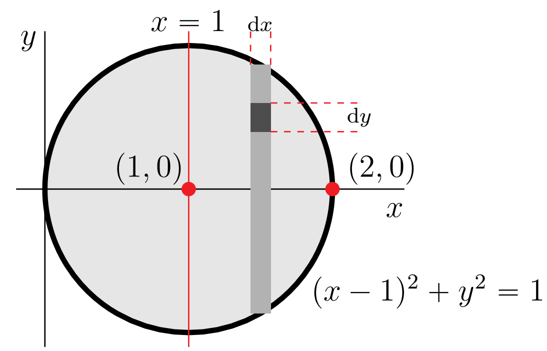
Observe that $y = \pm\sqrt{2x - x^2}$ is equivalent to

$$y^2 = 2x - x^2 = 1 - (x-1)^2 \iff (x-1)^2 + y^2 = 1$$

Our domain of integration is exactly the disk

$$\mathcal{R} = \left\{\ (x, y)\ \middle|\ (x-1)^2 + y^2 \leqslant 1\ \right\}$$

of radius 1 centred on $(1, 0)$. So $\mathcal{R}$ has area π and centre of mass $(\bar{x}, \bar{y}) = (1, 0)$ and



$$\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (2x+3y)\, dy\, dx = 2 \iint_{\mathcal{R}} x\, dx\, dy + 3 \iint_{\mathcal{R}} y\, dx\, dy$$
$$= 2\,\bar{x}\,\text{Area}(\mathcal{R}) + 3\,\bar{y}\,\text{Area}(\mathcal{R}) = 2\pi$$

Example 3.3.12

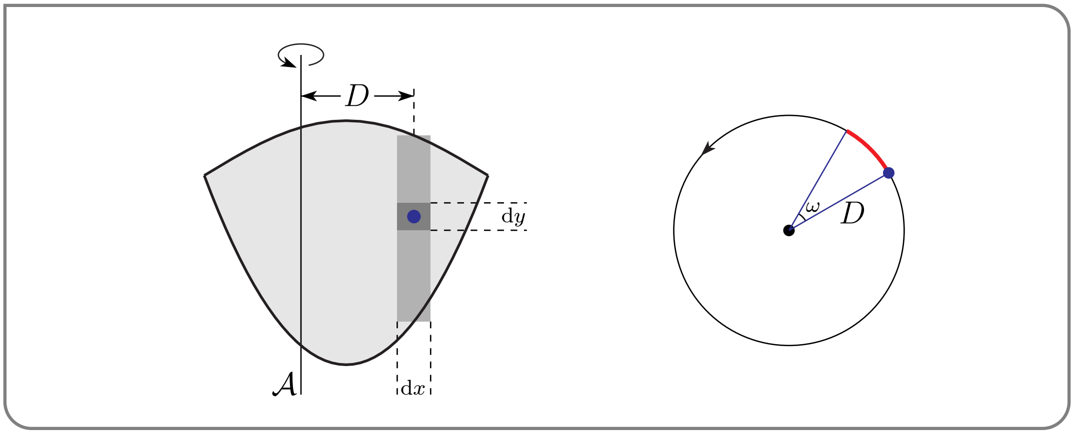
▸▸▸ Moment of Inertia

Consider a plate that fills the region $\mathcal{R}$ in the xy-plane, that has mass density $\rho(x,y)$ kg/m^2, and that is rotating at ω rad/s about some axis. Let's call the axis of rotation $\mathcal{A}$. We are now going to determine the kinetic energy of that plate. Recall[24] that, by definition, the kinetic energy of a point particle of mass m that is moving with speed v is $\frac{1}{2}mv^2$.

To get the kinetic energy of the entire plate, cut it up into tiny rectangles[25], say of size $dx \times dy$. Think of each rectangle as being (essentially) a point particle. If the point (x,y) on the plate is a distance $D(x,y)$ from the axis of rotation $\mathcal{A}$, then as the plate rotates, the point (x,y) sweeps out a circle of radius $D(x,y)$. The figure on the right below shows that circle as seen from high up on the axis of rotation. The circular arc that the point

24 If you don't recall, don't worry. We wouldn't lie to you. Or check it on Wikipedia. They wouldn't lie to you either.

25 The relatively small number of "rectangles" around the boundary of $\mathcal{R}$ won't actually be rectangles. But, as we have seen in the optional §3.2.4, one can still make things rigorous despite the rectangles being a bit squishy around the edges.



(x, y) sweeps out in one second subtends the angle ω radians, which is the fraction $\frac{\omega}{2\pi}$ of a full circle and so has length $\frac{\omega}{2\pi}\big(2\pi D(x, y)\big) = \omega\, D(x, y)$. Consequently the rectangle that contains the point (x, y)

- has speed $\omega\, D(x, y)$, and
- has area $\mathrm{d}x\,\mathrm{d}y$, and so
- has mass $\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$, and
- has kinetic energy

$$\frac{1}{2}\overbrace{\big(\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y\big)}^{m}\overbrace{\big(\omega\, D(x, y)\big)^2}^{v^2} = \frac{1}{2}\omega^2\, D(x, y)^2\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

So (via our usual Riemann sum limit procedure) the kinetic energy of $\mathcal{R}$ is

$$\iint_{\mathcal{R}} \frac{1}{2}\omega^2\, D(x, y)^2\,\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\omega^2 \iint_{\mathcal{R}} D(x, y)^2\,\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}I_{\mathcal{A}}\,\omega^2$$

where

Definition 3.3.13 (Moment of Inertia)**.**

$$I_{\mathcal{A}} = \iint_{\mathcal{R}} D(x, y)^2\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

is called the *moment of inertial* of $\mathcal{R}$ about the axis $\mathcal{A}$. In particular the moment of inertia of $\mathcal{R}$ about the y-axis is

$$I_y = \iint_{\mathcal{R}} x^2\,\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

and the moment of inertia of $\mathcal{R}$ about the x-axis is

$$I_x = \iint_{\mathcal{R}} y^2\,\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

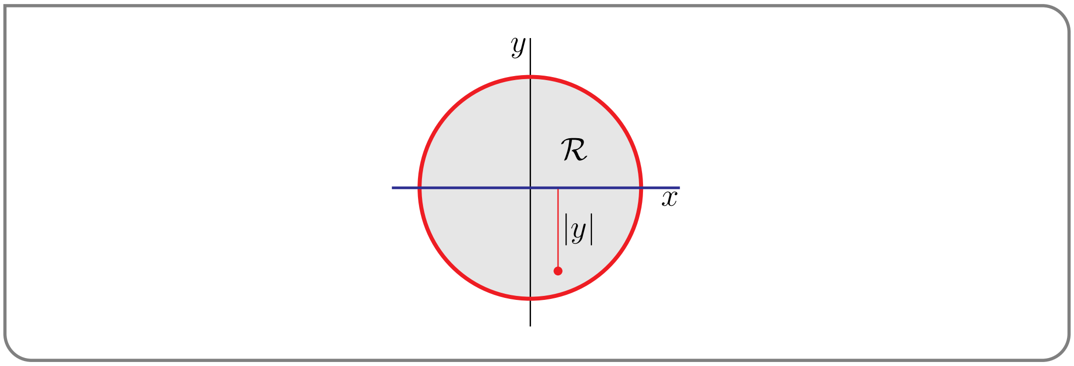
Notice that the expression $\frac{1}{2}I_{\mathcal{A}}\,\omega^2$ for the kinetic energy has a very similar form to $\frac{1}{2}mv^2$, just with the velocity v replaced by the angular velocity ω, and with the mass m replaced by $I_{\mathcal{A}}$, which can be thought of as being a bit like a mass.

So far, we have been assuming that the rotation was taking place in the xy-plane — a two dimensional world. Our analysis extends naturally to three dimensions, though the resulting integral formulae for the moment of inertia will then be triple integrals, which we have not yet dealt with. We shall soon do so, but let's first do an example in two dimensions.

Example 3.3.14 (Disk)

Find the moment of inertia of the interior, $\mathcal{R}$, of the circle $x^2 + y^2 = a^2$ about the x-axis. Assume that it has density one.

Solution. The distance from any point (x, y) inside the disk to the axis of rotation (i.e. the



x-axis) is $|y|$. So the moment of inertia of the interior of the disk about the x-axis is

$$I_x = \iint_{\mathcal{R}} y^2\, \mathrm{d}x\mathrm{d}y$$

Switching to polar coordinates[26],

$$\begin{aligned}
I_x &= \int_0^{2\pi} \mathrm{d}\theta \int_0^a \mathrm{d}r\; r\,\overbrace{(r\sin\theta)^2}^{y^2} = \int_0^{2\pi} \mathrm{d}\theta\; \sin^2\theta \int_0^a \mathrm{d}r\; r^3 \\
&= \frac{a^4}{4}\int_0^{2\pi} \mathrm{d}\theta\; \sin^2\theta = \frac{a^4}{4}\int_0^{2\pi} \mathrm{d}\theta\; \frac{1-\cos(2\theta)}{2} \\
&= \frac{a^4}{8}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{2\pi} \\
&= \frac{1}{4}\pi a^4
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta$, see Remark 3.3.5.

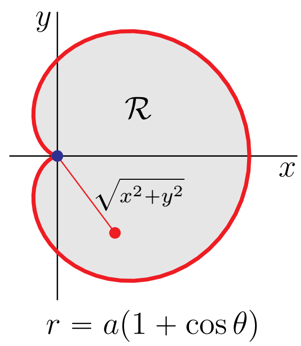
26 See how handy they are!

Example 3.3.14

Example 3.3.15 (Cardioid)

Find the moment of inertia of the interior, $\mathcal{R}$, of the cardiod $r = a(1 + \cos\theta)$ about the z-axis. Assume that the cardiod lies in the xy-plane and has density one.

Solution. We sketched the cardiod (with $a = 1$) in Example 3.2.3. As we said above, the



formula for $I_\mathcal{A}$ in Definition 3.3.13 is valid even when the axis of rotation is not contained in the xy-plane. We just have to be sure that our $D(x, y)$ really is the distance from (x, y) to the axis of rotation. In this example the axis of rotation is the z-axis so that $D(x, y) = \sqrt{x^2 + y^2}$ and that the moment of inertia is

$$I_\mathcal{A} = \iint_\mathcal{R} (x^2 + y^2)\, \mathrm{d}x\mathrm{d}y$$

Switching to polar coordinates using $\mathrm{d}x\mathrm{d}y = r\,\mathrm{d}r\mathrm{d}\theta$ and $x^2 + y^2 = r^2$,

$$\begin{aligned}
I_\mathcal{A} &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{a(1+\cos\theta)} \mathrm{d}r\; r \times r^2 = \int_0^{2\pi} \mathrm{d}\theta \int_0^{a(1+\cos\theta)} \mathrm{d}r\; r^3 \\
&= \frac{a^4}{4} \int_0^{2\pi} \mathrm{d}\theta\; \big(1 + \cos\theta\big)^4 \\
&= \frac{a^4}{4} \int_0^{2\pi} \mathrm{d}\theta\; \big(1 + 4\cos\theta + 6\cos^2\theta + 4\cos^3\theta + \cos^4\theta\big)
\end{aligned}$$

Now

$$\begin{aligned}
&\int_0^{2\pi} \mathrm{d}\theta\; \cos\theta = \sin\theta\Big|_0^{2\pi} = 0 \\
&\int_0^{2\pi} \mathrm{d}\theta\; \cos^2\theta = \int_0^{2\pi} \mathrm{d}\theta\; \frac{1 + \cos(2\theta)}{2} = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{2\pi} = \pi \\
&\int_0^{2\pi} \mathrm{d}\theta\; \cos^3\theta = \int_0^{2\pi} \mathrm{d}\theta\; \cos\theta\big[1 - \sin^2\theta\big] \overset{u=\sin\theta}{=} \int_0^0 \mathrm{d}u\; (1 - u^2) = 0
\end{aligned}$$

To integrate $\cos^4\theta$, we use the double angle formula

$$\begin{aligned}\cos^2\theta &= \frac{\cos(2\theta)+1}{2}\\ \implies \cos^4\theta &= \frac{\big(\cos(2\theta)+1\big)^2}{4} = \frac{\cos^2(2\theta)+2\cos(2\theta)+1}{4} = \frac{\frac{\cos(4\theta)+1}{2}+2\cos(2\theta)+1}{4}\\ &= \frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)\end{aligned}$$

to give

$$\int_0^{2\pi} \mathrm{d}\theta\ \cos^4\theta = \int_0^{2\pi}\mathrm{d}\theta\ \left[\frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)\right] = \frac{3}{8}\times 2\pi+\frac{1}{2}\times 0+\frac{1}{8}\times 0 = \frac{3}{4}\pi$$

All together

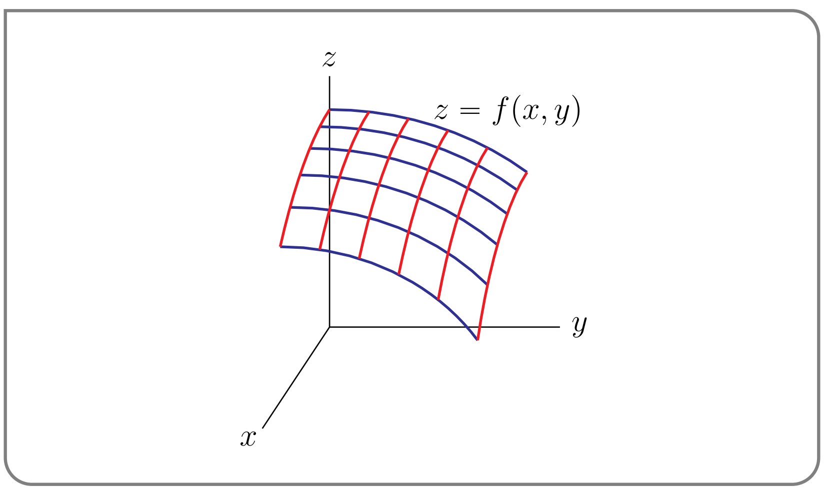
$$\begin{aligned}I_{\mathcal{A}} &= \frac{a^4}{4}\left[2\pi+4\times 0+6\times\pi+4\times 0+\frac{3}{4}\pi\right]\\ &= \frac{35}{16}\pi a^4\end{aligned}$$

Example 3.3.15

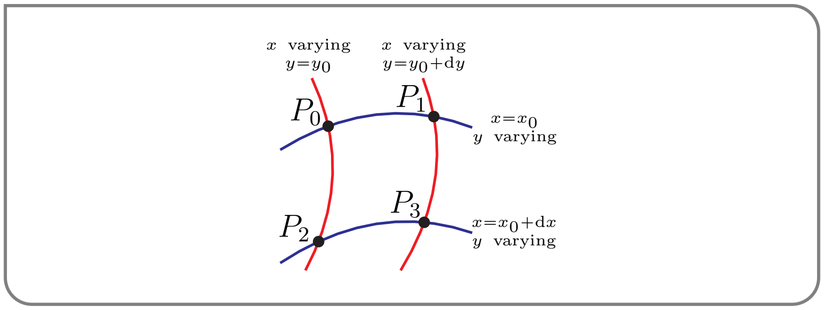
3.4▲ Surface Area

Suppose that we wish to find the area of part, S, of the surface $z = f(x,y)$. We start by cutting S up into tiny pieces. To do so,

- we draw a bunch of curves of constant x (the blue curves in the figure below). Each such curve is the intersection of S with the plane $x = x_0$ for some constant x_0. And we also
- draw a bunch of curves of constant y (the red curves in the figure below). Each such curve is the intersection of S with the plane $y = y_0$ for some constant y_0.



Concentrate on any one the tiny pieces. Here is a greatly magnified sketch of it, looking at it from above.



We wish to compute its area, which we'll call $\mathrm{d}S$. Now this little piece of surface need not be parallel to the xy-plane, and indeed need not even be flat. But if the piece is really tiny, it's almost flat. We'll now approximate it by something that is flat, and whose area we know. To start, we'll determine the corners of the piece. To do so, we first determine the bounding curves of the piece. Look at the figure above, and recall that, on the surface $z = f(x, y)$.

- The upper blue curve was constructed by holding x fixed at the value x_0, and sketching the curve swept out by $x_0\,\hat{\imath} + y\,\hat{\jmath} + f(x_0, y)\,\hat{\mathbf{k}}$ as y varied, and
- the lower blue curve was constructed by holding x fixed at the slightly larger value $x_0 + \mathrm{d}x$, and sketching the curve swept out by $(x_0 + \mathrm{d}x)\,\hat{\imath} + y\,\hat{\jmath} + f(x_0 + \mathrm{d}x, y)\,\hat{\mathbf{k}}$ as y varied.
- The red curves were constructed similarly, by holding y fixed and varying x.

So the four intersection points in the figure are

$$\begin{aligned}
P_0 &= x_0\,\hat{\imath} + y_0\,\hat{\jmath} + f(x_0, y_0)\,\hat{\mathbf{k}} \\
P_1 &= x_0\,\hat{\imath} + (y_0 + \mathrm{d}y)\,\hat{\jmath} + f(x_0, y_0 + \mathrm{d}y)\,\hat{\mathbf{k}} \\
P_2 &= (x_0 + \mathrm{d}x)\,\hat{\imath} + y_0\,\hat{\jmath} + f(x_0 + \mathrm{d}x, y_0)\,\hat{\mathbf{k}} \\
P_3 &= (x_0 + \mathrm{d}x)\,\hat{\imath} + (y_0 + \mathrm{d}y)\,\hat{\jmath} + f(x_0 + \mathrm{d}x, y_0 + \mathrm{d}y)\,\hat{\mathbf{k}}
\end{aligned}$$

Now, for any small constants $\mathrm{d}X$ and $\mathrm{d}Y$, we have the linear approximation[27]

$$f(x_0 + \mathrm{d}X, y_0 + \mathrm{d}Y) \approx f(x_0\,, y_0) + \frac{\partial f}{\partial x}(x_0\,, y_0)\,\mathrm{d}X + \frac{\partial f}{\partial y}(x_0\,, y_0)\,\mathrm{d}Y$$

Applying this three times, once with $\mathrm{d}X = 0$, $\mathrm{d}Y = \mathrm{d}y$ (to approximate P_1), once with $\mathrm{d}X = \mathrm{d}x$, $\mathrm{d}Y = 0$ (to approximate P_2), and once with $\mathrm{d}X = \mathrm{d}x$, $\mathrm{d}Y = \mathrm{d}y$ (to approximate P_3),

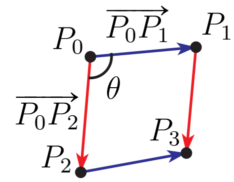
$$\begin{aligned}
P_1 &\approx P_0 \qquad\qquad + \mathrm{d}y\,\hat{\jmath} + \frac{\partial f}{\partial y}(x_0\,, y_0)\,\mathrm{d}y\,\hat{\mathbf{k}} \\
P_2 &\approx P_0 + \mathrm{d}x\,\hat{\imath} \qquad\qquad + \frac{\partial f}{\partial x}(x_0\,, y_0)\,\mathrm{d}x\,\hat{\mathbf{k}} \\
P_3 &\approx P_0 + \mathrm{d}x\,\hat{\imath} + \mathrm{d}y\,\hat{\jmath} + \left[\frac{\partial f}{\partial x}(x_0\,, y_0)\,\mathrm{d}x + \frac{\partial f}{\partial y}(x_0\,, y_0)\,\mathrm{d}y\right]\hat{\mathbf{k}}
\end{aligned}$$

27 Recall (2.6.1).

Of course we have only approximated the positions of the corners and so have introduced errors. However, with more work, one can bound those errors (like we in the optional §3.2.4) and show that in the limit $\mathrm{d}x, \mathrm{d}y \to 0$, all of the error terms that we dropped contribute exactly 0 to the integral.

The small piece of our surface with corners P_0, P_1, P_2, P_3 is approximately a parallelogram with sides

$$\overrightarrow{P_0P_1} \approx \overrightarrow{P_2P_3} \approx \mathrm{d}y\,\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\mathrm{d}y\,\hat{\mathbf{k}}$$
$$\overrightarrow{P_0P_2} \approx \overrightarrow{P_1P_3} \approx \mathrm{d}x\,\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\mathrm{d}x\,\hat{\mathbf{k}}$$



Denote by θ the angle between the vectors $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$. The base of the parallelogram, $\overrightarrow{P_0P_1}$, has length $|\overrightarrow{P_0P_1}|$, and the height of the parallelogram is $|\overrightarrow{P_0P_2}|\ \sin\theta$. So the area of the parallelogram is[28], by Theorem 1.2.23,

$$\begin{aligned}\mathrm{d}S = |\overrightarrow{P_0P_1}|\,|\overrightarrow{P_0P_2}|\ \sin\theta &= |\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}| \\ &\approx \left|\left(\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\hat{\mathbf{k}}\right) \times \left(\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\hat{\mathbf{k}}\right)\right| \mathrm{d}x\mathrm{d}y\end{aligned}$$

The cross product is easily evaluated:

$$\begin{aligned}\left(\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\hat{\mathbf{k}}\right) \times \left(\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\hat{\mathbf{k}}\right) &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 1 & \frac{\partial f}{\partial y}(x_0, y_0) \\ 1 & 0 & \frac{\partial f}{\partial x}(x_0, y_0)\end{bmatrix} \\ &= f_x(x_0, y_0)\,\hat{\boldsymbol{\imath}} + f_y(x_0, y_0)\,\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}\end{aligned}$$

as is its length:

$$\left|\left(\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial y}(x_0\,,\,y_0)\,\hat{\mathbf{k}}\right) \times \left(\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial x}(x_0\,,\,y_0)\,\hat{\mathbf{k}}\right)\right| = \sqrt{1 + f_x(x_0, y_0)^2 + f_y(x_0, y_0)^2}$$

Throughout this computation, x_0 and y_0 were arbitrary. So we have found the area of each tiny piece of the surface S.

28 As we mentioned above, the approximation below becomes exact when the limit $\mathrm{d}x, \mathrm{d}y \to 0$ is taken in the definition of the integral. See §3.3.5 in the CLP-4 text.

Equation 3.4.1.

For the surface $z = f(x, y)$,

$$dS = \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dxdy$$

Similarly, for the surface $x = g(y, z)$,

$$dS = \sqrt{1 + g_y(y, z)^2 + g_z(y, z)^2}\, dydz$$

and for the surface $y = h(x, z)$,

$$dS = \sqrt{1 + h_x(x, z)^2 + h_z(x, z)^2}\, dxdz$$

Consequently, we have

Theorem 3.4.2.

(a) The area of the part of the surface $z = f(x, y)$ with (x, y) running over the region $\mathcal{D}$ in the xy-plane is

$$\iint_{\mathcal{D}} \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dxdy$$

(b) The area of the part of the surface $x = g(y, z)$ with (y, z) running over the region $\mathcal{D}$ in the yz-plane is

$$\iint_{\mathcal{D}} \sqrt{1 + g_y(y, z)^2 + g_z(y, z)^2}\, dydz$$

(c) The area of the part of the surface $y = h(x, z)$ with (x, z) running over the region $\mathcal{D}$ in the xz-plane is

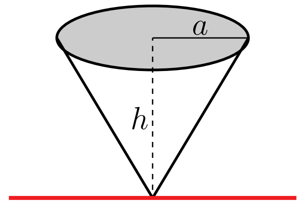
$$\iint_{\mathcal{D}} \sqrt{1 + h_x(x, z)^2 + h_z(x, z)^2}\, dxdz$$

Example 3.4.3 (Area of a cone)

As a first example, we compute the area of the part of the cone

$$z = \sqrt{x^2 + y^2}$$

with $0 \leqslant z \leqslant a$ or, equivalently, with $x^2 + y^2 \leqslant a^2$.



Note that $z = \sqrt{x^2 + y^2}$ is the side of the cone. It does not include the top.

To find its area, we will apply (3.4.1) to

$$z = f(x,y) = \sqrt{x^2 + y^2} \qquad \text{with } (x,y) \text{ running over } x^2 + y^2 \leqslant a^2$$

That forces us to compute the first order partial derivatives

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$
$$f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

Substituting them into the first formula in (3.4.1) yields

$$\begin{aligned}
\mathrm{d}S &= \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{1 + \Big(\frac{x}{\sqrt{x^2+y^2}}\Big)^2 + \Big(\frac{y}{\sqrt{x^2+y^2}}\Big)^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{1 + \frac{x^2+y^2}{x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

So

$$\text{Area} = \iint\limits_{x^2+y^2\leqslant a^2} \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\iint\limits_{x^2+y^2\leqslant a^2} \mathrm{d}x\,\mathrm{d}y = \sqrt{2}\pi a^2$$

because $\iint_{x^2+y^2\leqslant a^2} \mathrm{d}x\,\mathrm{d}y$ is exactly the area of a circular disk of radius a.

Example 3.4.3

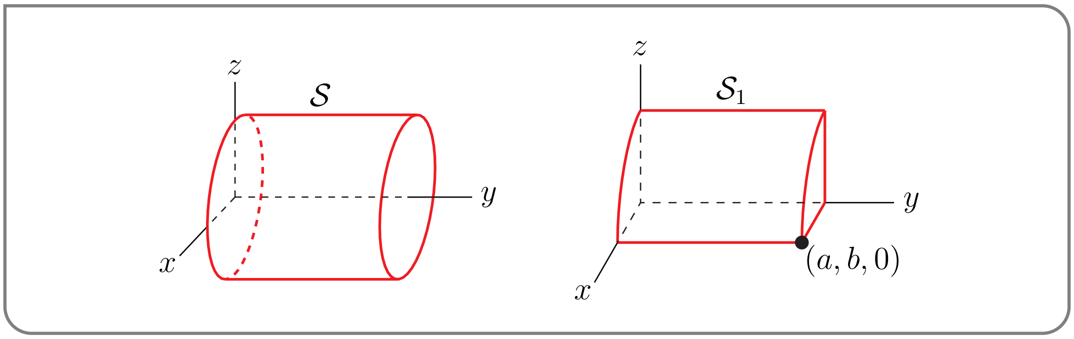
Example 3.4.4 (Area of a cylinder)

Let $a, b > 0$. Find the surface area of

$$\mathcal{S} = \big\{\, (x,y,z) \bigm| x^2 + z^2 = a^2,\ 0 \leqslant y \leqslant b \,\big\}$$

Solution. The intersection of $x^2 + z^2 = a^2$ with any plane of constant y is the circle of radius a centred on $x = z = 0$. So $\mathcal{S}$ is a bunch of circles stacked sideways. It is a cylinder on its side (with both ends open). By symmetry, the area of $\mathcal{S}$ is four times the area of the part of $\mathcal{S}$ that is in the first octant, which is

$$\mathcal{S}_1 = \Big\{(x, y, z) \Bigm| z = f(x, y) = \sqrt{a^2 - x^2},\ 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b\Big\}$$



Since

$$f_x(x, y) = -\frac{x}{\sqrt{a^2 - x^2}} \qquad f_y(x, y) = 0$$

the first formula in (3.4.1) yields

$$\begin{aligned}
\mathrm{d}S &= \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{1 + \Big(-\frac{x}{\sqrt{a^2 - x^2}}\Big)^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \sqrt{1 + \frac{x^2}{a^2 - x^2}}\,\mathrm{d}x\mathrm{d}y \\
&= \frac{a}{\sqrt{a^2 - x^2}}\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

So

$$\mathrm{Area}(\mathcal{S}_1) = \int_0^a \mathrm{d}x \int_0^b \mathrm{d}y\, \frac{a}{\sqrt{a^2 - x^2}} = ab \int_0^a \mathrm{d}x \frac{1}{\sqrt{a^2 - x^2}}$$

The indefinite integral of $\frac{1}{\sqrt{a^2-x^2}}$ is $\arcsin \frac{x}{a} + C$. (See the table of integrals in Appendix D. Alternatively, use the trig substitution $x = a \sin\theta$.) So

$$\mathrm{Area}(\mathcal{S}_1) = ab \left[\arcsin \frac{x}{a}\right]_0^a = ab\big[\arcsin 1 - \arcsin 0\big] = \frac{\pi}{2} ab$$

and

$$\mathrm{Area}(\mathcal{S}) = 4\mathrm{Area}(\mathcal{S}_1) = 2\pi ab$$

We could have also come to this conclusion by using a little geometry, rather than using calculus. Cut open the cylinder by cutting along a line parallel to the y-axis, and then flatten out the cylinder. This gives a rectangle. One side of the rectangle is just a circle of radius a, straightened out. So the rectangle has sides of lengths $2\pi a$ and b and has area $2\pi ab$.

