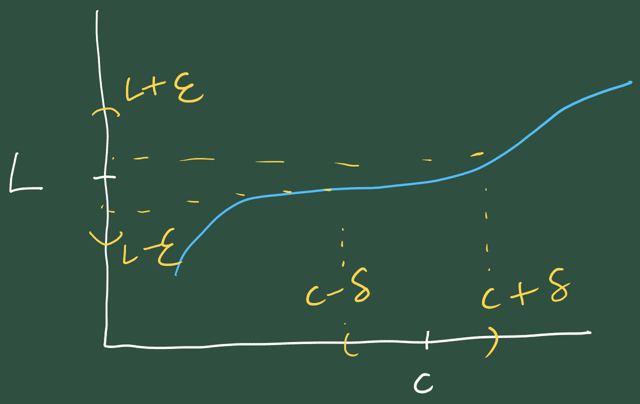
Quick reminder of def:

" $\lim_{x \to c} f(x) = L$ "

means $\forall \varepsilon > 0 \; \exists \delta > 0 \;\; |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$



1.3. Finding limits analytically

We can exploit properties of how the limiting process works to evaluate limits of complex functions, without having to use ε–δ proofs.

See Theorem 1.3.1. shows how to evaluate limits of compound functions built from two functions with known limits.

We will prove some of the cases of 1.3.1 using ε–δ argument.

Combine a few of these cases into a result about limits of linear combinations.

<u>Claim</u> If $\boxed{\lim_{x\to c} f(x) = L}$, $\boxed{\lim_{x\to c} g(x) = K}$

then for any pair of non-zero reals $\alpha, \beta \in \mathbb{R}$ we will have

$$\lim_{x\to c} (\alpha f + \beta g)(x) = \alpha L + \beta K.$$

<u>Proof</u>: First look at the inequality we are trying to achieve.

$$|(\alpha f + \beta g)(x) - (\alpha L + \beta K)| < \varepsilon$$

$$\Longleftrightarrow |\alpha f(x) + \beta g(x) - \alpha L - \beta K| < \varepsilon \qquad (*)$$

$$\Longleftrightarrow \boxed{|\alpha(f(x) - L) + \beta(g(x) - K)| < \varepsilon}$$

We can use the $\triangle$ inequality for abs. value, i.e. $|A+B| \leq |A| + |B|$

and the multiplicative property $|uv| = |u||v|$

$$\boxed{|\alpha(f(x) - L) + \beta(g(x) - K)|}$$

$$\leq |\alpha||f(x) - L| + |\beta||g(x) - K|$$

How can I arrange that this is $< \varepsilon$?

Look at it $\underbrace{\ }_{\text{want}}$

$$\overbrace{|\alpha|\,|f(x)-L|}^{<\varepsilon/2} + \overbrace{|\beta|\,|g(x)-K|}^{<\frac{\varepsilon}{2}} \overset{\text{want}}{<} \varepsilon$$

$$\underbrace{|f(x)-L|}_{<\frac{\varepsilon}{2|\alpha|}} \qquad \underbrace{|g(x)-K|}_{<\frac{\varepsilon}{2|\beta|}}$$

I can achieve these using the known limiting values of f, g.

$\boxed{\text{Let } \varepsilon>0}$ be given. assume $f(x)\to L$ and $g(x)\to K$ as $x\to c$, by the ε-δ def.

$\exists\ \delta_1>0$ s.t. $|x-c|<\delta_1 \Rightarrow |f(x)-L| < \dfrac{\varepsilon}{2|\alpha|}$

AND

$\exists\ \delta_2>0$ st $|x-c|<\delta_2 \Rightarrow |g(x)-K| < \dfrac{\varepsilon}{2|\beta|}$

If we can be sure that both of these hold true then by putting together all the above we will have (*) being true.

So we should choose $\delta=\min(\delta_1,\delta_2)$

then if $|x-c|<\delta$ then

$|(\alpha f+\beta g)(x) - (\alpha L+\beta K)| < \varepsilon$

This proves the claim!

Now for products.

