$$\frac{d^2y}{dt^2} = 0$$

This is simple enough to solve by repeated integration

$$\frac{dy}{dt} + C_1 = C_2$$

i.e. $\frac{dy}{dt} = C$, C incorporates both constants C_1, C_2

Integrate again.

$$y(t) = Ct + D$$

a solution paramatrised by two parameters C, D.

If we know that $y(0)=1$ & $y(1)=2$. this will determine C and D.

$y(0) = 1 = D$.

$$y(1) = 2 = C + D.$$
$$\implies C = 1$$

So $y(t) = t + 1$ is the unique solution.

Some general about "transforms"

Very often in mathematics when ~~pro~~ presented by a hard problem in one domain, can be transformed into an easier problem in another domain

e.g. use of logarithms and exponentials in pre-computer arithmetic

<u>Hard cases</u> $\xrightarrow{\log}$ <u>Easier</u>

a^b $\qquad$ $\log(a^b)$

$= b \log(a)$

$$ab \xleftarrow{\exp} \log(ab) = \underbrace{\log(a) + \log(b)}$$

The Laplace transform does this kind of job for ODEs.

Def:

$$\boxed{\mathcal{L}\{f(t)\} = \underbrace{\int_0^\infty e^{-st} f(t)\, dt}_{\text{is a function of this new variable } s.}}$$

$$= \bar{f}(s) = F(s)$$

We can see this integral transform as perhaps a generalization of the notion of a power series.

Powerseries

$(a_0, a_1, a_2, \dots)$ an ∞ sequence of $a_i \in \mathbb{R}$

$$\rightsquigarrow A(x) = \sum_{i=0}^{\infty} a_i x^i$$

$\hookrightarrow f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$

where $f(i) = a_i$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$e^{-st} = (e^{-s})^t$$

$f : \mathbb{R}_{\geq 0} \to \mathbb{R}$

Let's find some transforms.

Consider the constant function

$f(t) = 1$, for all $t \geq 0$.

$$\boxed{\mathcal{L}\{1\}} = \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

$$= \frac{-1}{s}\left(\lim_{t\to\infty}(e^{-st}) - 1\right)$$

$$\boxed{= \frac{1}{s}} \quad , \text{ for } s>0$$

$$e^{-st} = \frac{1}{e^{st}} \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

4.2.3 Consider f defined by

$f(t) = t$ for all $t \geq 0$.

$$\boxed{\mathcal{L}\{t\}} = \int_0^\infty e^{-st}\, t \, dt.$$

$$= \int_0^\infty \frac{d}{dt}\left(\frac{e^{-st}}{-s}\right) t \, dt.$$

$$= \left[\frac{e^{-st}}{-s} t\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \, dt.$$

, integration by parts.

$$= \frac{1}{-s}\left(\lim_{t\to\infty} (te^{-st}) - 0 \right)$$

$$+ \frac{1}{s}\int_0^\infty e^{-st}\, dt.$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\, dt \qquad \text{for } s>0$$

$$= \boxed{\frac{1}{s^2}}, \quad \text{for } s>0.$$

Th 4.1 $\mathcal{L}$ is linear

Pf: $\mathcal{L}\{\alpha f + \beta g\}$

$$= \int_0^\infty e^{-st} (\alpha f + \beta g)(t)\, dt$$

$$= \int_0^\infty e^{-st} (\alpha f(t) + \beta g(t))\, dt.$$

$$= \int_0^\infty \alpha\, \underline{e^{-st} f(t)} + \beta\, \underline{e^{-st} g(t)}\, dt$$

linearity of integration

$$\alpha \int_0^\infty e^{-st} f(t)\, dt + \beta \int_0^\infty e^{-st} g(t)\, dt.$$

$$= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}.$$

Application of th 4.4.3 in Example 4.4.4.

