Series

From an infinite sequence $\{x_n\}_{n=1}^{\infty}$

$$x_1, x_2, x_3, \ldots$$

we can form an infinite series which is the sum of all these numbers.

$$\sum_{i=1}^{\infty} x_i = x_1 + x_2 + x_3 + \ldots$$

$\sum$ "Sigma"

But there is a conceptual problem with this "adding up an infinite things" but this can be dealt with by using the formal def for "$\sum_{i=1}^{\infty} x_i$"

Def "Partial sum"

The k^{th} partial sum S_k of the series $\sum_{i=1}^{\infty} x_i$ is the finite sum

$$S_k = \sum_{i=1}^{k} x_i = x_1 + x_2 + \ldots + x_k.$$

and we say that $\sum_{i=1}^{\infty} x_i$ converges

if $\lim_{k\to\infty} S_k$ exists, in which case

we write

$$\sum_{i=1}^{\infty} x_i = \lim_{k\to\infty} S_k$$

$$= \lim_{k\to\infty} \left(\sum_{i=1}^{k} x_i\right)$$

So $\sum_{1}^{\infty}$ is defined as the limit of the sequence of its partial sums.

and if $\{S_k\}_{k=1}^{\infty}$ diverges then we say $\sum_{i=1}^{\infty} x_i$ is divergent also.

Examples

1. Suppose $x_m = m$ for all $m \geq 1$ then consider the series $\sum_{m=1}^{\infty} x_m$

$$\sum_{m=1}^{\infty} = 1+2+3+4+\dots$$

which is surely divergent. Its partial sums are.

$$S_k = \sum_{m=1}^{k} m = 1+2+3+\dots+k.$$

$$= \frac{k}{2}(k+1) = \frac{k^2}{2} + \frac{k}{2}$$

and clearly $S_k \to \infty$ as $k \to \infty$

2. Let $y_m = \left(\frac{1}{2}\right)^m = \frac{1}{2^m}$

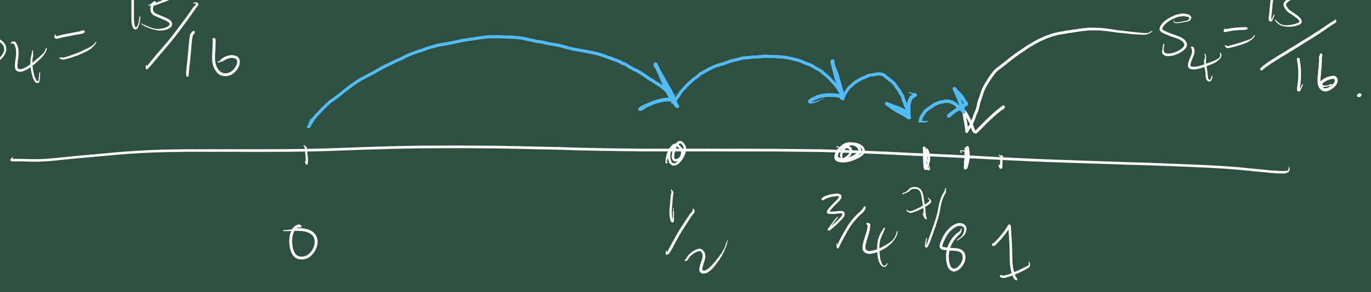
and consider $\sum_{m=1}^{\infty} y_m = \sum_{m=1}^{\infty} \frac{1}{2^m}$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

This will converge, let's plot its partial sums $S_1, S_2, S_3, \dots$

$S_1 = 1/2$, $S_2 = \frac{1}{2} + 1/4 = 3/4$, $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = 15/16$



and in general

$$S_k = \frac{2^k - 1}{2^k} \to 1 \text{ as } k \to \infty.$$

stepping "half the remaining distance" towards 1.

So we would say.

$$\sum_{m=1}^{\infty} \frac{1}{2^m} = 1.$$

3. This example will illustrate that it's not enough for the terms to get smaller and smaller in order for the series to converge.

Harmonic Series.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\cdots$$

Although these terms are getting smaller and $\frac{1}{n} \to 0$ as $n \to \infty$, but still the series diverges, as is shown by the following grouping argument.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$
$$+ \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$
$$+ \left(\frac{1}{9} + \cdots + \frac{1}{16}\right)$$
$$+ \left(\frac{1}{17} + \cdots + 1/32\right) + \cdots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots + \frac{1}{16}\right) + \dots$$

replacing each term from each group with the final term from that group

$$= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \frac{16}{32} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

This final series is clearly divergent

This demonstrates that $\sum_{n=1}^{\infty} \frac{1}{n}$ will grow without any upper bound (albeit very slowly)

Theorem (see 9.2.19).

