Manchester Metropolitan University

6G4Z3006 Calculus Mock Exam 01 Solutions

<u>Instructions to students</u>

- Answer **both** questions from **Section A** and **two** questions from **Section B**.
- You must show all of your working and explain your reasoning carefully to gain full marks.
- Marks awarded for each question part are shown in square brackets aligned to the right-hand margin.

<u>Permitted materials</u>

- Students are permitted to use their own calculators without mobile communication facilities.

Section A – answer both questions

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Use the concept of limiting value of a function to give the definition of the *derivative* of f at the point $a \in \mathbb{R}$. [3]

Solution: The function f is differentiable at a and has derivative $f'(a)$ if the following limiting value exits,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by [3]

$$f(x) = \begin{cases} -x + 2 & \text{when } x < -1, \\ x + 4 & \text{when } x \geq -1. \end{cases}$$

The function f is differentiable everywhere in its domain except for at one point. Explain briefly which point this is and why it is not differentiable there.

Solution: The function f is differentiable everywhere except at $a = -1$. This is because at $a = 0$ the value of $\frac{f(x)-f(a)}{x-a}$ for negative and positive x is -1 and $+1$ respectively. So there is no limit for this as $x \to a$. At other values of a only one of the cases of the definition of f plays a role in the limit as $x \to a$ so there is a limiting value.

(c) State and prove the *product rule* of differentiation. You can use without proof any relevant properties of the limiting values of functions, providing you clearly state these in your proof. [8]

Solution: Consider the derivative $(fg)'(a)$ of the product of functions f and g. The main steps are

$$\begin{aligned}
(fg)'(a) &= \lim_{x \to a} \left(\frac{f(x)g(x) - f(a)g(a)}{x - a} \right) \\
&= \lim_{x \to a} \left(\frac{g(x)\big(f(x) - f(a)\big) + f(a)\big(g(x) - g(a)\big)}{x - a} \right), \text{prep.} \\
&= \lim_{x \to a} \left(\frac{g(x)\big(f(x) - f(a)\big)}{x - a} \right) + f(a) \lim_{x \to a} \left(\frac{g(x) - g(a)}{x - a} \right), \text{linearity of lim} \\
&= g(a) \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right) + f(a) \lim_{x \to a} \left(\frac{g(x) - g(a)}{x - a} \right), \text{multiplicative prop. of lim} \\
&= g(a)f'(a) + f(a)g'(a), \text{def. of derivs}
\end{aligned}$$

(d) State the *chain rule* of differentiation. [3]

Solution: For functions f and g, differentiable at a and $f(a)$ respectively, the derivative of the composition $g \circ f$ is given by

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

(e) State the *quotient rule* of differentiation. [3]

Solution: For functions a and v, differentiable at a and v non-zero at and about a, the derivative of the quotient function u/v is given by

$$(u/v)'(a) = \frac{v(a)u'(a) - u(a)v'(a)}{v^2(a)}.$$

(f) Show how to use the product rule and the known derivative $a'(x) = 1$, for the function a defined by $a(x) = x$, to derive the derivative for the square function s, defined by $s(x) = x^2$. [5]

Solution: Note that we can express s as the product function $s = a \cdot a$. The main steps in obtaining s' are:

$$\begin{aligned}(s)'(x) &= (a \cdot a)'(x) \\ &= a(x)a'(x) + a'(x)a(x), \text{ by the prod. rule} \\ &= x \cdot 1 + 1 \cdot x, \text{ known formulas for } a \text{ and } a' \\ &= 2x.\end{aligned}$$

Section A – answer both questions

2. (a) State both parts of the *Fundamental Theorem of Calculus*. Suppose the function $F(x)$ is defined by [8]

$$F(x) = \int_0^x (t^3 + \sin(t))dt.$$

Use the Fundamental Theorem of Calculus to find $F'(\pi)$, i.e. the derivative of F at $x = \pi$.

