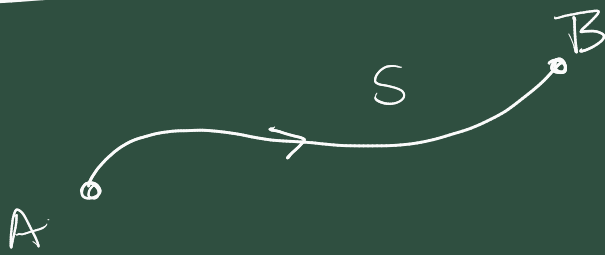
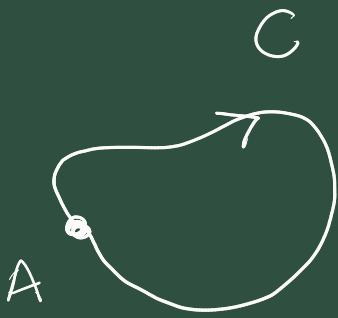


Green's Theorem



$$\int_S$$



$$\oint_C$$

indicates
that C is a closed curve

Example 3.8.3 $P(x,y)$ $Q(x,y)$

$$I = \oint_C \underbrace{3x^2y^2 dx + 2xy dy}_L$$

where C is the closed curve

DEFG as shown

We should break I down into
the sum of four sub-integrals

$$I = \int_{DE} L + \int_{EF} L + \int_{FG} L + \int_{GD} L$$

On DE: $y=1, dy=0, x:1 \rightarrow 2$

On EF: $x=2, dx=0, y:1 \rightarrow 2$

On FG: $y=2, dy=0, x:2 \rightarrow 1$

On GD: $x=1, dx=0, y:2 \rightarrow 1$

$$I = \int_1^2 3x^2 dx + \int_1^2 4y dy + \int_2^1 12x^2 dx$$

$$+ \int_2^1 2y dy$$

$$= \left[x^3 \right]_1^2 + \left[2y^2 \right]_1^2 + \left[4x^3 \right]_2^1$$

$$+ \left[y^2 \right]_2^1$$

$$= 8 - 1 + 8 - 2 + 4 - 32$$

$$+ 1 - 4 = -18$$

Keep this
mind.

Green's Theorem

First: Verify the theorem for the previous example.

In the previous example

$$P(x, y) = 3x^2 y^2$$

$$Q(x, y) = 2xy.$$

Green's theorem says that

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = -18$$

where R is the square region.

$$1 \leq x, y \leq 2.$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

mostly
negative
for $1 \leq x, y \leq 2$

$f(x, y)$

$$= \iint_R (2y - 6x^2 y) dx dy$$

$$= \int_1^2 \left(\int_1^2 (2y - 6x^2 y) dx \right) dy$$

$$= \int_1^2 \left[2yx - 2x^3 y \right]_{x=1}^{x=2} dy$$

$$= \int_1^2 4y - 16y - (2y - 2y) dy$$

$$= \int_1^2 -12y \, dy$$

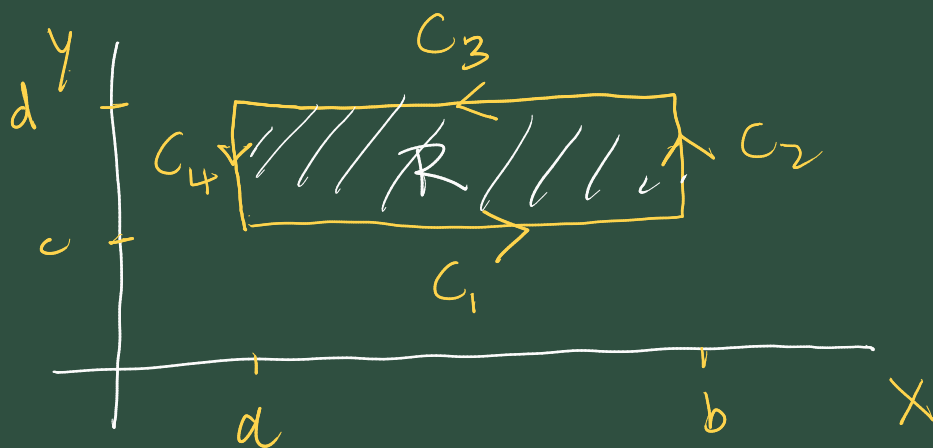
$$= - \left[6y^2 \right]_1^2$$

$$= - [24 - 6] = \underline{\underline{-18}},$$

as expected from statement
of Green's Theorem.

