Q9]

Consider $\sum_{m=0}^{\infty} a_m$

where $a_m = \dfrac{2^m}{(m+1)^2}$, $m \geq 0$

For the ratio test we examine the limit

$$\boxed{L = \lim_{m \to \infty} \frac{a_{m+1}}{a_m}}$$

$$= \lim_{m \to \infty} \left(\frac{2^{m+1}}{(m+2)^2} \cdot \frac{(m+1)^2}{2^m} \right)$$

$$= 2 \lim_{m \to \infty} \left(\frac{(m+1)^2}{(m+2)^2} \right)$$, cancelling the 2s and linearity of limit

$$= 2 \lim_{m \to \infty} \left(\frac{m^2 + 2m + 1}{m^2 + 4m + 4} \right)$$

$$= 2 \lim_{m \to \infty} \left(\frac{1 + 2/m + 1/m^2}{1 + 4/m + 4/m^2} \right)$$

$$= 2 \, \frac{\lim_{m \to \infty} (1 + 2/m + 1/m^2)}{\lim_{m \to \infty} (1 + 4/m + 4/m^2)}$$

$$= 2 \times \frac{1}{1} = \boxed{2}$$

since $1/m, 1/m^2 \to 0$ as $m \to \infty$
and by linearity of the limit.

Since $L = 2 > 1$ the ratio test says that $\sum_{m=0}^{\infty} \frac{2^m}{(m+1)^2}$ is a

divergent series.

Note: One could apply <u>the general term test</u> to reach the same conclusion.

as. $\frac{2^m}{(m+1)^2} \not\to 0$ as $m \to \infty$.

since $\frac{2^m}{(m+1)^2}$ is unbounded

as $m \to \infty$. So $\sum a_m$ diverges.

Q11.

Consider $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+2}{k(k^2+5)}$.

$$= \sum_{k=1}^{\infty} \frac{k^2+2}{k^3+5k} \quad a_k.$$

To select the comparison series. look at the dominant behaviour of numerator & denominator.

$$a_k \sim \frac{k^2}{k^3} = \frac{1}{k}.$$

So we will compare to the known divergent harmonic series. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$

$$\boxed{a_k = \frac{k^2+2}{k^3+5k}} \qquad \geq C \cdot \frac{1}{k}$$

$$> \frac{k^2}{k^3+5k}.$$

$$\geq \frac{k^2}{k^3+5k^3}, \quad = \text{only for } k=1, \text{ otherwise } >$$

$$= \frac{k^2}{6k^3}$$

$$= \frac{1}{6}\cdot\frac{1}{k}.$$

Note that $\sum_{k=1}^{\infty}\left(\frac{1}{6}\cdot\frac{1}{k}\right)$ is divergent as it is a constant multiple of the divergent harmonic series.

So by the comparison test, $\sum_{k=1}^{\infty} a_k$ diverges also.

Q15

Consider

$$\sum_{n=2}^{\infty} \frac{9}{n^2+n-2}$$

$$= \sum_{n=2}^{\infty} \frac{9}{(n-1)(n+2)}$$

We seek a partial fraction expansion of

$$\left[\frac{9}{(n-1)(n+2)}\right] = \frac{\alpha}{n-1} + \frac{\beta}{n+2}.$$

(, for some $\alpha, \beta \in \mathbb{R}$.)

$$= \frac{\alpha(n+2) + \beta(n-1)}{(n-1)(n+2)}.$$

$$= \left[\frac{\boxed{(\alpha+\beta)n + 2\alpha - \beta}}{(n-1)(n+2)}\right]$$

Comparing numerators we get that

$$9 = (\alpha+\beta)n + 2\alpha - \beta, \text{ for all } n \geq 2.$$

$$\Rightarrow \begin{cases} \alpha+\beta = 0 \\ 2\alpha - \beta = 9 \end{cases} \Rightarrow \alpha = -\beta$$

$$\Rightarrow \quad \alpha = -\beta$$
$$2\alpha + \alpha = 9 \Rightarrow \alpha = 3$$

$$\Rightarrow \beta = -3.$$

So our series term is:

$$\frac{3}{n-1} - \frac{3}{n+2}.$$

Remember $\sum_{n=2}^{\infty} x_n = \lim_{k \to \infty} \sum_{n=2}^{k} x_n.$

and

$$\sum_{n=2}^{k} \left(\frac{3}{n-1} + \frac{-3}{n+2} \right)$$

almost all terms cancel the $-\frac{3}{n+2}$ will cancel with a later (3 steps later) $3/n-1$

$$\frac{-3}{4}.$$

$$= \frac{3}{1} + \frac{3}{2} + \frac{3}{3} - \frac{3}{k} - \frac{3}{k+1} - \frac{3}{k+2}$$

$$= \frac{11}{2} - 3\left(\frac{1}{k} + \frac{1}{k+1} + \frac{1}{k+2} \right)$$

$$\longrightarrow \frac{11}{2}. \quad \text{as } k \to \infty.$$

since $\frac{1}{k}, \frac{1}{k+1}, \frac{1}{k+2} \to 0$

as $k \to \infty$.

So our original series is

$$\sum_{n=2}^{\infty} \frac{9}{n^2+n-2} = \frac{11}{2}.$$