Line Integrals

0.1 Introduction

Before continuing on to chapter 3 of CLP on multiple integrals we have a short section on another type of integral it is possible to do with functions of two or more variables. These are called *line integrals*, or sometimes *path integrals*.

0.2 Line integrals

An expression of the form,

$$L = P(x, y)\, dx + Q(x, y)\, dy, \tag{1}$$

is called a *linear differential form* in the variables x and y. Recall that for functions of a single variable we met integrals of the form

$$I = \int_a^b f(x)\, dx, \tag{2}$$

which involved integrating the function f over the interval from $x = a$ to $x = b$.

Similar integrals occur in the calculus of two-variable functions. These integrals take place over a line, or curve, C in the xy-plane. Their integrands are linear differential forms and they take the form

$$I = \int\limits_C P(x, y)\, dx + Q(x, y)\, dy. \tag{3}$$

One way of evaluating such integrals is to use a specification of the curve C to reduce I to an integral of a one-variable function, as the following example illustrates.

Example 0.1. We will integrate the form

$$L = 10x^2y\, dx + (3x + 2y)\, dy, \tag{4}$$

along the curve

$$C : \quad y = x^2, \tag{5}$$

from the point $(0, 0)$ to the point $(1, 1)$.

Solution. On the curve C, since $y = x^2$, we also have

$$dy = 2x\, dx, \tag{6}$$

and so making these substitutions in the integrand, the integral becomes

$$\begin{aligned} I &= \int\limits_C 10x^2y\, dx + (3x + 2y)\, dy, && (7)\\ &= \int\limits_C 10x^4 + 4x^3 + 6x^2\, dx, && (8)\\ &= \int_0^1 10x^4 + 4x^3 + 6x^2\, dx, && (9)\\ &= \left[2x^5 + x^4 + 2x^3\right]_0^1, && (10)\\ &= 5. && (11) \end{aligned}$$

□

Theorem 0.1 (Properties of line integrals)**.** *These properties are all generalizations of equivalent properties that hold for definite integrals of functions of a single variable.*

- *Line integrals are linear, and in particular*

$$\int\limits_C P(x,y)\,dx + Q(x,y)\,dy = \int\limits_C P(x,y)\,dx + \int\limits_C Q(x,y)\,dy. \tag{12}$$

- *If C is a curve and C' is the same curve, but running in the* opposite *direction, then for any linear differential form L,*

$$\int\limits_C L = -\int\limits_{C'} L. \tag{13}$$

- *If a path is split into components then the integral over the path is the sum of the integrals over the components. For example if AB denotes a curve from a point A to a point B, BC denotes a curve from B to C and AC denotes the union of the two curves, then for any linear differential form L*

$$\int\limits_{AC} L = \int\limits_{AB} L + \int\limits_{BC} L. \tag{14}$$

Example 0.2. Integrate the linear differential form

$$L = 10x^2y\,dx + (3x+2y)\,dy \tag{15}$$

from the point $A = (0,0)$ to the point $B = (1,1)$ along the path made up of the two straight line segments AD and DB, where D is the intermediate point $D = (1,0)$.

Solution. Let $C = AD + DB$ denote the overall path, then using theorem 0.1 we can write the integral as

$$\int\limits_C 10x^2y\,dx+(3x+2y)\,dy = \int\limits_{AD} 10x^2y\,dx+(3x+2y)\,dy + \int\limits_{DB} 10x^2y\,dx+(3x+2y)\,dy. \tag{16}$$

On the portion AD we have $y = 0$ and $dy = 0$ and so

$$\int\limits_{AD} 10x^2y\,dx + (3x+2y)\,dy = 0. \tag{17}$$

On the portion DB we have $x = 1$ and $dx = 0$ and so

$$\int\limits_{DB} 10x^2y\,dx + (3x+2y)\,dy = \int_0^1 3 + 2y\,dy = 4. \tag{18}$$

And so we have the result that $\int_C L = 4$. $\square$

0.3 Line integrals along parametrised paths

When the path of integration is *parametrised*, i.e. the x and y coordinates are given as functions of an independent variable t, then a line integral over such a path can often be evaluated by converting everything to the independent variable t to obtain a standard single-variable integral.