Sketch of one way to prove G.T.

Consider first, a rectangle aligned
with the coord. axes.



We'll prove G.T. for this.

$$C_R = C_1 + C_2 + C_3 + C_4$$

On C_1 : $y = c, \, dy = 0, \, x: a \rightarrow b$

" C_2 : $x = b, \, dx = 0, \, y: c \rightarrow d$

$$\therefore C_3 : y=d, dy=0, x=b \rightarrow a$$

$$\therefore C_4 : x=a, dx=0, y=d \rightarrow c.$$

So $\oint_{C_4} P dx + Q dy$ $P(x,y)$
 $Q(x,y)$

L.H.S. of GT.

$$= \int_a^b P(x, c) dx + \int_c^d Q(b, y) dy$$

$$+ \int_b^a P(x, d) dx + \int_d^c Q(a, y) dy$$

→ as far as we can evaluate it,
without knowing P, Q .

Now for the R.H.S of GT.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$= \iint_R \frac{\partial Q}{\partial x} dx dy - \iint_R \frac{\partial P}{\partial y} dx dy,$$

by linearity

$$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x} dx \right) dy$$

$$- \int_a^b \left(\int_c^d \frac{\partial P}{\partial y} dy \right) dx$$

, forming into repeated integrals

$$= \int_c^d \left[Q(x, y) \right]_a^b dy - \int_a^b \left[P(x, y) \right]_c^d dx$$

$$= \int_c^d (Q(b, y) - Q(a, y)) dy - \int_a^b (P(x, d) - P(x, c)) dx$$

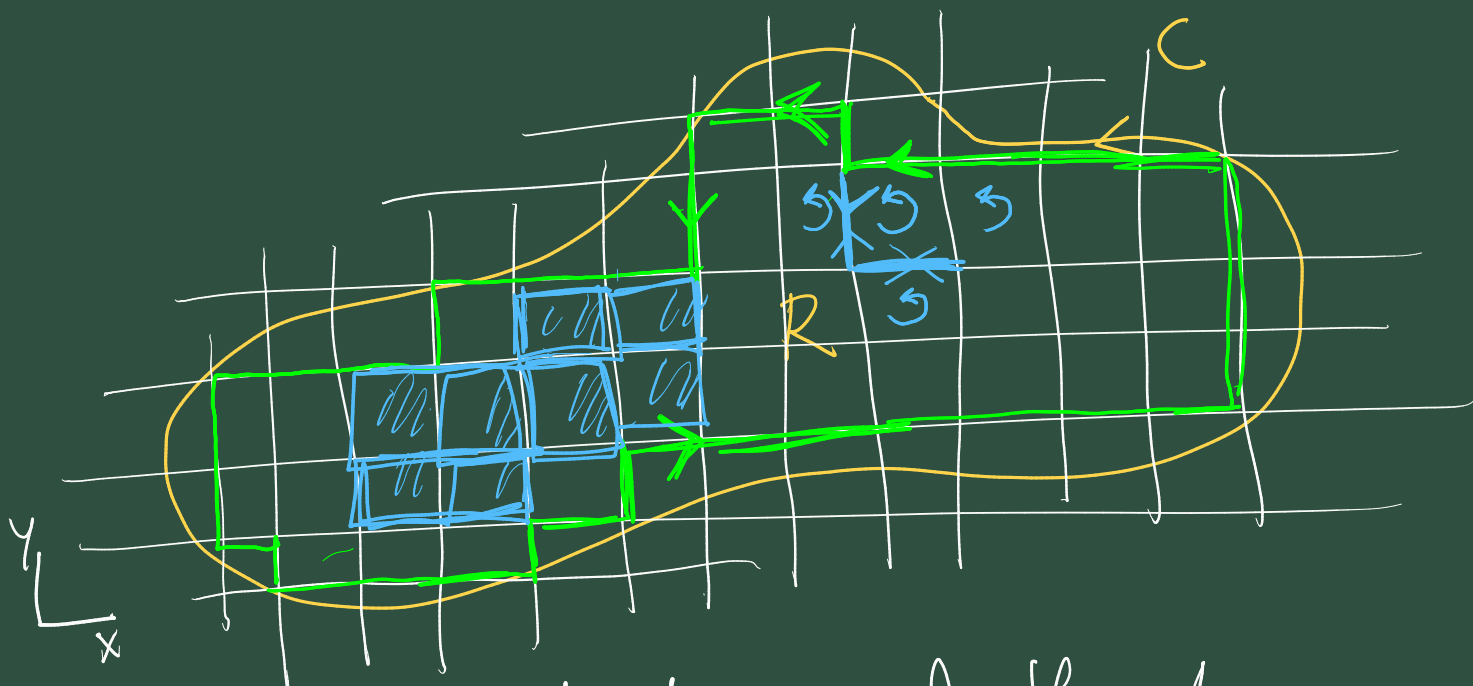
$$= \left[\int_c^d Q(b, y) dy + \int_d^c Q(a, y) dy + \int_b^a P(x, d) dx + \int_a^b P(x, c) dx \right]$$

