Chap 7 Applications of integration

Reading: 7.1 — 7.4.

7.5, 7.6 are not included.

7.1 Area between curves.

To find area between two curves as in figure 7.1.2.

For a given partition of $[a, b]$, with representative points x_i^* in the sub-intervals. then

$$\left(f(x_i^*) - g(x_i^*)\right) \underbrace{\Delta x_i}_{\text{width of } i^{th} \text{ subinterval}}$$

will be the area of one of the rectangles in the approximation.

So we get an approximation Q for the area given by

$$Q = \sum_{i=1}^{n} \left(f(x_i^*) - g(x_i^*)\right) \Delta x_i$$

and $$Q \approx \int_a^b f(x) - g(x)\, dx.$$

and in the limit we will have.

$$\text{area} = \lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} Q.$$

So thanks to our Fundamental Theorem of Calculus we can use anti-differentiation to determine such integrals and evaluate these areas.

Example 7.1.8

Determine area bounded by the three curves.

$$y = \sqrt{x} + 2, \quad y = -(x-1)^2 + 3.$$

$$y = 2.$$

as shown in figure 7.1.9.

From the figure we can guess approximate the points of intersection of these ~~~~ curves but we should confirm the exact points.

$(0,2)$ satisfies $y=2$ and $y=\sqrt{x}+2$

$(2,2)$ " " and $y=-(x-1)^2+3$

$(1,3)$ " $y=\sqrt{x}+2$ and " "

We should split the area into two according to the two boundary curves forming the upper boundary, giving the area Q as

$$Q=\int_0^1\left(\sqrt{x}+2-2\right)dx$$
$$\quad+\int_1^2\left(-(x-1)^2+3-2\right)dx$$

$$=\int_0^1\sqrt{x}\,dx+\int_1^2\left(-(x-1)^2+1\right)dx$$

(using anti-differentiation).

$$=\left[\frac{2x^{3/2}}{3}\right]_0^1+\left[-\frac{(x-1)^3}{3}+x\right]_1^2$$

$$=\frac{2}{3}+\left(-\frac{1}{3}+2-(1)\right)$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

We could also look at this from the "other direction"

$$\text{area} = \int \sim\sim \, dy.$$

$$y = \sqrt{x} + 2. \iff x = (y-2)^2 \qquad (1)$$

$$y = -(x-1)^2 + 3 \iff x = \sqrt{3-y} + 1 \qquad (2)$$

(2) provides the right edge for the partition rectangles.

(1) provides the left edge for the partition rectangles.

So

$$\text{area} = \int_2^3 \left(\sqrt{3-y} + 1 - (y-2)^2 \right) dy$$

$$= \left[-\frac{2}{3}(3-y)^{3/2} + y - \frac{(y-2)^3}{3} \right]_2^3$$

$$= 3 - \frac{1}{3} - \left(-\frac{2}{3} + 2\right)$$

~~oops wrong~~

$$= 1 + \frac{1}{3}$$

$$= \frac{4}{3}$$

7.2. Volumes by cross-sectional area.

General cylinder volume principle.

Volumes with a constant cross-sectional area A. have volume.

$$V = A \times h.$$

where h is the height that the ~~as~~ cross-sections are extended through.

This leads to so called 'disk' and 'washer' methods.

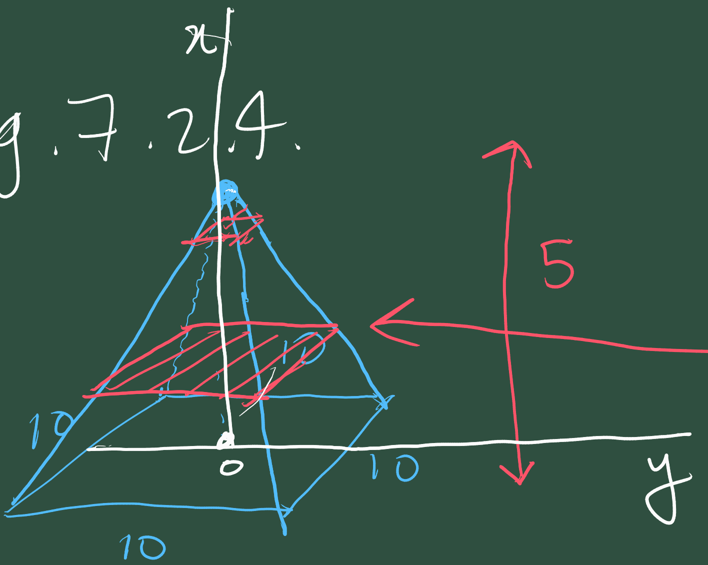
an "annular" region, ie. an area between two concentric circles.

eg in theorem 7.2.3 we can integrate the cross-sectional to get the volume.

