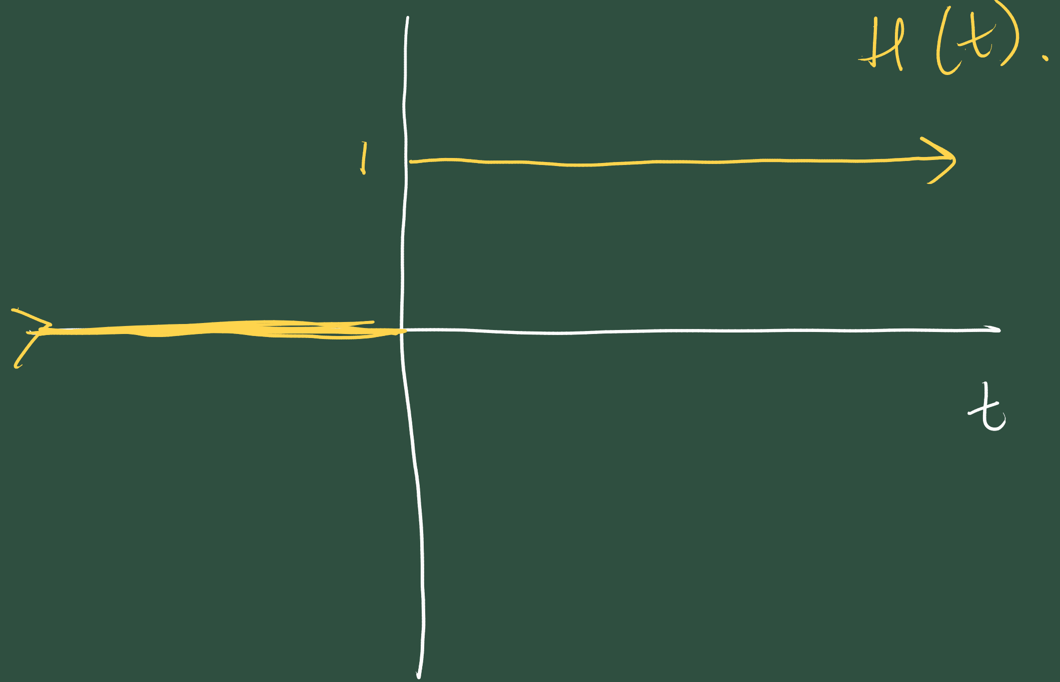
Dirac delta function δ

Heaviside step function H.



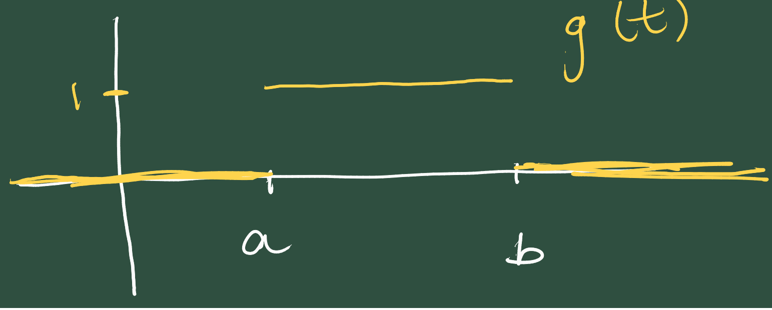
$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$$H(t-a) = \begin{cases} 0, & t-a < 0 \iff t < a \\ 1, & t-a \geq 0 \iff t \geq a \end{cases}$$

<u>Examples exploiting</u> H

1. $g(t) = H(t-a) - H(t-b)$

where $a < b$.