Example 3.4.4

Example 3.4.5 (Area of a hemisphere)

This time we compute the surface area of the hemisphere

$$x^2 + y^2 + z^2 = a^2 \qquad z \geqslant 0$$

(with $a > 0$). You probably know, from high school, that the answer is $\frac{1}{2} \times 4\pi a^2 = 2\pi a^2$. But you have probably not seen a derivation[29] of this answer. Note that, since $x^2 + y^2 = a^2 - z^2$ on the hemisphere, the set of (x, y)'s for which there is a z with (x, y, z) on the hemisphere is exactly $\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant a^2 \}$. So the hemisphere is

$$S = \left\{ (x, y, z) \,\middle|\, z = \sqrt{a^2 - x^2 - y^2},\ x^2 + y^2 \leqslant a^2 \right\}$$

We will compute the area of S by applying (3.4.1) to

$$z = f(x, y) = \sqrt{a^2 - x^2 - y^2} \qquad \text{with } (x, y) \text{ running over } x^2 + y^2 \leqslant a^2$$

The first formula in (3.4.1) yields

$$\begin{aligned}
\mathrm{d}S &= \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\mathrm{d}y \\
&= \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\,\mathrm{d}x\mathrm{d}y \\
&= \sqrt{1 + \frac{x^2 + y^2}{a^2 - x^2 - y^2}}\,\mathrm{d}x\mathrm{d}y \\
&= \sqrt{\frac{a^2}{a^2 - x^2 - y^2}}\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

So the area is $\iint_{x^2+y^2 \leqslant a^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,\mathrm{d}x\mathrm{d}y$. To evaluate this integral, we switch to polar

29 There is a pun hidden here, because you can (with a little thought) also get the surface area by differentiating the volume with respect to the radius.

coordinates, substituting $x = r\cos\theta$, $y = r\sin\theta$. This gives

$$\begin{aligned}
\text{area} &= \iint\limits_{x^2+y^2\leqslant a^2} \frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\mathrm{d}y = \int_0^a \mathrm{d}r\; r\int_0^{2\pi}\mathrm{d}\theta\;\frac{a}{\sqrt{a^2-r^2}}\\
&= 2\pi a\int_0^a \mathrm{d}r\;\frac{r}{\sqrt{a^2-r^2}}\\
&= 2\pi a\int_{a^2}^0 \frac{-\mathrm{d}u/2}{\sqrt{u}} \qquad \text{with } u = a^2-r^2,\ \mathrm{d}u = -2r\,\mathrm{d}r\\
&= 2\pi a\Big[-\sqrt{u}\Big]_{a^2}^0\\
&= 2\pi a^2
\end{aligned}$$

as it should be.

Example 3.4.5

Example 3.4.6

Find the surface area of the part of the paraboloid $z = 2 - x^2 - y^2$ lying above the xy-plane.

Solution. The equation of the surface is of the form $z = f(x,y)$ with $f(x,y) = 2 - x^2 - y^2$. So

$$f_x(x,y) = -2x \qquad f_y(x,y) = -2y$$

and, by the first part of (3.4.1),

$$\begin{aligned}
\mathrm{d}S &= \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,\mathrm{d}x\mathrm{d}y\\
&= \sqrt{1+4x^2+4y^2}\,\mathrm{d}x\mathrm{d}y
\end{aligned}$$

The point (x,y,z), with $z = 2-x^2-y^2$, lies above the xy-plane if and only if $z \geqslant 0$, or, equivalently, $2 - x^2 - y^2 \geqslant 0$. So the domain of integration is $\{\,(x,y)\bigm| x^2+y^2\leqslant 2\,\}$ and

$$\text{Surface Area} = \iint\limits_{x^2+y^2\leqslant 2} \sqrt{1+4x^2+4y^2}\,\mathrm{d}x\mathrm{d}y$$

Switching to polar coordinates,

$$\begin{aligned}
\text{Surface Area} &= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+4r^2}\;r\,\mathrm{d}r\,\mathrm{d}\theta\\
&= 2\pi\left[\frac{1}{12}\big(1+4r^2\big)^{3/2}\right]_0^{\sqrt{2}} = \frac{\pi}{6}[27-1]\\
&= \frac{13}{3}\pi
\end{aligned}$$

Example 3.4.6

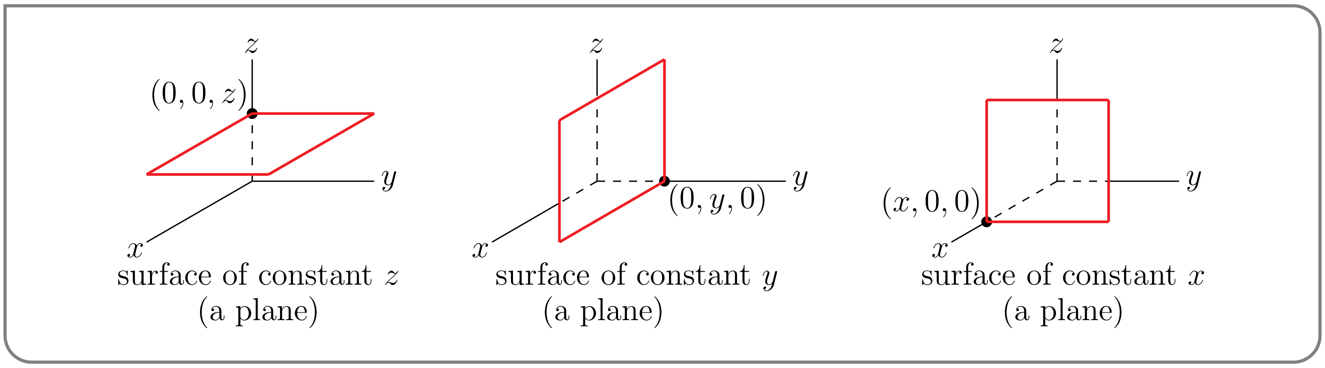
3.5▲ Triple Integrals

Triple integrals, that is integrals over three dimensional regions, are just like double integrals, only more so. We decompose the domain of integration into tiny cubes, for example, compute the contribution from each cube and then use integrals to add up all of the different pieces. We'll go through the details now by means of a number of examples.

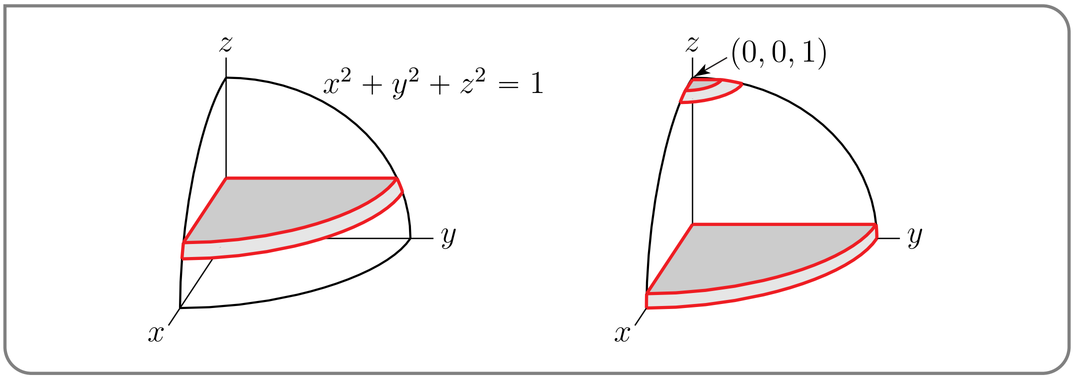
Example 3.5.1

Find the mass inside the sphere $x^2 + y^2 + z^2 = 1$ if the density is $\rho(x, y, z) = |xyz|$.

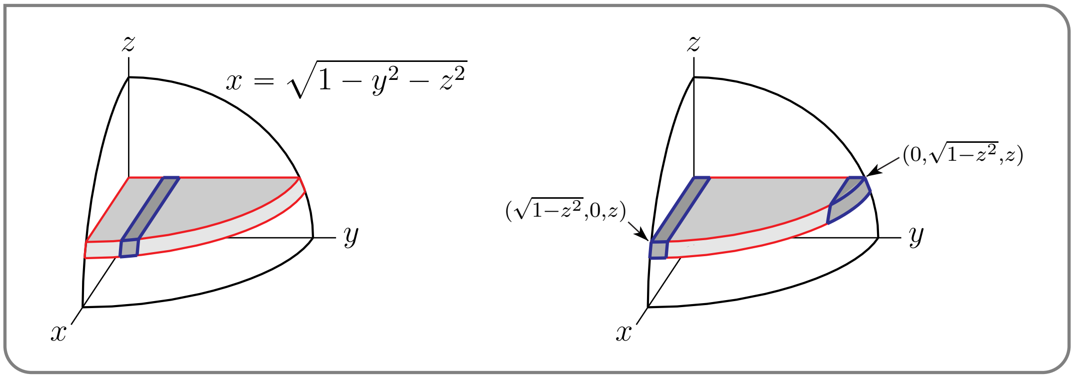
Solution. The absolute values can complicate the computations. We can avoid those complications by exploiting the fact that, by symmetry, the total mass of the sphere will be eight times the mass in the first octant. We shall cut the first octant part of the sphere into tiny pieces using Cartesian coordinates. That is, we shall cut it up using planes of constant z, planes of constant y, and planes of constant x, which we recall look like



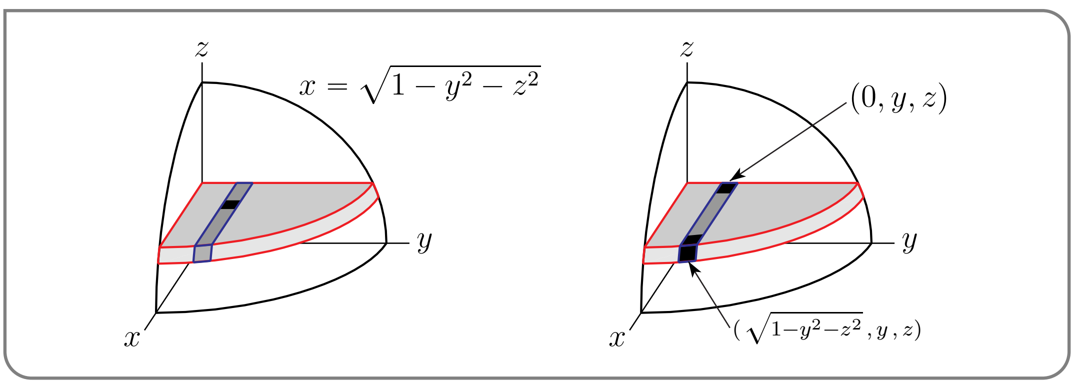
- First slice the (the first octant part of the) sphere into horizontal plates by inserting many planes of constant z, with the various values of z differing by dz. The figure on the left below shows the part of one plate in the first octant outlined in red. Each plate
  - has thickness dz,
  - has z almost constant throughout the plate (it only varies by dz), and
  - has (x, y) running over $x \geqslant 0$, $y \geqslant 0$, $x^2 + y^2 \leqslant 1 - z^2$.
  - The bottom plate starts at $z = 0$ and the top plate ends at $z = 1$. See the figure on the right below.



- Concentrate on any one plate. Subdivide it into long thin "square" beams by inserting many planes of constant y, with the various values of y differing by $\mathrm{d}y$. The figure on the left below shows the part of one beam in the first octant outlined in blue. Each beam
  - has cross-sectional area $\mathrm{d}y\,\mathrm{d}z$,
  - has z and y essentially constant throughout the beam, and
  - has x running over $0 \leqslant x \leqslant \sqrt{1-y^2-z^2}$.
  - The leftmost beam has, essentially, $y = 0$ and the rightmost beam has, essentially, $y = \sqrt{1-z^2}$. See the figure on the right below.



- Concentrate on any one beam. Subdivide it into tiny approximate cubes by inserting many planes of constant x, with the various values of x differing by $\mathrm{d}x$. The figure on the left below shows the top of one approximate cube in black. Each cube
  - has volume $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, and
  - has x, y and z all essentially constant throughout the cube.
  - The first cube has, essentially, $x = 0$ and the last cube has, essentially, $x = \sqrt{1-y^2-z^2}$. See the figure on the right below.



Now we can build up the mass.

- Concentrate on one approximate cube. Let's say that it contains the point (x, y, z).
  - The cube has volume essentially $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and
  - essentially has density $\rho(x, y, z) = xyz$ and so
  - essentially has mass $xyz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

- To get the mass of any one beam, say the beam whose y coordinate runs from y to $y + \mathrm{d}y$, we just add up the masses of the approximate cubes in that beam, by integrating x from its smallest value on the beam, namely 0, to its largest value on the beam, namely $\sqrt{1-y^2-z^2}$. The mass of the beam is thus

$$\mathrm{d}y\,\mathrm{d}z \int_0^{\sqrt{1-y^2-z^2}} \mathrm{d}x\; xyz$$

- To get the mass of any one plate, say the plate whose z coordinate runs from z to $z + \mathrm{d}z$, we just add up the masses of the beams in that plate, by integrating y from its smallest value on the plate, namely 0, to its largest value on the plate, namely $\sqrt{1-z^2}$. The mass of the plate is thus

$$\mathrm{d}z \int_0^{\sqrt{1-z^2}} \mathrm{d}y \int_0^{\sqrt{1-y^2-z^2}} \mathrm{d}x\; xyz$$

- To get the mass of the part of the sphere in the first octant, we just add up the masses of the plates that it contains, by integrating z from its smallest value in the octant, namely 0, to its largest value on the sphere, namely 1. The mass in the first octant is thus

$$\begin{aligned}
\int_0^1 \mathrm{d}z \int_0^{\sqrt{1-z^2}} \mathrm{d}y \int_0^{\sqrt{1-y^2-z^2}} \mathrm{d}x\; xyz &= \int_0^1 \mathrm{d}z \int_0^{\sqrt{1-z^2}} \mathrm{d}y\; yz \left[\int_0^{\sqrt{1-y^2-z^2}} \mathrm{d}x\; x\right] \\
&= \int_0^1 \mathrm{d}z \int_0^{\sqrt{1-z^2}} \mathrm{d}y\; \frac{1}{2} yz\big(1-y^2-z^2\big) \\
&= \int_0^1 \mathrm{d}z \int_0^{\sqrt{1-z^2}} \mathrm{d}y\; \left[\frac{z(1-z^2)}{2} y - \frac{z}{2} y^3\right] \\
&= \int_0^1 \mathrm{d}z\; \left[\frac{z(1-z^2)^2}{4} - \frac{z(1-z^2)^2}{8}\right] \\
&= \int_0^1 \mathrm{d}z\; z\, \frac{(1-z^2)^2}{8} \\
&= \int_1^0 \frac{\mathrm{d}u}{-2}\, \frac{u^2}{8} \qquad \text{with } u = 1-z^2,\ \mathrm{d}u = -2z\,\mathrm{d}z \\
&= \frac{1}{48}
\end{aligned}$$

- So the mass of the total (eight octant) sphere is $8 \times \frac{1}{48} = \frac{1}{6}$.

Example 3.5.1

Consider, for example, the limits of integration for the integral

$$\int_0^1 \mathrm{d}z \int_0^{\sqrt{1-z^2}} \mathrm{d}y \int_0^{\sqrt{1-y^2-z^2}} \mathrm{d}x\; xyz = \int_0^1 \left(\int_0^{\sqrt{1-z^2}} \left(\int_0^{\sqrt{1-y^2-z^2}} xyz\,\mathrm{d}x\right) \mathrm{d}y\right) \mathrm{d}z$$

that we have just evaluated.

- When we are integrating over the innermost integral, with respect to x, the quantities y and z are treated as constants. In particular, y and z may appear in the limits of integration for the x-integral, but x may not appear in those limits.
- When we are integrating over y, we have already integrated out x; x no longer exists. The quantity z is treated as a constant. In particular, z, but neither x nor y, may appear in the limits of integration for the y-integral.
- Finally, when we are integrating over z, we have already integrated out x and y; they no longer exist. None of x, y or z, may appear in the limits of integration for the z-integral.

Example 3.5.2

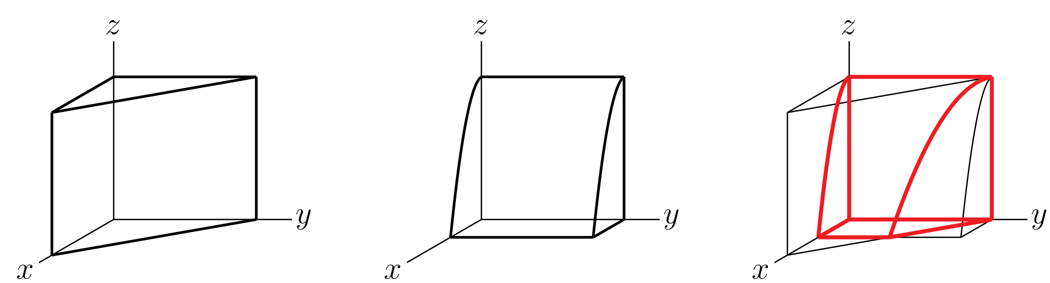
In practice, often the hardest part of dealing with a triple integral is setting up the limits of integration. In this example, we'll concentrate on exactly that.

Let $\mathcal{V}$ be the solid region in $\mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 0$, $y = 4 - x$, and the surface $z = 4 - x^2$. We are now going to write $\iiint_{\mathcal{V}} f(x, y, z)\ \mathrm{d}V$ as an iterated integral (i.e. find the limits of integration) in two different ways. Here f is just some general, unspecified, function.

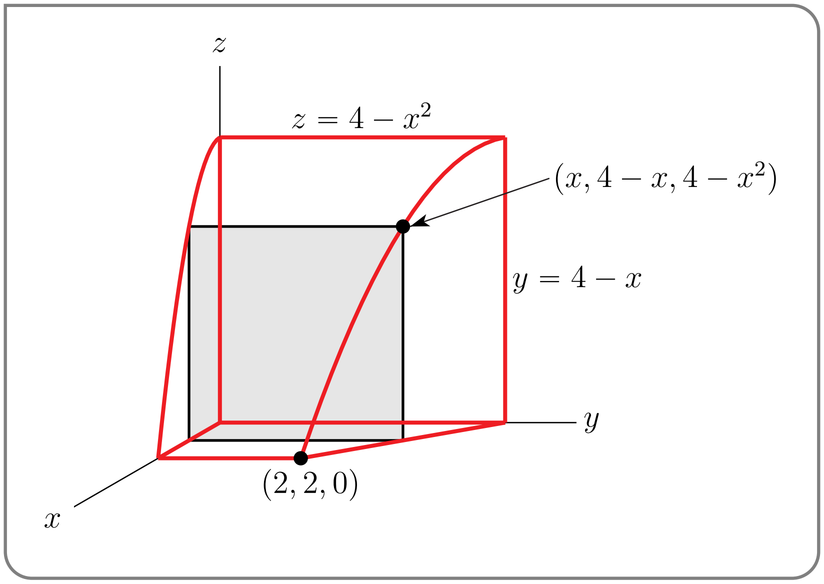
First, we'll figure out what $\mathcal{V}$ looks like. The following three figures show

- the part of the first octant with $y \leqslant 4 - x$ (except that it continues vertically upward)
- the part of the first octant with $z \leqslant 4 - x^2$ (except that it continues to the right)
- the part of the first octant with both $y \leqslant 4 - x$ and $z \leqslant 4 - x^2$. That's

$$\mathcal{V} = \left\{ (x, y, z) \,\middle|\, x \geqslant 0, y \geqslant 0,\ z \geqslant 0,\ x + y \leqslant 4,\ z \leqslant 4 - x^2 \right\}$$



The iterated integral $\iiint_{\mathcal{V}} f(x, y, z)\ \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x = \int \left(\int \left(\int f(x, y, z)\ \mathrm{d}z\right) \mathrm{d}y\right) \mathrm{d}x$*:* For this iterated integral, the outside integral is with respect to x, so we first slice up $\mathcal{V}$ using planes of constant x, as in the figure below.



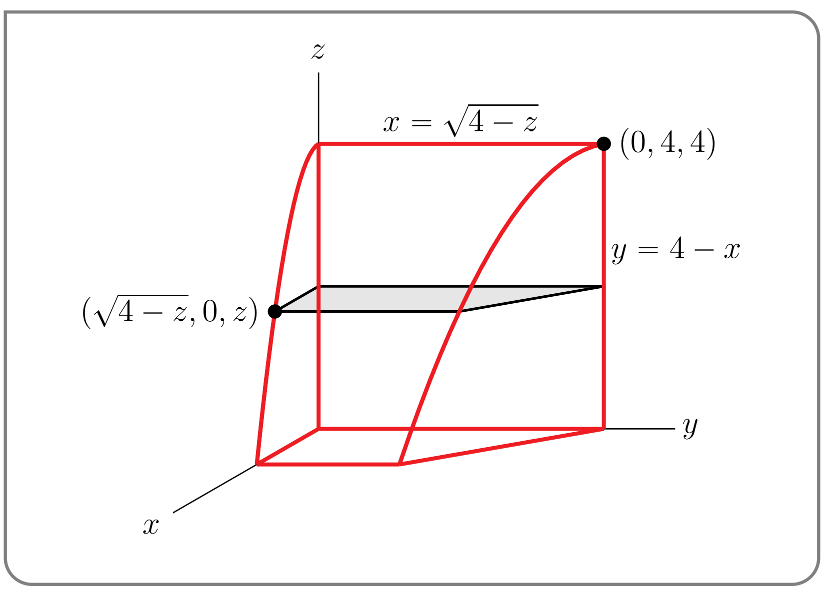
Observe from that figure that, on $\mathcal{V}$,

- x runs from 0 to 2, and
- for each fixed x in that range, y runs from 0 to $4 - x$ and
- for each fixed (x, y) as above, z runs from 0 to $4 - x^2$.

So

$$\iiint_{\mathcal{V}} f(x,y,z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x = \int_0^2 \mathrm{d}x \int_0^{4-x} \mathrm{d}y \int_0^{4-x^2} \mathrm{d}z\, f(x,y,z)$$
$$= \int_0^2 \int_0^{4-x} \int_0^{4-x^2} f(x,y,z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x$$

The iterated integral $\iiint_{\mathcal{V}} f(x,y,z)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z = \int \left(\int \left(\int f(x,y,z)\, \mathrm{d}y\right) \mathrm{d}x\right) \mathrm{d}z$*:* For this iterated integral, the outside integral is with respect to z, so we first slice up $\mathcal{V}$ using planes of constant z, as in the figure below.



Observe from that figure that, on $\mathcal{V}$,

- z runs from 0 to 4, and
- for each fixed z in that range, x runs from 0 to $\sqrt{4-z}$ and
- for each fixed (x, z) as above, y runs from 0 to $4 - x$.

So

$$\iiint_{\mathcal{V}} f(x,y,z)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z = \int_0^4 \mathrm{d}z \int_0^{\sqrt{4-z}} \mathrm{d}x \int_0^{4-x} \mathrm{d}y\; f(x,y,z)$$
$$= \int_0^4 \int_0^{\sqrt{4-z}} \int_0^{4-x} f(x,y,z)\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z$$

Example 3.5.2

Example 3.5.3

As was said in the last example, in practice, often the hardest parts of dealing with a triple integral concern the limits of integration. In this example, we'll again concentrate on exactly that. This time, we will consider the integral

$$I = \int_0^2 \mathrm{d}y \int_0^{2-y} \mathrm{d}z \int_0^{\frac{2-y}{2}} \mathrm{d}x\; f(x,y,z)$$

and we will reexpress I with the outside integral being over z. We will figure out the limits of integration for both the order $\int \mathrm{d}z \int \mathrm{d}x \int \mathrm{d}y\, f(x,y,z)$ and for the order $\int \mathrm{d}z \int \mathrm{d}y \int \mathrm{d}x\, f(x,y,z)$.

Our first task is to get a good idea as to what the domain of integration looks like. We start by reading off of the given integral that

- the outside integral says that y runs from 0 to 2, and
- the middle integral says that, for each fixed y in that range, z runs from 0 to $2 - y$ and
- the inside integral says that, for each fixed (y, z) as above, x runs from 0 to $\frac{2-y}{2}$.

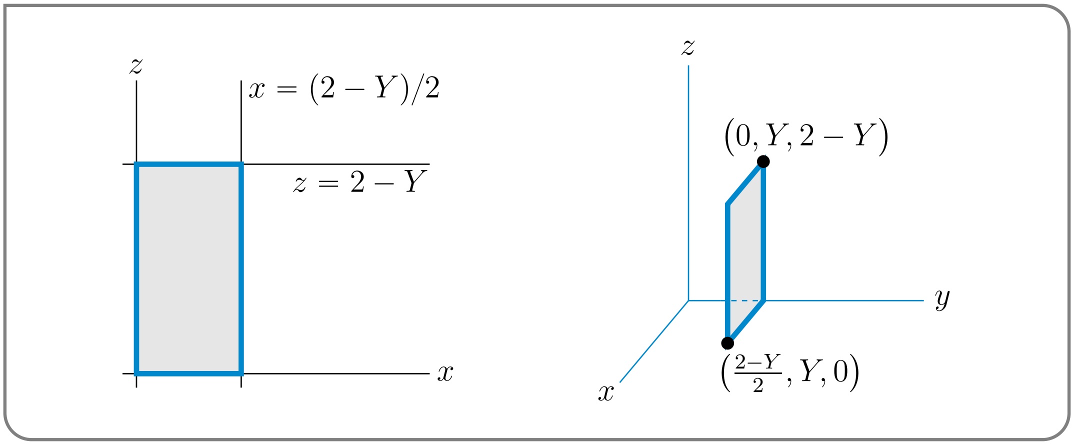
So the domain of integration is

$$V = \left\{\, (x,y,z) \;\middle|\; 0 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 2-y,\ 0 \leqslant x \leqslant \tfrac{2-y}{2} \,\right\} \tag{$*$}$$

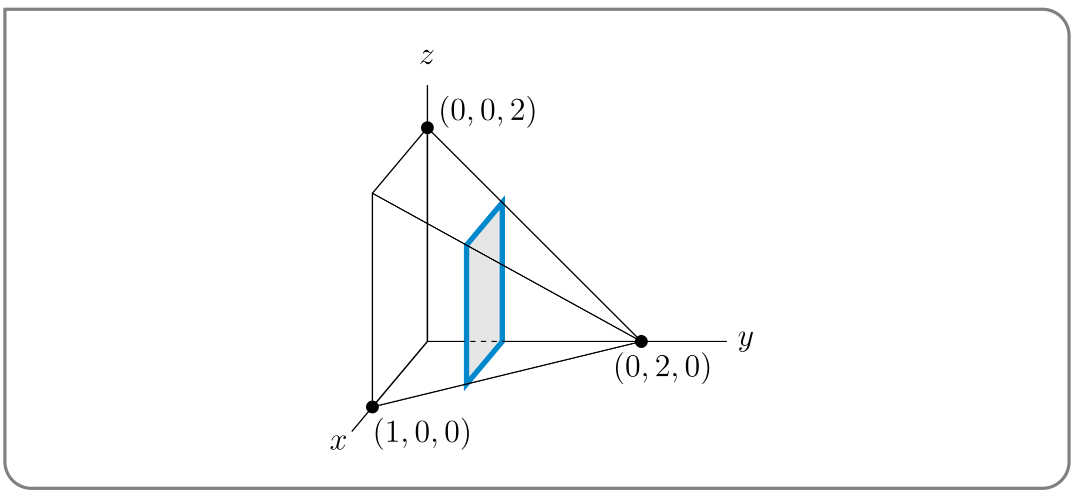
We'll sketch V shortly. Because it is generally easier to make 2d sketches than it is to make 3d sketches, we'll first make a 2d sketch of the part of V that lies in the vertical plane $y = Y$. Here Y is any constant between 0 and 2. Looking at the definition of V, we see that the point (x, Y, z) lies in V if and only if

$$0 \leqslant z \leqslant 2 - Y \qquad 0 \leqslant x \leqslant \frac{2-Y}{2}$$

Here, on the left, is a (2d) sketch of all (x, z)'s that obey those inequalities, and, on the right, is a (3d) sketch of all (x, Y, z)'s that obey those inequalities.



