Public Key Cryptography

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6G6Z0024 Applied Cryptography 2025/26

Lecture Week 07 – Mon 10 November 2025

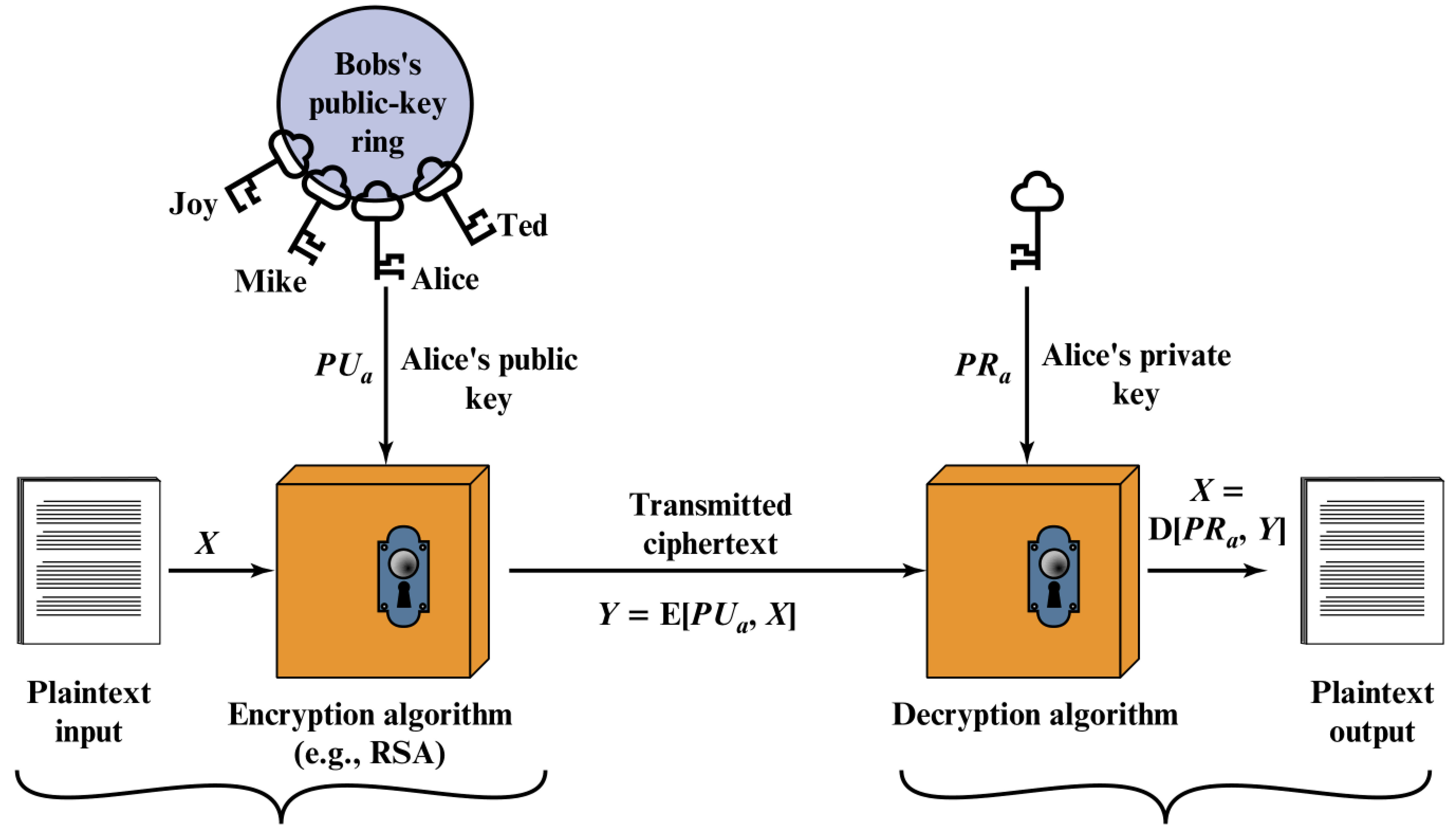
Introduction and initial promise

- Reading: Stallings, Chapter 9, *Public-key cryptography and RSA*
- Symmetric ciphers, such as DES, and then AES, can provide excellent security, but rely on the distribution of *secret keys* between the parties.
  - How are these secret keys to be distributed in a secure and efficient way?
- Public-key cryptography, an asymmetric approach (different keys for encryption and decryption) solves this by using two keys,
  - a **public key** (which does not need to be kept secret) to encrypt messages,
  - and a **private key** (which must be kept secret, but which only the receiver needs) to decrypt messages.
- Public-key cryptography promises and enables something, which seems almost paradoxical. Suppose that
  - two parties, Alice and Bob, wish to communicate,
  - they have never met or communicated before, and do not have any access to pre-arranged secret keys,
  - **ALL** their communications can be intercepted and inspected by the eavesdropper, Eve,
  - nevertheless, using public-key techniques, Alice and Bob can exchange some initial unencrypted communications, and then pass into secure encrypted communication,
  - even though **ALL** their initial unencrypted communications were intercepted, read and understood by Eve.

