Series

Comparison Tests.
Ratio test.
Telescoping series.

9.3. Comparison Tests

The direct or limit comparisons work by comparing a given series to a series with known convergence/divergence.

Theorem 9.3.7 Direct test

Suppose $\{a_n\}_{n=1}^{\infty}$ $\{b_n\}_{n=1}^{\infty}$ are two series of positive terms and that.

for some N and

for all $n \geq N$ $\quad a_n \leq b_n$

1. If $\{b_n\}$ is a convergent series then so is $\{a_n\}$ series.

2. If $\sum_{n=1}^{\infty} a_n$ is divergent then so is $\sum_{n=1}^{\infty} b_n$.

Eg. Consider the series $\sum_{n=1}^{\infty} \frac{1}{3^n+n^2}$.

Note: $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series.

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent. hyper-harmonic series.

Then note that

$$\frac{1}{3^n+n^2} < \frac{1}{3^n}$$

for all $n \geq 1$, as $3^n+n^2 > 3^n$.

Therefore by the comparison test.

$\sum_{n=1}^{\infty} \frac{1}{3^n+n^2}$ will be convergent also.

Theorem 9.3.3. Limit Comparison Test.

~~Eg~~ $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are assumed to be series of positive terms.

If.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \quad , \text{with } 0 < L < \infty$$

then $\sum a_n$ converges if and only if $\sum b_n$ converges.

and so $\sum a_n$ diverges if and only if $\sum b_n$ diverges.

If. $\sum b_n$ converges and $L=0$.
then $\sum a_n$ will also converge.

If $\sum b_n$ diverges and $L=\infty$.
then $\sum a_n$ diverges also.

Reason

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

Then eventually

$$a_n \approx L\, b_n.$$

and so $\sum a_n \approx L \sum b_n$.

<u>Example 9.3.16</u>:

"a_n"

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2-n+1}$.

To choose the series to compare to, look at the dominant behaviour in this quotient as $n \to \infty$.

"b_n"

$$\sim \frac{\sqrt{n}}{n^2} = \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}}.$$

So we will compare to the series. $\displaystyle\sum^{\infty} \frac{1}{n^{3/2}}$ which is

$n=1$

a known convergent hyper-harmonic series

$\left(\sum_{m=1}^{\infty} \frac{1}{m^a} \text{ converges iff } a > 1 \right).$

Evaluate the ~~lim~~ quotient limit. a_n / b_n

$$\lim_{n\to\infty} \left(\frac{\sqrt{n} + 3}{n^2 - n + 1} \cdot n^{3/2} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n^2 + 3n^{3/2}}{n^2 - n + 1} \right).$$

$$= \lim_{n\to\infty} \left(\frac{1 + 3/n^{1/2}}{1 - 1/n + 1/n^2} \right)$$

$$= \frac{\lim_{n\to\infty}\left(1 + 3/n^{1/2}\right)}{\lim_{n\to\infty}\left(1 - 1/n + 1/n^2\right)}$$

$$= \frac{1}{1} = 1 \quad \text{as } \frac{1}{n^{1/2}}, \frac{1}{n}, \frac{1}{n^2}$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

$$\parallel$$

$$L$$

Therefore, by the limit comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2-n+1}$ also converges.

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2. Ratio Test.

If $\displaystyle\sum_{n=1}^{\infty} a_n$ is a series of positive terms and that $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

then.

1. If $0 \leq L < 1$ then $\sum_{n=1}^{\infty} a_n$ is a convergent series

2. If. $L > 1$ then $\sum_{n=1}^{\infty} a_n$ is a divergent series.

3. If $L = 1$ then the ratio test gives no conclusion.

Reason:

If ~~lim~~ $\frac{a_{n+1}}{a_n} \to L$ as $n \to \infty$ then eventually.

$$a_{n+1} \approx L a_n.$$

and so eventually $\sum a_n$ will act like a geometric series with constant ratio L.

and so will converge ($L<1$) or diverge ($L>1$).

in a similar way to the geometric series.

