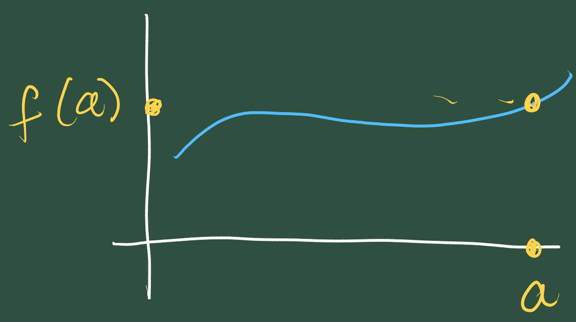
Derivative

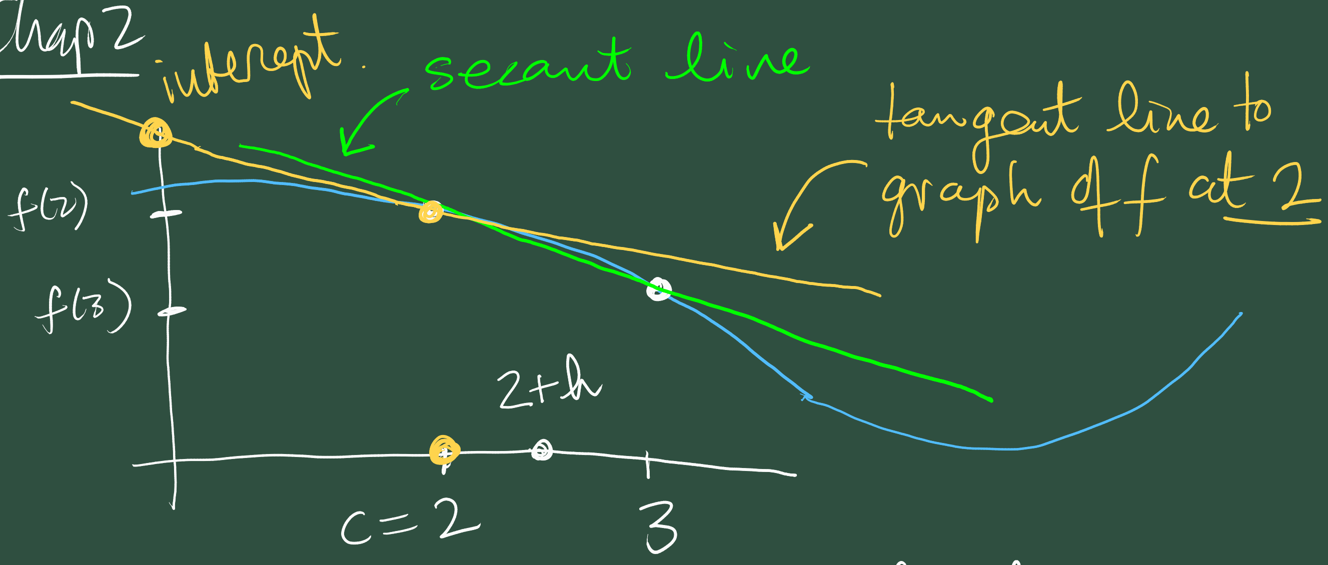
Defn We say a function $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$



and we say f is continuous on $\mathbb{R}$ if this is so for all points in $\mathbb{R}$.

Chap 2



we want to quantify how the function is changing near 2.

consider the change in height of this line.

$$f(3) - f(2)$$ "absolute change"

But this depends heavily on distance between 3 and 2. So instead consider the relative change in f

$$\frac{f(3) - f(2)}{3-2}$$

this also has a geometric meaning as the gradient/slope of the secant line.

But what about to capture the behaviour of f at 2. This can be done with a limiting value.

$$\boxed{\lim_{h \to 0} \left(\frac{f(2+h) - f(2)}{h} \right)}$$

this will the gradient of the so called tangent line, the limiting case of the secant lines.