Discovery

- Discovered by Whitefield Diffie and Martin Hellman at Stanford University in 1976.
- Though in 1997, UK government declassified material revealing that James Ellis, Clifford Cocks and Martin Williamson, working at GCHQ, made the same discoveries earlier in the 1970s.
- Two problems are solved by these methods
    - encrypted communications without the need for secretly pre-arranged keys,
    - **digital signatures** enabling the cryptographic proof that a message was authored by the claimed author.

- A pair of related keys are generated by the user Alice, a **public key** PU_a, and a **private key**, PR_a.
- The public key is published for all to see.
- The private key is kept private and secure.
- Messages can be encrypted using this public key, and communicated to Alice.
- The messages can be decrypted by Alice using her private key.
- Details of the public key, encryption and decryption algorithms are all public.



(a) Encryption with public key

The eavesdropper's/cryptanalsyt's task

- The cryptanalyst intercepts encrypted message Y and attempts to form estimates of the original plaintext X or the private key PR_b.

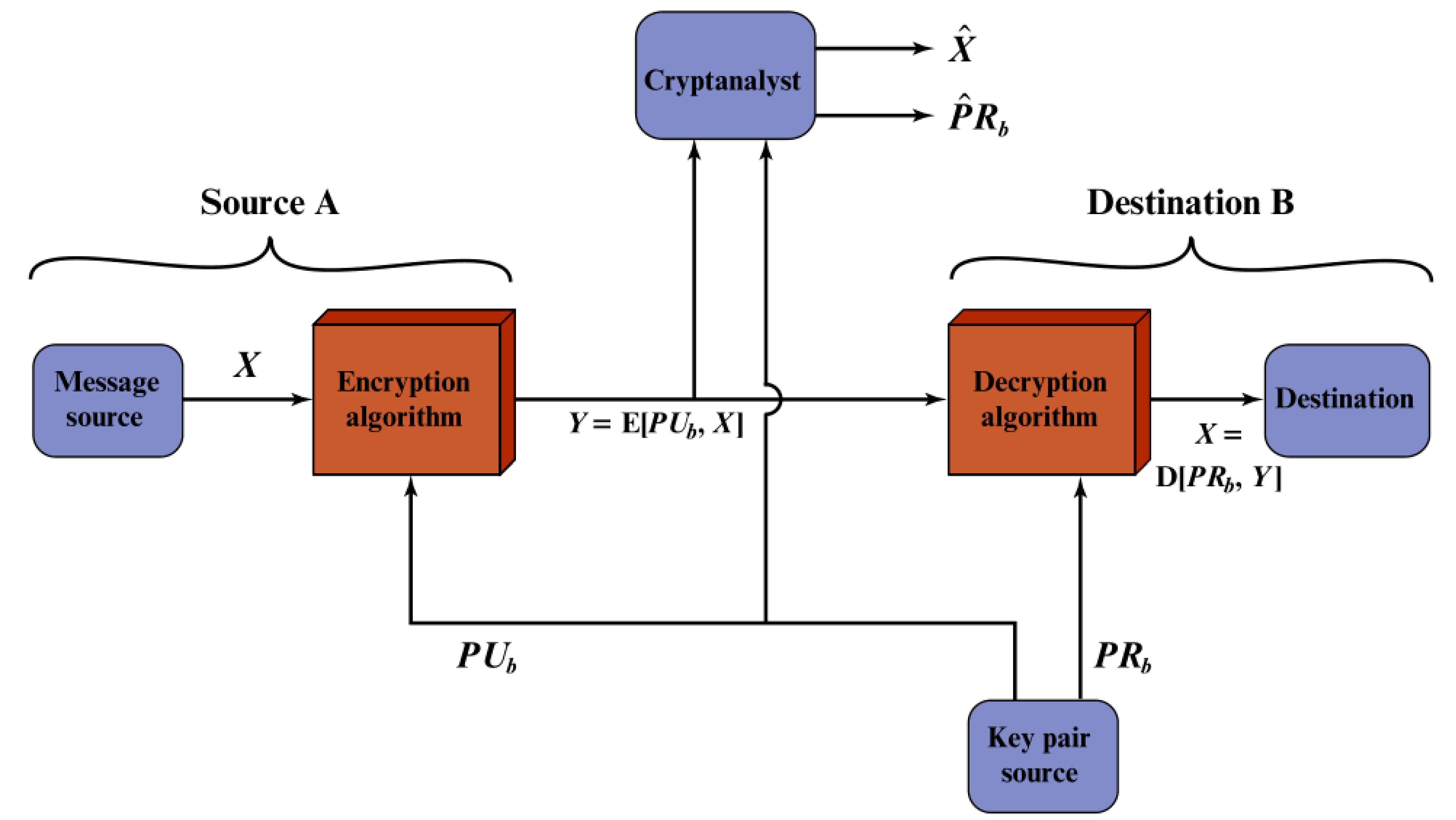


Figure 9.2 Public-Key Cryptosystem: Confidentiality

Outline of a digital signature approach

- Using keys of both sender and recipient can enable **authenticated** and encrypted communication.
- The receiver Bob is assured that only the holder of the private key corresponding to Alice's public key could have authored this message X.
- Alternatively, the middle encryption step can be skipped, and Alice can simply publish the encrypted message Y, which anyone can decrypt with her public key PU_a. Any such receiver is assured that only the holder of the private key corresponding to Alice's public key could have authored this message X.

