Fourier series

Briefly

A way to express functions as infinite sums (series) of "standard functions" (sines and cosines). Finite parts of these sums will provide approximations to our functions.

Motivation Reminders about concepts you already have:

Consider Taylor series. A Taylor expansion of a function f at some base point a is

$$f(a+x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{\text{coeffs depend on } n^{\text{th}} \text{ derivative of } f.} \underbrace{(x+a)^n}_{\text{polys. in variable } x}$$

ie. $f^{(n)} = \frac{d^n f}{dx^n}$.

approximations to f will be given by

$$f(a+x) \approx \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x+a)^n.$$

getting better and better as $k \to \infty$

Fourier series will have these features too, ~~us~~ instead of polys, they use a family of standard functions (basis)

$$F = \{ 1, \sin x, \sin(2x), \sin(3x), \ldots, \cos(x), \cos(2x), \cos(3x), \ldots \}$$

giving a Fourier series

$$f(x) = \frac{1}{2} a_0 1 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Fourier coefficients

a_0, a_n, b_n $n \geq 1$

$\llcorner$ will come certain integrals of f.

Functions from $\mathcal{F}$ are all periodic, repeating every 2π, so the Fourier series will in fact replicate the behaviour of f across the interval $(-\pi, \pi)$, and repeat this across the whole domain.

Recall some basic linear algebra concepts.

bases for a vector space, and associated coefficients for vectors with respect to the basis.

Consider the standard n-dim Euclidean vector space. $V = \mathbb{R}^n$

V has a standard basis

$B = \{\underline{e}_1, \ldots, \underline{e}_n\}$

$\underline{e}_1 = (1, 0, \ldots, 0)$

$\underline{e}_2 = (0, 1, 0, \ldots, 0)$

This basis B is orthonormal

ie. $\underline{e}_i \cdot \underline{e}_j = 0$

$\vdots$

$\underline{e}_n = (0, \ldots, 0, 1)$

for $i \neq j$, as $\underline{e}_i, \underline{e}_j$ perpendicular to one another.

and

$\underline{e}_i \cdot \underline{e}_i = 1.$

i.e. $\|\underline{e}_i\| = 1.$

And any vector $\underset{\sim}{x} \in V$ can be expressed as

$$\boxed{\underset{\sim}{x} = \sum_{i=1}^{n} \lambda_i \underline{e}_i}$$

, for some coeffs λ_i

and $\boxed{\lambda_i = \underset{\sim}{x} \cdot \underset{\sim}{e}_i}$

Fourier series will have similar features. We'll be dealing with

- an infinite dimensional vectorspace of ~~~~ "well behaved" functions f
- A basis for this space

will be the family of functions F we saw above

$$F = \{ 1, \sin(nx), \cos(nx) : n \geq 1 \}$$

- Any "vector" / function f can be expressed as a linear combination over F, ie the Fourier series

basis elements

$$f(x) = \frac{1}{2} a_0 1 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

coefficients of f w.r.t. basis F

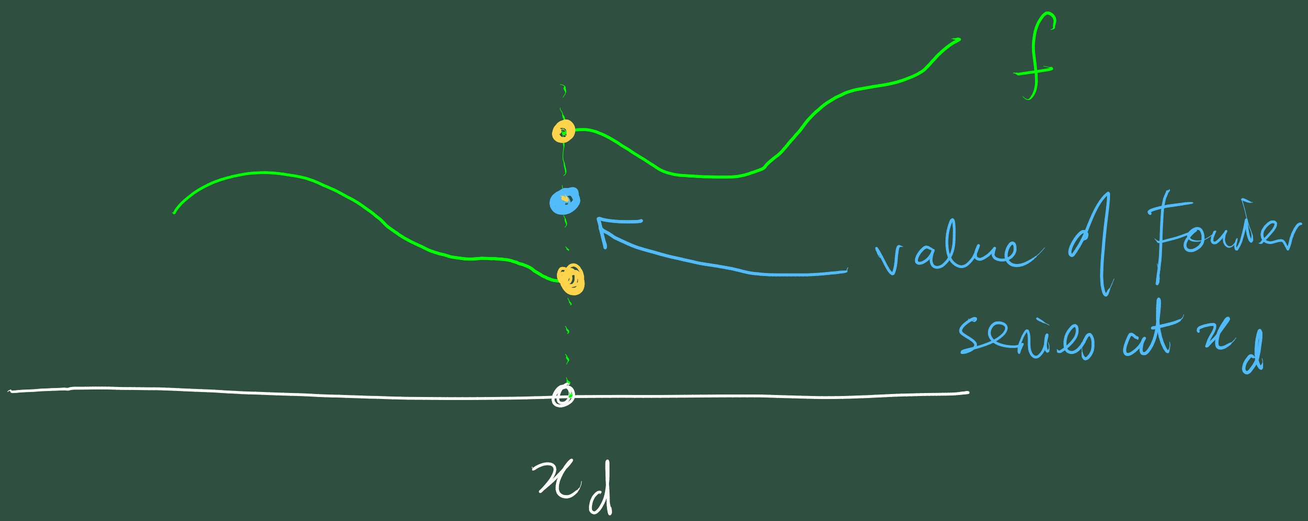
These coefficients will be given by expressions using the relevant standard inner product for such function spaces over the interval/domain $(-\pi, \pi)$ defined by

