# 6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 04 - SOLUTIONS

Solutions to questions 1 – 6 listed on the following pages under the heading of *Exercise 12*

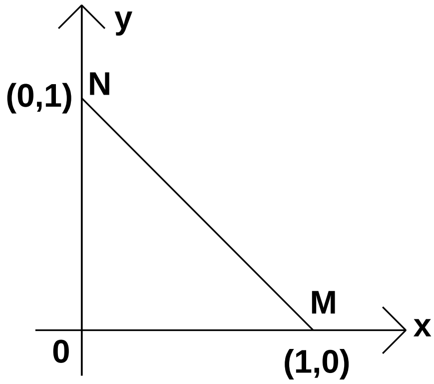
Solutions to questions 7 – 10 listed on the following pages under the heading of *Exercise 13*

BSc Mathematics/BSc Combined Honours

**MA2101 Mathematical Methods**

# Partial Integration Worked Solutions

## Exercise 12




Q1. On OM, y=0 so dy=0.

Hence $$\int_{OM} (2xydx + (3x^2+1)dy = 0$$

On MN x ranges from 1 to 0 and y=1-x so dy = -dx so

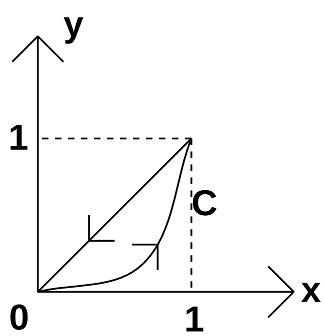
$$\int_{MN} (2xydx + (3x^2+1)dy = \int_1^0 2x(1-x)dx + (3x^2+1)(-dx)$$

$$= \int_0^1 5x^2 - 2x + 1\, dx = \frac{5}{3}$$

On NO x = 0 so dx = 0, y ranges from 1 to 0 so

$$\int_{NO} (2xydx + (3x^2+1)dy = \int_1^0 1\, dy = -1$$

Then $$\int_C (2xydx + (3x^2+1)dy = 0 + \frac{5}{3} - 1 = \frac{2}{3}$$



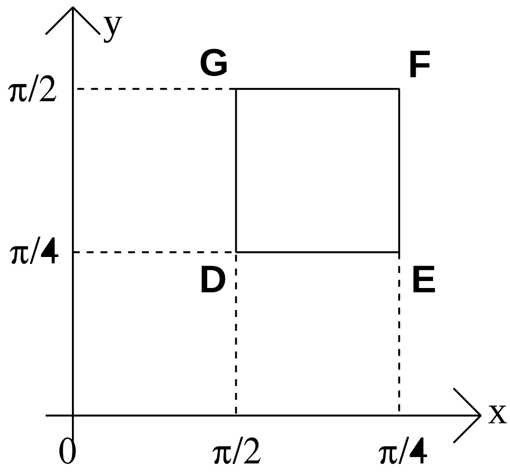

Q2. On $y = x^2$, $dy = 2xdx$ x ranges from 0 to 1.
On $y = x$, $dy = dx$ x ranges from 1 to 0 so

$$\int_C e^{y/x}dx + yx^3dy = \int_0^1 e^x + x^2.x^3.2xdx + \int_1^0 e + x.x^3dx$$

$$= \int_0^1 e^x + 2x^6\, dx + \int_1^0 e + x^4\, dx$$

$$= (e + \frac{2}{7} - 1) + (- e - \frac{1}{5}) = - \frac{32}{35}$$

Q3. Let the square have vertices DEFG as shown.




On DE $y = \pi/4$, $dy = 0$ and x ranges from $\pi/4$ to $\pi/2$ so

$$\int_{DE} \sin y\, dx + \cos x\, dy = \int_{\pi/4}^{\pi/2} \frac{1}{\sqrt{2}}\, dx = \frac{\pi}{4\sqrt{2}}$$

On EF $x = \pi/2$, $dx = 0$ and y ranges from $\pi/4$ to $\pi/2$ so

$$\int_{EF} \sin y\, dx + \cos x\, dy = \int_{\pi/4}^{\pi/2} 0\, dy = 0$$

On FG $y = \pi/2$, $dx = 0$ and y ranges from $\pi/2$ to $\pi/4$ so

$$\int_{FG} \sin y\, dx + \cos x\, dy = \int_{\pi/2}^{\pi/4} 1\, dx = -\frac{\pi}{4}$$

On GD $x = \pi/4$ so $dx = 0$ y ranges from $\pi/2$ to $\pi/4$ so

$$\int_{GD} \sin y\, dx + \cos x\, dy = \int_{\pi/2}^{\pi/4} \frac{1}{\sqrt{2}}\, dx = -\frac{\pi}{4\sqrt{2}}$$

