Taylor Series (for multi variable functions)

Recalling 1-variable case.

Given a nice function $f(x)$ (continuous, differentiable)

it has a Taylor series, based at point 0, given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad \text{where} \quad f^{(n)} = \frac{d^n f}{dx^n}$$

valid for x in some interval around 0.

Finite parts of this ~~se~~ series will provide approximations to $f(x)$

$$f(x) \approx \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n$$

Also called Mclaurin series for a Taylor series based at 0.

the larger k is, the better the approximation will be.

What do Taylor series look like for multi-variable functions?

Consider $f(x,y)$ and its behavior near a base point (a,b)

we'll write $h = \Delta x$, $k = \Delta y$ then the Taylor series for f, about (a,b), will be:

$$f(a+h, b+k) = \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} (D^n f)(a,b)}_{\text{poly. in } h,\, k.}$$

where D is the differential operator

$$D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$$

and D^n we mean the n^{th} iteration of D

$$\text{So } D^2 f = D(D(f)), \quad D^3 f = D(D^2 f) = D(D(D(f))).$$

eg. $D^0 f := f.$

$$Df = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

lin. comb. of operators

lin. comb of partial derivatives

$$\boxed{D^2 f} = D(Df) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$

$$\begin{matrix} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{matrix}$$

$$= h^2 \frac{\partial^2 f}{\partial x^2} + hk \frac{\partial^2 f}{\partial x \partial y} + kh \frac{\partial^2 f}{\partial y \partial x} + k^2 \frac{\partial^2 f}{\partial y^2}$$

$$\boxed{= h^2 \frac{\partial^2 f}{\partial x^2} + k^2 \frac{\partial^2 f}{\partial y^2} + 2hk \frac{\partial^2 f}{\partial x \partial y}}.$$

$$D^3 f = h^3 \frac{\partial^3 f}{\partial x^3} + 3kh^2 \frac{\partial^3 f}{\partial x^2 \partial y} + 3k^2 h \frac{\partial^3 f}{\partial x \partial y^2} + k^3 \frac{\partial^3 f}{\partial y^3}$$

In general

$$\boxed{D^n f = \sum_{j=0}^{n} \binom{n}{j} \underbrace{h^j k^{n-j}}_{\text{polys in } h,k} \frac{\partial^n f}{\partial x^j \partial y^{n-j}}}$$

$\to$ to be evaluated at (a,b)

where the binom. coefficients $\binom{n}{j} := \dfrac{n!}{j!\,(n-j)!}$

Example Consider $f(x,y) = \sin(x+3y) + \cos(3x+y)$

Find the beginning (up to incl. order 2 terms) of its Taylor series about the base point $(a,b) = (\pi/2, 0)$

To write down the poly approx. for $f(\pi/2 + h, k)$ we'll need the coefficients $D^0 f$, Df, $D^2 f$ all evaluated at $(\pi/2, 0)$

$D^0 f = f$, at $(\pi/2, 0) \rightsquigarrow f(\pi/2, 0) = \sin(\pi/2) + \cos(3\pi/2)$

$= 1 + 0 = \boxed{1}$

$$Df = h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}.$$

$$\frac{\partial f}{\partial x} = \Big[\cos(x+3y) - 3\sin(3x+y)\Big]_{\pi/2,\,0} = \cos(\pi/2) - 3\sin(3\pi/2) = 0 + 3 = \boxed{3}$$

$$\frac{\partial f}{\partial y} = \Big[3\cos(x+3y) - \sin(3x+y)\Big]_{(\pi/2,\,0)} = \boxed{1}$$

$$D^2 f = h^2 \frac{\partial^2 f}{\partial x^2} + \underline{2} hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \Big[-\sin(x+3y) - 9\cos(3x+y)\Big]_{(\pi/2,\,0)} = \boxed{-1}$$

$$\boxed{\frac{\partial^2 f}{\partial y^2}} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \Big[-9\sin(x+3y) - \cos(3x+y)\Big]_{(\frac{\pi}{2},0)} \boxed{= -9}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y}} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \Big[-3\sin(x+3y) - 3\cos(3x+y)\Big]_{(\frac{\pi}{2},0)} \boxed{= -3}$$

Putting all this together gives.

$$f\left(\frac{\pi}{2}+h, k\right) \approx 1 + 3h + k - \frac{h^2}{2} - \frac{9k^2}{2} - 3hk.$$

Let's answer this with some plotting.

These generalize to multi-variable setting.

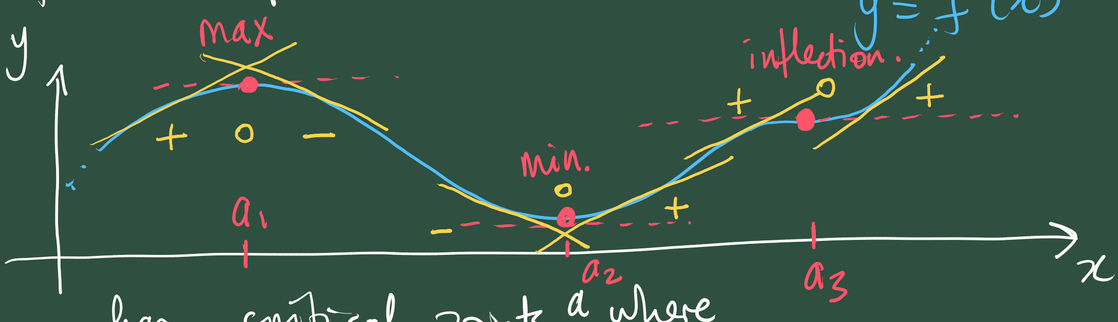
say $f(x_1,\dots,x_n)$, use $f(a_1+h_1, a_2+h_2, \dots, a_n+h_n)$

$$D = \sum_{j=1}^{n} h_j \frac{\partial}{\partial x_j}$$

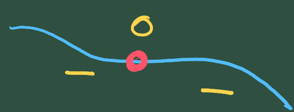
and
$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(D^n f\right)(a_1, a_2, \dots, a_n)$$

