Q9 from Tut Sheet 01

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

time t

"Heat Equation"

$u(x,t)$ is the heat at point x at time t.

Confirm that $u(x,t) = e^{-\beta t}\sin(\alpha x)$ is a solution of Heat equation when a certain relationship holds between α, β.

Idea: Work out $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$, put into Heat equation and draw a consequence for α, β.

$$\boxed{\frac{\partial u}{\partial t}} = \frac{\partial}{\partial t}\left(e^{-\beta t}\sin(\alpha x)\right)$$

$$= \sin(\alpha x)\frac{\partial}{\partial t}\left(e^{-\beta t}\right), \quad \text{linearity}$$

$$= \sin(\alpha x)\, e^{-\beta t}(-\beta)$$

$$= \boxed{-\beta \sin(\alpha x)\, e^{-\beta t}}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2}} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} \left(e^{-\beta t} \sin(\alpha x)\right)\right)$$

$$= e^{-\beta t} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\sin(\alpha x)\right)\right), \quad \text{lin.}$$

$$= \alpha e^{-\beta t} \frac{\partial}{\partial x}\left(\cos(\alpha x)\right)$$

$$= -\alpha^2 e^{-\beta t} \sin(\alpha x)$$

So if u is a solution then

for all x, t

$$-\beta e^{-\beta t} \sin(\alpha x) + \alpha^2 e^{-\beta t} \sin(\alpha x) = 0$$

$$\Leftrightarrow \quad (\alpha^2 - \beta)\, e^{-\beta t} \sin(\alpha x) = 0$$

$$\Leftrightarrow \quad \alpha^2 - \beta = 0$$

$$\Leftrightarrow \quad \beta = \alpha^2$$

So we've identified the family of solutions

$$u(x,t) = e^{-\alpha^2 t} \sin(\alpha x)$$

paramatrized by $\alpha \in \mathbb{R}$.

Chain rule. Generalization of 1-variable chain rule to multi-variable functions.

Begin by recalling how the 1-variable derivative provides a "small increments /changes" approximation.

For small increments Δx of the variable x, a differentiable function $f(x)$ will change in value, approximated by

$$\boxed{f(x+\Delta x) \approx f(x) + \Delta x f'(x)}$$

In a 2-variable setting we have the small increments formula for $f(x,y)$

$$\boxed{\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)}$$

$$\boxed{\approx \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y}}$$

where $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ evaluated at starting values x, y

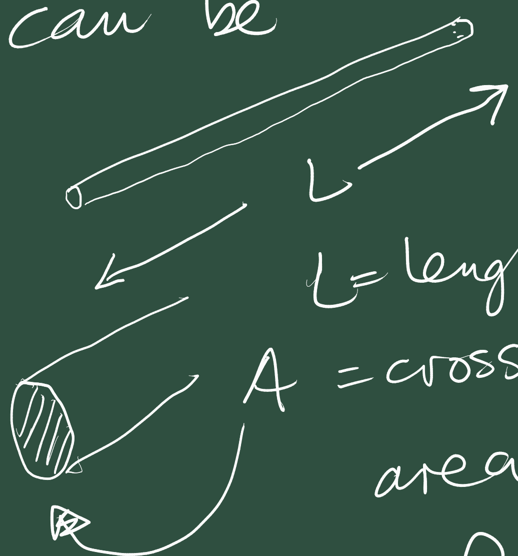
For more than two variables, has the clear extension

$f(x_1, \dots, x_n)$

$$\boxed{\Delta f \approx \sum_{i=1}^{n} \Delta x_i \frac{\partial f}{\partial x_i}}$$

Example The resistance R of a length of wire can be modeled as

$$\boxed{R = \rho \frac{L}{A}}$$



L = length of wire.

A = cross sectional area

ρ "rho"

$R(L, A)$

ρ = resistivity of the material.

Suppose that A decreases by 1% and L increases by 2.5%. Use the small increments formula to approximate ΔR, the resulting change in resistance.

$$\Delta R \approx \overbrace{\Delta L \frac{\partial R}{\partial L} + \Delta A \frac{\partial R}{\partial A}}^{\text{linear combination of } \Delta L,\ \Delta A.}$$

$$= \Delta L \, \rho \frac{1}{A} - \Delta A \, \rho \frac{L}{A^2}$$

$$= 0.025 \, L \rho \frac{1}{A} - (-0.01 \, A) \rho \frac{L}{A^2}$$

$$= 0.035 \underbrace{\rho \frac{L}{A}}_{R.}$$

So $\Delta R \approx 0.035 R$, ie. we see approx 3.5% increase in resistance of wire.

