Review

Introduced two things called integration.

1. Anti-differentiation.

Given a function $f(x)$, a function $F(x)$ is called an anti-derivative or indefinite integral of $f(x)$, if.

$$F'(x) = f(x)$$

We can write.

$$\int f(x)\, dx$$

for the set of indefinite integrals of f.

2. Definite integrals, which measure the area between the graph of a function $f(x)$ and the x-axis over an interval (a, b), written as

$$\int_a^b f(x)\, dx.$$

We described the Riemann sum approach to formally define and provide a way to compute such definite integrals.

These two concepts seemingly unrelated, but in fact are.

Fundamental Theorem of Calculus

Normally presented as two parts.

(1). For a function f, the definite integral.

$$F(x) = \int_a^x f(t)\,dt$$

is an anti-derivative of f.

i.e. $F'(x) = f(x)$

or.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

② For a function $f(x)$, with anti-derivative $F(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof of ②

Use the Riemann sum approach for developing the definite integral.

Let f be a function and consider

$\boxed{\int_a^b f'(x)\,dx}$, we'll show this is equal to $\boxed{f(b) - f(a)}$.

(Here we are using f as the anti-derivative for f'.)

We'll also employ the "linear approximation with error"(*) form of the def. of the derivative which says.

$$f(c+h) = \overbrace{f(c) + hf'(c)}^{\text{lin. approx.}} + \overbrace{h\varepsilon(h)}^{\text{error}} \quad (*)$$

where $\boxed{\varepsilon(h) \to 0 \text{ as } h \to 0}$

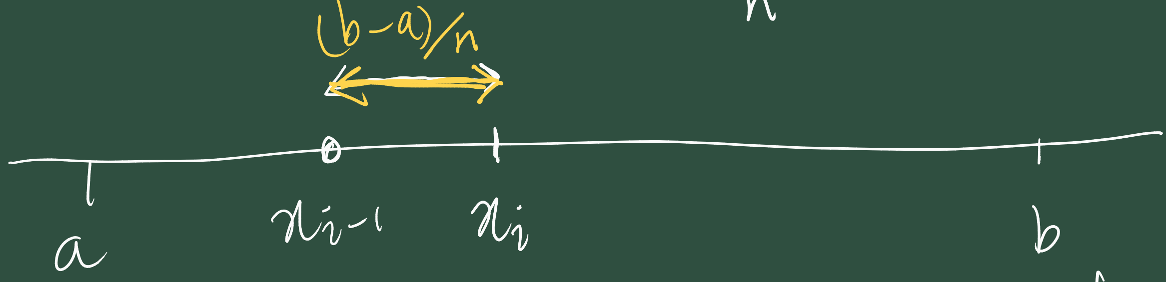
Use the equally spaced partition of $[a,b]$.

given by x_i, $i = 0, \dots, n$

$x_0 = a$, $x_n = b$

and $x_i = a + i\,\dfrac{b-a}{n}$, $i = 0, \dots, n$

$= x_{i-1} + \dfrac{b-a}{n}$, $i = 1, \dots, n$



Note the widths of rectangles are $\dfrac{b-a}{n}$.

and we'll use the start/left-endpoint of each interval as the representative point.

$$x_i^* = x_{i-1} = a + (i-1)\frac{b-a}{n}.$$

$$\boxed{\int_a^b f'(x)\,dx =}$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f'\Big(\underbrace{a+(i-1)\tfrac{b-a}{n}}_{c}\Big)\overbrace{\underbrace{\frac{b-a}{n}}}^{h}\right)$$

Substitute (*) into this to get.