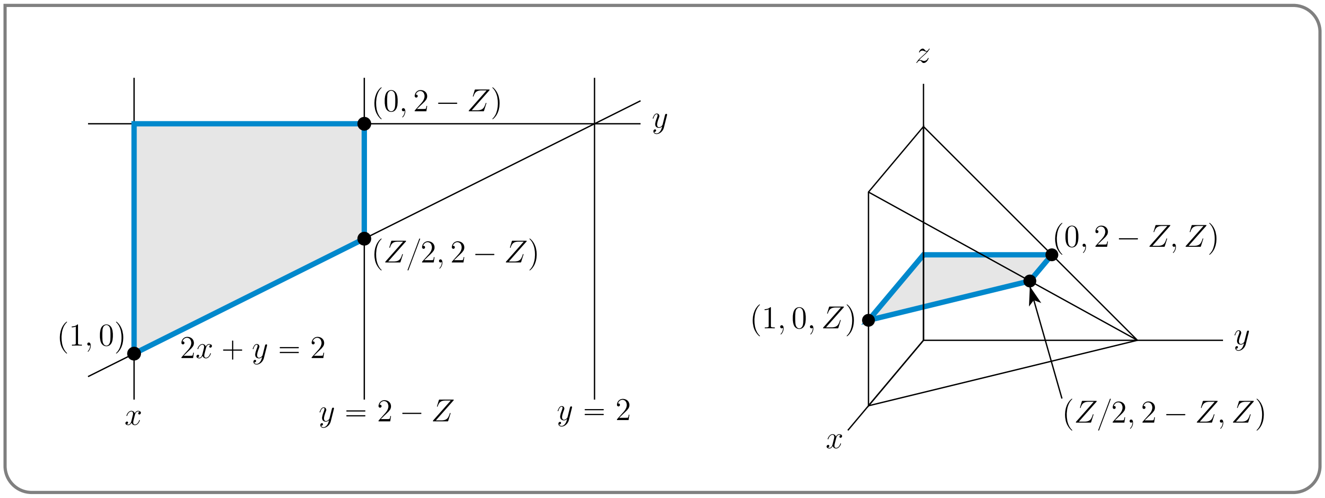
So our solid V consists of a bunch of vertical rectangles stacked sideways along the y-axis. The rectangle in the plane $y = Y$ has side lengths $\frac{2-Y}{2}$ and $2 - Y$. As we move from the plane $y = Y = 0$, i.e. the xz-plane, to the plane $y = Y = 2$, the rectangle decreases in size linearly from a one by two rectangle, when $Y = 0$, to a zero by zero rectangle, i.e. a point, when $Y = 2$. Here is a sketch of V together with a typical $y = Y$ rectangle.



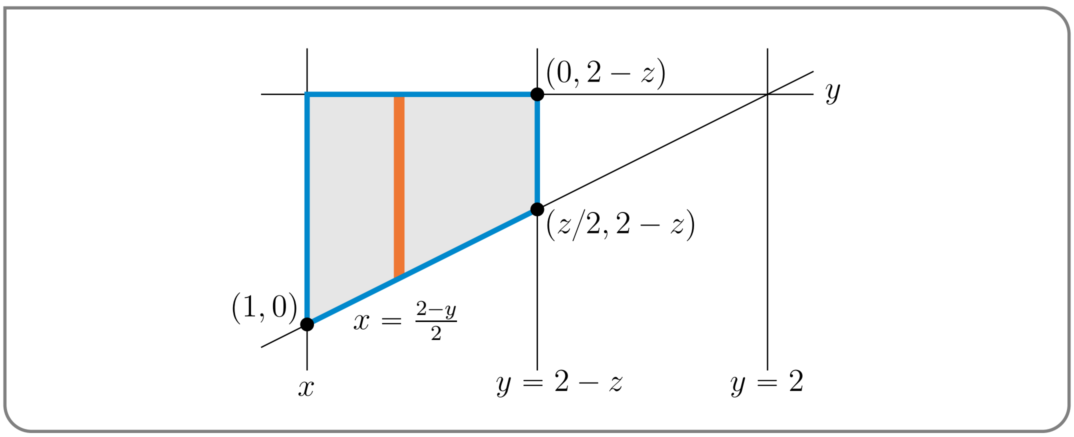
To reexpress the given integral with the outside integral being with respect to z, we have to slice up V into horizontal plates by inserting planes of constant z. So we have to figure out what the part of V that lies in the horizontal plane $z = Z$ looks like. From the figure above, we see that, in V, the smallest value of z is 0 and the biggest value of z is 2. So Z is any constant between 0 and 2. Again looking at the definition of V in $(*)$ above, we see that the point (x, y, Z) lies in V if and only if

$$y \geqslant 0 \qquad y \leqslant 2 \qquad y \leqslant 2 - Z \qquad x \geqslant 0 \qquad 2x + y \leqslant 2$$

Here, on the left, is a (2d) sketch showing the top view of all (x, y)'s that obey those inequalities, and, on the right, is a (3d) sketch of all (x, y, Z)'s that obey those inequalities.



To express I as an integral with the order of integration $\int \mathrm{d}z \int \mathrm{d}y \int \mathrm{d}x\, f(x, y, z)$, we subdivide the plate at height z into vertical strips as in the figure



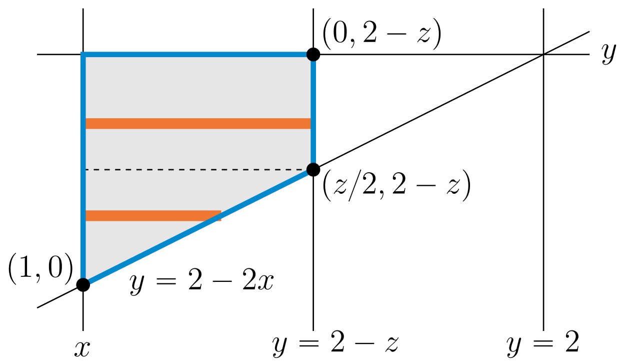
Since

- y is essentially constant on each strip with the leftmost strip having $y = 0$ and the rightmost strip having $y = 2 - z$ and
- for each fixed y in that range, x runs from 0 to $\frac{2-y}{2}$

we have

$$I = \int_0^2 \mathrm{d}z \int_0^{2-z} \mathrm{d}y \int_0^{\frac{2-y}{2}} \mathrm{d}x\, f(x, y, z)$$

Alternatively, to express I as an integral with the order of integration $\int \mathrm{d}z \int \mathrm{d}x \int \mathrm{d}y\, f(x, y, z)$, we subdivide the plate at height z into horizontal strips as in the figure



Since

- x is essentially constant on each strip with the first strip having $x = 0$ and the last strip having $x = 1$ and
- for each fixed x between 0 and $z/2$, y runs from 0 to $2 - z$ and
- for each fixed x between $z/2$ and 1, y runs from 0 to $2 - 2x$

we have

$$I = \int_0^2 \mathrm{d}z \int_0^{z/2} \mathrm{d}x \int_0^{2-z} \mathrm{d}y\, f(x,y,z) + \int_0^2 \mathrm{d}z \int_{z/2}^1 \mathrm{d}x \int_0^{2-2x} \mathrm{d}y\, f(x,y,z)$$

Example 3.5.3

3.6 Triple Integrals in Cylindrical Coordinates

Many problems possess natural symmetries. We can make our work easier by using coordinate systems, like polar coordinates, that are tailored to those symmetries. We will look at two more such coordinate systems — cylindrical and spherical coordinates.

3.6.1 Cylindrical Coordinates

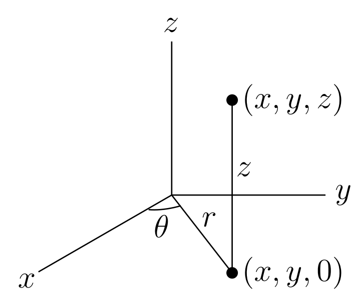
In the event that we wish to compute, for example, the mass of an object that is invariant under rotations about the z-axis[30], it is advantageous to use a natural generalization of polar coordinates to three dimensions. The coordinate system is called *cylindrical coordinates*.

30 like a pipe or a can of tuna fish

Definition 3.6.1.

Cylindrical coordinates are denoted[31] r, θ and z and are defined by

$$\begin{aligned} r &= \text{the distance from } (x,y,0) \text{ to } (0,0,0) \\ &= \text{the distance from } (x,y,z) \text{ to the } z\text{-axis} \\ \theta &= \text{the angle between the positive } x \text{ axis and the line joining } (x,y,0) \text{ to } (0,0,0) \\ z &= \text{the signed distance from } (x,y,z) \text{ to the } xy\text{-plane} \end{aligned}$$



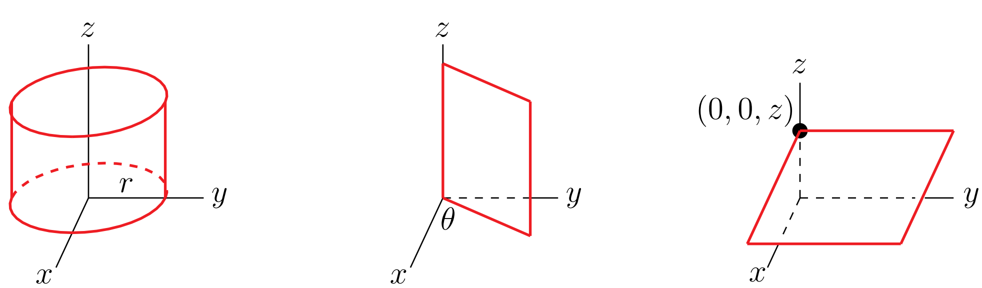
That is, r and θ are the usual polar coordinates and z is the usual z.

The Cartesian and cylindrical coordinates are related by[32]

Equation 3.6.2.

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta & z &= z \\ r &= \sqrt{x^2+y^2} & \theta &= \arctan\frac{y}{x} & z &= z \end{aligned}$$

Here are sketches of surfaces of constant r, constant θ, and constant z.



surface of constant r (a cylindrical shell) surface of constant θ (a plane) surface of constant z (a plane)

31 We are using the standard mathematics conventions for the cylindrical coordinates. Under the ISO conventions they are (ρ, ϕ, z). See Appendix G.

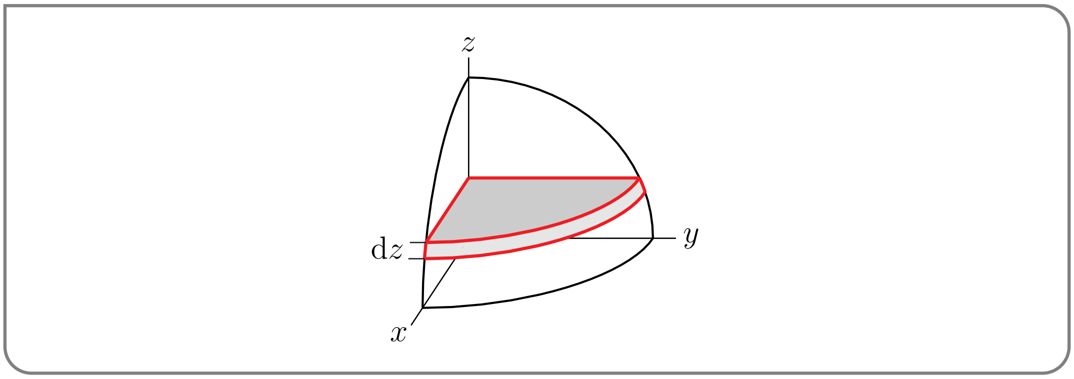
32 As was the case for polar coordinates, it is sometimes convenient to extend these definitions by saying that $x = r\cos\theta$ and $y = r\sin\theta$ even when r is negative. See the end of Section 3.2.1.

3.6.2 ▸▸ The Volume Element in Cylindrical Coordinates

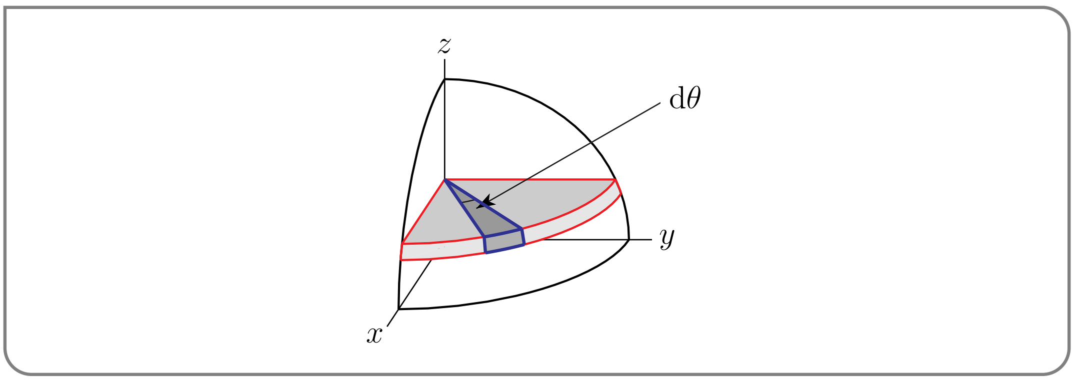
Before we can start integrating using these coordinates we need to determine the volume element. Recall that before integrating in polar coordinates, we had to establish that $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. In the arguments that follow we establish that $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$.

If we cut up a solid by

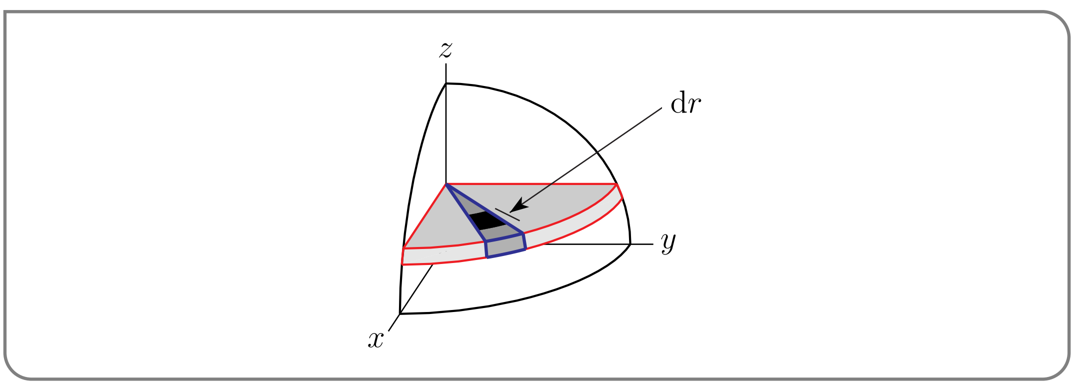
- first slicing it into horizontal plates of thickness $\mathrm{d}z$ by using planes of constant z,



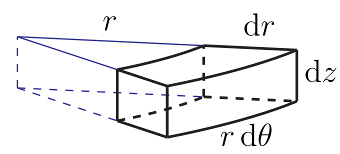
- and then subdividing the plates into wedges using surfaces of constant θ, say with the difference between successive θ's being $\mathrm{d}\theta$,



- and then subdividing the wedges into approximate cubes using surfaces of constant r, say with the difference between successive r's being $\mathrm{d}r$,



we end up with approximate cubes that look like



- When we introduced slices using surfaces of constant r, the difference between the successive r's was $\mathrm{d}r$, so the indicated edge of the cube has length $\mathrm{d}r$.
- When we introduced slices using surfaces of constant z, the difference between the successive z's was $\mathrm{d}z$, so the vertical edges of the cube have length $\mathrm{d}z$.
- When we introduced slices using surfaces of constant θ, the difference between the successive θ's was $\mathrm{d}\theta$, so the remaining edges of the cube are circular arcs of radius essentially[33] r that subtend an angle θ, and so have length $r\,\mathrm{d}\theta$. See the derivation of equation (3.2.5).

So the volume of the approximate cube in cylindrical coordinates is (essentially[34])

Equation 3.6.3.

$$\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

3.6.3 ▸▸ Sample Integrals in Cylindrical Coordinates

Now we can use (3.6.3) to handle a variant of Example 3.5.1 in which the density is invariant under rotations around the z-axis. Cylindrical coordinates are tuned to provide easier integrals to evaluate when the integrand is invariant under rotations about the z-axis, or when the domain of integration is cylindrical.

Example 3.6.4

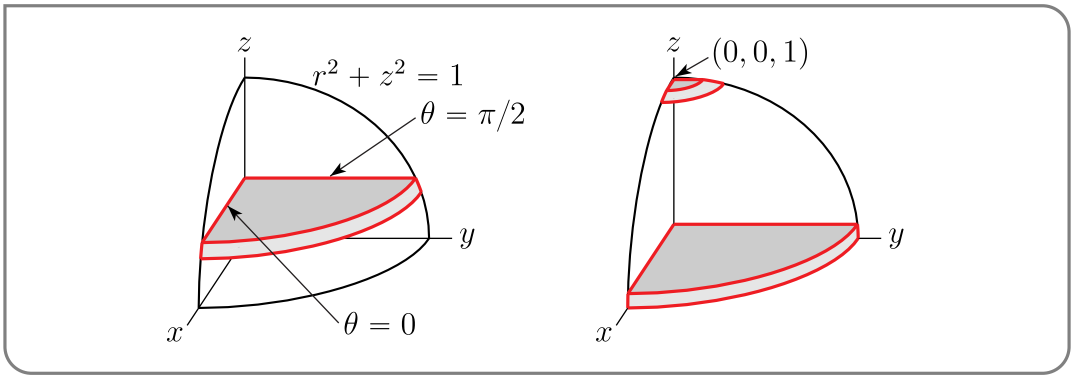
Find the mass of the solid body consisting of the inside of the sphere $x^2+y^2+z^2=1$ if the density is $\rho(x,y,z)=x^2+y^2$.

Solution. Before we get started, note that x^2+y^2 is the square of the distance from (x,y,z) to the z-axis. Consequently both the integrand, x^2+y^2, and the domain of integration, $x^2+y^2+z^2\leqslant 1$, and hence our solid, are invariant under rotations about the z-axis[35]. That makes this integral a good candidate for cylindrical coordinates.

Again, by symmetry the total mass of the sphere will be eight times the mass in the first octant. We shall cut the first octant part of the sphere into tiny pieces using cylindrical coordinates. That is, we shall cut it up using planes of constant z, planes of constant θ, and surfaces of constant r.

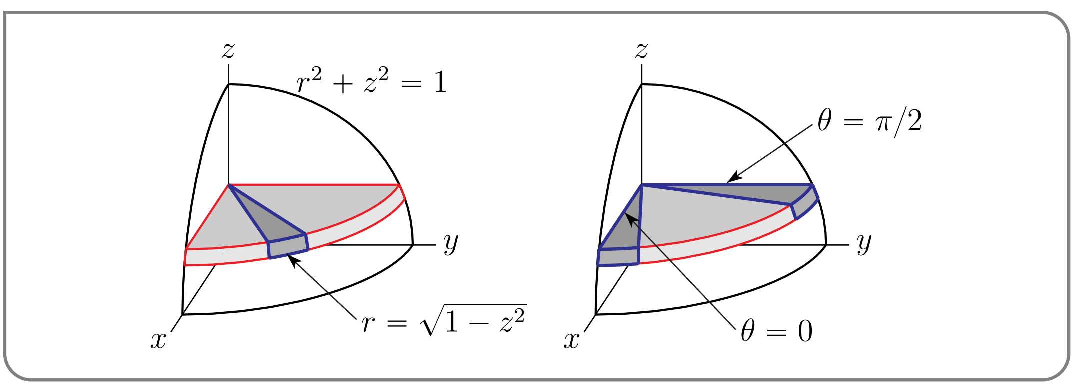
33 The inner edge has radius r, but the outer edge has radius $r+\mathrm{d}r$. However the error that this generates goes to zero in the limit $\mathrm{d}r, \mathrm{d}\theta, \mathrm{d}z \to 0$.
34 By "essentially", we mean that the formula for $\mathrm{d}V$ works perfectly when we take the limit $\mathrm{d}r, \mathrm{d}\theta, \mathrm{d}z \to 0$ of Riemann sums.
35 Imagine that you are looking that the solid from, for example, far out on the x-axis. You close your eyes for a minute. Your evil twin then sneaks in, rotates the solid about the z-axis, and sneaks out. You open your eyes. You will not be able to tell that the solid has been rotated.

- First slice the (the first octant part of the) sphere into horizontal plates by inserting many planes of constant z, with the various values of z differing by $\mathrm{d}z$. The figure on the left below shows the part of one plate in the first octant outlined in red. Each plate
  – has thickness $\mathrm{d}z$,
  – has z essentially constant on the plate, and
  – has (x, y) running over $x \geqslant 0$, $y \geqslant 0$, $x^2 + y^2 \leqslant 1 - z^2$. In cylindrical coordinates, r runs from 0 to $\sqrt{1 - z^2}$ and θ runs from 0 to $\pi/2$.
  – The bottom plate has, essentially, $z = 0$ and the top plate has, essentially, $z = 1$. See the figure on the right below.



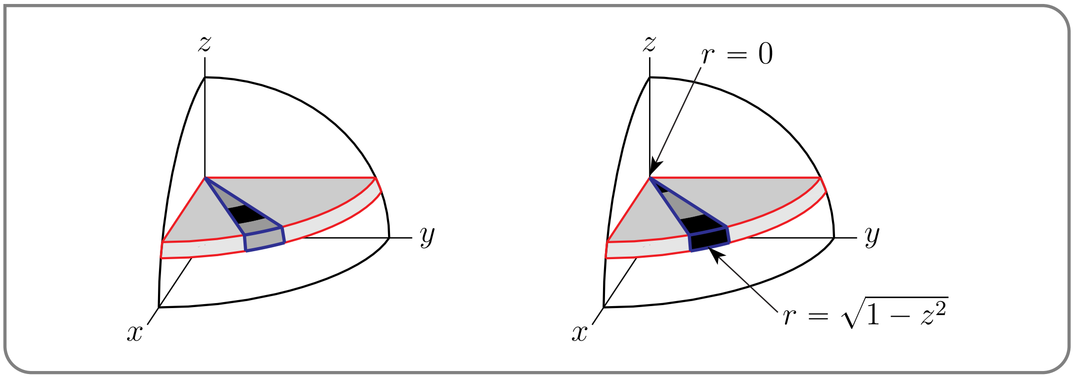
So far, this looks just like what we did in Example 3.5.1.

- Concentrate on any one plate. Subdivide it into wedges by inserting many planes of constant θ, with the various values of θ differing by $\mathrm{d}\theta$. The figure on the left below shows one such wedge outlined in blue. Each wedge
  – has z and θ essentially constant on the wedge, and
  – has r running over $0 \leqslant r \leqslant \sqrt{1 - z^2}$.
  – The leftmost wedge has, essentially, $\theta = 0$ and the rightmost wedge has, essentially, $\theta = \pi/2$. See the figure on the right below.



- Concentrate on any one wedge. Subdivide it into tiny approximate cubes by inserting many surfaces of constant r, with the various values of r differing by $\mathrm{d}r$. The figure on the left below shows the top of one approximate cube in black. Each cube
  – has volume $r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$, by (3.6.3), and

- has r, θ and z all essentially constant on the cube.
- The first cube has, essentially, $r = 0$ and the last cube has, essentially, $r = \sqrt{1-z^2}$. See the figure on the right below.



Now we can build up the mass.

- Concentrate on one approximate cube. Let's say that it contains the point with cylindrical coordinates r, θ and z.
  - The cube has volume essentially $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$ and
  - essentially has density $\rho(x,y,z) = \rho(r\cos\theta, r\sin\theta, z) = r^2$ and so
  - essentially has mass $r^3\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$. (See how nice the right coordinate system can be!)
- To get the mass any one wedge, say the wedge whose θ coordinate runs from θ to $\theta + \mathrm{d}\theta$, we just add up the masses of the approximate cubes in that wedge, by integrating r from its smallest value on the wedge, namely 0, to its largest value on the wedge, namely $\sqrt{1-z^2}$. The mass of the wedge is thus

$$\mathrm{d}\theta\,\mathrm{d}z \int_0^{\sqrt{1-z^2}} \mathrm{d}r\, r^3$$

- To get the mass of any one plate, say the plate whose z coordinate runs from z to $z + \mathrm{d}z$, we just add up the masses of the wedges in that plate, by integrating θ from its smallest value on the plate, namely 0, to its largest value on the plate, namely $\pi/2$. The mass of the plate is thus

$$\mathrm{d}z \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\sqrt{1-z^2}} \mathrm{d}r\, r^3$$

- To get the mass of the part of the sphere in the first octant, we just add up the masses of the plates that it contains, by integrating z from its smallest value in the octant, namely 0, to its largest value on the sphere, namely 1. The mass in the first octant is thus

$$\begin{aligned}\int_0^1 \mathrm{d}z \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\sqrt{1-z^2}} \mathrm{d}r\, r^3 &= \frac{1}{4}\int_0^1 \mathrm{d}z \int_0^{\pi/2} \mathrm{d}\theta\, (1-z^2)^2 \\ &= \frac{\pi}{8}\int_0^1 \mathrm{d}z\, (1-z^2)^2 = \frac{\pi}{8}\int_0^1 \mathrm{d}z\, (1-2z^2+z^4) \\ &= \frac{\pi}{8}\overbrace{\left[1-\frac{2}{3}+\frac{1}{5}\right]}^{8/15} \\ &= \frac{1}{15}\pi\end{aligned}$$

- So the mass of the total (eight octant) sphere is $8 \times \frac{1}{15}\pi = \frac{8}{15}\pi$.

Just by way of comparison, here is the integral in Cartesian coordinates that gives the mass in the first octant. (We found the limits of integration in Example 3.5.1.)

$$\int_0^1 \mathrm{d}z \int_0^{\sqrt{1-z^2}} \mathrm{d}y \int_0^{\sqrt{1-y^2-z^2}} \mathrm{d}x\, \left(x^2+y^2\right)$$

Example 3.6.4

In the next example, we compute the moment of inertia of a right circular cone. The Definition 3.3.13 of the moment of inertia was restricted to two dimensions. However, as was pointed out at the time, the same analysis extends naturally to the definition

Equation 3.6.5.

$$I_{\mathcal{A}} = \iiint_{\mathcal{V}} D(x,y,z)^2\, \rho(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

of the moment of inertia of a solid $\mathcal{V}$ in three dimensions. Here

- $\rho(x,y,z)$ is the mass density of the solid at the point (x,y,z) and
- $D(x,y,z)$ is the distance from (x,y,z) to the axis of rotation.

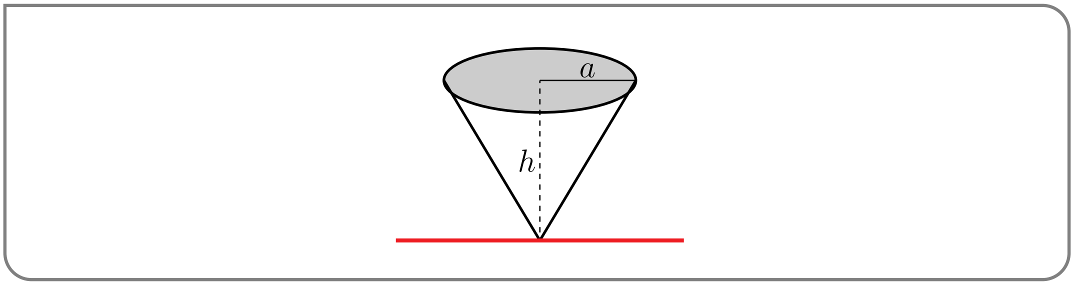
Example 3.6.6

Find the moment of inertia of a right circular cone

- of radius a,
- of height h, and
- of constant density with mass M

about an axis through the vertex (i.e. the tip of the cone) and parallel to the base.

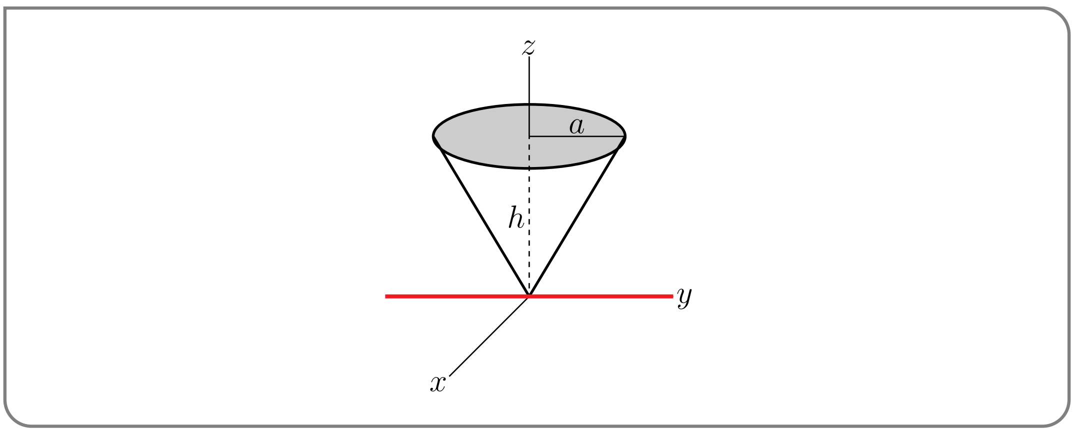
Solution. Here is a sketch of the cone.



