Manchester Metropolitan University

Advanced Encryption Standard (AES)

Killian O'Brien

6G6Z0024 Applied Cryptography 2025/26

Lecture Week 06 – Mon 02 November 2025

Introduction

- The **Advanced Encryption Standard** (AES), published by NIST in 2001.
- To replace DES due to security concerns over small key size and other considerations.
- AES designed to be more secure and fast.
- From 2008 on, chip manufacturers implement AES capabilities in low-level chip design.
- Now the most widely used cipher.

Polynomial arithmetic and the fields $GF(2^n)$

- Consider all polynomials

$$f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 = \sum_{i=0}^{n-1} a_i x^i, \quad a_i \in \mathbb{Z}_2.$$

  of degree $n-1$ or less.
- Arithmetic follows the rules of $+$ and $\cdot$ for polynomials, with arithmetic of the coefficients a_i carried out in $\mathbb{Z}_2$, i.e. addition of coefficients is the same as XOR.
- If multiplication results in a polynomial of degree greater than $n-1$ then the product is reduced modulo a specified irreducible polynomial $m(x)$ of degree n, the modulos polynomial.

AES and $\mathrm{GF}(2^8)$.

- The Advanceed Encryption Standard (AES) uses such a field $\mathrm{GF}(2^8)$, consisting of polynomials of degree less than or equal to 7, with binary coefficients and polynomial operations carried out modulo the irreducible polynomial

$$m(x) = x^8 + x^4 + x^3 + x + 1.$$

- The figure on the right shows the calculation of an example product in $\mathrm{GF}(2^8)$.
- AES uses this since it is designed to operate on 8-bit bytes.
    - Addition of bytes is just bit-wise XOR.
    - *Multiplication* of bytes is defined as multiplication in the finite field $\mathrm{GF}(2^8)$, modulo the irreducible polynomial $m(x) = x^8 + x^4 + x^3 + x + 1$.

The Advanced Encryption Standard (AES) uses arithmetic in the finite field $\mathrm{GF}(2^8)$, with the irreducible polynomial $m(x) = x^8 + x^4 + x^3 + x + 1$. Consider the two polynomials $f(x) = x^6 + x^4 + x^2 + x + 1$ and $g(x) = x^7 + x + 1$. Then

$$\begin{aligned} f(x) + g(x) &= x^6 + x^4 + x^2 + x + 1 + x^7 + x + 1 \\ &= x^7 + x^6 + x^4 + x^2 \end{aligned}$$

$$\begin{aligned} f(x) \times g(x) &= x^{13} + x^{11} + x^9 + x^8 + x^7 \\ &\quad + x^7 + x^5 + x^3 + x^2 + x \\ &\quad + x^6 + x^4 + x^2 + x + 1 \\ &= x^{13} + x^{11} + x^9 + x^8 + x^6 + x^5 + x^4 + x^3 + 1 \end{aligned}$$

$$\begin{array}{r|l} & x^5 + x^3 \\ \hline x^8 + x^4 + x^3 + x + 1 & x^{13} + x^{11} + x^9 + x^8 + x^6 + x^5 + x^4 + x^3 + 1 \\ & x^{13} + x^9 + x^8 + x^6 + x^5 \\ \hline & x^{11} + x^4 + x^3 \\ & x^{11} + x^7 + x^6 + x^4 + x^3 \\ \hline & x^7 + x^6 + 1 \end{array}$$

Therefore, $f(x) \times g(x) \bmod m(x) = x^7 + x^6 + 1$.

The structure of AES

- Operates on plaintext message blocks of 16 bytes = 128 bits.
- Various key lengths allowed, 16, 24 or 32 bytes. Ciphers referred to as AES-128, AES-192 or AES-256, depending on how many bits used in key.
- Throughout encryption (and decryption) the message block is maintained as a 4×4 array of bytes. This is referred to as the **state**.
- First four bytes form the first column, next four bytes the second column, and so on.
- An initial transformation of the state is followed by N rounds. Where N depends on the key length used.
    - $N = 10$ for 128 bit key.
    - $N = 12$ for 192 bit key.
    - $N = 14$ for 256 bit key.
- The key passes through a *key expansion* transformation to provide $N + 1$ sub-keys to be used in the initial transformation and N rounds.
- Each sub-key consists for four 4-byte **words**, which form the columns of the **round key matrix**.