> **Solution: First part:** If f is a continuous integrable function and F is defined by $F(x) = \int_a^x f(t)\,dt$ then F is differentiable and $F'(x) = f(x)$.
>
> **Second part:** If f is continuous and integrable and F is an antiderivative of f, i.e. $F'(x) = f(x)$, then
>
> $$\int_a^b f(x)\,dx = F(b) - F(a).$$
>
> For the function F in the question we will get
>
> $$F'(\pi) = (\pi)^3 + \sin(\pi) = \pi^3.$$

(b) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Give the (ϵ, N)-definition for the convergence of the sequence $\{x_n\}_{n=1}^{\infty}$. [3]

> **Solution:** The sequence $\{x_n\}$ converges to the limit L if and only if given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n - L| < \epsilon$.

(c) Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of real numbers, and that $\alpha, \beta \in \mathbb{R}$. Use the definition from the previous part to prove that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two convergent sequences with $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then the product sequence $\{a_nb_n\}_{n=1}^{\infty}$ is also convergent and [6]

$$a_nb_n \to ab, \text{ as } n \to \infty.$$

> **Solution:** Main steps in the argument are:
>
> - Choose $N_1, N_2 \in \mathbb{N}$ so that for all $n \geq N_1$ we have $|a_n - a| < \min(\sqrt{\epsilon/3}, \epsilon/(3|b|))$ and for all $n \geq N_2$ we have $|b_n - b| < \min(\sqrt{\epsilon/3}, \epsilon/(3|a|))$

- Set $N = \max(N_1, N_2)$. Then for all $n \geq N$ we have

$$\begin{aligned}
|a_n b_n - ab| &= |(a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a))|, \text{prep.} \\
&\leq |a_n - a||b_n - b| + |a||b_n - b| + |b||a_n - a|, \text{triang. ineq.} \\
&< \sqrt{\epsilon/3}\sqrt{\epsilon/3} + |a|\epsilon/(3|a|) + |b|\epsilon/(3|b|) \\
&= \epsilon
\end{aligned}$$

- Hence $a_n b_n \to ab,$ as $n \to \infty$.

(d) Use concepts of sequence convergence and partial sums to give the definition for the convergence of an infinite series $\sum_{n=1}^{\infty} a_n$. [3]

Solution: The partial sum, S_k, of the series is defined as

$$S_k = \sum_{n=1}^{k} a_n.$$

The series is said to converge to a sum L if and only if $S_k \to L$ as $k \to \infty$.

(e) Use part (d) and relevant properties of sequence convergence to prove that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two convergent series of real numbers with [5]

$$\sum_{n=1}^{\infty} a_n = a \text{ and } \sum_{n=1}^{\infty} b_n = b,$$

then the sum series $\sum_{n=1}^{\infty}(a_n + b_n)$ is convergent also and

$$\sum_{n=1}^{\infty}(a_n + b_n) = a + b.$$

Solution: This involves working on the partial sum S_k of the new series and using the linear combinations rule of sequence converge to prove convergence. The main steps are

$$\begin{aligned}
S_k &= \sum_{n=1}^{k}(a_n + b_n) \\
&= \left(\sum_{n=1}^{k} a_n\right) + \left(\sum_{n=1}^{k} b_n\right), \text{manip. of fin. sum} \\
&\to a + b, \text{ as } n \to \infty,
\end{aligned}$$

where the last step follows from the linear combinations property of sequence convergence.

End of Section A

Section B – answer any two questions

3. (a) Use the limit definition of the derivative to obtain the derivative of the function f defined by the formula [5]

$$f(x) = x^3.$$

Point out clearly where you make use of any properties of the limiting values of functions.

Solution: Implement the limit definition and evaluate as follows, for $a \in \mathbb{R}$,

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h} \\ &= \lim_{h\to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h} \\ &= \lim_{h\to 0} (3a^2 + 3ah + h^2) \\ &= 3a^2, \end{aligned}$$

where the last step is justified by multiple applications of the linearity form of the algebra of limits theorem for limiting values of functions.