Let's pick a coordinate system with

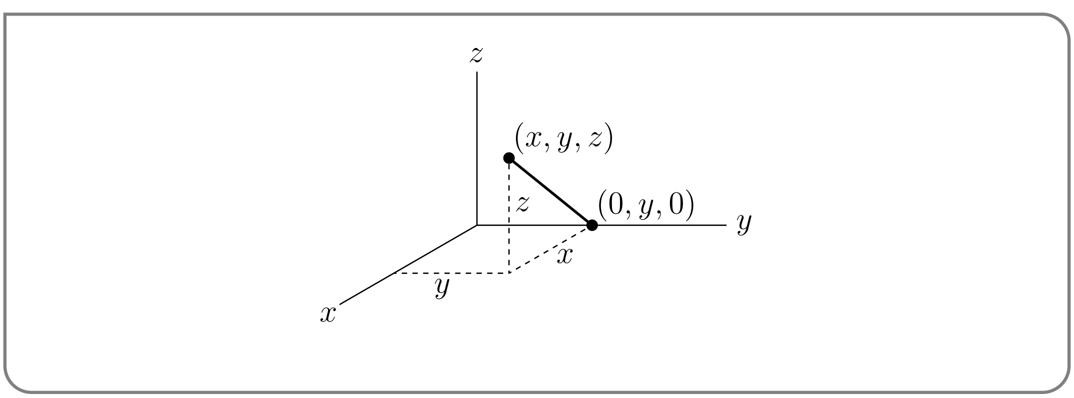
- the vertex at the origin,
- the cone symmetric about the z-axis and
- the axis of rotation being the y-axis.

and call the cone $\mathcal{V}$.



We shall use (3.6.5) to find the moment of inertia. In the current problem, the axis of rotation is the y-axis. The point on the y-axis that is closest to (x, y, z) is $(0, y, 0)$ so that the distance from (x, y, z) to the axis is just

$$D(x, y, z) = \sqrt{x^2 + z^2}$$



Our solid has constant density and mass M, so

$$\rho(x, y, z) = \frac{M}{\text{Volume}(\mathcal{V})}$$

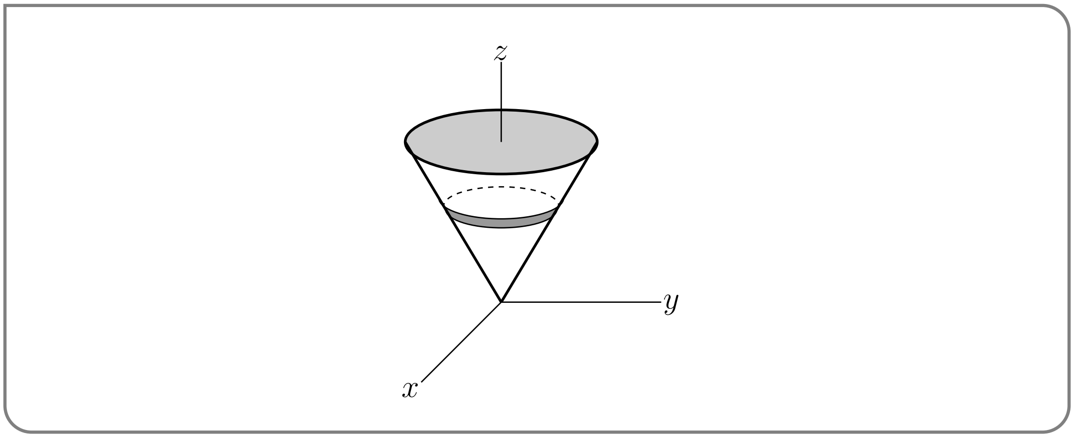
The formula

$$\text{Volume}(\mathcal{V}) = \frac{1}{3}\pi a^2 h$$

for the volume of a cone was derived in Example 1.6.1 of the CLP-2 text and in Appendix B.5.2 of the CLP-1 text. However because of the similarity between the integral Volume($\mathcal{V}$) $= \iiint_{\mathcal{V}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and the integral $\iiint_{\mathcal{V}}(x^2+z^2)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, that we need for our computation of $I_{\mathcal{A}}$, it is easy to rederive the volume formula and we shall do so.

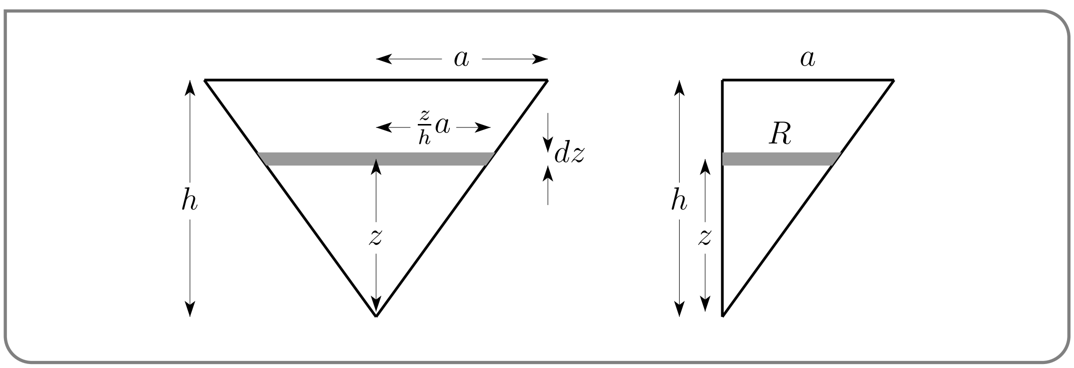
We'll evaluate both of the integrals above using cylindrical coordinates.

- Start by slicing the cone into horizontal plates by inserting many planes of constant z, with the various values of z differing by $\mathrm{d}z$.



  Each plate
  - is a circular disk of thickness $\mathrm{d}z$.
  - By similar triangles, as in the figure on the right below, the disk at height z has radius R obeying

$$\frac{R}{z} = \frac{a}{h} \implies R = \frac{a}{h}z$$



  - So the disk at height z has the cylindrical coordinates r running from 0 to $\frac{a}{h}z$ and θ running from 0 to 2π.
  - The bottom plate has, essentially, $z = 0$ and the top plate has, essentially, $z = h$.

- Now concentrate on any one plate. Subdivide it into wedges by inserting many planes of constant θ, with the various values of θ differing by $\mathrm{d}\theta$.
  - The first wedge has, essentially $\theta = 0$ and the last wedge has, essentially, $\theta = 2\pi$.

- Concentrate on any one wedge. Subdivide it into tiny approximate cubes[36] by inserting many surfaces of constant r, with the various values of r differing by $\mathrm{d}r$. Each cube
  - has volume $r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$, by (3.6.3).
  - The first cube has, essentially, $r = 0$ and the last cube has, essentially, $r = \frac{a}{h}z$.

So the two integrals of interest are

$$\begin{aligned}\iiint_{\mathcal{V}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^h \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{a}{h}z} \mathrm{d}r\, r \\ &= \int_0^h \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta\, \frac{1}{2}\left(\frac{a}{h}z\right)^2 = \frac{a^2\pi}{h^2}\int_0^h \mathrm{d}z\, z^2 \\ &= \frac{1}{3}\pi a^2 h\end{aligned}$$

as expected, and

$$\begin{aligned}\iiint_{\mathcal{V}} (x^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^h \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{a}{h}z} \mathrm{d}r\, r\, \overbrace{\left(r^2\cos^2\theta + z^2\right)}^{x^2+z^2} \\ &= \int_0^h \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \left[\frac{1}{4}\left(\frac{a}{h}z\right)^4 \cos^2\theta + \frac{1}{2}\left(\frac{a}{h}z\right)^2 z^2\right] \\ &= \int_0^h \mathrm{d}z \left[\frac{1}{4}\frac{a^4}{h^4} + \frac{a^2}{h^2}\right]\pi z^4 \\ &\qquad \text{since } \int_0^{2\pi} \cos^2\theta\,\mathrm{d}\theta = \pi \text{ by Remark 3.3.5} \\ &= \frac{1}{5}\left[\frac{1}{4}\frac{a^4}{h^4} + \frac{a^2}{h^2}\right]\pi h^5\end{aligned}$$

Putting everything together, the moment of inertia is

$$\begin{aligned}I_{\mathcal{A}} &= \iiint_{\mathcal{V}} \overbrace{(x^2+z^2)}^{D(x,y,z)^2}\, \overbrace{\frac{M}{\frac{1}{3}\pi a^2 h}}^{\rho(x,y,z)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= 3\frac{M}{\pi a^2 h}\,\frac{1}{5}\left[\frac{1}{4}\frac{a^4}{h^4} + \frac{a^2}{h^2}\right]\pi h^5 \\ &= \frac{3}{20}M\big(a^2 + 4h^2\big)\end{aligned}$$

Example 3.6.6

36 Again they are wonky cubes, but we can bound the error and show that it goes to zero in the limit $\mathrm{d}r, \mathrm{d}\theta, \mathrm{d}z \to 0$.

3.7▲ Triple Integrals in Spherical Coordinates

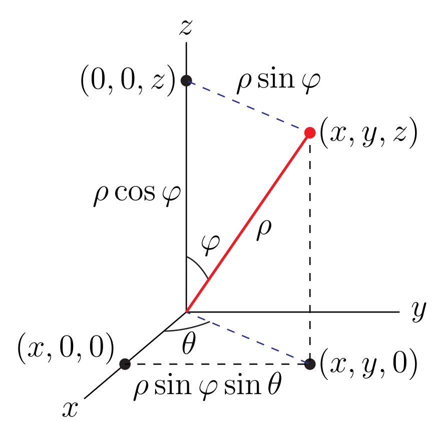
3.7.1 ▸▸ Spherical Coordinates

In the event that we wish to compute, for example, the mass of an object that is invariant under rotations about the origin, it is advantageous to use another generalization of polar coordinates to three dimensions. The coordinate system is called *spherical coordinates*.

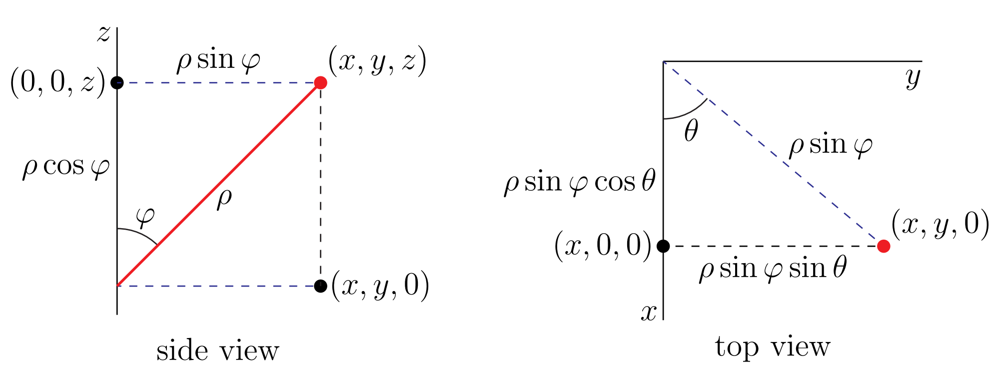
Definition 3.7.1.

Spherical coordinates are denoted[37] ρ, θ and φ and are defined by

$$\begin{aligned} \rho &= \text{ the distance from } (0,0,0) \text{ to } (x,y,z) \\ \varphi &= \text{ the angle between the } z \text{ axis and the line joining } (x,y,z) \text{ to } (0,0,0) \\ \theta &= \text{ the angle between the } x \text{ axis and the line joining } (x,y,0) \text{ to } (0,0,0) \end{aligned}$$



Here are two more figures giving the side and top views of the previous figure.



37 We are using the standard mathematics conventions for the spherical coordinates. Under the ISO conventions they are (r, ϕ, θ). See Appendix G.

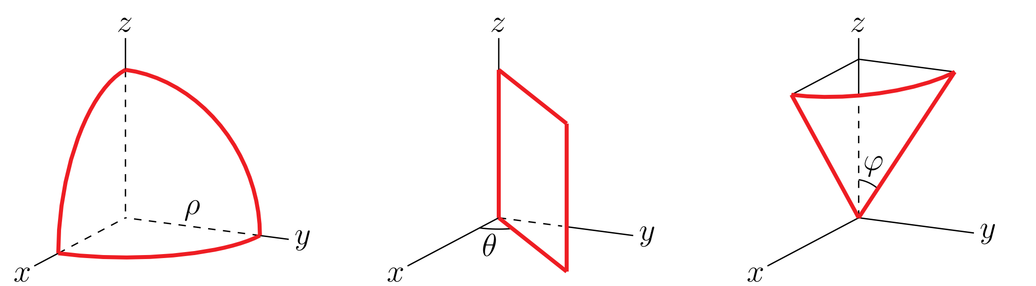
The spherical coordinate θ is the same as the cylindrical coordinate θ. The spherical coordinate φ is new. It runs from 0 (on the positive z-axis) to π (on the negative z-axis). The Cartesian and spherical coordinates are related by

Equation 3.7.2.

$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho\cos\varphi$$
$$\rho = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\frac{y}{x} \qquad \varphi = \arctan\frac{\sqrt{x^2+y^2}}{z}$$

Here are three figures showing

- a surface of constant ρ, i.e. a surface $x^2+y^2+z^2=\rho^2$ with ρ a constant (which looks like an onion skin),
- a surface of constant θ, i.e. a surface $y = x\,\tan\theta$ with[38] θ a constant (which looks like the page of a book), and
- a surface of constant φ, i.e. a surface $z = \sqrt{x^2+y^2}\,\tan\varphi$ with φ a constant (which looks a conical funnel).



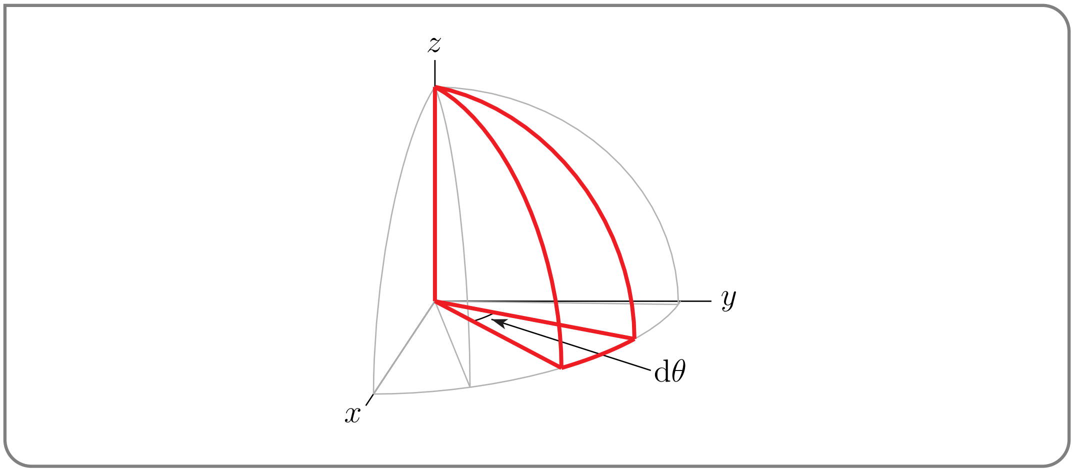
surface of constant ρ (a sphere) — surface of constant θ (a plane) — surface of constant φ (a cone)

3.7.2 ▸▸ The Volume Element in Spherical Coordinates

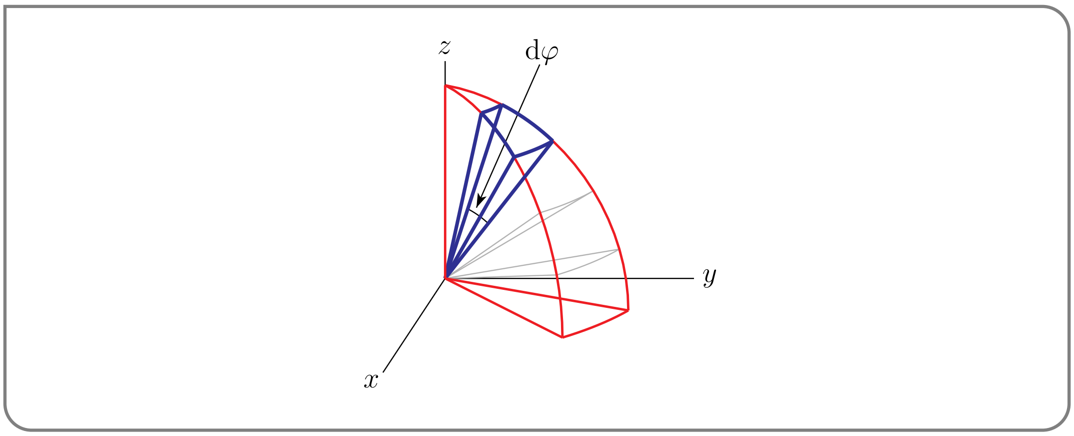
If we cut up a solid[39] by

- first slicing it into segments (like segments of an orange) by using planes of constant θ, say with the difference between successive θ's being $\mathrm{d}\theta$,

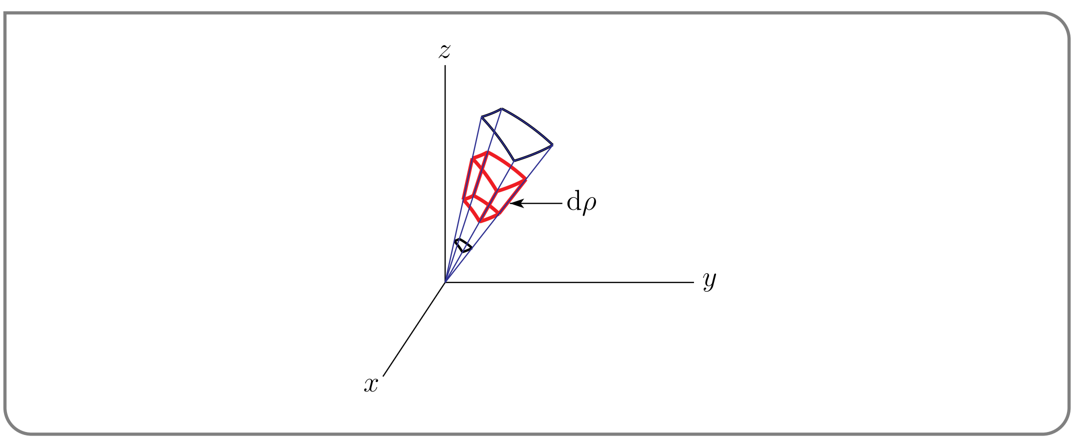
38 and with the sign of x being the same as the sign of $\cos\theta$
39 You know the drill.



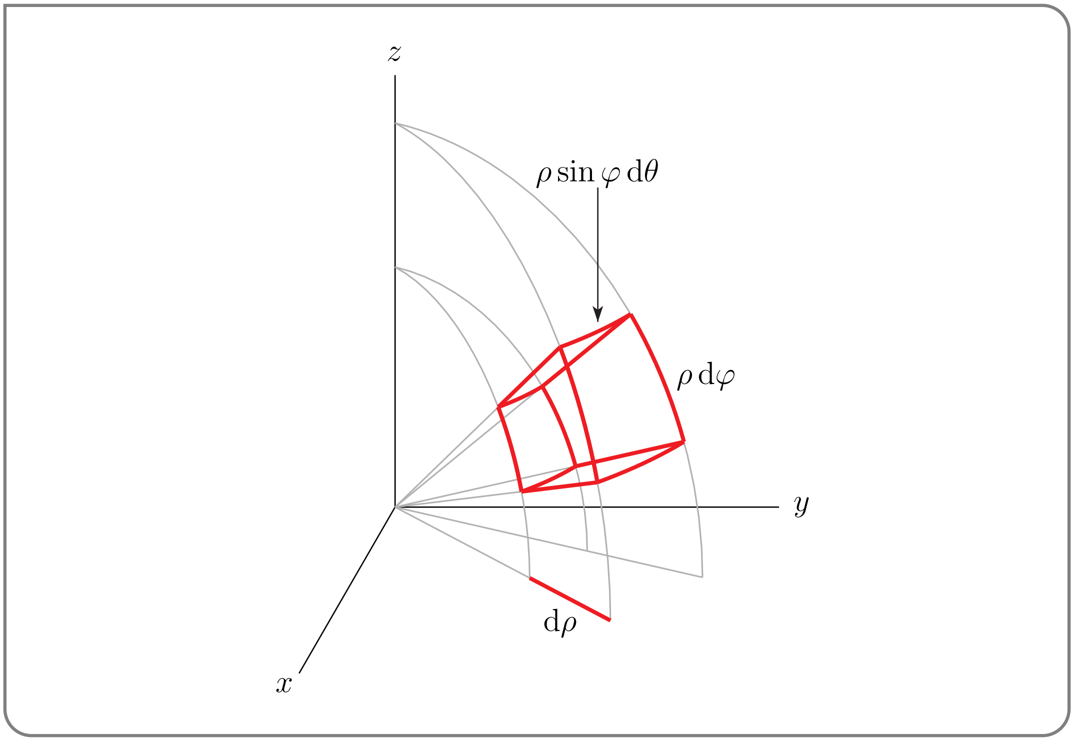
- and then subdividing the segments into "searchlights" (like the searchlight outlined in blue in the figure below) using surfaces of constant φ, say with the difference between successive φ's being $d\varphi$,



- and then subdividing the searchlights into approximate cubes using surfaces of constant ρ, say with the difference between successive ρ's being $d\rho$,



we end up with approximate cubes that look like



The dimensions of the approximate "cube" in spherical coordinates are (essentially) $\mathrm{d}\rho$ by $\rho\mathrm{d}\varphi$ by $\rho \sin\varphi\, \mathrm{d}\theta$. (These dimensions are derived in more detail in the next section.) So the approximate cube has volume (essentially)

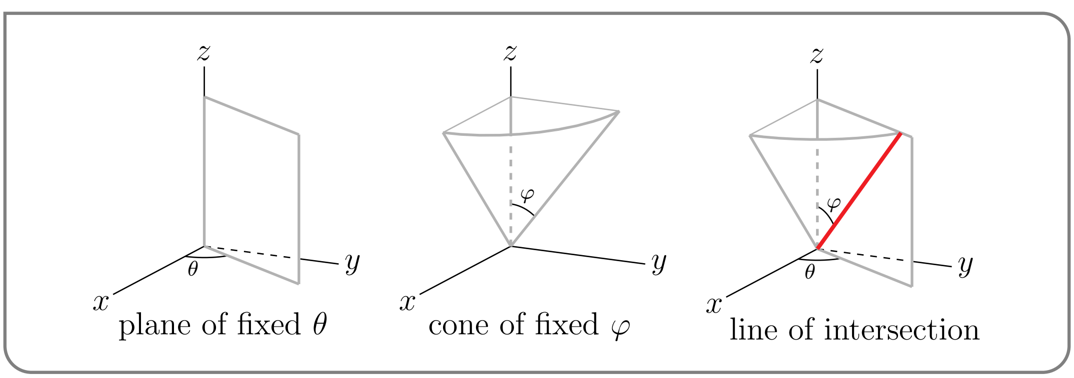
Equation 3.7.3.

$$\mathrm{d}V = \rho^2 \sin\varphi\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\varphi$$

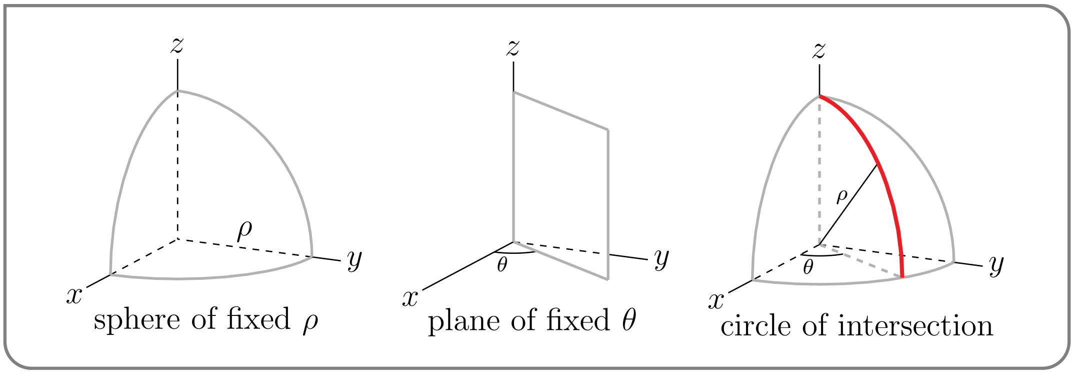
▸▸▸ The Details

Here is an explanation of the edge lengths given in the above figure. Each of the 12 edges of the cube is formed by holding two of the three coordinates ρ, θ, φ fixed and varying the third.

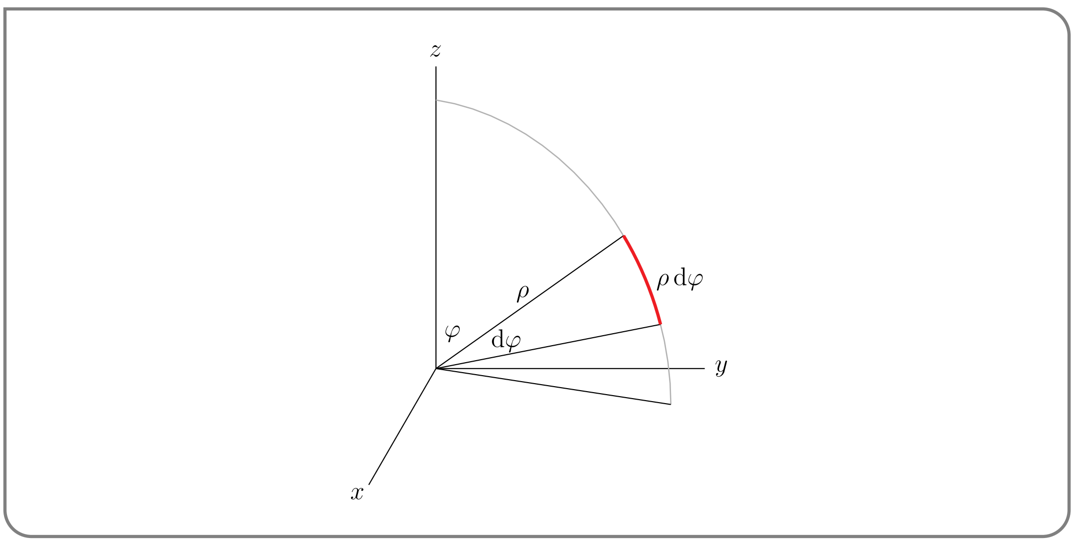
- Four of the cube edges are formed by holding θ and φ fixed and varying ρ. The intersection of a plane of fixed θ with a cone of fixed φ is a straight line emanating from the origin. When we introduced slices using spheres of constant ρ, the difference between the successive ρ's was $\mathrm{d}\rho$, so those edges of the cube each have length $\mathrm{d}\rho$.



- Four of the cube edges are formed by holding θ and ρ fixed and varying φ. The intersection of a plane of fixed θ (which contains the origin) with a sphere of fixed ρ (which is centred on the origin) is a circle of radius ρ centred on the origin. It is a line of longitude[40].

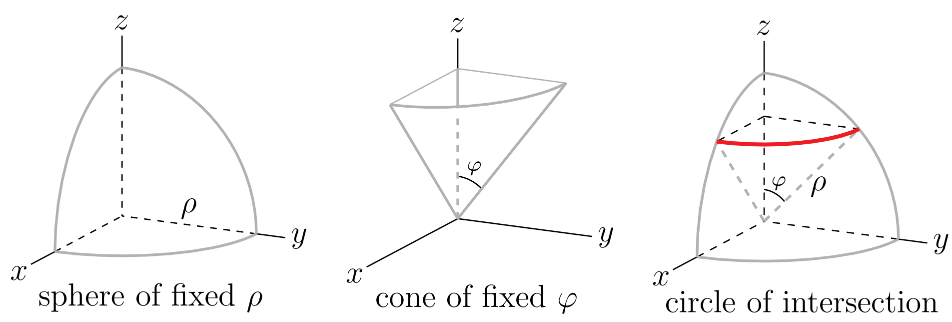


When we introduced searchlights using surfaces of constant φ, the difference between the successive φ's was $d\varphi$. Thus those four edges of the cube are circular arcs of radius essentially ρ that subtend an angle $d\varphi$, and so have length $\rho\, d\varphi$.