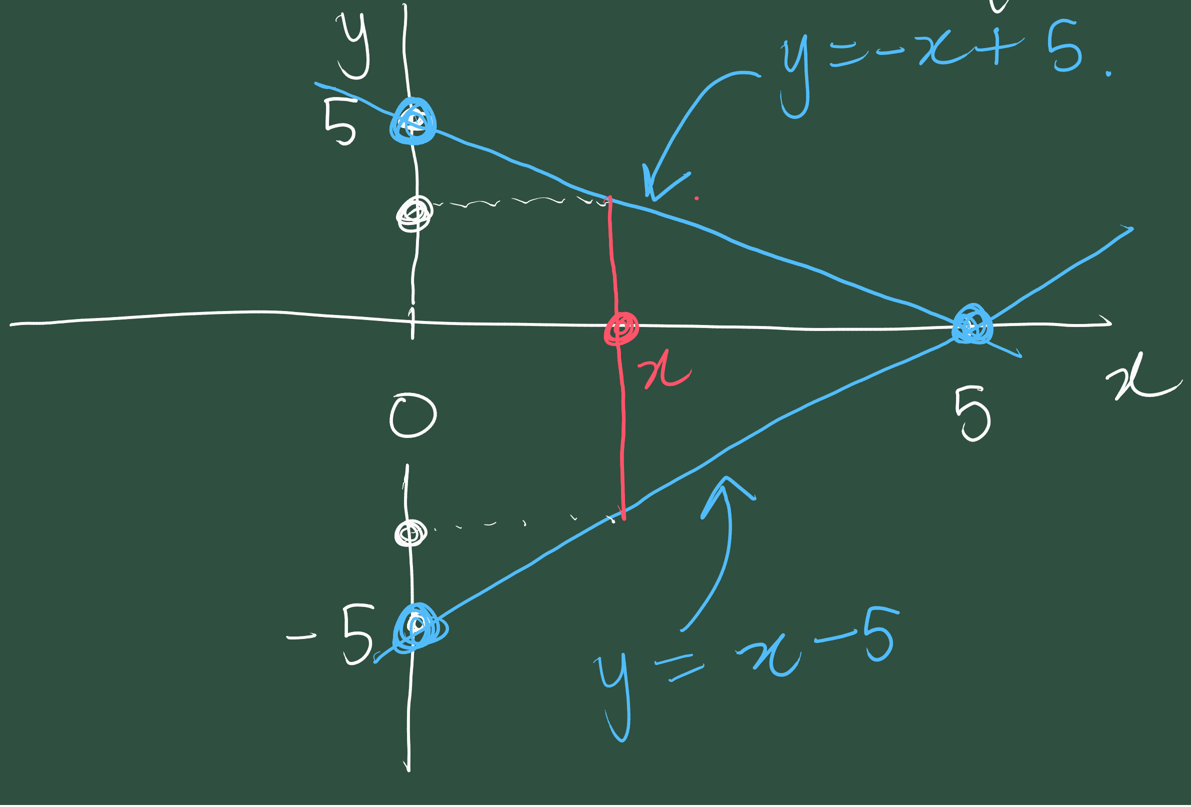
$$V = \int_a^b \underbrace{A(x)}\, dx$$

area of cross-section at height x.

eg. 7.2.4.



Consider a side-on view of this.