(b) Consider the function g defined by the formula [6]

$$g(x) = x^4 \sin(3x).$$

Use the linearity, product and chain rules to obtain the first and second order derivatives, $g'(x)$ and $g''(x)$, of g. Point out clearly where you are using each of the rules.

Solution: Using the product rule and chain rules we get

$$g'(x) = 4x^3 \sin(3x) + 3x^4 \cos(3x).$$

Then using product and chain rules again we get, before simplification,

$$g''(x) = 12x^2 \sin(3x) + 12x^3 \cos(3x) + 12x^3 \cos(3x) - 9x^4 \sin(3x)$$

which tidies up to

$$g''(x) = 24x^3 \cos(3x) + (12x^2 - 9x^4) \sin(3x).$$

(c) The strength, S, of a wooden beam is directly proportional to its cross-sectional width w and the square of its height h, that is, $S = kwh^2$, for some constant k. Given a circular log of wood with a diameter of 12 inches, what sized beam can be cut from the log, so that it has maximum strength? [10]

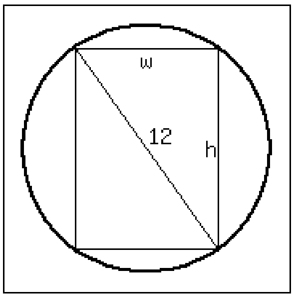

Figure: *The cross-section of the wooden beam of dimensions w and h that could be cut from the circular log.*

Solution: The strength of the beam increases with both w and h so its strength will be maximized when the beam *fills out* the log, i.e. when w and h are constrained by the relation

$$w^2 + h^2 = 144.$$

So we must find the maximum value of $S(w)$, given by,

$$S(w) = kwh^2 = kw(144 - w^2) = k(144w - w^3).$$

The stationary point(s) of S are given by the solutions of

$$\frac{dS}{dw} = k(144 - 3w^2) = 0,$$

which is equivalent to

$$w = \pm\sqrt{48} = \pm 4\sqrt{3}.$$

The width w must be positive, so there is one stationary point at $w = 4\sqrt{3}$. The second derivative $\frac{d^2S}{dw^2}$ is indeed negative so this is a local maximum of S. The *end point* beams of $w = 0$ or $h = 0$ have zero strength, so the maximum possible strength of the beam is achieved when $w = 4\sqrt{3}$.

(d) The *linear approximation with error* version of the definition of the derivative says that $f'(a)$ is the unique quantity that provides a linear approximation $f(a) + hf'(a)$ to the value of $f(a+h)$, so that the associated error function $\epsilon(h)$ that satisfies [4]

$$f(a+h) = f(a) + hf'(a) + \epsilon(h),$$

has the property that $\epsilon(h)/h \to 0$ as $h \to 0$.

Prove that if a different quantity $s \neq f'(a)$, is used here instead, i.e.

$$f(a+h) = f(a) + hs + \lambda(h),$$

then the error function $\lambda(h)$ will not have the property that $\lambda(h)/h \to 0$ as $h \to 0$.

Solution: Let us write s as

$$s = f'(a) + k,$$

for some $k \neq 0$. Then we can express the error function λ as

$$\begin{aligned}\lambda(h) &= f(a+h) - f(a) - sh, \\ &= f(a+h) - f(a) - f'(a)h - kh, \\ &= \epsilon(h) - kh.\end{aligned}$$

Then from this we can say $\lambda(h)/h = \epsilon(h)/h - k \to -k$ as $h \to 0$. Therefore, $\lambda(h)/h$ does not converge to 0 as $h \to 0$, as required.

Section B – answer any two questions

4. (a) Evaluate the following integrals. You can use various integration techniques and/or trigonometric formula but you must explain what you are doing and clearly reference any standard integrals you make use of.

