Fourier series

Odd / Even functions Sec. 5.4.

Def 5.4.1

A function $f : \mathbb{R} \to \mathbb{R}$ is even iff

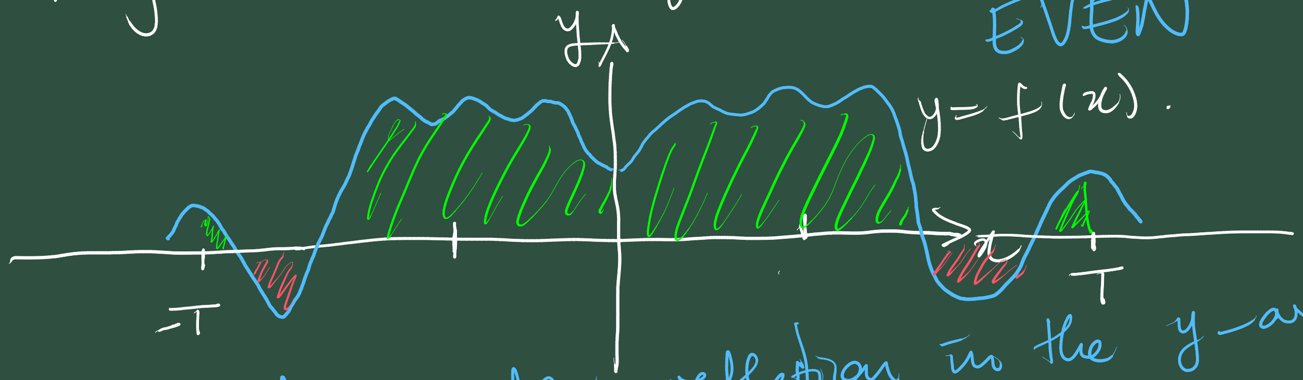
$$\forall x \in \mathbb{R} \qquad f(-x) = f(x)$$

" " " " odd iff

$$\forall x \in \mathbb{R} \qquad f(-x) = -f(x)$$

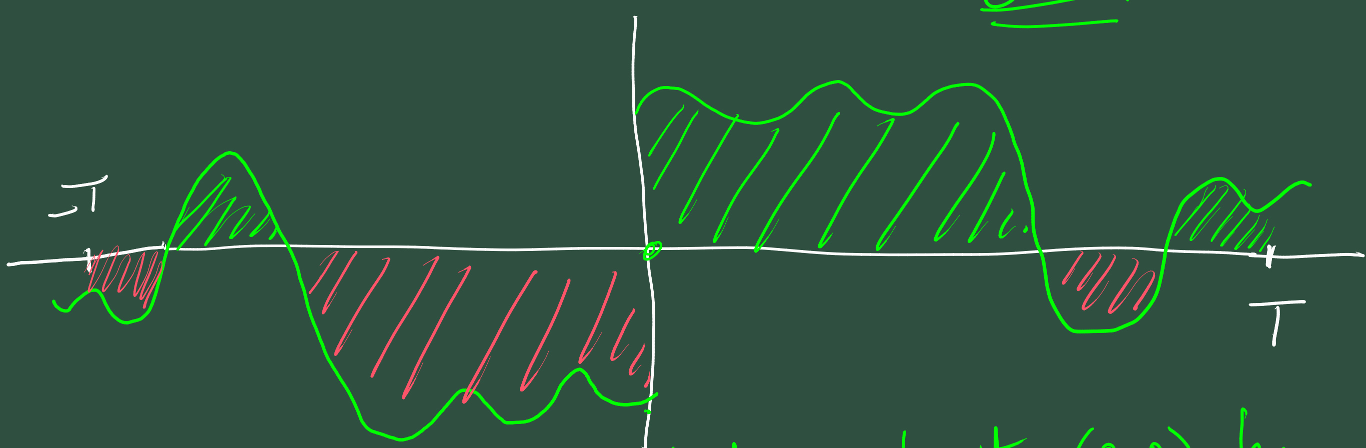
Very characteristic graphs.

EVEN



Symmetric under reflection in the y-axis

odd



Symmetric under rotation about $(0,0)$ by π radians.

odd / even functions, their Fourier series have particular forms, because of nature of sine & cosine.

cosine is an <u>even</u> function.

$\forall x \in \mathbb{R}$. $\cos(-x) = \cos(x)$

sine is <u>odd</u>

$\forall x \in \mathbb{R}$ $\sin(-x) = -\sin(x)$.

