6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 02 - SOLUTIONS

Solutions to questions 1 - 4 list on the following pages under the heading of *Exercise 6* questions 2 through 5. (*apologies for the renumbering*)

Then solutions to questions 5 - 10 listed on the following pages under the heading of *Exercise 7* questions 1 through 5. (*apologies for the renumbering*)

Exercise 6

Q1.a) $f(x,y) = x^2 - y^2$ so

$f(x+\delta x, y+\delta y) = (x+\delta x)^2 - (y+\delta y)^2$ and therefore

$f(x+\delta x, y+\delta y) - f(x,y) = (x+\delta x)^2 - (y+\delta y)^2 - (x^2 - y^2)$

i.e. $\delta f = 2x\delta x + (\delta x)^2 - 2y\delta y - (\delta y)^2$

$= 2x\delta x + \lambda\delta x - 2y\delta y - \mu\delta y$

where $\lambda = \delta x$ and $\mu = \delta y$ both $\to 0$ as δx and $\delta y \to 0$.

Hence $df = 2xdx - 2ydy$ i.e. f has a total derivative.

b) $f(x,y) = xy^3$

$f(x+\delta x, y+\delta y) = (x+\delta x)(y+\delta y)^3$

$f(x+\delta x, y+\delta y) - f(x,y) = (x+\delta x)(y+\delta y)^3 - xy^3$

i.e. $\delta f = (x+\delta x)[y^3+3y^2(\delta y)+3y(\delta y)^2+(\delta y)^3] - xy^3$

$= xy^3+3xy^2(\delta y)+3xy(\delta y)^2+x(\delta y)^3+y^3\delta x+3y^2(\delta y)\delta x+3y(\delta y)^2\delta x+(\delta y)^3\delta x - xy^3$

$= y^3\delta x + 3xy^2\delta y + \lambda\delta x + \mu\delta y$

where $\lambda = 3y^2(\delta y)+3y(\delta y)^2+(\delta y)^3$ and $\mu = 3xy\delta y + x(\delta y)^2$ both $\to 0$ as δx and $\delta y \to 0$.

Hence $df = y^3dx + 3xy^2dy$ i.e. f has a total derivative.

Q2. $f(x,y) = (x + 2y^2)^5$

Let δf be the change in f caused by small changes δx in x and δy in y.

$$\delta f \cong \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

$$= 5(x + 2y^2)^4\,\delta x + 5\times 4y\times(x + 2y^2)^4\,\delta y$$

Now if $x = 1$, $y = 1$, $\delta x = 0.01$ and $\delta = 0.01$

$$\delta f = 5(3)^4\,0.01 + 5(4)(3)^4\,0.01 = 20.25$$

In fact $f(1,1) = 3^5 = 243$ and

$f(1.01,1.01) = (1.01 + 2\ 1.01^2)^5 = 264.02$

so the true change is $264.02 - 243 = 21.02$.

Q3. $$P = \frac{DL^2}{3}$$

Let δP be the change in P caused by small changes δL in L and δD in D then

$$\delta P \cong \frac{\partial P}{\partial L}\delta L + \frac{\partial P}{\partial D}\delta D$$

$$= \frac{2LD}{3}\delta L + \frac{L^2}{3}\delta D$$

Now $\delta L = 2\%$ of $L = \frac{2}{100}L$ and $\delta D = \frac{1}{100}D$

$$\delta P = \frac{2LD}{3}\,\frac{2L}{100} + \frac{L^2}{3}\,\frac{1}{100}D$$

$$= \frac{DL^2}{3}\left(\frac{4}{100} + \frac{1}{100}\right)$$

$$= \frac{5}{100}P$$

so there is a 5% increase in the pressure, P.

