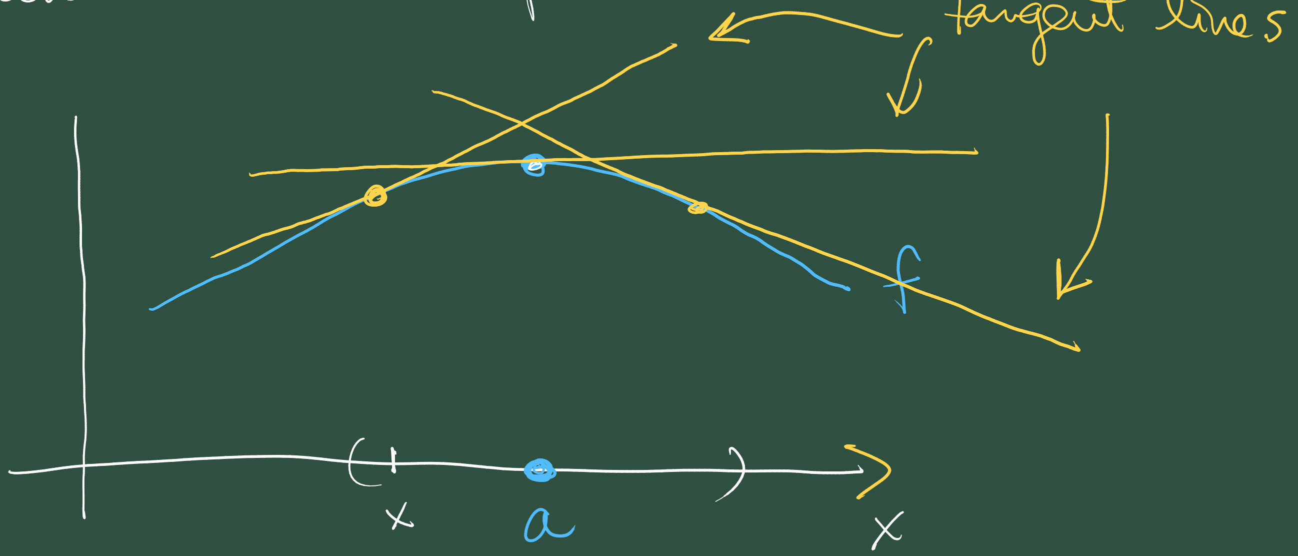
- Applying the derivative to find optimal solutions to various problems. ↳ (maximums, minimums).

- Stationary points / critical points of single variable functions.

- Think about how a continuous function can reach an optimal value.



f might have a (local) maximum at a

tangent line will be horizontal at a i.e. $\boxed{f'(a)=0}$ *

increasing at points to the left of a i.e. $f'(x)>0$

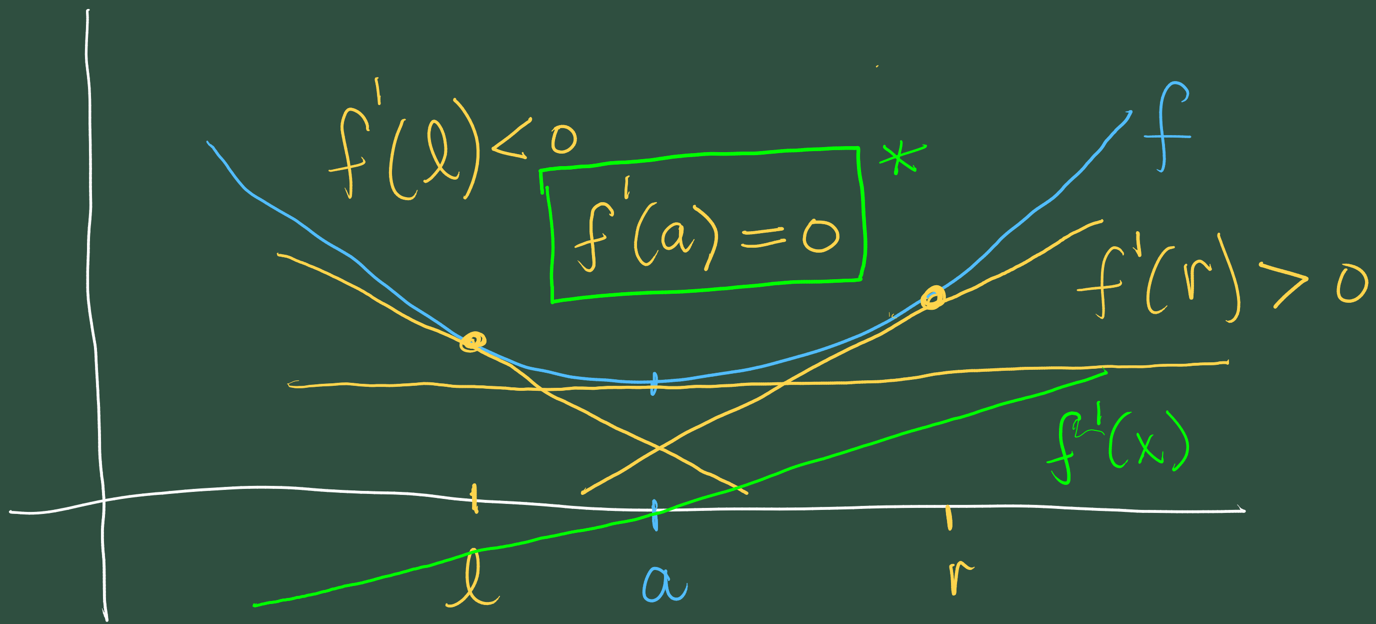
decreasing at points to the right of a. i.e. $f'(x)<0$