Example.

Consider. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n!}$ — a_n.

Examine.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{\cancel{2}}{(n+1)!}\cdot\frac{n!}{\cancel{2}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 = L$$

$$\left(\begin{aligned} n! &= n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\cdot 3\cdot 2\cdot 1 \\ (n+1)! &= (n+1)\,n\,(n-1)(n-2)\ldots 3\cdot 2\cdot 1\end{aligned}\right)$$

So by the ratio-test. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n!}$

is a convergent series. Since the limit of the ratio $\frac{a_{n+1}}{a_n}$ of consecutive terms is 0.

Example 9.4.3 (2). APEX.

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$.

ie. $a_n = 3^n/n^3$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{3^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{3n^3}{(n+1)^3}\right)$$

$$= 3\lim_{n\to\infty} \frac{n^3}{n^3+3n^2+3n+1}.$$

$$= 3 \cdot \lim_{n\to\infty} \frac{1}{1+3/n+3/n^2+1/n^3}$$

$= 3 \cdot 1 = 3$. as $1/n, 1/n^2, 1/n^3 \to 0$ as $n \to \infty$.

So we have.

$$\frac{a_{n+1}}{a_n} \to 3 \quad \text{as } n \to \infty.$$

and since $3 > 1$ by the ratio test $\sum_{n=1}^{\infty} a_n$ diverges.

Note: General Term Test also gives divergence here as.

$$\frac{3^n}{n^3} \not\to 0 \quad \text{as } n \to \infty.$$

in fact $3^n/n^3$ is unbounded as $n \to \infty$.

The inconclusive conclusion can be shown by the harmonic series.

<u>Recall:</u> $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

But the ratio-test cannot detect this as if $a_n = 1/n$, then.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{1}{n+1}\cdot\frac{n}{1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)$$

$$= 1,$$ as $1/n \to 0$ as $n \to \infty$.

So the ratio-test is silent on the harmonic series

<u>Telescoping series</u> (see 9.2. APEX).

Use partial fraction expansions to determine the exact sum of certain infinite series. This technique can work for certain series whose terms are rational functions of the summation index. (ratio of polys)

<u>Example</u> Q14 from tutorial sheet.

Determine $\sum_{n=1}^{\infty} b_n$

where $b_n = \dfrac{8}{n^2+6n+5}$

$$= \frac{8}{(n+5)(n+1)}.$$

(P.F. expansion) $= \dfrac{\alpha}{n+5} + \dfrac{\beta}{n+1}.$

$$= \frac{\alpha(n+1) + \beta(n+5)}{(n+5)(n+1)}$$

$$= \frac{(\alpha+\beta) n^2 + \alpha + 5\beta}{(n+5)(n+1)}.$$

$$\Rightarrow \quad \alpha+\beta = 0$$
$$\alpha + 5\beta = 8$$

$$\Rightarrow \quad \alpha = -\beta$$

$$4\beta = 8$$

$$\Rightarrow \beta = 2 \text{ and } \alpha = -2.$$

So $b_n = \dfrac{-2}{n+5} + \dfrac{2}{n+1}.$

Remember.

$$\sum_{n=1}^{\infty} b_n = \lim_{k\to\infty} \left(\sum_{n=1}^{k} b_n \right).$$

Consider the partial sums.

$$\sum_{n=1}^{k} b_n = \sum_{n=1}^{k} \left(\frac{-2}{n+5} + \frac{2}{n+1} \right)$$

$$= \left(\cancel{\frac{-2}{6}} + \frac{2}{2}\right) + \left(\cancel{\frac{-2}{7}} + \frac{2}{3}\right)$$

$$+ \left(\cancel{\frac{-2}{8}} + \frac{2}{4}\right) + \left(\cancel{\frac{-2}{9}} + \frac{2}{5}\right)$$

$$+ \left(\cancel{\frac{-2}{10}} + \cancel{\frac{2}{6}}\right) + \left(\cancel{\frac{-2}{11}} + \cancel{\frac{2}{7}}\right)$$

$$+ \cancel{\ldots\ldots}$$

$$+ \left(\frac{-2}{k+4} + \cancel{\frac{2}{k}}\right) + \left(\frac{-2}{k+5} + \cancel{\frac{2}{k+1}}\right)$$