(i) Show how integration by parts can be applied to evaluate the integral [6]

$$I_1 = \int_{-3\pi/2}^{3\pi/2} e^{2x}\cos(x)\,dx.$$

Solution: This requires integration by parts twice,

$$\begin{aligned} I_1 &= \left[e^{2x}\sin(x)\right]_{-3\pi/2}^{3\pi/2} - 2\int_{-3\pi/2}^{3\pi/2} e^{2x}\sin x\,dx, \\ &= (-e^{-3\pi} - e^{3\pi}) + 2\left[e^{2x}\cos(x)\right]_{-3\pi/2}^{3\pi/2} - 4\int_{-3\pi/2}^{3\pi/2} e^{2x}\cos(x), \\ &= (-e^{-3\pi} - e^{3\pi}) - 4I_1, \end{aligned}$$

which implies that

$$I_1 = \frac{1}{5}(-e^{3\pi} - e^{-3\pi}).$$

(ii) Use the substitution $u = x + 8$ to evaluate the integral [6]

$$I_2 = \int \frac{x^2 - 7x - 1}{x + 8}\,dx.$$

Solution: By substitution, using $u = x + 8$, i.e. $x = u - 8$, we have $dx = du$ and the integral becomes.

$$\begin{aligned} I_2 &= \int \frac{(u-8)^2 - 7(u-8) - 1}{u}\,du \\ &= \int \frac{u^2 - 23\,u + 119}{u}\,du \\ &= \int u - 23 + 119/u\,du \\ &= \frac{u^2}{2} - 23u + 119\ln(|u|) + C, \\ &= \frac{(x+8)^2}{2} - 23(x+8) + 119\ln(|x+8|) + C, \end{aligned}$$

for some constant C.

(b) The Riemann integral of a function $f(x)$ between the limits $x = a$ and $x = b$ is defined as [13]

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta x\to 0}} \left(\sum_{i=1}^{n} f(x_i^*)\Delta x_{i-1}, \right),$$

where the x_i $(0 \leq i \leq n)$ form a subdivision of the interval $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

$\Delta x_{i-1} = x_i - x_{i-1}$ is the width of the i^{th} subinterval,
$\Delta x = \max(\{\Delta x_i : 0 \leq i \leq n-1\})$ and x_i^* is a representative point from the i^{th} subinterval, i.e. $x_{i-1} \leq x_i^* \leq x_i$.

In this question you will evaluate the Riemann integral for the function $f(x) = x^3$ and the limits $a = 0$ and $b = 2$. You should use the equally spaced interval

$$x_i = i\frac{2}{n}, \quad (i = 0, \ldots, n),$$

and the start point of each subinterval as the representative point, i.e.

$$x_i^* = (i-1)\frac{2}{n}, \quad (i = 1, \ldots n).$$

Question 4 continues on the next page.

Set up and evaluate the limit in the Riemann integral, using known summation formulae, to show that

$$\int_0^2 x^3\,dx = 4.$$

You can use, without proof, any of the following summation formulae

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Solution: Applying the Riemann integration formula to the given set up results in

$$I = \int_0^2 x^3\,dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left((i-1)\frac{2}{n}\right)^3 \frac{2}{n}\right),$$

$$\begin{aligned}
I &= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{16}{n^4}\left((i-1)^3\right)\right),\\
&= 16\lim_{n\to\infty}\left(\frac{1}{n^4}\sum_{i=1}^{n-1} i^3\right),\\
&= 16\lim_{n\to\infty}\left(\frac{1}{n^4}\frac{(n-1)^2(n^2)}{4}\right),\\
&= 16\lim_{n\to\infty}\left(\frac{1}{n^4}\frac{n^4-2n^3+n^2}{4}\right),\\
&= 4\lim_{n\to\infty}\left(1-2/n+1/n^2\right),\\
&= 4
\end{aligned}$$

as required.

