Tutorial sheet: Sequences

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When possible you should work on the questions in advance and be prepared for discussions on them with your colleagues and tutor. Some questions are straightforward calculation while others require more discussion and thought. The computer icon indicates where computer calculation or programming may be of use. Further questions can be found in the lecture notes as well as the recommended unit resources (see Moodle).

Sequence notation and defining formulas

(1) List the initial five terms of the following sequences.

- (a) $\{a_n\}_{n=0}^{\infty}$, where $a_n = \frac{3n}{n^2+1}$.
- (b) $\{b_n\}_{n=1}^{\infty}$, where $b_n = \frac{2^n}{2+2^n}$.
- (c) $\{c_n\}_{n=1}^{\infty}$, where $c_n = \frac{(-1)^n}{3n}$.
- (d) $\{d_n\}_{n=1}^{\infty}$, where $d_n = \frac{(-1)^{n-1}}{2n}$.
- (e) $\{e_n\}_{n=0}^{\infty}$, where $e_0 = e_1 = 1$ and for all $n \geq 2$, $e_n = e_{n-1} + e_{n-2}$.
- (f) $\{f_n\}_{n=1}^{\infty}$, where $f_1 = 3$ and for all $n \geq 1$, $f_{n+1} = \dfrac{f_n}{n}$.
- (g) $\{g_n\}_{n=0}^{\infty}$, where $g_0 = 2$ and for all $n \geq 0$, $g_{n+1} = \dfrac{g_n}{1+g_n}$.
- (h) $\{h_n\}_{n=0}^{\infty}$, where $h_n = \dfrac{3^n}{(2n)!}$.

(2) For each of the sequences in question 1 find an expression for the $(2m+3)^{\text{rd}}$ element, simplified as far as possible.

Increasing / decreasing sequences

(3) Consider the sequences in question 1.

- (a) Decide whether they are increasing, decreasing or neither. Computer plots of sections of the sequence will help with this.
- (b) For those that are increasing or decreasing, try and construct a clear argument (proof) for your conclusion using the ratio and/or difference of consecutive terms.

Convergence definition for sequences

(4) How well do you understand the definition of convergence for a sequence?

- (a) Can you quote it accurately from memory?
- (b) Can you draw a suitable diagram illustrating sequence convergence and attach the parameters from the definition to it in the correct way?
- (c) Don't be too attached to the symbols x_n, N, ϵ used in the definition in the notes. Can you reliably quote the definition using entirely different symbols?

(5) Consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \frac{1}{n^2}$. Does this have a limit and if so prove this by showing that the definition of convergence is satisfied.

(6) Consider the following statements about a sequence $\{y_n\}_{n=1}^{\infty}$ which at first glance appear similar to the definition of sequence convergence.

S1: There exists a real value $\lambda > 0$ and a positive integer M such that for all $m > M$, $|y_m - L| < \lambda$.

S2: There exists an integer $M > 0$ such that for all real values $\lambda > 0$, if $m > M$ then $|y_m - L| < \lambda$.

For each of these statements can you give some examples of sequences (and values L) that satisfy them? For each statement can you characterize all the sequences that satisfy them?

Applying convergence results

(7) Determine the limit of the following convergent sequences. You might need the algebra of limits theorem and other results. Your solutions should explain clearly the reasoning used.

- (a) $\{a_n\}_{n=0}^{\infty}$, where $a_n = \frac{3n}{n^2+1}$.
- (b) $\{b_n\}_{n=1}^{\infty}$, where $b_n = \frac{2^n}{2+2^n}$.
- (c) $\{c_n\}_{n=1}^{\infty}$, where $c_n = \frac{5n^2-2}{3n^2-n}$.
- (d) $\{d_n\}_{n=1}^{\infty}$, where $d_n = \frac{2n}{3n-1} - \frac{n^2+1}{2-n^2}$.
- (e) $\{e_n\}_{n=1}^{\infty}$, where $e_n = \frac{n^3}{n^3+1}$.
- (f) $\{f_n\}_{n=1}^{\infty}$, where $f_n = \frac{\cos(3n)}{n^{1/3}+1}$.

Confirm your results by comparing to computer plots of the sequences.