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left[f\Big(a+i\,\tfrac{b-a}{n}\Big) - f\Big(a+(i-1)\,\tfrac{b-a}{n}\Big) - \frac{(b-a)}{n}\,\varepsilon\Big(\frac{b-a}{n}\Big)\right]\right)$$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left[f(x_i) - f(x_{i-1}) - \frac{b-a}{n}\,\varepsilon\Big(\frac{b-a}{n}\Big)\right]\right)$$

$$\cancel{f(x_1)}\ \boxed{-f(x_0)}$$
$$\cancel{f(x_2)} - \cancel{f(x_1)}$$
$$\cancel{f(x_3)} - \cancel{f(x_2)}$$
$$\cancel{f(x_4)} - \cancel{f(x_3)}$$
$$\vdots$$
$$\boxed{f(x_n)} - \cancel{f(x_{n-1})}$$

$$\cancel{n}\,\frac{b-a}{\cancel{n}}\,\varepsilon\Big(\frac{b-a}{n}\Big)$$

, all the $f(x_i)$ terms cancel with ~~one ano~~ each other

$$= \lim_{n\to\infty}\left(f(x_n) - f(x_0) - (b-a)\,\varepsilon\Big(\frac{b-a}{n}\Big)\right)$$

$$= \lim_{n\to\infty} \left(f(b) - f(a) - (b-a)\, \varepsilon\left(\frac{b-a}{n}\right)\right)$$

$$= f(b) - f(a) - (b-a) \lim_{n\to\infty}\left(\varepsilon\left(\frac{b-a}{n}\right)\right)$$

, by linearity of limits.

$$= \boxed{f(b) - f(a)}$$

since $\varepsilon(h) \to 0$ as $h \to 0$ and $\frac{b-a}{n} \to 0$ as $n \to \infty$.

F.T.C. 2

This gives us a better way to evaluate definite integrals in practice than Riemann sum approach, & using known derivative rules for familiar functions.

We're not studying 5.5.
But we do study chapter 6.

Chap 6 Techniques for anti-differentiation.

Give this a detailed reading and work with the many examples.

The main properties of differentiation: linearity, product rule, chain rule, quotient rule, derivatives of known functions also gives us equivalent results for integration.

Eg. "integration by substitution" comes from the chain rule.

Consider a function

$$f(x) = (x^2+3x-5)^{10}$$

$$= h(g(x)), \quad \text{where } h(x) = x^{10}, \quad g(x) = x^2+3x-5.$$

Chain rule would imply. $\quad h'(x) = 10x^9$

$$f'(x) = h'(g(x))\, g'(x)$$

$$= 10(x^2+3x-5)^9 (2x+3)$$

$$= (20x+30)(x^2+3x-5)^9.$$

So if we're confronted by an integral

$$I = \int (20x+30)(x^2+3x-5)^9\,dx.$$

Use a substitution.

$$u = x^2+3x-5$$

$$\Rightarrow \quad du = 2x.dx + 3\,dx$$

$$= (2x+3)\,dx,$$ from chain rule

Then: I becomes.

$$I = \int (20x+30)\,u^9 \frac{du}{2x+3}.$$

$$= \int 10u^9\,du$$

$$= u^{10} + C,$$ anti-differentiation. with C an arbitrary constant of integration

$$= (x^2+3x-5)^{10} + C.$$

Ex 6.1.3

$$I = \int x\sin(x^2+5)\,dx.$$

Try a substitution $u = x^2+5$

$$\Rightarrow du = 2x\,dx.$$

integral becomes.

$$I = \int x\sin(u)\frac{du}{2x}$$

$= \frac{1}{2} \int \sin(u)\, du$, using linearity

$= \frac{1}{2} \left(-\cos(u) + C \right)$, anti-differentiation.

$= -\frac{1}{2}\cos(u) + C$, for an arbitrary choice of integration constant C.

6.1, 6.2 are important

integration by parts is the integration technique corresponding to product rule of differentiation.

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} \quad , \text{ prod. rule}$$

$$\int \frac{d}{dx}(fg)\, dx = \int f\frac{dg}{dx}\, dx + \int g\frac{df}{dx}\, dx.$$

, integrating both sides and linearity.

$$\Rightarrow \quad fg = \int f\frac{dg}{dx}\, dx + \int g\frac{df}{dx}\, dx.$$

by F.T.C.

$$\Longleftrightarrow \boxed{\int f \frac{dg}{dx}\, dx = fg - \int g \frac{df}{dx}\, dx.}$$

integration by parts formula.
can be used when we can express our integrand as $f \frac{dg}{dx}$ for suitable f, g and where $\frac{df}{dx}$ is likely to simplify, compared to f.

