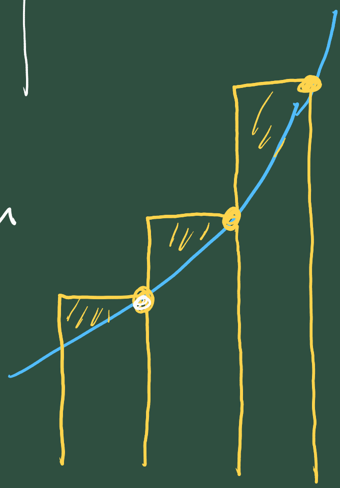
Section 5.3

Q35

$\int_0^1 x^3\,dx.$

Right hand rule – choose the right edge as the representative point x_j^* of the j^{th} rectangle



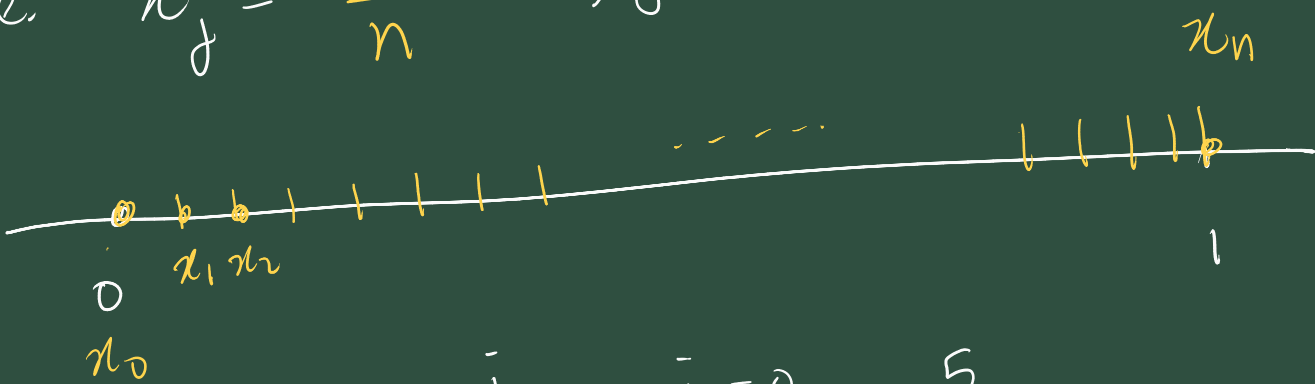
Use the equally spaced partition

$x_0, x_1, x_2, \ldots, x_n = 1.$

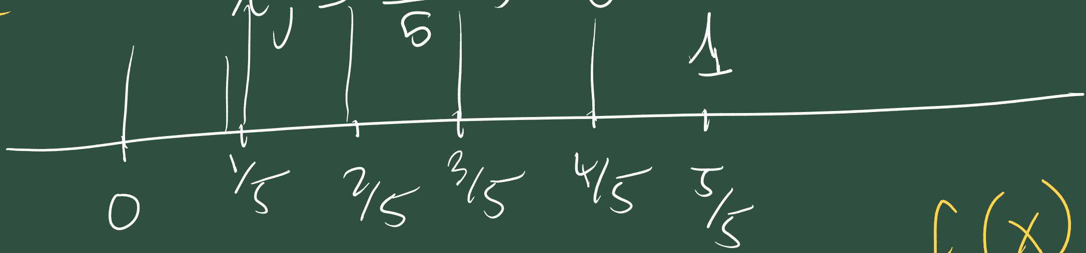
$\|$

0

i.e. $x_j = \frac{j}{n}$, $j = 0, \ldots, n$



$n=5$ $\quad x_j = \frac{j}{5}$, $j = 0, \ldots, 5$



$f(x) = x^3$

And $x_j^* = x_j = \frac{j}{n}$

~~And~~ each rectangle have width $\Delta x = 1/n$

The Riemann sum formula for integral

is $\displaystyle\int_0^1 x^3\,dx = \lim_{n\to\infty}\left(\sum_{j=1}^{n} \underbrace{f(x_j^*)}_{\substack{\text{height}\\ \text{of } j\text{th } \sqcap}} \underbrace{\Delta x_j}_{\substack{\text{width}\\ \text{of } j\text{th } \sqcap}}\right)$