We know $\mathcal{L}\{t\} = \frac{1}{s^2}$

$$\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\}$$

$$= -\frac{d}{ds} \mathcal{L}\{t\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2}\right)$$

$$= \frac{2}{s^3}$$

$$\boxed{\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\}} = ?$$

$$= \mathcal{L}\left\{ \frac{d}{dt}\left(\frac{dx}{dt}\right) \right\}.$$

$$= s\,\mathcal{L}\left\{\frac{dx}{dt}\right\} - \left.\frac{dx}{dt}\right|_{t=0}$$

$$= s\left(s\,\mathcal{L}\{x(t)\} - x(0) \right) - \left.\frac{dx}{dt}\right|_{t=0}$$

$$\boxed{= s^2\,\mathcal{L}\{x(t)\} - s\,x(0) - \left.\frac{dx}{dt}\right|_{t=0}}$$

We shall talk of the inverse transform $\mathcal{L}^{-1}$, as the reverse of $\mathcal{L}$.

NB: $\mathcal{L}^{-1}$ will be linear also.

Ex. 4.5.1 Solve. (in simpler notation).

$$\dot{y} = \frac{dy}{dt}$$

$$\ddot{y} - 5\dot{y} + 4y = 12.$$

$$\ddot{y} = \frac{d^2y}{dt^2}$$

Subject to the initial conditions.

$$y(0) = \dot{y}(0) = 0$$

1. $\mathcal{L}\{\ddot{y} - 5\dot{y} + 4y\} = \mathcal{L}\{12\}$

$\Rightarrow \mathcal{L}\{\ddot{y}\} - 5\mathcal{L}\{\dot{y}\} + 4\mathcal{L}\{y\} = 12\mathcal{L}\{1\}$, applying linearity

Write $\bar{y} = \mathcal{L}\{y\}$

$\Rightarrow s^2\bar{y} - s\,y(0) - \dot{y}(0) - 5\,(s\bar{y} - y(0)) + 4\bar{y}$

$= \dfrac{12}{s}$

2. $s^2\bar{y} - 5s\bar{y} + 4\bar{y} = \dfrac{12}{s}$

3. $(s^2 - 5s + 4)\,\bar{y} = \dfrac{12}{s}$

$$\Rightarrow \quad \bar{y} = \frac{12}{s\,(s^2 - 5s + 4)}$$

A solution for $\bar{y}$!!

Before reading inverse transform the table we need the partial fraction expansion of $\bar{y}$ in order to see it as a linear comb. of transforms appearing on the table.

$$\bar{y} = \frac{12}{s\,(s-4)(s-1)}$$

$$= \frac{\alpha}{s} + \frac{\beta}{s-4} + \frac{\gamma}{s-1}$$

for some, as yet unknown, constants α, β, γ.

$$= \frac{\alpha(s^2-5s+4) + \beta(s^2-s) + \gamma(s^2-4s)}{s(s-4)(s-1)}$$

$$= \frac{(\alpha+\beta+\gamma)s^2 + (-5\alpha-\beta-4\gamma)s + 4\alpha}{s(s-4)(s-1)}$$

This numerator poly. must equal 12 (for all s)

$$\Longrightarrow \quad \begin{aligned} \alpha+\beta+\gamma &= 0 \\ -5\alpha-\beta-4\gamma &= 0 \\ 4\alpha &= 12. \end{aligned}$$

$$\Rightarrow \alpha = 3 \quad \text{and}$$

$$\begin{array}{r} \beta + \gamma = -3 \\ -(\beta + 4\gamma = -15) \\ \hline -3\gamma = 12 \end{array}$$

$$\Rightarrow \gamma = -4.$$

$$\Rightarrow \beta = 1$$

So now we can say

$$\bar{y} = \frac{3}{s} + \frac{1}{s-4} - \frac{4}{s-1}.$$

4. We get the solution $y(t)$ by taking inverse transform of $\bar{y}$

$$\boxed{y(t)} = \mathcal{L}^{-1}\{\bar{y}\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{3}{s} + \frac{1}{s-4} - \frac{4}{s-1} \right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

, by linearity.

$$\boxed{= 3 + e^{4t} - 4e^{t}}$$