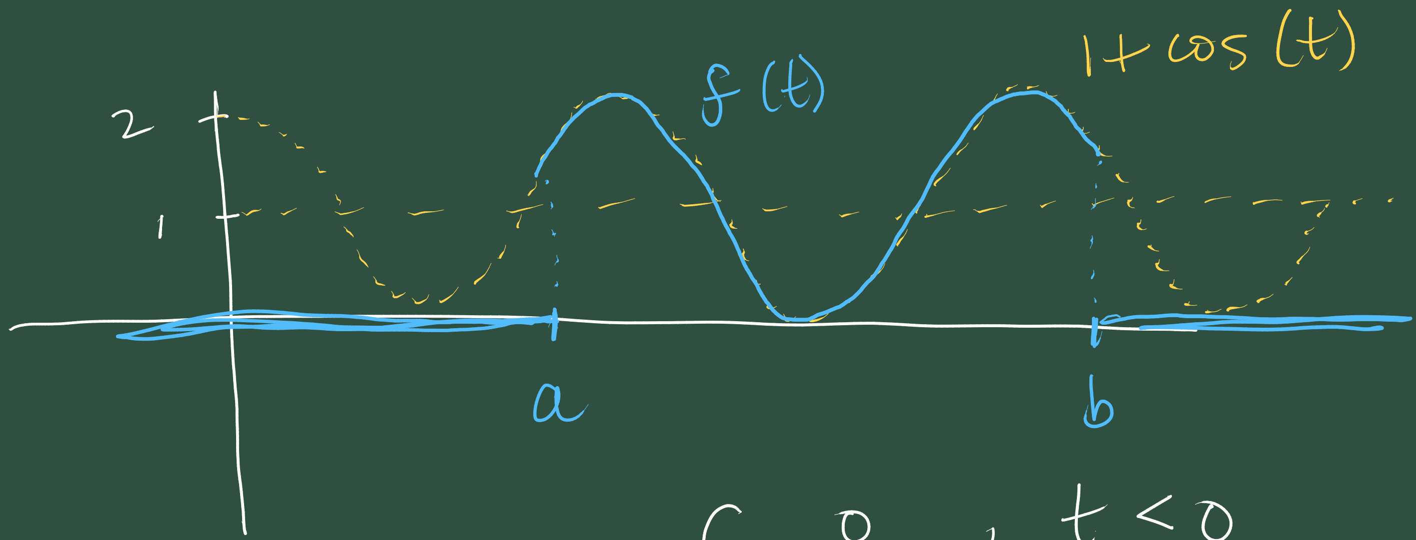


$$g(t) = \begin{cases} 0, & t < a \\ 1, & a \le t < b \\ 0, & t \ge b \end{cases}$$

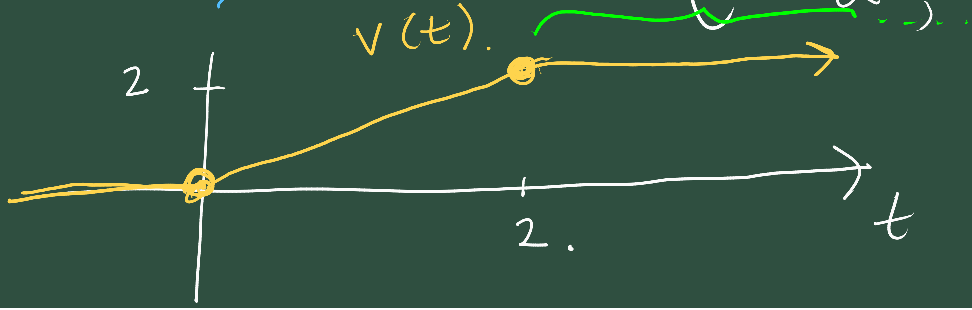
2. Let $a < b$

$$f(t) = (1+\cos(t))\big(H(t-a) - H(t-b)\big)$$

$$f(t) = \begin{cases} 0, & t < a \\ 1+\cos(t), & a \le t < b \\ 0, & t \ge b \end{cases}$$



3. $$v(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < 2 \\ 2, & t \ge 2 \end{cases}$$



