~~Series~~ Sequences.

Reading: See 9.1 in APEX.

Main object of study here are infinite ordered sequences of numbers.

$$x_1, x_2, x_3, x_4, \ldots$$

Can denote the whole sequence as

$$\{x_n\}_{n=1}^{\infty}$$

Typically defined by a function of formula on n

We want to formalize and understand the concepts of <u>convergence</u> and <u>limit</u> of a sequence.

But first an example exploring increasing/decreasing nature of a sequence.

<u>Example:</u>

Consider the sequence defined by $\{a_n\}_{n=1}^{\infty}$ and

$$\boxed{a_n = \frac{3n}{n^2+1}.}$$

Examining the initial terms of this it seems to be a decreasing sequence.

$$\frac{3}{2}, \frac{6}{5}, \frac{9}{10}, \frac{12}{17}, \ldots$$

To prove it is decreasing for all n we can build a proof by considering the difference of ratio of neighboring terms.

ie.

$$a_{n+1} - a_n \begin{cases} < 0, \text{ for all } n \geq 1 \text{ ie. decreasing} \\ > 0, \text{ for all } n \geq 1 \text{ ie. an increasing.} \end{cases}$$

(and if $a_n > 0$) for all n.

OR

$$\frac{a_{n+1}}{a_n} \begin{cases} < 1, \text{ for all } n \geq 1 \text{ ie. decreasing.} \\ > 1, \text{ for all } n \geq 1 \text{ ie. increasing.} \end{cases}$$

So back to our example.

$$\frac{a_{n+1}}{a_n} = \frac{3(n+1)}{(n+1)^2+1} \cdot \frac{n^2+1}{3n}.$$

$$= \frac{(3n+3)(n^2+1)}{(n^2+2n+2)3n}$$

$$= \frac{3n^3+3n^2+3n+3}{3n^3+6n^2+6n}.$$

$$= \frac{(3n^3+6n^2+6n)-3n^2-3n+3}{3n^3+6n^2+6n}$$

$$= 1 - \frac{3n^2+3n-3}{3n^3+6n^2+6n}.$$

< 1 since. $3n^3+6n^2+6n>0$ for all $n \geq 1$

and $3n^2+3n-3>0$ for all $n \geq 1$

for all $n \geq 1$.

So if $\dfrac{a_{n+1}}{a_n} < 1$ for all $n \geq 1$

$\Rightarrow a_{n+1} < a_n$ " " "

So $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence.

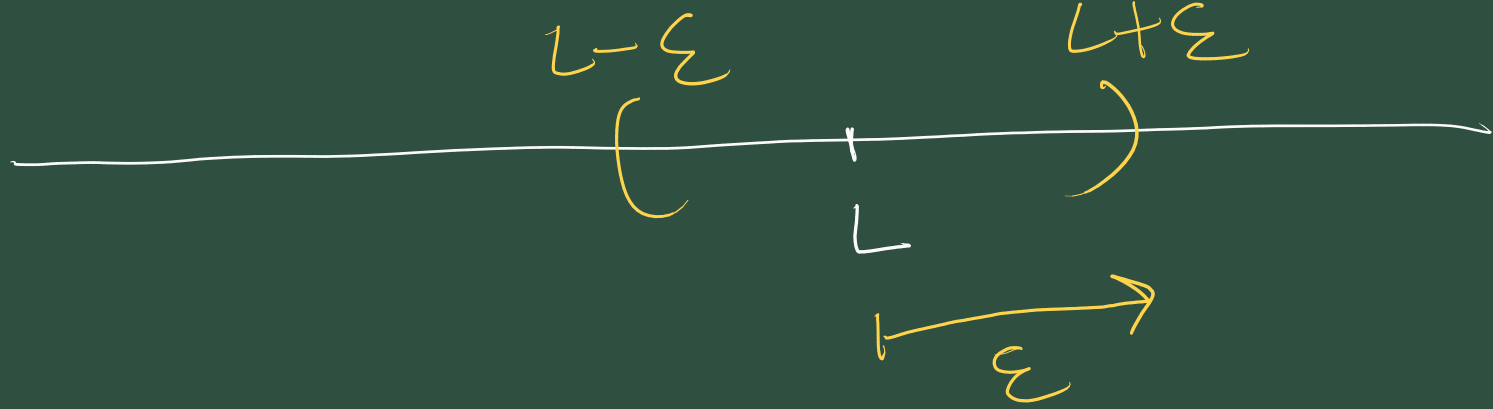
"limit" and "convergence" for sequences in def. 9.1.9.

