- Theorem 1.3.1 Quotient and composition cases

- Applications.

<u>Theorem 1.3.1.</u> Assume $f(x) \to L$, $g(x) \to K$ as $x \to c$.

<u>Notice</u> $\frac{f}{g} = f \cdot \left(\frac{1}{g}\right)$

$K \neq 0$

$$\boxed{\frac{1}{g(x)} \to \frac{1}{K} \text{ as } x \to c}$$

We only have to prove

$\boxed{\text{Let } \varepsilon > 0 \text{ be given.}}$

ie want to arrange that

$$\left| \frac{1}{g(x)} - \frac{1}{K} \right| < \varepsilon.$$

$|g(x) - K|$

$$\left| \frac{1}{g(x)} - \frac{1}{K} \right| = \left| \frac{K - g(x)}{g(x) K} \right|$$

$$= \frac{|g(x) - K|}{|g(x)||K|}$$

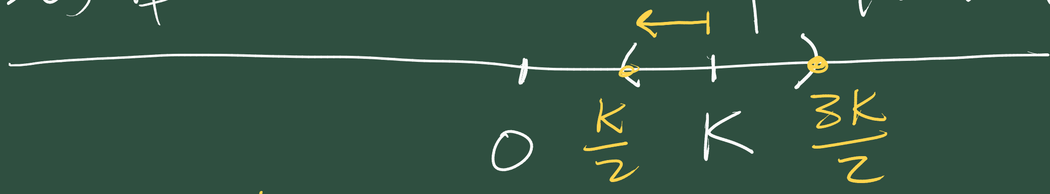
We need to "cap" or "bound" the effect

$\frac{1}{|g(x)|}$ factor.

$\frac{K}{2}$

(Assume $K>0$) If $K<0$ a v. similar proof will work.

$|K|=K$



For the radius $\frac{K}{2}$ $\exists\, \delta_1$ s.t.

$$|x-c|<\delta_1 \quad \boxed{|g(x)-K| < \frac{K}{2}}$$

since $g(x) \to K$ as $x \to c$.

Let $\varepsilon>0$ be given

$$\iff \frac{K}{2} < |g(x)| < \frac{3K}{2}$$

$$\iff \frac{2}{3K} < \frac{1}{|g(x)|} < \frac{2}{K}$$

So if $|x-c|<\delta_1$ ✓

$$\left|\frac{1}{g(x)} - \frac{1}{K}\right| = \frac{|g(x)-K|}{g(x) \quad K} < \frac{2}{K^2}|g(x)-K|$$

We also know $\exists\, \delta_2$ s.t.

$$|x-c|<\delta_2 \implies |g(x)-K| < \frac{K^2\varepsilon}{2}.$$

Set $\boxed{\delta = \min(\delta_1, \delta_2)}$

and if $\boxed{|x-c|<\delta}$ $\boxed{\text{then}}$

$$\boxed{\left|\frac{1}{g(x)} - \frac{1}{K}\right|} < \frac{2}{K^2}|g(x)-K| < \frac{2}{K^2}\frac{K^2\varepsilon}{2}$$

