Chap 5 Integration

APEX: 5.1 — 5.4, won't cover 5.3

5.3 material comes in 2nd year Numerical Methods unit.

Introduction

Initially we think of integration in <u>two</u> ways, seemingly unconnected.

①. integration is "reverse differentiation" OR "anti-differentiation" a kind of inverse operation to differentiation.

eg. $$f(x) = x^2 \xrightarrow{\text{diff.}} f'(x) = 2x.$$

(indefinite) integration

②. integration as the operation of summing up the total accumulated value of a function across some interval giving the "area beneath

the curve" of a function.
"definite integration"

It turns out these two processes are intimately connected through the "Fundamental Theorem of Calculus"

The anti-derivative

Def An anti-derivative of a function $f(x)$ is any function $F(x)$ satisfying.

$$\frac{d}{dx}F(x) = f(x). \qquad \int$$

or $F'(x) = f(x)$

We can use the notation. $\int f(x)\,dx$ to denote such a function $F(x)$.

eg. x^2 is an anti-derivative for $2x$
$F(x)$ $f(x)$

But also

$x^2 + 5$ " " " " " $2x$

indeed for any fixed constant $c \in \mathbb{R}$.

$x^2 + c$ " " " " " $2x$.

Theorem 5.1.4

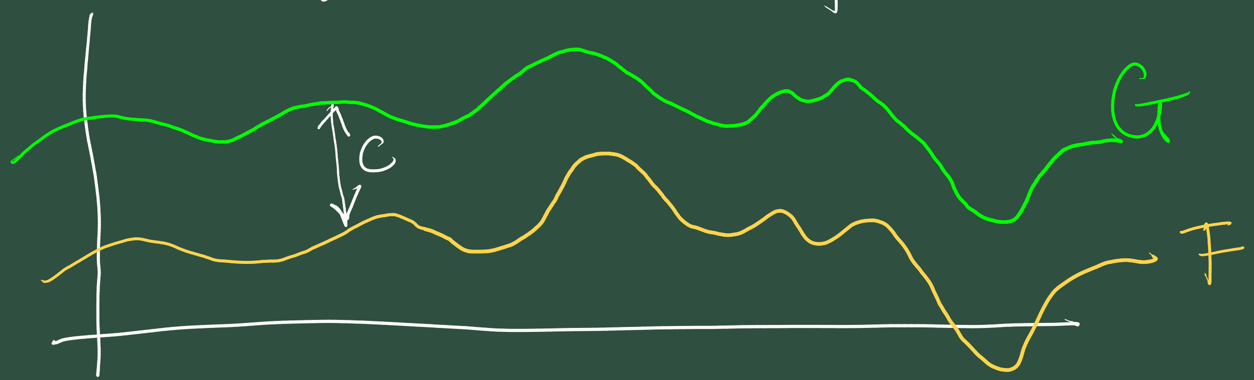
Any two anti-derivatives for $f(x)$ will differ by a constant.

So we generally capture this by saying

$$\int f(x)\,dx = F(x) + C, \text{ where } C \in \mathbb{R} \text{ is a constant.}$$

some particular anti-derivative of $f(x)$

Geometrically / graphically this is the statement if G, F are anti-derivatives for f.



In Theorem 5.1.8 we see lots of standard indefinite integrals and some basic properties of integration that come from what we know about differentiation

e.g. integration is linear

Lots more to say about antidifferentiation next week and in chap. 6.

The Definite Integral

1-dimensional motion.

Motivating example:

Consider an object travelling with constant velocity u, beginning at time t_1 and ending at time t_2, then the displacement of the object in this time interval is s, (total distance travelled) given by