So as x passes through a (left to right)

f' is decreasing, i.e. $\frac{d}{dx}f' < 0$, i.e. $\boxed{f''(a)<0}$

+ to −

A similar diagram for a (local) minimum.

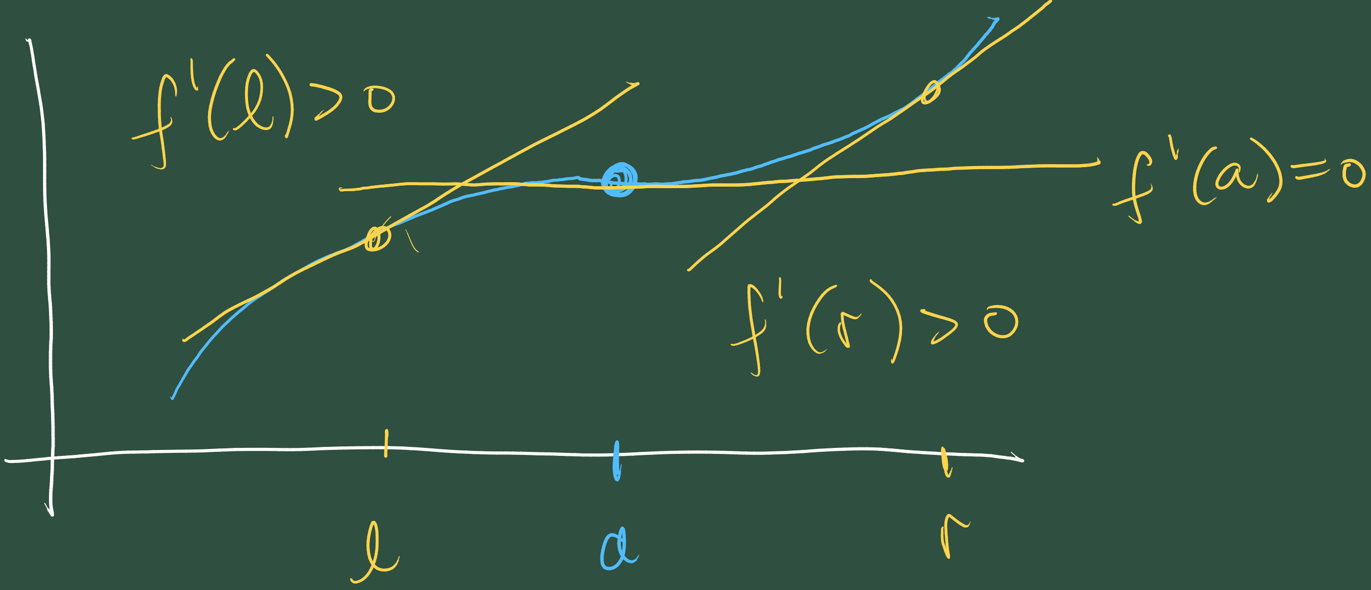


with signs of the derivatives flipped as appropriate.

