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Introduction to Number Theory

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6G6Z0024 Applied Cryptography

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Introduction to the unit

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- 6G6Z0024 Applied Cryptography (15 credits)
- Timetable
- Let’s look at the Moodle page for the unit.

Introduction to Number Theory

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We deal with the positive and negative *counting* numbers, more properly named the *integers*, and denoted by the symbol $\mathbb{Z}$, (coming from the German *Zahl*, for number)

- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Z}$ is an *infinite* set.
- $\mathbb{Z}$ obviously carries the operations of addition, $+$, and multiplication, $\cdot$, that you've known from primary school.

Modern cryptography relies heavily on techniques and facts from *number theory*, which is the mathematical study of the integers and their properties under $+$ and $\cdot$.

Topics we need to know

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- The **divisibility relation** on $\mathbb{Z}$.
- **Greatest Common Divisors** (gcd) and the **Euclidean Algorithm**.
- The **congruence relation** and **modular arithmetic**.
- **Prime** numbers and
    - The **Fundamental Theorem of Arithmetic** and **prime factorizations**
    - **Fermat's Little Theorem**
    - **Euler's totient** function
    - **Euler's theorem**
    - **Primality** testing, the **Miller-Rabin** test
- **The Chinese Remainder Theorem**
- **Discrete logarithms**

All these covered in Stallings, Chapter 2: Introduction to Number Theory.

The divisibility relation

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- Recall, a *relation* in computer science / mathematics is a formula $A(x_1, \ldots, x_n)$, so that when values are supplied for the variables $x_1, \ldots, x_n$, results in a *statement* $A(x_1, \ldots, x_n)$, i.e. something which is true or false.
- For a pair of integers a, b, with $b \neq 0$, we say b *divides* a, and write $b|a$ if there exists an integer c such that

$$a = b \cdot c,$$

  and if no such integer c exists then we say b does *not divide* a, and can write $b \not| a$.
- So $b|a$ is a binary relation on a, b, i.e. a statement that is true or false, depending on the values of a, b.
- If $b|a$ then we say b is a *factor* or *divisor* of a.

Examples

- $3|15, 5|15, 1|15, 15|15$.
- $3 \not| 10, 17 \not| 20$.

Properties of divisibility

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The divisibility relation enjoys the following properties, which can all be demonstrated (and proved) using its definition and basic properties of the integers.

- If $a|1$ then $a = \pm 1$, i.e. $a = -1$ or $a = +1$.
- If $a|b$ and $b|a$ then $a = \pm b$.
- For all non-zero integers b, we have $b|0$, i.e. *everything divides 0*.
- If $a|b$ and $b|c$ then $a|c$, i.e. the divisibility relation is *transitive*, it travels through intermediaries.
- If $x|y$ and $x|z$ then for all pairs of integer coefficients α, β, we have

$$x|(\alpha \cdot y + \beta \cdot z),$$

  i.e. x divides all *linear combinations* of y and z.

To familiarise yourself with these, work through some examples of the transitivity of divisibility and the divisibility of linear combinations.

The integer division algorithm

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Do you remember this kind of thing from primary school?

- 20 divided by 3, goes in 6 times, with remainder 2.
- $20 = 6 \cdot 3 + 2$

The *integer division algorithm* is simply a formalization of this. It is:

- Given any postitive integer n and any non-negative integer a, we can divide a by n to get an integer quotient q and remainder r that satisfy
- $a = qn + r$, and $0 \leq r < n$, and $q = \lfloor a/n \rfloor$
- $\lfloor x \rfloor$ is defined as the largest integer less than x, the so-called *floor* function.

Greatest Common Divisors (gcd)

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We write $\gcd(a, b)$ for the *greatest common divisor of a and b*. So $\gcd$ is defined by

- $\gcd(a, b) = d$, where d is the alrgest integer that divides both a and b.
- For neatness, we also define $\gcd(0, 0) = 0$.

For example

- $\gcd(60, 24) = 12, \gcd(100, 75) = 25, \gcd(15, 32) = 1$.
- Note that, by its definition, $\gcd$ will always be non-negative, i.e. $\gcd(-60, 24) = 12$.

For small arguments a, b, we can calculate $\gcd(a, b)$ *in our heads*, so to speak.

- $\gcd(25, 3) =?, \gcd(99, 27) =?, \ldots$.
- But what about $\gcd(12349878973245, 324765)$?

The Euclidean Algorithm

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In fact there is a classic algorithm that can quickly determine $\gcd$, and establishes the following, non-obvious fact,

- $\gcd(a, b)$ is the smallest postitive integer d that can be written in the form

$$d = x \cdot a + y \cdot b,$$

for integer coefficients x, y.

The Euclidean algortihm was known to ancient mathematicians and has severl important uses and generalisations in mathematics and cryptography.

The Euclidean Algorithm

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A detailed treatment is given in Stallings. The algorithm depends on the following property of $\gcd$.

- If $a = qn + r$ then $\gcd(a, n) = \gcd(n, r)$.