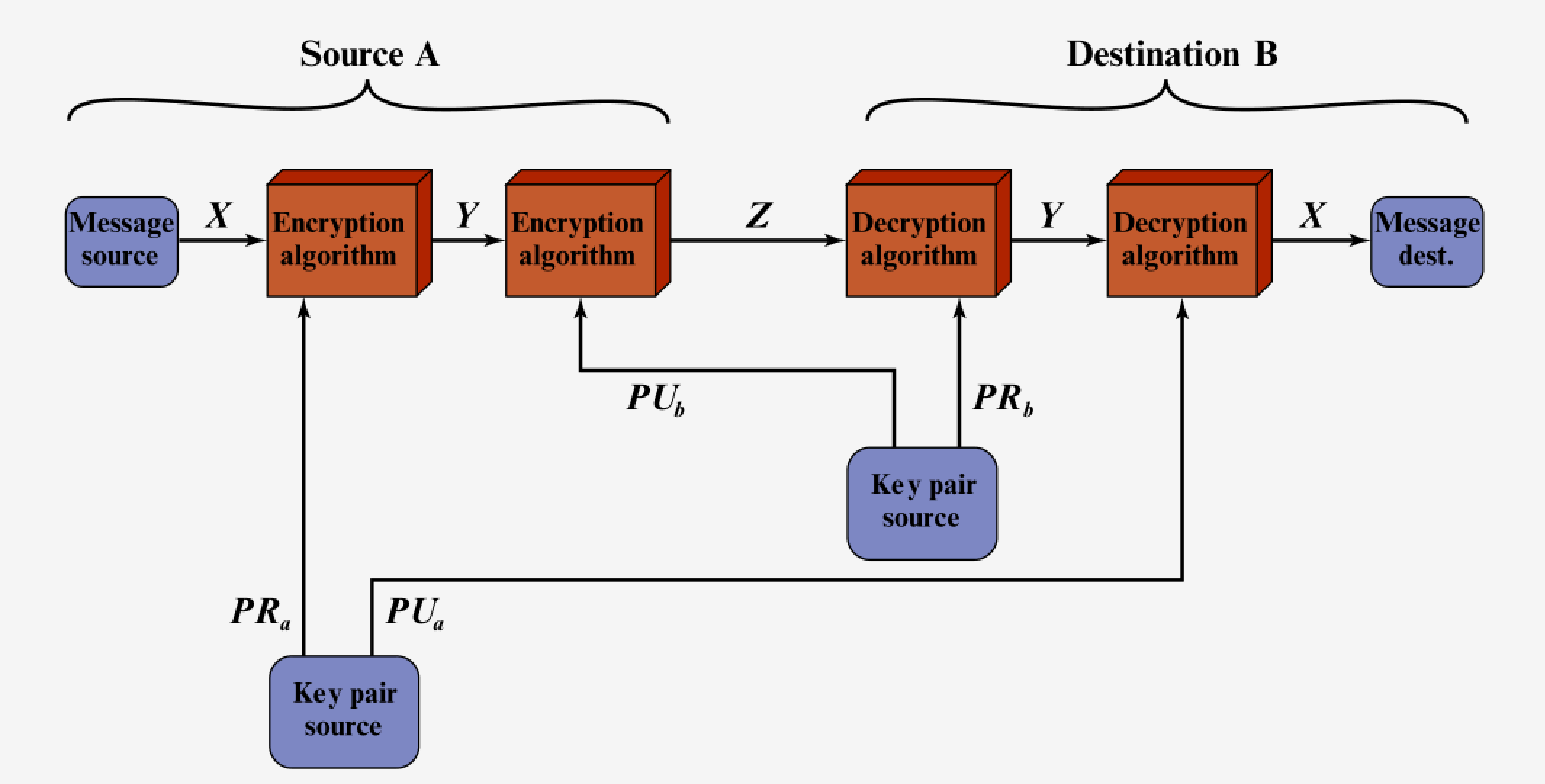


Figure 9.4 Public-Key Cryptosystem: Authentication and Secrecy

Firming up the requirements (Stallings, pg. 294)

- It is computationally easy for a party B to generate keys pairs PU_b and PR_b.
- It is computationally easy for a sender A, with the public key PU_b and plaintext M, to generate the corresponding ciphertext

$$C = E(PU_b, M).$$

- It is computationally easy for the receiver B to decrypt C using PR_b, to recover M as

$$M = D(PR_b, C) = D\big(PR_b, E(PU_b, M)\big).$$

- It is computationally infeasible for an adversary, knowing the public key PU_b, to determine the private key PR_b.
- It is computationally infeasible for an adversary, knowing the public key PU_b and ciphertext C, to recover the original message M.

While not essential, the following useful property is possessed by the RSA implementation of public-key cryptography.

- The two keys can be applied in either order, i.e.

$$M = D\big(PR_b, E(PU_b, M)\big) = D\big(PU_b, E(PR_b, M)\big).$$

This is all very nice to describe, but what exactly is the technology that can enable such a scheme?

The RSA system

- Discovered in 1978 at MIT by Ron **R**ivest, Ade **S**hamir and Len **A**dleman.
- It remains one of the most widely used general purpose public-key schemes.
- It deals with messages, or message blocks, encoded as integers in the range 0 to $n-1$, for some suitably large n.
- Typicall size for n might be 1024 bits, or around 309 decimal digits.
- RSA makes use of exponentials in modular arithmetic.
- The message M is an integer in the range $0 \leq M \leq n-1$.
- The receiver chooses integers e and d, with the property that

$$ed \equiv 1 \pmod{\phi(n)},$$

i.e. e and d are multiplicative inverses of each other modulo the Euler totient function value $\phi(n)$.
