Q5 (b). Find and classify critical points of $g(x,y) = (x^2-4)(4-y^2)$.

Critical points (a,b) are simultaneous solutions to

$$\frac{\partial g}{\partial x} = 0 \quad \text{and} \quad \frac{\partial g}{\partial y} = 0.$$

$$\Leftrightarrow \quad \frac{\partial g}{\partial x} = 2x(4-y^2) = 0 \quad \text{and} \quad -2y(x^2-4) = 0.$$

$$\frac{\partial g}{\partial x} = 0 \Leftrightarrow x = 0 \text{ or } (y=-2 \text{ or } y = 2).$$

$(0,0)$

$(2,-2)$, $(-2,-2)$

$(2,2)$, $(-2,2)$

So three cases

Assume $x=0$ then $\frac{\partial g}{\partial y} = 8y = 0$
$\Leftrightarrow y = 0$. Yields 1 critical point

Assume $y=-2$ then $\frac{\partial g}{\partial y} = 4(x^2-4) = 0$
$\Leftrightarrow x = -2$ or $x = 2$. Yields two critical points

Now assume $y=2$ then $\frac{\partial g}{\partial y} = -4(x^2-4) = 0$
$\Leftrightarrow x = -2, 2$ Yields two critical points

So five critical points in total.

These are classified by Hessian determinant.

$$D = \frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial y^2} - \left(\frac{\partial^2 g}{\partial x \partial y}\right)^2$$

$$= 2(4-y^2)(-2(x^2-4)) - (-4xy)^2$$

$D(0,0) = 64 > 0$ and $\frac{\partial^2 g}{\partial x^2}(0,0) = 8 > 0$ So $(0,0)$ is a local minimum

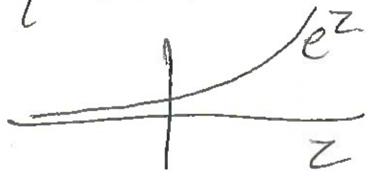
So quickly see that

$$D(\pm 2, \pm 2) < \cancel{a} - 16^2 = \underline{\underline{-256 < 0}}$$

So all four of $(\pm 2, \pm 2)$ are <u>saddles</u>.

Q 6 (a). $g(x,y) = e^{x+y}(x^2 - xy + y^2)$.

Critical points found by solving the pair of equations.



$$\frac{\partial g}{\partial x} = 0 \quad , \quad \frac{\partial g}{\partial y} = 0.$$

$$\frac{\partial g}{\partial x} = e^{x+y}(2x - y) + e^{x+y}(x^2 - xy + y^2) \text{ , by prod. rule.}$$

$$= e^{x+y}(x^2 + 2x - xy - y + y^2) = 0.$$

$$\frac{\partial g}{\partial y} = e^{x+y}(y^2 + 2y - xy - x + x^2) \text{ , done quickly due to symmetry of } g$$

Consider $\frac{\partial g}{\partial x} = 0$ Notice that $e^{x+y} > 0$ for ~~even~~ all x, y.

$$\text{So } \frac{\partial g}{\partial x} = 0 \iff x^2 + 2x - xy - y + y^2 = 0. \quad \text{—(1)}$$

$$\text{And } \frac{\partial g}{\partial g} = 0 \iff y^2 + 2y - xy - x + x^2 = 0. \quad \text{—(2)}$$

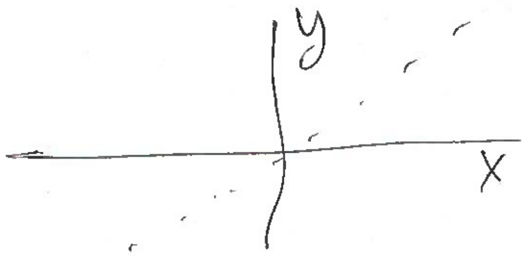
Notice that

$$\left(\frac{\partial g}{\partial x} = 0 \;\&\; \frac{\partial g}{\partial y} = 0\right) \Rightarrow \quad (1) - (2)$$

$$3x - 3y = 0$$

$$\iff x = y.$$

So ~~all~~ any critical points must be of the form (a,a).



So now assume that $x=y$.

And now

$$\frac{\partial g}{\partial x}=0 \iff \cancel{x^2}+\cancel{2x}-\cancel{x^2}-x+x^2=0$$

$$\iff x^2+x=0$$

$$\iff x(x+1)=0$$

$$\iff x=0 \text{ or } x=-1.$$

~~Also~~

And notice when $x=y$ and $x=0=y$.

$$\frac{\partial g}{\partial y}=0 \checkmark$$

and when $x=y$ and $x=-1=y$.

then $\frac{\partial g}{\partial y}=0$ ✓

So in summary the only critical points are $(0,0)$ and $(-1,-1)$.