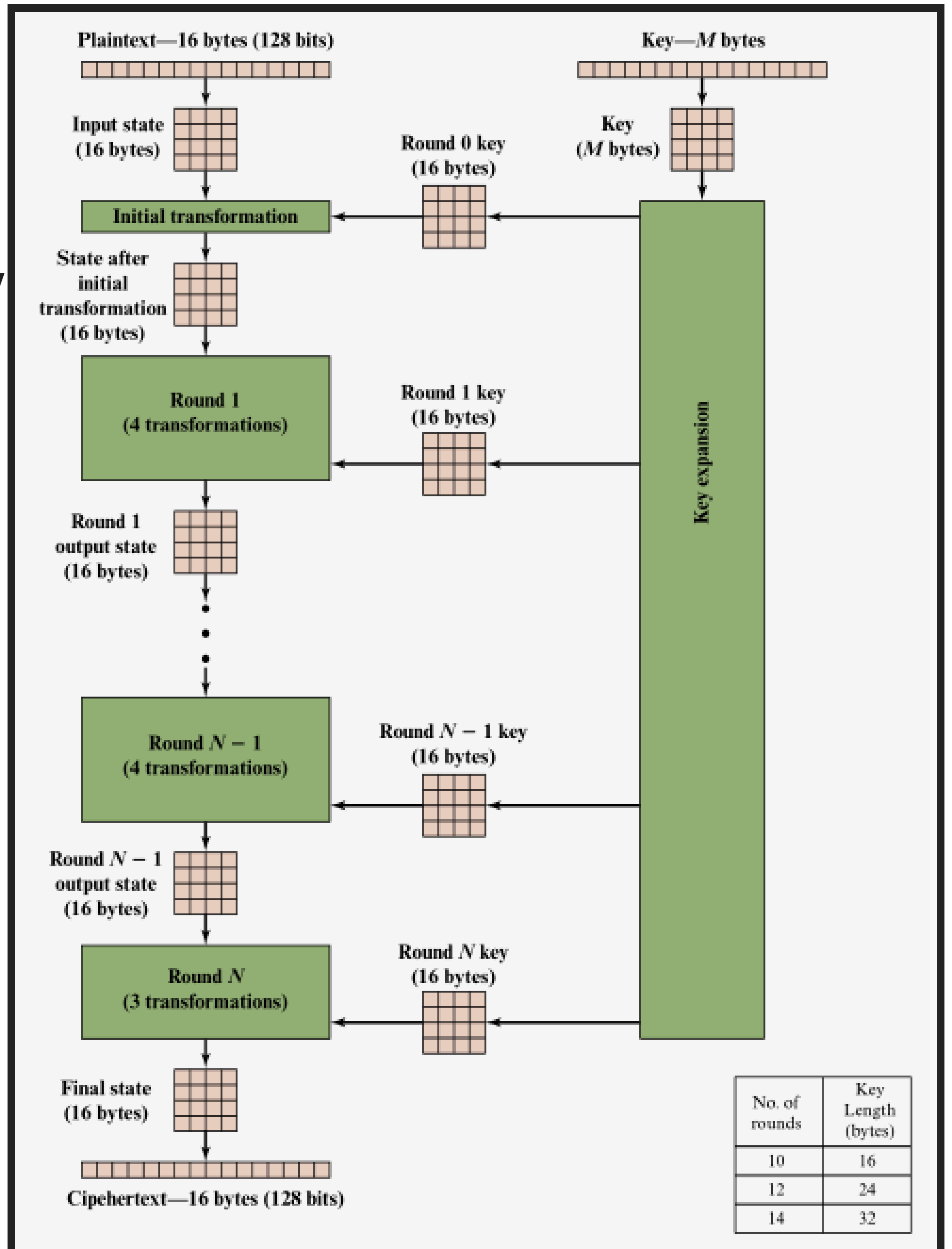


Figure 6.1 AES Encryption Process

AES detail

- This figure exposes the transformations within each round, for AES-128. The other schemes are similar.
- Note that is departs from the Feistel design. There is no notion of dividing the block into halves.
- Rounds 1 - 9 consists of four transformations
    - **Substitute bytes**: and S-box type permutation of the bytes of the state.
    - **Shift rows**: a simple permutation of the bytes within each row of the state.
    - **Mix columns**: a transformation that combines the bytes within each column of the state. This transformation uses the $\mathrm{GF}(2^8)$ field.
    - **Add round key**: bit-wise XOR of the state with the appropriate round key matrix.
- The decryption algorithm reverses all the transformations. At each horizontal level, the intermediate states of the encryption and decryption algorithms are the same.