Notice that almost every negative term $-\frac{2}{n+5}$ cancels with a later positive term $\frac{2}{n+1}$, which leaves.

$$\sum_{n=1}^{k} b_n = \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5}$$

$$- \frac{2}{k+2} - \frac{2}{k+3} - \frac{2}{k+4} - \frac{2}{k+5}$$

$$\longrightarrow \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5}$$

$$+ 0 + 0 + 0 + 0$$

as $\frac{1}{k+2}, \frac{1}{k+3}, \ldots, \frac{1}{k+5}$

$\longrightarrow 0$ as $k \to \infty$.

So $$\sum_{n=1}^{\infty} b_n = \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5}$$

$$= \frac{60 + 40 + 30 + 24}{60}$$

$$= 154/60 = 77/30$$

This behaviour is known as "telescoping behaviour".

Mock Exam

Q1.

(a). The derivative, $f'(a)$, of f at $x=a$ is the limit.

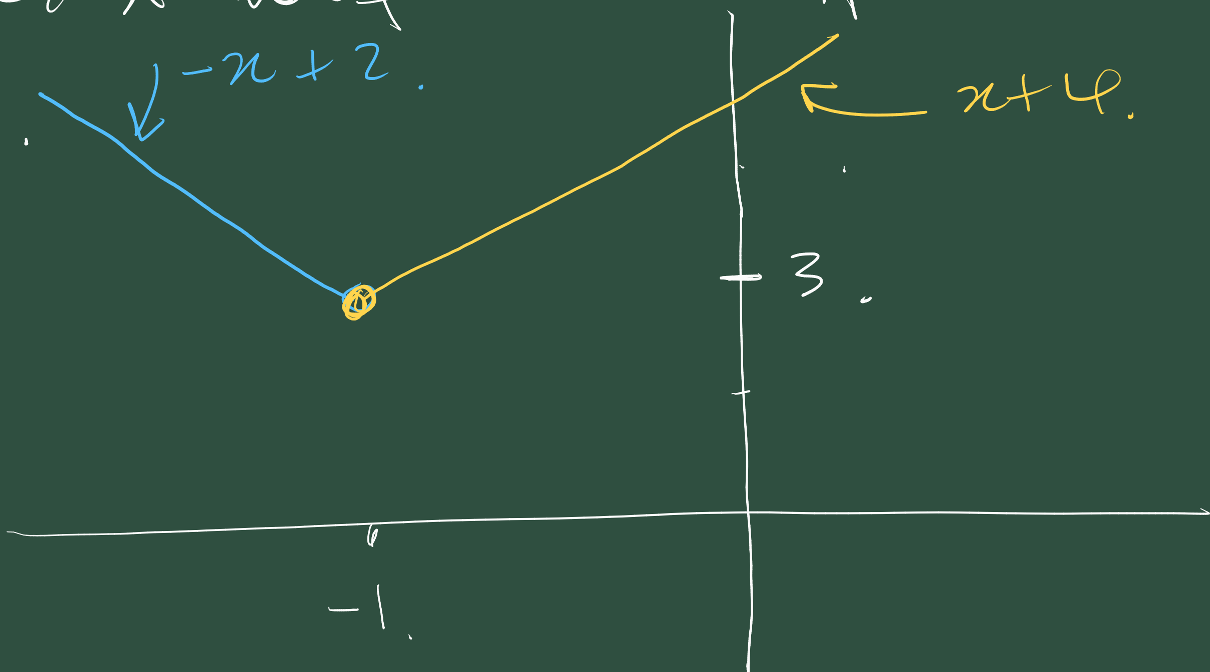
$$f'(a) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)$$

OR

$$= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

provided this limit exists. If it does not exist then f is not differentiable at a.

