6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 02

Small increments formula questions

(1) Use the small increments formula to estimate the change in the function

$$f(x,y) = \left(x + 2y^2\right)^5$$

when x and y both increase from 1 to 1.01. Check your answer by direct evaluation of the values of f.

(2) The pressure exerted by a column of gas of density D and length L is $P = \frac{1}{3}DL^2$. Find the percentage change in pressure caused by a 1% increase in density and a 2% increase in length.

(3) The surface area S and the volume V of a right circular cone of base radius r and height h are given by

$$S = \pi r^2 + \pi r\left(r^2 + h^2\right)^{\frac{1}{2}}$$

$$V = \frac{1}{3}\pi r^2 h$$

(a) What percentage increase in h will cancel out a 2% decrease in r if the volume is to required to remain the same.

(b) In an attempt to estimate the value of S the radius r is measured as 5.00 ± 0.01mm and the height h as 12.00 ± 0.01mm. Estimate the value of S and quantify the possible error.
If it is possible to measure just one of the quantities h and r more accurately, which one would you choose? Justify your answer.

(4) Consider the function f of n variables defined by

$$f(x_1, x_2, \ldots, x_n) = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}.$$

Suppose that for each r there is an i_r% increase in the value of x_r. Show that the total increase in the value of f is $p_1 i_1 + p_2 i_2 + \cdots + p_n i_n$.

Chain rule and Jacobian questions

(5) Show that if the Cartesian coordinates (x, y) are transformed to the polar coordinates (r, θ) and $f(x, y)$ is a differentiable function of $x$and y then

$$x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \theta}.$$

(6) Consider the transformation from the coordinates (x, y) to a new pair (s, t) defined by

$$s = xy, \quad t = \frac{1}{y}.$$

If $f(x, y)$ is a differentiable function of x and y show that

$$y\frac{\partial f}{\partial y}\left(x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right) = t\frac{\partial f}{\partial t}\left(s\frac{\partial f}{\partial s} - t\frac{\partial f}{\partial t}\right).$$

(7) Show that the Jacobian of the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ) is given by

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

(8) Use the chain rule to prove that if a coordinate transformation from (x, y) to (s, t) is composed with one from (s, t) to (u, v) then

$$\frac{\partial(x, y)}{\partial(s, t)} \frac{\partial(s, t)}{\partial(u, v)} = \frac{\partial(x, y)}{\partial(u, v)}.$$

Hence establish the reciprocal rule that if a transformation from (x, y) to (s, t) is composed with the inverse transformation from (s, t) to (x, y) then

$$\frac{\partial(x, y)}{\partial(s, t)} \frac{\partial(s, t)}{\partial(x, y)} = 1.$$

(9) Show that the Jacobian of the transformation from polar coordinates (r, θ) to Cartesian coordinates (x, y) is given by

$$\frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}.$$

(10) The Cartesian coordinates (x, y) are transformed by the mapping

$$s = y - x, \quad t = (y - x)^2.$$

Complete the following table of coordinate values

x	0	0	1	1	2	0	2	1	2	3	0	3	1	3	2	3
y	0	1	0	1	0	2	1	2	2	0	3	1	3	2	3	3
s																
t																

Plot the resulting points (s, t) from the table and comment on what you notice. Find the Jacobian $\frac{\partial(s,t)}{\partial(x,y)}$. What can be said about the inverse transformation?