$$\langle \phi, \psi \rangle = \int_{-\pi}^{\pi} \phi(x) \psi(x) \, dx$$

i.e. $a_n \longleftrightarrow \langle f(x), \cos(nx)\rangle$

$b_n \longleftrightarrow \langle f(x), \sin(nx)\rangle$

$a_0 \longleftrightarrow \langle 1, f(x)\rangle$



Claim: F is orthogonal wrt $\langle , \rangle$.

~~About~~ 5 types of case to check.

Ex 5.1.2

Prove $\langle \cos(mx), \sin(nx)\rangle = 0$

$$\langle \cos(mx), \sin(nx)\rangle = \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx.$$

use trig. form.

$$\cos(A)\sin(B) = \frac{1}{2}\left(\sin(A+B) + \sin(B-A)\right)$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left[\sin((m+n)x) + \sin((n-m)x)\right] dx$$

assume $n \neq m$

$$= \frac{1}{2}\left[\frac{-\cos((m+n)x)}{m+n} - \frac{\cos((n-m)x)}{n-m}\right]_{-\pi}^{\pi}$$

Note cosine is an even function

i.e. $\cos(z) = \cos(-z)$

$= 0$, as required

When $n = m$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin((m+n)x)\, dx$$

$= 0$, for the same reason as above.

There are four other general cases to check.

i.e. $\langle \cos(nx), \cos(mx) \rangle$

$= \langle \sin(nx), \sin(mx) \rangle$

$$= \langle 1, \cos(nx) \rangle$$

$$= \langle 1, \sin(nx) \rangle \quad = 0$$

What if we take

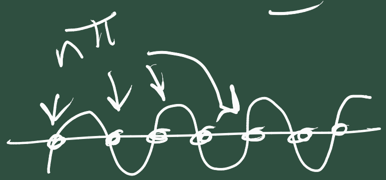
Ex 5.1.3

$$\langle \sin(nx), \sin(nx) \rangle$$

$$= \int_{-\pi}^{\pi} \sin^2(nx)\, dx$$

use trig. formula.

$$\sin^2(A) = \frac{1}{2}(1 - \cos(2A))$$



sin(x)

$$= \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos(2nx)\, dx.$$

$$= \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} 2\pi,$$

$\sin(m\pi) = 0$ for all $m \in \mathbb{Z}$

$$= \pi$$

Can also check that

$$\langle \cos(nx), \cos(nx) \rangle = \pi.$$

and

$$\langle 1, 1 \rangle = 2\pi.$$

<u>Ex 5.2.1</u> Can use existence of the Fourier series and linearity properties of the inner product to justify the coefficient formulas.

Consider

$$\langle f, \sin(nx) \rangle$$

$$= \left\langle \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos(mx) + \sum_{m=1}^{\infty} b_m \sin(mx), \sin(nx) \right\rangle$$

$$= \tfrac{1}{2}a_0 \underbrace{\langle 1, \sin(nx) \rangle}_{=0} + \sum_{m=1}^{\infty} a_m \underbrace{\langle \cos(mx), \sin(nx) \rangle}_{=0}$$

$$+ \sum_{m=1}^{\infty} b_m \langle \sin(mx), \sin(nx) \rangle$$

by linearity of $\langle \, , \, \rangle$.

$$= b_n \langle \sin(nx), \sin(nx) \rangle$$

, as all other $\langle \, , \, \rangle = 0$ by orthogonality of F.

$$= b_n \pi$$

, by previous example.

Therefore.

