Sections 6.1, 6.2, 6.5

tinyurl.com/apexcalculus.

Q20.

$$I = \int \frac{x^2+13x+3}{x^2+6x+12}\,dx.$$

idea is to reexpress the integrand as a linear combination of simpler rational functions.

Firstly degree numerator = degree denominator

$$x^2+13x+3 = (x^2+6x+12) + (7x-9)$$

$$\Rightarrow \frac{x^2+13x+3}{x^2+6x+12} = \frac{(\qquad)+(\qquad)}{x^2+6x+12}$$

$$= 1 + \frac{7x-9}{x^2+6x+12}.$$

Note that denominator is irreducible polynomial.

$$I = \int \left(1 + \frac{7x-9}{x^2+6x+12}\right) dx$$

$$= x + \int \frac{7x-9}{x^2+6x+12}\,dx.$$

we notice that $7x-9$ is a degree 1 polynomial, as is the derivative of the denominator, $2x+6$.

note that

$$\frac{d}{dx}\left[\ln(x^2+6x+12)\right] = \frac{2x+6}{x^2+6x+12}$$

To take advantage of this rewrite numerator as

$$7x-9 = \frac{7}{2}(2x+6) - 30$$

$$I = x + \int \frac{7}{2} \cdot \frac{2x+6}{x^2+6x+12} - \frac{30}{x^2+6x+12}\, dx$$

$$= x + \frac{7}{2}\int \frac{2x+6}{x^2+6x+12}\, dx - 30\int \frac{1}{x^2+6x+12}\, dx$$

$$= x + \frac{7}{2}\ln(x^2+6x+12) - 30\int \frac{1}{x^2+6x+12}\, dx$$

Reexpress denominator by "completing the square"

$$x^2+6x+12 = (x+3)^2+3$$

So our remaining integral is

$$\int \frac{1}{(x+3)^2+3}\,dx.$$

$$= \int \frac{1}{3\left[\left(\frac{x}{\sqrt{3}}+\sqrt{3}\right)^2+1\right]}\,dx.$$

$$= \frac{\sqrt{3}}{3}\int \frac{1/\sqrt{3}}{\left(\frac{x}{\sqrt{3}}+\sqrt{3}\right)^2+1}\,dx.$$

using atan result.

$$= \frac{\sqrt{3}}{3}\,\text{atan}\left(\frac{x}{\sqrt{3}}+\sqrt{3}\right)$$

So returning to I.

$$I = x + \frac{7}{2}\ln\left(x^2+6x+12\right)$$
$$-10\sqrt{3}\ \operatorname{atan}\left(\frac{x}{\sqrt{3}}+\sqrt{3}\right)$$

$$\operatorname{atan}(bx+a)$$

$$\Big\downarrow \ {}^{d}/_{dx}$$

$$\frac{b}{(bx+a)^2+1}.$$

$$\int \frac{b}{(bx+a)^2+1}\,dx = \operatorname{atan}(bx+a)$$