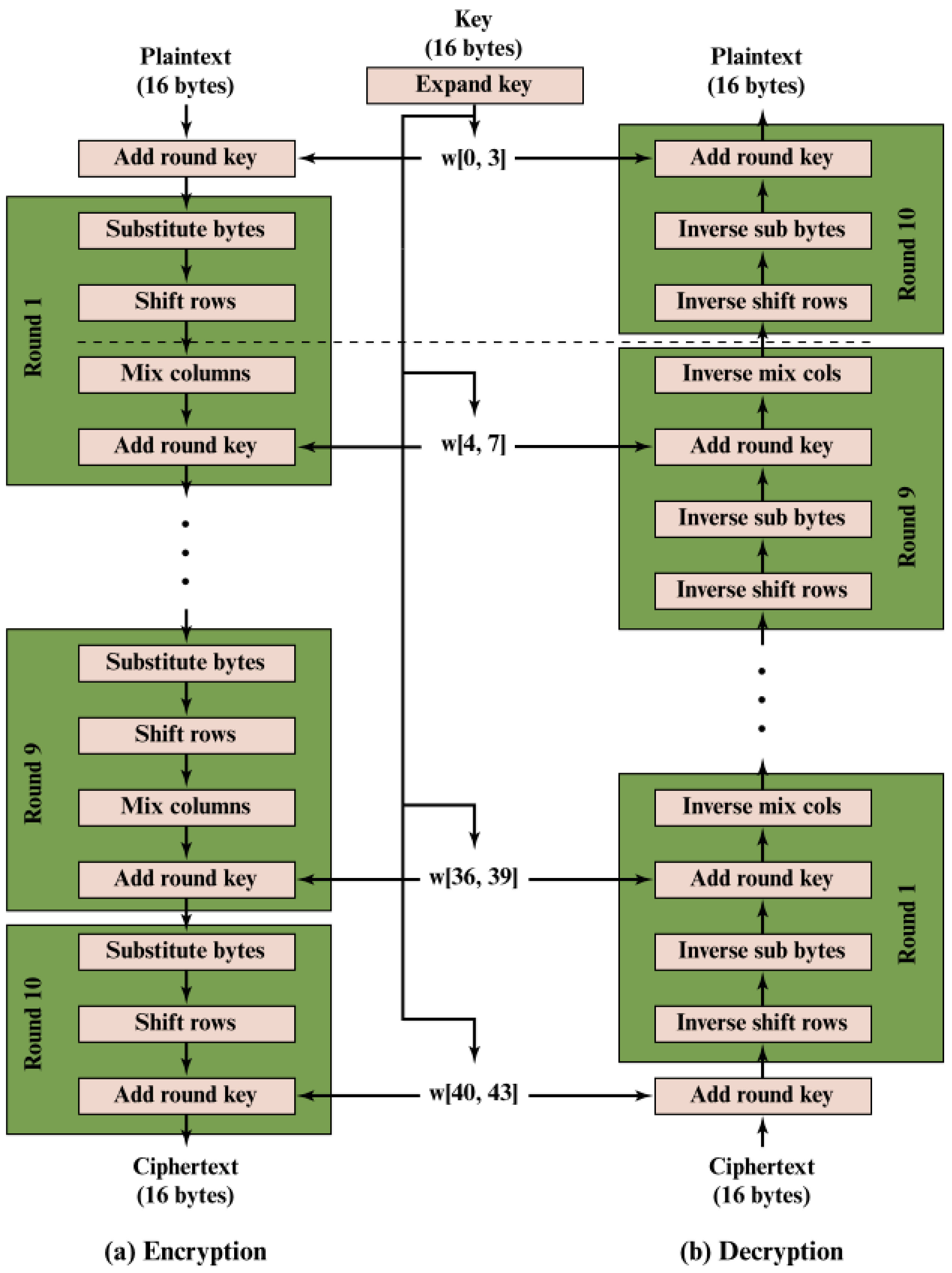


Figure 6.3 AES Encryption and Decryption

- This figure visualizes the four transformations within a typical round.
- Here the state matrix is shown laid out as a row of 16 bytes.

