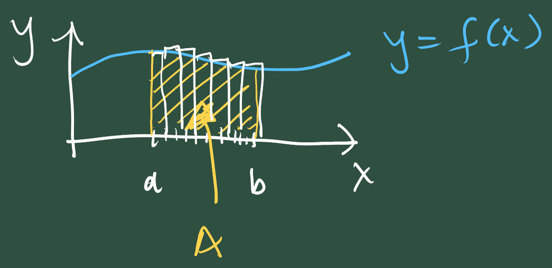
Multiple integrals

Focus first on functions of two variables

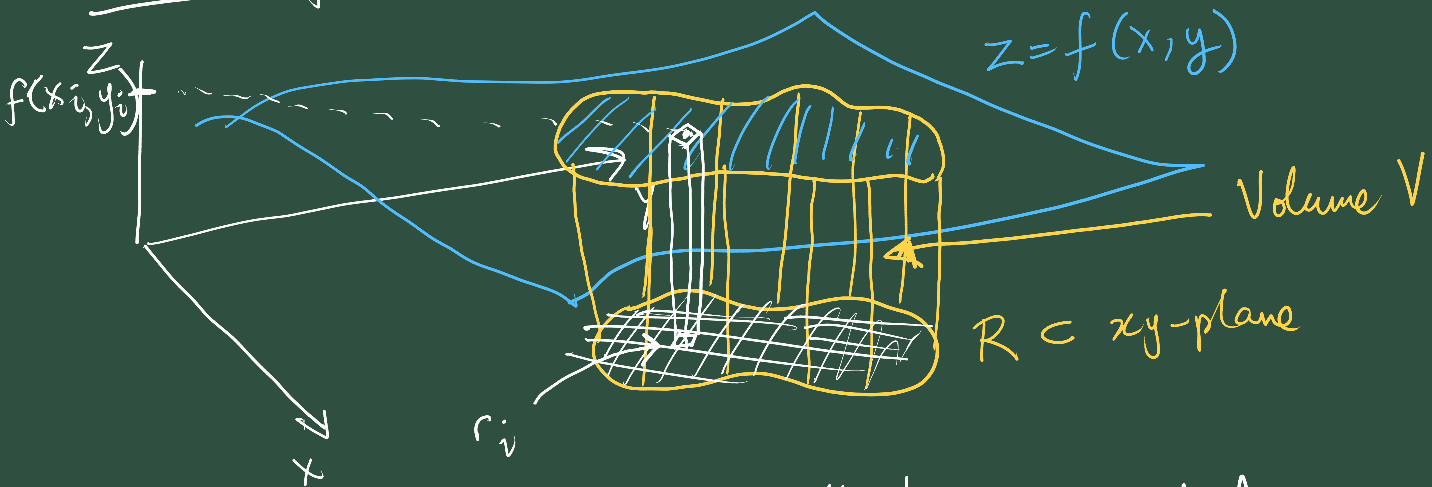
Double integrals

Intro. This type of integral most closely resembles single-variable integrals, but in 1-dimension higher.



Area $A = \underbrace{\int_a^b f(x)\,dx}_{\text{formally defined by means of limits of Riemann sums}}$.

2-variable functions



The volume V here will be represented as a "double integral"

$$V = \iint_R f(x,y)\,dx\,dy$$

The formal rigorous definition uses limits and rectangular boxes, like in 1-variable case.

We can approximate R as a union of rectangles

$$R \approx \bigcup_i r_i$$

and as size of the $r_i \to 0$ this approximate will become exact (in the limit).

Over each rectangle r_i we position a rectangular box with base r_i and height $f(x_i, y_i)$ where (x_i, y_i) is some point within r_i.

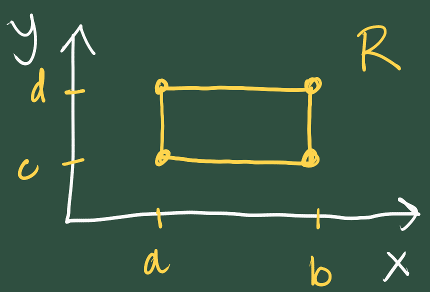
this rect. box has volume $= (\text{area } r_i) \cdot f(x_i, y_i)$ which leads to a def. for the double integral as