We say $\{a_n\}_{n=1}^{\infty}$ converges to the limit L if and only if given any $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$ we have.

$$|a_n - L| < \varepsilon$$

ie. eventually (for all $n \geq N$)

the sequence lives in the interval $(L-\varepsilon, L+\varepsilon)$



Associated notation

- $a_n \to L$ as $n \to \infty$.

"a_n converges to L as n tends to infinity"

- $\lim_{n \to \infty} a_n = L$

"the limit of a_n, as n tends to infinity, is L"

<u>Example</u> Use this ε-N definition to prove that

$$\boxed{\frac{1}{n} \to 0 \text{ as } n \to \infty.}$$

$\{a_n\}_{n=1}^{\infty}$, $a_n = \frac{1}{n}$

<u>Proof</u> $\boxed{\text{Let } \varepsilon > 0 \text{ be given}}$.

The claimed limit is $L = 0$.

$$|a_n - L| = \left|\frac{1}{n} - 0\right|$$

$$= \left|\frac{1}{n}\right|$$

$$= \frac{1}{n}, \text{ since } n \geq 1$$

So now we can say:

$$|a_n - L| < \varepsilon \iff \frac{1}{n} < \varepsilon$$

$$\iff n > \frac{1}{\varepsilon}$$

So we can choose N to be any integer greater than $\frac{1}{\varepsilon}$

So that if $n \geq N$ then

$$n > \frac{1}{\varepsilon}$$

and so.

$$\boxed{|a_n - L| < \varepsilon}$$

So this shows the def of convergence is satisfied, and so $\frac{1}{n} \to 0$ as

required.

We can also use the definition to establish rules/properties for convergence of sequences.

see theorem 9.1.20.

We will prove a linearity result that combines 1, 4 of theorem 9.1.20.

"Algebra of limits theorem".

Claim: If $a_n \to L$, $b_n \to K$ as $n \to \infty$ then for any pair $\alpha, \beta \neq 0$. the new sequence $\{\alpha a_n + \beta b_n\}_{n=1}^{\infty}$ is also convergent and

$\alpha a_n + \beta b_n \to \alpha L + \beta K$. as $n \to \infty$.

<u>Proof</u> $\boxed{\text{Let } \varepsilon > 0 \text{ be given.}}$

Let's investigate the goal inequality.

$$|(\alpha a_n+\beta b_n)-(\alpha L+\beta K)|<\varepsilon$$

$$\Leftrightarrow \boxed{|\alpha(a_n-L)+\beta(b_n-K)|<\varepsilon}$$

and

$$\boxed{|\alpha(a_n-L)+\beta(b_n-K)|} \le \boxed{|\alpha(a_n-L)|+|\beta(b_n-K)|},$$

$\triangle$ inequality.

$$|u+v|\le|u|+|v|$$

$$= |\alpha|\overbrace{|a_n-L|}^{\frac{\varepsilon}{2|\alpha|}} + |\beta|\overbrace{|b_n-K|}^{\frac{\varepsilon}{2|\beta|}},$$

$$|uv|=|u||v|$$

$$\underbrace{\quad}_{<\varepsilon/2} \qquad \underbrace{\quad}_{\varepsilon/2}$$

$$< \quad \varepsilon \qquad \text{GOAL.}$$

So by the definition of convergence there exists points N_1, N_2

such that

$$n \geq N_1 \implies |a_n - L| < \frac{\varepsilon}{2|\alpha|}$$

$$n \geq N_2 \implies |b_n - K| < \frac{\varepsilon}{2|\beta|}$$

So finally choose $\boxed{N = \max(N_1, N_2)}$

And recognize now that

if $\boxed{n \geq N}$ then

$$\boxed{|(\alpha a_n + \beta b_n) - (\alpha L + \beta K)| < \varepsilon}$$

(seen from above investigations).

So by the def. of convergence

$$\alpha a_n + \beta b_n \to \alpha L + \beta K$$

as $n \to \infty$ as required.

Can construct proofs for products and ratios cases of theorem 9.1.20 in a similar way (take inspiration for these from corresponding proofs for limiting values of functions).