velocity

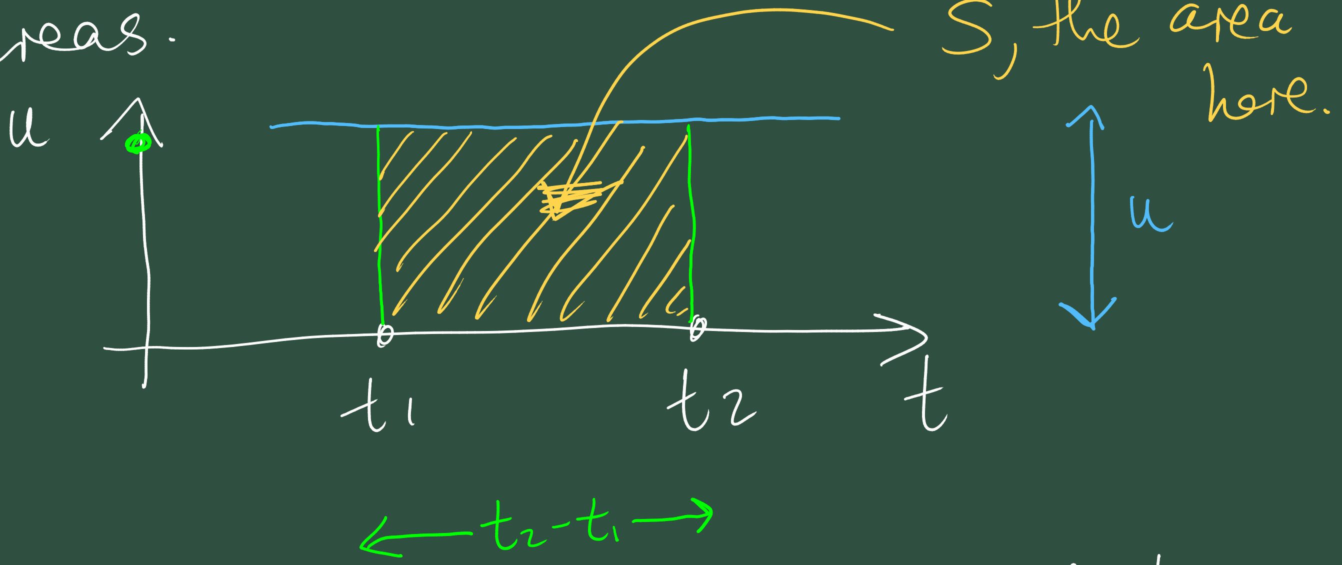
$$s = u(t_2 - t_1)$$

start.

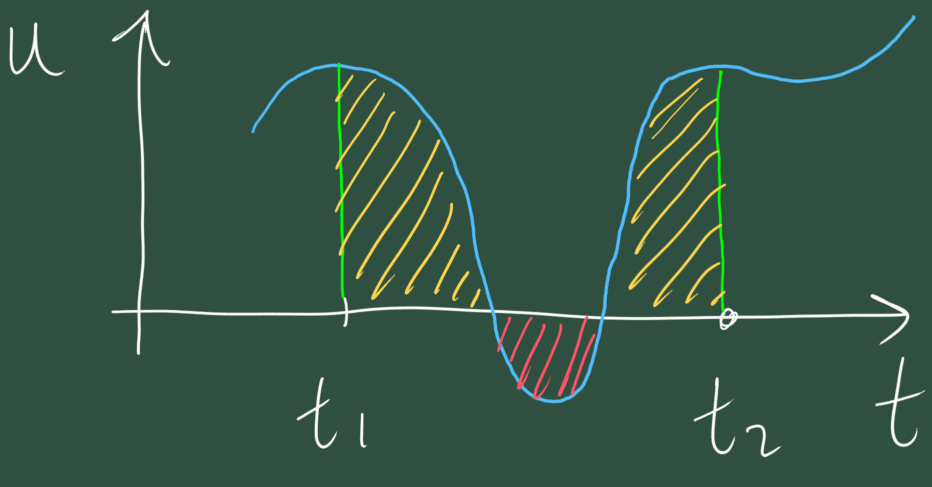
displacement.

end

Key idea: products like this can often be conceived of as 2-dimensional areas.



This concept can be extended to situations with changing velocity.



The displacement of the object will be the signed area between $u(t)$ between t_1 and t_2, with areas above the line $(u>0)$ counting positively.

and areas below the line ($u < 0$) counting negatively, towards the displacement

This is known as the definite integral of the velocity function. Def 5.2.6.

and written as.

$$\int_{t_1}^{t_2} u(t)\, dt$$

Q? Given a variable velocity function $u(t)$. How do we compute $u(t)$?

Sometimes we can exploit what we know about areas of rectangles, triangles, circles, ... But in general how might we do it?

From this definition of def. integral as area we can establish some basic properties in Th 5.2.11 .

Riemann sums / integration

is a general method for evaluating definite integrals.
We'll make good use of summation notation.

upper limit

$$\sum_{i=1}^{m} a_i = a_1 + a_2 + \dots + a_m$$

a_2

summand

lower limit.

summation index i

Σ is upper-case Greek sigma (σ is the lower case version).