$$\boxed{= \varepsilon.}$$

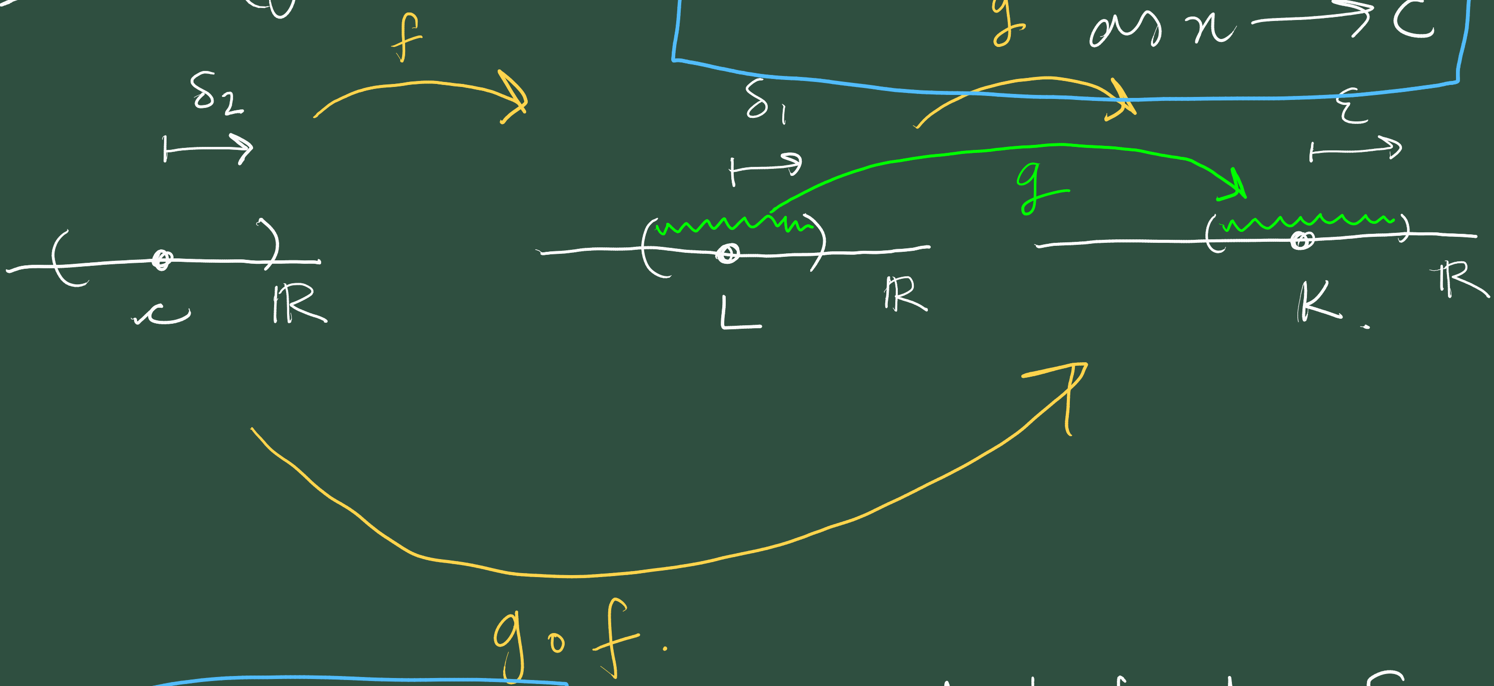
This is the ε-δ def for $\frac{1}{g(x)} \to \frac{1}{K}$

as $x \to c$.

Compound rule

Assume
$f(x) \to L$ as $x \to c$
$g(x) \to K$ as $x \to L$
$g(L) = K$

Prove $(g \circ f)(x) = \boxed{g(f(x)) \to K \text{ as } x \to c}$



$\boxed{\text{Let } \varepsilon > 0 \text{ be given.}}$ we need to find a $\delta > 0$ st $|x - c| < \delta$ then $|g(f(x)) - K| < \varepsilon$.

we know $\exists\ \delta_1 > 0$ st

If $|y - L| < \delta_1$ then $|g(y) - K| < \varepsilon$

we know $\boxed{\exists\, \delta_2 > 0}$ st.

if $\boxed{|x - c| < \delta_2}$ then $|f(x) - L| < \delta_1$

$$\boxed{\Longrightarrow |g(f(x)) - K| < \varepsilon}$$

This is the ε-δ proof for $g(f(x)) \to K$ as $x \to c$.

<u>Q11</u> Using comp. rule of Th. 1.3.1

$$\lim_{x \to 9} \underbrace{g(\underbrace{f(x)}_{\downarrow\, 6})}_{\downarrow\, 3} = ?$$

Can't decide.

Since $g(6) = 9$ not 3

9

<u>Q13</u>

$$\lim_{x \to 6} g(\underbrace{f(\overbrace{f(x)}^{9 \uparrow})}_{\downarrow\, 6}) =$$

3 Ooops!!!

we're missing

a condition.

we would need to also know

that $g(6) = 3$

but infact, $g(6) = 9$.

$$\boxed{\lim_{x \to 6} g(x) = 3 \ \& \ g(6) = 9}$$

i.e. g is not continuous at 6.