6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 07 - SOLUTIONS

Solutions to questions 1 – 5 listed on the following pages under the heading of *Exercise 1*

Solutions to questions 6 – 7 listed on the following pages under the heading of *Exercise 2*

BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Laplace Transform Worked Solutions

Exercise 1

Q1. If $\frac{d^3y}{dx^3} = 6$

then $\frac{d^2y}{dx^2} = 6x + C$

$\frac{dy}{dx} = 3x^2 + Cx + D$

and the general solution is

$$y = x^3 + C\frac{x^2}{2} + Dx + E$$

If $y(0) = 0$, then $E = 0$

If $y(1) = 0$, then $1 + C/2 + D = 0$

If $y(2) = 1$, then $8 + 2C + 2D = 1$

Solving simultaneous equations, $C = -5/2$ and $D = 3/2$

Hence $y = x^3 - \frac{5x^2}{2} + \frac{3x}{2}$

Q2. $L(e^{at}) = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{(a-s)t}dt = \left[\frac{e^{(a-s)t}}{a - s}\right]_0^\infty = \frac{1}{s - a}$

Now

$L(\sinh at) = \frac{1}{2}\{L(e^{at}) - L(^{-at})\} = \frac{1}{2}\{\frac{1}{s - a} - \frac{1}{s + a}\} = \frac{a}{s^2 - a^2}$

Q3. $L(t^2) = \int_0^\infty t^2e^{-st}dt = \left[\frac{t^2e^{-st}}{-s}\right]_0^\infty - \int_0^\infty 2t\frac{e^{-st}}{-s}dt$ by parts

$= 0 + \frac{2}{s}\int_0^\infty te^{-st}\,dt = \frac{2}{s}\{\left[\frac{te^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt\}$ by parts

$= \frac{2}{s}\{0 + \frac{1}{s}\int_0^\infty e^{-st}dt\} = \frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{2}{s^3}$

Q4.a) From tables Laplace Transform is $\frac{1}{s} + \frac{2}{s^2} + \frac{6}{s^3}$ (b) $\frac{5s}{s^2 + 1}$

c) From tables Laplace Transform is $4\times\frac{3}{(s - 2)^2 + 3^2} = \frac{12}{s^2 - 4s + 13}$.

Q5.a) From tables Inverse Transform of $7\times\frac{1}{s + 4}$ is $7e^{-4t}$.

b) $\frac{4s + 3}{s^2} = \frac{4}{s} + \frac{3}{s^2}$ so from tables Inverse Transform is $4 + 3t$.

c) $\frac{2s + 8}{s+4)^2 + 100} = \frac{2(s + 4)}{(s+4)^2 + 10^2}$ so from tables Inverse Transform is $2e^{-4t}\cos 10t$.

d) By partial fractions $\frac{3}{(s + 1)(s - 2)} = \frac{1}{s - 2} - \frac{1}{s + 1}$ so from tables Inverse Transform is $e^{2t} - e^{-t}$.

Exercise 2

Q1. a) Taking Laplace Transforms

$$sL(y) - y_o + L(y) = \frac{1}{s}$$

so $$(s + 1)L(y) = \frac{1}{s}$$

and $$L(y) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$ by partial fractions

so $$y = 1 - e^{-t}$$

b) Taking Laplace Transforms

$$sL(y) - y_o - 2L(y) = \frac{4}{s+2}$$

so $$(s - 2)L(y) - 2 = \frac{4}{s+2}$$

and $$L(y) = \frac{2s+8}{(s+2)(s-2)} = \frac{3}{s-2} - \frac{1}{s+2}$$ by partial fractions

so $$y = 3e^{2t} - e^{-2t}$$

c) Taking Laplace Transforms

$$s^2L(y) - sy_0 - y_1 + 2L(y) = \frac{3}{s^2}$$

so $$(s^2 + 2)L(y) - 7s - 1 = \frac{3}{s^2}$$

and $$L(y) = \frac{3 + s^2 + 7s^3}{s^2(s^2+2)}$$

$$= \frac{3}{2s^2} + \frac{7s - 1/2}{s^2+2}$$ by partial fractions

$$= \frac{3}{2s^2} + 7\frac{s}{s^2+2} - \frac{1}{2\sqrt{2}}\frac{\sqrt{2}}{s^2+2}$$