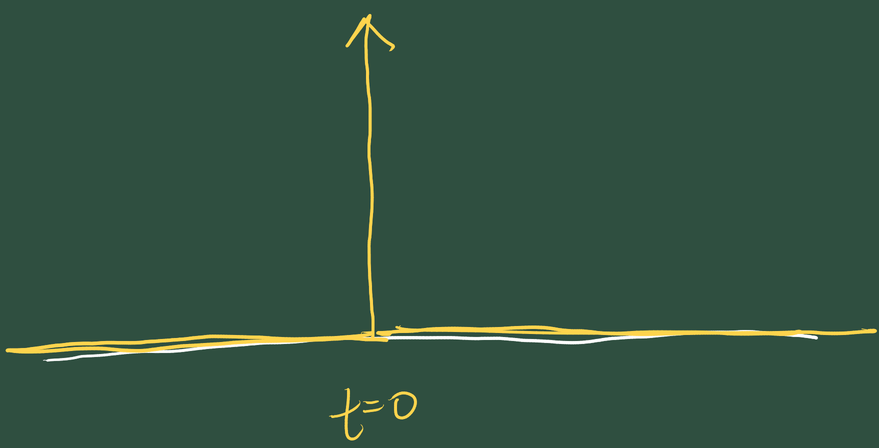
$$v(t) = t\left(H(t) - H(t-2)\right) + 2H(t-2)$$

$$= t\,H(t) - (t-2)\,H(t-2).$$

same shifted

same shifted argument

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$



To solve ODEs featuring H, δ we need to understand their transforms.

<u>Theorem 4.7.9</u>

$$\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}$$

Pf: Using the definition

$$\mathcal{L}\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\, dt.$$

Assuming $a>0$

$$= \int_0^a e^{-st} \cdot 0 \, dt + \int_a^\infty e^{-st} \cdot 1 \, dt$$

$$= \int_a^\infty e^{-st}\, dt.$$

$$= \left[\frac{e^{-st}}{-s} \right]_{t=a}^{t\to\infty}$$

$$= 0 - \frac{e^{-sa}}{-s} \quad , \text{ for } s>0$$

$$= \frac{e^{-as}}{s} \quad , \text{ as claimed. } \blacksquare$$

$$\int_0^\infty e^{-st} f(t)\, dt$$

Theorem 4.7.10

If $\mathcal{L}\{f(t)\} = F(s)$ then.

$$\mathcal{L}\{f(t-a) H(t-a)\} = e^{-as} F(s)$$

Proof: Again, using def of $\mathcal{L}$.

$\mathcal{L}\{f(t-a)H(t-a)\}$, assuming $a>0$.

$$= \int_0^\infty e^{-st} f(t-a) \underbrace{H(t-a)}_{0 \text{ or } 1} dt.$$

$$= \overbrace{\int_0^a e^{-st} f(t-a)\cdot 0 \, dt}^{=0} + \int_a^\infty e^{-st} f(t-a)\, dt$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt.$$

$$\left(\begin{array}{l} \text{Perform the substitution} \\ \qquad u = t-a. \\ \text{as } t \to \infty, \quad u \to \infty \\ \text{when } t=a, \quad u=0 \end{array} \right)$$

$$= \int_0^\infty e^{-s(u+a)} f(u)\, du$$

$$= \int_0^\infty e^{-su} \underbrace{e^{-sa}}_{\text{indep. of } u} f(u)\, du$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-sa} F(s).$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Ex 4.7.12

rl. Sometimes some prep. is needed before using the theorem.

$$R(t) = t^2 H(t-2)$$

$$\mathcal{L}\{R(t)\} = ?$$

We know

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}$$

Can't immediately apply the previous theorem. First we must express t^2 in terms of the argument $t-2$.

$$t^2 = (t-2)^2 + 4t - 4.$$

$$= (t-2)^2 + 4(t-2) + 4$$

$= g(t-2)$

where g is the polynomial defined by $g(t) = t^2 + 4t + 4$

This allows to write ready for Th 4.7.10

$$R(t) = \boxed{g(t-2)\,H(t-2)}$$

So $\mathcal{L}\{R(t)\} = \mathcal{L}\{g(t-2)\,H(t-2)\}$

$$= e^{-2s}\,G(s)$$

where $G(s) = \mathcal{L}\{g(t)\}$

$$= \mathcal{L}\{t^2 + 4t + 4\}$$

$$= \mathcal{L}\{t^2\} + 4\mathcal{L}\{t\} + 4\mathcal{L}\{1\}$$

$$= \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$$

So $\mathcal{L}\{R(t)\}$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

Ex 4.7.13 $\quad \dot{y} = \frac{dy}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$

