- Some more proofs for theorem 1.3.1
- Selection of exercises from chapt. 1.

tinyurl.com/apexcalculus

Since we've product rule, to get quotient rule we only need to prove

assume $g(x) \to K$

$$\frac{1}{g(x)} \longrightarrow \frac{1}{K} \quad \text{as } x \to c. \quad K \neq 0$$

we have to prove under certain conditions

the $\left| \frac{1}{g(x)} - \frac{1}{K} \right| < \varepsilon.$

Consider

$$\left| \frac{1}{g(x)} - \frac{1}{K} \right| = \left| \frac{K - g(x)}{g(x) K} \right|$$

$$= \frac{|g(x) - K|}{|g(x)||K|}$$

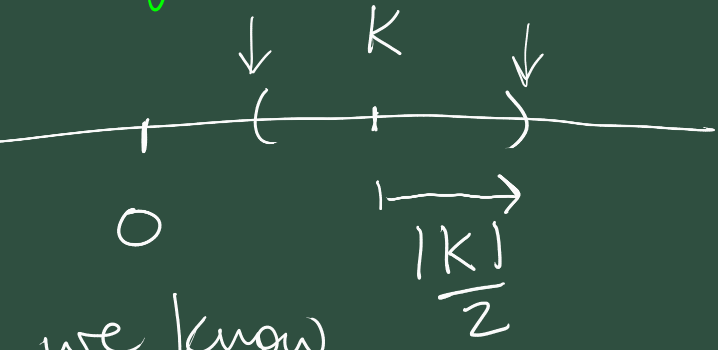
want $< \varepsilon$

Worry about $\frac{1}{|g(x)|}$ being large.

we want to stop $\frac{1}{|g(x)|}$ getting too big.

Assume $K>0$

($K<0$ can be done in a similar way)



Since $g(x) \to K$ we know

$\exists\, \delta_1$ s.t $|x-c|<\delta_1$ $|g(x)-K| < \frac{|K|}{2}$

$$\Leftrightarrow \quad \frac{K}{2} < g(x) < \frac{3K}{2}$$

$$\Leftrightarrow \quad \frac{2}{3K} < \frac{1}{g(x)} < \frac{2}{K}$$

$\boxed{\text{Let } \varepsilon > 0 \text{ be given.}}$

If $|x-c|<\delta_1$ then

$$\left| \frac{1}{g(x)} - \frac{1}{K} \right| = \frac{|g(x)-K|}{|g(x)|\,|K|}$$

$$< \frac{2}{K^2} |g(x)-K|$$

So $\exists\, \delta_2$ st. if $|x-c|<\delta_2$

then $|g(x)-K| < \frac{\varepsilon K^2}{2}$

So let $\boxed{\delta = \min(\delta_1, \delta_2)}$

and note that if $\boxed{|x-c|<\delta}$

$$\boxed{\left|\frac{1}{g(x)} - \frac{1}{K}\right|} < \frac{2}{K^2}\,|g(x)-K| < \frac{2}{K^2}\,\frac{\varepsilon K^2}{2}$$

$$\boxed{= \varepsilon.}$$

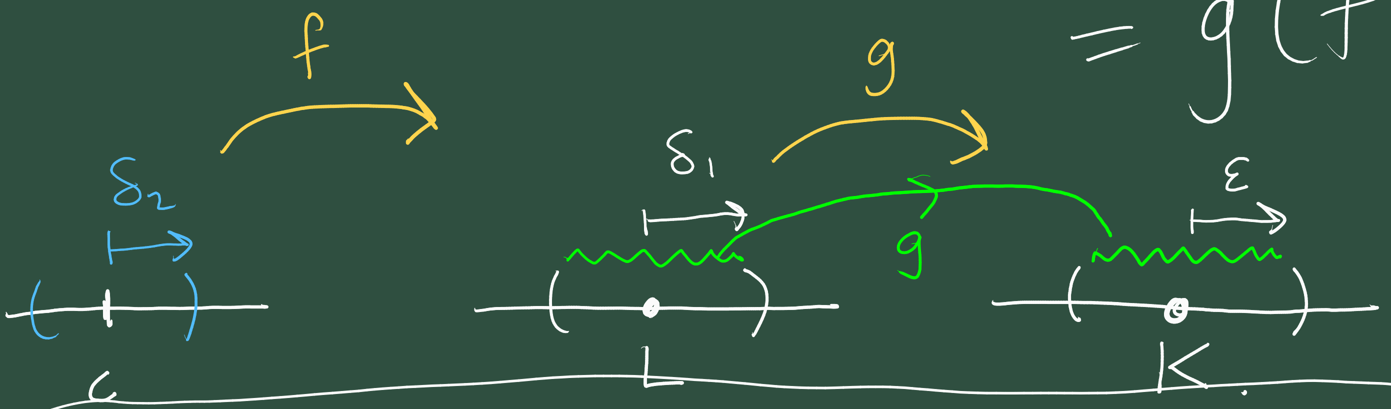
$$\frac{f}{g} = f \times \frac{1}{g}$$

g composed with f "

Compositions case. "g after f /

Considering function $(g \circ f)(x)$

$$= g(f(x))$$



Assume $\boxed{\lim_{x\to c} f(x) = L}$, $\boxed{\lim_{x\to L} g(x) = K}$,

$g(L) = K$

So prove that $\boxed{\lim_{x\to c} g(f(x)) = K.}$

Let $\varepsilon > 0$ be given.

we need to make $|g(f(x)) - K| < \varepsilon$.

$\exists\, \delta_1$ s.t. $|x - L| < \delta_1 \Rightarrow |g(x) - K| < \varepsilon$ $(*)$

$\exists\, \delta_2$ s.t. $|x - c| < \delta_2 \Rightarrow |f(x) - L| < \delta_1$

$\Rightarrow |g(f(x)) - K| < \varepsilon$, from $(*)$

This (in pink) is the ε-δ definition for $\lim\limits_{x \to c} g(f(x)) = K$.

Sec 1.3

Q11 $\lim\limits_{x \to 9} g(f(x)) = 3$

as $x \to 9$ $f(x) \to 6$

as $y \to 6$ $g(y) \to 3$

Q13

$$\lim_{x \to 6} \underbrace{g(\underbrace{f(\underbrace{f(x)}_{\to 9})}_{\to 6})}_{\to 3} = 3.$$

Q14

$$\lim_{x \to 6} \left(f(x)g(x) - f(x)^2 + g(x)^2 \right).$$

using thm 1.3.1

$$= \left(\lim f(x)\right)\left(\lim g(x)\right) - \left(\lim f(x)\right)^2 + \left(\lim g(x)\right)^2$$

$$= 9 \cdot 3 - 9^2 + 3^2,$$ using assumptions

$$= -45.$$