(note change of direction in 2nd and 3rd integrals)

== L.H.S. of GT above.

So G.T. is true for these rectangles.

Consider now a general simply connected region R .



Consider a sub-division of the plane into rectangles. The union of the rectangles inside R , approximates R .
As the size of the rectangles $\rightarrow 0$
this approx $\rightarrow R$.

RHS of G.T.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\approx \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

union
of rectangles

$$= \sum_{\square} \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$= \sum_{\square} \oint P dx + Q dy, \quad \begin{array}{l} \text{by the} \\ \text{simple} \\ \text{case} \\ \text{of G.T.} \\ \text{proved} \\ \text{above} \end{array}$$

$$\approx \oint_C P dx + Q dy, \quad \text{L.H.S. of G.T.}$$

note that integrals along all edges of interior rectangles cancel out with the integrals of neighboring rectangles except for those near the boundary.

So we have an approx. form
of G.T. which becomes =
in the limit as size of
rectangles $\rightarrow 0$ (i.e. their number
 $\rightarrow \infty$).

Changing coordinate system in a double integral

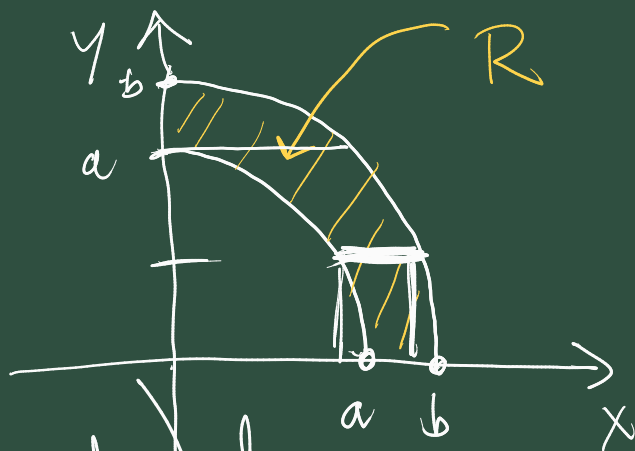
Consider this integral

$$I = \iint_R \frac{1}{x^2 + y^2} dx dy$$

"between circles"

over the region R , the annular region in the upper-right quadrant between circles of radius a and b .