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n}\left(\left(\frac{j}{n}\right)^3 \frac{1}{n}\right)\right)$$

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{j^3}{n^4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^4}\left(\sum_{j=1}^{n} j^3\right)\right)$$, factorising the sum.

$$= \lim_{n\to\infty}\left(\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^2(n^2+2n+1)}{4n^4}\right)$$

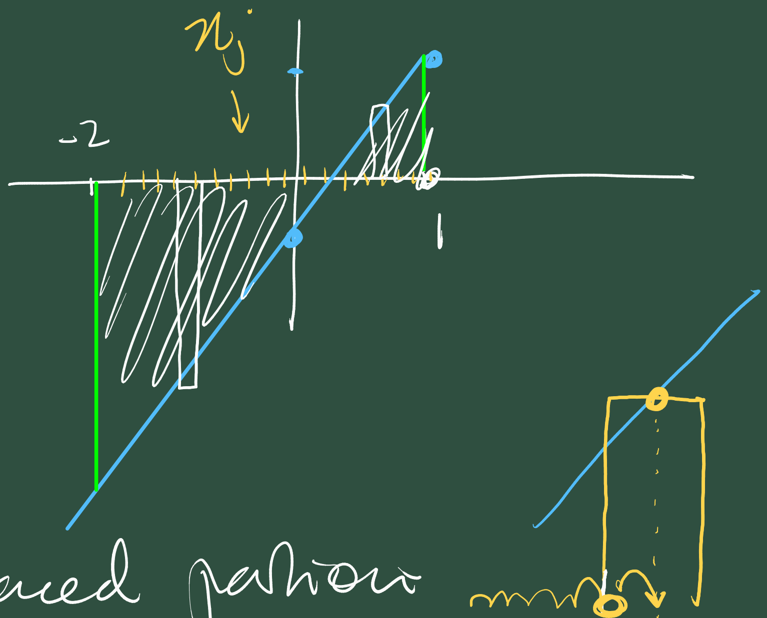
$$= \lim_{n\to\infty}\left(\frac{n^4+2n^3+n^2}{4n^4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right).$$

$= \frac{1}{4}$, by linearity of limits and $\frac{1}{2n}, \frac{1}{4n^2} \to 0$ as $n \to \infty$.

Q37

$$\int_{-2}^{1} 5x - 2 \, dx.$$



Use equally spaced partition

$$x_j = -2 + j\frac{3}{n}, \quad j = 0, \ldots, n.$$

and the representative points are the midpoints of each segment.

$$x_j^* = -2 + (j-1)\frac{3}{n} + \frac{3}{2n}$$
$$= -2 + \frac{6j - 3}{2n}$$

Each rectangle has width $\Delta x = \frac{3}{n}$

Apply the Riemann sum formula for the integral.

$$\int_{-2}^{1} (5x-2)\,dx$$

$$= \lim_{n\to\infty} \left(\sum_{j=1}^{n} \left(\left(5\left(-2 + \frac{6j-3}{2n}\right) - 2 \right) \frac{3}{n} \right) \right)$$

$$= 3 \lim_{n\to\infty} \left(\frac{1}{n} \sum_{j=1}^{n} \left(-10 + \frac{30j-15}{2n} - 2 \right) \right)$$

$$= 3 \lim_{n\to\infty} \left(\frac{1}{n} \left[-12n + \frac{15}{n} \left(\sum_{j=1}^{n} j \right) - \frac{15}{2} \right] \right)$$

$$= 3 \lim_{n\to\infty} \left(-12 + \frac{15}{n^2} \frac{n(n+1)}{2} - \frac{15}{2n} \right)$$

$$= 3 \lim_{n\to\infty} \left(-12 + \frac{15(n^2+n)}{2n^2} - \frac{15}{2n} \right)$$

$$= 3 \lim_{n\to\infty} \left(-12 + \frac{15}{2} + \cancel{\frac{15}{2n}} - \cancel{\frac{15}{2n}} \right)$$

$$= 3 \left(-12 + \frac{15}{2} \right)$$

$$= 3\, \frac{-24+15}{2} = 3 \cdot \left(-\frac{9}{2} \right) = \frac{-27}{2}.$$