Q9

$\underset{\sim}{u} = (1, 0, 1)$, $\underset{\sim}{v} = (1, 0, -1)$,

$\underline{w} = (0, 3, 4)$.

$\{\underline{u}, \underline{v}, \underline{w}\}$ are linearly independent, and so form a basis for $V_3(\mathbb{R})$.

But they're of variable length, and not mutually orthogonal.

The G-S orthogonalization process can take $\{\underline{u}, \underline{v}, \underline{w}\}$ and produce an associated set $\{\underline{a}, \underline{b}, \underline{c}\}$ which is an orthogonal basis, which we can then normalize to give an <u>orthonormal basis</u>.

The key thing about G-S process is it preserves the intermediate subspaces generated by the basis $\{\underline{u}, \underline{v}, \underline{w}\}$

i.e. $\text{span}\{\underline{u}\} = \text{span}\{\underline{a}\}$

$\text{span}\{\underline{u}, \underline{v}\} = \text{span}\{\underline{a}, \underline{b}\}$

$\vdots$

$\text{span}\{\underline{u}, \underline{v}, \underline{w}\} = \text{span}\{\underline{a}, \underline{b}, \underline{c}\}$

Firstly,

$$\boxed{\underline{a} = \underline{u} = (1, 0, 1).}$$

In this context the inner product to use is the regular dot/scalar product.

We'll need $\boxed{\underset{\sim}{a} \cdot \underset{\sim}{a} = 2}$

and $\underline{v} \cdot \underline{a} = (1, 0, -1) \cdot (1, 0, 1)$

$= 0$

So the second new basis vector $\underline{b}$ will be

$$\underline{b} = \underline{v} - \frac{\underline{v} \cdot \underline{a}}{\underline{a} \cdot \underline{a}} \underline{a}$$

$$= \underline{v} \quad , \quad \text{since } \underline{v} \cdot \underline{a} = 0$$

$$\boxed{= (1, 0, -1).}$$

For the third vector $\underline{c}$ we'll need the dot products. $\underline{b} \cdot \underline{b} = 2$

and $\underline{w} \cdot \underline{a} = (0, 3, 4) \cdot (1, 0, 1) = 4.$

$\underline{w} \cdot \underline{b} = (0, 3, 4) \cdot (1, 0, -1) = -4.$

So the third new basis vector is

$$\boxed{\underline{c}} = \underline{w} - \frac{\underline{w}\cdot\underline{a}}{\underline{a}\cdot\underline{a}}\underline{a} - \frac{\underline{w}\cdot\underline{b}}{\underline{b}\cdot\underline{b}}\underline{b}$$

$$= (0,3,4) - \frac{4}{2}(1,0,1) - \frac{-4}{2}(1,0,-1).$$

$$= (0,3,4) - (2,0,2) + (2,0,-2)$$

$$= \boxed{(0,3,0)}$$

So we have the (hopefully) orthogonal new basis $\{\underline{a},\underline{b},\underline{c}\}$

$$\underline{a} = (1,0,1), \quad \underline{b} = (1,0,-1)$$

$$\underline{c} = (0,3,0).$$

and indeed $\underline{a}\cdot\underline{b} = \underline{a}\cdot\underline{c} = \underline{b}\cdot\underline{c} = 0$

so this confirms orthogonality.

To obtain the associated orthonormal basis, we normalize/rescale each vector to have length 1.

$$\boxed{\underline{e} = \frac{1}{|\underline{a}|}\underline{a} = \frac{1}{\sqrt{2}}(1,0,1)}$$

$$\boxed{\underline{f} = \frac{1}{|\underline{b}|}\underline{b}} = \boxed{\frac{1}{\sqrt{2}}(1,0,-1).}$$

$$\boxed{\underline{g} = \frac{1}{|\underline{c}|}\underline{c}} = \frac{1}{3}(0,3,0) = \boxed{(0,1,0)}$$

And $\{\underline{e}, \underline{f}, \underline{g}\}$ is an orthonormal basis for $V_3(\mathbb{R})$.

Q10 Apply G-S process to obtain the orthogonal basis for $P_3(\mathbb{R})$ associated to the basis $\{1, x, x^2, x^3\}$ w.r.t. the inner product

$$\phi(p_1, p_2) = \int_{-1}^{1} (1-x^2)\, p_1(x)\, p_2(x)\, dx.$$

New orthogonal basis will be $\{a(x), b(x), c(x), d(x)\}$

$$\boxed{a(x) = 1}$$

$$\phi(a(x), a(x)) = \int_{-1}^{1} (1-x^2)\, dx$$

$$= \left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= 2 - \frac{1}{3}(2) = 4/3$$

$$\phi(x, a(x)) = \int_{-1}^{1} (1-x^2)x\, dx.$$

$$= \int_{-1}^{1} x - x^3 \, dx. \qquad = 0$$

$$= \left[x^2 - \frac{x^4}{4} \right]_{-1}^{1} = 0$$

So $$b(x) = x - \frac{\phi(x, a(x))}{\phi(a(x), a(x))} \cdot a(x)$$

$$\Rightarrow \boxed{b(x) = x.}$$

Obtaining $c(x)$ we'll need.