<u>Chain rule</u> 1-variable case.

Suppose $y = f(x)$

but in turn $x = g(t)$

So really $y = f(g(t)) = (f \circ g)(t)$.

Chain rule tells us.

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

$$\text{also } = \frac{df}{dx} \cdot \frac{dg}{dt}$$

or in the other notation-

$$y'(t) = f'(g(t))\, g'(t)$$

Multi-variable case

Suppose. we have

$$y = f(x_1, \dots, x_n)$$

and the x_i are given by

$$x_i = u_i(z_1, \dots, z_n)$$

We need to know how a derivative $\frac{\partial y}{\partial z_i}$ depends on

$$\frac{\partial y}{\partial x_j} \quad \text{and} \quad \frac{\partial u_j}{\partial z_l}$$

Focus on 2-variable case.

$z = f(x, y)$ and $x = u(s, t)$

and $y = v(s, t)$

Start with small increments formula.

$$\Delta z \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

$$\Rightarrow \quad \frac{\Delta z}{\Delta s} \approx \frac{\partial f}{\partial x} \frac{\Delta x}{\Delta s} + \frac{\partial f}{\partial y} \frac{\Delta y}{\Delta s}.$$

Here, consider Δs as the fundamental increment, $\Delta x, \Delta y$ the resulting increments in x, y, and Δz the resulting increment in z. Take the limit of both sides as $\Delta s \to 0$ which induces limits.

$$\frac{\Delta z}{\Delta s} \longrightarrow \frac{\partial z}{\partial s}, \quad \frac{\Delta x}{\Delta s} \longrightarrow \frac{\partial x}{\partial y}$$

$\frac{\Delta y}{\Delta s} \longrightarrow \frac{\partial y}{\partial s}$ and the $\approx$ will become equality. Giving

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}}$$

and similarly can show

$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

↳ These are the chain rule equations. for this change in coords $(x,y) \longrightarrow (s,t)$.

How it appears in the general multi-variable case above.

$$\boxed{\frac{\partial y}{\partial z_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial z_i}, \quad i = 1, \dots, n}$$

CLP Ex 2.4.5

Q6

Firstly without chain rule

$w = x^2 + y^2 + z^2$

where $x = st$, $y = s\cos(t)$

$z = s\sin(t)$.

by direct substitution

$$w = (st)^2 + s^2\cos^2(t) + s^2\sin^2(t)$$

$$= s^2t^2 + s^2(\underbrace{\cos^2(t) + \sin^2(t)}_{=1.})$$

$$= s^2(t^2+1)$$

And so

$$\frac{\partial w}{\partial s} = 2s(t^2+1)$$

$$\frac{\partial w}{\partial t} = 2s^2 t$$

But, secondly, done with chain rule. For instance.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= 2x\,t + 2y\cos(t) + 2z\sin(t)$$

$$= 2st\,t + 2s\underbrace{\cos(t)\cos(t)}$$
$$+ 2s\underbrace{\sin(t)\sin(t)}$$

$$= 2st^2 + 2s$$

$$= 2s(t^2+1)$$

Similarly and show the other chain rule equation

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= \cdots$$

$$= 2s^2t.$$

Jacobians (Describe some here, apply later in integration of multi-variable functions).

Consider the chain rule

$$\boxed{\frac{\partial y}{\partial z_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial z_i}, \quad i=1,\dots,n}$$

The RHS may look familiar...

$$\square_i = \sum_{j=1}^{n} \square_j \, \boxed{\;i\;}_j$$

Matrix multiplication!

$$(AB)_{li} = \sum_{j=1}^{n} A_{lj} B_{ji}$$

2-variable case. Chain rule equations can be expressed.

$$\begin{pmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \underbrace{\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}}$$

This 2×2 matrix of partial derivatives of (x,y) wrt (s,t) is called the $\boxed{\text{Jacobian matrix}}$

of the transformation from (x,y) to (s,t).

index the rows. index the columns

The term "Jacobian" usually refers to its determinant.

(which we'll apply when we

come to integration)