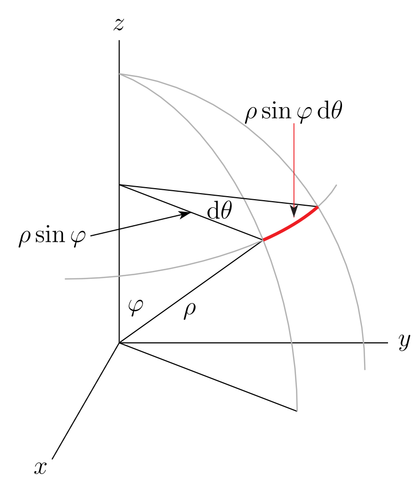


- Four of the cube edges are formed by holding φ and ρ fixed and varying θ. The intersection of a cone of fixed φ with a sphere of fixed ρ is a circle. As both ρ and φ are fixed, the circle of intersection lies in the plane $z = \rho\cos\varphi$. It is a line of latitude. The circle has radius $\rho\sin\varphi$ and is centred on $\big(0, 0, \rho\cos\varphi\big)$.

40 The problem of finding a practical, reliable method for determining the longitude of a ship at sea was a very big deal for a period of several centuries. Among the scientists who worked in this were Galileo, Edmund Halley (of Halley's comet) and Robert Hooke (of Hooke's law).



When we introduced segments using surfaces of constant θ, the difference between the successive θ's was $\mathrm{d}\theta$. Thus these four edge of the cube are circular arcs of radius essentially $\rho \sin\varphi$ that subtend an angle $\mathrm{d}\theta$, and so have length $\rho\sin\varphi\,\mathrm{d}\theta$.



3.7.3 ▸▸ Sample Integrals in Spherical Coordinates

Example 3.7.4 (Ice Cream Cone)

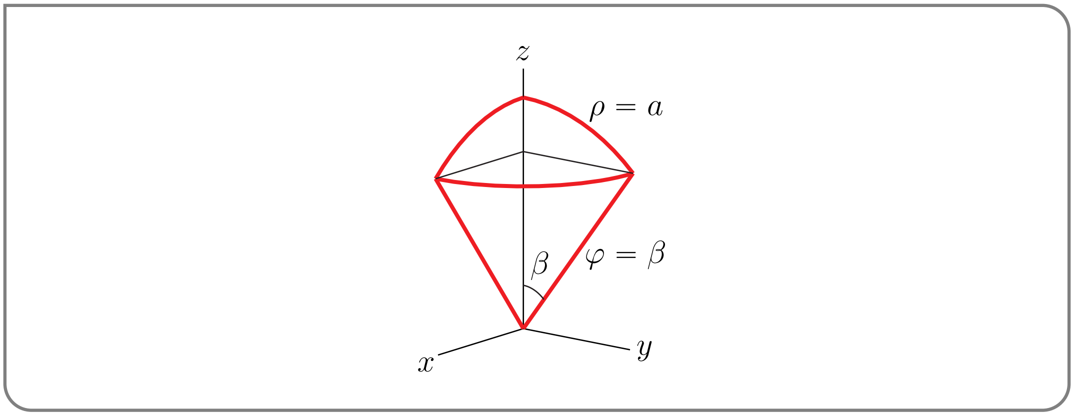
Find the volume of the ice cream[41] cone that consists of the part of the interior of the sphere $x^2+y^2+z^2=a^2$ that is above the xy-plane and that is inside the cone $x^2+y^2=b^2z^2$. Here a and b are any two strictly positive constants.

Solution. Note that, in spherical coordinates

$$x^2+y^2=\rho^2\sin^2\varphi \qquad z^2=\rho^2\cos^2\varphi \qquad x^2+y^2+z^2=\rho^2$$

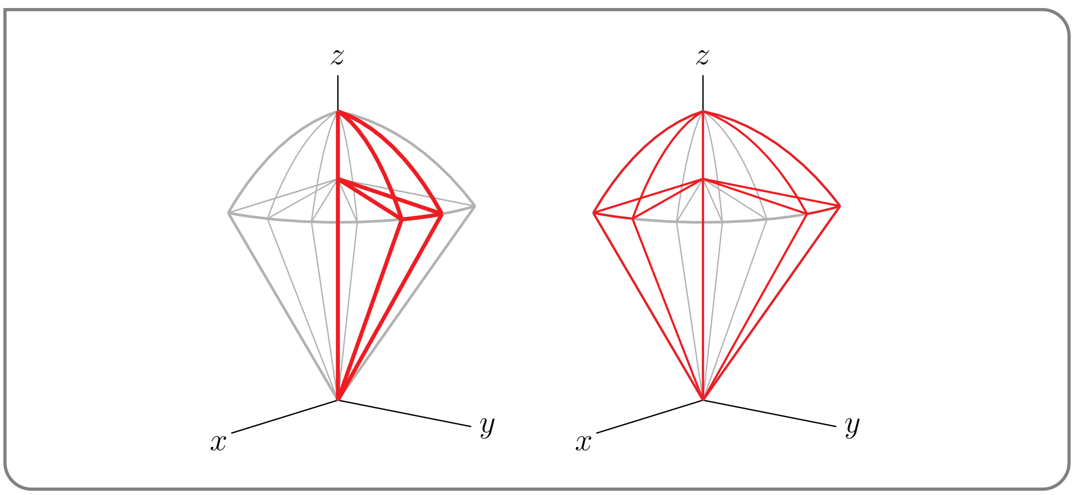
Consequently, in spherical coordinates, the equation of the sphere is $\rho=a$, and the equation of the cone is $\tan^2\varphi=b^2$. Let's write $\beta=\arctan b$, with $0<\beta<\pi/2$. Here is a sketch of the part of the ice cream cone in the first octant. The volume of the full ice cream cone will be four times the volume of the part in the first octant.

41 A very mathematical ice cream. Rocky-rho'd? Choculus?

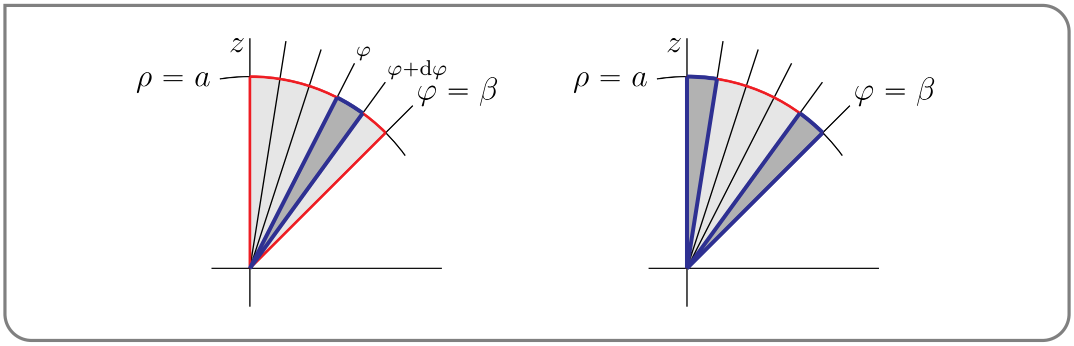


We shall cut the first octant part of the ice cream cone into tiny pieces using spherical coordinates. That is, we shall cut it up using planes of constant θ, cones of constant φ, and spheres of constant ρ.

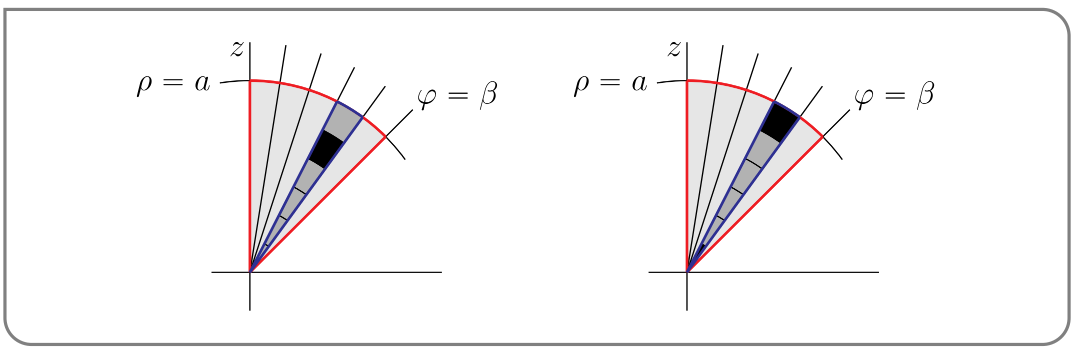
- First slice the (the first octant part of the) ice cream cone into segments by inserting many planes of constant θ, with the various values of θ differing by $d\theta$. The figure on the left below shows one segment outlined in red. Each segment
  - has θ essentially constant on the segment, and
  - has φ running from 0 to β and ρ running from 0 to a.
  - The leftmost segment has, essentially, $\theta = 0$ and the rightmost segment has, essentially, $\theta = {}^{\pi}/_{2}$. See the figure on the right below.



- Concentrate on any one segment. A side view of the segment is sketched in the figure on the left below. Subdivide it into long thin searchlights by inserting many cones of constant φ, with the various values of φ differing by $d\varphi$. The figure on the left below shows one searchlight outlined in blue. Each searchlight
  - has θ and φ essentially constant on the searchlight, and
  - has ρ running over $0 \leqslant \rho \leqslant a$.
  - The leftmost searchlight has, essentially, $\varphi = 0$ and the rightmost searchlight has, essentially, $\varphi = \beta$. See the figure on the right below.



- Concentrate on any one searchlight. Subdivide it into tiny approximate cubes by inserting many spheres of constant ρ, with the various values of ρ differing by $\mathrm{d}\rho$. The figure on the left below shows the side view of one approximate cube in black. Each cube
  - has ρ, θ and φ all essentially constant on the cube and
  - has volume $\rho^2 \sin\varphi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\varphi$, by (3.7.3).
  - The first cube has, essentially, $\rho = 0$ and the last cube has, essentially, $\rho = a$. See the figure on the right below.



Now we can build up the volume.

- Concentrate on one approximate cube. Let's say that it contains the point with spherical coordinates ρ, θ, φ. The cube has volume essentially $\mathrm{d}V = \rho^2 \sin\varphi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\varphi$, by (3.7.3).

- To get the volume any one searchlight, say the searchlight whose φ coordinate runs from φ to $\varphi + \mathrm{d}\varphi$, we just add up the volumes of the approximate cubes in that searchlight, by integrating ρ from its smallest value on the searchlight, namely 0, to its largest value on the searchlight, namely a. The volume of the searchlight is thus

$$\mathrm{d}\theta \, \mathrm{d}\varphi \int_0^a \mathrm{d}\rho \; \rho^2 \sin\varphi$$

- To get the volume of any one segment, say the segment whose θ coordinate runs from θ to $\theta + \mathrm{d}\theta$, we just add up the volumes of the searchlights in that segment, by integrating φ from its smallest value on the segment, namely 0, to its largest value on the segment, namely β. The volume of the segment is thus

$$\mathrm{d}\theta \int_0^\beta \mathrm{d}\varphi \; \sin\varphi \int_0^a \mathrm{d}\rho \; \rho^2$$

- To get the volume of $\mathcal{V}_1$, the part of the ice cream cone in the first octant, we just add up the volumes of the segments that it contains, by integrating θ from its smallest value in the octant, namely 0, to its largest value on the octant, namely $\pi/2$.

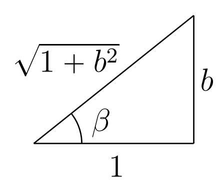
- The volume in the first octant is thus

$$\begin{aligned}\text{Volume}(\mathcal{V}_1) &= \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\beta} \mathrm{d}\varphi\ \sin\varphi \int_0^a \mathrm{d}\rho\ \rho^2 \\ &= \frac{a^3}{3}\int_0^{\pi/2} \mathrm{d}\theta \int_0^{\beta} \mathrm{d}\varphi\ \sin\varphi \\ &= \frac{a^3}{3}\big[1-\cos\beta\big]\int_0^{\pi/2} \mathrm{d}\theta \\ &= \frac{\pi a^3}{6}\big[1-\cos\beta\big]\end{aligned}$$

- So the volume of $\mathcal{V}$, the total (four octant) ice cream cone, is

$$\text{Volume}(\mathcal{V}) = 4\,\text{Volume}(\mathcal{V}_1) = \frac{4\pi a^3}{6}\big[1-\cos\beta\big]$$

We can express β (which was not given in the statement of the original problem) in terms of b (which was in the statement of the original problem), just by looking at the triangle



The right hand and bottom sides of the triangle have been chosen so that $\tan\beta = b$, which was the definition of β. So $\cos\beta = \frac{1}{\sqrt{1+b^2}}$ and the volume of the ice cream cone is

$$\text{Volume}(\mathcal{V}) = \frac{2\pi a^3}{3}\left[1 - \frac{1}{\sqrt{1+b^2}}\right]$$

Note that, as in Example 3.2.11, we can easily apply a couple of sanity checks to our answer.

- If $b=0$, so that the cone is just $x^2+y^2=0$, which is the line $x=y=0$, the total volume should be zero. Our answer does indeed give 0 in this case.

- In the limit $b\to\infty$, the angle $\beta\to\pi/2$ and the ice cream cone opens up into a hemisphere of radius a. Our answer does indeed give the volume of the hemisphere, which is $\frac{1}{2}\times\frac{4}{3}\pi a^3$.

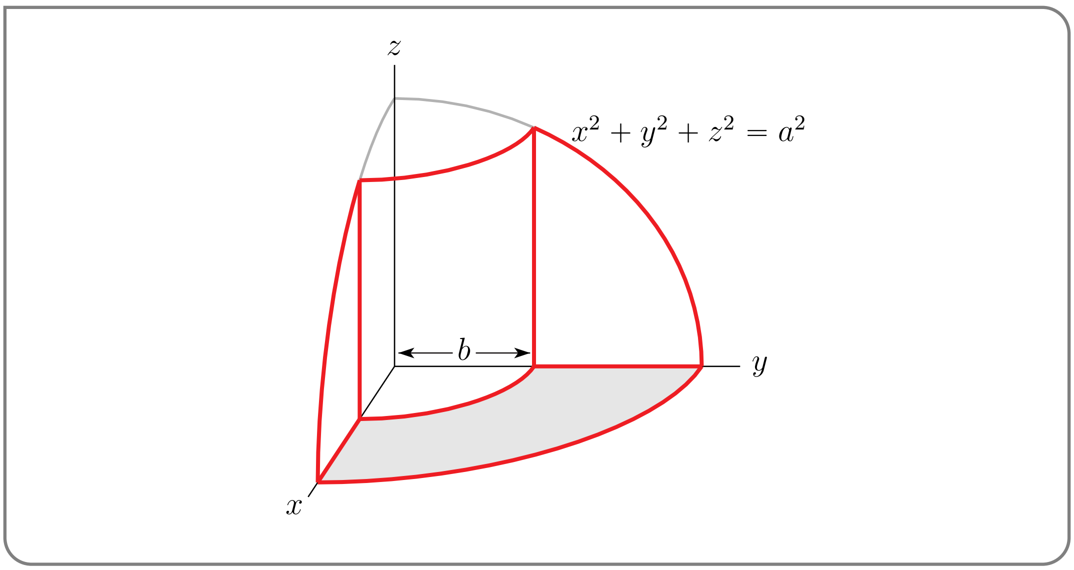
Example 3.7.4

Example 3.7.5 (Cored Apple)

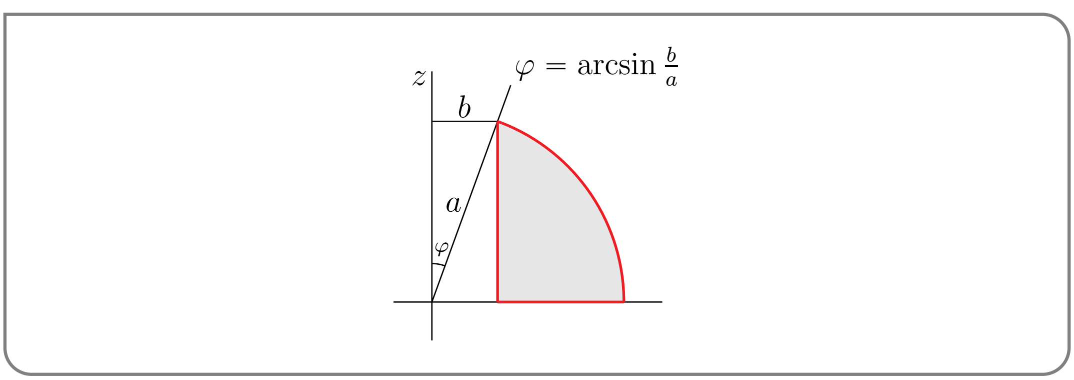
A cylindrical hole of radius b is drilled symmetrically through a perfectly spherical apple of radius $a \geqslant b$. Find the volume of apple that remains.

Solution. In Example 3.2.11 we computed the volume removed, basically using cylindrical coordinates. So we could get the answer to this question just by subtracting the answer of Example 3.2.11 from $\frac{4}{3}\pi a^3$. Instead, we will evaluate the volume remaining as an exercise in setting up limits of integration when using spherical coordinates.

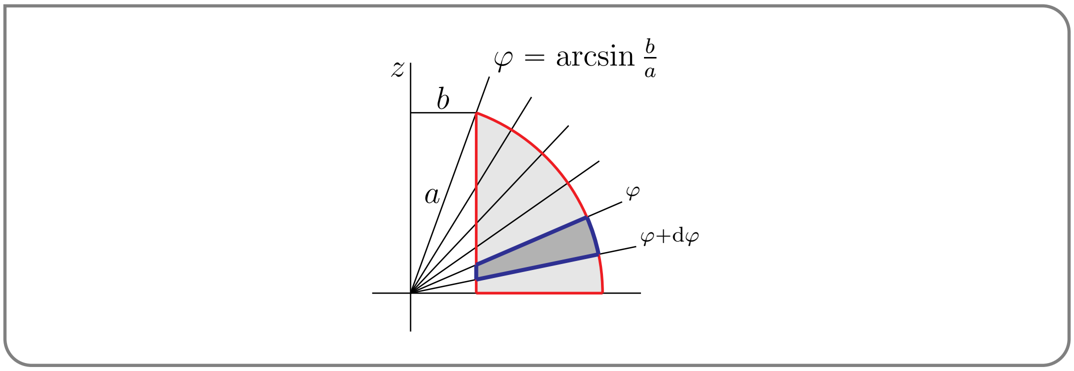
As in Example 3.2.11, let's use a coordinate system with the sphere centred on $(0,0,0)$ and with the centre of the drill hole following the z-axis. Here is a sketch of the apple that remains in the first octant. It is outlined in red. By symmetry the total amount of apple remaining will be eight times the amount from the first octant.



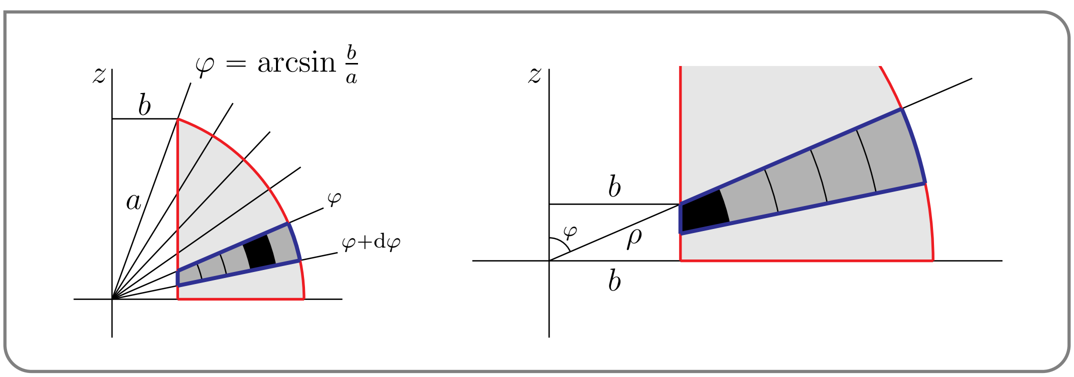
- First slice the first octant part of the remaining apple into segments by inserting many planes of constant θ, with the various values of θ differing by $d\theta$. The leftmost segment has, essentially, $\theta = 0$ and the rightmost segment has, essentially, $\theta = \pi/2$.
- Each segment, viewed from the side, looks like



Subdivide it into long thin searchlights by inserting many cones of constant φ, with the various values of φ differing by $\mathrm{d}\varphi$. The figure on below shows one searchlight outlined in blue. Each searchlight
  - has θ and φ essentially constant on the searchlight.
  - The top searchlight has, essentially, $\varphi = \arcsin\frac{b}{a}$ and the bottom searchlight has, essentially, $\varphi = \pi/2$.



- Concentrate on any one searchlight. Subdivide it into tiny approximate cubes by inserting many spheres of constant ρ, with the various values of ρ differing by $\mathrm{d}\rho$. The figure on the left below shows the side view of one approximate cube in black. Each cube
  - has ρ, θ and φ all essentially constant on the cube and
  - has volume $\mathrm{d}V = \rho^2 \sin\varphi\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\varphi$, by (3.7.3).
  - The figure on the right below gives an expanded view of the searchlight. From it, we see (after a little trig) that the first cube has, essentially, $\rho = \frac{b}{\sin\varphi}$ and the last cube has, essentially, $\rho = a$ (the radius of the apple).



Now we can build up the volume.

- Concentrate on one approximate cube. Let's say that it contains the point with spherical coordinates ρ, θ, φ. The cube has volume essentially $\mathrm{d}V = \rho^2 \sin\varphi\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\varphi$, by (3.7.3).

- To get the volume any one searchlight, say the searchlight whose φ coordinate runs from φ to $\varphi + \mathrm{d}\varphi$, we just add up the volumes of the approximate cubes in that searchlight, by integrating ρ from its smallest value on the searchlight, namely $\frac{b}{\sin\varphi}$, to its largest value on the searchlight, namely a. The volume of the searchlight is thus

$$d\theta\, d\varphi \int_{\frac{b}{\sin\varphi}}^{a} d\rho\ \rho^2 \sin\varphi$$

- To get the volume of any one segment, say the segment whose θ coordinate runs from θ to $\theta + d\theta$, we just add up the volumes of the searchlights in that segment, by integrating φ from its smallest value on the segment, namely $\arcsin\frac{b}{a}$, to its largest value on the segment, namely $\frac{\pi}{2}$. The volume of the searchlight is thus

$$d\theta \int_{\arcsin\frac{b}{a}}^{\frac{\pi}{2}} \int_{\frac{b}{\sin\varphi}}^{a} d\rho\ \rho^2 \sin\varphi$$

- To get the volume of the remaining part of the apple in the first octant, we just add up the volumes of the segments that it contains, by integrating θ from its smallest value in the octant, namely 0, to its largest value on the octant, namely $\pi/2$. The volume in the first octant is thus

$$\text{Volume}(\mathcal{V}_1) = \int_0^{\pi/2} d\theta \int_{\arcsin\frac{b}{a}}^{\frac{\pi}{2}} d\varphi \int_{\frac{b}{\sin\varphi}}^{a} d\rho\ \rho^2 \sin\varphi$$

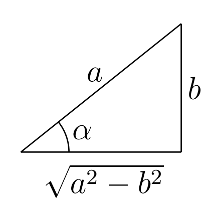
- Now we just have to integrate

$$\begin{aligned}
\text{Volume}(\mathcal{V}_1) &= \frac{1}{3}\int_0^{\pi/2} d\theta \int_{\arcsin\frac{b}{a}}^{\frac{\pi}{2}} d\varphi\ \sin\varphi \left[a^3 - \frac{b^3}{\sin^3\varphi}\right] \\
&= \frac{1}{3}\int_0^{\pi/2} d\theta \int_{\arcsin\frac{b}{a}}^{\frac{\pi}{2}} d\varphi\ \left[a^3\sin\varphi - b^3\csc^2\varphi\right] \\
&= \frac{1}{3}\int_0^{\pi/2} d\theta\ \Big[-a^3\cos\varphi + b^3\cot\varphi\Big]_{\arcsin\frac{b}{a}}^{\frac{\pi}{2}} \\
&\qquad\qquad \text{since} \int \csc^2\varphi\, d\varphi = -\cot\varphi + C \\
&= \frac{\pi}{6}\Big[-a^3\cos\varphi + b^3\cot\varphi\Big]_{\arcsin\frac{b}{a}}^{\frac{\pi}{2}}
\end{aligned}$$

Now $\cos\frac{\pi}{2} = \cot\frac{\pi}{2} = 0$ and, if we write $\alpha = \arcsin\frac{b}{a}$,

$$\text{Volume}(\mathcal{V}_1) = \frac{\pi}{6}\left[a^3\cos\alpha - b^3\cot\alpha\right]$$

From the triangle below, we have $\cos\alpha = \frac{\sqrt{a^2-b^2}}{a}$ and $\cot\alpha = \frac{\sqrt{a^2-b^2}}{b}$.



So

$$\text{Volume}(\mathcal{V}_1) = \frac{\pi}{6}\left[a^2\sqrt{a^2-b^2} - b^2\sqrt{a^2-b^2}\right] = \frac{\pi}{6}\left[a^2-b^2\right]^{3/2}$$

The full (eight octant) volume of the remaining apple is thus

$$\text{Volume}(\mathcal{V}) = 8\text{Volume}(\mathcal{V}_1) = \frac{4}{3}\pi\left[a^2-b^2\right]^{3/2}$$

We can, yet again, apply the sanity checks of Example 3.2.11 to our answer.

- If the radius of the drill bit $b = 0$, no apple is removed at all. So the total volume remaining should be $\frac{4}{3}\pi a^3$. Our answer does indeed give this.
- If the radius of the drill bit $b = a$, the radius of the apple, then the entire apple disappears. So the remaining apple should have volume 0. Again, our answer gives this.

As a final check note that the sum of the answer to Example 3.2.11 and the answer to this Example is $\frac{4}{3}\pi a^3$, as it should be.

Example 3.7.5

3.8▲ Optional— Integrals in General Coordinates

One of the most important tools used in dealing with single variable integrals is the change of variable (substitution) rule

Equation 3.8.1.

$$x = f(u) \qquad \mathrm{d}x = f'(u)\,\mathrm{d}u$$

See Theorems 1.4.2 and 1.4.6 in the CLP-2 text. Expressing multivariable integrals using polar or cylindrical or spherical coordinates are really multivariable substitutions. For example, switching to spherical coordinates amounts replacing the coordinates x, y, z with the coordinates ρ, θ, φ by using the substitution

$$\mathbf{X} = \mathbf{r}(\rho,\theta,\varphi) \qquad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\,\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

where

$$\mathbf{X} = \langle x\,,\,y\,,\,z\rangle \qquad \text{and} \qquad \mathbf{r}(\rho,\theta,\varphi) = \langle \rho\cos\theta\sin\varphi\,,\,\rho\sin\theta\sin\varphi\,,\,\rho\cos\varphi\rangle$$

We'll now derive a generalization of the substitution rule (3.8.1) to two dimensions. It will include polar coordinates as a special case. Later, we'll state (without proof) its generalization to three dimensions. It will include cylindrical and spherical coordinates as special cases.

Suppose that we wish to integrate over a region, $\mathcal{R}$, in $\mathbb{R}^2$ and that we also wish[42] to use two new coordinates, that we'll call u and v, in place of x and y. The new coordinates u, v are related to the old coordinates x, y, by the functions[43]

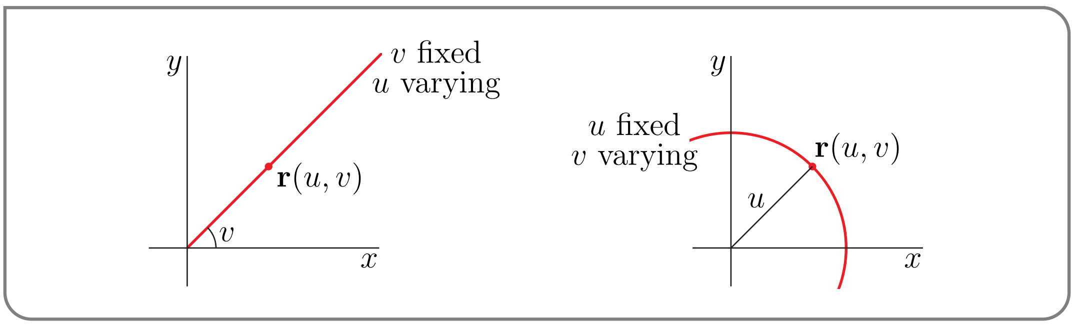
$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v) \end{aligned}$$

To make formulae more compact, we'll define the vector valued function $\mathbf{r}(u, v)$ by

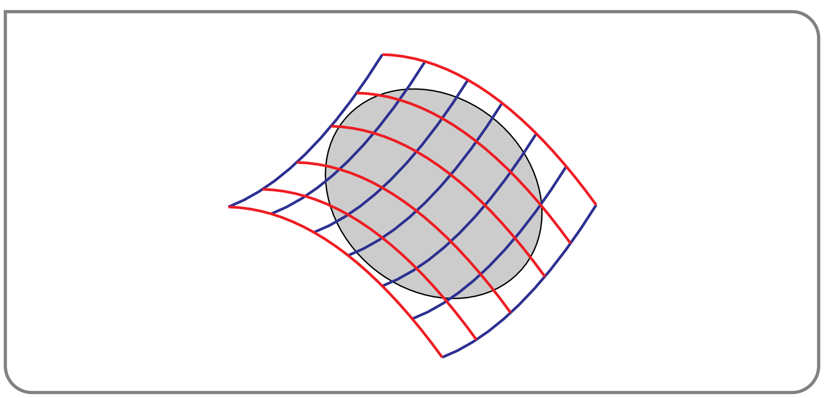
$$\mathbf{r}(u, v) = \langle x(u, v)\,, y(u, v) \rangle$$

As an example, if the new coordinates are polar coordinates, with r renamed to u and θ renamed to v, then $x(u, v) = u \cos v$ and $y = u \sin v$.

