6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 09 - SOLUTIONS

Solutions to questions 1 – 4 listed on the following pages under the heading of *Exercise 14*

Solutions to questions 5 – 13 listed on the following pages under the heading of *Exercise 15*

BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Fourier Series Worked Solutions

Exercise 14

Q1. $\langle 1, \sin nx\rangle = \int_{-\pi}^{\pi} 1\sin nx\, dx = \left[\frac{-\cos nx}{n}\right]_{-\pi}^{\pi} = -\frac{\cos n\pi}{n} + \frac{\cos n(-\pi)}{n} = 0$

Let $\frac{A+B}{2} = mx$ and $\frac{A-B}{2} = nx$

Adding gives $A = (m+n)x$

Subtracting gives $B = (m-n)x$

Then

$$\langle \cos mx, \cos nx\rangle = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx$$

$$= \int_{-\pi}^{\pi} \cos \frac{A+B}{2} \cos \frac{A-B}{2}\, dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}(\cos A + \cos B)dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}(\cos(m+n)x + \cos(m-n)x)dx$$

Now if $m \neq n$ integrating gives

$$\langle \cos mx, \cos nx\rangle = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0 \text{ as } \sin x = 0 \text{ whenever } x \text{ is a multiple of } \pi$$

if however $m=n$, $\cos(m-n)x=1$ so when we integrate we get

$$\langle \cos mx, \cos nx\rangle = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + x\right]_{-\pi}^{\pi} = \frac{1}{2}(0 + \pi - (0 + -\pi)) = \pi$$

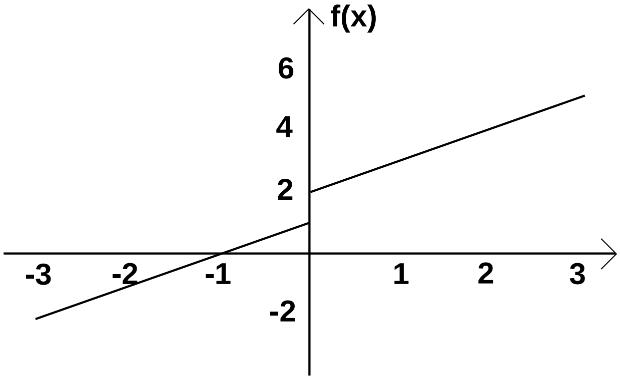
Q2. a) $f(x)$ is continuous except at $\pm\pi/2$ where it has two finite discontinuities. It has no maxima or minima in $(-\pi, \pi)$ so it satisfies Dirichlet's condition and will have a Fourier series in this interval.

b) $\cos x$ has maxima when $x = 2\pi n$ where n is an integer so $\cos(1/x)$ has maxima when $1/x = 2\pi n$. Then $x = 1/2\pi n$ i.e. maxima occur at $1/2\pi$, $1/4\pi$, $1/6\pi$ etc. There are therefore an infinite number of maxima (and similarly of minina) in $(-\pi, \pi)$ so the function might not have a Fourier series.

c) $8x^4 - 8x^2 + 1$ is continuous throughout $(-\pi, \pi)$ and being a polynomial of degree 4 has at most 3 turning points. Hence it satisfies Dirichlet's condition and will have a Fourier series in this interval.

d) $\tan x$ has an infinite discontinuity when $x = n\pi + \pi/2 = \pi(n + 1/2)$ so $\tan(1/x)$ has an infinite discontinuity when $1/x = \pi(n + 1/2)$ i.e. when $x = 1/\pi(n + 1/2)$. There are an infinite number of these in the interval $(-\pi, \pi)$ so the function might not have a Fourier series.