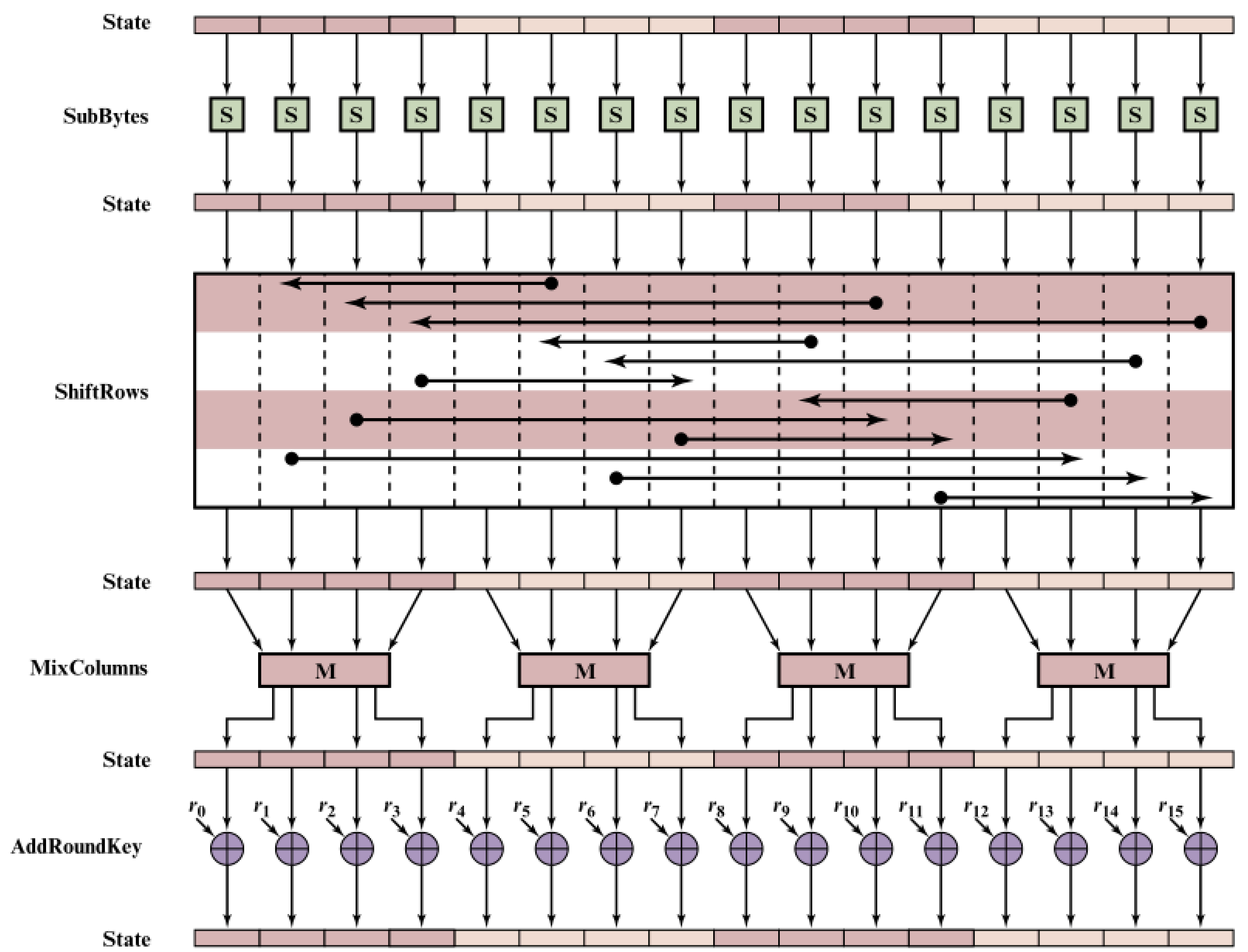
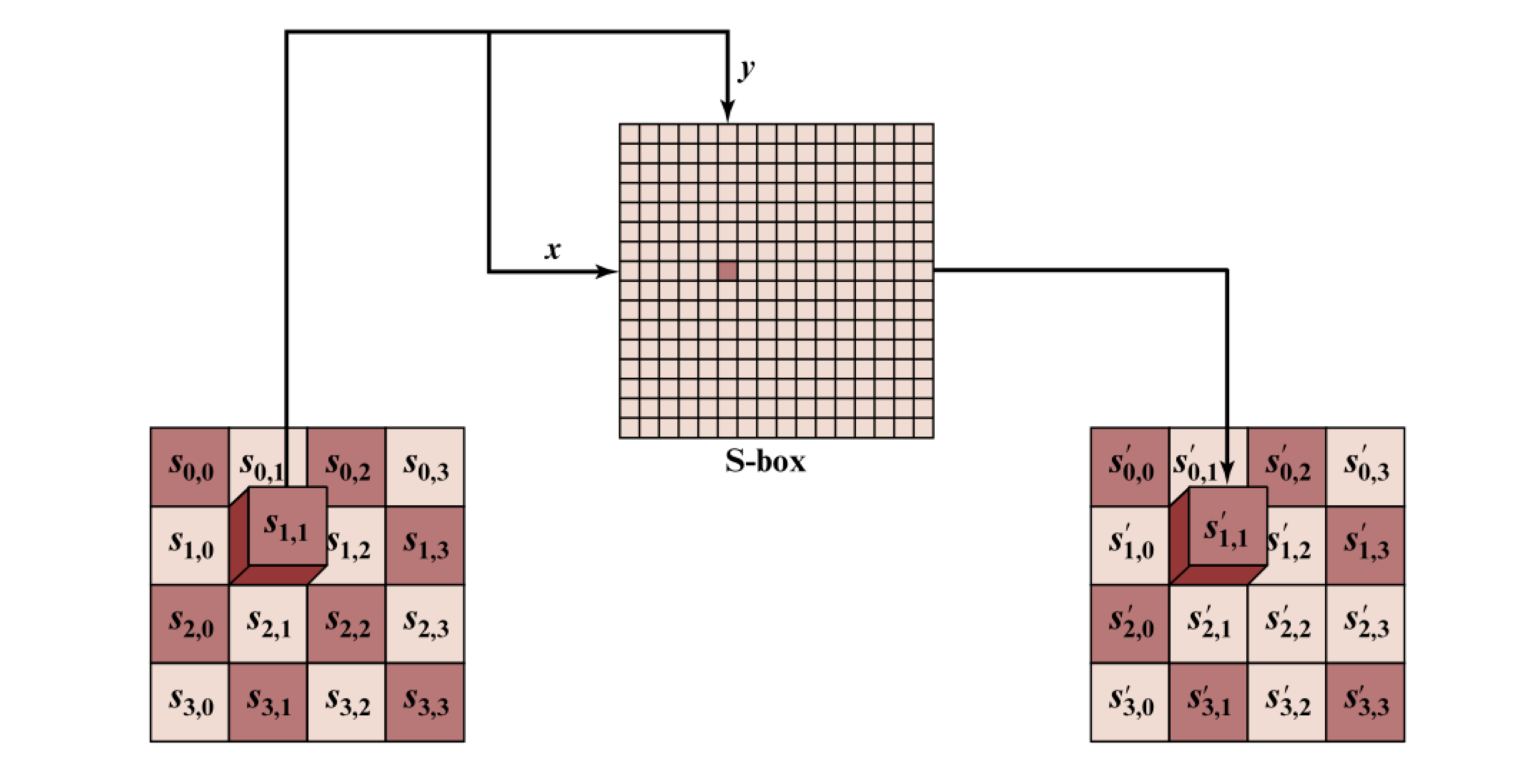


Figure 6.4 AES Encryption Round

- This figure shows how the substitute bytes transformation is defined.
- For each entry of the incoming state matrix, i.e. for each byte
    - the first four bits denote the row index x of the S-box
    - the second four bits denote the column index y of the S-box
    - the S-box entry at that row and column is a byte that replaces the original byte of the state.
- After replacing each entry of the incoming state matrix, we get the outgoing state matrix.



(a) Substitute byte transformation

The S-box itself

- This figure shows the S-box.
- Remember
  - a four bit block is denoted by a hexadecimal digit `0,1,...9,a,b,c,d,e,f`.
  - a single byte (i.e. a 8-bit block) is denoted by a two-digit hexadecimal number.
- A corresponding inverse S-box table is used in the decryption algorithm.
- Lots of detail in Stallings on the construction of this S-box table.
  - Designed like this to minimize any correlation between incoming and outgoing bits.
  - The construction of the S-box table involves the process of taking multiplicative inverses in $\mathrm{GF}(2^8)$, after interpreting the 8-bit bytes as polynomials formed from that sequence of 8 binary coefficients.