Note that if we hold v fixed and vary u, then $\mathbf{r}(u, v)$ sweeps out a curve. For example, if $x(u, v) = u \cos v$ and $y = u \sin v$, then, if we hold v fixed and vary u, $\mathbf{r}(u, v)$ sweeps out a straight line (that makes the angle v with the x-axis), while, if we hold $u > 0$ fixed and vary v, $\mathbf{r}(u, v)$ sweeps out a circle (of radius u centred on the origin).



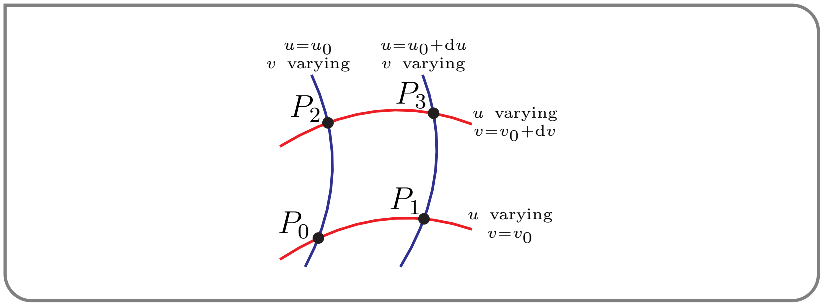
We start by cutting $\mathcal{R}$ (the shaded region in the figure below) up into small pieces by drawing a bunch of curves of constant u (the blue curves in the figure below) and a bunch of curves of constant v (the red curves in the figure below).



Concentrate on any one of the small pieces. Here is a greatly magnified sketch.

42 We'll keep our third wish in reserve.

43 We are abusing notation a little here by using x and y both as coordinates and as functions. We could write $x = f(u, v)$ and $y = g(u, v)$, but it is easier to remember $x = x(u, v)$ and $y = y(u, v)$.



For example, the lower red curve was constructed by holding v fixed at the value v_0, varying u and sketching $\mathbf{r}(u, v_0)$, and the upper red curve was constructed by holding v fixed at the slightly larger value $v_0 + \mathrm{d}v$, varying u and sketching $\mathbf{r}(u, v_0 + \mathrm{d}v)$. So the four intersection points in the figure are

$$\begin{aligned} P_2 &= \mathbf{r}(u_0, v_0 + \mathrm{d}v) & P_3 &= \mathbf{r}(u_0 + \mathrm{d}u, v_0 + \mathrm{d}v) \\ P_0 &= \mathbf{r}(u_0, v_0) & P_1 &= \mathbf{r}(u_0 + \mathrm{d}u, v_0) \end{aligned}$$

Now, for any small constants $\mathrm{d}U$ and $\mathrm{d}V$, we have the linear approximation[44]

$$\mathbf{r}(u_0 + \mathrm{d}U, v_0 + \mathrm{d}V) \approx \mathbf{r}(u_0\,,\, v_0) + \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)\,\mathrm{d}U + \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)\,\mathrm{d}V$$

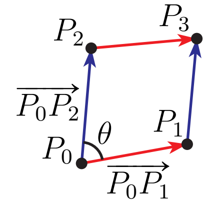
Applying this three times, once with $\mathrm{d}U = \mathrm{d}u$, $\mathrm{d}V = 0$ (to approximate P_1), once with $\mathrm{d}U = 0$, $\mathrm{d}V = \mathrm{d}v$ (to approximate P_2), and once with $\mathrm{d}U = \mathrm{d}u$, $\mathrm{d}V = \mathrm{d}v$ (to approximate P_3),

$$\begin{aligned} P_0 &= \mathbf{r}(u_0\,,\, v_0) \\ P_1 &= \mathbf{r}(u_0 + \mathrm{d}u, v_0) &&\approx \mathbf{r}(u_0\,,\, v_0) + \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)\,\mathrm{d}u \\ P_2 &= \mathbf{r}(u_0, v_0 + \mathrm{d}v) &&\approx \mathbf{r}(u_0\,,\, v_0) \qquad\qquad\qquad\qquad + \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)\,\mathrm{d}v \\ P_3 &= \mathbf{r}(u_0 + \mathrm{d}u, v_0 + \mathrm{d}v) &&\approx \mathbf{r}(u_0\,,\, v_0) + \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)\,\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)\,\mathrm{d}v \end{aligned}$$

We have dropped all Taylor expansion terms that are of degree two or higher in $\mathrm{d}u$, $\mathrm{d}v$. The reason is that, in defining the integral, we take the limit $\mathrm{d}u, \mathrm{d}v \to 0$. Because of that limit, all of the dropped terms contribute exactly 0 to the integral. We shall not prove this. But we shall show, in the optional §3.8.1, why this is the case.

The small piece of $\mathcal{R}$ surface with corners P_0, P_1, P_2, P_3 is approximately a parallelogram with sides

$$\begin{aligned} \overrightarrow{P_0P_1} \approx \overrightarrow{P_2P_3} &\approx \frac{\partial \mathbf{r}}{\partial u}(u_0\,,\, v_0)\,\mathrm{d}u = \left\langle \frac{\partial x}{\partial u}(u_0, v_0)\,,\, \frac{\partial y}{\partial u}(u_0, v_0) \right\rangle \mathrm{d}u \\ \overrightarrow{P_0P_2} \approx \overrightarrow{P_1P_3} &\approx \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\, v_0)\,\mathrm{d}v = \left\langle \frac{\partial x}{\partial v}(u_0, v_0)\,,\, \frac{\partial y}{\partial v}(u_0, v_0) \right\rangle \mathrm{d}v \end{aligned}$$



44 Recall (2.6.1).

Here the notation, for example, $\overrightarrow{P_0P_1}$ refers to the vector whose tail is at the point P_0 and whose head is at the point P_1. Recall, from (1.2.17) that

$$\text{area of parallelogram with sides } \langle a,b\rangle \text{ and } \langle c,d\rangle = \left|\det\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right| = |ad-bc|$$

So the area of our small piece of $\mathcal{R}$ is essentially

Equation 3.8.2.

$$\mathrm{d}A = \left|\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{bmatrix}\right| \mathrm{d}u\,\mathrm{d}v$$

Recall that det M denotes the determinant of the matrix M. Also recall that we don't really need determinants for this text, though it does make for nice compact notation.

The formula (3.8.2) is the heart of the following theorem, which tells us how to translate an integral in one coordinate system into an integral in another coordinate system.

Theorem 3.8.3.

Let the functions $x(u,v)$ and $y(u,v)$ have continuous first partial derivatives and let the function $f(x,y)$ be continuous. Assume that $x = x(u,v)$, $y = y(u,v)$ provides a one-to-one correspondence between the points (u,v) of the region $\mathcal{U}$ in the uv-plane and the points (x,y) of the region $\mathcal{R}$ in the xy-plane. Then

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{U}} f\big(x(u,v)\,,\,y(u,v)\big)\left|\det\begin{bmatrix} \frac{\partial x}{\partial u}(u,v) & \frac{\partial y}{\partial u}(u,v) \\ \frac{\partial x}{\partial v}(u,v) & \frac{\partial y}{\partial v}(u,v)\end{bmatrix}\right| \mathrm{d}u\,\mathrm{d}v$$

The determinant

$$\det\begin{bmatrix} \frac{\partial x}{\partial u}(u,v) & \frac{\partial y}{\partial u}(u,v) \\ \frac{\partial x}{\partial v}(u,v) & \frac{\partial y}{\partial v}(u,v)\end{bmatrix}$$

that appears in (3.8.2) and Theorem 3.8.3 is known as the Jacobian[45].

Example 3.8.4 ($\mathrm{d}A$ for $x \leftrightarrow y$)

We'll start with a pretty trivial example in which we simply rename x to Y and y to X. That is

$$x(X,Y) = Y$$
$$y(X,Y) = X$$

45 It is not named after the Jacobin Club, a political movement of the French revolution. It is not named after the Jacobite rebellions that took place in Great Britain and Ireland between 1688 and 1746. It is not named after the Jacobean era of English and Scottish history. It is named after the German mathematician Carl Gustav Jacob Jacobi (1804–1851). He died from smallpox.

Since

$$\frac{\partial x}{\partial X} = 0 \qquad \frac{\partial y}{\partial X} = 1$$
$$\frac{\partial x}{\partial Y} = 1 \qquad \frac{\partial y}{\partial Y} = 0$$

(3.8.2), but with u renamed to X and v renamed to Y, gives

$$\mathrm{d}A = \left|\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right| \mathrm{d}X\,\mathrm{d}Y = \mathrm{d}X\,\mathrm{d}Y$$

which should really not be a shock.

Example 3.8.4

Example 3.8.5 (dA for Polar Coordinates)

Polar coordinates have

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$

Since

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

(3.8.2), but with u renamed to r and v renamed to θ, gives

$$\begin{aligned} \mathrm{d}A &= \left|\det \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}\right| \mathrm{d}r\mathrm{d}\theta = \left(r\cos^2\theta + r\sin^2\theta\right)\mathrm{d}r\mathrm{d}\theta \\ &= r\,\mathrm{d}r\,\mathrm{d}\theta \end{aligned}$$

which is exactly what we found in (3.2.5).

Example 3.8.5

Example 3.8.6 (dA for Parabolic Coordinates)

Parabolic[46] coordinates are defined by

$$x(u,v) = \frac{u^2 - v^2}{2}$$
$$y(u,v) = uv$$

46 The name comes from the fact that both the curves of constant u and the curves of constant v are parabolas.

Since

$$\frac{\partial x}{\partial u} = u \qquad\qquad \frac{\partial y}{\partial u} = v$$
$$\frac{\partial x}{\partial v} = -v \qquad\qquad \frac{\partial y}{\partial v} = u$$

(3.8.2) gives

$$\mathrm{d}A = \left|\det\begin{bmatrix} u & v \\ -v & u\end{bmatrix}\right| \mathrm{d}u\mathrm{d}v = (u^2+v^2)\,\mathrm{d}u\,\mathrm{d}v$$

Example 3.8.6

In practice applying the change of variables Theorem 3.8.3 can be quite tricky. Here is just one simple (and rigged) example.

Example 3.8.7

Evaluate

$$\iint_{\mathcal{R}} \frac{y}{1+x}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{where } \mathcal{R} = \big\{\, (x,y) \bigm| 0 \leqslant x \leqslant 1,\ 1+x \leqslant y \leqslant 2+2x \,\big\}$$

Solution. We can simplify the integrand considerably by making the change of variables

$$s = x \qquad\qquad x = s$$
$$t = \frac{y}{1+x} \qquad\qquad y = t(1+x) = t(1+s)$$

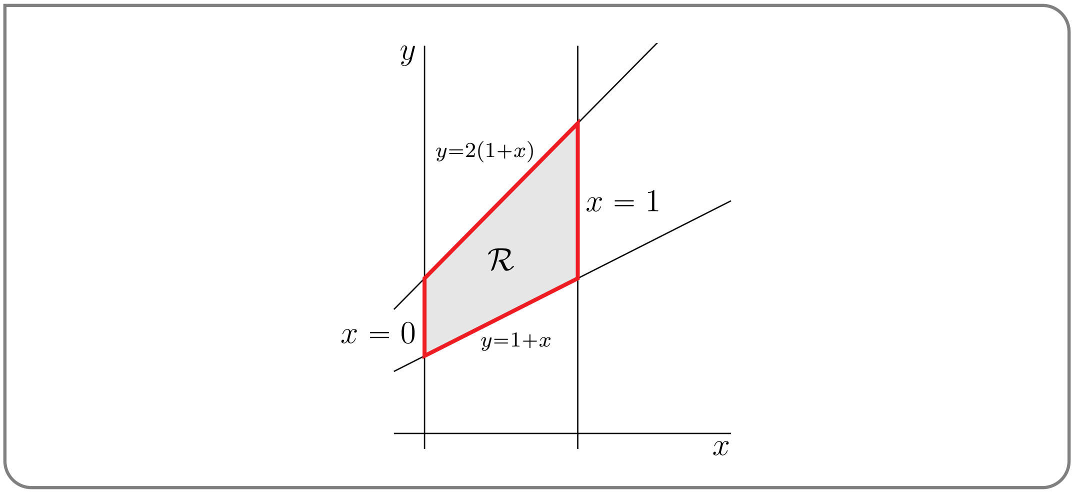
Of course to evaluate the given integral by applying Theorem 3.8.3 we also need to know

- the domain of integration in terms of s and t and
- $\mathrm{d}x\,\mathrm{d}y$ in terms of $\mathrm{d}s\,\mathrm{d}t$.

By (3.8.2), recalling that $x(s,t) = s$ and $y(s,t) = t(1+s)$,

$$\mathrm{d}x\,\mathrm{d}y = \left|\det\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t}\end{bmatrix}\right| \mathrm{d}s\,\mathrm{d}t = \left|\det\begin{bmatrix} 1 & t \\ 0 & 1+s\end{bmatrix}\right| \mathrm{d}s\,\mathrm{d}t = (1+s)\,\mathrm{d}s\,\mathrm{d}t$$

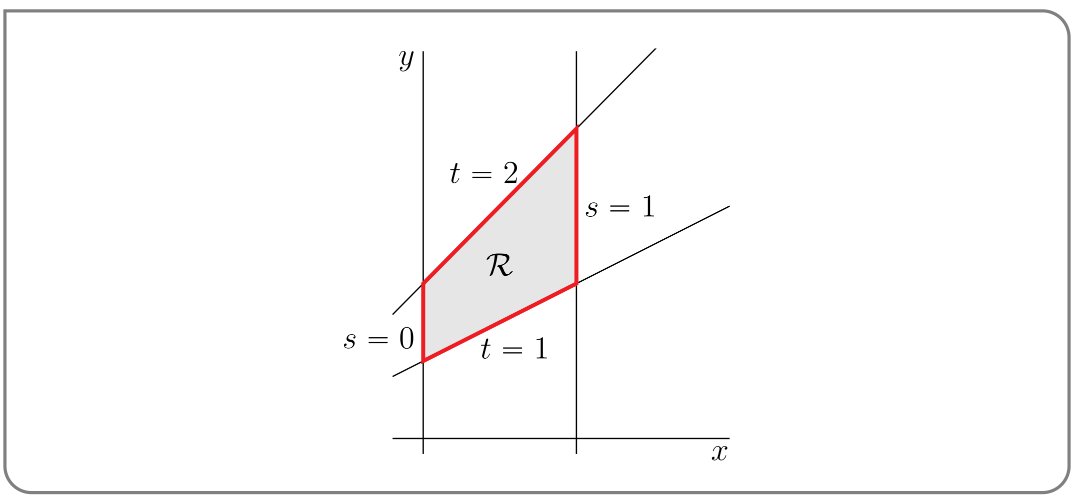
To determine what the change of variables does to the domain of integration, we'll sketch $\mathcal{R}$ and then reexpress the boundary of $\mathcal{R}$ in terms of the new coordinates s and t. Here is the sketch of $\mathcal{R}$ in the original coordinates (x,y).



The region $\mathcal{R}$ is a quadrilateral. It has four sides.

- The left side is part of the line $x = 0$. Recall that $x = s$. So, in terms of s and t, this line is $s = 0$.
- The right side is part of the line $x = 1$. In terms of s and t, this line is $s = 1$.
- The bottom side is part of the line $y = 1 + x$, or $\frac{y}{1+x} = 1$. Recall that $t = \frac{y}{1+x}$. So, in terms of s and t, this line is $t = 1$.
- The top side is part of the line $y = 2(1 + x)$, or $\frac{y}{1+x} = 2$. In terms of s and t, this line is $t = 2$.

Here is another copy of the sketch of $\mathcal{R}$. But this time the equations of its four sides are expressed in terms of s and t.



So, expressed in terms of s and t, the domain of integration $\mathcal{R}$ is much simpler:

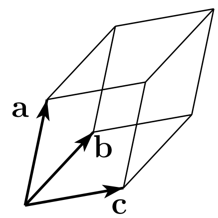
$$\big\{ (s,t) \bigm| 0 \leqslant s \leqslant 1,\ 1 \leqslant t \leqslant 2 \big\}$$

As $\mathrm{d}x\,\mathrm{d}y = (1+s)\,\mathrm{d}s\,\mathrm{d}t$ and the integrand $\frac{y}{1+x} = t$, the integral is, by Theorem 3.8.3,

$$\iint_{\mathcal{R}} \frac{y}{1+x}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}s \int_1^2 \mathrm{d}t\ (1+s)t = \int_0^1 \mathrm{d}s\ (1+s)\ \left[\frac{t^2}{2}\right]_1^2 = \frac{3}{2}\left[s + \frac{s^2}{2}\right]_0^1 = \frac{3}{2} \times \frac{3}{2}$$
$$= \frac{9}{4}$$

Example 3.8.7

There are natural generalizations of (3.8.2) and Theorem 3.8.3 to three (and also to higher) dimensions, that are derived in precisely the same way as (3.8.2) was derived. The derivation is based on the fact, discussed in the optional Section 1.2.4, that the volume of the parallelepiped (three dimensional parallelogram) determined by the three vectors



$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is given by the formula

$$\text{volume of parallelepiped with edges } \mathbf{a},\ \mathbf{b},\ \mathbf{c} \ = \left|\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right|$$

where the determinant of a 3×3 matrix can be defined in terms of some 2×2 determinants by

$$\begin{aligned} \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} &= a_1 \det \begin{bmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ \cancel{b_1} & b_2 & b_3 \\ \cancel{c_1} & c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ b_1 & \cancel{b_2} & b_3 \\ c_1 & \cancel{c_2} & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} \cancel{a_1} & \cancel{a_2} & \cancel{a_3} \\ b_1 & b_2 & \cancel{b_3} \\ c_1 & c_2 & \cancel{c_3} \end{bmatrix} \\ &= a_1\,(b_2c_3 - b_3c_2) \quad - a_2\,(b_1c_3 - b_3c_1) \quad + a_3\,(b_1c_2 - b_2c_1) \end{aligned}$$

If we use

$$\begin{aligned} x &= x(u, v, w) \\ y &= y(u, v, w) \\ z &= z(u, v, w) \end{aligned}$$

to change from old coordinates x, y, z to new coordinates u, v, w, then

Equation 3.8.8.

$$\mathrm{d}V = \left|\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix}\right| \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}w$$

Example 3.8.9 (dV for Cylindrical Coordinates)

Cylindrical coordinates have

$$\begin{aligned} x(r,\theta,z) &= r\cos\theta \\ y(r,\theta,z) &= r\sin\theta \\ z(r,\theta,z) &= z \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta & \frac{\partial y}{\partial r} &= \sin\theta & \frac{\partial z}{\partial r} &= 0 \\ \frac{\partial x}{\partial \theta} &= -r\sin\theta & \frac{\partial y}{\partial \theta} &= r\cos\theta & \frac{\partial z}{\partial \theta} &= 0 \\ \frac{\partial x}{\partial z} &= 0 & \frac{\partial y}{\partial z} &= 0 & \frac{\partial z}{\partial z} &= 1 \end{aligned}$$

(3.8.8), but with u renamed to r and v renamed to θ, gives

$$\begin{aligned} \mathrm{d}V &= \left|\det\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\ &= \left|\cos\theta\det\begin{bmatrix} r\cos\theta & 0 \\ 0 & 1 \end{bmatrix} - \sin\theta\det\begin{bmatrix} -r\sin\theta & 0 \\ 0 & 1 \end{bmatrix} + 0\det\begin{bmatrix} -r\sin\theta & r\cos\theta \\ 0 & 0 \end{bmatrix}\right| \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\ &= \left(r\cos^2\theta + r\sin^2\theta\right)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\ &= r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \end{aligned}$$

which is exactly what we found in (3.6.3).

Example 3.8.9

Example 3.8.10 (dV for Spherical Coordinates)

Spherical coordinates have

$$\begin{aligned} x(\rho,\theta,\varphi) &= \rho\,\cos\theta\,\sin\varphi \\ y(\rho,\theta,\varphi) &= \rho\,\sin\theta\,\sin\varphi \\ z(\rho,\theta,\varphi) &= \rho\,\cos\varphi \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial x}{\partial \rho} &= \cos\theta\,\sin\varphi & \frac{\partial y}{\partial \rho} &= \sin\theta\,\sin\varphi & \frac{\partial z}{\partial \rho} &= \cos\varphi \\ \frac{\partial x}{\partial \theta} &= -\rho\,\sin\theta\,\sin\varphi & \frac{\partial y}{\partial \theta} &= \rho\,\cos\theta\,\sin\varphi & \frac{\partial z}{\partial \theta} &= 0 \\ \frac{\partial x}{\partial \varphi} &= \rho\,\cos\theta\,\cos\varphi & \frac{\partial y}{\partial \varphi} &= \rho\,\sin\theta\,\cos\varphi & \frac{\partial z}{\partial \varphi} &= -\rho\,\sin\varphi \end{aligned}$$

(3.8.8), but with u renamed to ρ, v renamed to θ and w renamed to φ, gives

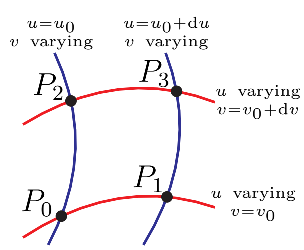
$$\begin{aligned}
\mathrm{d}V &= \left|\det\begin{bmatrix} \cos\theta\,\sin\varphi & \sin\theta\,\sin\varphi & \cos\varphi \\ -\rho\,\sin\theta\,\sin\varphi & \rho\,\cos\theta\,\sin\varphi & 0 \\ \rho\,\cos\theta\,\cos\varphi & \rho\,\sin\theta\,\cos\varphi & -\rho\,\sin\varphi \end{bmatrix}\right| \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \left|\cos\theta\,\sin\varphi\det\begin{bmatrix} \rho\,\cos\theta\,\sin\varphi & 0 \\ \rho\,\sin\theta\,\cos\varphi & -\rho\,\sin\varphi \end{bmatrix} - \sin\theta\,\sin\varphi\det\begin{bmatrix} -\rho\,\sin\theta\,\sin\varphi & 0 \\ \rho\,\cos\theta\,\cos\varphi & -\rho\,\sin\varphi \end{bmatrix}\right. \\
&\qquad\qquad \left. + \cos\varphi\det\begin{bmatrix} -\rho\,\sin\theta\,\sin\varphi & \rho\,\cos\theta\,\sin\varphi \\ \rho\,\cos\theta\,\cos\varphi & \rho\,\sin\theta\,\cos\varphi \end{bmatrix}\right| \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \rho^2\big| -\cos^2\theta\sin^3\varphi - \sin^2\theta\sin^3\varphi - \sin\varphi\cos^2\varphi\big|\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \rho^2\big| -\sin\varphi\sin^2\varphi - \sin\varphi\cos^2\varphi\big|\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi
\end{aligned}$$

which is exactly what we found in (3.7.3).

Example 3.8.10

3.8.1 ▸▸ Optional — Dropping Higher Order Terms in $\mathrm{d}u, \mathrm{d}v$

In the course of deriving (3.8.2), that is, the $\mathrm{d}A$ formula for



we approximated, for example, the vectors

$$\overrightarrow{P_0P_1} = \mathbf{r}(u_0+\mathrm{d}u, v_0) - \mathbf{r}(u_0\,,\,v_0) = \frac{\partial\mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u + E_1 \approx \frac{\partial\mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u$$

$$\overrightarrow{P_0P_2} = \mathbf{r}(u_0, v_0+\mathrm{d}v) - \mathbf{r}(u_0\,,\,v_0) = \frac{\partial\mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v + E_2 \approx \frac{\partial\mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v$$

where $\mathbf{E}_1$ is bounded[47] by a constant times $(\mathrm{d}u)^2$ and E_2 is bounded by a constant times $(\mathrm{d}v)^2$. That is, we assumed that we could just ignore the errors and drop E_1 and E_2 by setting them to zero.

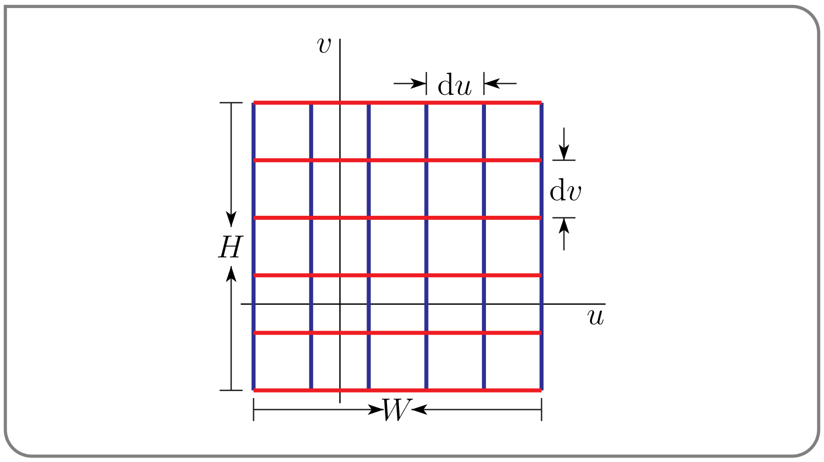
47 Remember the error in the Taylor polynomial approximations. See (2.6.13) and (2.6.14).

So we approximated

$$\begin{aligned}\left|\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}\right| &= \left|\left[\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u + \mathbf{E}_1\right] \times \left[\frac{\partial \mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v + \mathbf{E}_2\right]\right| \\ &= \left|\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u \times \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v + \mathbf{E}_3\right| \\ &\approx \left|\frac{\partial \mathbf{r}}{\partial u}(u_0\,,\,v_0)\,\mathrm{d}u \times \frac{\partial \mathbf{r}}{\partial v}(u_0\,,\,v_0)\,\mathrm{d}v\right|\end{aligned}$$

where the length of the vector $\mathbf{E}_3$ is bounded by a constant times $(\mathrm{d}u)^2\,\mathrm{d}v + \mathrm{d}u\,(\mathrm{d}v)^2$. We'll now see why dropping terms like $\mathbf{E}_3$ does not change the value of the integral at all[48].

Suppose that our domain of integration consists of all (u,v)'s in a rectangle of width W and height H, as in the figure below.



Subdivide the rectangle into a grid of $n \times n$ small subrectangles by drawing lines of constant v (the red lines in the figure) and lines of constant u (the blue lines in the figure). Each subrectangle has width $\mathrm{d}u = \frac{W}{n}$ and height $\mathrm{d}v = \frac{H}{n}$. Now suppose that in setting up the integral we make, for each subrectangle, an error that is bounded by some constant times

$$(\mathrm{d}u)^2\,\mathrm{d}v + \mathrm{d}u\,(\mathrm{d}v)^2 = \left(\frac{W}{n}\right)^2\frac{H}{n} + \frac{W}{n}\left(\frac{H}{n}\right)^2 = \frac{WH(W+H)}{n^3}$$

Because there are a total of n^2 subrectangles, the total error that we have introduced, for all of these subrectangles, is no larger than a constant times

$$n^2 \times \frac{WH(W+H)}{n^3} = \frac{WH(W+H)}{n}$$

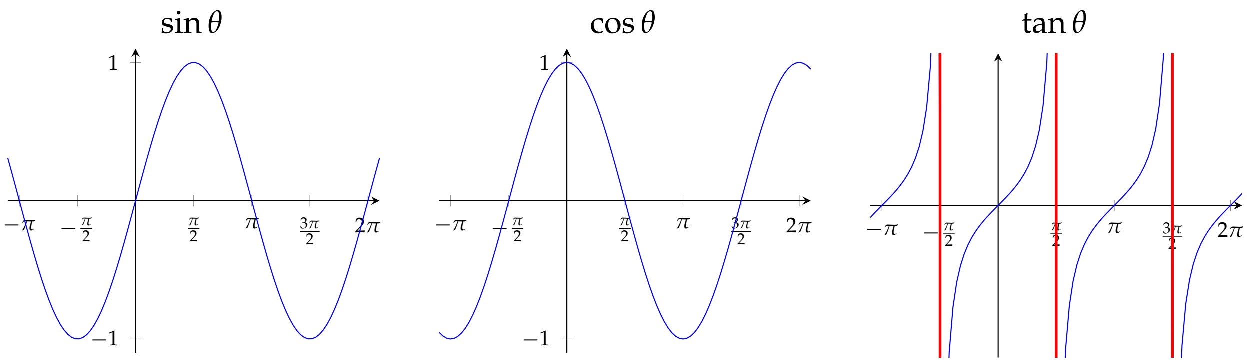
When we define our integral by taking the limit $n \to 0$ of the Riemann sums, this error converges to exactly 0. As a consequence, it was safe for us to ignore the error terms when we established the change of variables formulae.