6.5 Partial fractions.

When integrating "rational functions" which are quotients of polynomials. It can be useful to replace the rational function with a sum of simpler rational functions, which we can integrate.

Eg 6.5.6

Consider $\int \frac{1}{(x-1)(x+2)^2} \, dx$.

The rational function $r(x) = \frac{1}{(x-1)(x+2)^2}$

Following 6.5.2. we expect r to be expressed as

$$r(x) = \frac{A_1}{x-1} + \frac{A_2}{x+2} + \frac{A_3}{(x+2)^2} \quad (*)$$

for suitable constants A_i which we can determine.

$$r(x) = \frac{A_1(x+2)^2 + A_2(x-1)(x+2) + A_3(x-1)}{(x-1)(x+2)^2}$$

$$= \frac{x^2(A_1 + A_2) + x(4A_1 + A_2 + A_3) + \begin{matrix} 4A_1 \\ -2A_2 \\ -A_3 \end{matrix}}{(x-1)(x+2)^2}.$$

Comparing this to $r(x) = \frac{1}{(x-1)(x+2)^2}$

equating the numerator polynomials.

we get.

$$\left\{\begin{aligned} A_1 + A_2 &= 0 \\ 4A_1 + A_2 + A_3 &= 0 \\ 4A_1 - 2A_2 - A_3 &= 1 \end{aligned}\right\}$$

system of 3 equations in 3 unknowns A_1, A_2, A_3

From our lin. alg. we can solve this using Gaussian elimination.

The Augmented matrix is:

$$\begin{array}{cccc} A_1 & A_2 & A_3 & \text{rhs} \\ \end{array}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 \\ 4 & -2 & -1 & 1 \end{pmatrix}$$

use row operations to reduce this to reduced row echelon form, where the solution can be read off.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & \boxed{-3} & 1 & 0 \\ 0 & -6 & -1 & 1 \end{pmatrix}$$

$r_2' = r_2 - 4r_1$

$r_3' = r_3 - 4r_1$

$$\begin{pmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & \boxed{-3} & 1 \end{pmatrix}$$

$r_2' = -\frac{1}{3} r_2$.

$r_1' = r_1 - r_2'$

$r_3' = r_3 + 6r_2'$

$$\left(\begin{array}{cccc} 1 & 0 & 0 & 1/9 \\ 0 & 1 & 0 & -1/9 \\ 0 & 0 & 1 & -1/3 \end{array}\right) \quad \begin{array}{l} r_3' = -\frac{1}{3} r_3 \\ r_1' = r_1 - \frac{1}{3} r_3' \\ r_2' = r_2 + \frac{1}{3} r_3' \end{array}$$

The solution can be read off from this reduced row-echelon form augmented matrix.

$$A_1 = 1/9, \quad A_2 = -1/9, \quad A_3 = -1/3.$$

So our integrand $r(x)$ becomes

$$r(x) = \frac{1/9}{x-1} - \frac{1/9}{x+2} - \frac{1/3}{(x+2)^2}$$

So $\int r(x)\, dx$

$$= \int \left(\frac{1/9}{x-1} - \frac{1/9}{x+2} - \frac{1/3}{(x+2)^2} \right) dx$$

$$= \frac{1}{9} \int \frac{1}{x-1}\, dx - \frac{1}{9} \int \frac{1}{x+2}\, dx$$

$$- 1/3 \int \frac{1}{(x+2)^2}\, dx.$$

, by linearity
and use anti-derivatives of known functions.

$$= \frac{1}{9} \ln(|x-1|) - \frac{1}{9} \ln(|x+2|)$$

$$- \frac{1}{3} \frac{-1}{x+2}.$$

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

$$\Rightarrow \varepsilon(h) = \frac{f(c+h) - f(c)}{h} - f'(c)$$

$$\Leftrightarrow h\varepsilon(h) = f(c+h) - f(c) - hf'(c)$$

$$\Leftrightarrow f(c+h) = f(c) + hf'(c) + h\varepsilon(h)$$