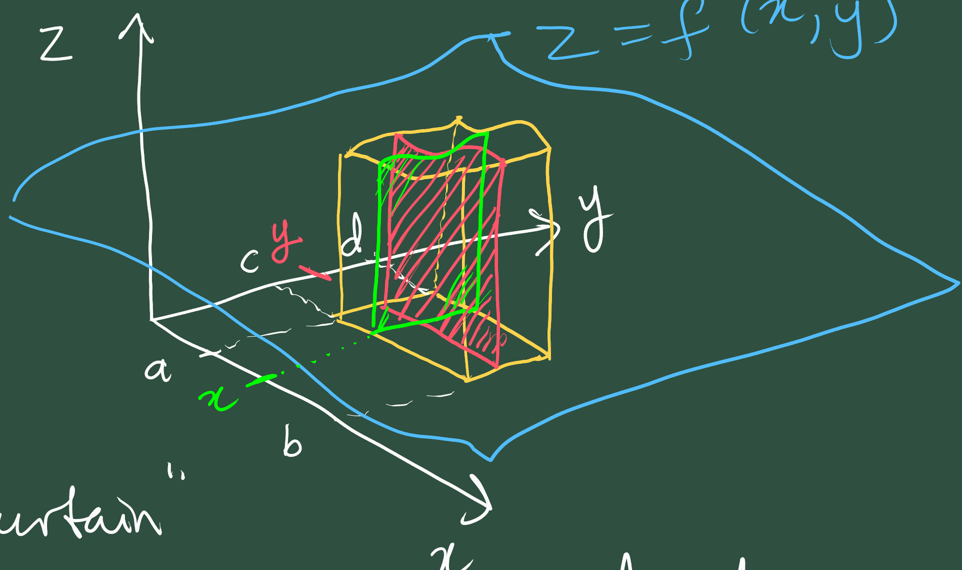
$$V = \iint_R f(x,y)\, dx\, dy$$

$$= \lim_{\text{size } r_i \to 0} \sum_i \underbrace{(\text{area } r_i) \cdot f(x_i, y_i)}_{\text{volume of rectangular box over } r_i}$$

We don't use this def. directly to evaluate such integrals, but rather see them as "repeated integrals"

$\boxed{\text{Simplest case}}$ is when R itself is a rectangle, aligned with x,y-axes





Consider the cross-sectional "curtain" shown in diagram, which is located at some y value $c \le y \le d$

$$\text{area of curtain} = \int_a^b f(x,y)\,dx$$

this is the location of the curtain

Idea is to express the volume V as the <u>continuous sum</u> of the <u>areas of these cross-sections over every y value from c to d</u>.

$$V = \int_c^d \underline{\qquad\qquad} \, dy$$

$$= \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx \right)} dy.$$

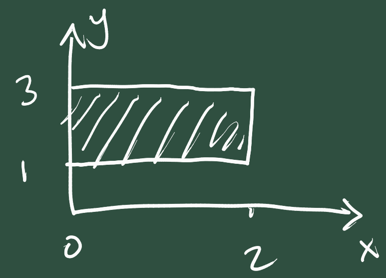
"repeated integral"

y treated as a constant

These are typically evaluated from inside out

<u>Example</u> Consider $f(x,y) = 2xy + 4x + 3y + 1$

Integrate this over the region R, the rectangle over $0 \le x \le 2$, $1 \le y \le 3$



The volume lying over R, and under $z = f(x,y)$ given by

$$V = \iint_R f(x,y)\,dx\,dy$$

$$= \int_1^3 \left(\int_0^2 (2xy + 4x + 3y + 1)\,dx \right) dy$$

$$= \int_1^3 \left(\left[x^2 y + 2x^2 + 3xy + x \right]_{x=0}^{x=2} \right) dy$$

$$= \int_1^3 4y + 8 + 6y + 2 \quad dy$$

$$= \int_1^3 (10y + 10)\,dy$$

$$= \left[5y^2 + 10y \right]_1^3$$

$$= 45 + 30 - (5 + 10)$$

$$= 75 - 15$$

$$= 60$$

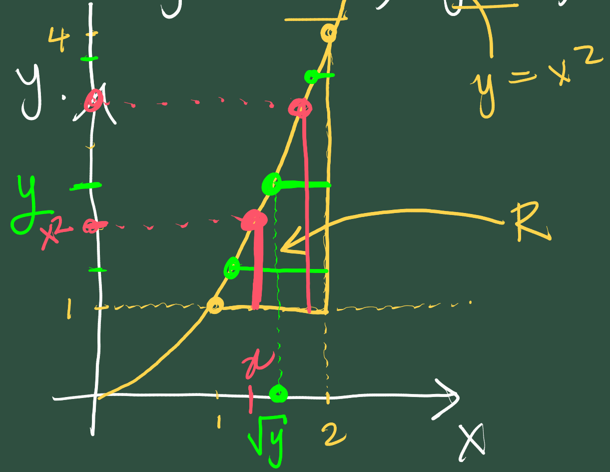
We could also ~~ev~~ evaluate this double integral as the repeated integral with inner integral wrt y and outer wrt x

$$\iint_R f(x,y)\,dx\,dy = \int_0^2 \left(\int_1^3 f(x,y)\,dy \right) dx$$

<u>Exercise</u> show this also evaluates to 60.

