6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 03 - SOLUTIONS

Solutions to questions 1 - 6 listed on the following pages under the heading of *Exercise 8*

Q5. $\frac{\partial(r,\theta)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(r,\theta)}{\partial(r,\theta)}$

$$= \begin{vmatrix} \frac{\partial r}{\partial r} & \frac{\partial r}{\partial \theta} \\ \frac{\partial \theta}{\partial r} & \frac{\partial \theta}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Hence $\frac{\partial(r,\theta)}{\partial(x,y)} \cdot r = 1$

so $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$

Q6. The graph is of $t = s^2$.

Now $$\frac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -2(y-x) & 2(y-x) \end{vmatrix} = 0$$

so the inverse does not exist. Hence there is no inverse transformation. This arises because s and t are not independent variables.

Exercise 8

Q1. $f(x,y) = \ln(x + y^2)$.

Using Taylor's theorem

$$f(1+h, 0+k) = f(1,0) + h\frac{\partial f}{\partial x}(1,0) + k\frac{\partial f}{\partial y}(1,0) + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}(1,0) + \frac{2hk}{2!}\frac{\partial^2 f}{\partial x \partial y}(1,0) + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}(1,0) + \text{higher order terms}$$

Now $f(1,0) = \ln(1) = 0$

$$\frac{\partial f}{\partial x} = \frac{1}{x+y^2} = 1 \quad \text{at } (1,0)$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x+y^2} = 0 \quad \text{at } (1,0)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{(x+y^2)^2} = -1 \quad \text{at } (1,0)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-2y}{(x+y^2)^2} = 0 \quad \text{at } (1,0)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x+y^2)2 - 2y(2y)}{(x+y^2)^2} = 2 \quad \text{at } (1,0)$$

Then $$f(1+h,k) = 0 + h.1 + k.0 + \frac{h^2}{2!}(-1) + \frac{2hk}{2!}.0 + \frac{k^2}{2!}.2 + \text{higher order terms}$$

$$\cong h + \frac{1}{2}(-h^2 + 2k^2) \quad \text{if h and k are small.}$$

Q2. $f(x,y) = e^{xy}$

Using Taylor's theorem

$$f(0+h, 0+k) = f(0,0) + h\frac{\partial f}{\partial x}(0,0) + k\frac{\partial f}{\partial y}(0,0) + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}(0,0) + \frac{2hk}{2!}\frac{\partial^2 f}{\partial x \partial y}(0,0) + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}(0,0) + \text{higher order terms}$$

Now $\quad f(0,0) = e^0 = 1$

$$\frac{\partial f}{\partial x} = ye^{xy} = 0 \quad \text{at } (0,0)$$

$$\frac{\partial f}{\partial y} = xe^{xy} = 0 \quad \text{at } (0,0)$$

$$\frac{\partial^2 f}{\partial x^2} = y^2e^{xy} = 0 \quad \text{at } (0,0)$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{xy} + xye^{xy} = 1 \quad \text{at } (0,0)$$

$$\frac{\partial^2 f}{\partial y^2} = x^2e^{xy} = 0 \quad \text{at } (0,0)$$

Then
$$f(h,k) = 1 + h.0 + k.0 + \frac{h^2}{2!}.0 + \frac{2hk}{2!}.1 + \frac{k^2}{2!}.0 + \text{higher order terms}$$

$$\cong 1 + hk \qquad \text{if } h \text{ and } k \text{ are small.}$$

Q3.a) $\quad f(x, y) = (x^3 + 3x)(y^2 - 6y)$

$$\frac{\partial f}{\partial x} = (3x^2 + 3)(y^2 - 6y) = 0 \ \text{ only if } \ y = 0 \ \text{ or } \ 6.$$

$$\frac{\partial f}{\partial y} = (x^3 + 3)(2y - 6) = 0 \ \text{ only if } \ x = 0 \ \text{ or } \ y = 3.$$

So the stationary points are (0,0) and (0,6).

Now
$$\frac{\partial^2 f}{\partial x^2} = 6x(y^2 - 6y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = (3x^2 + 3)(2y - 6)$$

$$\frac{\partial^2 f}{\partial y^2} = 2(3x^2 + 3)$$

Then
$$\Delta = \left[\frac{\partial^2 f}{\partial x \partial y}\right]^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = (3x^2 + 3)^2(2y - 6)^2 - 6x(y^2 - 6y)2(3x^2 + 3)$$

At (0,0), $\Delta = 324 > 0$, so this point is a saddle point.

At (0,6), $\Delta = 324 > 0$, so this point is also a saddle point.

b) $\quad g(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 + 1$

$$\frac{\partial g}{\partial x} = 4x^3 + 8xy^2 - 4x = 4x(x^2 + 2y^2 - 1)$$

$$\frac{\partial g}{\partial y} = 8x^2y + 4y = 4y(2x^2 + 1)$$

$$= 0 \quad \text{if and only if } \ y = 0.$$

Then
$$\frac{\partial g}{\partial x} = 0 \quad \text{when} \quad 4x(x^2 - 1) = 0 \ \text{ so } x = 0 \text{ or } \pm 1$$

and the stationary points are at (0,0), (1,0) and (-1,0).