So as x passes through a f' is <u>increasing</u> $-\ 0\ +$, ie $\left(\frac{d}{dx}f'\right)(a)>0$

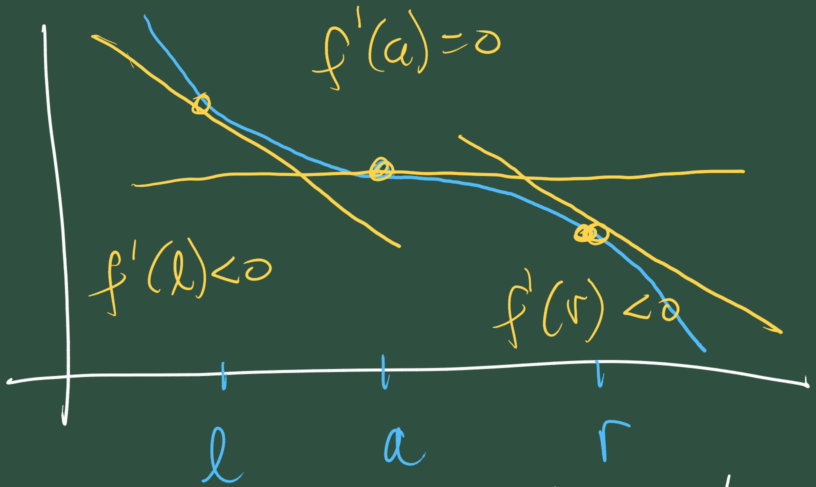
i.e. $\boxed{f''(a)>0}$.

There is a third case with horizontal tangent lines, so called <u>inflection point</u>, a kind of mixture of minimum and maximum.



So as x passes through a, we see f' has a local minimum at $x=a$.
i.e. $f''(a)=0$ (a is a critical point for f' also).

There's also a mirrored version



So as x passes through a, we see f' has a local maximum at $x=a$
i.e. $f''(a)=0$.

All these considerations apply to 'isolated critical points' of f'

i.e. points a for which $\boxed{f'(a) = 0}$ and in some ~~not~~ interval a, there are no other critical points.

<u>Ex 4.3.2</u>

Rectangle with perimeter constrained to be 100 feet.

$$2h + 2w = 100$$

Area $= A(h, w) = hw$ a function of <u>two</u> variables. But with the constraint these aren't independent

$$2h + 2w = 100$$

$$\iff h + w = 50$$

$$\iff h = 50 - w \quad \text{for } 0 \leq w \leq 50$$

So A becomes a single-variable function.

$$A = hw = (50 - w)\,w$$

$$\implies A(w) = -w^2 + 50w.$$

The edge cases.

≈ 50

w near 0

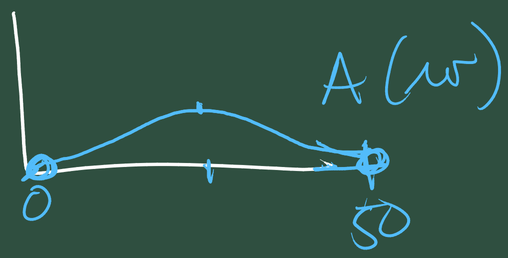
$A \approx 0$

$h \approx 0$

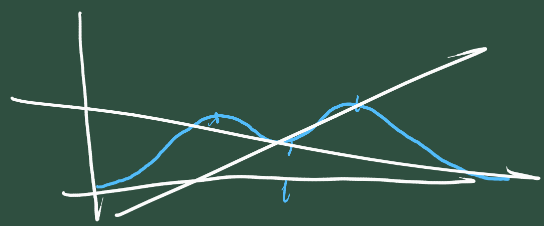
$w \approx 50$

$A \approx 0$.

So our intuition is that there is some intermediate case(s) with maximal area.