Hence

$$\int_{C} \sin y\, dx + \cos x\, dy = \frac{\pi}{4\sqrt{2}} - \frac{\pi}{4} - \frac{\pi}{4\sqrt{2}} = -\frac{\pi}{4}$$

Now if $P = \sin y$ and $Q = \cos x$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos y + \sin x$$

and if R is the region enclosed by C then

$$\iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\pi/4}^{\pi/2}\int_{\pi/4}^{\pi/2} \cos y + \sin x\, dxdy$$

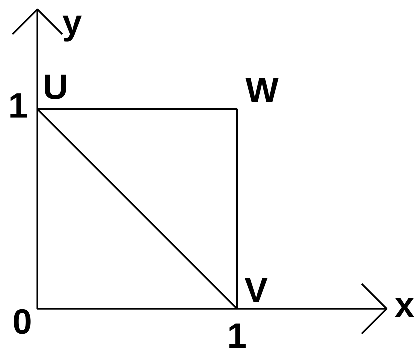
$$= \int_{\pi/4}^{\pi/2} \Big[\, x\cos y - \cos x \,\Big]_{\pi/4}^{\pi/2} dy$$

$$= \int_{\pi/4}^{\pi/2} \frac{\pi}{4}\cos y + \frac{1}{\sqrt{2}}\, dy$$

$$= \left[ \frac{\pi}{4}\sin y + \frac{y}{\sqrt{2}} \right]_{\pi/4}^{\pi/2} = \frac{\pi}{4}$$

So Green's Theorem is verified in this instance.

Q4. Let the triangle have vertices UVW as shown.
On UV, $y = 1 - x$, so $dy = - dx$ and x ranges from 0 to 1




$$\int_{UV} (x^3y + x^4)dx + (x + y + 1)dy = \int_0^1 (x^3(1-x) + x^4)dx + 2(-dx)$$

$$= \int_0^1 (x^3 - 2)\, dx = \frac{-7}{4}$$

On VW, $x = 1$, so $dx = 0$ and y ranges from 0 to 1

$$\int_{VW} (x^3y + x^4)dx + (x + y + 1)dy = \int_0^1 (y + 2)\, dy = \frac{5}{2}$$

On WU, $y = 1$, so $dy = 0$ and x ranges from 1 to 0

$$\int_{WU} (x^3y + x^4)dx + (x + y + 1)dy = \int_1^0 (x^3 + x^4)\, dx = -\frac{9}{20}$$

Then

$$\int_{C} (x^3y + x^4)dx + (x + y + 1)dy = \frac{5}{2} - \frac{7}{4} - \frac{9}{20} = \frac{3}{10}$$

Now if $P = x^3y + x^4$ and $Q = x + y + 1$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - x^3$$

and if R is the region enclosed by C then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^1 \int_{1-x}^1 (1 - x^3)\, dydx$$

$$= \int_0^1 \left[ y(1-x^3) \right]_{1-x}^{1} dx$$

$$= \int_0^1 x(1-x^3)\, dx$$

$$= \frac{3}{10}$$

So Green's theorem is verified in this case.

Q5. The region R lies between 2 consecutive circles centre 0 and with radii 1 and √3 and is therefore not simply connected.
Also the point (-1, 1) lies in R because there $x^2 + y^2 = 2$ which is between 1 and √3. But there $xy + 1 = 0$ so the integrand $\ln(xy + 1)$ is not finite.

Q6. Let $P = x^2y \cos x + 2xy \sin x - y^2e^x$
and $Q = x^2 \sin x - 2ye^x$
Then $\frac{\partial P}{\partial y} = x^2 \cos x + 2x \cos x - 2ye^x$
and $\frac{\partial Q}{\partial x} = 2x \sin x + x^2 \cos x - 2ye^x$
Then by Green's theorem if R is the region enclosed by C

$$\int_C (x^2y \cos x + 2xy \sin x - y^2e^x)dx + (x^2 \sin x - 2ye^x)dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dxdy = 0$$

BSc Mathematics/BSc Combined Honours

**MA2101 Mathematical Methods**

# Partial Integration Worked Solutions

## Exercise 13

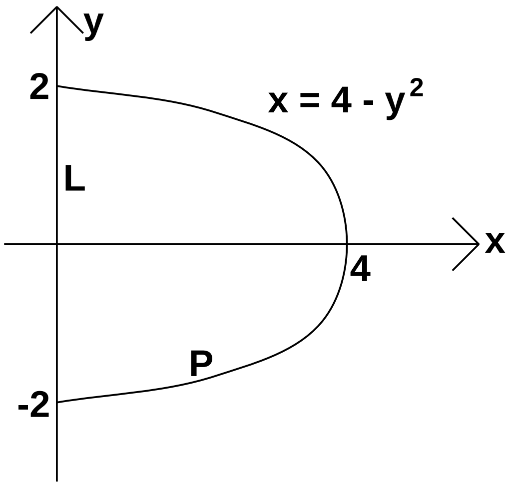
Q1. Let $x = r\cos\theta$ and $y = r\sin\theta$
Then, $dx = -r\sin\theta\, d\theta$, $dy = r\cos\theta\, d\theta$ and
as $\theta$ varies from 0 to $2\pi$, the point $(x, y)$ traverses the circle $x^2 + y^2 = r^2$