Example 0.3. Evaluate the line integral

$$I = \int_C x^2\,dy - yx\,dx, \tag{19}$$

where C is the semi-circular path, of radius 1 and traversed clockwise, from the point $(-1,0)$ to $(1,0)$.

Solution. The path C is parametrised by

$$x(t) = \cos(t),\ y(t) = \sin(t),\ t = \pi \to 0. \tag{20}$$

Therefore

$$dx = -\sin(t)\,dt,\ dy = \cos(t)\,dt, \tag{21}$$

and the integral becomes

$$\begin{aligned} I &= \int_C x^2\,dy - yx\,dx, && (22)\\ &= \int_\pi^0 \cos^2(t)\cos(t) + \sin(t)\cos(t)\sin(t)\,dt, && (23)\\ &= \int_\pi^0 \cos(t)\left(\cos^2(t) + \sin^2(t)\right)\,dt && (24)\\ &= \int_\pi^0 \cos(t)\,dt, && (25)\\ &= [\sin(t)]_\pi^0, && (26)\\ &= 0. && (27) \end{aligned}$$

□

0.4 An application of line integrals to physics

When a force acting on a body causes it to move then we have the concept of the *work* done by the force. It is a concept associated to the energy that is required to move the body.

In general terms the work done by the force when it moves a body in the same direction of the force is

$$\text{work} = \text{force} \times \text{displacement}. \tag{28}$$

When the body moves in a direction which is not the same as that of the force then we have to consider the *component* of the force in the direction of movement. Suppose a force in three dimensions is modelled using the vector quantity $\mathbf{F}$,

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}, \tag{29}$$

and the position of a body is given by the position vector $\mathbf{r}$,

$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}. \tag{30}$$

Then the work done when the body moves along a path C is given by the path integral

$$\begin{aligned} \text{work done} &= \int_C \mathbf{F}.d\mathbf{r} && (31)\\ &= \int_C F_x\,dx + F_y\,dy + F_z\,dz. && (32) \end{aligned}$$

Example 0.4. A two-dimensional example. Find the work done by a force

$$\mathbf{F} = (3x^2 + 4y^2)\mathbf{i} + (xy)\mathbf{j}, \tag{33}$$

when it moves a particle along a path C which is defined by the relationship $y = x^3$ for $0 \leq x \leq 2$.

Exercises 0.1. Tutorial exercises.

1. A, B and D are the points $(0,0)$, $(2,0)$ and $(2,1)$ respectively. Evaluate the path integral

$$\int_C (x^2 + 2y + 4)\,dx + (x^2 + 2y + 4)\,dy$$

when (a) C is the straight line sgment AD and (b) when C is the path made up of the straight line segments AB and BD.

2. Evaluate the path integral

$$\int_C (x^2 + 2y)\,dx + (x + y^2)\,dy$$

where C is the segment of the line $y = 2x + 1$ from $(1,3)$ to $(3,7)$.

3. Evaluate the path integral

$$\int_C x\,dy + (y + 1)\,dx$$

where C is

a) the segment of the curve $y = \sin x$ from $(0,0)$ to $(\frac{\pi}{2}, 1)$,

b) the segment of the line $y = \frac{2x}{\pi}$ from $(0,0)$ to $(\frac{\pi}{2}, 1)$,

c) any other path from $(0,0)$ to $(\frac{\pi}{2}, 1)$.

4. When a force $\mathbf{F}$ moves along a path C in the plane then the total workis given by

$$\int_C \mathbf{F}.\,d\mathbf{r}$$

where $\mathbf{r}(x,y)$ is the position vector $x\mathbf{i} + y\mathbf{j}$.

Show that when the force $\mathbf{F}$, given by $\mathbf{F}(x,y) = xy\mathbf{i} + y^2\mathbf{j}$, moves along the path C, defined by $t\mathbf{i} + t^2\mathbf{j}$ where $0 \le t \le 1$, the work done by the force is $\frac{7}{12}$.