or maybe



So let's find and classify critical points of $A(w) = -w^2 + 50w$

$$A'(w) = -2w + 50$$

and $$A'(w) = -2w + 50 = 0$$

$$\Rightarrow w = 25$$

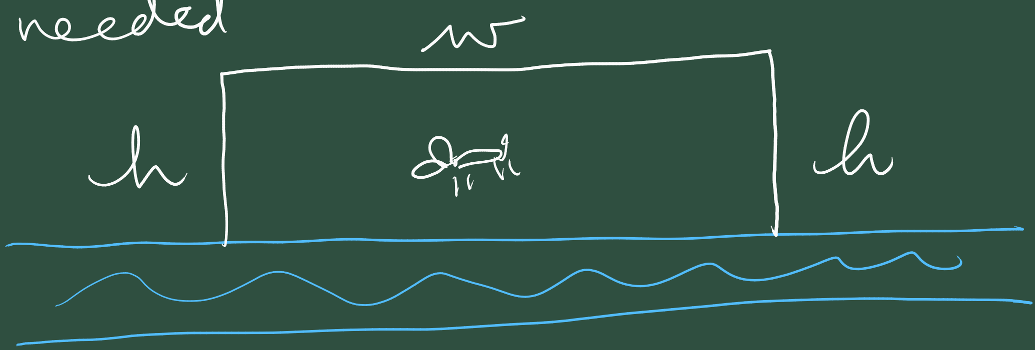
So only <u>one</u> critical point

And we suspect it is a local maximum.

$$A''(w) = -2 < 0 \quad \text{at } w = 25$$

So $w = 25$ is a local maximum of A and the maximal such field is the square field of side length 25 feet.

Eg 4.3.5 Still perimeter fencing constrained to be 100 feet, but only 3 sides are needed



Perimeter constraint is $2h + w = 100$

$$\iff w = 100 - 2h$$

$$A(h, w) = hw$$

So area becomes

$$A(h) = h(100 - 2h) = -2h^2 + 100h$$

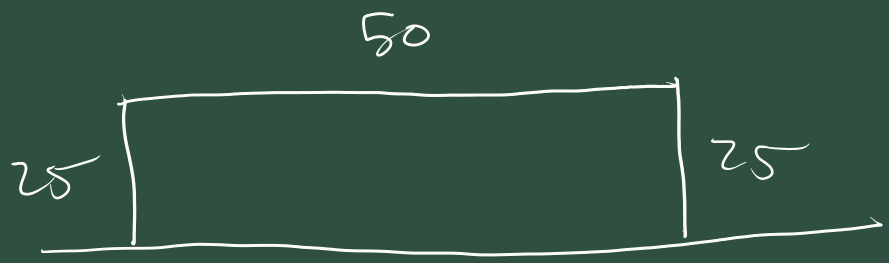
Proceed as before

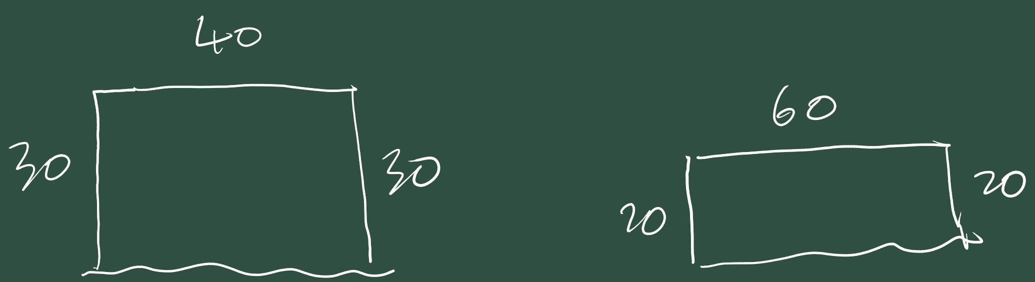
$$A'(h) = -4h + 100 = 0$$

$$\iff h = 25$$

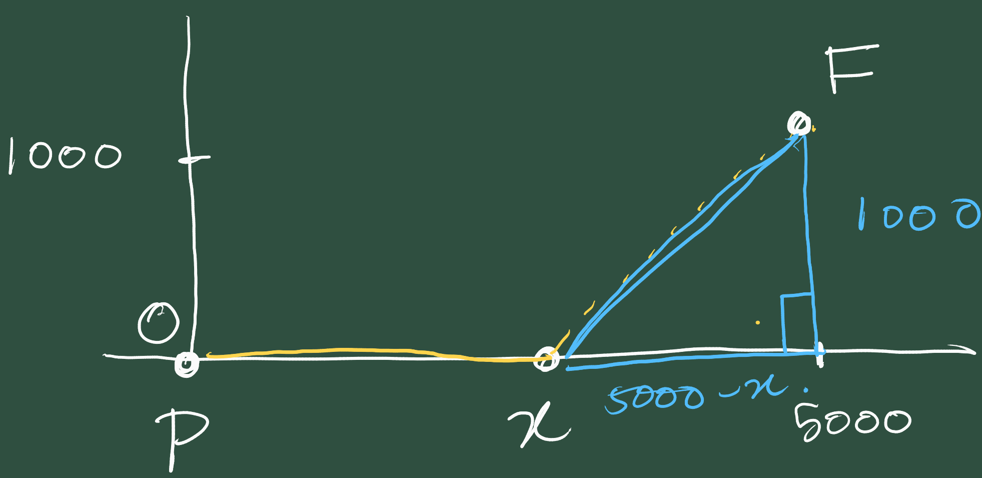
So again only one critical point.