Cross-sectional area is

$$A(x) = \left(-x+5-(x-5)\right)^2$$

$$= (-2x+10)^2$$

So by theorem 7.2.3 the volume of the pyramid is given by

$$V = \int_0^5 (-2x+10)^2\,dx$$

$$= \left[\frac{(-2x+10)^3}{-6}\right]_0^5$$

$$= -\frac{10^3}{-6}$$

$$= \frac{500}{3}$$

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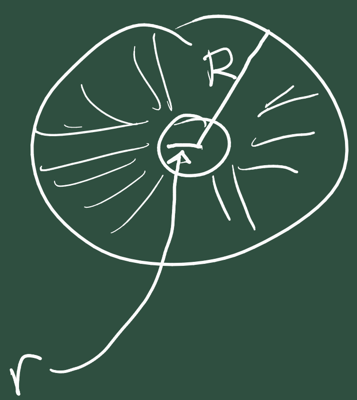
area of a circular disk is πr^2 gives rise to the 'disk' method

$$V = \pi \int_a^b R(x)^2\,dx$$

where $R(x)$ is the radius of the disk cross-section at height x.

However if the cross-sections are not disks but annular regions / 'washers'

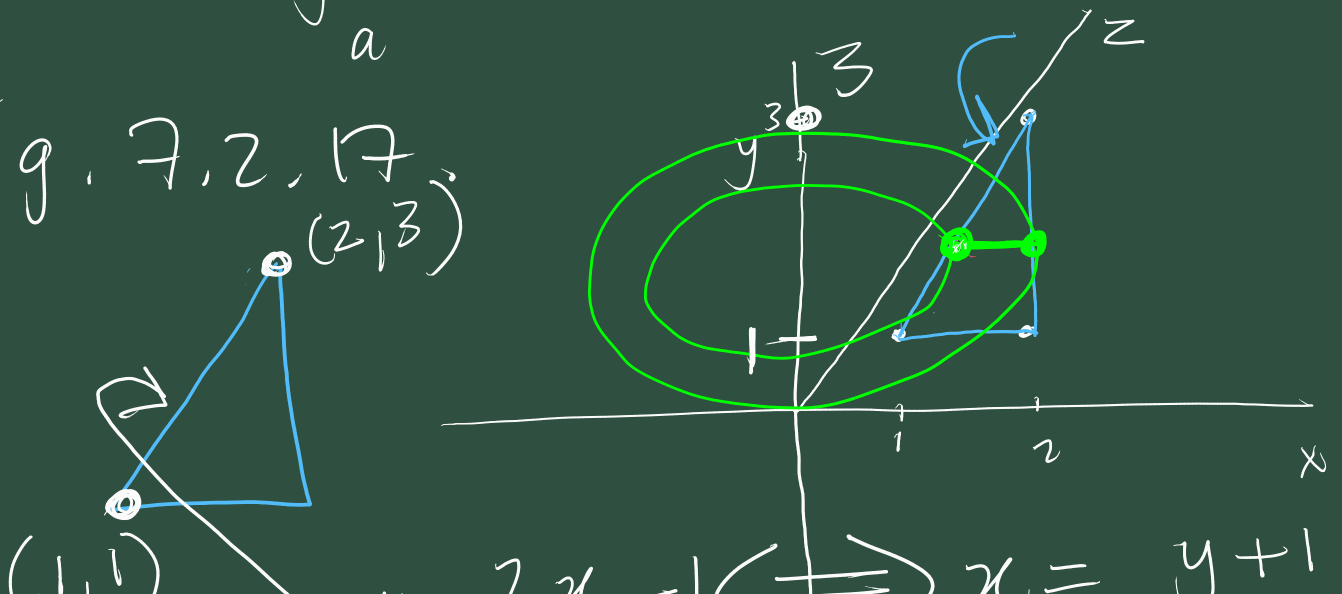
cross-section area $= \pi R^2 - \pi r^2$.