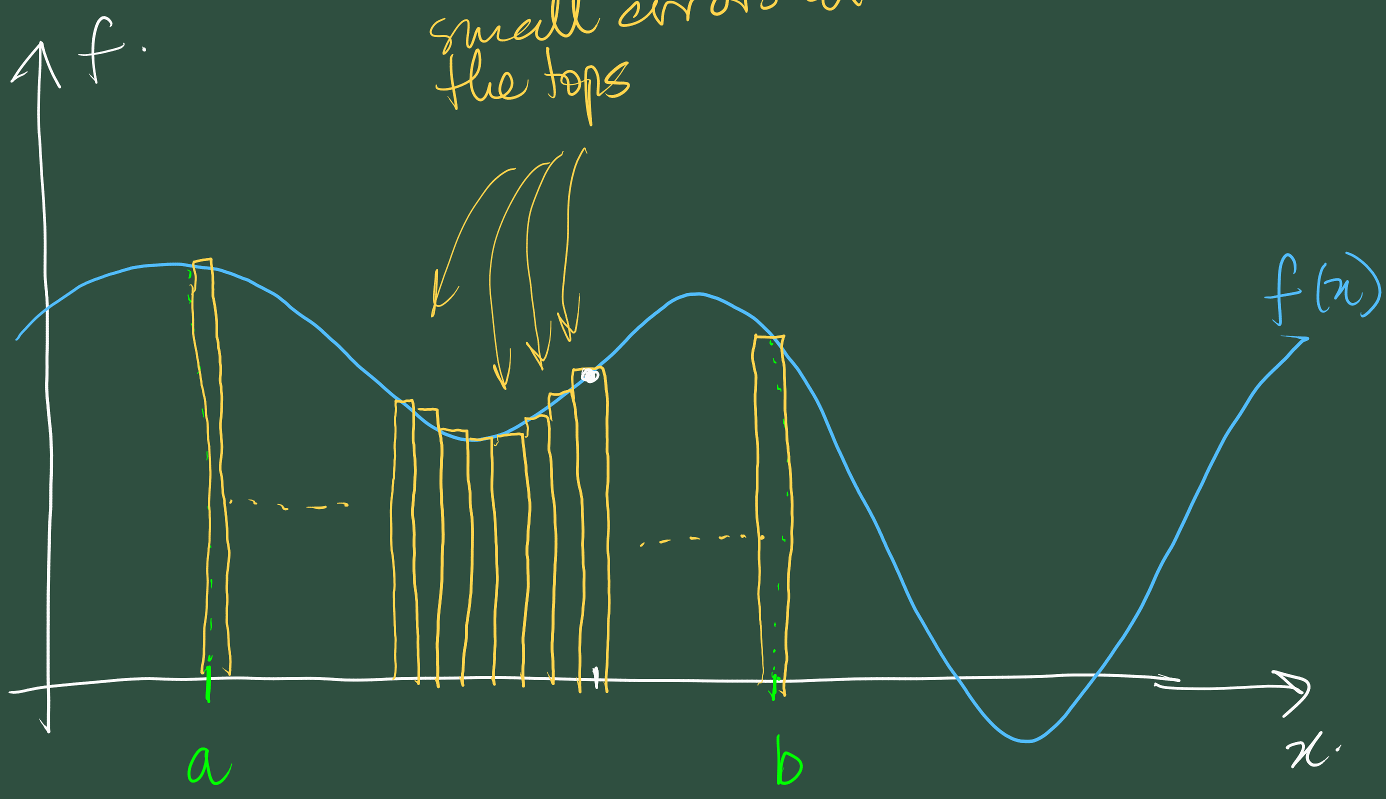
Similarly uppercase pi Π is used for products.

$$\prod_{i=1}^{m} a_i = a_1 a_2 \cdots a_m$$

Properties in Theorem 5.3.9.

1, 2, 3, 4 general properties.

5, 6, 7 Specific interesting sums



$\int_a^b f(x)\,dx$, key idea, approximate this area with lots of thin rectangles, and take a limit as their number $\to \infty$,

OR their width $\to 0$.

Use a partition/sub-division of $[a,b]$.

- $x_0 = a, x_1, x_2, x_3, \ldots, \ldots, x_{n-1}, x_n = b$

  where $x_i < x_{i+1}$

we write $\Delta x_{i-1} = x_i - x_{i-1}$, $\boxed{\text{width of } i^{th} \text{ rectangle}}$

$\Delta x_0 = x_1 - x_0, \ \Delta x_1 = x_2 - x_1, \ldots$

- for each i, we let x_i^* be a representative point in i^{th} interval.

  so $x_{i-1} \le x_i^* \le x_i$, for each $i = 1, \ldots, n$.

- $\Delta x = \max\limits_{i=1,\ldots,n} (\Delta x_i)$

Definition of the Riemann (definite) integral

$$\int_a^b f(x)\,dx = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \left(\sum_{i=1}^{n} \overbrace{\left(f(x_i^*) \Delta x_{i-1} \right)}^{\text{area of } i^{th} \text{ rectangle.}} \right)$$

height · width.