Q4. a) $V = \dfrac{\pi h r^2}{3}$

Let δV be the change in V caused by small changes δh in h and δr in r then

$$\delta V \cong \frac{\partial V}{\partial h}\,\delta h + \frac{\partial V}{\partial r}\,\delta r$$

$$= \frac{\pi r^2}{3}\,\delta h + \frac{2\pi r h}{3}\,\delta r$$

Now if there is no change in V, $\delta V = 0$ and if there is a 2% decrease in r, $\delta r = -\dfrac{2}{100}\,r$. Then

$$0 = \frac{\pi r^2}{3}\,\delta h - \frac{2\pi r h}{3}\,\frac{2}{100}$$

and rearraging this gives

$$\delta h = \frac{4}{100}\,h$$

so there must be a 4% increase in the height, h.

b) $S = \pi r^2 + \pi r\sqrt{(r^2+h^2)}$

Let δS be the change in S caused by small changes δh in h and δr in r then

$$\delta S \cong \frac{\partial S}{\partial h}\,\delta h + \frac{\partial S}{\partial r}\,\delta r$$

$$= \left[2\pi r + \pi\sqrt{(r^2+h^2)} + \frac{\pi r\,(1/2)2r}{\sqrt{(r^2+h^2)}}\right]\delta r + \frac{\pi r\,(1/2)2r}{\sqrt{(r^2+h^2)}}\,\delta h$$

Then when r = 5, h = 12, and $\delta r = \delta h = 0.01$,

$$S = 25\pi + 5\pi\times 13 = 90\pi = 282.74$$

and

$$\delta S = \left[10\pi + 13\pi + \frac{25\pi}{13}\right] 0.01 + \frac{60\pi}{13}\,0.01$$

$$= 0.783 + 0.145 = 0.928$$

Hence $S = 282.74 \pm 0.93$

Most of the error arises from the error in r so this measurement needs to be more accurate.

Q5. Let δf be the change in f caused by small changes δx_r in x_r for r = 1, 2, 3, ..., n.

Then

$$\delta f = \frac{\partial f}{\partial x_1}\,\delta x_1 + \frac{\partial f}{\partial x_2}\,\delta x_2 + \dots\dots\dots + \frac{\partial f}{\partial x_n}\,\delta x_n.$$

But $\dfrac{\partial f}{\partial x_1} = p_1 x_1^{p_1-1} x_2^{p_2} \dots\dots x_n^{p_n}$

and, if δx_r is an i_r% change in x_r, $\delta x_r = \dfrac{i_r}{100}\,x_r$.

So

$$\frac{\partial f}{\partial x_1}\,\delta x_1 = p_1 x_1^{p_1-1} x_2^{p_2} \dots\dots x_n^{p_n}\,\frac{i_1}{100}\,x_1$$

$$= \frac{p_1 i_1 x_1^{p_1} x_2^{p_2} \dots\dots x_n^{p_n}}{100}$$

$$= \frac{p_1 i_1 f}{100}$$

Similarly for x_2, x_3, ... x_n.

Then

$$\delta f = \frac{f}{100}\,(p_1 i_1 + p_2 i_2 + \dots\dots + p_n i_n)$$

so the percentage change in f is $p_1 i_1 + p_2 i_2 + \dots\dots + p_n i_n$.

BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Partial Differentiation Worked Solutions

Exercise 7

Q1. Using the chain rule

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

Now $x = r\cos\theta$ and $y = r\sin\theta$ so

$\dfrac{\partial x}{\partial \theta} = -r\sin\theta = -y$ and $\dfrac{\partial y}{\partial \theta} = r\cos\theta = x$

Hence $$\frac{\partial f}{\partial \theta} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

Q2. Using the chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial x} = y\frac{\partial f}{\partial s}$$

so $$x\frac{\partial f}{\partial x} = xy\frac{\partial f}{\partial s} = s\frac{\partial f}{\partial s}.$$