(8) Consider the following sequence (which continues in the suggested way)

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}},\ldots.$$

(a) Formulate this as a sequence defined by a recurrence relation.
(b) With the aid of the Monotone Convergence Theorem prove that the sequence is convergent.
(c) Determine the limit.

(9) Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by the recurrence relation

$$a_0 = 1, \qquad \text{for all } n \geq 1,\ a_n = 3 - \frac{1}{a_{n-1}}.$$

Prove that this sequence is increasing, bounded and hence convergent. Determine its limit.

(10) Consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation

$$x_0 = 2, \qquad \text{for all } n \geq 1,\ x_n = \frac{1}{3 - x_{n-1}}.$$

Is this sequence convergent? If so, determine the limit.

(11) Analyze the convergence/divergence of the sequence $\{y_n\}_{n=0}^{\infty}$ defined by $y_n = nr^n$, where $r \in \mathbb{R}$. Your conclusion should say it diverges for these values of r, converges for those, and preferably determine the limit when it exists.

More convergence results

(12) Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and that $y_n \to 0$ as $n \to \infty$. Prove that the product sequence $\{x_n y_n\}_{n=1}^{\infty}$ also converges to 0, i.e. $x_n y_n \to 0$ as $n \to \infty$.

(13) Suppose that $\{n_j\}_{j=1}^{\infty} = n_1, n_2, n_3, \ldots$ is a strictly increasing sequence of positive integers, i.e. for all $j \geq 1$, $n_{j+1} > n_j \geq 1$. Prove that if $a_m \to L$ as $m \to \infty$ then the *subsequence* $\left\{a_{n_j}\right\}_{j=1}^{\infty}$ also converges to the same limit L, i.e.

$$a_{n_j} \to L \text{ as } j \to \infty.$$

Estimating π

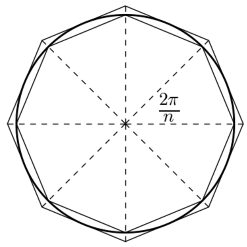

Figure 1: Unit circle with inscribed and circumscribed n-sided regular polygons, shown here with $n = 8$.

(14) In this question we can apply some of the sequence convergence ideas to approximate π using a precise geometric method based on the technique used by Archimedes to estimate π.
Consider figure 1 which shows a unit circle (i.e. with radius 1) together with two n-sided regular polygons. The inner polygon is the largest such one that fits inside the circle, i.e. its vertices lie on the circle, and the outer polygon is the smallest such one containing the circle, i.e. the midpoints of its sides lie on the circle. The angle formed at the centre of the circle by two adjacent polygon vertices is thus $2\pi/n$ as shown. The circle has area π. Let a_n denote the area of the inner n-sided polygon and b_n denote the area of the outer polygon. By studying a_n and b_n we will obtain a sequence of successively narrower intervals containing π.

(a) Make sure you appreciate why for every $n \geq 3$ we have $a_n < \pi < b_n$, and that as n increases we will obtain successively narrower intervals estimating π.
(b) Carefully consider the triangles making up the polygons and prove that

$$a_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right) \qquad \text{and} \qquad b_n = n\tan\left(\frac{\pi}{n}\right).$$

(c) With the help of various trigonometric formulas derive two recurrence relations, one giving a_{2n} in terms of a_n, the other giving b_{2n} in terms of b_n. The recurrence relations you are aiming for are quadratic in nature, i.e. involving constants, the usual arithmetic operations of addition/subtraction/multiplication/division and squaring and square roots.
(d) By considering the inscribed and circumscribed squares prove the initial values $a_4 = 2$ and $b_4 = 4$ and then use your recurrence formulas to generate some interval estimates of π.
(e) Code your recurrence formulas in a suitable programming language (Matlab, Sage, Python, Excel, ...) and see how accurately you can estimate π.

Suggestions for further work

You may get curious results as you compute a_{2^m} and b_{2^m} for larger and larger m, such as the sequence failing to consistenly increase/decrease or failing to return a value at all. This is because you have run up against rounding errors due to the computer only being able to store numbers to a certain precision (i.e. number of digits accuracy). Increase the accuracy of your estimates by increasing the precision of the numbers in your computation. Some information available from:
SageMath – `http://doc.sagemath.org/pdf/en/reference/rings_numerical/rings_numerical.pdf`
Matlab – `http://uk.mathworks.com/help/symbolic/numbers-and-precision.html`