<u>Claim</u> If $f(x) \to L$, $g(x) \to K$ as $x \to c$ then

$$f(x)g(x) \longrightarrow LK$$

<u>Proof</u>: Examine the goal inequality

$$\boxed{|f(x)g(x) - LK| < \varepsilon} \quad (*)$$

How does $|f(x)g(x) - LK|$ depend on $|f(x) - L|$, $|g(x) - K|$?

$$|f(x)g(x) - LK|$$

$$= |(f(x) - L)(g(x) - K) \overbrace{-2LK}^{-LK \ -LK}$$

$$+ Kf(x) + Lg(x)|$$

$$= |(f(x) - L)(g(x) - K) +$$

$$K(f(x)-L) + L(g(x)-K)\,|$$

(Apply Δ inequality and mult. prop. of abs val.)

$$\leq \underbrace{\underbrace{|f(x)-L|}_{<\sqrt{\frac{\varepsilon}{3}}}\;\underbrace{|g(x)-K|}_{<\sqrt{\frac{\varepsilon}{3}}}}_{<\varepsilon/3} + \underbrace{|K|\,\underbrace{|f(x)-L|}_{<\frac{\varepsilon}{3|K|}}}_{<\varepsilon/3}$$

$$+ \underbrace{|L|\,\underbrace{|g(x)-K|}_{<\frac{\varepsilon}{3|L|}}}_{<\frac{\varepsilon}{3}}.$$

wait

$< \varepsilon$

$\boxed{\text{Let } \varepsilon > 0 \text{ be given.}}$

Assume $f(x) \to L$, $g(x) \to K$, as $x \to c$, then from the ε-δ definition we get

$\exists \delta_1$ s.t. $|x-c| < \delta_1 \Rightarrow |f(x)-L| < \min\left(\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3|K|}\right)$

$\exists \delta_2$ s.t. $|x-c| < \delta_2 \Rightarrow |g(x)-K| < \min\left(\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3|L|}\right)$

So let $\boxed{\delta = \min(\delta_1, \delta_2)}$

and now

$\boxed{\text{If } |x-c| < \delta}$, all above becomes true

and we get $\boxed{|f(x)g(x) - LK| < \varepsilon.}$

which proves the claim for products.

Ex 1.3.3. Assume $\lim_{x\to 2} f(x) = 2$,

~~Then~~ $\lim_{x\to 2} g(x) = 3$, $p(x) = 3x^2 - 5x + 7$.

a). $\lim_{x\to 2} (f(x) + g(x)) = 2 + 3 = 5$.

b) $\lim_{x\to 2} \left(5f(x) + g(x)^2\right) = \lim(5f(x)) + \lim(g(x)^2)$

$$= 5\lim(f(x)) + \lim(g(x))^2$$

$$= 5\cdot 2 + 3^2 = 19$$

(All steps above, justified by Theorem 1.3.1).