$A''(h) = -4 < 0$, so this need a maximum for the area





Eg 4.3.7. Feeling is that ~~those~~ the cheapest installation is some intermediate case as depicted.



We need to identify the cost function

$C(x)$ $\qquad 130\left(1000^2 + (5000-x)^2\right)^{1/2}$

$$C(x) = \underset{\sim\sim}{50x} + \underset{\sim\sim\sim\sim\sim\sim\sim\sim}{130\sqrt{1000^2 + (5000-x)^2}}$$

Let's find and locate critical point(s) of C.

$$C'(x) = 50 + \frac{130}{\cancel{2}}\left(10^6 + (5000-x)^2\right)^{-1/2}\left(-\cancel{2}(5000-x)\right)$$

, using chain rules, power rules and linearity

$$= 50 + \frac{130(x-5000)}{\left(10^6 + (5000-x)^2\right)^{1/2}}.$$

Solve $C'(x) = 0$.

$$\iff \frac{130(x-5000)}{(\qquad)^{1/2}} = -50.$$

Square both sides.

$$\Rightarrow \frac{130^2(x-5000)^2}{10^6 + (5000-x)^2} = 2500$$

$$\iff 130^2(x-5000)^2 = 2500\left(10^6 + (5000-x)^2\right)$$

which is a quadratic equation.

Write $y = x - 5000$

$$\Leftrightarrow \quad 130^2 y^2 = 2500\left(10^6 + y^2\right)$$

$$\Leftrightarrow \quad 130^2 y^2 = 2500 \cdot 10^6 + 2500\, y^2$$

$$\Leftrightarrow \quad y^2 = \frac{2500 \cdot 10^6}{14400} = \frac{1562500}{9}.$$

$$\Leftrightarrow \quad y = \frac{-1250}{3}$$, taking negative square root to undo our original squaring

$$\Leftrightarrow \quad x^* = 5000 - \frac{1250}{3}$$

$$= 4583\tfrac{1}{3} \text{ feet.}$$

In base 3 notation $\frac{1}{3}$ is $0{\cdot}1$

Is this critical point a local minimum. Well, our reasoning indicate so. To formally check

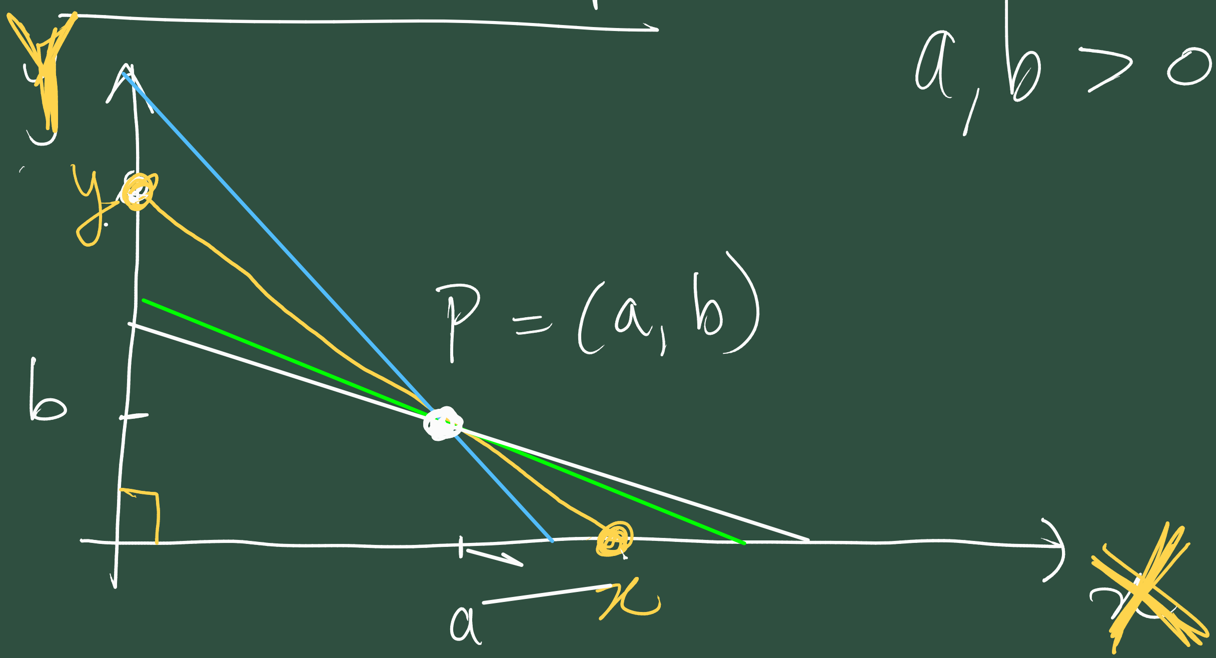
we can check the C'' value.
Evaluating this on the computer gives

