Q1.
$a(x) = x^2 + 2x + 9$
$b(x) = x^3 + 11x^2 + x + 7.$

Find the product $a(x) \cdot b(x)$, but noting the coeffs are understood to live in $\mathbb{Z}_{11}$.

Multiply as normal and reduce coeffs mod 11 afterwards.

$$\boxed{a(x) \cdot b(x)} = x^5 + 11x^4 + x^3 + 7x^2$$
$$2x^4 + 22x^3 + 2x^2 + 14x.$$
$$9x^3 + 99x^2 + 9x + 63$$

(adding up)

$$= x^5 + 13x^4 + 32x^3 + 108x^2 + 23x + 63$$

(reduce coeffs modulo 11)

$$\boxed{= x^5 + 2x^4 + 10x^3 + 9x^2 + x + 8}$$

(since $108 \equiv 9 \pmod{11}$ etc)

Second ex

Polys with coeffs from $\mathbb{Z}_2$

$m(x) = x^8 + x^4 + x^3 + x + 1.$
$p(x) = x^5 + x^3 + x^2 + 1.$

First, apply the E.A. and confirm that
$\gcd(m(x), p(x)) = 1.$

E.A. proceeds with a sequence of

"poly. divisions with remainder"
the first of which will be

$$\boxed{m(x) = q_1(x)\, p(x) + r_1(x)}$$

for some polys q_1, r_1 and $\deg(r_1) < \deg(p)$

$$\boxed{x^8 + x^4 + x^3 + x + 1} = x^3 \overbrace{(x^5 + x^3 + x^2 + 1)}^{p} + \sim\sim$$

$$= \underbrace{(x^8 + x^6 + x^5 + x^3)}_{x^3 p} + \boxed{(x^6 + x^5 + x^4 + x + 1)}$$

$$= x^3 p(x) + x(x^5 + x^3 + x^2 + 1) + \sim\sim$$

$$= x^3 p(x) + \underbrace{(x^6 + x^4 + x^3 + x)}_{x\,p(x)} + x^5 + x^3 + 1.$$

$$= (x^3 + x)\, p(x) + x^5 + x^3 + 1$$

$$= \boxed{\underbrace{(x^3 + x + 1)\, p(x) + x^2}_{q_1(x) \cdot p(x) + r_1(x)}} \quad (1)$$

$q_1(x) = x^3 + x + 1$

$r_1(x) = x^2$

Now obtain the second poly. div. with rem.

$$p(x) = q_2(x)\, r_1(x) + r_2(x)$$

$$\boxed{x^5 + x^3 + x^2 + 1 = \underbrace{(x^3 + x + 1)}_{q_2(x)} \underbrace{x^2}_{r_1(x)} + \overbrace{1}^{r_2(x)}.} \quad (2)$$

The fact we obtain a remainder poly. 1 confirms that $\gcd(m(x), p(x)) = 1$

Now for $p^{-1}(x)$ in $GF(2^8)$.

So we seek a poly p^{-1} such that

$$p^{-1}(x)\, p(x) \equiv 1 \mod m(x)$$

From ② we get

$$1 = p(x) + q_2(x)\, x^2$$

then from eq. ①, I replace x^2 to get.

$$1 = p(x) + q_2(x)\big(m(x) + q_1(x)\, p(x)\big)$$

$$= q_2(x)\, m(x) + \underbrace{\big(1 + q_2(x)\, q_1(x)\big)}_{p^{-1}(x)}\, p(x)$$

So $\big(1 + q_2(x)\, q_1(x)\big)\, p(x) \equiv 1 \mod m(x)$

So $p^{-1}(x) = 1 + q_2(x)\, q_1(x)$

$$= 1 + (x^3+x+1)(x^3+x+1)$$

$$= 1 + x^6 + x^4 + x^3 + x^4 + x^2 + x$$
$$+ x^3 + x + 1$$

$$= x^6 + x^2.$$

So finally we can say

$$p^{-1}(x) = x^6 + x^2.$$

We could go an confirm

$$(x^6+x^2)(x^5+x^3+x^2+1)$$

$$= x^{11} + \dots\dots + x^2$$

$$= (\underbrace{\qquad\qquad}_{\text{some qoutient poly.}})\, m(x) + 1.$$

$$\equiv 1 \pmod{m(x)}$$