Section B – answer any two questions

5. (a) Let the sequence $\{b_m\}_{m=2}^{\infty}$ be defined by the formula [5]

$$b_m = \frac{2m+1}{m}.$$

By considering the ratio or difference of its consecutive terms, prove that this sequence is increasing for all $m \geq 2$.

Solution: Examination the ratio and comparing it to 1 we get

$$\begin{aligned}\frac{b_{m+1}}{b_m} &= \frac{2m+3}{m+1}\frac{m}{2m+1}\\ &= \frac{2m^2+3m}{2m^2+3m+1}\\ &< 1, \text{ for all } m \geq 2,\end{aligned}$$

since the numerator is less than the denominator. Therefore, $b_{m+1} < b_m$ for all $m \geq 2$ and the sequence is decreasing.

(b) Let the sequence $\{c_m\}_{m=1}^{\infty}$ be defined by the formula [7]

$$c_m = \frac{7m^3 - 5m^2 + 1}{4m^3 + 7}.$$

Show how the algebra of limits theorem for convergent sequences can be used to prove that $\{c_m\}$ is a convergent sequence and to determine its limit. Make it clear in your answer where you are using each part of the theorem that you rely on.

Solution:

$$\begin{aligned}\lim_{m\to\infty} c_m &= \lim_{m\to\infty} \frac{7m^3 - 5m^2 + 1}{4m^3 + 7}\\ &= \lim_{m\to\infty} \frac{7 - 5/m + 1/m^3}{4 + 7/m^3}\\ &= \frac{7}{4},\end{aligned}$$

where the last step uses the quotient and linear combinations form of the algebra of limits theorem and the fact that $1/m^3, 1/m \to 0$ as $m \to \infty$.

(c) Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_0 = 1$ and the recurrence relation

$$a_n = 3 - \frac{1}{a_{n-1}},$$

for $n \geq 1$.

(i) Use the recurrence relation above to prove, by induction, that for all $n \geq 0$ the sequence elements satisfy $a_n \geq 1$. Then explain why the sequence elements satisfy $a_n < 3$, for all $n \geq 0$. [6]

Solution: Base case is true as $a_0 = 1$. Assume that $a_n \geq 1$. Then $0 < 1/a_n \leq 1$ and so $a_{n+1} = 3 - 1/a_n \geq 2 > 1$. Hence, by induction, for all $n \geq 1$, $a_n \geq 1$.

Knowing this we can say that $a_n < 3$ as it is defined as 3 minus a positive number.

(ii) Use the recurrence relation to prove that the sequence $\{a_n\}_{n=0}^{\infty}$ is an increasing sequence. [2]

Solution: Using proof by induction. The base case of $a_2 > a_1$ is true as $a_1 = 1$ and $a_2 = 2$. Then if $a_{n+1} > a_n$ we get

$$\begin{aligned} a_{n+1} > a_n &\Rightarrow \frac{1}{a_{n+1}} < \frac{1}{a_n}, \\ &\Rightarrow -\frac{1}{a_{n+1}} > -\frac{1}{a_n}, \\ &\Rightarrow 3 - \frac{1}{a_{n+1}} > 3 - \frac{1}{a_n}, \\ &\Rightarrow a_{n+2} > a_{n+1}, \end{aligned}$$

and so, by induction, the sequence is increasing for all n.

(iii) Briefly explain how the Monotone Convergence Theorem can be used to prove that $\{a_n\}_{n=0}^{\infty}$ is a convergent sequence. [2]

Solution: From parts (i) and (ii), the sequence is increasing and bounded above by 3. Therefore, by the Monotone Convergence Theorem, it is convergent (and has a limit ≤ 3).