$$\begin{aligned} x^2 + y^2 &= a^2 \\ x^2 + y^2 &= b^2 \end{aligned}$$



$$I = \int_0^a \left(\int_{\sqrt{a^2 - y^2}}^{\sqrt{b^2 - y^2}} \frac{1}{x^2 + y^2} dx \right) dy$$

$$+ \int_a^b \left(\int_0^{\sqrt{b^2 - y^2}} \frac{1}{x^2 + y^2} dx \right) dy$$

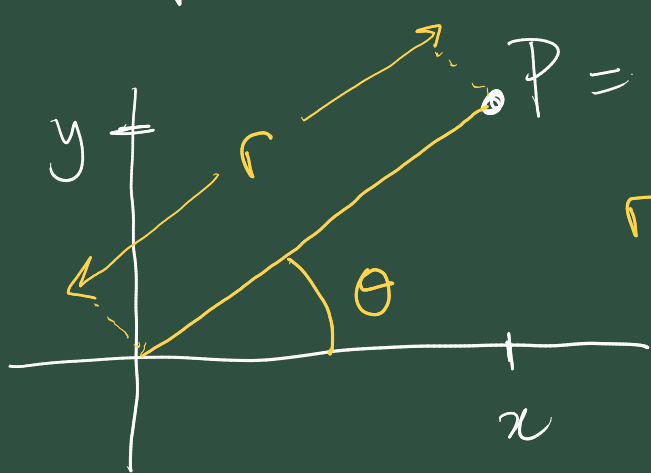
= my heart sinks.... looks awkward. Is there a better way?

Observation: everything about this integral look "circular"

The region is "circular"
and $x^2 + y^2$ is the (radial distance)² of (x, y) from $(0, 0)$

We should adopt the coordinate best suited to describing circles.

The polar coordinates (r, θ) .



r = length of radius defined by P

θ = angle between radius and x -axis.

Transformation equations

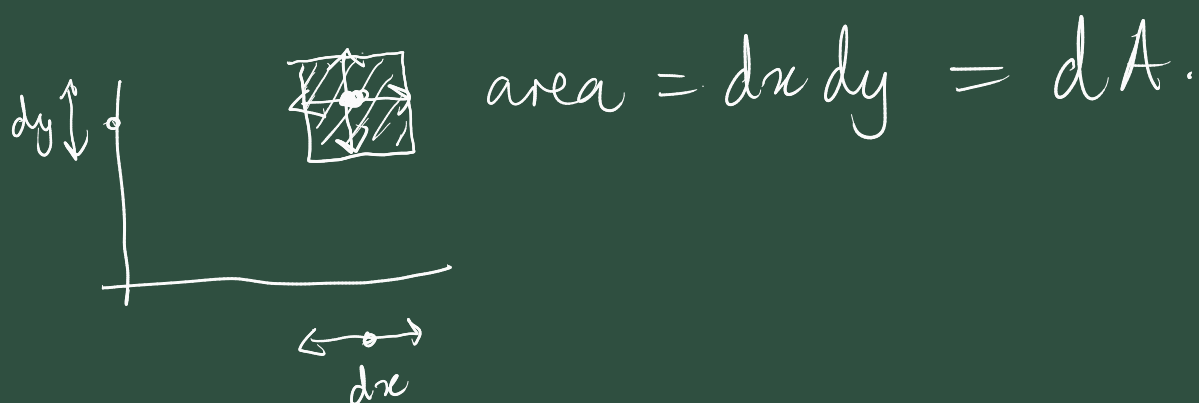
$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

We'd like to "lower" the integral into r, θ .

