6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 05 - SOLUTIONS

Solutions to questions 1 – 5 listed on the following pages under the heading of *Exercise 10*

Solutions to questions 6 – 8 listed on the following pages under the heading of *Exercise 11*

BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Partial Integration Worked Solutions

Exercise 10

Q1. $$\int_1^2\int_0^3 (x^2y + y^2x)\,dx\,dy = \int_1^2 \left[\frac{x^3}{3}y + y^2\frac{x^2}{2} \right]_0^3 dy$$

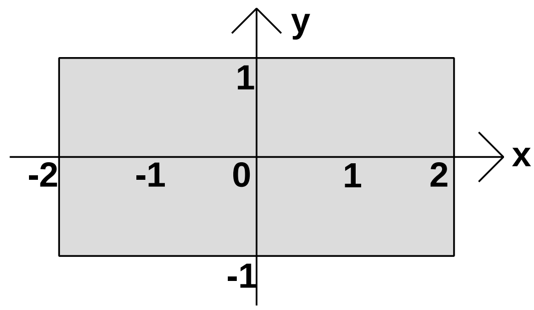
$$= \int_1^2 9y + \frac{9}{2}y^2 \, dy = \left[\frac{9}{2}y^2 + \frac{3}{2}y^3 \right]_1^2$$

$$= (18 + 12) - \left(\frac{9}{2} + \frac{3}{2}\right)$$

$$= 24.$$

This is the volume under the surface, $z = x^2y + y^2x$ above the rectangle $0 \leqslant x \leqslant 1$, $1 \leqslant y \leqslant 2$.

Q2. Region, R is the shaded area shown in the diagram.



$$\int_{-1}^1\int_{-2}^2 (4xy + \sin x + \cos y)\,dx\,dy = \int_{-1}^1 \left[(2x^2y - \cos x + x\cos y) \right]_{-2}^2 dy$$

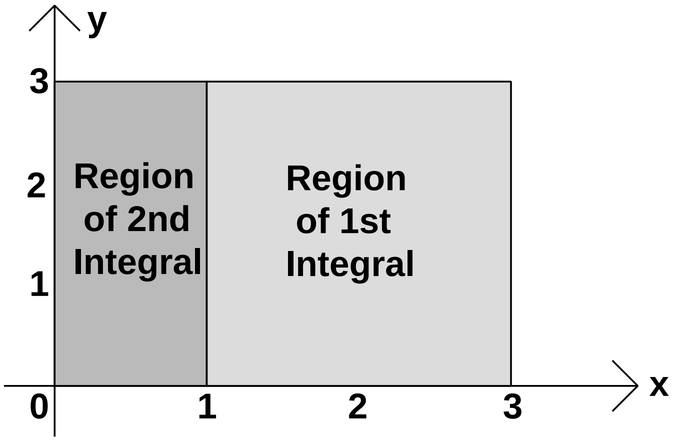
$$= \int_{-1}^1 \left[(8y - \cos 2 + 2\cos y) - (8y - \cos(-2) - 2\cos y) \right] dy$$

$$= \int_{-1}^1 4\cos y \, dy$$

$$= \left[4\sin y \right]_{-1}^1$$

$$= 8\sin 1$$

Q3.



In the first integral x ranges from 1 to 3 and y from 0 to 3. In the second integral y ranges from 0 to 3 and x from 0 to 1. So for combined region x ranges from 0 to 3 and y ranges from 0 to 3.

Combined integral = $\int_0^3 \int_0^3 (9x^2y^2 + 4xy + 5)dxdy$

$$= \int_0^3 \Big[3x^3y^2 + 2x^2y + 5x \Big]_0^3 dy$$

$$= \int_0^3 \Big[81y^2 + 18y + 15 \Big] dy$$

$$= \Big[27y^3 + 9y^2 + 15y \Big]_0^3$$

$= 855$

Q4.a) $\int_0^1 \int_0^{0.5} ye^{xy}\, dydx = \int_0^{0.5} \int_0^1 ye^{xy}\, dxdy$

$$= \int_0^{0.5} \Big[e^{xy} \Big] dy$$

$$= \int_0^{0.5} (e^y - 1)dy$$

$$= \Big[e^y - y \Big]_0^{0.5} = 0.149$$

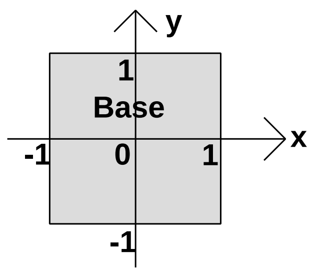
Q4.b) $$\int_0^1 \int_0^{0.5} x \sin xy \, dx dy = \int_0^{0.5} \int_0^1 x \sin xy \, dy dx$$

$$= \int_0^{0.5} \Big[-\cos xy \Big]_0^1 dx$$

$$= \int_0^{0.5} (1 - \cos x) dx$$

$$= \Big[x - \sin x \Big]_0^{0.5} = 0.021$$

Q5. Average height of surface $= \dfrac{\text{Volume of Surface}}{\text{Area of Base}}$



