6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 8

Qs 1 − 4 on working with the Heaviside step function and Dirac delta function.
Qs 5 − 9 on transforms of Heaviside step function and Dirac delta function and solving ODEs featuring them.

(1) Sketch each of the following functions and express them in terms of Heaviside functions.

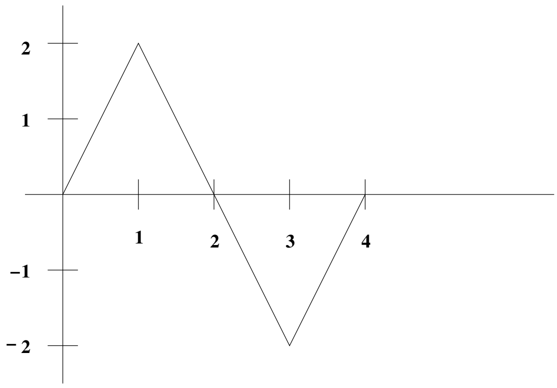
(a)
$$f(t) = \begin{cases} 6, & 2 < t \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

(b)
$$f(t) = \begin{cases} 0, & t \leq 1 \\ t-1, & t > 1 \end{cases}$$

(c)
$$f(t) = \begin{cases} 0, & t \leq 2 \\ t-2, & 2 < t \leq 6 \\ 4, & t > 6 \end{cases}$$

(d)
$$f(t) = \begin{cases} 3t, & 0 < t \leq 1 \\ 4-t, & 1 < t \leq 4 \\ 0, & t > 4 \end{cases}$$

(2) Express the function correspoding to the graph below in terms of Heaviside functions.



(3) Sketch graphs of the following functions.

(a) $\mathrm{H}(t) - \mathrm{H}(t-3)$

(b) $t\,(\mathrm{H}(t-2) - \mathrm{H}(t-4))$

(c) $\delta(t) + t^2\mathrm{H}(t) - (t^2-4)\mathrm{H}(t-2) - 4\mathrm{H}(t-4) + \delta(t-4)$

(4) Prove that if $F(s)$ is the Laplace transform of the function $f(t)$ then the transform $f(t-a)\mathrm{H}(t-a)$ is $e^{-as}F(s)$.

(5) Find the Laplace transform of the following functions.

(a) $3\delta(t-1) + 4\mathrm{H}(t+2)$

(b) $t\mathrm{H}(t-4)$

(c) $\cos(t-3)\mathrm{H}(t-3)$

(d) $t^2\mathrm{H}(t-1)$

(6) Find the inverse Laplace transform of the following functions

(a) $2e^{-4s} + 5\dfrac{e^{-5s}}{s}$

(b) $\left(\frac{3}{s+4} + \frac{s}{s^2+1}\right) e^{-s}$

(7) (a) Solve the following differential equations subject, in both cases, to the intial conditions $y(0) = y'(0) = 0$.

(i)

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 5y = 4\delta(t-2)$$

(ii)

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = (t-1)\mathrm{H}(t-1)$$

(b) Given the intial condition $y(0) = 0$ solve the differential equation

$$\frac{dy}{dt} - y = f(t),$$

where f is the function defined by

$$f(t) = \begin{cases} t, & 0 < t \leq 1 \\ 1, & 1 < t \leq 2 \\ 3-t, & 2 < t \leq 3 \\ 0, & \text{elsewhere} \end{cases}.$$

(8) Find the inverse Laplace transforms of the functions given by the following expressions. Identify the steaty state and transient parts of the function and show that in each case the initial and final value theorems hold.

(a)

$$\frac{4}{s+2} + \frac{7}{s-3}$$

(b)

$$\frac{5s+17}{s^2+6s+10}$$

(c)

$$\frac{25+14s}{s^2+6s+8}e^{-s}$$

(9) Solve the differential eqaution

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 16$$

subject to the intial conditions $y(0) = 3$ and $y'(0) = 4$. Identify the steady state and transient parts of the solution. Verify that the intial and final value theorems hold for the transform of y.