(iv) Finally, use the algebra of limits theorem, together with the bounds from part (i),to prove that $a_n \to \sqrt{5}/2 + 3/2$ as $n \to \infty$. [3]

Solution: Let a denote the limit, i.e. $a_n \to a$ as $n \to \infty$. Taking the limit of both sides of the recurrence relation and applying the algebra of limits theorem, gives us

$$a = 3 - 1/a,$$

which is equivalent to

$$a^2 - 3a + 1 = 0,$$

so $a = \frac{3 \pm \sqrt{5}}{2}$. The smaller root is not possible since we know from before that $a \geq 1$. Hence the limit of the sequence is $a = \frac{3+\sqrt{5}}{2}$

Section B – answer any two questions

6. (a) Consider the infinite series [8]

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^{n+2}}.$$

Use the result about the convergence of geometric series to prove that the series defined above is convergent and determine its sum. You will also need to make use of the theorem about linear combinations of convergent series.

> **Solution:** Split the series up into two geometric series as follows and appeal to the geometric series result and theorem 2.2
>
> $$\begin{aligned}\sum_{n=1}^{\infty} \frac{1+2^n}{3^{n+2}} &= \sum_{n=1}^{\infty} \frac{1}{3^3}\left(\frac{1}{3}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{2}{3^3}\left(\frac{2}{3}\right)^{n-1}, \\ &= \frac{1/3^3}{1-1/3} + \frac{2/3^3}{1-2/3}, \\ &= \frac{5}{18}.\end{aligned}$$

(b) (i) Use a suitable comparison test to prove that the series [5]

$$\sum_{m=1}^{\infty} \frac{3n}{7+n^3},$$

is a convergent series. You can make use of the known convergence of any hyper-harmonic series.

> **Solution:** We compare to the convergent series $\sum 1/n^2$, using the limit comparison test. The limit to examine is
>
> $$\begin{aligned}\lim_{n\to\infty}\left(\frac{3n}{7+n^3}\cdot n^2\right) &= \lim_{n\to\infty}\frac{3n^3}{7+n^3} \\ &= \lim_{n\to\infty}\frac{3}{7/n^3+1} \\ &= 1, \text{ since } 7/n^3 \to 0 \text{ as } n\to\infty.\end{aligned}$$
>
> Therefore by the limit comparison test and the fact that $\sum 1/n^2$ is a convergent hyper-harmonic series, the original series converges also.

(ii) Use the ratio test to prove that the series [5]

$$\sum_{m=1}^{\infty} \frac{2^m}{(m+5)!},$$

is a convergent series. *Note that* ! *denotes the factorial operation.*

Solution: Taking the limit of the ratio of consecutive terms $x_m = \frac{2^m}{(m+5)!}$ we see

$$\begin{aligned}\lim_{m\to\infty} \frac{x_{m+1}}{x_m} &= \lim_{m\to\infty} \left(\frac{2^{m+1}}{(m+6)!} \frac{(m+5)!}{2^m} \right) \\ &= \lim_{m\to\infty} \left(\frac{2}{m+6} \right) \\ &= 0, \text{ by algebra of limits theorem.}\end{aligned}$$

Since the limit is 0, the original series converges by the ratio test as the limit of the ratio of consecutive terms has absolute value less than 1.

(c) Obtain the partial fraction expansion of [7]

$$\frac{2}{m^2+2m},$$

and use it to obtain the sum of the infinite series

$$\sum_{m=1}^{\infty} \frac{2}{m^2+2m},$$

by considering cancellation effects in the partial sums of this series.

Solution: The required partial fraction expansion is

$$\frac{2}{m^2+m} = \frac{2}{m} - \frac{2}{m+1},$$

so in the partial sum we get cancellation of all but one term at the beginning and one at the end,

$$\begin{aligned}S_k &= \sum_{m=1}^{k} \frac{2}{m^2+m}, \\ &= \sum_{m=1}^{k} \left(\frac{2}{m} - \frac{2}{m+1} \right), \\ &= \frac{2}{1} - \frac{2}{k+1}.\end{aligned}$$

So

$$\sum_{m=1}^{\infty} \frac{2}{m^2+m} = \lim_{k\to\infty} S_k = 2,$$

by applying the algebra of limits theorem to the limit of S_k.

End of Section B

End OF QUESTIONS