Q1 $T: V_3(\mathbb{R}) \longrightarrow V_2(\mathbb{R})$

Judging i), ii), iii) against the def of a L.T.

$T: U \longrightarrow V$ is a L.T.

iff. $\boxed{\begin{array}{l} \underline{\forall}\ \underline{u}_1, \underline{u}_2 \in U \text{ and } \underline{\forall}\ \alpha \in \mathbb{R} \\ T(\alpha \underline{u}_1 + \underline{u}_2) = \alpha T(\underline{u}_1) \\ \qquad\qquad\qquad\qquad + T(\underline{u}_2) \end{array}}$

(i) T defined by

$$T\big((a,b,c)\big) = (a+b-c\ ,\ 2a+b)$$

Well try and prove it's a L.T.

Let (a_1, b_1, c_1), $(a_2, b_2, c_2) \in V_3(\mathbb{R})$

(ie. $a_i, b_i, c_i \in \mathbb{R}$) and let $\alpha \in \mathbb{R}$

$T(\alpha(a_1, b_1, c_1) + (a_2, b_2, c_2))$

vector. ops in V_3

$$= T((\alpha a_1 + a_2, \alpha b_1 + b_2, \alpha c_1 + c_2))$$

$$= \left(\alpha a_1 + a_2 + \alpha b_1 + b_2 - (\alpha c_1 + c_2),\ 2(\alpha a_1 + a_2) + \alpha b_1 + b_2\right)$$ applying T.

$\downarrow$ algebraically rewriting both components

$$= \Big(\overbrace{\alpha(a_1 + b_1 - c_1) + (a_2 + b_2 - c_2)},\ \overbrace{\alpha(2a_1 + b_1) + (2a_2 + b_2)}\Big)$$

vector ops in V_2.

$$= \alpha(a_1 + b_1 - c_1, 2a_1 + b_1) + (a_2 + b_2 - c_2, 2a_2 + b_2)$$

$$= \alpha T((a_1, b_1, c_1)) + T((a_2, b_2, c_2)),$$

application of def of T.

(ii) Consider T def. by

$$T((a,b,c)) = (|a|, 0)$$

Presence of abs. value suggest this is not a linear transformation.

So we need a counter-example showing this.

$\underline{u}_1 = (1, 0, 0)$, $\underline{u}_2 = (0, 0, 0)$

$\alpha = -1$

$T(\alpha \underline{u}_1 + \underline{u}_2) = T((-1, 0, 0))$

$= (1, 0)$

But. $\not\parallel$

$\alpha T(\underline{u}_1) + T(\underline{u}_2) = -1(1, 0) + (0, 0)$

$= (-1, 0)$

So T is <u>not</u> a linear transformation.

(iii) Consider T def by

$T((a, b, c)) = (ab, ba)$?

I think it's not ~~an~~ a L.T.

Counter-example.

$\underline{u}_1 = (1, 1, 0)$, $\underline{u}_2 = (0, 0, 0)$

$\alpha = 2$

$T(\alpha \underline{u}_1 + \underline{u}_2) = T((2,2,0))$

$\neq \quad = (4,4)$

But

$\alpha T(\underline{u}_1) + T(\underline{u}_2) = 2(1,1) + (0,0)$

$= (2,2)$

So it's not a L.T.

Q3 $\quad T: M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$

$M_2(\mathbb{R})$ is v. space of all 2×2 real matrices under addition and scalar multiplication.

(i) T defined by

$T(A) = AS$, where S is some fixed matrix.

Think it is a L.T.