$$C''(x^*) = \frac{216}{274625} > 0$$

So x^* is a local minimum of
Cost function C.

Another example

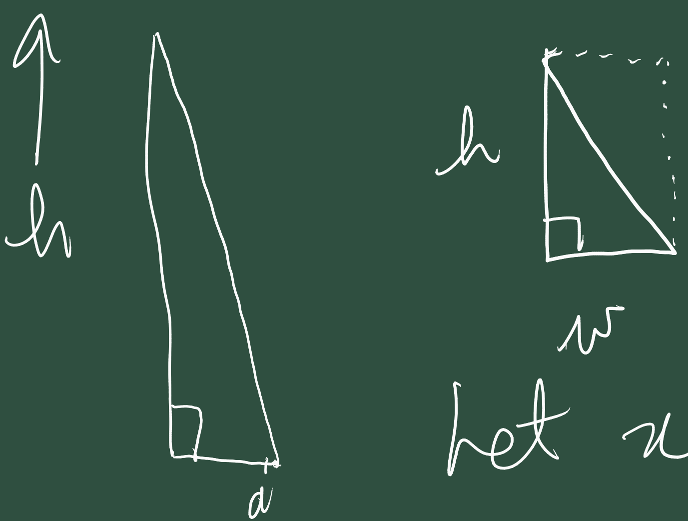
$a, b > 0$



Consider all triangles with sides of the two axes and hypoteneuse through P. Find the one with minimal area

There's no maximum area as hypotenuse that start close to a.

produce triangles with width just over a , ~~th~~ and very high



Area $= \frac{1}{2} w h$.

Let x be point on X axis where hypotenouse starts $x \in (a, \infty)$ and y the point on y axis. $y \in (b, \infty)$

$A(x, y) = \frac{1}{2} x y$, but x, y are not independent, as the line must go through P.

Consider gradient of hypotenouse.

$$\frac{-y}{x} = \frac{-b}{x-a} \quad)$$

$$\Leftrightarrow \quad y = \frac{xb}{x-a}$$

So $A(x,y) = \frac{1}{2}xy$

$$\Rightarrow \boxed{A(x) = \frac{1}{2}\frac{x^2 b}{x-a}}$$

$$\boxed{a < x < \infty}$$

The edge case triangles:

as $x \longrightarrow a$, or $x \longrightarrow \infty$

have unbounded area

Find and classify critical point(s) of A.

$$A'(x) = \frac{b}{2}\frac{2(x-a)x - x^2}{(x-a)^2}$$

, by quotient rule.

$$= \frac{b}{2} \frac{x^2 - 2ax}{(x-a)^2}$$

$$A'(x) = 0$$

$$\iff x^2 - 2ax = 0$$

$$\iff x(x-2a) = 0$$

$$\iff x = 0 \quad \underline{\text{OR}} \quad x = 2a$$

out of scope
out of model

So only one critical point
$x = 2a$ in the valid interval
for $x \in (a, \infty)$
This is a local minimum
from our consideration of edge

cases or we can check

$A''(2a)$ for confirmation.

$A''(2a) = \frac{b}{a}$, from computer

> 0

This confirms it's a minimum.

and $A(2a) = \frac{1}{2} \cdot \frac{4a^2 b}{a}$.

$= 2ab$.

Section 4.3 exercises for the tutorial.