c) $\lim_{x\to 2} p(x) = \lim_{x\to 2} (3x^2 - 5x + 7)$

$$= 3\lim(x)^2 - 5\lim(x) + 7.$$

$$= 3\cdot 2^2 - 5\cdot 2 + 7$$

$$= 9$$

The Quotient case $K \neq 0$

If $f(x) \to L$, $g(x) \to K$ then $\frac{f(x)}{g(x)} \to \frac{L}{K}$

Since we already have proved the product rule, and notice that,

$$\frac{f(x)}{g(x)} = f(x) \cdot \left(\frac{1}{g(x)} \right)$$

so all we have to prove is the reciprocal property,

$$\frac{1}{g(x)} \longrightarrow \frac{1}{K}$$

Be thinking about this in meantime,

$$\left| \frac{1}{g(x)} - \frac{1}{K} \right| \qquad \text{want} < \varepsilon$$

discover how to express this in terms of $|\underbrace{g(x) - K}|$

$$\left| \frac{1}{g(x)} - \frac{1}{K} \right|$$

$|a-b|$
$=|b-a|$

$$= \left| \frac{K - g(x)}{g(x)K} \right|$$

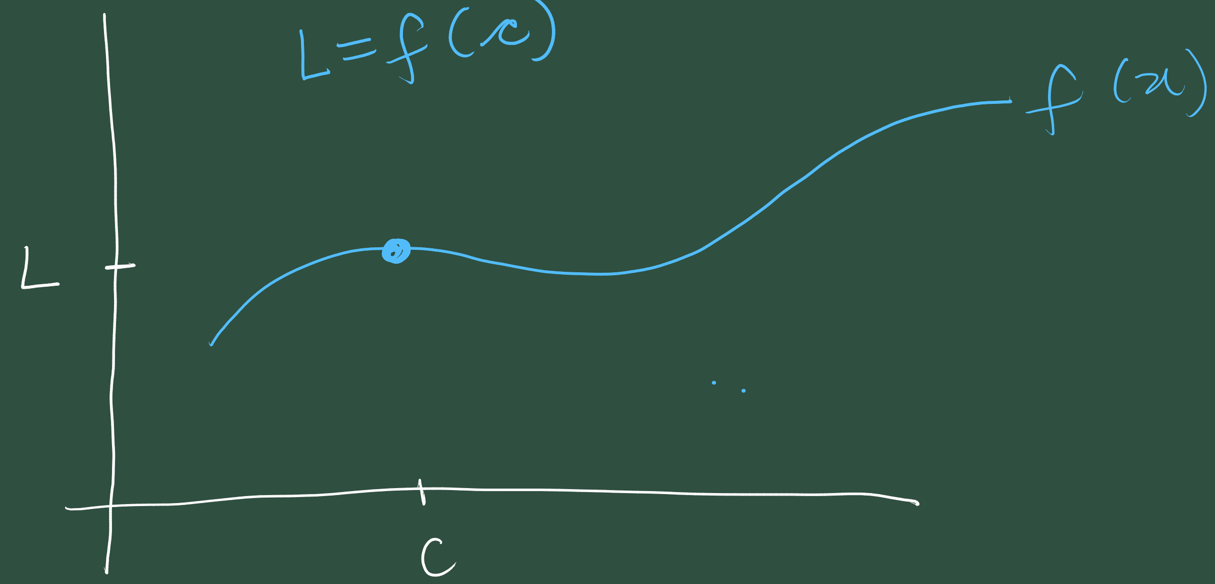
$$= \frac{|K - g(x)|}{|g(x)||K|} \text{, props of } |\ |$$

$$= \underbrace{\frac{|g(x) - K|}{|g(x)||K|}}$$

Want. $< \varepsilon$

This could get tiny.

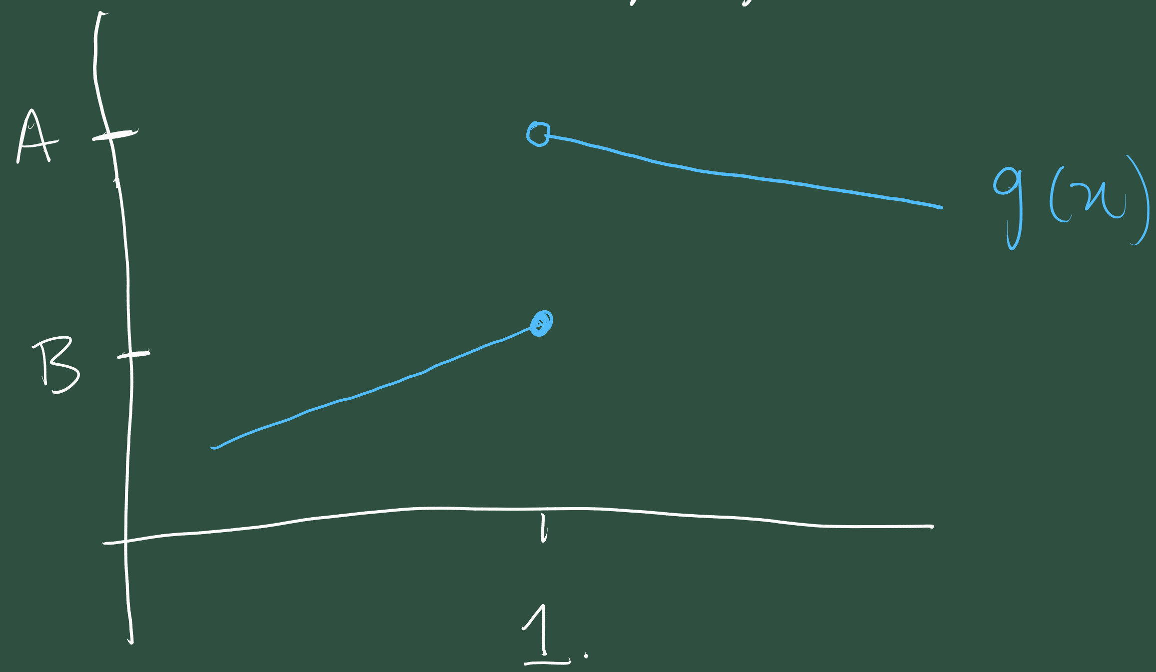
Section 1.5 Continuity



"Continuous at c"

A function f is "continuous at c"

if $\quad f(c) = \lim_{x \to c} f(x)$.



This function has a point of discontinuity at 1, since $\lim_{x \to 1} g(x)$ does not exist, as its left and right handed limits are different.

$$\lim_{x\to 1^-} g(x) = B \neq A = \lim_{x\to 1^+} g(x)$$

So g is <u>not</u> continuous at 1.

Also think about composition rule for limits. $(g\circ f)(x) = g(f(x))$

$$\lim_{x\to c} g(f(x)) = K$$

provided $\lim_{x\to c} f(x) = L$

$\lim_{x\to L} g(x) = K$, $g(L) = K$.