<u>Theorem 5.4.3</u>

Proofs of these are straightforward and just rely on tracking the − signs carefully.

eg. Suppose g_1, g_2 are both <u>even</u>.

1. $\boxed{(g_1 g_2)(-x)} = g_1(-x)\, g_2(-x)$

$$= g_1(x)\, g_2(x)$$

$$= \boxed{(g_1 g_2)(x)}$$

So $g_1 g_2$ is even.

3. If f is <u>odd</u> and g is <u>even</u>

$\boxed{(fg)(-x)} = f(-x)\, g(-x)$

$$= -f(x)\cdot g(x)$$
$$\boxed{= -(fg)(x)}$$

So fg is odd.

Theorem 5.4.4 Integrating odd/even functions on intervals centred at 0.

Suppose g is even and h is odd

- $\displaystyle\int_{-T}^{T} g(x)\,dx = 2\int_0^T g(x)\,dx.$
- $\displaystyle\int_{-T}^{T} h(x)\,dx = 0$

Proof Can really be "seen" straight away from our earlier graph sketches.

Let's prove the odd case formally.

So h is odd, ie $\forall x\ h(-x) = -h(x)$

$$\int_{-T}^{T} h(x)\,dx = \int_{-T}^{0} h(x)\,dx + \int_0^T h(x)\,dx$$

on the first integral we use a substitution $\boxed{t=-x}$

so $dt = -dx$, $x = -T \Rightarrow t = T$
$x = 0 \Rightarrow t = 0$

$$= \int_T^0 h(-t)(-dt) + \int_0^T h(x)\,dx$$

$$= \int_T^0 -h(t)(-dt) + \int_0^T h(x)\,dx,$$
, since h is odd.

$$= \int_T^0 h(t)\,dt + \int_0^T h(x)\,dx$$

$$= -\underbrace{\int_0^T h(t)\,dt}_{} + \underbrace{\int_0^T h(x)\,dx}_{},$$
changing direction of integration.

$= 0$, these two integrals are the same

A similar approach will prove the even case.

Fourier series of odd/even functions.

Theorem 5.4.5 g is even $\Rightarrow$ $b_n = 0$, $n \geq 1$

__Theorem 5.4.6__ h is odd $\Rightarrow a_n = 0$, $n \geq 0$

These results can save us "half the work"

__Proofs__ Just apply the previous properties.

For instance: If g is __even__ then.

for all $n \geq 1$ $\boxed{g(x) \sin(nx)}$ will be an $\boxed{\text{odd}}$ function, since $\sin(nx)$ is __odd__

$$\Rightarrow \boxed{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)\,dx$$

$$\boxed{= 0}, \text{ by previous theorem}$$

Similar argument for odd case __h__

__Example 5.4.7__

Consider $\boxed{f(x)} = x^2 = (-x)^2 \boxed{= f(-x)}$

So this f is an __even__ function.

So its b_n Fourier coefficients are all $= 0$, $n \geq 1$. So we only need to evaluate the a_n coefficients.

$$\boxed{a_0} = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\boxed{= \frac{2}{3}\pi^2}$$

For $n \geq 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx.$$

apply int. by. parts twice to this to

$\leadsto$ $\int \cos = -\sin$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \frac{d}{dx}\left(\frac{\sin(nx)}{n}\right)dx.$$

$$= \frac{2}{\pi}\left(\underbrace{\left[\frac{x^2 \sin(nx)}{n} \right]_0^{\pi}}_{=0} - \int_0^{\pi} \frac{2x \sin(nx)}{n}\, dx \right)$$

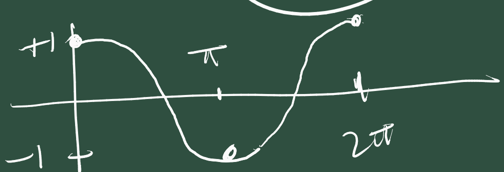
$$= \frac{-4}{n\pi} \int_0^{\pi} x \sin(nx)\, dx.$$

$$= \frac{-4}{n\pi} \int_0^{\pi} x \frac{d}{dx}\left(-\frac{\cos(nx)}{n} \right) dx.$$

$$= \frac{-4}{n\pi}\left(\left[\frac{x(-\cos(nx))}{n} \right]_0^{\pi} - \int_0^{\pi} -\frac{\cos(nx)}{n}\, dx. \right)$$

$$= \frac{-4}{n\cancel{\pi}}\left(-\frac{\cancel{\pi}\cos(n\pi)}{n} + \frac{1}{n} \underbrace{\left[\frac{\sin(nx)}{n} \right]_0^{\pi}}_{=0} \right)$$

$$= \frac{4}{n^2}(-1)^n = a_n \qquad a_0 = \frac{2}{3}\pi^2$$



$$b_n = 0$$

So the Fourier series for x^2

is.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$$

as required.