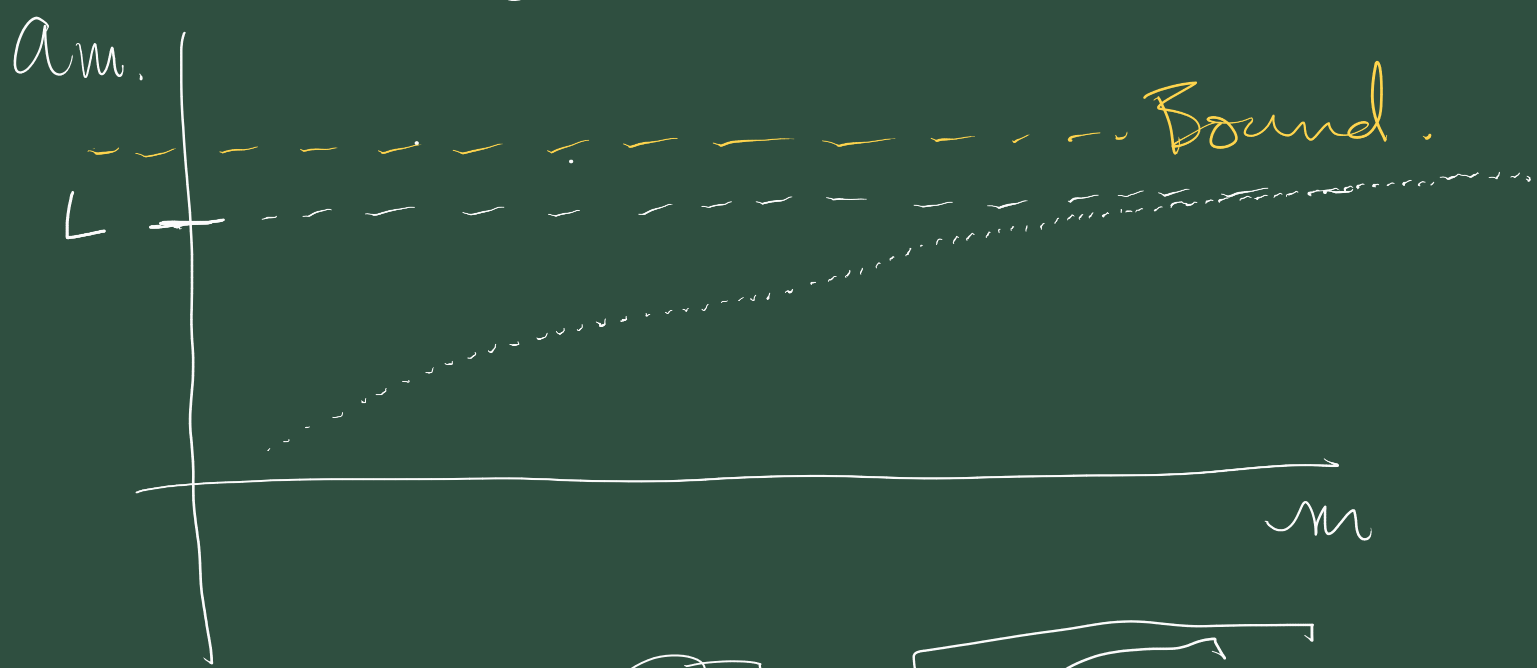
Bounded sequences & Monotone Convergence Theorem.

See def 9.1.28.

A sequence is monotonic if it is always increasing or always decreasing.

MCT (Monotone Convergence Theorem) 9.1.32 says that bounded monotone sequences are convergent.

IDEA $\{a_n\}_{n=1}^{\infty}$ monotone increasing.



Q8. $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$

a). Call the sequence $\{a_n\}_{n=1}^{\infty}$

$\boxed{a_1 = \sqrt{2}}$, $a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}}$

$a_3 = \sqrt{2\sqrt{2\sqrt{2}}} = \sqrt{2a_2}$.

and so on.

So $\boxed{a_{n+1} = \sqrt{2a_n}}$ for $n \geq 1$

(b) By investigating initial values.
we see the sequence seems to
be increasing. and all < 2.

First we'll prove sequence is
bounded above by 2.

Use "proof by induction"

Certainly $a_1 = \sqrt{2} < 2$

"Base case of induction"

Let's assume that $\boxed{a_n < 2.}$

$$\boxed{a_{n+1}} = \sqrt{2a_n}$$
$$= \sqrt{2}\sqrt{a_n}$$
$$< \sqrt{2}\sqrt{2}, \text{ since } a_n < 2.$$
$$\boxed{= 2}$$

This proves.

$$a_n < 2 \Rightarrow a_{n+1} < 2.$$

Known as the "induction step"

By the principle of induction

for all $n \geq 1$ $a_n < 2$.

