tinyurl.com/apexcalculus

1.1

We have a first def. of limiting value for functions.

We say "limit of $f(x)$ as x approaches a is L"

to mean the values $f(x)$ get closer and closer to L as x gets closer and closer to a.

In notation it is.

$$\text{"} \lim_{x \to a} f(x) = L \text{"}$$

OR

$$\text{"} f(x) \to L \quad \text{as } x \to a \text{"}$$

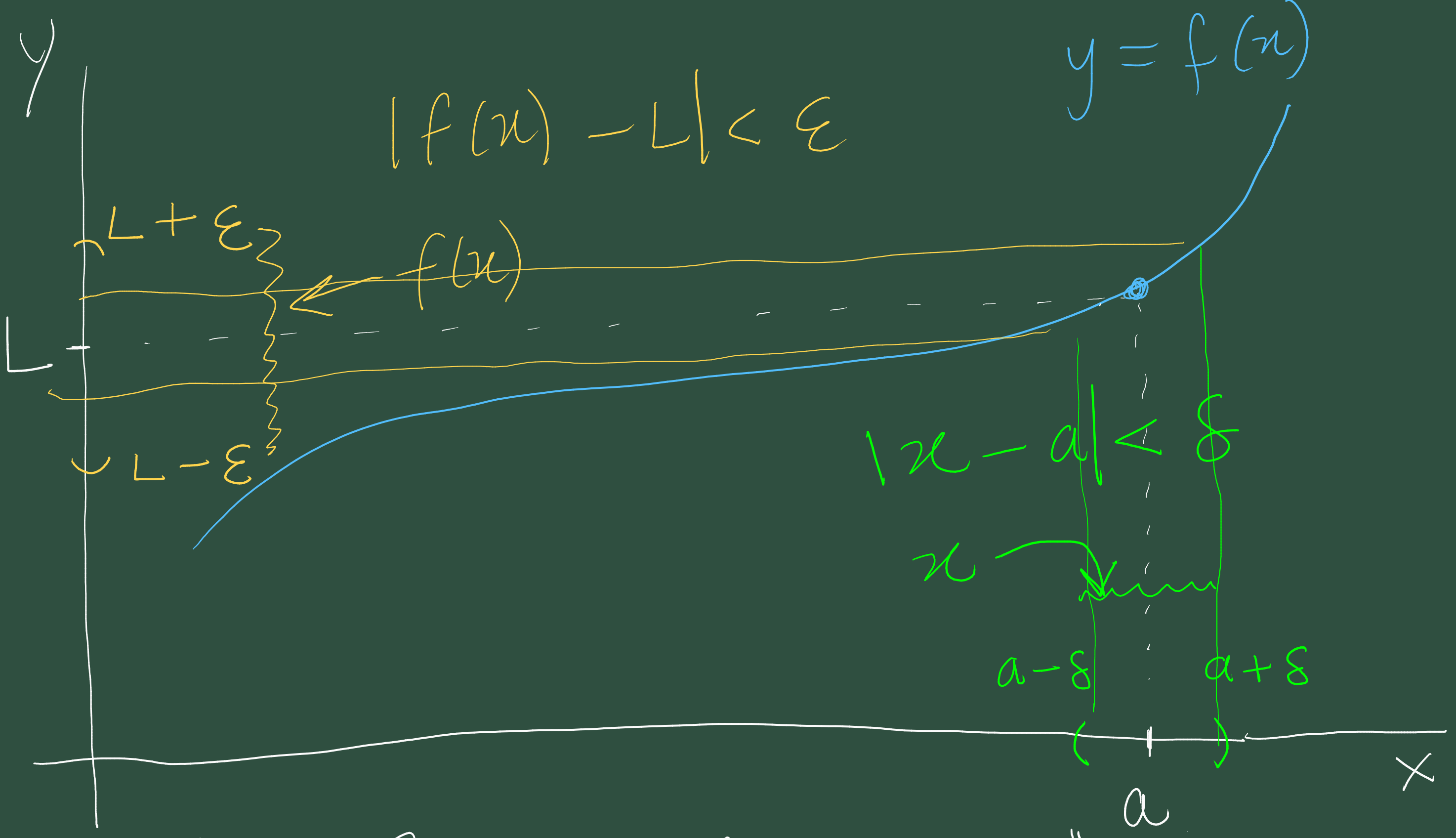
"closer and closer" what exactly does this mean? How "fast" is this happening? Suggests a need for a more precise definition

Section 1.2

The epsilon, delta definition.