Area of base $= 2 \times 2 = 4$

$$\text{Volume of surface} = \int_{-1}^{1} \int_{-1}^{1} (x^2 + y^2) \, dx dy$$

$$= \int_{-1}^{1} \Big[\frac{x^3}{3} + xy^2) \Big]_{-1}^{1} dy$$

$$= \int_{-1}^{1} (\frac{2}{3} + 2y^2) \, dy$$

$$= \Big[\frac{2y}{3} + \frac{2y^3}{3} \Big]_{-1}^{1}$$

$$= \frac{8}{3}$$

Therefore the average height of the surface $= \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}$

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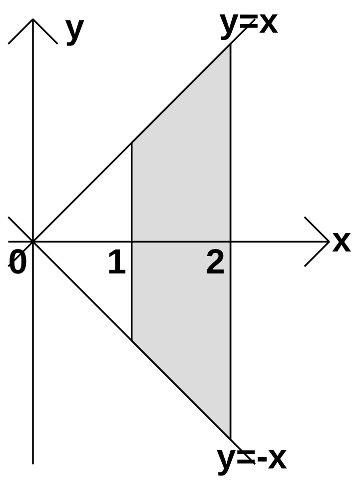
Exercise 11



Q1. The region of integration is the shaded area shown in the diagram.

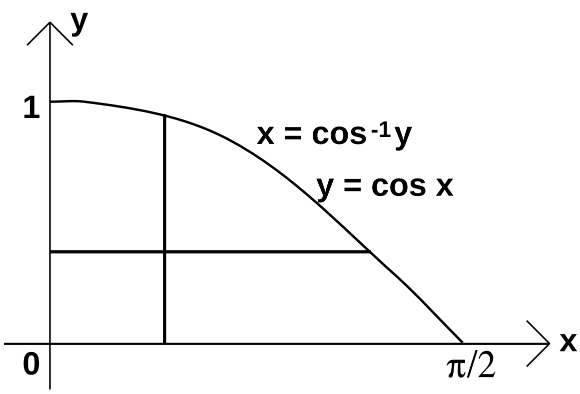
$$\int_1^2\int_{-x}^{x}\left(\frac{y}{x}+1\right)dydx = \int_1^2\left[\frac{y^2}{2x}+y\right]_{-x}^{x}dx$$

$$= \int_1^2\left(\frac{x}{2}+x\right)-\left(\frac{x}{2}-x\right)dx$$

$$= \int_1^2 2x\,dx = \Big[\;x^2\;\Big]_1^2 = 3$$

Q2.a) If $x = \cos^{-1}y$, then $y = \cos x$.

For the given integral x ranges from 0 to $\cos^{-1}y$ and then y ranges from 0 to 1 so the region of integration is that shown in the diagram. Equivalently y ranges from 0 to cos x and then x ranges from 0 to $\pi/2$. Then



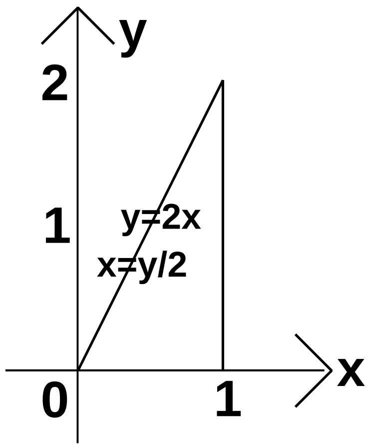
$$\int_0^1\int_0^{\cos^{-1}y}\sec x\,dxdy = \int_1^{\pi/2}\int_0^{\cos x}\sec x\,dydx$$

$$= \int_0^{\pi/2}\Big[\;y\sec x\;\Big]_0^{\cos x}dx$$

$$= \int_0^{\pi/2}(\cos x\sec x - 0)\,dx$$

$$= \int_0^{\pi/2}1\,dx = \pi/2$$

Q2.b) For the given integral x ranges from y/2 to 1 and then y ranges from 0 to 2 so the region of integration is that shown in the diagram. Equivalently y ranges from 0 to 2x and then x ranges from 0 to 1. Then



