Tut sheet 9

Q 10

Let's formulate the function f.

$$f(x) = \begin{cases} -x-\pi, & \text{for } -\pi < x < 0 \\ -x+\pi, & \text{for } 0 < x < \pi \quad \leftarrow \end{cases}$$

Consider the oddness / even-ness of f?

It appears that f is odd.

Pf: If $0 < x < \pi$

$$\boxed{f(-x)} = -(-x) - \pi$$
$$= x - \pi$$
$$= -(-x+\pi)$$
$$= \boxed{-f(x).} \quad \text{So } f \text{ is odd.}$$

So the $\boxed{a_0, a_n = 0}$ for $n \geq 1$ and f will have a sine Fourier series.

So let's evaluate the b_n.

For $n \geq 1$

$$\boxed{b_n} = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx.$$

$$= \frac{2}{\pi} \int^{\pi} (-x+\pi)\, \sin(nx)\, dx.$$

(prepare. for int.$^{\text{o}}$ by parts).

$$= \frac{2}{\pi} \int_0^{\pi} (-x+\pi) \frac{d}{dx}\left(\frac{-\cos(nx)}{n} \right) dx.$$

$$= \frac{2}{n\pi} \left(\Big[(-x+\pi)(-\cos(nx)) \Big]_0^{\pi} - \int_0^{\pi} (-\cos(nx))(-1)\, dx \right)$$

$$= \frac{2}{n\pi} \left(\pi - \underbrace{\left[\frac{\sin(nx)}{n} \right]_0^{\pi}}_{=0 \text{ as } \sin(n\pi)=0 \text{ for } n\in\mathbb{Z}.} \right)$$

$$= \boxed{\frac{2}{n}}$$

So the Fourier series for f will be

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$$

Checking the behaviour at $x=0$, the point of discontinuity, the F.S. evaluates to

$$\sum_{n=1}^{\infty} \frac{2}{n} \underbrace{(\sin(0))}_{=0}$$

$$= 0.$$

as expected, as $\lim_{x \to 0_-} f(x) = -\pi$

and $\lim_{x \to 0_+} f(x) = \pi$

From Tut Sheet 10

Q1. Here the function is

$$f(x) = \begin{cases} x+\pi, & \text{for } -\pi < x < 0 \\ -x+\pi, & \text{for } 0 < x < \pi \end{cases}$$

This function f is an even function on $(-\pi, \pi)$ since if $0 < x < \pi$

$$\boxed{f(-x)} = -(-x) + \pi$$

$$= x + \pi$$

$$\boxed{= f(x)} \quad \text{i.e. } f \text{ is even}$$

So the $b_n = 0$ for all $n \geq 1$.

So f will have a cosine Fourier series.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Let's determine these coeffs.

$$\boxed{a_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\pi^2 = \boxed{\pi},$$

area under the graph is the area of a square of side length π.

Then for $n \geq 1$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(-x+\pi)\cos(nx)\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\frac{d}{dx}\left(\frac{\sin(nx)}{n}\right)dx$$

$$+ \frac{1}{\pi}\int_{0}^{\pi}(-x+\pi)\frac{d}{dx}\left(\frac{\sin(nx)}{n}\right)dx.$$

$$= \frac{1}{n\pi}\left(\underbrace{\Big[(x+\pi)\sin(nx)\Big]_{-\pi}^{0}}_{=0} - \int_{-\pi}^{0}\sin(nx)\,dx.\right)$$

$$+\frac{1}{n\pi}\left(\underbrace{\Big[(-x+\pi)\sin(nx)\Big]_0^{\pi}}_{=0}-\int_0^{\pi}(-1)\sin(nx)\,dx\right)$$

$$=\frac{1}{n\pi}\left[\frac{\cos(nx)}{n}\right]_{-\pi}^{0}+\frac{1}{n\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}$$

$$=\frac{1}{n^2\pi}\Big[1-\cos(n\pi)\Big]+\frac{1}{n^2\pi}\Big[-\cos(n\pi)+1\Big]$$

$$=\frac{2}{n^2\pi}\Big[1-\cos(n\pi)\Big]$$

$1, -1, 1, -1, 1, -1$

$$=\begin{cases} 0, & \text{for } n\equiv 0 \pmod 2, \text{ ie even} \\ \dfrac{4}{n^2\pi}, & \text{for } n\equiv 1 \pmod 2, \text{ ie. odd.} \end{cases}$$

$$= a_n$$

So the Fourier series for f is

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(nx)$$

$$=\frac{1}{2}\pi+\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{4}{n^2\pi}\cos(nx)$$

$$= \frac{1}{2}\pi + \sum_{m=1}^{\infty} \frac{4}{(2m-1)^2\pi}\cos\left((2m-1)x\right)$$

Evaluate the series at $x=0$, $\cos(0)=1$.

$$\pi = \frac{1}{2}\pi + \sum_{m=1}^{\infty} \frac{4}{(2m-1)^2\pi}$$

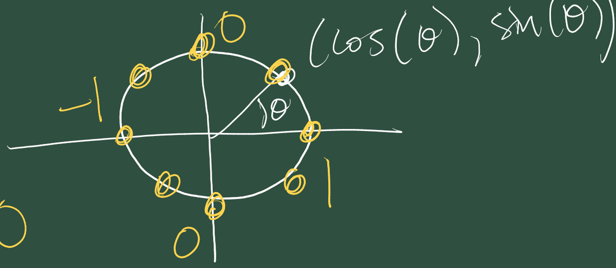
$$\boxed{\frac{\pi^2}{8} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}}$$

A nice formula with π

$$\cos\left(n\frac{\pi}{2}\right)$$

$$= 1, 0, -1, 0, 1, 0, -1, 0$$

$n=0$, $n=1$, ...



$$(4-4t)\,H(t-1)$$

$$= -4(t-1)\,H(t-1)$$

$$= f(t-1)\,H(t-1)$$

where f is the function defined by

$$f(t) = -4t$$