(b).



At every point except $x=-1$, f is a straight line and has derivative.

$$f'(a) = \begin{cases} -1, & \text{for } a < -1. \\ +1, & \text{for } a > -1 \\ \text{undefined}, & \text{for } a = -1 \end{cases}$$

f is not differentiable at $a=-1$.

for $\lim\limits_{x \to -1^-} \left(\dfrac{f(x)-f(a)}{x-a} \right) = -1$

and $\lim\limits_{x \to -1^+} \left(\dfrac{f(x)-f(a)}{x-a} \right) = +1$

So there is no well defined value for

$$\lim_{x \to -1} \frac{f(x)-f(a)}{x-a}.$$

(f).

Product rule

If f, g are differentiable functions.

then.
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Let a be the function defined by
$$a(x) = x.$$

with derivative.
$$a'(x) = 1.$$

Consider the square function s defined by $s(x) = x^2$

Note $s = a \cdot a$

So using the product rule we would get.
$$\begin{aligned} s'(x) &= (a \cdot a)'(x) \\ &= a'(x)a(x) + a(x)a'(x) \\ &= 1 \cdot x + x \cdot 1 \end{aligned}$$

$= 2x.$

So $S'(x) = 2x$, as expected.

$\boxed{Q2}$ F.T.C. can be given as.

1. If $F(x) = \int_0^x f(t)\,dt.$

then $F'(x) = f(x).$

2. If. $\frac{d}{dx} F(x) = f(x)$, i.e. F is an anti-derivative of f.

then. $$\int_a^b f(x)\,dx = F(b) - F(a).$$

For $F(x) = \int_0^x \left(t^3 + \sin(t)\right) dt$

$F'(\pi) = \pi^3 + \sin(\pi)$, by F.T.C. 1.

$= \pi^3.$

(b). We say x_n converges to a limit L as n tends to infinity, written as $x_n \to L$ as $n \to \infty$,

to mean:

for all $\varepsilon > 0$ there exists a natural number N such that for all $n \geq N$ we have

$$|x_n - L| < \varepsilon.$$

(c). Let $\varepsilon > 0$ be given.

We need to show that there exists a natural N such that

$$n \geq N \Rightarrow |a_n b_n - ab| < \varepsilon.$$

Consider this inequality.

$$|a_n b_n - ab|$$

$$= |(a_n - a)(b_n - b) - 2ab + ab_n + ba_n|$$

$$= |(a_n - a)(b_n - b) + b(a_n - a) + a(b_n - b)|$$

$$\leq |a_n - a||b_n - b| + |b||a_n - a| + |a||b_n - b|$$

, using the $\triangle$ inequality and $|uv| = |u||v|$

$$\underbrace{\qquad\qquad}_{< \varepsilon/3} \qquad \underbrace{\qquad\qquad}_{< \varepsilon/3} \qquad \underbrace{\qquad\qquad}_{< \varepsilon/3}$$

Since $a_n \to a$ and $b_n \to b$ ~~we~~ know there exist natural numbers N_1, N_2 such that,

$$n \geq N_1 \Rightarrow |a_n - a| < \min\left(\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3|b|}\right)$$

$$n \geq N_2 \Rightarrow |b_n - b| < \min\left(\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3|a|}\right)$$

So now let $\boxed{N = \max(N_1, N_2)}$

so for all $\boxed{n \geq N}$ we will have.

$$\boxed{|a_n b_n - ab|} < \sqrt{\frac{\varepsilon}{3}}\sqrt{\frac{\varepsilon}{3}} + \cancel{|b|}\frac{\varepsilon}{3\cancel{|b|}} + \cancel{|a|}\frac{\varepsilon}{3\cancel{|a|}}$$

$$= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

In the paragraph we have shown the definition of convergence for

$a_n b_n \to ab$ as $n \to \infty$.