Now $\quad \dfrac{\partial^2 g}{\partial x^2} = 12x^2 + 8y^2 - 4$

$\dfrac{\partial^2 g}{\partial x \partial y} = 8x^2 + 4$

$\dfrac{\partial^2 g}{\partial y^2} = 16xy$

so $\quad \Delta = (8x^2 + 4)^2 - (12x^2 + 8y^2 - 4)(16xy)$

$= 16 > 0$ at (0,0) so this point is a saddle point,

$= -96 < 0$ at (1,0) where $\dfrac{\partial^2 g}{\partial x^2} = 8 > 0$ so this point is a minimum,

$= -96 < 0$ at (1,0) where $\dfrac{\partial^2 g}{\partial x^2} = 8 > 0$ so this point is a minimum,

Q4. $\quad \dfrac{\partial q}{\partial x} = 2ax + 2hy + f$

$\dfrac{\partial q}{\partial y} = 2by + 2hx + g$

so for a stationary point at (1,1)

$2a + 2h + f = 0$

and $\quad 2b + 2h + g = 0$

$\dfrac{\partial^2 q}{\partial x^2} = 2a$

$\dfrac{\partial^2 q}{\partial x \partial y} = 2h$

$\dfrac{\partial^2 q}{\partial x^2} = 2b$

For a saddle point, $\Delta = (2h)^2 - (2a)(2b) > 0$ i.e. $h^2 > ab$.

For a minimum, $h^2 < ab$ and $a > 0$.

For a maximum, $h^2 < ab$ and $a < 0$.

Q5.a) $\quad \dfrac{\partial f}{\partial x} = 2x(y^2 - 4) = 0$ if $x = 0$ or $y = \pm 2$.

$\dfrac{\partial f}{\partial y} = 2y(x^2 -4)$

if $x = 0$, $\quad \dfrac{\partial f}{\partial y} = 0$ only if $y = 0$,

if $y = 2$, $\quad \dfrac{\partial f}{\partial y} = 0$ only if $x = \pm 2$,

if $y = -2$, $\quad \dfrac{\partial f}{\partial y} = 0$ only if $x = \pm 2$.

Hence the stationary points are (0,0), (-2,-2), (-2,2), (2,-2) and (2,2).

Now $\quad \dfrac{\partial^2 f}{\partial x^2} = 2(y^2 - 4)$

$\dfrac{\partial^2 f}{\partial x \partial y} = 4xy$

and $\quad \dfrac{\partial^2 f}{\partial x^2} = 2(x^2 - 4)$

so $\quad \Delta = 16x^2y^2 - 4(x^2 - 4)(y^2 - 4)$

At (0,0) $\Delta < 0$, and $\dfrac{\partial^2 f}{\partial x^2} = -8 < 0$ so this point is a maximum,

At the other points $\Delta < 0$ so these are saddle points.

A similar analysis may be used to solve (ii), (iii) and (iv).

Q6.a) $g(x,y) = e^{x+y}(x^2 - xy + y^2)$

At a stationary point

$$\frac{\partial g}{\partial x} = e^{x+y}(x^2 - xy + y^2) + e^{x+y}(2x - y) = 0 \qquad (1)$$

and

$$\frac{\partial g}{\partial y} = e^{x+y}(x^2 - xy + y^2) + e^{x+y}(2y - x) = 0 \qquad (2)$$

Subtracting (2) from (1) $e^{x+y}(3x - 3y) = 0$ so $y = x$.

Then substituting for y,

$$e^{2x}(x^2 + x) = 0 \text{ so } x = 0 \text{ or } -1.$$

Hence stationary points are at (0,0) and (-1,-1).

Now

$$\frac{\partial^2 g}{\partial x^2} = e^{x+y}(x^2 - xy + y^2 + 2x - y)) + e^{x+y}(2x - y + 2)$$

$$\frac{\partial^2 g}{\partial x \partial y} = e^{x+y}(x^2 - xy + y^2 + 2x - y) + e^{x+y}(2y - x - 1)$$

$$\frac{\partial^2 g}{\partial x^2} = e^{x+y}(x^2 - xy + y^2 + 2y - x) + e^{x+y}(2y - x + 2)$$

At (0,0) $\Delta = (-1)^2 - 2\times2 = -3 < 0$, and here $\frac{\partial^2 g}{\partial x^2} = 2 > 0$ so this point is a minimum,

Here $g(x,y) = 0$.

At (-1,-1) $\Delta = (-2e^{-2})^2 - e^{-2}\times e^{-2} = 3e^{-4} > 0$, so this point is a saddle point.

b) $h(x,y) = 6\ln(x + y) - 2xy - 4x - 6y + x^3 + 7$

At a stationary point

$$\frac{\partial h}{\partial x} = \frac{6}{x+y} - 2y - 4 + 3x^2 = 0 \qquad (1)$$

and

$$\frac{\partial h}{\partial y} = \frac{6}{x+y} - 2x - 6 = 0 \qquad (2)$$

Subtracting (2) from (1)

$$-2y - 4 + 3x^2 - (-2x - 6) = 0$$

so

$$y = \frac{1}{2}(3x^2 + 2x + 2) \qquad (3)$$

Also rearranging (2)

$$y = \frac{6}{2x+6} - x .$$

Hence

$$\frac{1}{2}(3x^2 + 2x + 2) = \frac{6}{2x+6} - x .$$

and now by algebraic manipulation this becomes

$$6x^3 + 26x^2 + 28x = 0$$

i.e.

$$2x(3x + 7)(x + 2) = 0$$

so $x = 0, -7/3$ or -2 and from (3) the corresponding values of y are 1, 41/6 , and 5.

Now

$$\frac{\partial^2 h}{\partial x^2} = \frac{-6}{(x + y)^2} + 6x$$

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{-6}{(x + y)^2} - 2$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{-6}{(x + y)^2}$$

At (0,1) $\Delta = (-8)^2 - (-6)(-6) = 28 > 0$, so this point is a saddle point.

At (-7/3,41/6) $\Delta = 1.037 > 0$, so this point is a saddle point.

At (-2, 5) $\Delta = -4/3 < 0$ and here $\frac{\partial^2 h}{\partial x^2} = -38/3 < 0$ so this point is a maximum,

Here $h(x,y) = 6\ln(3) - 3 = 3.59$