Let $A_1, A_2 \in M_2(\mathbb{R})$ and let $\alpha \in \mathbb{R}$

$T(\alpha A_1 + A_2) = (\alpha A_1 + A_2)S$, def of T

$= (\alpha A_1)S + A_2 S$, m. mult. is distributive

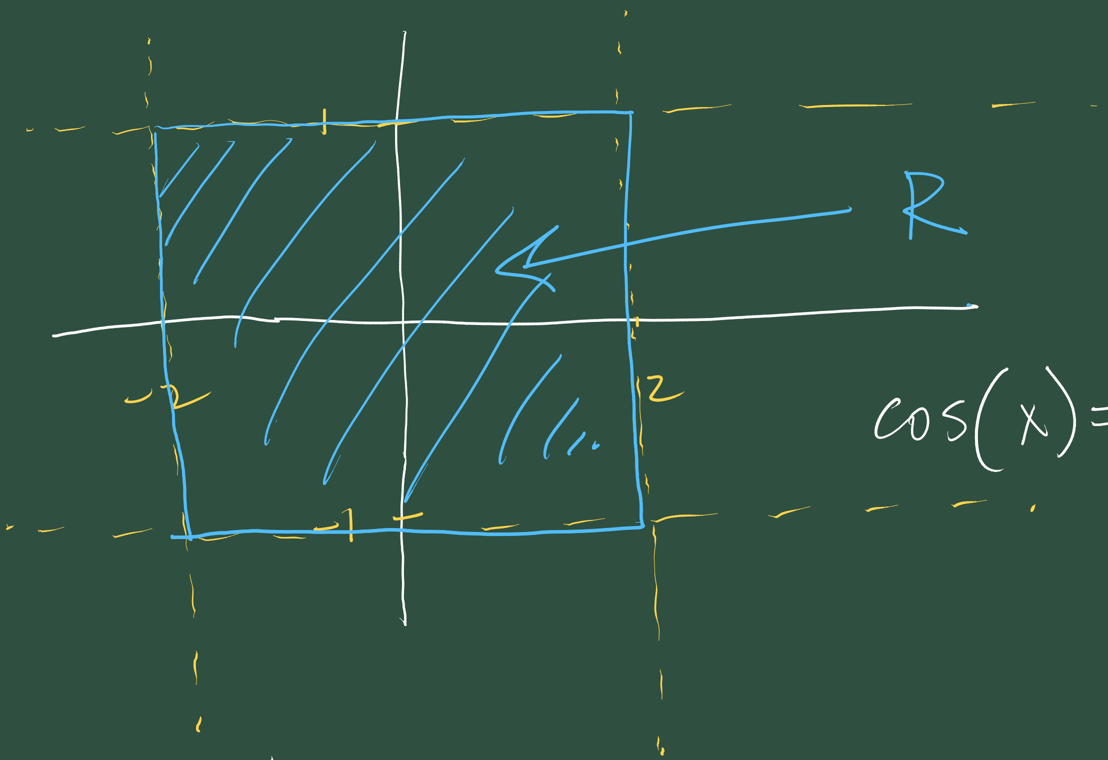
$$= \alpha (A_1 S) + A_2 S,$$ prop. of scalars

$$= \alpha T(A_1) + T(A_2)$$ within matrix products.

Q2.

