Tutorial sheet: Series

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When possible you should work on the questions in advance and be prepared for discussions on them with your colleagues and tutor. Some questions are straightforward calculation while others require more discussion and thought. The computer icon 🖳 indicates where computer calculation or programming may be of use. Further questions can be found in the lecture notes as well as the recommended unit resources (see Moodle).

Series notation and defining formulas

(1) Write down some initial terms of the following series

$$\text{(i)} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}, \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{n+1}{n^3+2}, \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{3}{n!}, \quad \text{(iv)} \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} i^{-2} \right).$$

Series convergence definition

(2) Make sure you can accurately quote the definition of convergence for a series.
(3) Can you confidently give the proof that the harmonic series $\sum_{j=1}^{\infty} 1/j$ diverges?
(4) Can you prove the General Term Test using the definition of convergence?

Geometric series test

(5) Ensure that you can derive the partial sum formula for a general geometric series of the form

$$\sum_{n=1}^{\infty} ar^{n-1}.$$

Can you then prove the convergence/divergence of this series for all possible combinations of values of the initial term a and the common ratio r?

(6) Use the analysis of the ratio test to determine the convergence status and sum if possible, of the series

$$\text{(i)} \sum_{n=1}^{\infty} \frac{2}{3} \left(\frac{1}{3} \right)^{n+1}, \quad \text{(ii)} \sum_{n=1}^{\infty} 3 \left(\frac{1}{2} \right)^{n}, \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{1}{5} \left(\frac{6}{5} \right)^{n+2}.$$

(7) Consider the number a with decimal expansion

$$a = 0.1234123412341234\ldots,$$

continuing in the suggested way. Use the geometric series summation formula to express a as a rational number, i.e. a ratio of integers.

Application of the ratio test

(8) Determine the convergence or divergence of the series

$$\sum_{m=0}^{\infty} \frac{2^m}{m!}.$$

(9) Determine the convergence or divergence of the series

$$\sum_{m=0}^{\infty} \frac{2^m}{(m+1)^2}.$$

(10) Does the ratio test give a conclusion about the series

$$\sum_{n=1}^{\infty} \frac{n}{n+1}?$$

If not, determine the convergence/divergence of this series by more direct means.

Application of comparison tests

(11) Use a suitable comparison test to determine the convergence status of the series

$$\sum_{k=1}^{\infty} \frac{k^2+2}{k(k^2+5)}.$$

(12) Use a suitable comparison test to determine the convergence status of the series

$$\sum_{k=1}^{\infty} \frac{k+3}{k(k^2+2)}.$$

Application of partial fraction expansions

(13) Consider the series $\sum_{n=1}^{\infty} a_n$ where the term is defined by

$$a_n = \frac{9\,n^2 + 33\,n + 12}{(n+4)(n+3)(n+1)n}.$$

Obtain the partial fraction expansion of a_n and use this to obtain the sum of the series.

(14) Consider the series $\sum_{n=1}^{\infty} b_n$ where the term is defined by

$$b_n = \frac{8}{n^2 + 6n + 5}.$$

Obtain the partial fraction expansion of b_n and use this to obtain the sum of the series.

(15) Consider the series $\sum_{n=2}^{\infty} c_n$ where the term is defined by

$$c_n = \frac{9}{n^2 + n - 2}.$$

Obtain the partial fraction expansion of c_n and use this to obtain the sum of the series. (n.b. take note of the starting value of the index n)

(16) Consider the series $\sum_{n=1}^{\infty} d_n$ where the term is defined by

$$d_n = \frac{11 - n}{n^2 + 6n + 5}.$$

Obtain the partial fraction expansion of d_n and use this to decide whether the series is convergent or divergent.