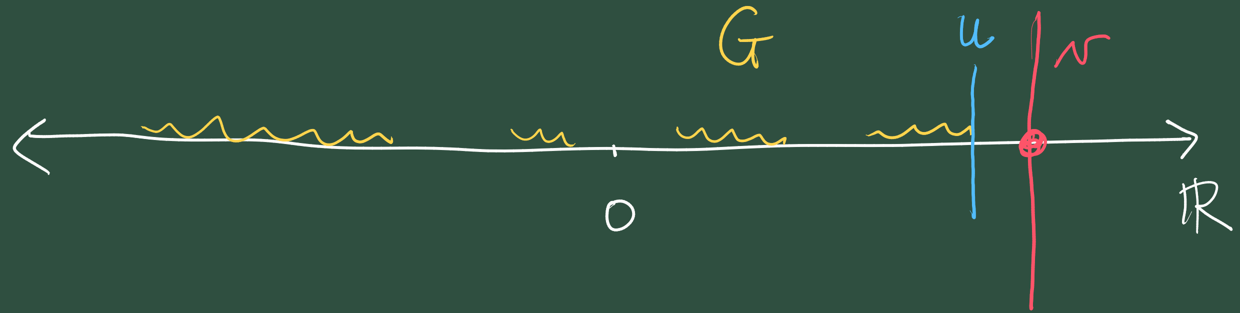
We'll be studying real valued functions.

typically defined as

$$f : \mathbb{R} \to \mathbb{R}.$$

What's the basis for doing maths in $\mathbb{R}$

$\mathbb{R}$ is defined via a system of Axioms.

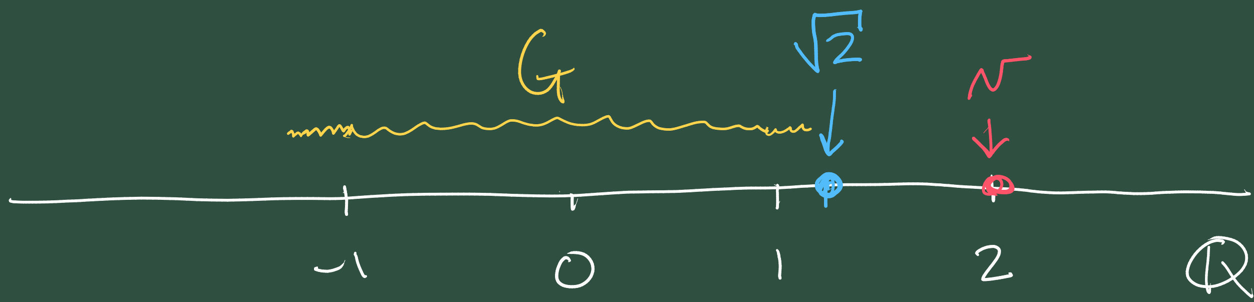


Axioms 1–15 are also true of $\mathbb{Q}$

The system $\mathbb{Q}$ does not satisfy

axiom 16

eg. Consider $G = \{ q \in \mathbb{Q} : q^2 < 2 \}$



G is bounded above, for example by $v = 2$

But G has no least upper bound in $\mathbb{Q}$.

$\sqrt{2}$ might be a least upper bound for G, but $\sqrt{2} \notin \mathbb{Q}$.

$$\sqrt{2} \neq \frac{a}{b}, \text{ for any } a, b \in \mathbb{Z}$$

So in a sense $\mathbb{Q}$ lack certain quantities.

Section 1.1

We speak of "limiting values for functions"

we'll write.

$$\boxed{\lim_{x \to a} f(x) = L}$$

to mean, that as x approaches a, the value $f(x)$, gets closer and closer to L.

ε

L informal definition of limiting value

eg. $y = f(x) = \dfrac{\sin x}{x}$

$\lim_{x \to 1} f(x) = ?$

well $f(1) \simeq 0.84$, and from the graph it's reasonable to say

$\lim_{x \to 1} f(x) \simeq 0.84$

But what about $\lim_{x \to 0} f(x) = ?$ $\quad f(x) = \dfrac{\sin x}{x}$

Maybe from the graph we could say $\lim_{x \to 0} f(x) = 1$?

Or we can perform calculations..

So the informal definition is good but perhaps we need something better.