Q3. a) $x = 1$ is a point of continuity so the series has the same value as the function i.e. the value $f(1) = 3$.



b) $x = 0$ is a point of discontinuity so the series has the average of $\lim_{x \uparrow 0} f(x)$ and $\lim_{x \downarrow 0} f(x)$ i.e. the value $\frac{1+2}{2} = \frac{3}{2}$

Q4. Assuming the series is uniformly convergent so we can integrate it term by term,

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{1}{2} a_0 dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \, dx$$

$$= \frac{1}{2} 2\pi a_0 = \pi a_0 \quad \text{since the other definite integrals are zero.}$$

The formula for a_0 follows immediately.
If we first multiply the series by cos mx and then integrate we get

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} a_0 \cos mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx$$

Now since the cos and sin terms are orthogonal all the terms on the right hand side are zero except the one in the cosine series when m = n. Hence

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_m \int_{-\pi}^{\pi} \cos mx \cos mx \, dx$$

$$= a_m \int_{-\pi}^{\pi} \frac{1 + \cos 2mx}{2} dx = \pi a_m \quad \text{and the formula for } a_m \text{ follows immediately.}$$

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Fourier Series Worked Solutions

Exercise 15

Q1. If $f(x) = \frac{1}{2}a_o + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$\pi a_0 = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx = \int_{-\pi}^{0} (-1)\,dx + \int_{0}^{\pi} (+1)\,dx = -\pi + \pi = 0.$$

$$\pi a_m = \int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{0} f(x)\cos mx\,dx + \int_{0}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{0} -1\cos mx\,dx + \int_{0}^{\pi} +1\cos mx\,dx$$

$$= \left[\frac{-\sin mx}{m}\right]_{-\pi}^{0} + \left[\frac{\sin mx}{m}\right]_{0}^{\pi} = 0$$

$$\pi b_m = \int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{0} f(x)\sin mx\,dx + \int_{0}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{0} -1\sin mx\,dx + \int_{0}^{\pi} +1\sin mx\,dx$$

$$= \left[\frac{\cos mx}{m}\right]_{-\pi}^{0} + \left[\frac{-\cos mx}{m}\right]_{0}^{\pi} = \frac{1-\cos(-m\pi)}{m} + \frac{(-\cos m\pi + 1)}{m} = \frac{2}{m}(1-(-1)^m)$$

so

$b_m = 0$ if m is even

$= \frac{4}{\pi m}$ if m is odd

The series for f(x) is therefore $4 \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{\sin mx}{\pi m} = \frac{4}{\pi}(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots)$

Q2. If we assume the usual notation

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0.$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos mx\,dx = \frac{1}{\pi}\left[\frac{x\sin mx}{m} + \frac{\cos mx}{m^2}\right]_{-\pi}^{\pi} \quad \text{by parts}$$

$$= \frac{\cos m\pi - \cos(-m\pi)}{\pi m^2} = 0$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin mx\,dx = \frac{1}{\pi}\left[\frac{-x\cos mx}{m} + \frac{\sin mx}{m^2}\right]_{-\pi}^{\pi} \quad \text{- by parts}$$

$$= \frac{-\pi\cos m\pi - \pi\cos(-m\pi)}{\pi m} = 2\frac{(-1)^{m+1}}{m}$$

The series for f(x) is therefore $2\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx = 2(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - .....)$

Q3.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi} x\cos mx\,dx = \frac{1}{\pi}\left[\frac{x\sin mx}{m} + \frac{\cos mx}{m^2}\right]_0^{\pi} \quad \text{by parts}$$

$= \frac{1}{\pi}\frac{(\cos m\pi - 1)}{m^2}$

$= 0$ if m is even

$= \frac{-2}{\pi m^2}$ if m is odd

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi} x\sin mx\,dx = \frac{1}{\pi}\left[\frac{-x\cos mx}{m} + \frac{\sin mx}{m^2}\right]_0^{\pi} - \text{ by parts}$$

$= \frac{1}{\pi}\frac{(-\pi\cos m\pi)}{m} = \frac{(-1)^{m+1}}{m}$

Hence

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{\cos mx}{m^2} + \frac{1}{\pi}\sum_{m=1}^{\infty} (-1)^{m+1}\frac{\sin mx}{m}$$

$$= \frac{\pi}{4} - \frac{2}{\pi}(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots) + \frac{1}{\pi}(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots)$$

Q4.(a) $(-x)^3 = -x^3$ so x^3 is and odd function.