<u>Non-rectangular regions</u>, and non-aligned rectangles careful consideration of boundary ~~an~~ curves is needed.

<u>Example</u> Integrate $f(x,y) = 4x^3 + 4y^3$ over R, where R is the region bounded by $x=2$, $y=1$, and $\boxed{y = x^2} \iff x = \sqrt{y}$. Sketch the region R.



$$I = \iint_R 4x^3 + 4y^3 \,dx\,dy$$

$$= \int_1^4 \left(\int_{\sqrt{y}}^2 (4x^3 + 4y^3)\,dx \right) dy$$

$$= \int_1^4 \left[x^4 + 4xy^3 \right]_{x=\sqrt{y}}^{x=2} dy$$

$$= \int_1^4 \left(16 + 8y^3 - \left(y^2 + 4y^{7/2} \right) \right) dy$$

non-rectangular regions are characterised by having non-constant limits on the inner integral.

$$= \left[16y + 2y^4 - \frac{y^3}{3} - \frac{8}{9} y^{9/2} \right]_1^4$$

$$= \left(64 + 512 - \frac{64}{3} - \frac{8}{9} 2^9 \right) - \left(16 + 2 - \frac{1}{3} - \frac{8}{9} \right)$$

$$= \frac{745}{9}$$

How would it appear as a repeated integral in the opposite order.

$$I = \int_1^2 \left(\int_1^{x^2} (4x^3 + 4y^3)\, dy \right) dx$$

<u>Exercise</u> Carry this out and obtain $\frac{745}{9}$ also.

<u>Example</u> Shows the need to sometimes break apart the double integral into a sum of two or more sub-integrals.

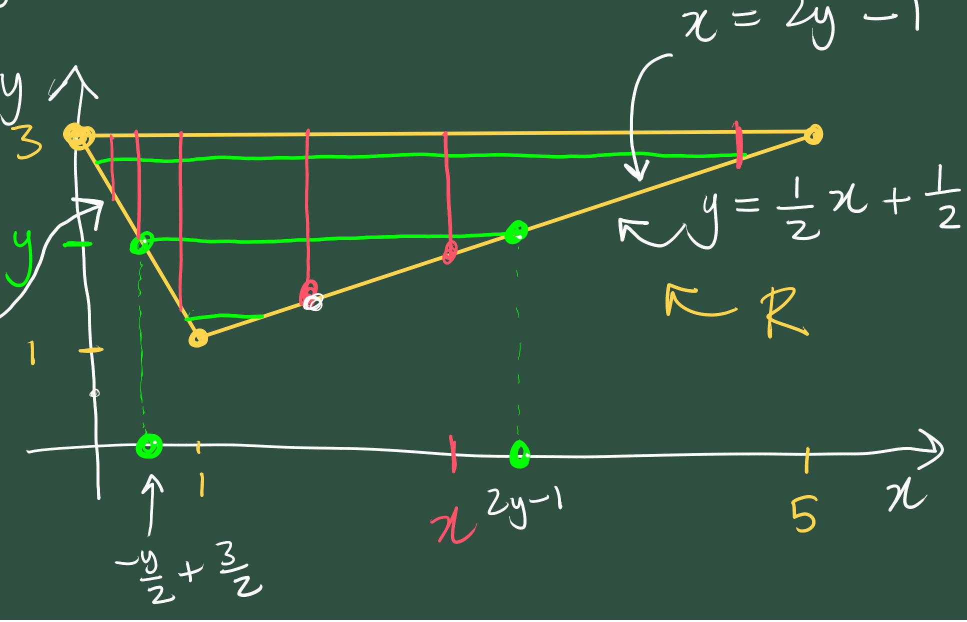
Consider integrating $f(x,y)$ over the triangle with vertices at $(1,1)$, $(5,3)$, $(0,3)$. Investigate the repeated integrals in both orders.

Sketch the region

$$I = \iint_R f(x,y)\, dx\, dy$$

$y = -2x + 3$

$\Leftrightarrow x = -\frac{y}{2} + \frac{3}{2}$



$$= \int_1^3 \left(\int_{-\frac{y}{2}+\frac{3}{2}}^{2y-1} f(x,y)\, dx \right) dy$$

(express as repeated integral in opposite order).