48 See the optional §1.1.6 of the CLP-2 text for an analogous argument concerning Riemann sums.

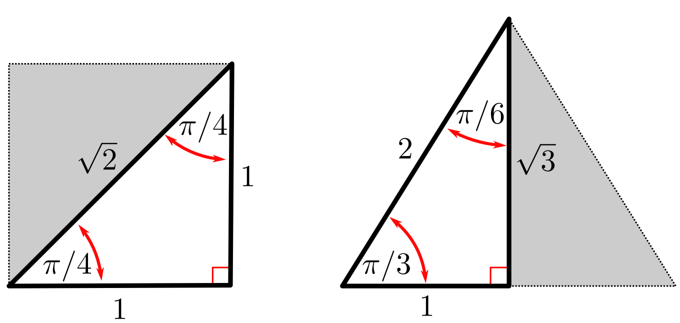
Appendix A

TRIGONOMETRY

A.1 Trigonometry — Graphs



A.2 Trigonometry — Special Triangles



From the above pair of special triangles we have

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$
$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

A.3▲ Trigonometry — Simple Identities

- Periodicity

$$\sin(\theta + 2\pi) = \sin(\theta) \qquad \cos(\theta + 2\pi) = \cos(\theta)$$

- Reflection

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$

- Reflection around $\pi/4$

$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$

- Reflection around $\pi/2$

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$

- Rotation by π

$$\sin(\theta + \pi) = -\sin\theta \qquad \cos(\theta + \pi) = -\cos\theta$$

- Pythagoras

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

- sin and cos building blocks

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

A.4▲ Trigonometry — Add and Subtract Angles

- Sine

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

- Cosine

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

- Tangent

$$\begin{aligned}
\tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
\tan(\alpha - \beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}
\end{aligned}$$

- Double angle

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
&= 2\cos^2(\theta) - 1 \\
&= 1 - 2\sin^2(\theta) \\
\tan(2\theta) &= \frac{2\tan(\theta)}{1 - \tan^2\theta} \\
\cos^2\theta &= \frac{1 + \cos(2\theta)}{2} \\
\sin^2\theta &= \frac{1 - \cos(2\theta)}{2} \\
\tan^2\theta &= \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}
\end{aligned}$$

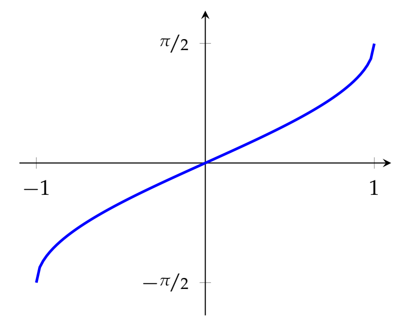
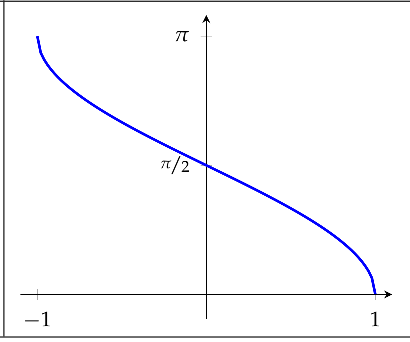
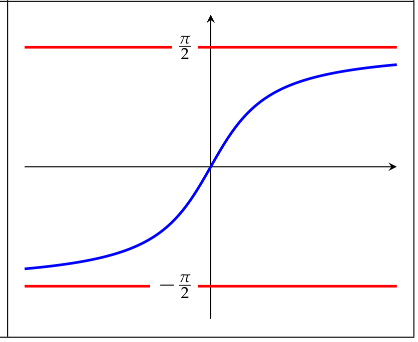
- Products to sums

$$\begin{aligned}
\sin(\alpha)\cos(\beta) &= \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2} \\
\sin(\alpha)\sin(\beta) &= \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2} \\
\cos(\alpha)\cos(\beta) &= \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}
\end{aligned}$$

- Sums to products

$$\begin{aligned}
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2} \\
\sin\alpha - \sin\beta &= 2\cos\frac{\alpha + \beta}{2}\sin\frac{\alpha - \beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha + \beta}{2}\sin\frac{\alpha - \beta}{2}
\end{aligned}$$

A.5▲ Inverse Trigonometric Functions

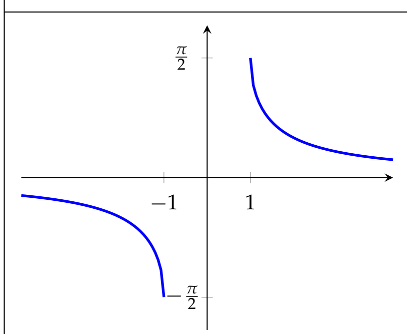
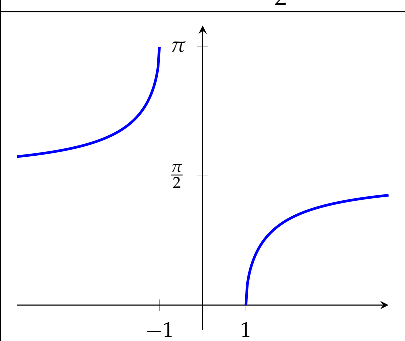
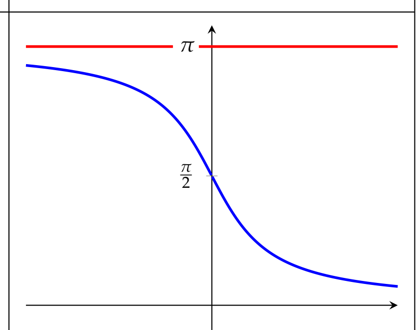
$\arcsin x$	$\arccos x$	$\arctan x$
Domain: $-1 \leqslant x \leqslant 1$ Range: $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$	Domain: $-1 \leqslant x \leqslant 1$ Range: $0 \leqslant \arccos x \leqslant \pi$	Domain: all real numbers Range: $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$
		

Since these functions are inverses of each other we have

$$
\begin{aligned}
\arcsin(\sin\theta) &= \theta & -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \\
\arccos(\cos\theta) &= \theta & 0 \leqslant \theta \leqslant \pi \\
\arctan(\tan\theta) &= \theta & -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}
\end{aligned}
$$

and also

$$
\begin{aligned}
\sin(\arcsin x) &= x & -1 \leqslant x \leqslant 1 \\
\cos(\arccos x) &= x & -1 \leqslant x \leqslant 1 \\
\tan(\arctan x) &= x & \text{any real } x
\end{aligned}
$$

$\operatorname{arccsc} x$	$\operatorname{arcsec} x$	$\operatorname{arccot} x$
Domain: $\lvert x\rvert \geqslant 1$ Range: $-\frac{\pi}{2} \leqslant \operatorname{arccsc} x \leqslant \frac{\pi}{2}$ $\operatorname{arccsc} x \neq 0$	Domain: $\lvert x\rvert \geqslant 1$ Range: $0 \leqslant \operatorname{arcsec} x \leqslant \pi$ $\operatorname{arcsec} x \neq \frac{\pi}{2}$	Domain: all real numbers Range: $0 < \operatorname{arccot} x < \pi$
		

Again

$$\begin{aligned} \operatorname{arccsc}(\csc\theta) &= \theta && -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2},\ \theta \neq 0 \\ \operatorname{arcsec}(\sec\theta) &= \theta && 0 \leqslant \theta \leqslant \pi,\ \theta \neq \frac{\pi}{2} \\ \operatorname{arccot}(\cot\theta) &= \theta && 0 < \theta < \pi \end{aligned}$$

and

$$\begin{aligned} \csc(\operatorname{arccsc} x) &= x && |x| \geqslant 1 \\ \sec(\operatorname{arcsec} x) &= x && |x| \geqslant 1 \\ \cot(\operatorname{arccot} x) &= x && \text{any real } x \end{aligned}$$

Appendix B

Powers and Logarithms

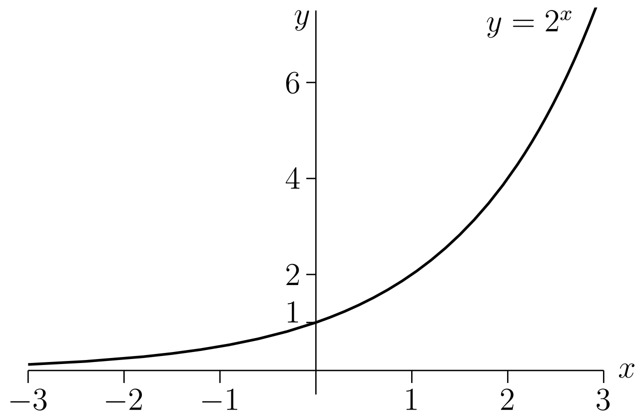
B.1▲ Powers

In the following, x and y are arbitrary real numbers, q is an arbitrary constant that is strictly bigger than zero and e is 2.7182818284, to ten decimal places.

- $e^0 = 1, \quad q^0 = 1$
- $e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad q^{x+y} = q^x q^y, \quad q^{x-y} = \frac{q^x}{q^y}$
- $e^{-x} = \frac{1}{e^x}, \quad q^{-x} = \frac{1}{q^x}$
- $\left(e^x\right)^y = e^{xy}, \quad \left(q^x\right)^y = q^{xy}$
- $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x, \quad \frac{\mathrm{d}}{\mathrm{d}x} e^{g(x)} = g'(x) e^{g(x)}, \quad \frac{\mathrm{d}}{\mathrm{d}x} q^x = (\ln q)\, q^x$
- $\int e^x \,\mathrm{d}x = e^x + C, \quad \int e^{ax} \,\mathrm{d}x = \frac{1}{a} e^{ax} + C$ if $a \neq 0$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0$

  $\lim_{x\to\infty} q^x = \infty, \quad \lim_{x\to-\infty} q^x = 0$ if $q > 1$

  $\lim_{x\to\infty} q^x = 0, \quad \lim_{x\to-\infty} q^x = \infty$ if $0 < q < 1$
- The graph of 2^x is given below. The graph of q^x, for any $q > 1$, is similar.



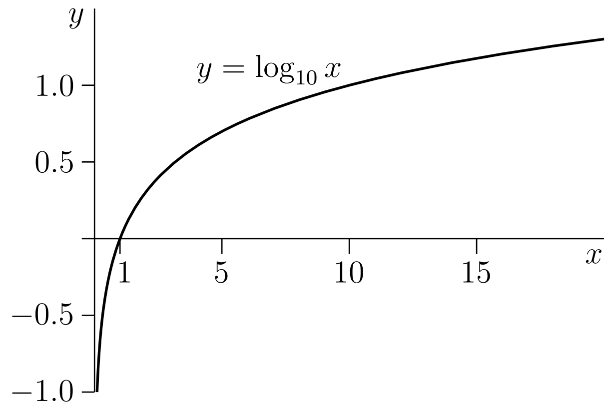
B.2▲ Logarithms

In the following, x and y are arbitrary real numbers that are strictly bigger than 0 (except where otherwise specified), p and q are arbitrary constants that are strictly bigger than one, and e is 2.7182818284, to ten decimal places. The notation $\ln x$ means $\log_e x$. Some people use $\log x$ to mean $\log_{10} x$, others use it to mean $\log_e x$ and still others use it to mean $\log_2 x$.

- $e^{\ln x} = x, \quad q^{\log_q x} = x$
- $\ln\left(e^x\right) = x, \quad \log_q\left(q^x\right) = x \quad$ for all $-\infty < x < \infty$
- $\log_q x = \frac{\ln x}{\ln q}, \quad \ln x = \frac{\log_p x}{\log_p e}, \quad \log_q x = \frac{\log_p x}{\log_p q}$
- $\ln 1 = 0, \quad \ln e = 1$

  $\log_q 1 = 0, \quad \log_q q = 1$
- $\ln(xy) = \ln x + \ln y, \quad \log_q(xy) = \log_q x + \log_q y$
- $\ln\left(\frac{x}{y}\right) = \ln x - \ln y, \quad \log_q\left(\frac{x}{y}\right) = \log_q x - \log_q y$
- $\ln\left(\frac{1}{y}\right) = -\ln y, \quad \log_q\left(\frac{1}{y}\right) = -\log_q y$
- $\ln(x^y) = y \ln x, \quad \log_q(x^y) = y \log_q x$
- $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}, \quad \frac{\mathrm{d}}{\mathrm{d}x} \log_q x = \frac{1}{x \ln q}$
- $\int \ln x \, \mathrm{d}x = x \ln x - x + C, \quad \int \log_q x \, \mathrm{d}x = x \log_q x - \frac{x}{\ln q} + C,$
- $\lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0+} \ln x = -\infty$

  $\lim_{x\to\infty} \log_q x = \infty, \quad \lim_{x\to 0+} \log_q x = -\infty$
- The graph of $\log_{10} x$ is given below. The graph of $\log_q x$, for any $q > 1$, is similar.


y
1.0
0.5
−0.5
−1.0
$y = \log_{10} x$
1
5
10
15
x

Appendix C

TABLE OF DERIVATIVES

Throughout this table, a and b are constants, independent of x.

$F(x)$	$F'(x) = \frac{dF}{dx}$
$af(x) + bg(x)$	$af'(x) + bg'(x)$
$f(x) + g(x)$	$f'(x) + g'(x)$
$f(x) - g(x)$	$f'(x) - g'(x)$
$af(x)$	$af'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$f(x)g(x)h(x)$	$f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
$\frac{1}{g(x)}$	$-\frac{g'(x)}{g(x)^2}$
$f(g(x))$	$f'(g(x))g'(x)$

$F(x)$	$F'(x) = \frac{dF}{dx}$
a	0
x^a	ax^{a-1}
$g(x)^a$	$ag(x)^{a-1}g'(x)$
$\sin x$	$\cos x$
$\sin g(x)$	$g'(x)\cos g(x)$
$\cos x$	$-\sin x$
$\cos g(x)$	$-g'(x)\sin g(x)$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
e^x	e^x
$e^{g(x)}$	$g'(x)e^{g(x)}$
a^x	$(\ln a)\, a^x$

$F(x)$	$F'(x) = \frac{dF}{dx}$
$\ln x$	$\frac{1}{x}$
$\ln g(x)$	$\frac{g'(x)}{g(x)}$
$\log_a x$	$\frac{1}{x \ln a}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arcsin g(x)$	$\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\arctan g(x)$	$\frac{g'(x)}{1+g(x)^2}$
$\operatorname{arccsc} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\operatorname{arcsec} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$

Appendix D

TABLE OF INTEGRALS

Throughout this table, a and b are given constants, independent of x and C is an arbitrary constant.

$f(x)$	$F(x) = \int f(x)\,\mathrm{d}x$
$af(x) + bg(x)$	$a\int f(x)\,\mathrm{d}x + b\int g(x)\,\mathrm{d}x + C$
$f(x) + g(x)$	$\int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x + C$
$f(x) - g(x)$	$\int f(x)\,\mathrm{d}x - \int g(x)\,\mathrm{d}x + C$
$af(x)$	$a\int f(x)\,\mathrm{d}x + C$
$u(x)v'(x)$	$u(x)v(x) - \int u'(x)v(x)\,\mathrm{d}x + C$
$f\big(y(x)\big)y'(x)$	$F\big(y(x)\big)$ where $F(y) = \int f(y)\,\mathrm{d}y$
a	$ax + C$
x^a	$\frac{x^{a+1}}{a+1} + C$ if $a \neq -1$
$\frac{1}{x}$	$\ln\|x\| + C$
$g(x)^a g'(x)$	$\frac{g(x)^{a+1}}{a+1} + C$ if $a \neq -1$

$f(x)$	$F(x) = \int f(x)\,\mathrm{d}x$
$\sin x$	$-\cos x + C$
$g'(x)\sin g(x)$	$-\cos g(x) + C$
$\cos x$	$\sin x + C$
$\tan x$	$\ln\lvert\sec x\rvert + C$
$\csc x$	$\ln\lvert\csc x - \cot x\rvert + C$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + C$
$\cot x$	$\ln\lvert\sin x\rvert + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$

$f(x)$	$F(x) = \int f(x)\,\mathrm{d}x$
e^x	$e^x + C$
$e^{g(x)}g'(x)$	$e^{g(x)} + C$
e^{ax}	$\frac{1}{a}\,e^{ax} + C$
a^x	$\frac{1}{\ln a}\,a^x + C$
$\ln x$	$x\ln x - x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\frac{g'(x)}{\sqrt{1-g(x)^2}}$	$\arcsin g(x) + C$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin \frac{x}{a} + C$
$\frac{1}{1+x^2}$	$\arctan x + C$
$\frac{g'(x)}{1+g(x)^2}$	$\arctan g(x) + C$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\frac{x}{a} + C$
$\frac{1}{x\sqrt{x^2-1}}$	$\operatorname{arcsec} x + C \quad (x > 1)$

Appendix E

TABLE OF TAYLOR EXPANSIONS

Let $n \geqslant$ be an integer. Then if the function f has $n+1$ derivatives on an interval that contains both x_0 and x, we have the Taylor expansion

$$f(x) = f(x_0) + f'(x_0)\,(x-x_0) + \frac{1}{2!}f''(x_0)\,(x-x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)\,(x-x_0)^n$$
$$+\frac{1}{(n+1)!}f^{(n+1)}(c)\,(x-x_0)^{n+1} \qquad \text{for some } c \text{ between } x_0 \text{ and } x$$

The limit as $n \to \infty$ gives the Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

for f. When $x_0 = 0$ this is also called the Maclaurin series for f. Here are Taylor series expansions of some important functions.

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{1}{n!}x^n && \text{for } -\infty < x < \infty \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} && \text{for } -\infty < x < \infty \\
&= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} && \text{for } -\infty < x < \infty \\
&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n && \text{for } -1 \leqslant x < 1 \\
&= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots
\end{aligned}$$

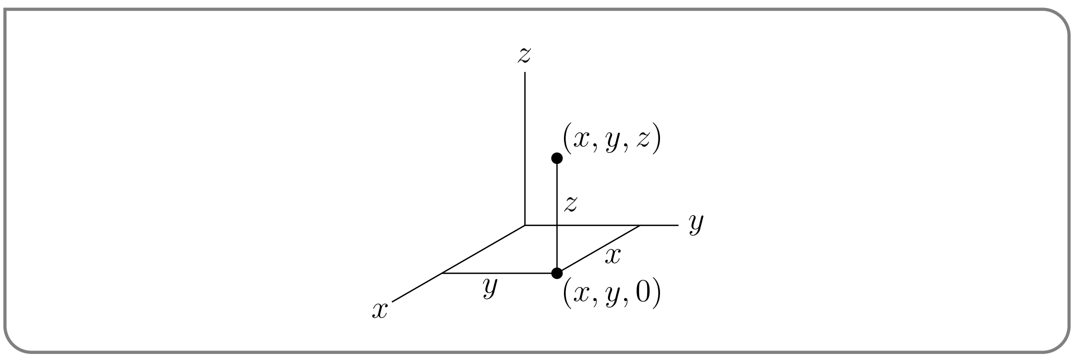
$$
\begin{aligned}
\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n && \text{for } -1 < x \leqslant 1 \\
&= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \\
\ln(1-x) &= -\sum_{n=1}^{\infty} \frac{1}{n} x^n && \text{for } -1 \leqslant x < 1 \\
&= -x - \tfrac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots - \frac{1}{n}x^n - \cdots \\
\ln(1+x) &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n && \text{for } -1 < x \leqslant 1 \\
&= x - \tfrac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots - \frac{(-1)^n}{n}x^n - \cdots \\
(1+x)^p &= 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \\
&\qquad + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n + \cdots
\end{aligned}
$$

Appendix F

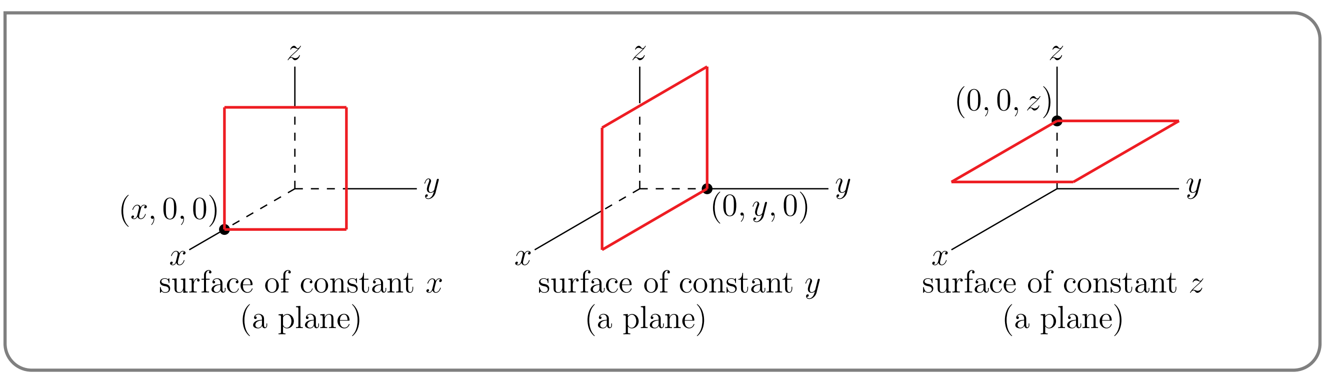
3D Coordinate Systems

F.1▲ Cartesian Coordinates

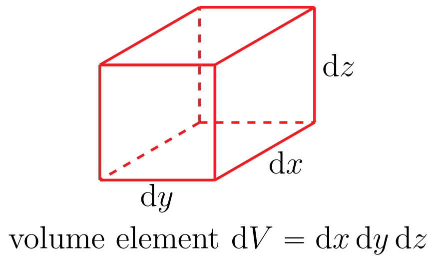
Here is a figure showing the definitions of the three Cartesian coordinates (x, y, z)



and here are three figures showing a surface of constant x, a surface of constant x, and a surface of constant z.



Finally here is a figure showing the volume element $\mathrm{d}V$ in cartesian coordinates.



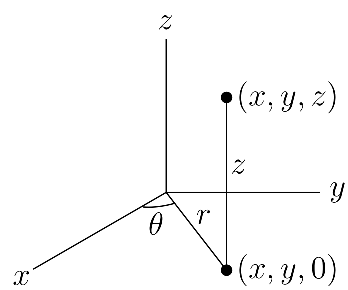
F.2▴ Cylindrical Coordinates

Here is a figure showing the definitions of the three cylindrical coordinates

$r =$ distance from $(0, 0, 0)$ to $(x, y, 0)$

$\theta =$ angle between the the x axis and the line joining $(x, y, 0)$ to $(0, 0, 0)$

$z =$ signed distance from (x, y, z) to the xy–plane

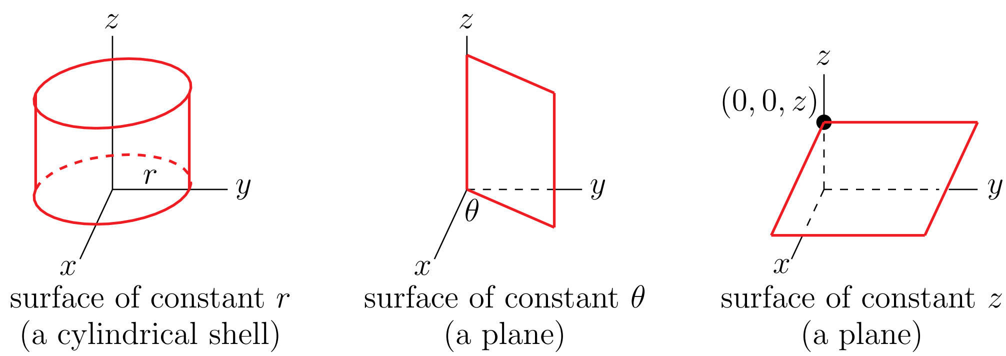


The cartesian and cylindrical coordinates are related by

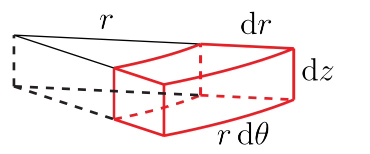
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\frac{y}{x} \qquad z = z$$

Here are three figures showing a surface of constant r, a surface of constant θ, and a surface of constant z.



Finally here is a figure showing the volume element dV in cylindrical coordinates.



volume element $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

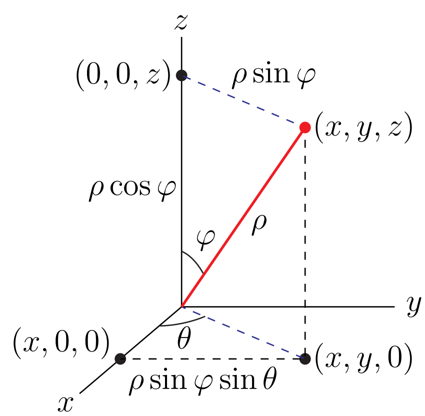
F.3▲ Spherical Coordinates

Here is a figure showing the definitions of the three spherical coordinates

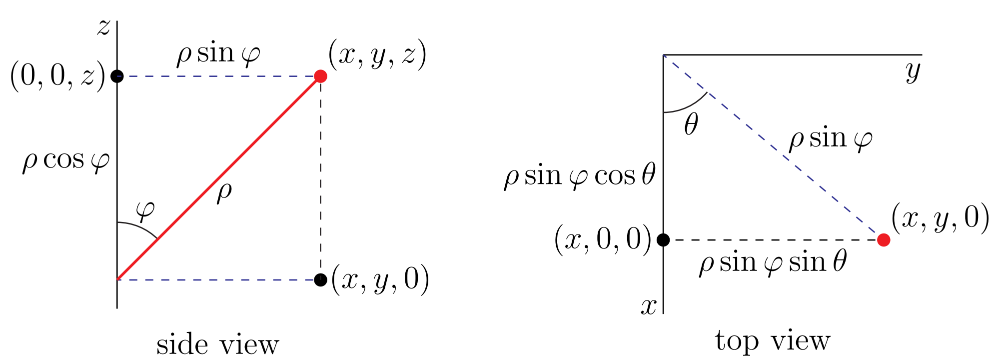
$\rho =$ distance from $(0,0,0)$ to (x,y,z)

$\varphi =$ angle between the z axis and the line joining (x,y,z) to $(0,0,0)$

$\theta =$ angle between the x axis and the line joining $(x,y,0)$ to $(0,0,0)$



and here are two more figures giving the side and top views of the previous figure.