$$\int_0^2\int_{y/2}^1 e^{x^2}\,dxdy = \int_0^1\int_0^{2x} e^{x^2}\,dydx$$

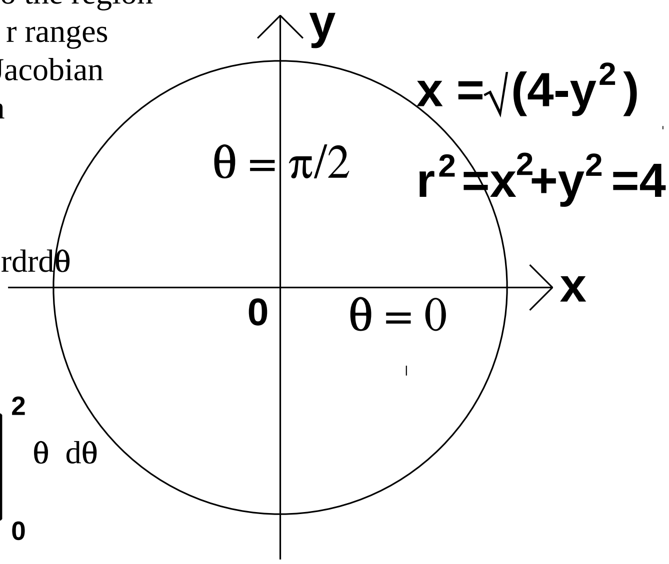
$$= \int_0^1 \Big[ye^{x^2}\Big]_0^{2x}\,dx$$

$$= \int_0^1 2xe^{x^2}\,dx$$

$$= \int_0^1 e^u du \quad \text{where } u = x^2 \text{ so that } du = 2x\,dx$$

$$= e - 1$$

Q3.a) Since $r^2 = x^2 + y^2$, $(x^2 + y^2)^{5/2} = r^5$. Also $\tan^{-1}(y/x) = \theta$. If $x = \sqrt{(4-y^2)}$, $r^2 = x^2 + y^2 = 4$ so $r = 2$. For the given integral x ranges from 0 to $\sqrt{(4-y^2)}$ and then y ranges from 0 to 2 so the region of integration is that shown in the diagram. Equivalently r ranges from 0 to 2 and then θ ranges from 0 to $\pi/2$. Finally the Jacobian from (x,y) to (r,θ) is r so dxdy is replaced by $rdrd\theta$. Then

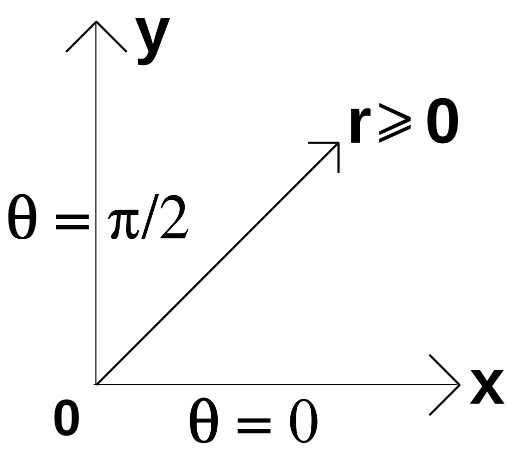


$$\int_0^2\int_0^{\sqrt{(4-y^2)}} (x^2 + y^2)^{5/2}\tan^{-1}(y/x)\,dxdy = \int_0^{\pi/2}\int_0^2 r^5\theta\, rdrd\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^7}{7}\right]_0^2 \theta\, d\theta$$

$$= \frac{128}{7}\int_0^{\pi/2} \theta\, d\theta = \frac{16\pi^2}{7}$$

Q3.b) For the given integral x and y both range from 0 to ∞. The region of integration is therefore the first quadrant where $0 \leqslant \theta \leqslant \pi/2$ and r ranges from 0 to ∞.
Also $r^2 = x^2 + y^2$ and the Jacobian from (x,y) to (r,θ) is r.
Then



$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dxdy = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\infty} \frac{e^{-u}}{2}\,du\,d\theta$$ using the substitution $u = r^2$.

$$= \int_0^{\pi/2}\left[\frac{-e^{-u}}{2}\right]_0^{\infty} d\theta$$

$$= \int_0^{\pi/2}\frac{1}{2}\,d\theta = \frac{\pi}{4}$$