MOCK EXAM 02
Faculty Of Science & Engineering
Department Of Computing And Mathematics
MATHEMATICS UNDERGRADUATE NETWORK
Level 4

6G4Z3006: Calculus Mock exam 02

Instructions to students

- You need to answer **four** questions. Answer **both** questions from **Section A** and **two** questions from **Section B**.
- You must show all of your working and explain your reasoning carefully to gain full marks.
- Marks awarded for each question part are shown in square brackets aligned to the right-hand margin.

Permitted materials

- Students are permitted to use their own calculators without mobile communication facilities.

Section A – answer both questions

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Use the concept of limiting value of a function to give the definition of the *derivative* of f at the point $a \in \mathbb{R}$. [4]

   (b) The *sum rule* of differentiation states that if f and g are two differentiable functions then the derivative $(f+g)'$ of the sum function $f+g$ is given by [8]

$$(f+g)'(x) = f'(x) + g'(x).$$

   Prove this sum rule using the limiting value definition of the derivative you have in part (a). You can use without proof any relevant properties of the limiting values of functions, providing you clearly state these in your proof.

   (c) State the *chain rule* of differentiation. [4]

   (d) State the *quotient rule* of differentiation. [4]

   (e) Let r be the reciprocal function $r : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by [5]

$$r(x) = \frac{1}{x}.$$

   Its derivative r' is the function defined by

$$r'(x) = \frac{-1}{x^2}.$$

   Let s be the square-root function, defined by $s(x) = \sqrt{x}$, which has derivative s' defined by

$$s'(x) = \frac{1}{2\sqrt{x}}.$$

   Show how to use the chain rule along with these two results to show that the function $\sigma : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by

$$\sigma(x) = \frac{1}{\sqrt{x}},$$

   has the derivative σ' defined by

$$\sigma'(x) = \frac{-1}{2x^{3/2}}.$$

   You will need to clearly point out where the chain rule of the derivative is being used.

Section A – answer both questions

2. (a) State both parts of the *Fundamental Theorem of Calculus*. Suppose the function $F(x)$ is defined by [8]

$$F(x) = \int_0^x (t + \cos(t))dt.$$

Use the Fundamental Theorem of Calculus to find $F'(\pi)$, i.e. the derivative of F at π.

(b) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Give the (ϵ, N)-definition for the convergence of the sequence $\{x_n\}_{n=1}^{\infty}$. [3]

(c) Consider the sequence $\{a_m\}_{m=1}^{\infty}$ defined by $a_m = 1/m$ for all $m \geq 1$. Use the definition of sequence convergence from part (b) to prove that $a_m \to 0$ as $m \to \infty$. [6]

(d) Use concepts of sequence convergence and partial sums to give the definition for the convergence of an infinite series $\sum_{n=1}^{\infty} a_n$. [3]

(e) Consider the hyper-harmonic series [5]

$$\sum_{m=1}^{\infty} \frac{1}{m^2}.$$

Use the integral test and knowledge of the integral of $1/t^2$ to show that this is a convergent series.

End of Section A

Section B – answer any two questions

3. (a) Use the limit definition of the derivative to obtain the derivative of the function f defined by the formula [5]

$$f(x) = x^2.$$

Point out clearly where you make use of any properties of the limiting values of functions.

(b) Consider the function g defined by the formula [6]

$$g(x) = \left(\cos(x) + x^3\right)^3.$$

Use the linearity, product and chain rules to obtain the first and second order derivatives, $g'(x)$ and $g''(x)$, of g. Point out clearly where you are using each of the rules. There is no need to simplify your answers, just carry out the differentiation using the relevant properties.

(c) A standard aluminium soft drinks can is roughly cylindrical and holds a volume of 330cm^3 of liquid. What dimensions should the cylinder have to minimize the material needed to produce the can? [10]

If a cyclinder has height h and the radius of its circular cross-section is r then its surface area, S, is given by

$$S = 2\pi rh + 2\pi r^2,$$

and its volume, V, is givne by

$$V = \pi r^2 h.$$

(d) Suppose that the function $f(x)$ is differentiable at a point $x = a$. Show how to derive the *linear approximation with error* result, [4]

$$f(a + h) = f(a) + hf'(a) + E(h),$$

which shows how $f(a) + hf'(a)$ gives a linear approximation to the value $f(a + h)$ for small h, with associated error term $E(h)$. Your derivation should show how $E(h)$ is defined and make clear the convergence property it has as $h \to 0$.