ε - 'epsilon' E

δ - 'delta' Δ

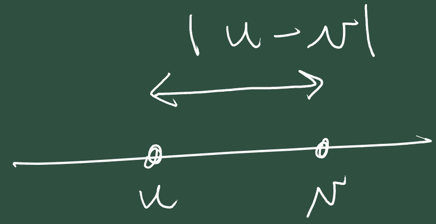


Formal definition of "$\lim_{x \to a} f(x) = L$"

is

"for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every x satisfying $|x - a| < \delta$, we have $|f(x) - L| < \varepsilon$."

- $|u - v|$ is the distance between u and v on the number line



Two things we'll get from this definition

- the ability, in some cases, to directly prove limiting values for functions.
- the ability to prove general statements about the concept of limiting value such as theorem 1.3.1.

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Compact notation for formal def.

$$\lim_{x \to a} f(x) = L$$

$\Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Consider Q8 from Sec. 1.2

Use ε-δ definition to prove.

$$\lim_{x \to 4} (x^2 + x - 5) = 15.$$

So f is the function $f(x) = x^2 + x - 5$.

We'll begin by investigating

$$|f(x) - 15| < \varepsilon$$

$\Leftrightarrow |x^2 + x - 20| < \varepsilon$

$\Leftrightarrow |(x-4)(x+5)| < \varepsilon$.

$\Leftrightarrow |x-4||x+5| < \varepsilon$, using $\forall u, v \; |uv| = |u||v|$

$\Leftrightarrow |x-4| < \dfrac{\varepsilon}{|x+5|}$

is this the δ to use? Not quite.

But note $\dfrac{\varepsilon}{|x+5|}$ still depends on x!

We need to control the effect of $\frac{1}{|x+5|}$ and somehow get rid of it.

Let's assume that $|x-4|<1$

$\iff \quad 3<x<5$

$\iff \quad 8<x+5<10$

$\iff \quad 8<|x+5|<10$

$\iff \quad \frac{1}{10}<\frac{1}{|x+5|}<\frac{1}{8}$

So if $|x-4|<1$ then $\frac{1}{10}<\frac{1}{|x+5|}$

and so $\frac{\varepsilon}{10}<\frac{\varepsilon}{|x+5|}$

Combine all this into the statement.

Let $\delta=\min\left(1,\frac{\varepsilon}{10}\right)$

then

$|x-4|<\delta \implies \left(|x-4|<1 \text{ and } |x-4|<\frac{\varepsilon}{10}\right)$

$$\Rightarrow |f(x) - 15| < \varepsilon$$

This proves that

$f(x) \longrightarrow 15$ as $x \longrightarrow L$

Theorem 1.3.1

Sum case.

Assume as $x \to c$, $f(x) \longrightarrow L$,

$g(x) \to K$.

We will prove. $f(x) + g(x) \longrightarrow L + K$.

Investigate.

$$|f(x) + g(x) - (L+K)| < \varepsilon$$

$$\Leftrightarrow |(f(x) - L) + (g(x) - K)| < \varepsilon$$

Apply the triangle inequality

$$|u + v| \leq |u| + |v|$$

Idea

So if I can arrange

$$\underbrace{|f(x)-L|}_{<\varepsilon/2} + \underbrace{|g(x)-K|}_{<\varepsilon/2} < \varepsilon$$

then I'll know

$$|f(x)-L+g(x)-K| < \varepsilon$$

By definition

$\exists\, \delta_1 > 0$ such that

$$|x-c| < \delta_1 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2}$$

$\exists\, \delta_2 > 0$ such that

$$|x-c| < \delta_2 \Rightarrow |g(x)-K| < \frac{\varepsilon}{2}$$

So choose $\delta = \min(\delta_1, \delta_2)$

So now, if $|x-c| < \delta$

then

$$\boxed{|f(x)-L+g(x)-K|}$$

$$\leq |f(x) - L| + |g(x) - K|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \boxed{\varepsilon}$$

$$\Longleftrightarrow$$

$$|f(x) + g(x) - (L + K)| < \varepsilon$$

from investigations above.

Tutorial

Section 1.2 exercises.