Def 2.17

For f continuous on open interval around c. The <u>derivative of f at c</u> is the limit.

$$\lim_{h \to 0} \left(\frac{f(c+h) - f(c)}{h} \right)$$

"f prime"

Use notation $f'(c)$ for this.

Def 2.19 The equation of the tangent line to f at c is

$$l(x) = f'(c)(x-c) + f(c).$$

$$= \underbrace{f'(c)}\, x + \underbrace{f(c) - f'(c)c}.$$

$$y = m \; x + \text{intercept}.$$

When f is differentiable (i.e. limit exists) at all points in its domain we say it's differentiable.

Eg. 2.1.22

Consider $f(x) = 3x^2 + 5x - 7$.

Find $f'(x)$ using the limit formulae.

From the def.

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3(x+h)^2 + 5(x+h) - 7 - (3x^2+5x-7)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3(\cancel{x^2} + 2xh + h^2) + \cancel{5x} + 5h \cancel{-7} \cancel{-3x^2} - \cancel{5x} + \cancel{7}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{6xh + 3h^2 + 5h}{h}\right)$$

$$= \lim_{h\to 0}\left(6x + 3h + 5\right)$$

$= 6x + 5$, by linearity of limits and $h \to 0$

Eg 2.1.23 Consider $f(x) = \frac{1}{x}$

Find $f'(x)$, assume $x \neq 0$.

Along similar lines to previous example.

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\frac{1}{x+h} - \frac{1}{x}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{\frac{x - (x+h)}{x(x+h)}}{h} \right)$$

$$\frac{\frac{a}{b}}{c} = \frac{a}{bc}.$$

$$= \lim_{h \to 0} \left(\frac{-\cancel{h}}{x(x+h)\cancel{h}} \right)$$

$$= \lim_{h \to 0} \left(\frac{-1}{x^2 + xh} \right)$$

$$= \frac{-1}{\lim_{h \to 0} (x^2 + xh)},$$ by quotient rule for limits

$$= \frac{-1}{x^2},$$ by linearity of limits.

linearity.

$$\boxed{f(y) = \frac{1}{y}}$$

$$f(\underbrace{x+h}_{y}) = \frac{1}{x+h}$$

Se. 2.3 Basic rules of the derivative.

Want more general principles/rules for how to find derivatives of complex functions.

Generalise above example to find.

for $f(x) = ax^2+bx+c$ we get.

$$f'(x) = 2ax+b.$$

A simple. for constant functions.

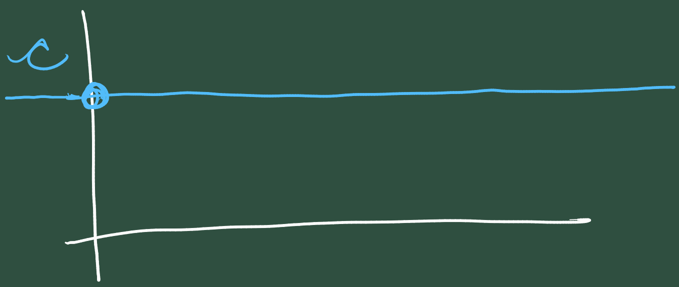
ie. $f(x) = c$.

we have $f'(x) = 0$

Alternative notation

$$\frac{df}{dx}$$

$$\frac{d}{dx}(f)$$



$$\left(f'(a) = \frac{df}{dx}\bigg|_{x=a}. \right)$$

Pf $\displaystyle \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\displaystyle = \lim_{h\to 0} \frac{c-c}{h}$

$\displaystyle = \lim_{h\to 0} 0$

$= 0$

Power rule for any integer $n>0$

If $f(x)=x^n$ then $f'(x)=nx^{n-1}$.