so $$y = \frac{3}{2}t + 7\cos\sqrt{2}t - \frac{1}{2\sqrt{2}}\sin\sqrt{2}t$$

d) Taking Laplace Transforms

$$s^2L(y) - sy_0 - y_1 + 2(sL(y) - y_o) + L(y) = \frac{3}{s}$$

so $$(s^2 + 2s + 1)L(y) - s - 4 - 2 = \frac{3}{s}$$

and hence $$L(y) = \frac{3 + 6s + s^2}{s(s+1)^2}$$

$$= \frac{3}{s} - \frac{2}{s+1} + \frac{2}{(s+1)^2}$$ by partial fractions

so $$y = 3 - 2e^{-t} + 2te^{-t}$$

Q2.a) Taking Laplace Transforms of both equations

$$L(x) + sL(y) - y_0 = \frac{2}{s} \quad (1)$$

$$sL(x) - x_0 - 6L(x) - 13L(y) = \frac{1}{s} \quad (2)$$

Then from (1) $$L(x) + sL(y) = \frac{2+3s}{s} \quad (3)$$

and from (2) $$(s - 6)L(x) - 13L(y) = \frac{1 - 10s}{s} \quad (4)$$

$13\times(3) + s\times(4)$ gives

$$(s^2 - 6s + 13)L(x) = \frac{26 + 39s}{s} + 1 - 10s$$

so that
$$L(x) = \frac{26 + 40s - 10s^2}{s(s^2 - 6s + 13)}$$

$$= \frac{2}{s} - \frac{(12s - 52)}{s^2 - 6s + 13} \quad \text{by partial fractions}$$

Hence
$$L(x) = \frac{2}{s} - \frac{12(s-3)}{(s-3)^2 + 2^2} + \frac{8\times 2}{(s-3)^2 + 2^2}$$

so that
$$x = 2 - 12e^{3t}\cos 2t + 8e^{3t}\sin 2t \qquad (5)$$

Now from the second differential equation

$$13y = \frac{dx}{dt} - 6x - 1$$

but from (5)
$$\frac{dx}{dt} = 48e^{3t}\sin 2t - 20e^{3t}\cos 2t$$

so
$$13y = 48e^{3t}\sin 2t - 20e^{3t}\cos 2t - 6(2 - 12e^{3t}\cos 2t + 8e^{3t}\sin 2t) - 1$$
$$= 52e^{3t}\cos 2t - 13$$

and
$$y = 4e^{3t}\cos 2t - 1$$

b) Taking Laplace Transforms of both equations

$$s^2L(y) - sy_0 - y_1 + 2(sL(x) - x_o) + L(y) = 0 \qquad (1)$$
$$sL(y) - y_0 - (sL(x) - x_o) - 2L(y) + 2L(x) = \frac{1}{s^2 + 1} \qquad (2)$$

Then from (1)
$$(s^2 + 1)L(y) + 2sL(x) = 0 \qquad (3)$$

and from (2)
$$(s - 2)L(y) - (s - 2)L(x) = \frac{1}{s^2 + 1}$$

so that
$$L(y) - L(x) = \frac{1}{(s^2 + 1)(s - 2)} \qquad (4)$$

$(3) + 2s\times(4)$ gives

$$(s^2 + 2s + 1)L(y) = \frac{2s}{(s^2 + 1)(s - 2)}$$

so
$$L(y) = \frac{2s}{(s^2 + 1)(s - 2)(s + 1)^2}$$

$$= \frac{4}{45(s-2)} + \frac{1}{9(s+1)} + \frac{1}{3(s+1)^2} - \frac{s}{5(s^2+1)} - \frac{2}{5(s^2+1)} \quad \text{by partial fractions}$$

so
$$y = \frac{4}{45}e^{2t} + \frac{1}{9}e^{-t} + \frac{1}{3}te^{-t} - \frac{1}{5}\cos t - \frac{2}{5}\sin t$$

From (4)
$$L(x) = L(y) - \frac{1}{(s^2 + 1)(s - 2)}$$

$$= \frac{1}{9(s+1)} + \frac{1}{3(s+1)^2} - \frac{1}{9(s-2)}$$

so
$$x = \frac{1}{9}e^{-t} + \frac{1}{3}te^{-t} - \frac{1}{9}e^{2t}$$