(b) $e^x \neq e^{-x}$ so the function is not even and $e^x \neq -e^{-x}$ so the function is not odd.

(c) Since $|-x| = |x|$, $e^{|-x|} = e^{|x|}$ so the function is even.

(d) $-x\cos(-x) = -(x\cos x)$ since cos is even. Hence xcos x is and odd function.

(e) $\cos(-x)\sin^2(-x) = \cos x\,(-\sin x)^2$ since cos is even and sin is odd,
$= \cos x \sin^2 x$ so $\cos x \sin^2 x$ is an even function.

Q5. Suppose first that f(x) and g(x) are even and h(x) = f(x)g(x).
Then f(-x) = f(x) and g(-x) = g(x).
Hence h(-x) = f(-x)g(-x) = f(x)g(x) = h(x) so h(x) is even.
Now suppose f(x) and g(x) are both odd so that f(-x) = -f(x) and g(-x) = -g(x).
Then h(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = h(x) so h(x) is even.

Q6. Let the function be f(x). Then

$$f(x) = \pi - x \qquad \text{if } 0<x<\pi$$

and f(x) is odd. Hence $a_m = 0$ for all m and

$$b_m = \frac{2}{\pi}\int_0^{\pi} (\pi - x) \sin mx \, dx = \frac{2}{\pi}\left[(\pi - x)\left(\frac{-\cos mx}{m}\right) - \frac{\sin mx}{m^2} \right]_0^{\pi} = \frac{2}{\pi}\frac{\pi}{m} = \frac{2}{m} \quad \text{by parts}$$

so $f(x) = 2 \sum_{m=1}^{\infty} \frac{\sin mx}{m}$

At x = 0 the value of the series is 0 and $\frac{1}{2}\{f(x+0) + f(x-0)\} = \frac{1}{2}\{\pi + (-\pi)\} = 0$.

Q7. Since h(x) is even, h(-x) = h(x). Then

$$\int_{-a}^{a} h(x)\, dx = \int_{-a}^{0} h(x)\, dx + \int_{0}^{a} h(x)\, dx$$

$$= \int_{-a}^{0} h(-u)\,(-du) + \int_{0}^{a} h(x)\, dx \qquad \text{putting } u = -x$$

$$= \int_{0}^{a} h(u)\, du + \int_{0}^{a} h(x)\, dx = 2\int_{0}^{a} h(x)\, dx$$

Q8. For a half range cosine series $b_m = 0$ for every m > 0,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \pi$$

and
$$a_m = \frac{2}{\pi}\int_0^{\pi} x \cos mx \, dx = \frac{2}{\pi}\left[\frac{x\sin mx}{m} + \frac{\cos mx}{m^2} \right]_0^{\pi} \qquad \text{by parts}$$

$$= \frac{-4}{\pi m^2} \quad \text{if m is odd}$$

$$= 0 \quad \text{if m is even}$$

so the half range series for cos x is $\pi - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - \ldots\right)$

Q9. For a half range sine series $a_m = 0$ for every $m \geqslant 0$ and

$$b_m = \frac{2}{\pi}\int_0^{\pi} x^2 \sin mx = \frac{2}{\pi}\left[\frac{-x^2\cos mx}{m} + \frac{2x\sin mx}{m^2} + \frac{2\cos mx}{m^3} \right]_0^{\pi} \qquad \text{by parts}$$

$$= \frac{2}{\pi}\left[(-1)^m\left(\frac{2}{m^3} - \frac{\pi^2}{m}\right) - \frac{2}{m^3} \right]$$

so
$$f(x) = \frac{2}{\pi}\sum_{m=1}^{\infty}\left[(-1)^m\left(\frac{2}{m^3} - \frac{\pi^2}{m}\right) - \frac{2}{m^3} \right] \sin mx$$