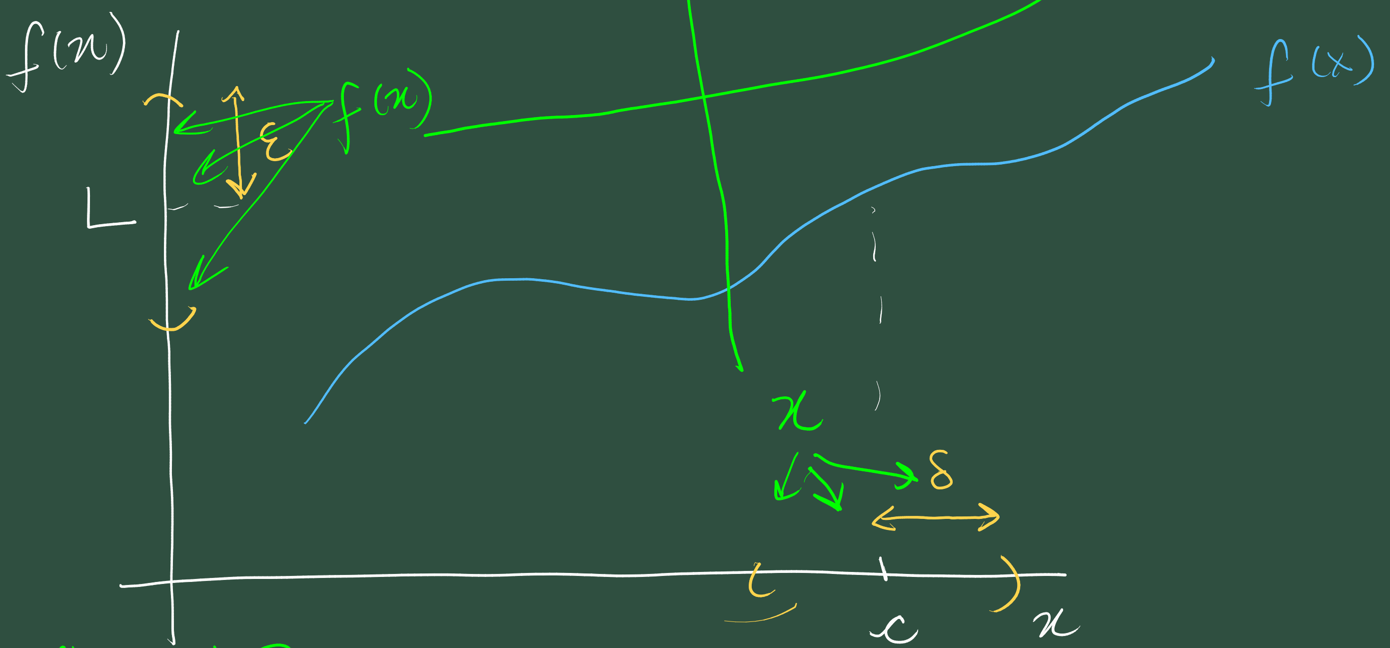
tinyurl.com/apexcalculus

"$\lim_{x \to c} f(x) = L$"

"$f(x) \to L$ as $x \to c$"

$$\boxed{\forall \varepsilon > 0 \ \exists \delta > 0 \quad |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon}$$



Section 1.2.

Q7

$$\lim_{\underset{c}{x \to 3}} \underbrace{x^2 - 3}_{f(x)} = \underbrace{6}_{L}$$

$\boxed{\text{Let } \varepsilon > 0}$ be given.