Solve $\quad \ddot{y} - 4\dot{y} + 3y = 6H(t-5)$

subject to initial conditions $y(0) = \dot{y}(0) = 0$

Follow the 4-steps.

$$\mathcal{L}\{\ddot{y} - 4\dot{y} + 3y\} = \mathcal{L}\{6H(t-5)\}$$

By linearity of $\mathcal{L}$

$$\mathcal{L}\{\ddot{y}\} - 4\mathcal{L}\{\dot{y}\} + 3\mathcal{L}\{y\} = 6\mathcal{L}\{H(t-5)\}$$

We'll write $\bar{y} = \mathcal{L}\{y\}$ and use transform of derivatives properties.

$$s^2\bar{y} - sy(0) - \dot{y}(0) - 4(s\bar{y} - y(0))$$

$$+ 3\bar{y} = \frac{6e^{-5s}}{s}$$

Inserting the initial conditions on $y, \dot{y}$ gives.

$$s^2\bar{y} - 4s\bar{y} + 3\bar{y} = \frac{6e^{-5s}}{s}$$

$$\Rightarrow (s^2 - 4s + 3)\bar{y} = \frac{6e^{-5s}}{s}$$

$$\Rightarrow \bar{y} = \frac{6e^{-5s}}{s(s^2 - 4s + 3)}$$

The solution to the ODE will be

$$y(t) = \mathcal{L}^{-1}\{\bar{y}\}$$

$$= \mathcal{L}^{-1}\Big\{ e^{-5s} \underbrace{\boxed{\frac{6}{s(s-3)(s-1)}}}_{\bar{f}(s)} \Big\}$$

So by Theorem 4.7.10

$$y(t) = \underbrace{\boxed{f(t-5)}}_{?} H(t-5)$$

So we'll find f by finding

$$f(t) = \mathcal{L}^{-1}(\bar{f}(s))$$

$$= \mathcal{L}^{-1}\Big\{ \frac{6}{s(s-3)(s-1)} \Big\}$$

First we need the partial fraction expansion of $\bar{f}(s)$

$$\bar{f}(s) = 6 \frac{1}{s(s-3)(s-1)} \qquad (*)$$

$$= 6\left(\frac{\alpha}{s} + \frac{\beta}{s-3} + \frac{\gamma}{s-1} \right)$$

$$= 6\left(\frac{\alpha(s^2-4s+3)+\beta(s^2-s)+\gamma(s^2-3s)}{s(s-3)(s-1)}\right)$$

$$= 6\left(\frac{(\alpha+\beta+\gamma)s^2+(-4\alpha-\beta-3\gamma)s+3\alpha}{s(s-3)(s-1)}\right)$$

$**$

So numerator poly of $(**)$ $= 1$.

$$\Rightarrow \begin{cases} \alpha+\beta+\gamma = 0 \\ -4\alpha-\beta-3\gamma=0 \\ 3\alpha = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha = 1/3 \\ \beta+\gamma = -1/3 \\ -\beta-3\gamma = 4/3 \end{cases}$$

$$\Rightarrow -2\gamma = 1$$

$$\Rightarrow \gamma = -1/2$$

$$\Rightarrow \beta - 1/2 = -1/3$$

$$\Rightarrow \beta = -\tfrac{1}{3} + \tfrac{1}{2} = \frac{-2+3}{6}$$

$$= \tfrac{1}{6}.$$

So

$$\bar{f}(s) = 6\left(\frac{1/3}{s} + \frac{1/6}{s-3} - \frac{1/2}{s-1} \right)$$

So $\boxed{f(t)} = \mathcal{L}^{-1}\{\bar{f}(s)\}$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$-3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$, by linearity

$$= \boxed{2 + e^{3t} - 3e^{t}}$$, reading transforms from table.

So finally we can express our ODE solution as

$$y(t) = f(t-5)\,H(t-5)$$

$$= \left(2 + e^{3t-15} - 3e^{t-5}\right)H(t-5).$$