- The public key is $PU = (e, n)$, the private key is $PR = (d, n)$.
- The plaintext M is encrypted as

$$C = (M^e \bmod n).$$

- The ciphertext C is decrypted as

$$(C^d \bmod n) = ((M^e)^d \bmod n) = (M^{ed} \bmod n) = (M^1 \bmod n) = M.$$

- The security comes from the fact that computing $\phi(n)$ from n is **hard**. And $\phi(n)$ is needed in order to compute the pair e, d with the $ed \equiv 1 \pmod{\phi(n)}$ property.

Euler's Theorem:
$M^{\phi(n)} \equiv 1 \pmod{n}$

- Figure on the right, from Stallings, outlines the procedure.

Key Generation by Alice

Select p, q	p and q both prime, $p \neq q$
Calculate $n = p \times q$	
Calculate $\phi(n)=(p-1)(q-1)$	
Select integer e	$\gcd(\phi(n), e) = 1; 1 < e < \phi(n)$
Calculate d	$d \equiv e^{-1} \pmod{\phi(n)}$
Public key	$PU = \{e, n\}$
Private key	$PR = \{d, n\}$

Encryption by Bob with Alice's Public Key

Plaintext:	$M < n$
Ciphertext:	$C = M^e \bmod n$

Decryption by Alice with Alice's Private Key

Ciphertext:	C
Plaintext:	$M = C^d \bmod n$

Figure 9.5 The RSA Algorithm

A small n example

- Extract from Stallings pg. 298, shows the calculations for an example based on a small n. Remember a typical size for n from real usage is circa 309 decimal digits.
- The Euler totient function value $\phi(n)$, when $n = pq$, for distinct primes p and q, is given by

$$\phi(n) = \phi(pq) = (p-1) \cdot (q-1).$$

- The reason that computing $\phi(n)$ from n is **hard** is that factoring n into the product $p \cdot q$ is hard. Given such a large n there is no easy way to discover its prime factors.
    - the best known algorithms for factoring integers will take a **long** time to factor n, given any realistic amount of computing power available.