Section B – answer any two questions

4. (a) Demonstrate the integration by substitution technique by making a suitable substitution and using the known integrals of rational powers of a variable to evaluate the indefinite integral I, given by [6]

$$I = \int \frac{x}{\sqrt{x+3}}\, dx.$$

(b) Consider the indefinite integral J given by [6]

$$J = \int x e^{-2x}\, dx.$$

Carry out the integral J by using integration by parts. You can make use of the known integral of the exponential function.

(c) The Riemann integral of a function $f(x)$ between the limits $x = a$ and $x = b$ is defined as [13]

$$\int_a^b f(x)\, dx = \lim_{\substack{n\to\infty \\ \Delta x \to 0}} \left(\sum_{i=1}^{n} f(x_i^*)\Delta x_{i-1}, \right),$$

where the x_i $(0 \le i \le n)$ form a subdivision of the interval $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

$\Delta x_{i-1} = x_i - x_{i-1}$ is the width of the i^{th} subinterval,
$\Delta x = \max(\{\Delta x_i \,:\, 0 \le i \le n-1\})$ and x_i^* is a representative point from the i^{th} subinterval, i.e. $x_{i-1} \le x_i^* \le x_i$.

In this question you will evaluate the Riemann integral for the function $f(x) = x^2$ and the limits 0 and b, i.e.

$$\int_0^b x^2\, dx.$$

You should use the equally spaced interval

$$x_i = i\frac{b}{n}, \quad (i = 0, \ldots, n),$$

and the start point of each subinterval as the representative point, i.e.

$$x_i^* = (i-1)\frac{b}{n}.$$

Question 4 continues on the next page.

Set up and evaluate the limit in the Riemann integral, using known summation formulae, to show that

$$\int_0^b x^2\,dx = \frac{b^3}{3}.$$

You can use, without proof, any of the following summation formulae

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Section B – answer any two questions

5. (a) Let the sequence $\{b_m\}_{m=1}^{\infty}$ be defined by the formula [5]

$$b_m = \frac{m}{m+5}.$$

By considering the ratio or difference of its consecutive terms, prove that this sequence is increasing for all $m \geq 1$.

(b) Let the sequence $\{c_m\}_{m=1}^{\infty}$ be defined by the formula [7]

$$c_m = \frac{7m^3 - m^2 + 1}{-3m^3 + m}.$$

Show how the algebra of limits theorem for convergent sequences can be used to prove that $\{c_m\}$ is a convergent sequence and to determine its limit. Make it clear in your answer where you are using each part of the theorem that you rely on.

(c) Consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation

$$x_0 = 2, \quad \text{for all } n \geq 1,\ x_n = \frac{1}{3 - x_{n-1}}.$$

Is this sequence convergent? If so, determine the limit.

(i) Use the recurrence relation above to prove, by induction, that for all $n \geq 0$ the sequence elements are bounded by 0 and 2. [6]

(ii) Use the result of part (i) to prove that the sequence $\{x_n\}_{n=0}^{\infty}$ is a decreasing sequence. [2]

(iii) Briefly explain how the Monotone Convergence Theorem can be used to prove that $\{x_n\}_{n=0}^{\infty}$ is a convergent sequence. [2]

(iv) Finally, use the algebra of limits theorem to prove that $x_n \to \frac{3-\sqrt{5}}{2}$ as $n \to \infty$. [3]

Section B – answer any two questions

6. (a) Consider the infinite series [6]

$$\sum_{n=1}^{\infty} \frac{5n^3}{n(n^2+2)}.$$

Show that the series diverges by applying the General Term Test.

(b) (i) Use a suitable comparison test to prove that the series [6]

$$\sum_{m=1}^{\infty} \frac{n}{1+n^3},$$

is a convergent series. You can make use of the known convergence of any hyper-harmonic series.

(ii) Use the ratio test to prove that the series [6]

$$\sum_{m=1}^{\infty} \frac{2m}{m!},$$

is a convergent series. *Note that* ! *denotes the factorial operation*.

(c) Consider carefully the partial sums of the series [7]

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right).$$

Make use of the logarithmic property

$$\ln(ab) = \ln(a) + \ln(b),$$

and its generalisation to more terms, to show that the series above is divergent.

End of Section B

End OF QUESTIONS