	y															
x	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	63	7C	77	7B	F2	6B	6F	C5	30	01	67	2B	FE	D7	AB	76
1	CA	82	C9	7D	FA	59	47	F0	AD	D4	A2	AF	9C	A4	72	C0
2	B7	FD	93	26	36	3F	F7	CC	34	A5	E5	F1	71	D8	31	15
3	04	C7	23	C3	18	96	05	9A	07	12	80	E2	EB	27	B2	75
4	09	83	2C	1A	1B	6E	5A	A0	52	3B	D6	B3	29	E3	2F	84
5	53	D1	00	ED	20	FC	B1	5B	6A	CB	BE	39	4A	4C	58	CF
6	D0	EF	AA	FB	43	4D	33	85	45	F9	02	7F	50	3C	9F	A8
7	51	A3	40	8F	92	9D	38	F5	BC	B6	DA	21	10	FF	F3	D2
8	CD	0C	13	EC	5F	97	44	17	C4	A7	7E	3D	64	5D	19	73
9	60	81	4F	DC	22	2A	90	88	46	EE	B8	14	DE	5E	0B	DB
A	E0	32	3A	0A	49	06	24	5C	C2	D3	AC	62	91	95	E4	79
B	E7	C8	37	6D	8D	D5	4E	A9	6C	56	F4	EA	65	7A	AE	08
C	BA	78	25	2E	1C	A6	B4	C6	E8	DD	74	1F	4B	BD	8B	8A
D	70	3E	B5	66	48	03	F6	0E	61	35	57	B9	86	C1	1D	9E
E	E1	F8	98	11	69	D9	8E	94	9B	1E	87	E9	CE	55	28	DF
F	8C	A1	89	0D	BF	E6	42	68	41	99	2D	0F	B0	54	BB	16

(a) S-box

- An example substitute bytes transformation is shown here

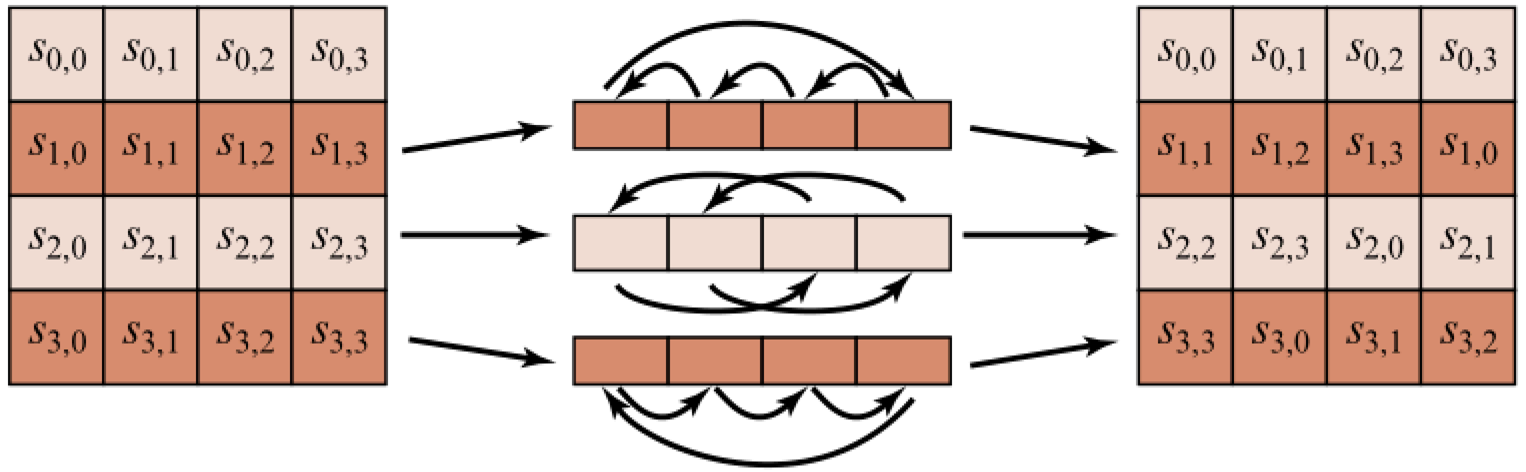
EA	04	65	85
83	45	5D	96
5C	33	98	B0
F0	2D	AD	C5

→

87	F2	4D	97
EC	6E	4C	90
4A	C3	46	E7
8C	D8	95	A6

The shift rows transformation

- In each of the second, third and fourth rows of state, permute the bytes in each row as shown.
- This has a significant effect on the original positions of the bits within the 128 bit message block.



(a) Shift row transformation

Mix columns transformation

- The equation shows how the incoming state matrix $s_{i,j}$ is multiplied by the matrix of constants to get the outgoing state matrix $s'_{i,j}$, shown on the right of the equation.

$$\begin{bmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{bmatrix} \begin{bmatrix} s_{0,0} & s_{0,1} & s_{0,2} & s_{0,3} \\ s_{1,0} & s_{1,1} & s_{1,2} & s_{1,3} \\ s_{2,0} & s_{2,1} & s_{2,2} & s_{2,3} \\ s_{3,0} & s_{3,1} & s_{3,2} & s_{3,3} \end{bmatrix} = \begin{bmatrix} s'_{0,0} & s'_{0,1} & s'_{0,2} & s'_{0,3} \\ s'_{1,0} & s'_{1,1} & s'_{1,2} & s'_{1,3} \\ s'_{2,0} & s'_{2,1} & s'_{2,2} & s'_{2,3} \\ s'_{3,0} & s'_{3,1} & s'_{3,2} & s'_{3,3} \end{bmatrix} \quad \textbf{(6.3)}$$

