Euclid's proof of the infinite amount of primes

Theorem: Any finite list of primes can be extended with a new prime, not already on the list.

Proof Let $p_1, \dots, p_j$ be a finite list of primes.

Consider the integer P.

$$P = (p_1 p_2 \cdots p_j) + 1.$$

Maybe P is prime, and note that for all $1 \le i \le j$ $P > p_i$. So P is a new prime.

otherwise, P is not prime, but has a prime factor q. We claim that q is not any of the p_i, for $1 \le i \le j$. Let's suppose that $q = p_k$, say, for some $1 \le k \le j$.

Notice that we can write.

$$1 = P - (p_1 \cdots p_j)$$

this implies that q is a factor of 1.

So $q = 1$. But primes, by definition, are not equal to 1.

So our earlier statement $q = p_k$ must be false. So q is a new prime, not on the original list.

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