$$\phi(b(x), b(x)) = \int_{-1}^{1} (1 - x^2) x^2 \, dx$$

$$= \int_{-1}^{1} x^2 - x^4 \, dx$$

$$= \left[x^3/3 - x^5/5 \right]_{-1}^{1}$$

$$= \frac{2}{3} - \frac{2}{5}$$

$$= 4/15$$

$$\phi(x^2, a(x))$$

$$= \int_{-1}^{1} (1 - x^2) x^2 \, dx.$$

$$= 4/15$$

$$\phi(x^2, b(x)) = \int_{-1}^{1} (1 - x^2) x^2 x \, dx$$

$$= \int_{-1}^{1} x^3 - x^5 \, dx = 0$$

Giving $c(x)$ as

$$c(x) = x^2 - \frac{\phi(x^2, a(x))}{\phi(a(x), a(x))} \cdot a(x) - \frac{\phi(x^2, b(x))}{\phi(b(x), b(x))} b(x)$$

$$= x^2 - \frac{4/15}{4/3} 1.$$

$$\boxed{c(x) = x^2 - 1/5.}$$, agrees with sols.

But could check that $\phi(a(x), c(x)) = 0$
and $\phi(b(x), c(x)) = 0$.

Find $d(x)$ using a similar approach.

$$\bar{x} = \frac{\quad}{s \cdots s}$$

$$= \quad + \quad e \quad \frac{\quad}{s \cdots s}$$

$$x(t) = \mathcal{L}^{-1}\{\bar{x}\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\cdots}{p}\right\} + \mathcal{L}^{-1}\left\{e^{\cdots}\frac{\cdots}{\cdots}\right\}$$

Chap 2 Q 26.

Find / classify critical points of

$$f(x,y) = x^3 + xy^2 - x.$$

$$\frac{\partial f}{\partial x} = 3x^2 + y^2 - 1$$

$$\frac{\partial f}{\partial y} = 2xy$$

Critical points are solutions (a,b) to the simultaneous equations

$$\frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0.$$

$$\Longleftrightarrow \boxed{\begin{cases} 3x^2 + y^2 - 1 = 0 & \text{— ①} \\ 2xy = 0. & \text{— ②} \end{cases}}$$

The second equation is simpler so focus first on it

② is true iff $\boxed{x=0 \ \underline{\text{OR}} \ y=0}$.

So this tells us that any critical points must be of the form $(0, y)$ or $(x, 0)$, for some $x, y \in \mathbb{R}$.

Now impose these conditions on ①.

Firstly, suppose $\boxed{x=0}$.

then ① becomes $y^2 - 1 = 0$

and so ① is true iff $y^2 = 1$

i.e. $y = -1$ or 1.

So we've found two critical points

$(0, -1)$ and $(0, 1)$.

Secondly, now suppose $\boxed{y=0}$, then

① become $3x^2 - 1 = 0$.

and this has solutions $3x^2 = 1$

$$x = -\frac{1}{\sqrt{3}}, \ +\frac{1}{\sqrt{3}}$$

So we've found two more critical points

$\left(-\frac{1}{\sqrt{3}}, 0\right)$ and $\left(\frac{1}{\sqrt{3}}, 0\right)$

$$\frac{\partial g}{\partial x} = poly(y)\ \sin(x) = 0$$

$$\boxed{\frac{\partial g}{\partial y} = poly(y)\ \cos(x) = 0}$$

When is $\dfrac{\partial g}{\partial y} = 0$?

This happens when

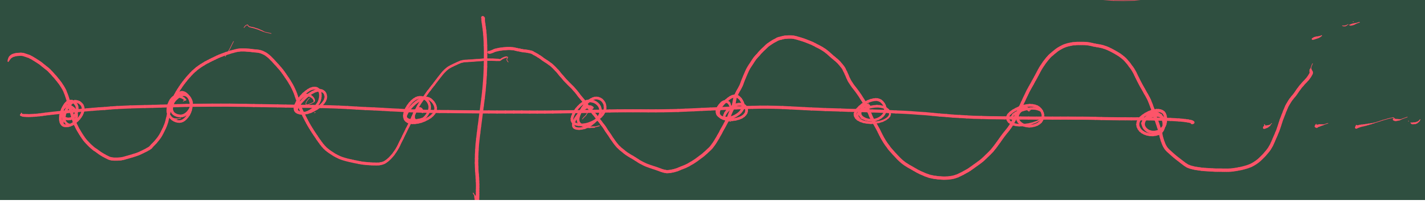
$poly(y) = 0 \rightarrow y = \ldots\ldots$

OR

$\cos(mx) = 0$ $\quad n \in \mathbb{Z}$

∞ sols.

$\rightarrow x = \ldots\ldots\ldots\ldots$



Cons:

Consider $\dfrac{\partial g}{\partial x} = 0$

under the assumption

$$x = \left(n + \frac{1}{2}\right)\pi \ , \quad n \in \mathbb{Z}$$

simplify $\Rightarrow$ ~~~~ $= 0$.

$y = \ldots\ldots$

$$D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$$

$$D^2(f) = D(D(f))$$

$$= h^2 \frac{\partial^2 f}{\partial x^2} + k^2 \frac{\partial^2 f}{\partial y^2} + \ldots$$

$$D^6(f) = h^6 \frac{\partial^6 f}{\partial x^6} + k^6 \frac{\partial^6 f}{\partial y^6} + \cdots\cdots$$

$$= D(D(D(\cdots(f))\cdots))$$

$$= D(D^5(f)).$$

$$Df,\ D(Df),\ D(\qquad),$$

$$D(\qquad)$$