- This results in the following transformations within the j^{th} column of the state

$$\begin{aligned} s'_{0,j} &= (2 \cdot s_{0,j}) \oplus (3 \cdot s_{1,j}) \oplus s_{2,j} \oplus s_{3,j} \\ s'_{1,j} &= s_{0,j} \oplus (2 \cdot s_{1,j}) \oplus (3 \cdot s_{2,j}) \oplus s_{3,j} \\ s'_{2,j} &= s_{0,j} \oplus s_{1,j} \oplus (2 \cdot s_{2,j}) \oplus (3 \cdot s_{3,j}) \\ s'_{3,j} &= (3 \cdot s_{0,j}) \oplus s_{1,j} \oplus s_{2,j} \oplus (2 \cdot s_{3,j}) \end{aligned}$$

- The operations $\oplus$ and $\cdot$ shown here are the operations from $\mathrm{GF}(2^8)$, carried out on the entries of the state, i.e. on the bytes, i.e. on the 8-bit blocks which are interpreted as the coefficients of degree 7 polynomials.
- So $\oplus$ is bitwise XOR and $\cdot$ is the multiplication obtained from the multiplication of these polynomials, modulo the polynomial $m(x) = x^8 + x^4 + x^3 + x + 1$.
- The design of this Mix Columns transformation ensures good mixing of the bytes within a column, and the use of the constants `01`, `02` and `03` results in efficient implementation of the encryption algorithm.

Add round key transformation

- Perhaps the most straightforward, this is just bitwise XOR amongst the bit entries of the state and round key matrices.
- Consider the examples shown here of

$$\text{incoming state} \oplus \text{round key} = \text{outgoing state}$$

$$\begin{array}{|c|c|c|c|}\hline 47 & 40 & \text{A3} & \text{4C} \\ \hline 37 & \text{D4} & 70 & \text{9F} \\ \hline 94 & \text{E4} & \text{3A} & 42 \\ \hline \text{ED} & \text{A5} & \text{A6} & \text{BC} \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|}\hline \text{AC} & 19 & 28 & 57 \\ \hline 77 & \text{FA} & \text{D1} & \text{5C} \\ \hline 66 & \text{DC} & 29 & 00 \\ \hline \text{F3} & 21 & 41 & \text{6A} \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline \text{EB} & 59 & \text{8B} & \text{1B} \\ \hline 40 & \text{2E} & \text{A1} & \text{C3} \\ \hline \text{F2} & 38 & 13 & 42 \\ \hline \text{1E} & 84 & \text{E7} & \text{D6} \\ \hline \end{array}$$

A summary flowchart for AES encryption

- This figure summarizes a typical encryption round.
- We still need to describe the key expansion process for the derivation of the round keys from the original key.

