Chain rule, quotient rule, and an alternative view on def. of the derivative

Quotient rule

warning $\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$

it is:

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

See examples in APEX.

We'll prove this, with the aid of the chain rule, and our product rule.

Chain rule How to differentiate 'chains' or compositions of functions

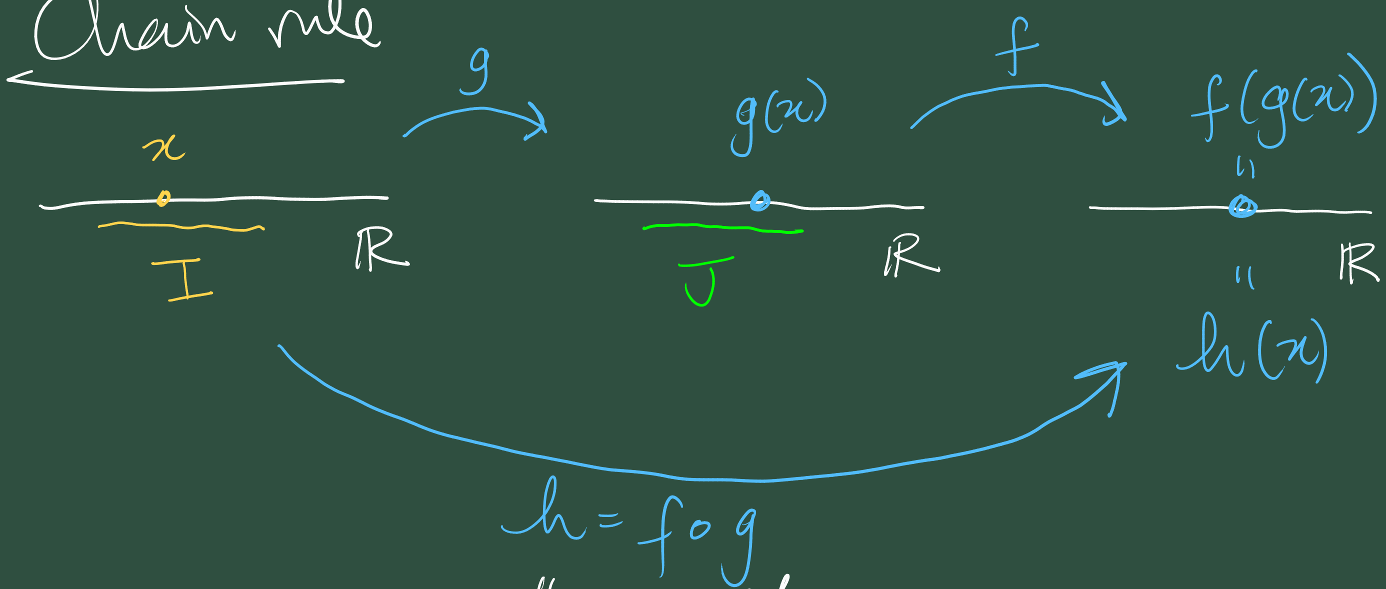
Consider a function h defined as "f after g"

$$h = f \circ g$$

i.e. $h(x) = f(g(x))$

Q? How does h' depend on f, g and their derivatives Q?

Chain rule



Assuming g differentiable on an interval I, $x \in I$, and f is differentiable on J containing $g(x)$, then h' given by

$$h'(x) = f'(g(x))\, g'(x).$$

Eg. 2.5.8 (2)

$$y = \ln(4x^3 - 2x^2).$$

We see y as the composition

$$y = \ln(g(x)),$$ where

g is the poly. function

$$g(x) = 4x^3 - 2x^2$$

$$y' = \frac{1}{g(x)} \cdot g'(x) \quad \text{, by chain rule}$$

$$= \frac{1}{4x^3 - 2x^2} \cdot (12x^2 - 4x)$$

$$= \frac{12x^2 - 4x}{4x^3 - 2x^2}$$

We will prove the chain rule.

But first: let's prove quotient rule using it.

Consider a quotient of functions $\frac{f}{g}$ viewed in terms of a product and a composition

$$\frac{f}{g} = f \cdot \left(\frac{1}{g}\right)$$

composition

product

$$= f \cdot (r \circ g)$$

where r is the reciprocal function defined by $\boxed{r(x) = \frac{1}{x}}$.

with derivative $\boxed{r'(x) = -\frac{1}{x^2}}$

Confirm this.

$$\left(f\cdot(r\circ g)\right)(x) = f(x)\cdot(r\circ g)(x)$$
$$= f(x)\cdot\left(r(g(x))\right)$$
$$= f(x)\cdot\frac{1}{g(x)}$$
$$= \frac{f(x)}{g(x)} = \left(\frac{f}{g}\right)(x)$$

Proof of the quotient rule.

$$\boxed{\left(\frac{f}{g}\right)'(x)} = \left(f\cdot(r\circ g)\right)'(x)$$
$$= f'(x)\,(r\circ g)(x) + f(x)\,(r\circ g)'(x)$$

, by product rule.

$$= f'(x)\cdot r(g(x)) + f(x)\cdot r'(g(x))\,g'(x)$$

, def. of comp. and chain rule

$$= f'(x)\cdot\frac{1}{g(x)} + f(x)\left(\frac{-1}{g(x)^2}\right)\cdot g'(x)$$
$$= \frac{f'(x)\,g(x)}{g(x)^2} + \frac{-f(x)\,g'(x)}{g(x)^2}$$

$$= \boxed{\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}}$$

which is the claimed quotient rule!

For the proof of chain rule, we'll use an alternative view/version of the def. of derivative.

Recall the def.

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

So we can formulate/define an "error function" called ε.