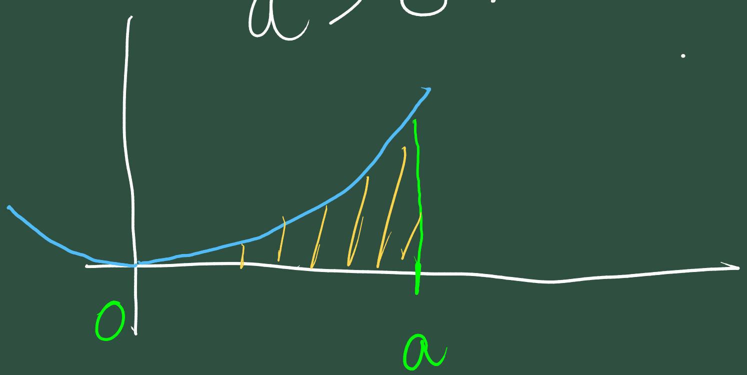
ambiguity over selection of x_i^*

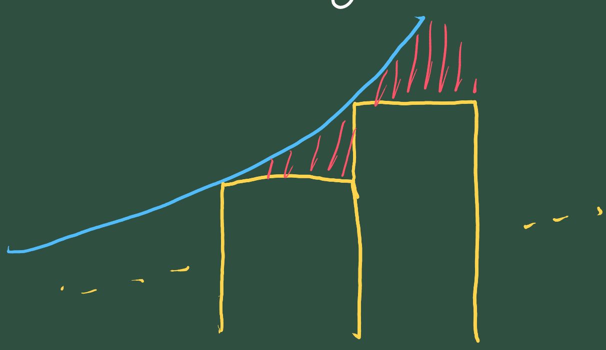
total area of the rectangles!

This def will applied both to evaluate certain integrals and to prove the F.T.C. which will allow us to use anti-differentiation to evaluate definite integrals.

<u>Eg.</u> Let's evaluate with the Riemann integral $I=\int_0^a x^2\,dx$.

$a>0$.





Use the equally spaced partition.

ie. $x_0 = 0$, $x_n = a$,

$$x_i = \frac{i}{n}a, \quad i = 0, \dots, n.$$

$$\Delta x_i = \frac{a}{n} = \Delta x$$

For the points x_i^*, use the left edges of the intervals. So $x_i^* = x_{i-1} = \frac{i-1}{n}a$

for each $i = 1, \dots, n$

Let's go!

$$\int_0^a x^2\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\Bigg(\overbrace{\frac{(i-1)a}{n}}^{x_i}\Bigg)^2 \overset{\Delta x_i}{\frac{a}{n}}\right)$$

$$= a^3 \lim_{n\to\infty}\left(\frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2\right)$$

, by linearity of lim, Σ

$$= a^3 \lim_{n\to\infty} \left(\frac{1}{n^3} \sum_{j=1}^{n-1} j^2 \right).$$

$$= a^3 \lim_{n\to\infty} \left(\frac{1}{n^3} \frac{(n-1)(n)(2n-1)}{6} \right)$$

, using known Σ formula for $\sum_{j=1}^{m} j^2$

$$= \frac{a^3}{6} \lim_{n\to\infty} \left(\frac{(n-1)(2n^2-n)}{n^3} \right)$$

$$= \frac{a^3}{6} \lim_{n\to\infty} \left(\frac{2n^3 - 3n^2 + n}{n^3} \right)$$

$$= \frac{a^3}{6} \lim_{n\to\infty} \left(2 - 3/n + 1/n^2 \right)$$

$$= \frac{a^3}{6} \left(2 - 0 + 0 \right)$$, by linearity of limits and $-3/n,\ 1/n^2 \to 0$

as $n \to \infty$

$$= \frac{a^3}{3}.$$

So $\displaystyle\int_0^a x^2\,dx = \frac{a^3}{3}.$

- A successful application of the Riemann integral.
- The $\frac{a^3}{3}$ reminds us of the antiderivative $\frac{x^3}{3}$ of x^2.
- This suggests some link between the definite integral and anti-differentiation.
- Again, note in the APEX example.

$$\int_{-1}^{5} x^3\,dx = 156.$$

Use the anti-derivative method.

$$\left[\frac{x^4}{4}\right]_{-1}^{5} = \frac{5^4}{4} - \frac{1}{4}$$

$$= \frac{625}{4} - \frac{1}{4}$$

$$= \frac{624}{4}$$

$$= 156$$

Both these examples point to Fundamental Theorem of Calculus. Which will prove.