Some basic properties of series convergence.

Forming linear combinations of convergent series.

Suppose we have two convergent series. $\sum_{m=1}^{\infty} a_m = A$, $\sum_{m=1}^{\infty} b_m = B$

then ~~the~~ for any $\alpha, \beta \in \mathbb{R}$ the series.

$$\sum_{m=1}^{\infty} (\alpha a_m + \beta b_m) \text{ is again}$$

convergent and

$$\sum_{m=1}^{\infty} (\alpha a_m + \beta b_m) = \alpha A + \beta B$$

Proof: (using the partial sum definition)

$$\boxed{\alpha A + \beta B} = \alpha \lim_{k \to \infty} \left(\sum_{m=1}^{k} a_m \right) + \beta \lim_{k \to \infty} \left(\sum_{m=1}^{k} b_m \right)$$

$$= \lim_{k\to\infty} \left(\alpha\left(\sum_{m=1}^{k} a_m\right) + \beta\left(\sum_{m=1}^{k} b_m\right) \right)$$

, algebra of limits theorem for sequences.

$$= \lim_{k\to\infty} \left(\sum_{m=1}^{k} (\alpha a_m + \beta b_m) \right)$$

, rewriting the finite sums

$$\boxed{= \sum_{m=1}^{\infty} (\alpha a_m + \beta b_m)}$$

In this and the following sections there a number of "series convergence tests" which are tools to help us assess the convergence status of given series

General Term Test (can sometimes be used to declare divergence)

Two equivalent versions (which are contrapositives of each other.

$$A \Rightarrow B \quad \equiv (\neg B) \Rightarrow (\neg A)$$

$\neg B$ is not B

- $\lim\limits_{m\to\infty} a_m \neq 0 \implies \sum\limits_{m=1}^{\infty} a_m$ diverges.
- $\boxed{\sum\limits_{m=1}^{\infty} a_m \text{ converges}} \implies \boxed{\lim\limits_{m\to\infty} a_m = 0.}$

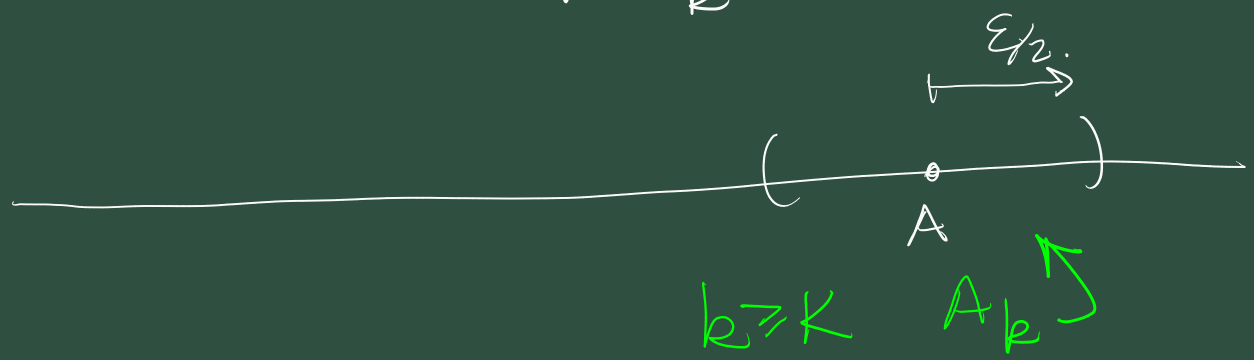
<u>Proof</u>

Suppose $\sum\limits_{m=1}^{\infty} a_m$ converges. Let A_k denote the partial sums, $A_k = \sum\limits_{m=1}^{k} a_m$

So we're assuming $A_k \to A$, as $k \to \infty$ for some A.

$\boxed{\text{So given any } \varepsilon > 0}$ there exists K such that for $k \geq K$

we have $|A_k - A| < \varepsilon/2$



$\implies$ for $k \geq K$ $|A_k - A_{k+1}| < \varepsilon$

$\implies$ for $k \geq K$ $|\quad -a_{k+1} \quad| < \varepsilon$

$\Rightarrow$ for $k \geq K$ $\boxed{|a_{k+1}| < \varepsilon}$

This last statement is saying

$$\boxed{a_m \to 0 \text{ as } m \to \infty.}$$

Geometric Series Test

A geometric series is one of the form:

$$\sum_{m=1}^{\infty} ar^{m-1} = a + a\underline{\underline{r}} + a\underline{\underline{r}}^2 + a\underline{\underline{r}}^3 + \dots$$

initial term

r is the common ratio between terms.