leads to the 'washer' method

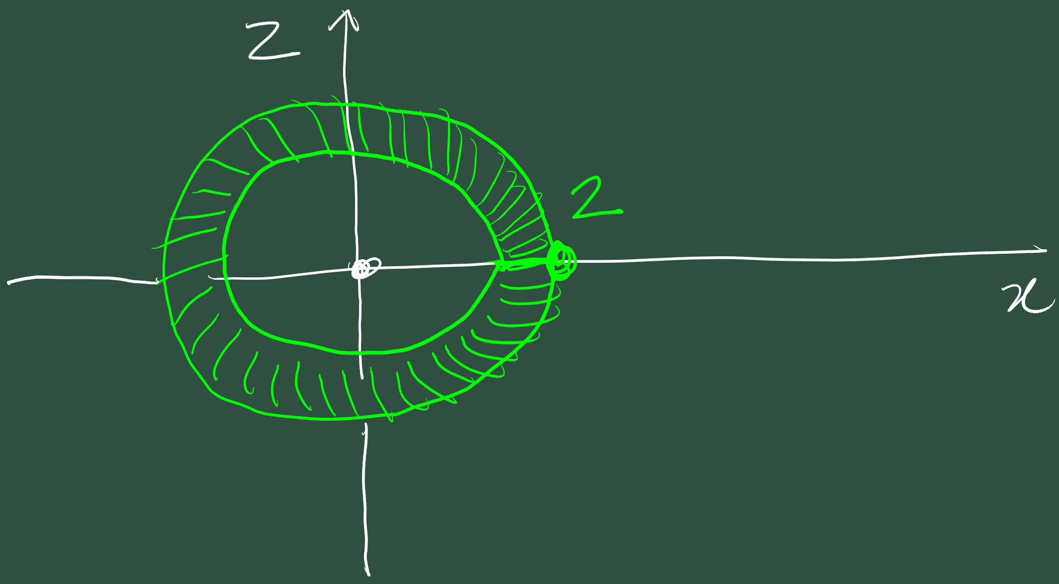
$$V = \pi \int_a^b \left(R(x)^2 - r(x)^2 \right) dx.$$

Eg. 7.2.17.



$$y = 2x - 1 \iff x = \frac{y+1}{2}$$

Consider the top-down view. (y-axis coming out of the page)



Consider taking cross sections through volume parallel to the xz-plane (perpendicular to y-axis). these cross-sections appear as washers, as shown with the outer radius of $R(y) = 2$ and an inner radius of $r(y) = \frac{y+1}{2}$.

So applying washer method gives the

$$V = \pi \int_1^3 R^2 - r^2 \, dy$$

$$= \pi \int_1^3 \left[2^2 - \left(\frac{y+1}{2} \right)^2 \right] dy.$$

$$= \pi \int_1^3 \left(4 - \frac{1}{4}(y+1)^2\right) dy$$

$$= \pi \left[4y - \frac{1}{12}(y+1)^3 \right]_1^3$$

$$= \pi \left(12 - \frac{1}{12} 4^3 - \left(4 - \frac{1}{12} 2^3\right)\right)$$

$$= \pi \left(8 - \frac{1}{12}\left(4^3 - 2^3\right)\right)$$

$$= \pi \left(8 - \frac{1}{12}(64 - 8)\right)$$

$$= \pi \left(8 - \frac{56}{12} \right)$$

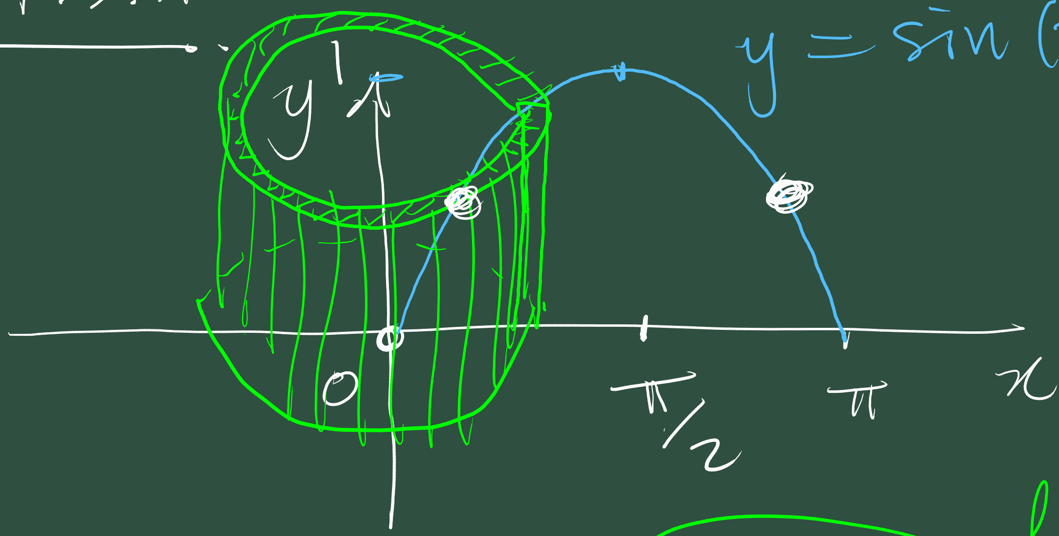
$$= \pi \left(8 - \frac{14}{3} \right)$$