$$I = \int_{-1}^{1}\left(\int_{-2}^{2} 4xy + \sin x + \cos y \, dx\right) dy.$$

Sketch the region



$$\cos(x) = \cos(-x)$$

$$I = \int_{-1}^{1} \left[2x^2y - \cos x + x\cos y\right]_{-2}^{2} dy$$

$$-\cos(2) - -\cos(-2)$$

$$= \int_{-1}^{1} 4\cos(y)\, dy$$

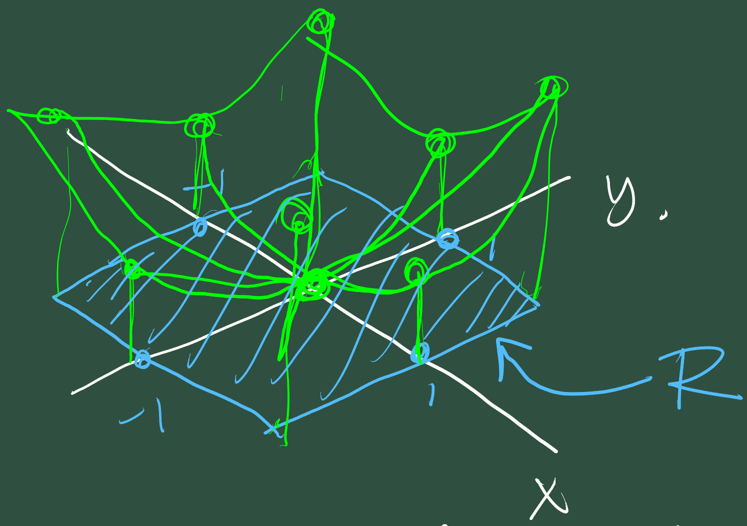
$$\sin(-y) = -\sin(y)$$

$$= 4\int_{-1}^{1} \cos(y)\, dy$$

$$= 4\left[\sin(y)\right]_{-1}^{1}$$

$$= 8\sin(1) = \ldots$$

Q5.



Average height of this surface.

$$= \frac{1}{\text{area } R} \iint_R x^2+y^2 \, dx\,dy$$

$$= \frac{1}{4} \iint_R x^2+y^2 \, dx\,dy$$

$$= \frac{1}{4}\int_{-1}^{1} \left(\int_{-1}^{1} (x^2+y^2)\,dx \right) dy$$

$$= \frac{1}{4}\int_{-1}^{1} \left[\frac{x^3}{3} + xy^2 \right]_{-1}^{1} dy.$$

$$= \frac{1}{4}\int_{-1}^{1} \left(\frac{2}{3} + 2y^2 \right) dy$$

$$= \frac{1}{4}\left[\frac{2}{3}y + \frac{2}{3}y^3 \right]_{-1}^{1}$$

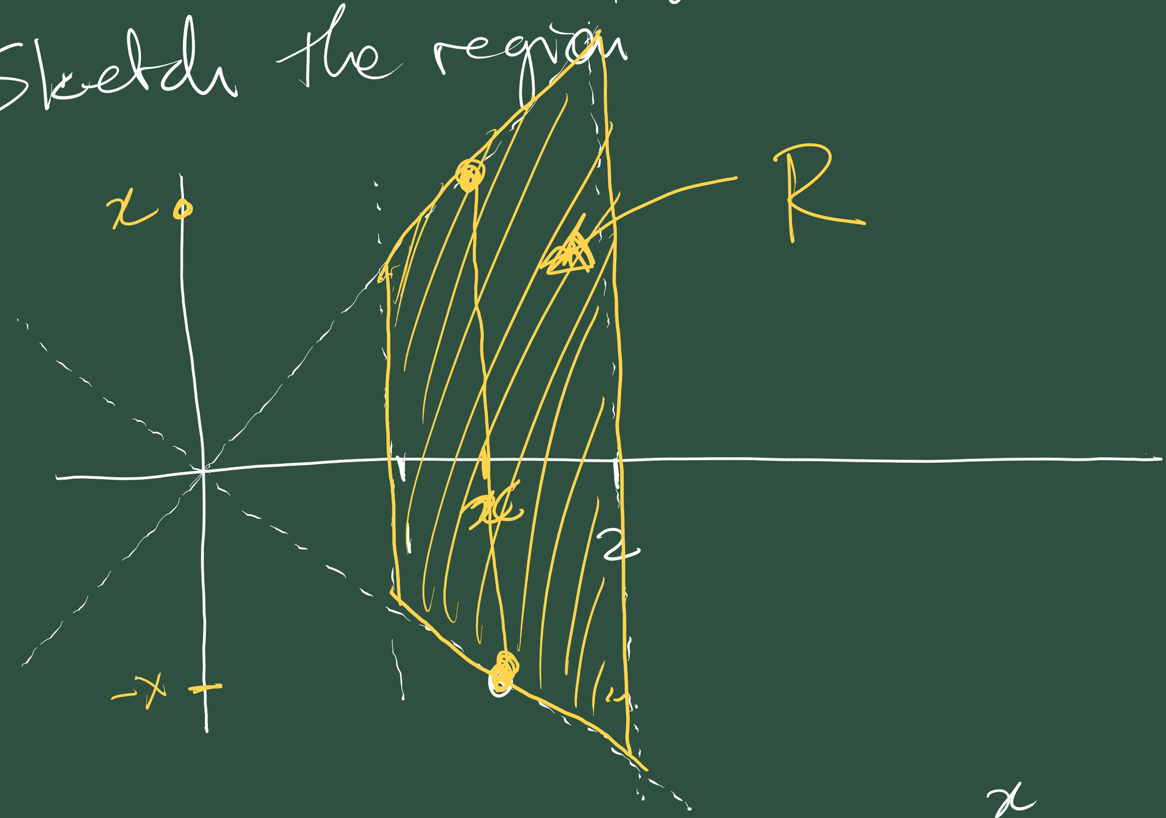
$$= \frac{1}{4}\left[\frac{4}{3} + \frac{4}{3}\right]$$