$$\text{Area of circle} = \frac{1}{2}\int_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_C r\cos\theta\,(r\cos\theta\, d\theta) - r\sin\theta\,(r\sin\theta\, d\theta)$$

$$= \frac{r^2}{2}\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\, d\theta = \pi r^2$$

Q2. $x = 4 - y^2$ is a parabola, P say, meeting the y axis when $4 - y^2 = 0$ i.e. $y = \pm 2$ so the required region is that shown in the diagram. C, the contour of this region is made up of P and L, the section of the y axis from $y = -2$ to $y = 2$.




On P, $x = 4 - y^2$ so $dx = -2y\,dy$ and y ranges from -2 to 2.

$$\int_P x\,dy - y\,dx = \int_{-2}^{2} (4 - y^2)dy - y(-2y)dy$$

$$= \int_{-2}^{2} (4 + y^2)dy = 64/3$$

On L, $x = 0$ so $dx = 0$ and $x\,dy - y\,dx = 0$. Hence the integral on L is 0.

$$\text{Area of region} = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_P x\,dy - y\,dx + \frac{1}{2}\int_L x\,dy - y\,dx$$

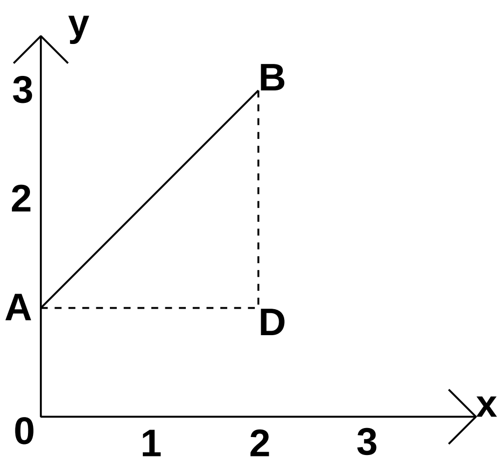
$$= \frac{1}{2}(64/3 + 0) = 32/3$$

Q3. Let $P = 2xy - y^4 + 3$ and $Q = x^2 - 4xy^3$

Then $\dfrac{\partial P}{\partial y} = 2x - 4y^3 = \dfrac{\partial Q}{\partial x}$

so the integral is independent of path.

Let D be the point (2,1). Then on AD, x ranges from 0 to 2 and $y = 1$ so $dy = 0$. Hence




$$\int_{AD} (2xy - y^4 + 3)dx + (x^2 - 4xy^3)dy = \int_0^2 (2x - 1 + 3)dx$$

$$= 8$$

On DB y ranges from 1 to 3 and $x = 2$ so $dx = 0$. Hence

$$\int_{DB} (2xy - y^4 + 3)dx + (x^2 - 4xy^3)dy = \int_1^3 (4 - 8y^3)dy = -152$$

Then

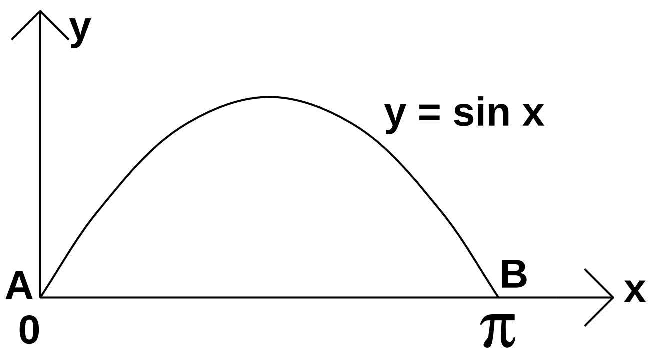
$$\int_{AB} (2xy - y^4 + 3)dx + (x^2 - 4xy^3)dy = -152 + 8 = -144$$

Q4. Let $P = 3x^2 + ye^y$ and $Q = x(1 + y)e^y$.

Then $\dfrac{\partial Q}{\partial x} = ye^y + e^y = \dfrac{\partial P}{\partial y}$

so the integral is independent of path.

We can therefore take the path from A to B to be $y = 0$ from $x = 0$ to $x = \pi$ and on this line $dy = 0$.




$$\int_{AB} (3x^2 + ye^y)dx + x(1 + y)e^y dy = \int_0^{\pi} 3x^2\, dx$$

$$= \pi^3$$