Series tutorial sheet.

Q9.

Consider $$\sum_{m=0}^{\infty} \frac{2^m}{(m+1)^2}$$ — a_m

For the ratio test we examine the limit

$\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdots\cancel{2}\cdot 2$

$$\lim_{m\to\infty} \frac{a_{m+1}}{a_m}$$

$$= \lim_{m\to\infty}\left(\frac{2^{m+1}}{(m+2)^2}\cdot\frac{(m+1)^2}{2^m}\right)$$ — $\cancel{2}\cdot\cancel{2}\cdots\cancel{2}$

$$= 2\lim_{m\to\infty}\left(\frac{m^2+2m+1}{m^2+4m+4}\right)$$, cancelling the 2s and linearity

$$= 2\lim_{m\to\infty}\left(\frac{1+2/m+1/m^2}{1+4/m+4/m^2}\right)$$

$$= 2\,\frac{\lim_{m\to\infty}(1+2/m+1/m^2)}{\lim_{m\to\infty}(1+4/m+4/m^2)}$$, Quotient rule for limits

$= 2$, as $1/n, 1/n^2 \to 0$ as $n \to \infty$ and by using linearity of limits

Since this limit 2 is strictly greater than 1 the series $\sum_{n=0}^{\infty} a_n$ diverges by the ratio test.

$\boxed{Q12}$ Consider $\sum_{k=1}^{\infty} \frac{k+3}{k(k^2+2)}$

$$= \sum_{k=1}^{\infty} \frac{k+3}{k^3+2k}$$

From the dominant terms in the ratio we will compare this series to the series $\sum_{k=1}^{\infty} 1/k^2$

which is a known convergent series (hyper-harmonic series).

Use the direct comparison test. For $k>3$

$$\boxed{\frac{k+3}{k^3+2k}} < \frac{k+k}{k^3+2k}$$

$$= \frac{2k}{k^3+2k}.$$

$$< \frac{2k}{k^3}$$

$$\boxed{= 2 \cdot \frac{1}{k^2}.}$$

A since $\sum_{k=4}^{\infty} \frac{2}{k^2}$ is a known convergent series by the direct comparison test.

$$\sum_{k=4}^{\infty} \frac{k+3}{k^3+2k}$$ converges also.

Then so does the original series. $\sum_{k=1}^{\infty} \frac{k+3}{k^3+2k}$

converge, with the three extra terms at the beginning.

Q16. Let $d_n = \dfrac{11-n}{n^2+6n+5}$.

$$= \frac{\boxed{11-n}}{(n+5)(n+1)}$$

$$= \frac{\alpha}{n+5} + \frac{\beta}{n+1}, \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

$$= \frac{\alpha(n+1)+\beta(n+5)}{(n+5)(n+1)}$$

$$= \frac{\boxed{(\alpha+\beta)n + (5\beta+\alpha)}}{(n+5)(n+1)}$$

These numerators must be the same which are both polys in n.

$$\Rightarrow \begin{cases} \alpha+\beta = -1 \\ \alpha+5\beta = 11 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha = -1-\beta \\ -1-\beta+5\beta = 11 \end{cases}$$

$$\Rightarrow \quad 4\beta = 12 \Rightarrow \beta = 3$$

$$\Rightarrow \alpha = -4.$$

So $d_n = \dfrac{-4}{n+5} + \dfrac{3}{n+1}.$

$$= \frac{-1}{n+5} + \frac{-3}{n+5} + \frac{3}{n+1}$$

So examining the partial sums.

$$\sum_{n=1}^{k} d_n = \sum_{n=1}^{k}\left(\underbrace{-\frac{1}{n+5}}_{③} + \underbrace{\frac{-3}{n+5}}_{①} + \underbrace{\frac{3}{n+1}}_{②} \right)$$

The terms ① will cancel with the later (4 steps later) terms ②.
and then all terms ③ survive.

So

$$\sum_{n=1}^{k} d_n. = \left(\sum_{n=1}^{k} -\frac{1}{n+5}\right) + \frac{3}{2} + \frac{3}{3} + \frac{3}{4} + \frac{3}{5} - \frac{3}{k+2} - \frac{3}{k+3} - \frac{3}{k+4} - \frac{3}{k+5}.$$

By the def for sum of an infinite series.

$$\sum_{n=1}^{\infty} d_n = \lim_{k \to \infty} \sum_{n=1}^{k} d_n.$$

Note that $-\frac{3}{k+i} \to 0$ as $k \to \infty$ for $i = 2, 3, 4, 5$

$$\frac{3}{2} + \frac{3}{3} + \frac{3}{4} + \frac{3}{5} = C, \text{ some constant.}$$

But as $k \to \infty$, $\sum_{n=1}^{k} -\frac{1}{n+5}$

will diverge, as it is a harmonic series.

$$\sum_{n=1}^{\infty} -\frac{1}{n+5} = \sum_{m=6}^{\infty} -\frac{1}{m}$$

$$= -\sum_{m=6}^{\infty} 1/m$$

which is divergent, just like the whole harmonic series.

$$\sum_{m=1}^{\infty} \frac{1}{m}$$

So $\sum_{n=1}^{k} d_n$ diverges as $k \to \infty$

so $\sum_{n=1}^{\infty} d_n$ is divergent.

Note: Could also get this conclusion from a comparison test on $\sum_{n=1}^{\infty} d_n$ comparing to the divergent harmonic series.

$$\sum_{n=1}^{\infty} \frac{1}{n},$$