Optimisation "Finding and classifying critical points on surface / functions of 2-variables"

1-variable case



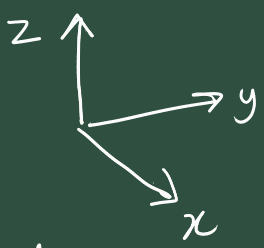
A function / curve has critical points a where



$\left.\frac{df}{dx}\right|_{x=a} = 0$ and classified according to

$$\left.\frac{d^2f}{dx^2}\right|_{x=a} = \begin{cases} >0, & \text{local min.} \\ <0, & \text{local max.} \\ =0, & \text{inflection point.} \end{cases}$$

<u>2-variable case</u> Consider $g(x,y)$ and its surface defined by $z = g(x,y)$



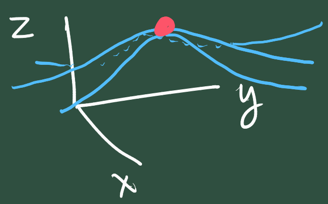
<u>Def</u> A critical point of g is a point (a,b) where both partial derivatives vanish, $=0$.

$$\frac{\partial g}{\partial x}(a,b) = 0 \quad \& \quad \frac{\partial g}{\partial y}(a,b) = 0.$$

ie. geometrically ⇝ the tangent plane to surface at (a,b) is horizontal, ie. parallell to xy-plane.

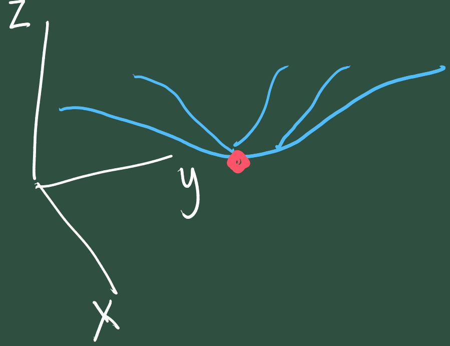
Classified into three types.

<u>local max</u>



"top of a hill"

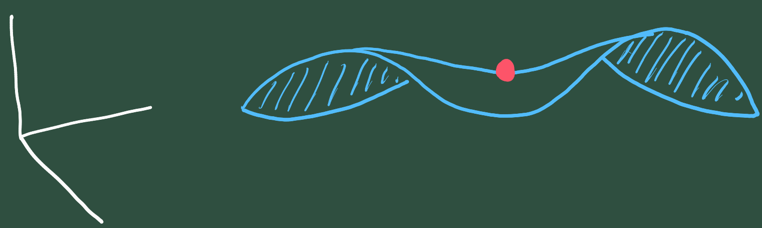
<u>local min.</u>



"bottom of a depression"

<u>saddle point</u>

"centre of a saddle"
"mixture of a min & max"



and classified by a second-order differential method

<u>Hessian determinant</u> $D :=$

$$D := \begin{vmatrix} \dfrac{\partial^2 g}{\partial x^2} & \dfrac{\partial^2 g}{\partial x \partial y} \\[2ex] \dfrac{\partial^2 g}{\partial y \partial x} & \dfrac{\partial^2 g}{\partial y^2} \end{vmatrix}$$

$$= \frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial y^2} - \left(\frac{\partial^2 g}{\partial x \partial y}\right)^2$$

according to

if $D(a,b) < 0$ then (a,b) is a saddle.

If $D(a,b) > 0$ then $\begin{cases} \left.\frac{\partial^2 g}{\partial x^2}\right|_{(a,b)} > 0, & (a,b) \text{ is a min.} \\ \left.\frac{\partial^2 g}{\partial x^2}\right|_{(a,b)} < 0, & (a,b) \text{ is a max.} \end{cases}$

This classification can be justified with the use of Taylor polys.

Example Consider

$$f(x,y) = x^3 + 3xy^2 - 15x - 12y.$$

Find its critical points and their classification.

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 15 = 0$$

&

$$\frac{\partial f}{\partial y} = 6xy - 12 = 0$$

Proceed by examining the simpler equation

$$\frac{\partial f}{\partial y} = 0 \iff \boxed{y = \frac{2}{x}} \text{ and } x \neq 0.$$

So under this condition.

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 15 = 3x^2 + \frac{12}{x^2} - 15 = 0$$

$$\iff 3x^4 - 15x^2 + 12 = 0$$

$$\iff x^4 - 5x^2 + 4 = 0$$

$$\iff (x^2-4)(x^2-1)=0$$

$$\iff \boxed{x=-2,2,-1 \text{ or } 1.}$$

So f has four critical points

$$(a,b) = (-2,-1), \quad (2,1), \quad (-1,-2), \quad (1,2)$$

and

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$= 6x\,6x - (6y)^2$$

$$= 36x^2 - 36y^2$$

$$= 36(x^2-y^2)$$

So $D(-1,-2) = D(1,2) = {}^{-3\cdot 36} < 0$

So $(-1,-2)$ and $(1,2)$ are saddles.

$D(2,1) = D(-2,-1) = 36\cdot 3 > 0$

and $\dfrac{\partial^2 f}{\partial x^2} = 6x \begin{cases} >0 & \text{at } (2,1), \text{ so is a } \underline{\text{min}} \\ <0 & \text{at } (-2,-1), \text{ so is a } \underline{\text{max}}. \end{cases}$

<u>Exercise</u> Obtain surface plots of f close to these critical points and "see" the min, max or saddle nature of these points.