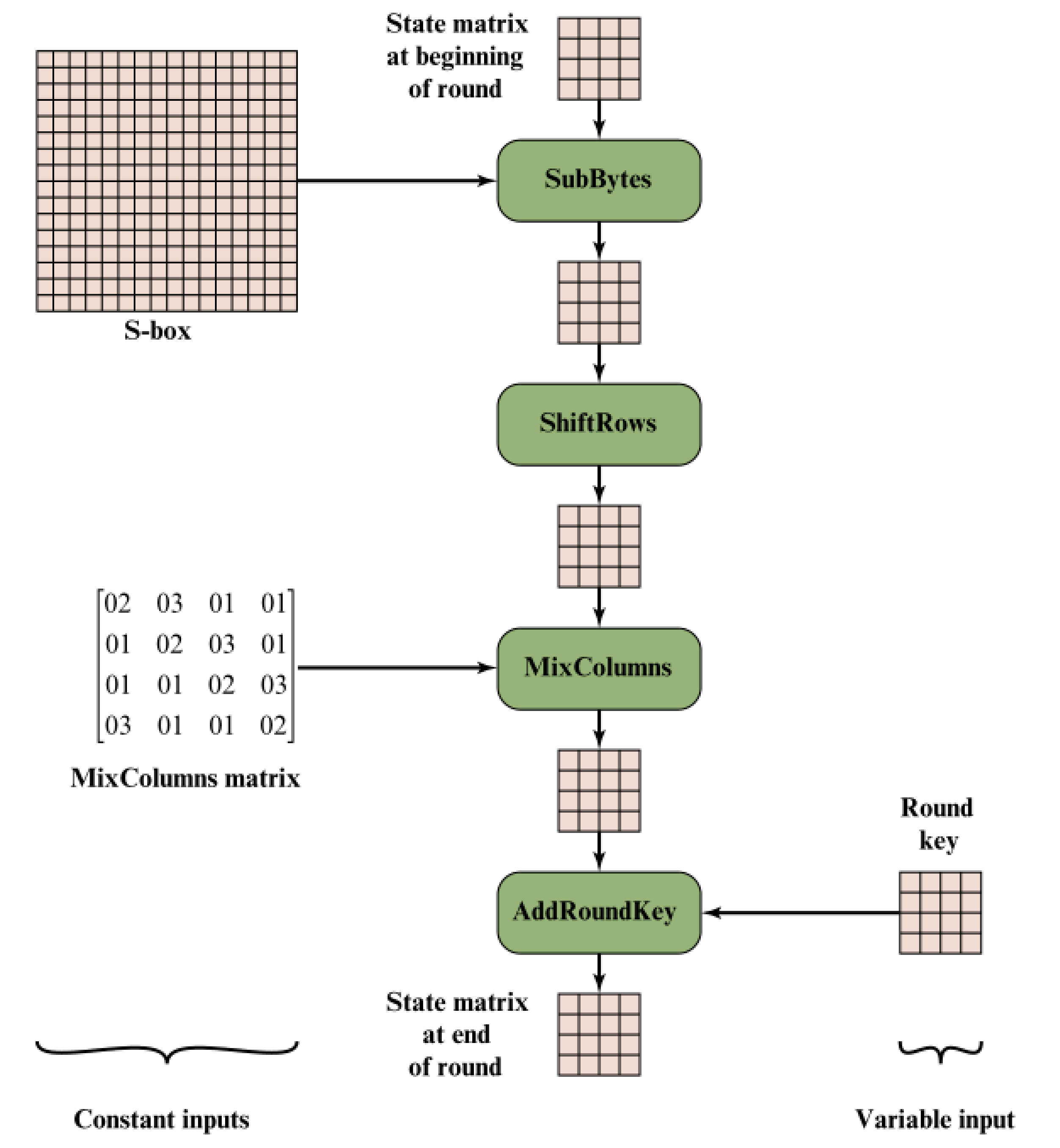


Figure 6.8 Inputs for Single AES Round

Key expansion in AES

- **Key expansion** is the process where the initial key is expanded to produce the $N+1$ round keys for the initial transformation and the N rounds of AES.
- Each round key consists of four 4-byte words, i.e. the four columns of the round key matrix.

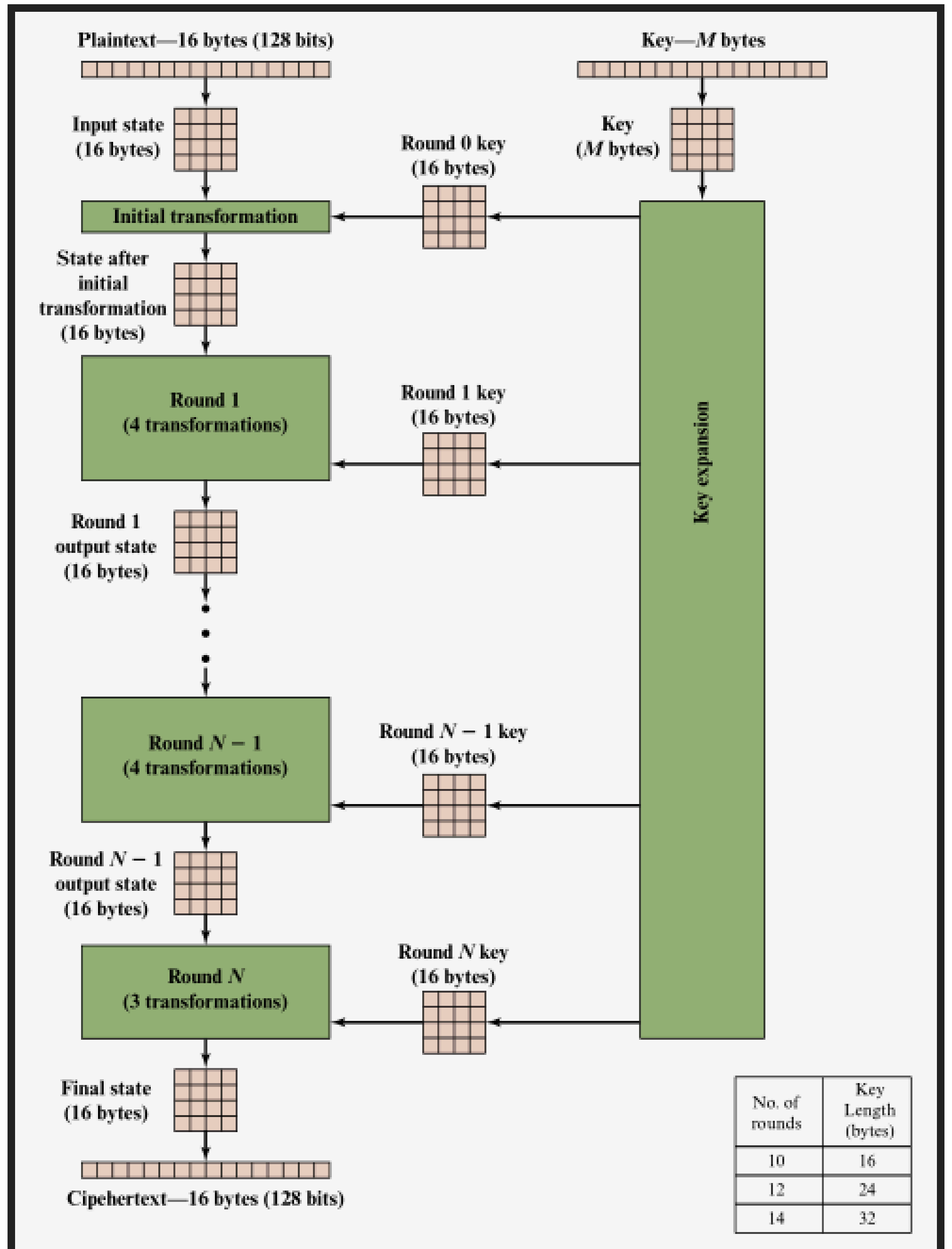


Figure 6.1 AES Encryption Process

Key expansion in AES-128

- The i^{th} of the $N+1$ round keys consist of the four keywords

$$w_{4i+0}, w_{4i+1}, w_{4i+2}, w_{4i+3}.$$

- The 16 bytes, $k_0, k_1, \ldots, k_{15}$, of the initial key form the first four key words w_0, w_1, w_3, w_3 as shown.
- An iterative process creates the 40 subsequent key words.
  - for $j = 1, 2, 3$, and $i = 1, \ldots, 10$,