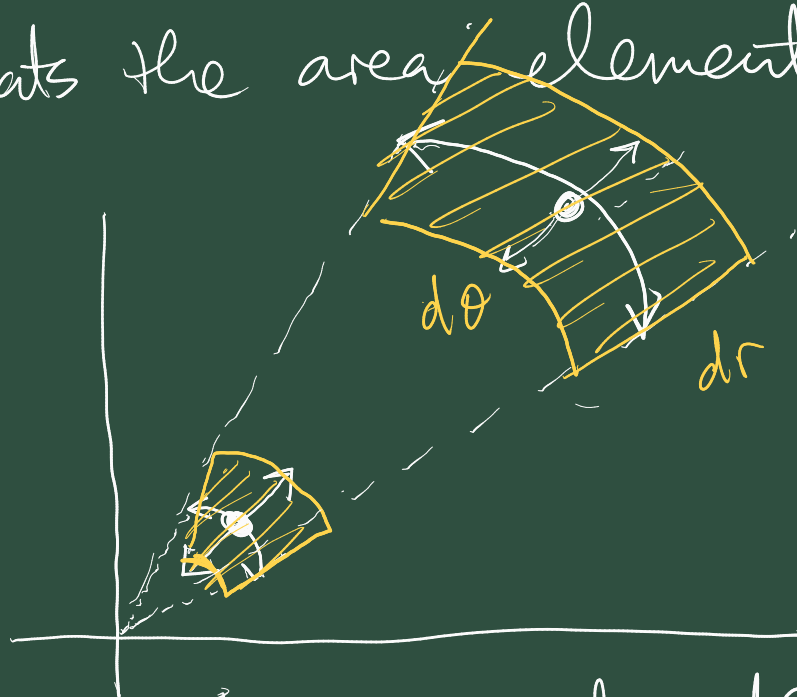
What happens to the differentials $dx dy$?

$dx dy$ is the "area element"

in Cartesian coordinates.



What's the area element in (r, θ) ?



The same $dr, d\theta$ produce a larger area, the further we are from the origin.

It turns out that the correct

area element for varying r, θ is

$$dA = r dr d\theta$$

In general the relation ~~between~~
between area elements for
coord systems (x, y) and (s, t)
is given by the following formula
featuring the Jacobian determinant

$$dx dy = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} ds dt$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

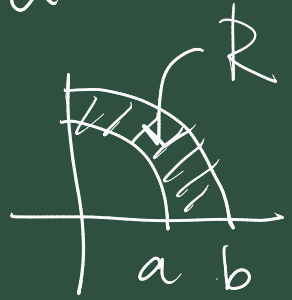
Jacobian determinant
of $(x, y) \rightarrow (s, t)$

For polar coords.

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$\begin{aligned}
 &= r \cos^2 \theta + r \sin^2 \theta \\
 &= r (\cos^2 \theta + \sin^2 \theta) \\
 &= r.
 \end{aligned}$$

Let's return to the initial integral



$$I = \iint_R \frac{1}{x^2 + y^2} \underbrace{dx dy}_1$$

$$= \iint_R \frac{1}{r^2} r dr d\theta$$

$$= \iint_R \frac{1}{r} dr d\theta$$

$$= \int_0^{\pi/2} \left(\int_a^b \frac{1}{r} dr \right) d\theta,$$

looks
much
nicer

$$= \int_0^{\pi/2} \left[\ln(r) \right]_a^b d\theta$$

$$= \int_0^{\pi/2} (\ln(b) - \ln(a)) d\theta$$

$$= \int_0^{\pi/2} \ln\left(\frac{b}{a}\right) d\theta$$

$$= \ln\left(\frac{b}{a}\right) \int_0^{\pi/2} d\theta, \text{ linearity.}$$

$$= \frac{\pi}{2} \ln\left(\frac{b}{a}\right).$$

This concept of translating integrals into other coord systems extends to higher dimensions too.

$$\int \dots \int_R f(x_1, \dots, x_n) dx_1 \dots dx_n$$

$$= \int \dots \int_R f(\dots, x_i(u_1, \dots, u_n), \dots) \dots \left| \frac{\partial(x_1, \dots, x_n)}{\partial(u_1, \dots, u_n)} \right| du_1 \dots du_n$$

for the transformation
 $(x_1, \dots, x_n) \longrightarrow (u_1, \dots, u_n)$

and

$$\frac{\gamma(x_1, \dots, x_n)}{\delta(u_1, \dots, u_n)} = \underbrace{\left| \frac{\partial x_i}{\partial u_j} \right|}_{\text{entry in } i^{\text{th}} \text{ row and } j^{\text{th}} \text{ col.}}$$

Look out for videos
on the 3-d coord systems

- spherical coordinates
- cylindrical coordinates.