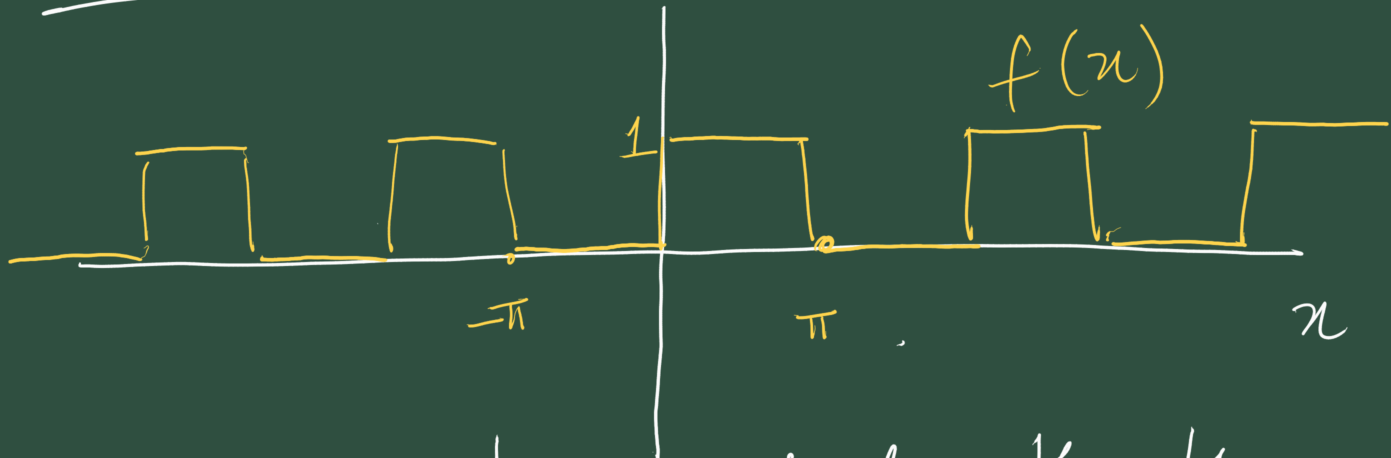
$$\langle f(x), \sin(nx) \rangle = b_n \pi$$

or in other words

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Other formulas are established in a similar way.

Ex 5.2.2. Our first Fourier series square wave function.



We'll want to evaluate the three integral formulas for a_0, a_n, b_n

$$\boxed{a_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\int_0^{\pi} dx = \frac{1}{\pi}\pi \boxed{= 1.}$$

For $n \geq 1$

$$\boxed{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} \boxed{= 0}$$

as $\sin(m\pi) = 0$ for all $m \in \mathbb{Z}$.

For $n \geq 1$

$$\boxed{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

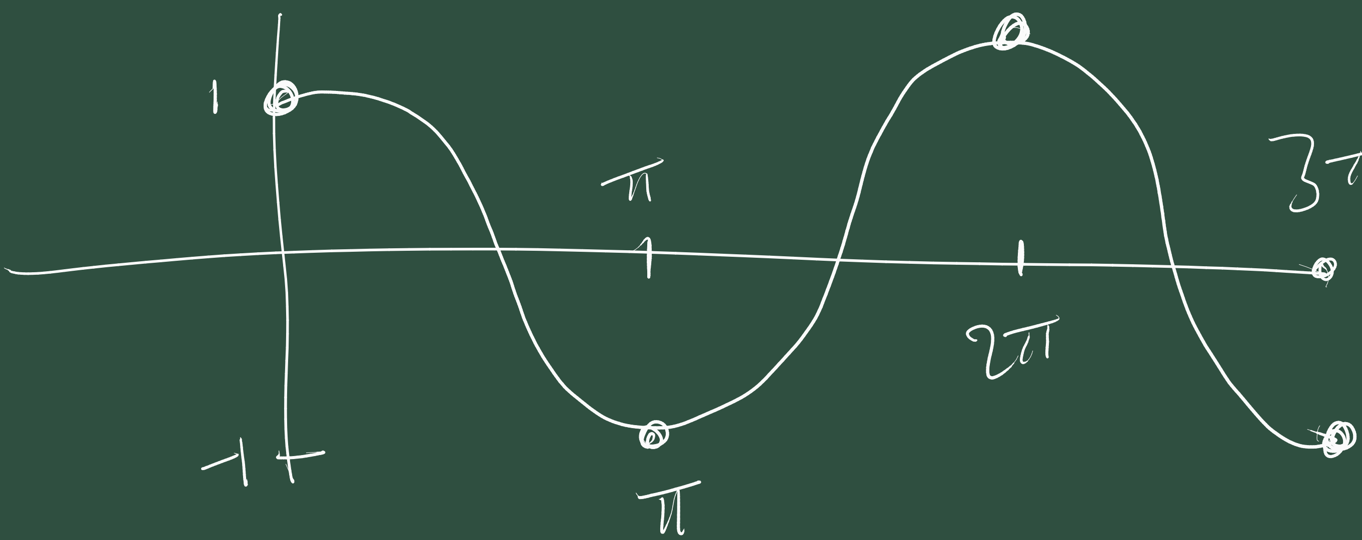
$$= \frac{1}{\pi}\int_0^{\pi} \sin(nx)\,dx.$$

$$= \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi}$$

$$= \frac{1}{n\pi}\left(-\cos(n\pi) + \cos(0)\right)$$

$$= \frac{1}{n\pi}\left(1 - \cos(n\pi)\right)$$

$$= \boxed{\begin{cases} 0 & , n \text{ even} \\ \dfrac{2}{n\pi} & , n \text{ odd} \end{cases}}$$



$$\cos(n\pi) = (-1)^n$$

So the Fourier series will be

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n\pi} \sin(nx)$$

$$= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2}{(2m-1)\pi} \sin\left((2m-1)x\right)$$

So we found the F.S. we expected.

What does it <u>look</u> like?