$$= \pi \cdot \frac{10}{3}$$

7.3 The Shell Method

Eg. 7.3.11

$\Longleftrightarrow x = \arcsin(y)$.



$$V = 2\pi \int_0^{\pi} \underbrace{x}_{r(x)} \overbrace{\sin(x)}^{h(x)} \, dx.$$

(prepare for integration by parts).

$$= 2\pi \int_0^{\pi} x \frac{d}{dx}\left(-\cos(x)\right) dx$$

$$= 2\pi \left\{ \left[-x \cos(x) \right]_0^{\pi} \right.$$

$$-\int_0^{\pi} -\cos(x)\,dx.\Big\}$$

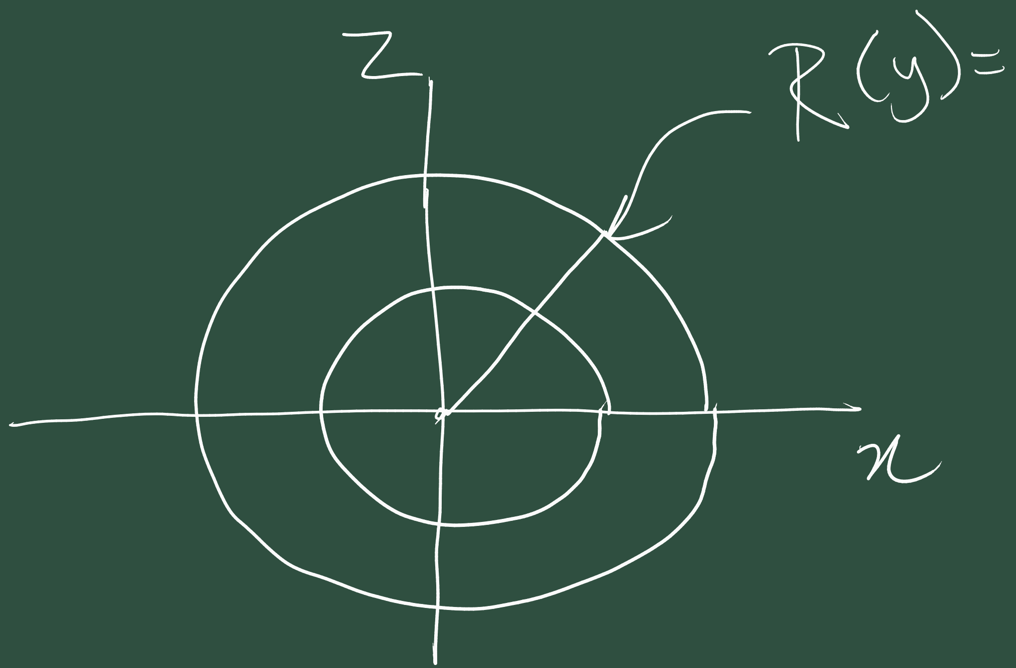
$$= 2\pi\left\{\pi + \int_0^{\pi}\cos(x)\,dx\right\}$$

$$= 2\pi\left\{\pi + \left[\sin(x)\right]_0^{\pi}\right\}$$

$$= 2\pi^2.$$

We could also try the washer method for this, by integrating cross-sections taken perpendicular to the y-axis.

$$V = \pi \int_0^1 \left(R(y)^2 - r(y)^2 \right) dx.$$



$$= \pi \int_0^1 \left(\arcsin(y)^2 - \arcsin(y)^2 \right) dx.$$

upper value in range $\left[\frac{\pi}{2}, \pi\right]$

lower arcsin(y) value from range $[0, \pi/2]$.

Sympy's asin function gives these values →