We completely understand the convergence of such series in terms of a, r.

if $a = 0$, then trivially $\sum_{m=1}^{\infty} ar^{m-1}$ converges and $\sum_{m=1}^{\infty} ar^{m-1} = 0$

So now assume $a \neq 0$

if $r = 1$ then $\sum_{m=1}^{\infty} ar^{m-1} = a + a + a + a + \dots$ which is clearly divergent.

So now can assume $a \neq 0$, and $r \neq 1$.

then the test says $\sum_{m=1}^{\infty} ar^{m-1}$ is convergent if and only if $|r|<1$ in which case $\sum_{m=1}^{\infty} ar^{m-1} = \frac{a}{1-r}$

and if $|r| \geq 1$ then series diverges.

Proof

$$S_k = \sum_{m=1}^{k} ar^{m-1} = a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{k-1}}$$

$$rS_k = \dots = \cancel{ar} + \cancel{ar^2} + \dots \cancel{+} ar^k$$

$$S_k - rS_k = a - ar^k$$

$$\Leftrightarrow (1-r)S_k = a(1-r^k)$$

$$\Leftrightarrow S_k = \frac{a(1-r^k)}{1-r}$$

Consider $\lim_{k\to\infty} S_k$

ie. $\lim_{k\to\infty} \frac{a(1-r^k)}{1-r}$

$$= \frac{a\left(1 - \lim_{k\to\infty} r^k\right)}{1-r},$$ by algebra of limits for sequences

Note that $r^k \to 0$ as $k \to \infty$, whenever $|r|<1$ and r^k is divergent whenever $|r|>1$ or $r=-1$

Therefore $\{S_k\}_{k=1}^{\infty}$ is convergent whenever $|r|<1$ and divergent whenever $|r|\geq 1$.

and if $|r|<1$ and so $r^k \to 0$ as $k\to\infty$, then from above

$$\lim_{k\to\infty} S_k = \frac{a}{1-r},$$ as required.

<u>Examples</u> Our previous example

$\sum_{m=1}^{\infty} \frac{1}{2^m}$, ~~[illegible]~~ is a geometric series which in standard form looks like.

$$\sum_{m=1}^{\infty} \frac{1}{2^m} = \sum_{m=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{m-1}$$

which is convergent (since $|\frac{1}{2}|<1$)

and has limit.

$$\sum_{m=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{m-1} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2}$$

$$= 1$$

All coming from Geometric Series Test.

Example 2

Consider $\sum_{n=1}^{\infty} x_n$ where

$$x_n = \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{4}\left(\frac{2}{3}\right)^n.$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} + \frac{1}{6}\left(\frac{2}{3}\right)^{n-1}$$

So we see $\sum_{n=1}^{\infty} x_n$ is the sum

of two geometric series

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{2}{3}\right)^{n-1}$$

both of which are convergent since $\left|\frac{1}{2}\right|, \left|\frac{2}{3}\right| < 1$

$$= \frac{1/3}{1-1/2} + \frac{1/6}{1-2/3}$$

(by the geometric series test.)