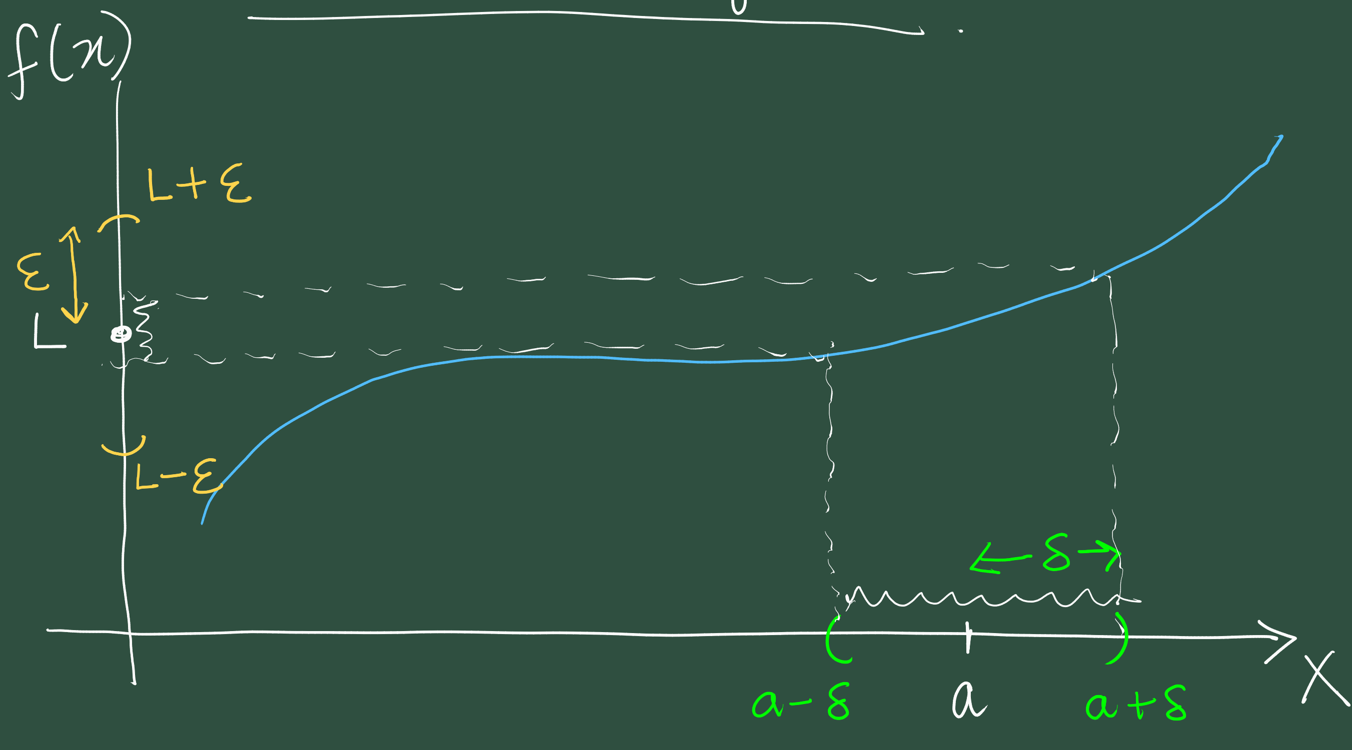
- we want to be able to rigorously calculate limits
- we want to be able to prove theorems about limiting value.

  for instance Theorem 1.3.1, showing many properties of limits.

Section 1.2 ε epsilon, δ delta

The Formal Definition

The ε-δ definition

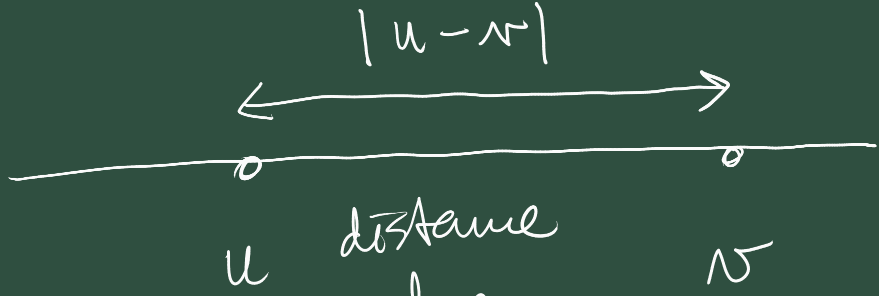


" $\lim_{x \to a} f(x) = L$ " **Formal definition**

means $\boxed{\text{For every positive } \varepsilon \text{, there exists an associated positive } \delta \text{ such that, whenever } x \text{ is within a distance } \delta \text{ of } a\text{, } f(x) \text{ will be within a distance } \varepsilon \text{ of } L.}$

More compact symbolic form is

$$\boxed{\forall \varepsilon > 0 \ \exists \delta > 0 \quad |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon}$$



distance between u and v on number line.

Ex 1.2.4. $f(x) = \sqrt{x}$.

" $\lim_{x \to 4} \sqrt{x} = 2.$ "

Exercises Sec. 1.2.

Q8. Claim $\displaystyle\lim_{x \to 4} \overbrace{(x^2+x-5)}^{f(x)} = \overbrace{15}^{L}$

We will prove this using ε-δ def. Let $\varepsilon > 0$ be given.

Investigate $\boxed{|f(x) - L| < \varepsilon}$.

$$|x^2+x-5 - 15| < \varepsilon$$

$$\iff |x^2+x-20| < \varepsilon$$

$$\iff |(x-4)(x+5)| < \varepsilon$$

$$\iff |x-4|\,|x+5| < \varepsilon,$$

abs. value is multiplicative, ie. $|uv| = |u|\,|v|$

$$\iff \boxed{|x-4| < \frac{\varepsilon}{|x+5|}},$$

provided that $|x+5| \neq 0$

is this our δ? But it still depends on x

We need to somehow eliminate $|x+5|$ from here.

Let us assume for the moment that $|x-4| < 1$

$$\iff -1 < x-4 < 1$$

$$\iff \quad 3 < x < 5$$

$$\iff \quad 8 < x+5 < 10$$

$$\iff \quad 8 < |x+5| < 10$$

since $|x+5| = x+5$

for $3 < x < 5$

because $x+5 > 0$.

This means that if $|x-4| < 1$

then $\dfrac{\varepsilon}{10} < \dfrac{\varepsilon}{|x+5|}$.

All these conditions can be arranged to be true by setting

$$\delta = \min\left(1, \frac{\varepsilon}{10}\right).$$

Then notice that if

$|x-4| < \delta$, then $|x-4| < \dfrac{\varepsilon}{10} < \dfrac{\varepsilon}{|x+5|}$,

i.e. $|\overbrace{x^2+x-5}^{f(x)} - \overbrace{15}^{L}| < \varepsilon$

Thus we've proved the ε-δ def of limiting value, and so we can say

$$\boxed{\lim_{x \to 4} x^2 + x - 5 = 15.}$$