With this we can prove $\{a_n\}_{n=1}^{\infty}$ is increasing.

$$\boxed{a_n} = \sqrt{a_n}\sqrt{a_n}$$

$$< \sqrt{2}\sqrt{a_n}, \text{ since } a_n < 2$$

$$= \sqrt{2a_n}$$

$$= \boxed{a_{n+1}}$$

So $\{a_n\}$ is increasing.

So by Monotone Convergence

Theorem $\{a_n\}$ must be convergent. Let's call the limit L.

(C). $a_n \to L$ as $n \to \infty$.

Consider the recurrence relation.

$$a_{n+1} = \sqrt{2a_n}, \quad n \geq 1.$$

Let's take the limit of both sides

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\sqrt{2a_n}\right).$$

$$\Rightarrow \boxed{L} = \sqrt{2} \lim_{n\to\infty} \left(\sqrt{a_n}\right)$$

, by linearity.

$$= \boxed{\sqrt{2}\sqrt{L},}$$

will follow by a continuity result, $\sqrt{\ }$ is continuous.

$$\Rightarrow \quad L^2 = 2L.$$

$$\Rightarrow \quad L = 2.$$

$\boxed{Q10}$ Consider $\{x_n\}_{n=0}^{\infty}$

with $x_0 = 2$

$$x_n = \frac{1}{3 - x_{n-1}} \text{ for all } n \geq 1.$$

Experimentation suggests $\{x_n\}$ is decreasing and convergent.

<u>Boundedness</u> (use induction)

Claim: for all $n \geq 0$ $\quad 0 < x_n \leq 2.$

Certainly true for $x_0 = 2$.

$$\boxed{0 < x_0 \leq 2.}$$

$$\boxed{\text{Now assume} \quad 0 < x_n \leq 2.}$$

So $x_n \leq 2$.

$$\Rightarrow \quad -x_n \geq -2.$$

$$\Rightarrow \quad 3 - x_n \geq 3 - 2. = 1.$$

$$\Rightarrow \quad \frac{1}{3 - x_n} \leq 1$$

$$\Rightarrow \quad x_{n+1} \leq 1 < 2.$$

So $x_{n+1} \leq 2$ also.

Secondly. $x_n > 0$.

$$\Rightarrow \quad -x_n < 0$$

$$\Rightarrow \quad 3 - x_n < 3$$

$$\Rightarrow \quad \frac{1}{3 - x_n} > \frac{1}{3}$$

$$\Rightarrow \quad x_{n+1} > \frac{1}{3} > 0$$

So $x_{n+1} > 0$.

So In summary

$$\boxed{0 < x_{n+1} \leq 2.}$$

So we've proved the induction step.

So by induction, we've proved,

for all $n \geq 0$ $\boxed{0 < x_n \leq 2.}$

Now claim $\{x_n\}$ is decreasing

(using proof by induction)

$x_0 = 2, \quad x_1 = 1.$

So $\boxed{x_0 > x_1}$

Now assume that

$$\boxed{x_n > x_{n+1}}$$

$\Rightarrow \quad -x_n < -x_{n+1}$

$\Rightarrow \quad 3 - x_n < 3 - x_{n+1}$

$\Rightarrow \quad \dfrac{1}{3 - x_n} > \dfrac{1}{3 - x_{n+1}}$

$$\Rightarrow \quad \boxed{x_{n+1} > x_{n+2}}$$

So by induction, for all $n \geq 0$

$x_n > x_{n+1}$, i.e. the sequence

is decreasing.

The Monotone Convergence

Theorem says that

$x_n \longrightarrow L$ as $n \to \infty$ for

some limit L.

$L = ?$

Consider the recurrence

relation.

$$x_{n+1} = \frac{1}{3 - x_n}.$$

Take the limit of both sides to see

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{3 - x_n}.$$

$$\Rightarrow \quad L = \frac{1}{3-L}.$$

, by algebra of limits theorem 9.1.20.

$$\Rightarrow \quad 3L - L^2 = 1.$$

$$\Rightarrow \quad L^2 - 3L + 1 = 0.$$

$\Rightarrow$ There are solutions, given by the quadratic formula.

$$L = \frac{3 \pm \sqrt{5}}{2}$$

the solution $L = \frac{3+\sqrt{5}}{2} > 2$.

So this is excluded since

$0 < x_n \leq 2$ so $0 \leq L \leq 2$

So it must be that

$$L = \frac{3 - \sqrt{5}}{2}$$