This is true because

- if d is a common divisor of a and n, then since $r = a - qn$, i.e. r is a linear combination of a and n, then d divides r also. And so d is a common divisor of n and r.
- Similarly we can show that if e is a common divisor of n and r, then e divides a also. And so e will be a common divisor of a and n.
- So the pairs (a, n) and (n, r) have the exact same set of common divisors.
- Therefore,

$$\gcd(a, n) = \gcd(n, r).$$

The Euclidean Algorithm

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The algorithm works by repeatedly applying the property from the last slide, to a sequence of integer divisions, until the $\gcd$ is clear. Best seen with a worked example

- What is $\gcd(710, 310)$?
- $710 = 2 \cdot 310 + 90$ so $\gcd(710, 310) = \gcd(310, 90)$,
- $310 = 3 \cdot 90 + 40$ so $\gcd(310, 90) = \gcd(90, 40)$,
- $90 = 2 \cdot 40 + 10$ so $\gcd(90, 40) = \gcd(40, 10)$,
- $40 = 4 \cdot 10 + 0$ so $\gcd(40, 10) = \gcd(10, 0) = 10$.

Note that

- The algorithm will terminate, since the remainders are a strictly decreasing sequence of non-negative integers.
- By definition of divisibility, $\gcd(x, 0) = x$, for all integers x.
- The $\gcd$ equations associated to the integer divisions all link together.
- So we can conclude that

$$\gcd(710, 310) = 10.$$

See Stallings for the full detail, a flowchart specification of the algorithm, and more examples.

The mod operator and the congruence relation

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For an integer a and a positive integer n we say that a *modulo* n is the remainder r in the integer division of a by n.

- $a = qn + r, 0 \leq r < n$
- We write $(a \mod n) = r$.
- n is called the *modulus* in this expression.

For example

- $(11 \mod 7) = 4$ and $(-11 \mod 4) = 1$.

There is an associated binary relation here. We say that two integers a and b are *congruent modulo* n, written as

$$a \equiv b \pmod{n},$$

if

- $(a \mod n) = (b \mod n)$
- That is, if a and b *leave the same remainder*, after division by n.

The mod operator and the congruence relation

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Examples

- $23 \equiv 8 \pmod{5}$
- $-11 \equiv 5 \pmod{8}$
- $81 \equiv 0 \pmod{27}$

The congruence relation has the following properties

- $a \equiv b \pmod{n}$ if and only if $n | (a - b)$
- $a \equiv a \pmod{n}$, called *reflexivity*
- $a \equiv b \pmod{n}$ implies that $b \equiv a \pmod{n}$, called *symmetry*
- If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$, called *transitivity*
- These last three properties mean congruence modulo n is an *equivalence relation* on $\mathbb{Z}$.

Modular arithmetic

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- The mod operator $(a \mod n)$ maps all integers a into the set

$$\mathbb{Z}_n = \{0, 1, 2, 3, \ldots, n-1\}.$$

- This is the set of *residues*, or *remainders*, modulo n.
- The familiar operations of $+$ and $\cdot$ on $\mathbb{Z}$ extend to $\mathbb{Z}_n$ in a natural way.

$$(a \mod n) + (b \mod n) := ((a+b) \mod n)$$

$$(a \mod n) \cdot (b \mod n) := ((a \cdot b) \mod n)$$

This means that $\mathbb{Z}_n$, with the operations of $+$ and $\cdot$ will form a *closed system* with respect to these operations, i.e. for any pair x, y from $\mathbb{Z}_n$, $x + y$ and $x \cdot y$ will again be elements of $\mathbb{Z}_n$.

See Stallings for worked examples of $\mathbb{Z}_8$ under $+$ and $\cdot$.

Modular arithmetic

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- So given x from $\mathbb{Z}_n$, x will have an *additive inverse*, $n - x$, which satisfies

$$x + (n - x) \equiv 0 \pmod{n}.$$

- Given x from $\mathbb{Z}_n$, if there exists a y in $\mathbb{Z}_n$ which satisfies

$$x \cdot y \equiv 1 \pmod{n},$$

  then we say y is the *multiplicative inverse of x modulo n*, and vice versa. We can write $y \equiv x^{-1} \pmod{n}$.
- But multiplicative inverses do not necessarily exist for every element of $\mathbb{Z}_n$.

Modular arithmetic

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This is connected to the issue of cancellation in $\mathbb{Z}_n$.

- If $(a+b) \equiv (a+c) \pmod{n} then b \equiv c \pmod{n}$.
- If $(a \cdot b) \equiv (a \cdot c) \pmod{n}$ then it's not neccessarily true that $b \equiv c \pmod{n}$.
- However if $a^{-1} \pmod{n}$ exists then we can cancel from products as

$$a^{-1}(a \cdot b) \equiv a^{-1}(a \cdot c) \pmod{n}$$

and so

$$(a^{-1}a) \cdot b \equiv (a^{-1}a) \cdot c \pmod{n}$$

and so

$$b \equiv c \pmod{n}.$$

Extended Euclidean algorithm and multiplicative inverses

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Using linear combinations and the Euclidean algorithm we can show that

- for a in $\mathbb{Z}_n$, a multiplicative inverse of a modulo n will exists if and only if $\gcd(a, n) = 1$.

Terminology

- If $\gcd(x, y) = 1$ then x, y are said to be *relatively prime*, or *coprime*.

See Stallings chapter 2 for details.

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