$$= \frac{2}{3}$$

Q6 $\qquad I = \int_1^2 \left(\int_{-x}^{x} \left(\frac{y}{x}+1\right) dy \right) dx$

This is a non-rectangular region

Sketch the region



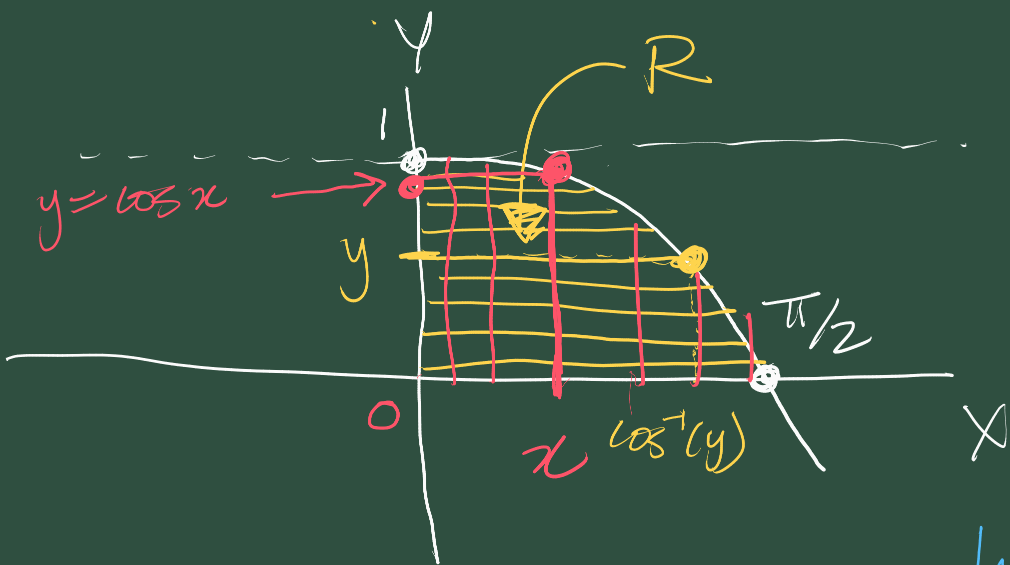
$$I = \int_1^2 \left[\frac{y^2}{2x} + y \right]_{-x}^{x} dx$$

$$= \int_1^2 2x \, dx$$

$$= \left[x^2 \right]_1^2 = 3 .$$

Q7 a), b). First sketch the region.

a)



$x = \cos^{-1} y \iff y = \cos(x)$

harder?

$$I = \int_0^1 \left(\int_0^{\cos^{-1} y} \sec x \, dx \right) dy$$

(change the order)

$$= \int_0^{\pi/2} \left(\int_0^{\cos(x)} \sec x \, dy \right) dx$$

$$= \int_0^{\pi/2} \sec x \left(\int_0^{\cos(x)} dy \right) dx$$

$$= \int_0^{\pi/2} \sec x \cos x \, dx$$

$$= \int_0^{\pi/2} dx = \pi/2.$$

do-able