Also $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial y} = x\frac{\partial f}{\partial s} - \frac{1}{y^2}\frac{\partial f}{\partial t}$$

so $$y\frac{\partial f}{\partial y} = xy\frac{\partial f}{\partial s} - \frac{1}{y}\frac{\partial f}{\partial t} = s\frac{\partial f}{\partial s} - t\frac{\partial f}{\partial t}.$$

Hence $$x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = t\frac{\partial f}{\partial t}$$

and so $$y\frac{\partial f}{\partial y}\left(x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right) = t\frac{\partial f}{\partial t}\left(s\frac{\partial f}{\partial s} - t\frac{\partial f}{\partial t}\right).$$

Q3. $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

Q4. $$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \quad \text{and} \quad \frac{\partial(s,t)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{vmatrix}$$

Now

$$\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} \cdot \frac{\partial s}{\partial u} + \frac{\partial x}{\partial t} \cdot \frac{\partial t}{\partial u} & \frac{\partial x}{\partial s} \cdot \frac{\partial s}{\partial v} + \frac{\partial x}{\partial t} \cdot \frac{\partial t}{\partial v} \\ \frac{\partial y}{\partial s} \cdot \frac{\partial s}{\partial u} + \frac{\partial y}{\partial t} \cdot \frac{\partial t}{\partial u} & \frac{\partial y}{\partial s} \cdot \frac{\partial s}{\partial v} + \frac{\partial y}{\partial t} \cdot \frac{\partial t}{\partial v} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \quad \text{by the chain rule.}$$

and taking determinants of both sides $$\frac{\partial(x,y)}{\partial(s,t)} \cdot \frac{\partial(s,t)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(u,v)}.$$

Q5. $\dfrac{\partial(r,\theta)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,\theta)} = \dfrac{\partial(r,\theta)}{\partial(r,\theta)}$

$$= \begin{vmatrix} \dfrac{\partial r}{\partial r} & \dfrac{\partial r}{\partial \theta} \\ \dfrac{\partial \theta}{\partial r} & \dfrac{\partial \theta}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Hence $\dfrac{\partial(r,\theta)}{\partial(x,y)} \cdot r = 1$

so $\dfrac{\partial(r,\theta)}{\partial(x,y)} = \dfrac{1}{r}$

Q6. The graph is of $t = s^2$.

Now $\dfrac{\partial(s,t)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y} \\ \dfrac{\partial t}{\partial x} & \dfrac{\partial t}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ -2(y-x) & 2(y-x) \end{vmatrix} = 0$

so the inverse does not exist. Hence there is no inverse transformation. This arises because s and t are not independent variables.

Exercise 8

Q1. $f(x,y) = \ln(x + y^2)$.
Using Taylor's theorem

$$f(1+h, 0+k) = f(1,0) + h\frac{\partial f}{\partial x}(1,0) + k\frac{\partial f}{\partial y}(1,0) + \frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2}(1,0) + \frac{2hk}{2!}\frac{\partial^2 f}{\partial x \partial y}(1,0) + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}(1,0) + \text{higher order terms}$$

Now $f(1,0) = \ln(1) = 0$

$\dfrac{\partial f}{\partial x} = \dfrac{1}{x+y^2} = 1$ at $(1,0)$

$\dfrac{\partial f}{\partial y} = \dfrac{2y}{x+y^2} = 0$ at $(1,0)$

$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{-1}{(x+y^2)^2} = -1$ at $(1,0)$

$\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{-2y}{(x+y^2)^2} = 0$ at $(1,0)$

$\dfrac{\partial^2 f}{\partial y^2} = \dfrac{(x+y^2)2 - 2y(2y)}{(x+y^2)^2} = 2$ at $(1,0)$

Then $f(1+h,k) = 0 + h.1 + k.0 + \dfrac{h^2}{2!}(-1) + \dfrac{2hk}{2!}.0 + \dfrac{k^2}{2!}.2 + \text{higher order terms}$

$\cong h + \dfrac{1}{2}(-h^2 + 2k^2)$ if h and k are small.