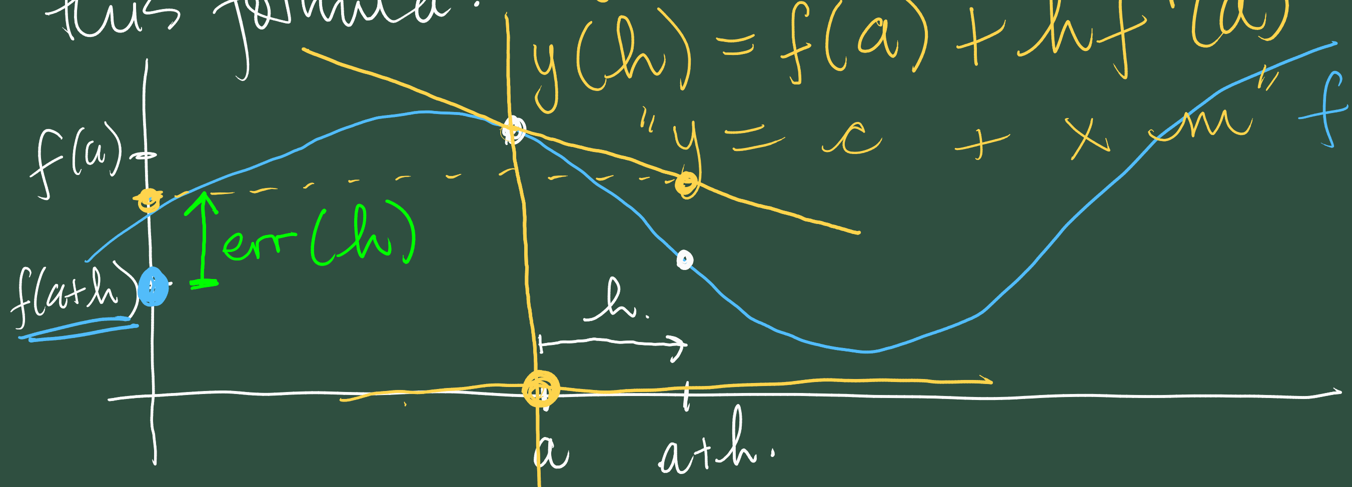
$$\varepsilon(h) = \frac{f(a+h) - f(a)}{h} - f'(a).$$

we know that $\varepsilon(h) \to 0$ as $h \to 0$.

Now rearrange this to get:

$$\underbrace{f(a+h)}_{\text{new value of } f} = \underbrace{f(a) + hf'(a)}_{\text{linear approximation}} + \underbrace{h\,\varepsilon(h)}_{\substack{\text{associated error term} \\ err(h)}}$$

Graphical / geometric interpretation of this formula.



We call this the "linear approximation with error" form for $f(a+h)$.

$\boxed{\text{We can define the derivative } f'(a) \text{ as the unique number } m \text{ such that the linear approximation } f(a) + hm \text{ for } f(a+h) \text{, has an associated error term } err(h) \text{ with the convergence property } \dfrac{err(h)}{h} \to 0 \text{ as } h \to 0.}$

L a new def. of $f'(a)$.

Chain rule proof: Given $y = f \circ g$. Find $y'(a)$.

Assuming g differentiable at a, f differentiable at $g(a)$.

So we bring in "lin. approx with error" forms for both f and g

1) $\boxed{g(a+h)} = \boxed{g(a) + h g'(a)} + \boxed{h\varepsilon(h)}$

2) $\boxed{f(g(a)+l)} = \boxed{f(g(a)) + l f'(g(a))}$
$+ \boxed{l\mu(l)}$

where ε, μ have the convergence props $\varepsilon(h) \to 0$ as $h \to 0$, $\mu(l) \to 0$ as $l \to 0$

Use 1) and 2) to derive the "lin. approx with error" form for $y(a+h)$ and in doing so discover what $y'(a)$ is

$\boxed{y(a+h)} = f(g(a+h))$, defn. of y.

$= f(g(a) + \underbrace{h g'(a) + h\varepsilon(h)}_{l})$, by 1).

$$= f(g(a)) + \underbrace{\left(hg'(a) + h\varepsilon(h)\right)}_{\text{, by (2) with } l \text{ as}} f'(g(a)) + l\mu(l).$$

$$= f(g(a)) + hf'(g(a))g'(a) + \underbrace{h\varepsilon(h)f'(g(a)) + l\mu(l)}_{err(h)},$$

$$= \boxed{y(a) + hf'(g(a))g'(a)} + \boxed{err(h)}$$

This is a linear approx for $y(a+h)$ with associated error function $err(h)$, with the expected chain rule expression sitting in gradient position! ✓

We will have proved that

$$\{y'(a) = f'(g(a))\,g'(a)\} \quad \text{"chain rule"}$$

once we've shown that

$$\frac{err(h)}{h} \longrightarrow 0 \text{ as } h \longrightarrow 0.$$

$$\frac{err(h)}{h} = \cancel{h}\,\varepsilon(h) f'(g(a)) + (\cancel{h} g'(a) + \cancel{h}\varepsilon(h))\,\mu(l)$$

From this we consider behaviour of $\frac{err(h)}{h}$ as $h \to 0$.

Note that

- $\varepsilon(h) f'(g(a)) \to 0$ as $h \to 0$
- $(g'(a) + \varepsilon(h)) \to g'(a)$ as $h \to 0$
- Note that $l \to 0$ as $h \to 0$

so therefore $\mu(l) \to 0$ as $h \to 0$

All these together imply that

$$\frac{err(h)}{h} \to 0 \text{ as } h \to 0.$$

This proves the chain rule!

Tutorial Fri: Applications of all these and rest of chapter 2.

- do bring Questions to tutorials.

Next week we'll look at chapt 4, focussing on Optimization.