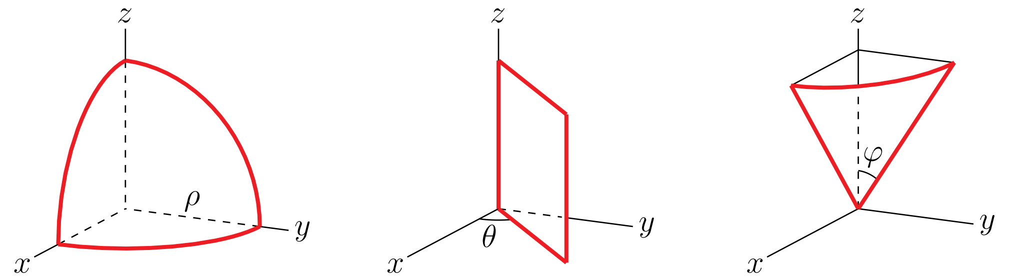


The cartesian and spherical coordinates are related by

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

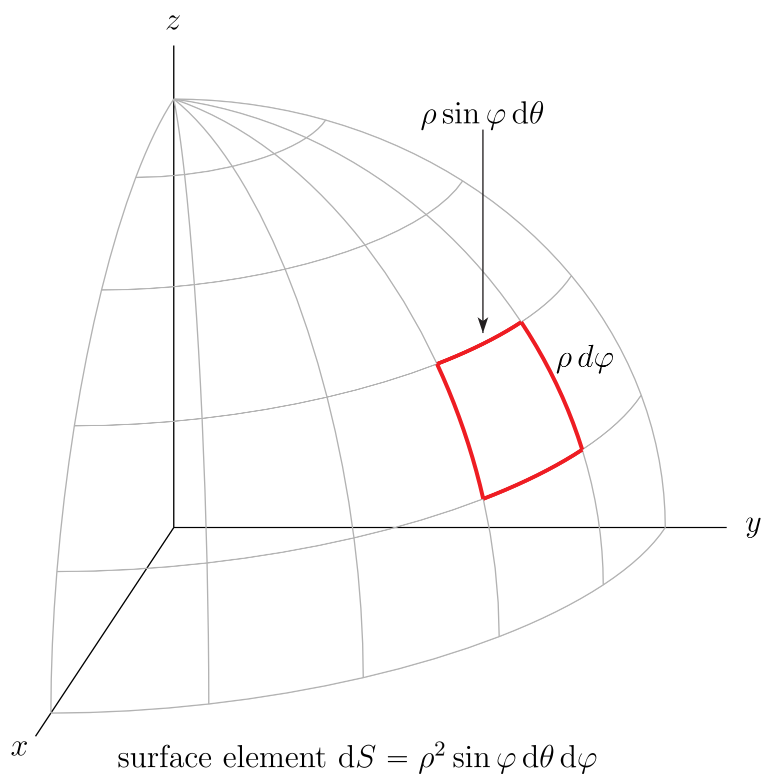
$$\rho = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\frac{y}{x} \qquad \varphi = \arctan\frac{\sqrt{x^2+y^2}}{z}$$

Here are three figures showing a surface of constant ρ, a surface of constant θ, and a surface of constant φ.



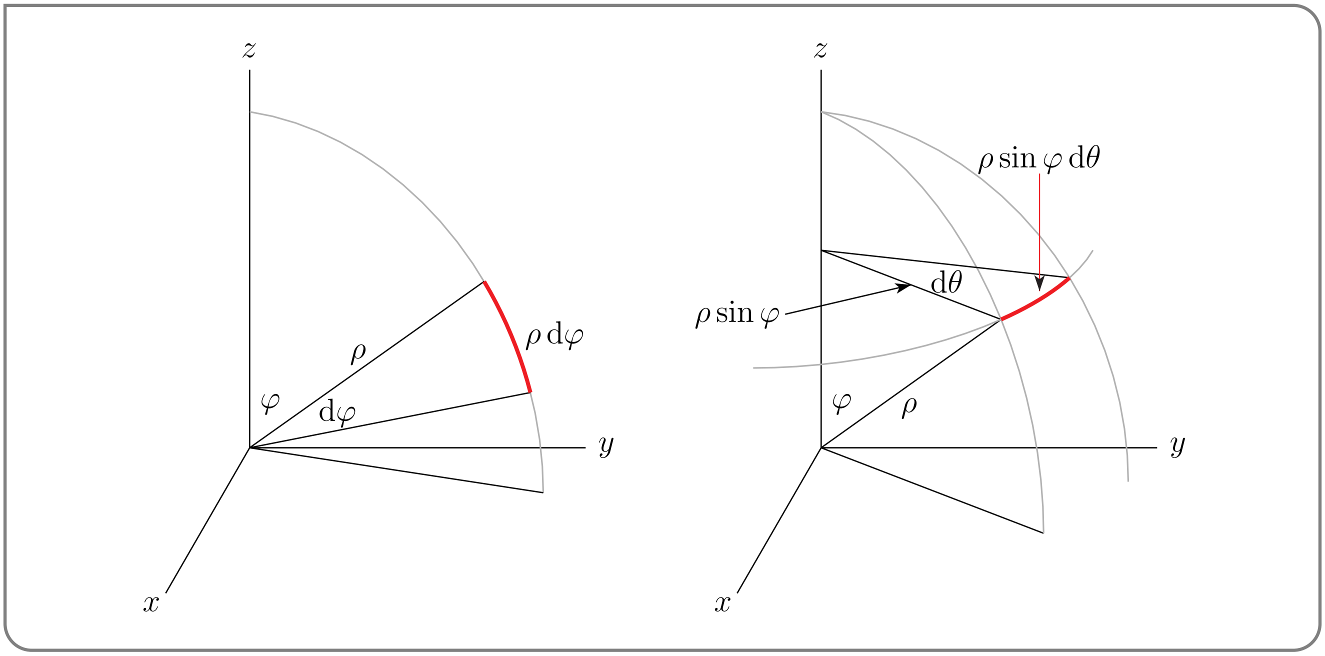
surface of constant ρ (a sphere) surface of constant θ (a plane) surface of constant φ (a cone)

Here is a figure showing the surface element dS in spherical coordinates

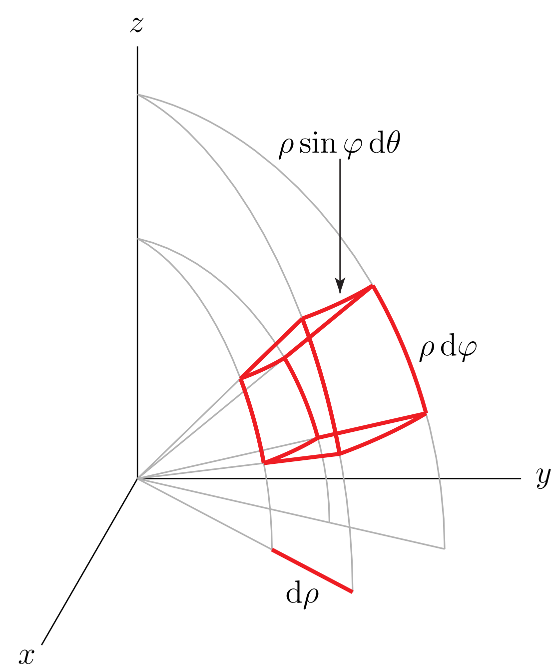


surface element $\mathrm{d}S = \rho^2 \sin\varphi\, \mathrm{d}\theta\, \mathrm{d}\varphi$

and two extracts of the above figure to make it easier to see how the factors $\rho\ \mathrm{d}\varphi$ and $\rho \sin\varphi\ \mathrm{d}\theta$ arise.



Finally, here is a figure showing the volume element $\mathrm{d}V$ in spherical coordinates



volume element $\mathrm{d}V = \rho^2 \sin\varphi\ \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

Appendix G

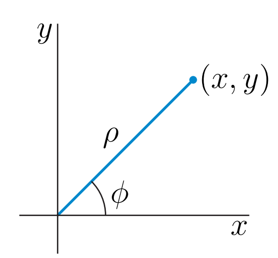
ISO Coordinate System Notation

In this text we have chosen symbols for the various polar, cylindrical and spherical coordinates that are standard for mathematics. There is another, different, set of symbols that are commonly used in the physical sciences and engineering. Indeed, there is an international convention, called ISO 80000-2, that specifies those symbols[1]. In this appendix, we summarize the definitions and standard properties of the polar, cylindrical and spherical coordinate systems using the ISO symbols.

G.1 Polar Coordinates

In the ISO convention the symbols ρ and ϕ are used (instead of r and θ) for polar coordinates.

$$\begin{aligned} \rho &= \text{the distance from } (0,0) \text{ to } (x,y) \\ \phi &= \text{the (counter-clockwise) angle between the } x \text{ axis and the line joining } (x,y) \text{ to } (0,0) \end{aligned}$$



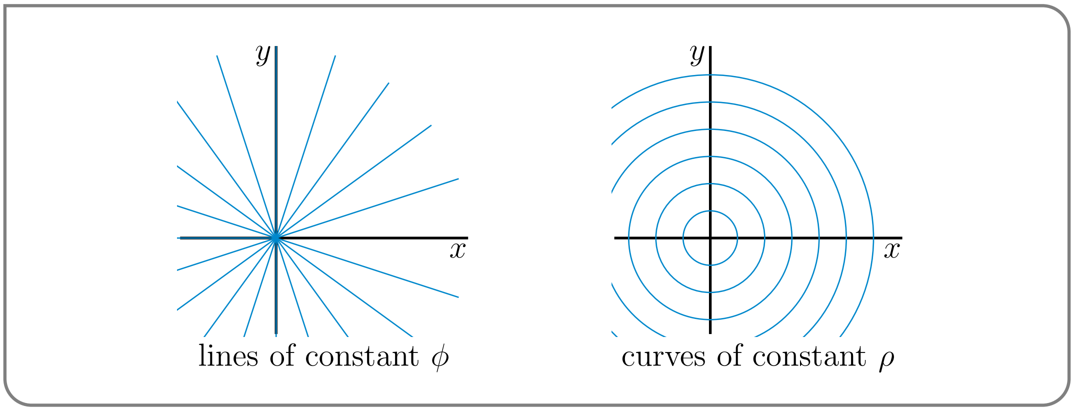
Cartesian and polar coordinates are related by

$$x = \rho\cos\phi \qquad y = \rho\sin\phi$$

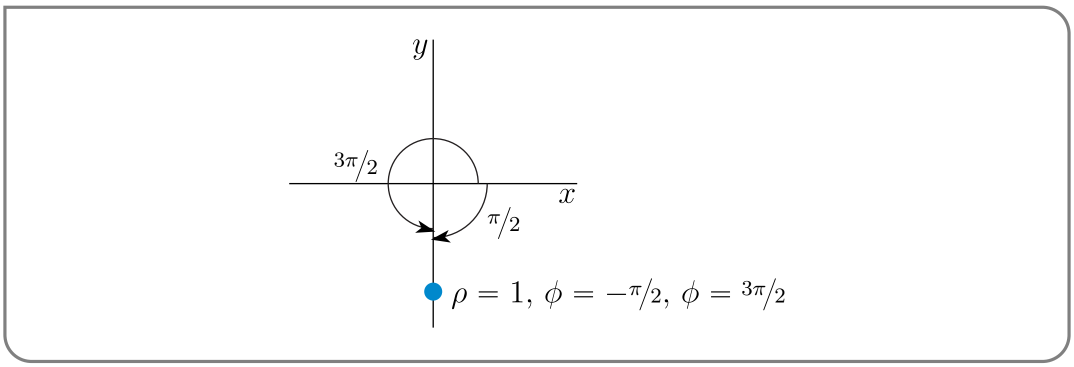
$$\rho = \sqrt{x^2+y^2} \qquad \phi = \arctan\frac{y}{x}$$

1 It specifies more than just those symbols. See https://en.wikipedia.org/wiki/ISO_31-11 and https://en.wikipedia.org/wiki/ISO/IEC_80000. The full ISO 80000-2 is available at https://www.iso.org/standard/64973.html — for $$.

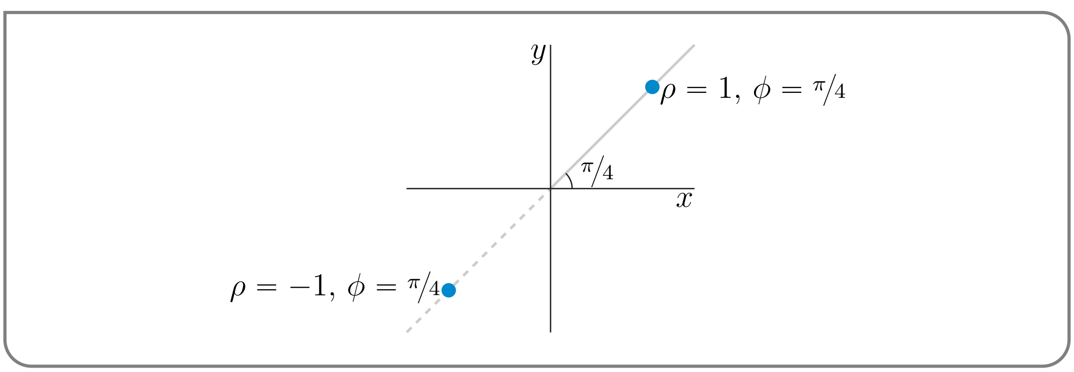
The following two figures show a number of lines of constant ϕ, on the left, and curves of constant ρ, on the right.



Note that the polar angle ϕ is only defined up to integer multiples of 2π. For example, the point $(1, 0)$ on the x-axis could have $\phi = 0$, but could also have $\phi = 2\pi$ or $\phi = 4\pi$. It is sometimes convenient to assign ϕ negative values. When $\phi < 0$, the counter-clockwise angle ϕ refers to the clockwise angle $|\phi|$. For example, the point $(0, -1)$ on the negative y-axis can have $\phi = -\frac{\pi}{2}$ and can also have $\phi = \frac{3\pi}{2}$.



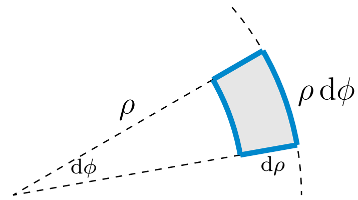
It is also sometimes convenient to extend the above definitions by saying that $x = \rho \cos\phi$ and $y = \rho \sin\phi$ even when ρ is negative. For example, the following figure shows (x, y) for $\rho = 1$, $\phi = \pi/4$ and for $\rho = -1$, $\phi = \pi/4$. Both points lie on the line through



the origin that makes an angle of $45°$ with the x-axis and both are a distance one from the origin. But they are on opposite sides of the the origin.

The area element in polar coordinates is

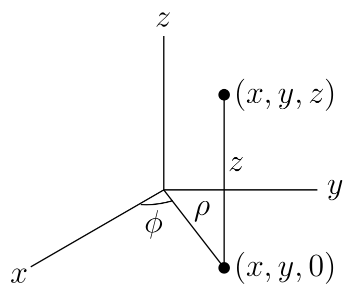
$$dA = \rho \, d\rho \, d\phi$$



G.2▲ Cylindrical Coordinates

In the ISO convention the symbols ρ, ϕ and z are used (instead of r, θ and z) for cylindrical coordinates.

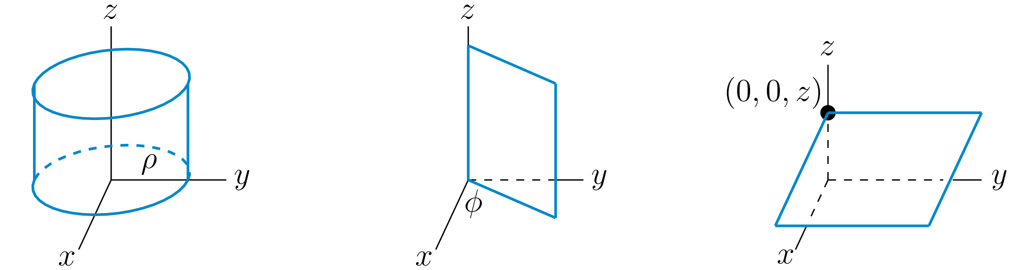
$$\begin{aligned} \rho &= \text{ distance from } (0,0,0) \text{ to } (x,y,0) \\ \phi &= \text{ angle between the the } x \text{ axis and the line joining } (x,y,0) \text{ to } (0,0,0) \\ z &= \text{ signed distance from } (x,y,z) \text{ to the } xy\text{-plane} \end{aligned}$$



The cartesian and cylindrical coordinates are related by

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$
$$\rho = \sqrt{x^2+y^2} \qquad \phi = \arctan\frac{y}{x} \qquad z = z$$

Here are three figures showing a surface of constant ρ, a surface of constant ϕ, and a surface of constant z.

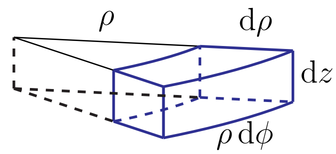


surface of constant ρ (a cylindrical shell)

surface of constant ϕ (half a plane)

surface of constant z (a plane)

Finally here is a figure showing the volume element $\mathrm{d}V$ in cylindrical coordinates.

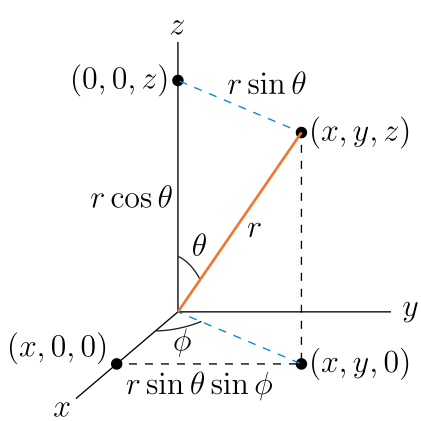


volume element $\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$

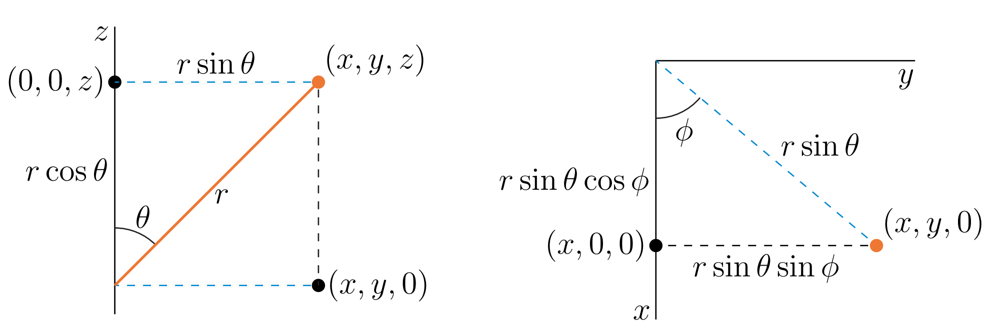
G.3▲ Spherical Coordinates

In the ISO convention the symbols r (instead of ρ), ϕ (instead of θ) and θ (instead of ϕ) are used for spherical coordinates.

$$\begin{aligned} r &= \text{ distance from } (0,0,0) \text{ to } (x,y,z) \\ \theta &= \text{ angle between the } z \text{ axis and the line joining } (x,y,z) \text{ to } (0,0,0) \\ \phi &= \text{ angle between the } x \text{ axis and the line joining } (x,y,0) \text{ to } (0,0,0) \end{aligned}$$



Here are two more figures giving the side and top views of the previous figure.

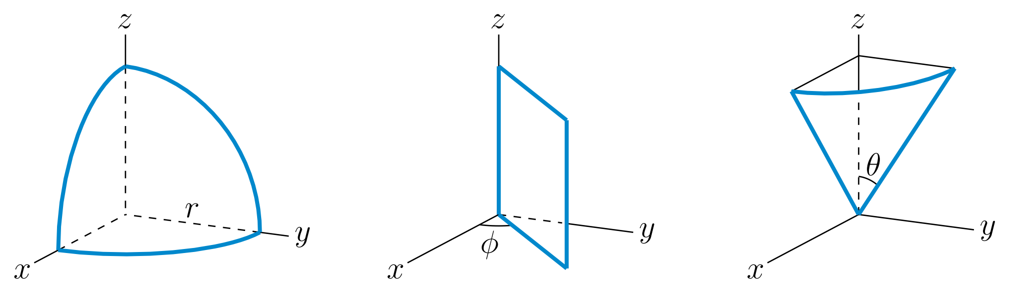


The cartesian and spherical coordinates are related by

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

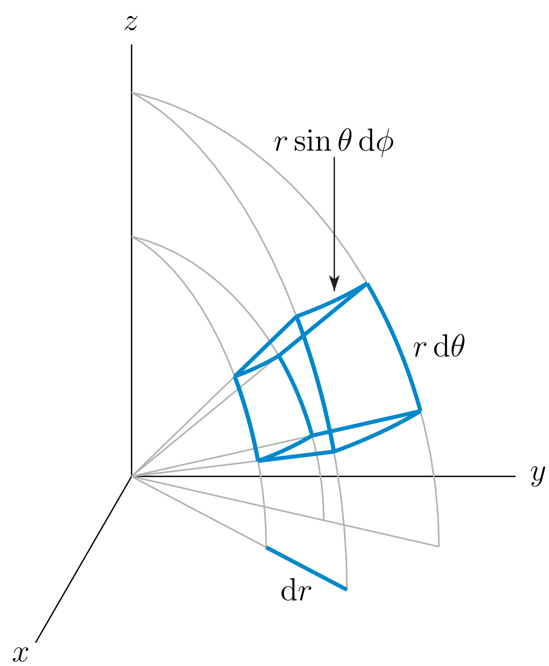
$$r = \sqrt{x^2+y^2+z^2} \qquad \phi = \arctan\frac{y}{x} \qquad \theta = \arctan\frac{\sqrt{x^2+y^2}}{z}$$

Here are three figures showing a surface of constant r, a surface of constant ϕ, and a surface of constant θ.



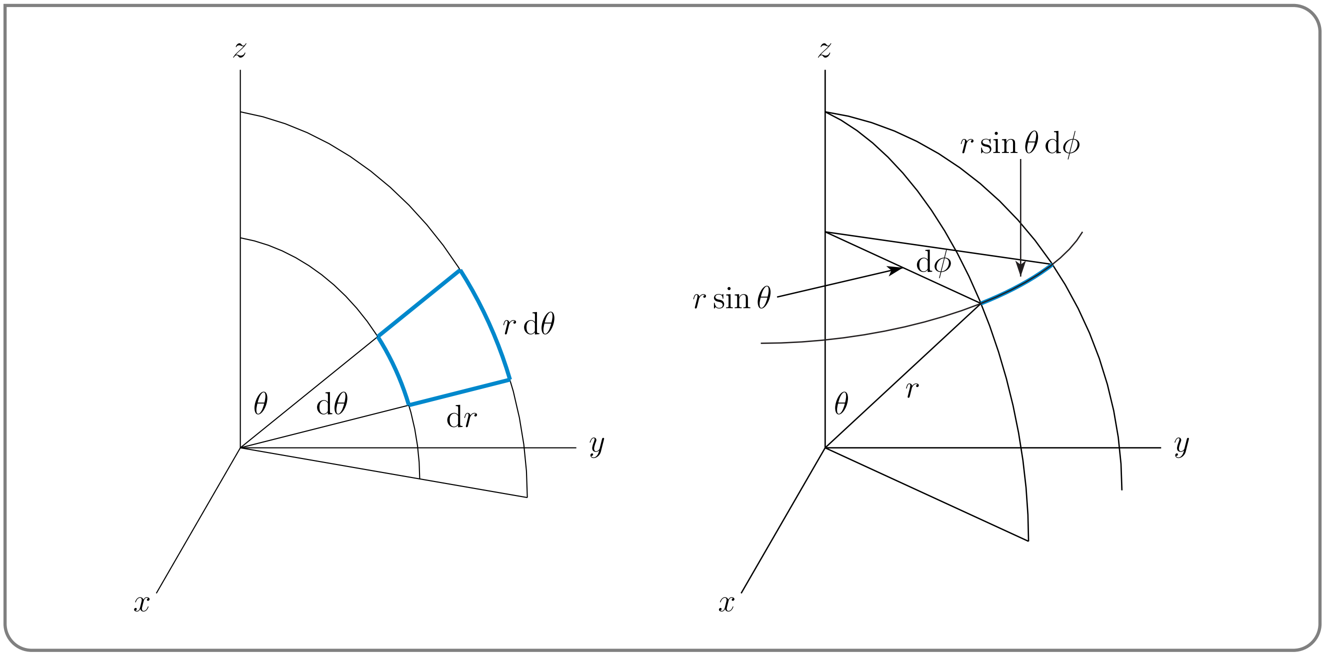
surface of constant r (a sphere) — surface of constant ϕ (half a plane) — surface of constant θ (a cone)

Finally, here is a figure showing the volume element dV in spherical coordinates



volume element $\mathrm{d}V = r^2\sin\theta\;\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

and two extracts of the above figure to make it easier to see how $r\;\mathrm{d}\theta$ and $r\sin\theta\;\mathrm{d}\phi$ arise.

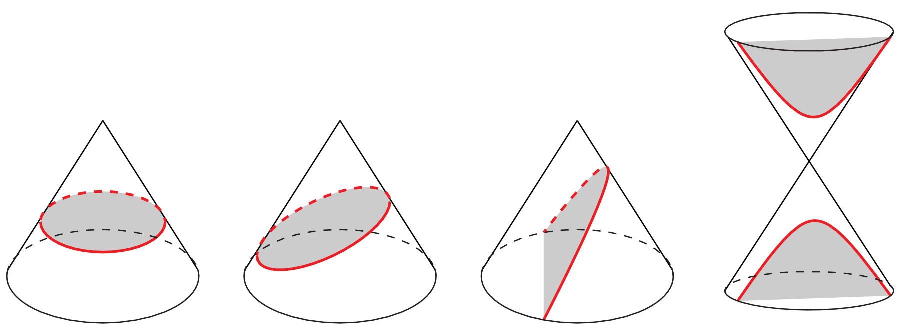


z
θ
$\mathrm{d}\theta$
$r\,\mathrm{d}\theta$
$\mathrm{d}r$
y
x
z
$r\sin\theta\,\mathrm{d}\phi$
$\mathrm{d}\phi$
$r\sin\theta$
θ
r
y
x

Appendix H

Conic Sections and Quadric Surfaces

A conic section is the curve of intersection of a cone and a plane that does not pass through the vertex of the cone. This is illustrated in the figures below. An equivalent[1] (and often



used) definition is that a conic section is the set of all points in the xy–plane that obey $Q(x, y) = 0$ with

$$Q(x,y) = Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

being a polynomial of degree two[2]. By rotating and translating our coordinate system the equation of the conic section can be brought into one of the forms[3]

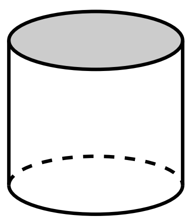
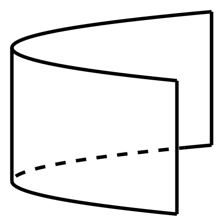
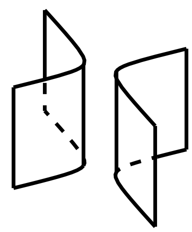
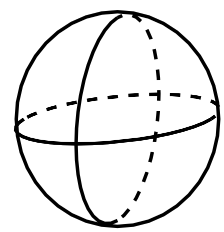
- $\alpha x^2 + \beta y^2 = \gamma$ with $\alpha, \beta, \gamma > 0$, which is an ellipse (or a circle),
- $\alpha x^2 - \beta y^2 = \gamma$ with $\alpha, \beta > 0$, $\gamma \neq 0$, which is a hyperbola,
- $x^2 = \delta y$, with $\delta \neq 0$ which is a parabola.

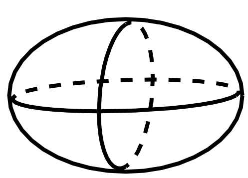
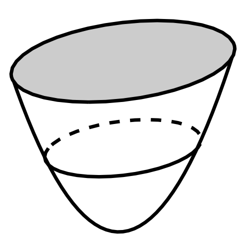
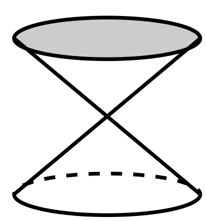
1 It is outside our scope to prove this equivalence.

2 Technically, we should also require that the constants A, B, C, D, E, F, are real numbers, that A, B, C are not all zero, that $Q(x, y) = 0$ has more than one real solution, and that the polynomial can't be factored into the product of two polynomials of degree one.

3 This statement can be justified using a linear algebra eigenvalue/eigenvector analysis. It is beyond what we can cover here, but is not too difficult for a standard linear algeba course.

The three dimensional analogs of conic sections, surfaces in three dimensions given by quadratic equations, are called quadrics. An example is the sphere $x^2 + y^2 + z^2 = 1$. Here are some tables giving all of the quadric surfaces.

name	elliptic cylinder	parabolic cylinder	hyperbolic cylinder	sphere
equation in standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y = ax^2$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x^2 + y^2 + z^2 = r^2$
$x =$ constant cross–section	two lines	one line	two lines	circle
$y =$ constant cross–section	two lines	two lines	two lines	circle
$z =$ constant cross–section	ellipse	parabola	hyperbola	circle
sketch				

name	ellipsoid	elliptic paraboloid	elliptic cone
equation in standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
$x =$ constant cross–section	ellipse	parabola	two lines if $x = 0$ hyperbola if $x \neq 0$
$y =$ constant cross–section	ellipse	parabola	two lines if $y = 0$ hyperbola if $y \neq 0$
$z =$ constant cross–section	ellipse	ellipse	ellipse
sketch			

name	hyperboloid of one sheet	hyperboloid of two sheets	hyperbolic paraboloid
equation in standard form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$	$\frac{y^2}{b^2}-\frac{x^2}{a^2}=\frac{z}{c}$
$x=$ constant cross–section	hyperbola	hyperbola	parabola
$y=$ constant cross–section	hyperbola	hyperbola	parabola
$z=$ constant cross–section	ellipse	ellipse	two lines if $z=0$ hyperbola if $z\neq 0$
sketch