$$w_{4i+j} = w_{4i+j-1} \oplus w_{4(i-1)+j}.$$

  - for $j = 0$ and $i = 1 \ldots 10$,

$$w_{4i} = g(w_{4i-1}) \oplus w_{4(i-1)}.$$

  - the function g is the composition of
    - a circular left-shift of the word bytes
    - an S-box byte substitution using the same S-box table as in the AES encryption rounds
    - a bitwise XOR with the word formed by the bytes RC$_j$, 00, 00, 00. The round constants RC$_j$ are shown on table on next slide.

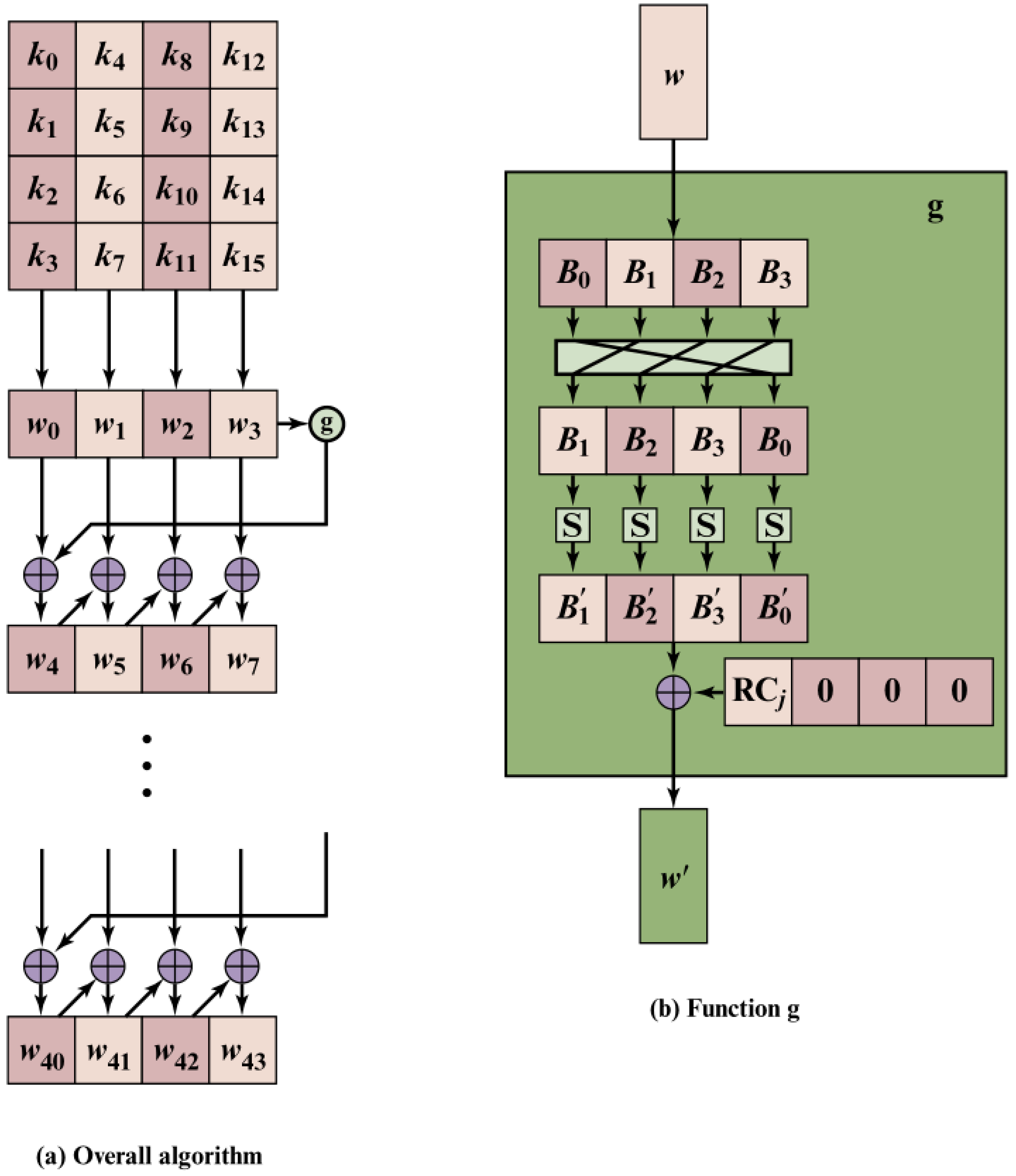


Figure 6.9 AES Key Expansion

Design considerations of the key expansion algorithm

- The round constants RC_j.

j	1	2	3	4	5	6	7	8	9	10
RC[j]	01	02	04	08	10	20	40	80	1B	36

- These transformations were chosen to ensure these features, amongst others,
  - speedy implementations in software and chip hardware,
  - partial knowledge of the original key or intermediate round keys will not enable determination of many other bits of other keywords,
  - use of different round constants eliminates any potential symmetries in the round key generations,
  - diffusion, i.e. each bit of the original key effects many round key bits.
- Further details, worked examples, and illustrations of bit diffusion from plaintext and key differences can be found in [Stallings, Chapter 6: AES]