Consider $|x^2 - 3 - 6| < \varepsilon$

$\iff |x^2 - 9| < \varepsilon$

"Looking for $|x-3|$ in this expression"

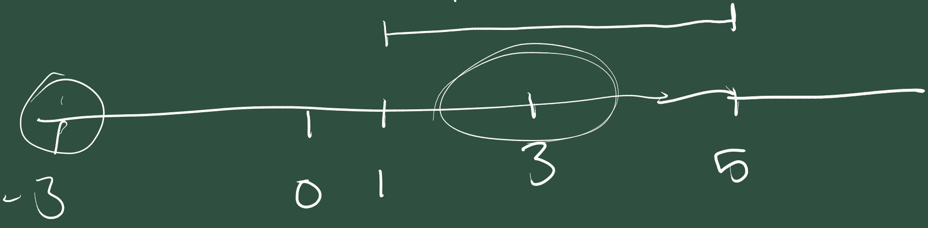
$$\iff |(x-3)(x+3)| < \varepsilon$$

$$\iff |x-3||x+3| < \varepsilon$$

$$\iff |x-3| < \frac{\varepsilon}{|x+3|},$$

note $|x+3|>0$ we should assume $|x+3|>0$

Consider possible values of $\frac{\varepsilon}{|x+3|}$?



Let's assume that $|x-3|<2$

$$\iff 1 < x < 5.$$

$$\iff 4 < x+3 < 8, \quad \text{so here } |x+3| = x+3$$

$$\iff \frac{1}{8} < \frac{1}{x+3} < \frac{1}{4}$$

$$\iff \boxed{\frac{\varepsilon}{8} < \frac{\varepsilon}{|x+3|} < \frac{\varepsilon}{4}}$$

$3<5$

$\frac{1}{3} > \frac{1}{5}$

So we set $\boxed{\delta = \min\left(2, \frac{\varepsilon}{8}\right)}$

So now, $\boxed{\text{if } |x-3| < \delta}$

then $|x-3| < 2$ AND $|x-3| < \frac{\varepsilon}{8}$

$$\Rightarrow \quad |x-3| < \frac{\varepsilon}{8} < \frac{\varepsilon}{|x+3|}$$

$$\Rightarrow \quad \boxed{|(x^2-3) - 6| < \varepsilon,}$$
$$|f(x) - L| < \varepsilon$$

from work above.

This is the formal def. proved

So $\lim_{x \to 3} x^2 - 3 = 6$.

$\lim_{x \to 4} 2x+5 = 13$

Q5 Let $\varepsilon > 0$ be given.

$$|2x+5-13| < \varepsilon$$

$$\Leftrightarrow \quad |2x-8| < \varepsilon$$

$$\Leftrightarrow \quad 2|x-4| < \varepsilon$$

$$\Leftrightarrow \quad |x-4| < \frac{\varepsilon}{2} \quad ✓$$

So we should assign $\delta = \frac{\varepsilon}{2}$.

So if $|x-4| < \delta$ then $|2x+5-13| < \varepsilon$.

So the limit is proved.

Q 13

$$\lim_{x \to 1} \frac{1}{x} = 1.$$

"Look for $|x-1|$ within this"

Let $\varepsilon > 0$ be given

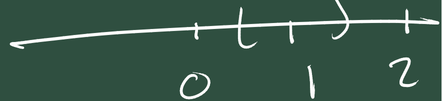
Consider $$\left| \frac{1}{x} - 1 \right| < \varepsilon$$

$$\iff \left| \frac{1-x}{x} \right| < \varepsilon$$

$$\iff \frac{|1-x|}{|x|} < \varepsilon.$$

$$\iff \frac{|x-1|}{|x|} < \varepsilon$$

$$\iff |x-1| < |x|\varepsilon$$



Let's assume $|x-1| < \frac{1}{2}.$

$$\iff -\tfrac{1}{2} < x-1 < \frac{1}{2}$$

$$\iff 1 - \tfrac{1}{2} < x < 1 + \tfrac{1}{2}$$

$$\iff \tfrac{1}{2} < |x| < \tfrac{3}{2}$$

So now assign $\boxed{\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)}$

Because now if $\boxed{|x-1| < \delta}$

then $\boxed{|x-1| < \frac{\varepsilon}{2} = \frac{1}{2}\varepsilon < |x|\varepsilon}$

and so $\boxed{\left|\frac{1}{x} - 1\right| < \varepsilon}$, from work above.

This proves the claimed limit.