$$= \frac{1/3}{1/2} + \frac{1/6}{1/3}$$

$$= \frac{2}{3} + 1/2.$$

$$= 7/6.$$

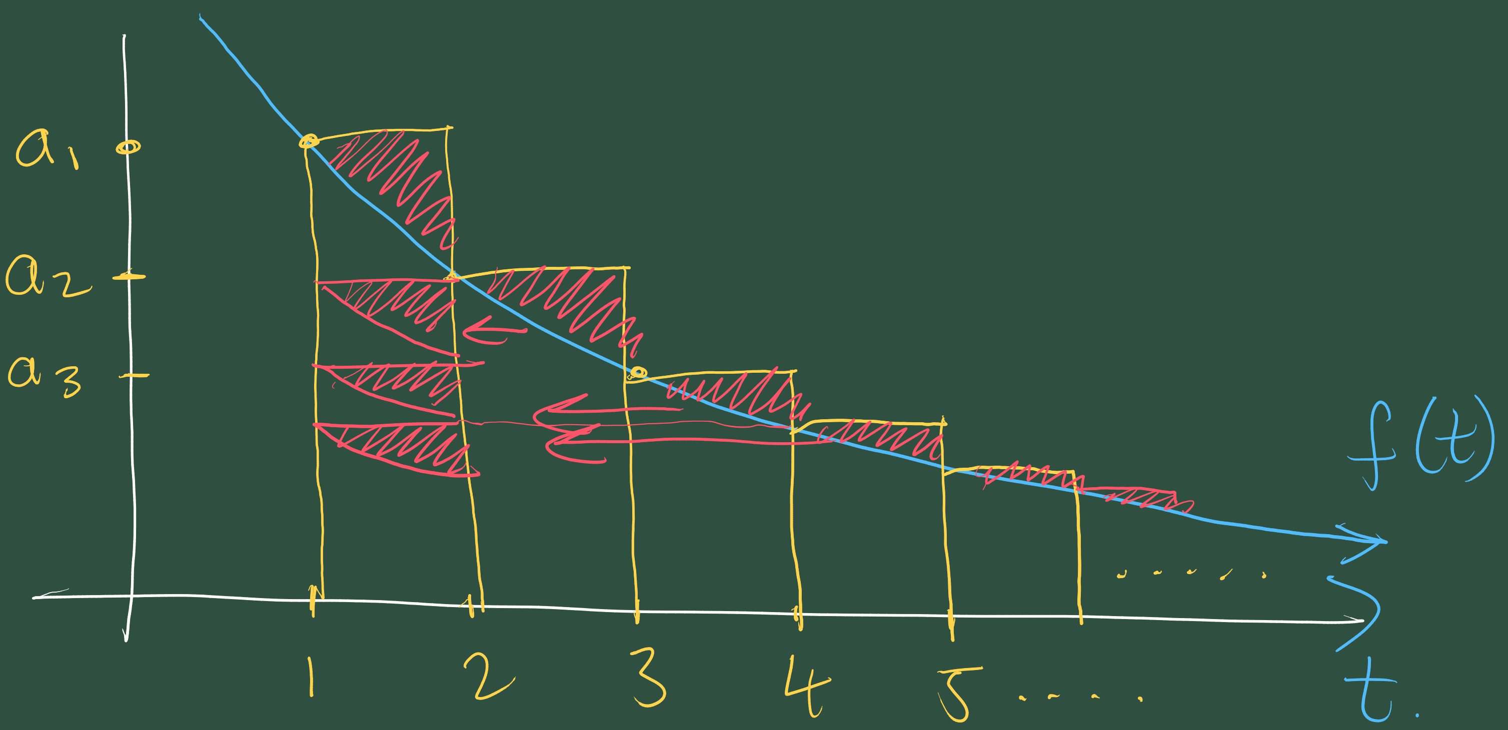
Integral Test

Consider $\sum_{m=1}^{\infty} a_m$ where $a_m = f(m)$

and f is an integrable, decreasing positive function.

Then $\sum_{m=1}^{\infty} a_m$ converges $\Longleftrightarrow$ $\int_1^{\infty} f(t)\,dt$ is finite.

Proof (best seen from this diagram)



Consider the rectangles of width 1 and heights $f(m) = a_m$

Observation $\sum_{m=1}^{\infty} a_m$ is the sum of the areas of all rectangles. and this is approximately $\int_1^{\infty} f(t)\,dt$.

with the errors shown in red above. Slide errors to the left and we see that they will all fit in the first rectangle without overlapping.

Therefore, total error $< a_1$, i.e. finite.

and

$$\sum_{m=1}^{\infty} a_m = \int_1^{\infty} f(t)\,dt + \text{total error}$$

finite number

and

So $\sum_{m=1}^{\infty} a_m$ will be finite (i.e. it will converge) if and only if $\int_1^{\infty} f(t)\,dt$ is also finite.

$\blacksquare$

Application of the integral test.

Consider the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^a}$$

Compare with the behaviour of integral.

$$\int_1^{\infty} \frac{1}{t^a}\, dt.$$

$$= \int_1^{\infty} t^{-a}\, dt.$$

$$= \left[\frac{t^{-a+1}}{-a+1} \right]_1^{\infty}, \text{ for } a \neq 1.$$

$$= \frac{1}{1-a}\left(\left(\lim_{t\to\infty} t^{-a+1} \right) - \frac{1}{1-a} \right)$$

Note that

$$t^{-a+1} \to \infty \text{ for } |a|<1.$$

as $t \to \infty$

But when $a>1$ then

$$t^{-a+1} = \frac{1}{t^{a-1}} \longrightarrow 0$$

as $t \to \infty$.

So by the integral test.

$$\sum_{m=1}^{\infty} \frac{1}{m^a} \text{ converges for } a>1$$

and diverges for $a<1$

(we already know it diverges for $a=1$, the harmonic series $\sum_{m=1}^{\infty} \frac{1}{m}$)

The series $\sum_{n=1}^{\infty} \frac{1}{n^a}$ are known as the hyper-harmonic series.

So eg. $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and so on are all convergent

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Still to consider the "ratio test", "comparison tests" and "telescoping series".