An interesting series result will come from this.
If we evaluate this Fourier series at $x=0$ we get something interesting

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

ie. $\boxed{1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots\cdots = \frac{\pi^2}{12}}$

Also, look back at series for square wave function. in example 5.2.2.

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left((2n-1)x\right)$$

n

$m=1$

$$f(x) = \begin{cases} 0, & \text{for } -\pi < x < 0 \\ 1 & \text{for } \underline{0 \le x < \pi} \end{cases}$$

For this, we can evaluate at $x = \pi/2$.

$$1 = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \underbrace{\sin\left((2n-1)\frac{\pi}{2}\right)}_{= \pm 1 \; = (-1)^{n+1}}$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}.$$

$$\Rightarrow \boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}.}$$

Convergence of F.S.

$$\frac{d}{dx}\left(f_1(x) + f_2(x)\right)$$

$$= \frac{df_1}{dx} + \frac{df_2}{dx}$$

differentiation is linear.

Does not imply that

$$\frac{d}{dx}\left(\sum_{i=1}^{\infty} f_i(x)\right) = \sum_{i=1}^{\infty} \frac{df_i}{dx}$$

It depends on the nature of the function convergence

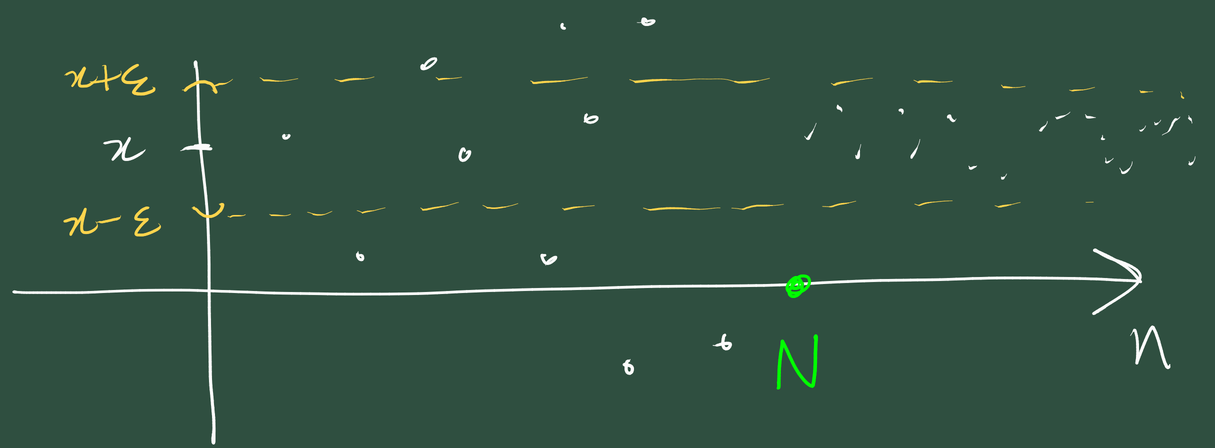
Recall the basic def. of convergence of sequences

$\{x_n\}_{n=1}^{\infty} = x_1, x_2, x_3, \ldots$

a sequence of real numbers.

We say $x_n \to x$ iff.

$\forall \varepsilon > 0 \quad \exists N \quad n \geq N \Rightarrow |x_n - x| < \varepsilon$



What's the definition for convergence of functions? $f_n \to f$.

Well it can happen in two ways.

$f_n \to f$ pointwise if at each argument a, $f_n(a) \to f(a)$, considered as a seouence.

$$\forall a \in \mathbb{R} \ \forall \varepsilon > 0 \ \exists N \ n \geq N \Rightarrow |f_n(a) - f(a)| < \varepsilon.$$

This allows for very fast convergue at some arguments a, and slow convergence at others.

If we insist that these convergences happen at some bounded rate across the whole domain, we have "uniform convergence"

$$\boxed{\forall \varepsilon > 0 \ \exists N \ \forall a \in \mathbb{R} \ n \geq N \Rightarrow |f_n(a) - f(a)| < \varepsilon}$$

A good example of pointwise, but non-uniform convergence, is provided by the Fourier series of square wave function.

Where we have "Gibb's phenomena" around the points of discontinuity.

eg $x=0$.