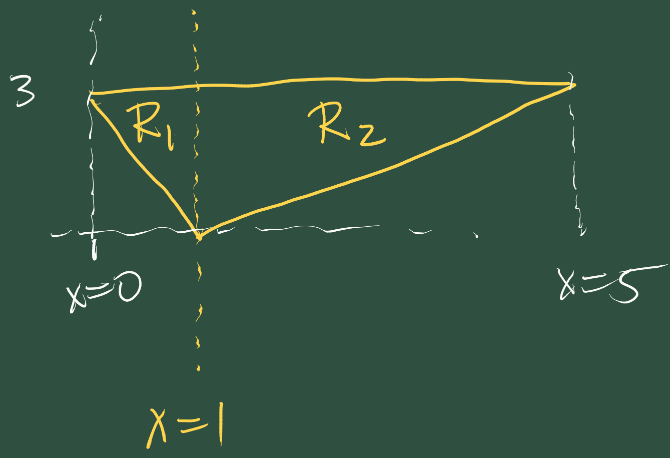
$$= \int_0^5 \left(\int_{?}^{3} f(x,y)\, dy \right) dx$$

0

no single expression in x that provides lower limit here

Resolution.

$$\iint_R = \iint_{R_1} + \iint_{R_2}$$



$$= \int_0^1 \left(\int_{-2x+3}^{3} f(x,y)\, dy \right) dx \Big\} R_1$$

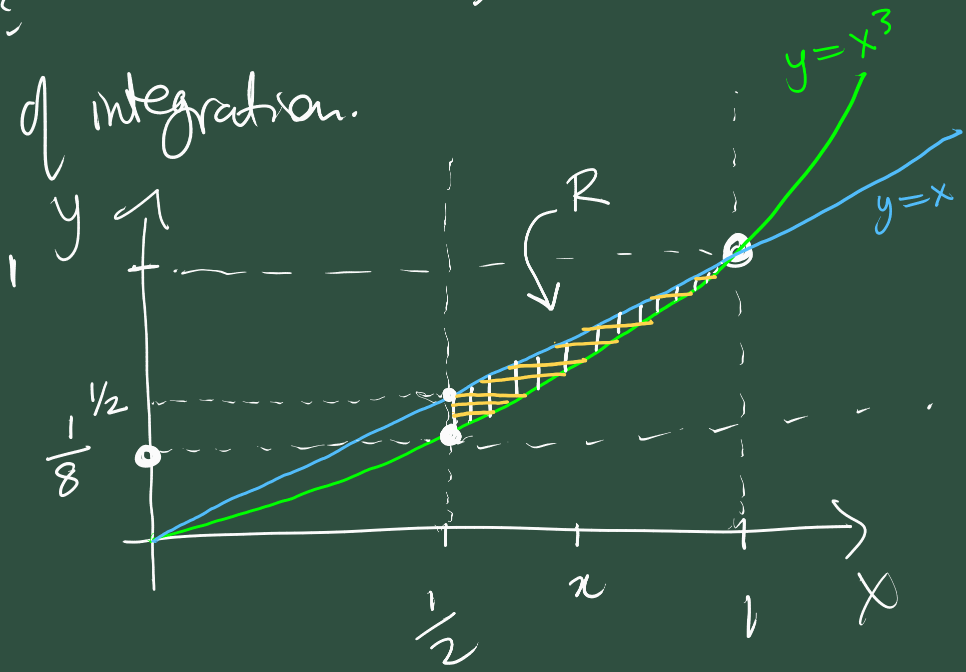
$$+ \int_1^5 \left(\int_{\frac{1}{2}x+\frac{1}{2}}^{3} f(x,y)\, dy \right) dx \Big\} R_2$$

<u>Example</u> Consider the following repeated integral. Reverse the order of integration

$$I = \int_{1/2}^{1} \left(\int_{x^3}^{x} f(x,y)\, dy \right) dx$$

$$= \int_{?}^{?} \left(\int_{?}^{?} f(x,y) \, dx \right) dy$$

Sketch the region of integration.



$$= \int_{1/8}^{1} \left(\int_{?}^{y^{1/3}} f(x,y) \, dx \right) dy$$

$$= \int_{1/8}^{1/2} \left(\int_{1/2}^{y^{1/3}} f(x,y) \, dx \right) dy$$

$$+ \int_{1/2}^{1} \left(\int_{y}^{y^{1/3}} f(x,y) \, dx \right) dy$$