1. Select two prime numbers, $p = 17$ and $q = 11$.
2. Calculate $n = pq = 17 \times 11 = 187$.
3. Calculate $\phi(n) = (p-1)(q-1) = 16 \times 10 = 160$.
4. Select e such that e is relatively prime to $\phi(n) = 160$ and less than $\phi(n)$; we choose $e = 7$.
5. Determine d such that $de \equiv 1 \pmod{160}$ and $d < 160$. The correct value is $d = 23$, because $23 \times 7 = 161 = (1 \times 160) + 1$; d can be calculated using the extended Euclid's algorithm (Chapter 2).

The resulting keys are public key $PU = \{7, 187\}$ and private key $PR = \{23, 187\}$. The example shows the use of these keys for a plaintext input of $M = 88$. For encryption, we need to calculate $C = 88^7 \bmod 187$. Exploiting the properties of modular arithmetic, we can do this as follows.

$$\begin{aligned}
88^7 \bmod 187 &= [(88^4 \bmod 187) \times (88^2 \bmod 187) \\
&\quad \times (88^1 \bmod 187)] \bmod 187 \\
88^1 \bmod 187 &= 88 \\
88^2 \bmod 187 &= 7744 \bmod 187 = 77 \\
88^4 \bmod 187 &= 59{,}969{,}536 \bmod 187 = 132 \\
88^7 \bmod 187 &= (88 \times 77 \times 132) \bmod 187 = 894{,}432 \bmod 187 = 11
\end{aligned}$$

For decryption, we calculate $M = 11^{23} \bmod 187$:

$$\begin{aligned}
11^{23} \bmod 187 &= [(11^1 \bmod 187) \times (11^2 \bmod 187) \times (11^4 \bmod 187) \\
&\quad \times (11^8 \bmod 187) \times (11^8 \bmod 187)] \bmod 187 \\
11^1 \bmod 187 &= 11 \\
11^2 \bmod 187 &= 121 \\
11^4 \bmod 187 &= 14{,}641 \bmod 187 = 55 \\
11^8 \bmod 187 &= 214{,}358{,}881 \bmod 187 = 33 \\
11^{23} \bmod 187 &= (11 \times 121 \times 55 \times 33 \times 33) \bmod 187 \\
&= 79{,}720{,}245 \bmod 187 = 88
\end{aligned}$$

Modular arithmetic reminder

- RSA involves using c, d that are multiplicative inverses of each other modulo $\phi(n)$.
- Multiplicative inverses are found using the extended Euclidean algorithm
  - If a is coprime to a modulus m, i.e. $\gcd(a, m) = 1$,
  - Run the extended Euclidean algorithm to find integer coefficients x, y satisfying

$$xa + ym = 1.$$

  - Then the inverse is given by

$$a^{-1} \bmod m = (x \bmod m),$$

    because

$$xa = 1 - ym \equiv 1 \pmod{m}.$$

Factorization and the choice of p, q

- Factorization of large n is computationally **hard**
  - even when using advanced *number field sieve* factoring algorithms.
- But computational power increases and theoretical advancements should be expected to continue.
- The reaction to both these possibilities is to increase the size of n, to make factoring harder.
- Recent advice from standards agencies
  - NIST 2015 recommends key lengths of 2048 bits or longer.
  - EU Agency for Network ad Information Security 2014 recommends 3072 bits for future developments.
- Other guidance on choice of p, q is
  - p and q should be of similar digit length. So for a 1024-bit key, they should be chosen in the range

$$10^{75} \leq p, q \leq 10^{100}.$$

  - both $p - 1$ and $q - 1$ should contain a large prime factor
  - $\gcd(p - 1, q - 1)$ should be small.
- However finding large primes is computationally hard, similar to factoring.
  - In practice, for choosing such large primes, probabilistic prime tests, such as the Miller-Rabin test, need to be used.
  - This test allows one to choose an integer which is *probably* a prime.
  - But this probability can be made arbitrarily close to 1, i.e. as near certain as one would like. (See chapter 2 of Stallings for details on Miller-Rabin test)