Pf: $\displaystyle f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$$= \lim_{h\to 0}\left(\frac{(x+h)^n - x^n}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\left(\sum_{j=0}^{n}\binom{n}{j} x^{n-j}h^j\right) - x^n}{h}\right)$$

[using binomial expansion of $(x+h)^n$
where $\binom{n}{j}$ is the binomial coefficient.]

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}, \quad a! = a(a-1)(a-2)\cdots 3\cdot 2\cdot 1$$

$$\binom{n}{0} = 1 \qquad \boxed{0! = 1} \text{ — definition}$$

$$\binom{n}{n} = 1, \quad \binom{n}{1} = n$$

$$= \lim_{h\to 0} \frac{\cancel{x^n} + \binom{n}{1} x^{n-1} \cancel{h} + \binom{n}{2} x^{n-2} h^{\cancel{2}\,1} + \cdots + \binom{n}{n} h^{\cancel{n}\,n-1} \cancel{- x^n}}{\cancel{h}}$$

$$= \lim_{h\to 0} \left(\underline{n x^{n-1}} + \binom{n}{2} x^{n-2} h + \cdots + \binom{n}{n} h^{n-1} \right)$$

$= n x^{n-1}$, using linearity of the limit and the fact that all terms after the first have a factor of h, so $\to 0$ as $h \to 0$.

<u>Theorem 2.3.6</u> Differentiation is linear.

Let's prove the sum rule, using what we know about <u>the linearity of the limiting process</u>.

Let f, g be two differentiable functions.

$$(f+g)'(x)$$

$$= \lim_{h \to 0} \left(\frac{(f+g)(x+h) - (f+g)(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right)$$

$$= \underbrace{\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) + \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right)}$$

, by linearity of limits

$$= f'(x) + g'(x)$$

, by def. of derivatives.

<u>Exercise</u> Prove the multiple rule from theorem 2.3.6.

<u>Polynomials all dealt with</u>

Combining linearity with the power rule. means we can differentiate any polynomial.

eg. $g(x) = 5x^{11} - 7x^3 + 3$

$g'(x) = 5 \cdot 11 x^{10} - 7 \cdot 3x^2 + 0$

$= 55x^{10} - 21x^2$

<u>Product rule</u>

Suppose a function h is defined as $h = f \cdot g$, of two differentiable functions.

$$h' = (f \cdot g)' \quad \overset{??}{\underset{??}{\sim\!\sim\!\sim}} \quad f' \quad g'$$

Warning differentiation is not multiplicative $(f \cdot g)' \neq f' \cdot g'$

Theorem 7.4.2 Product rule

$$(f \cdot g)'(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

Proof:

$$(f \cdot g)'(x) = \lim_{h \to 0} \left(\frac{(f \cdot g)(x+h) - (f \cdot g)x}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h} \right).$$

$-f(x)g(x)$

$$= \lim_{h \to 0} \left(\frac{f(x+h) g(x+h) - f(x+h) g(x) + f(x+h) g(x)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{f(x+h)\left(g(x+h) - g(x)\right) + g(x)\left(f(x+h) - f(x)\right)}{h} \right)$$

$$= \lim_{h \to 0} \left(f(x+h) \frac{g(x+h) - g(x)}{h} \right)$$

$$+ \lim_{h \to 0} \left(g(x) \frac{f(x+h) - f(x)}{h} \right)$$

, splitting fraction and sum rule for limits.

$$= \lim_{h \to 0} \left(f(x+h) \right) \boxed{\lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right)}$$

$$+ \quad g(x) \boxed{\lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)}$$

, using prod. rule for limits

$$= f(x) g'(x) + g(x) f'(x)$$, as expected.

$$\frac{a \cdot b}{c} = a\,\frac{b}{c}$$

$$= b\,\frac{a}{c}$$

Two important rules
to conn.

- Quotient rule $\quad \frac{f}{g}$
- composition rule / chain rule.

$$(f \circ g)(x) = f(g(x))$$