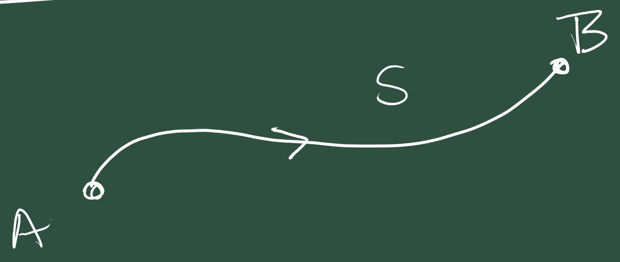
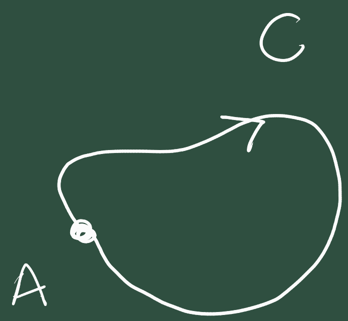
Green's Theorem



$$\int_S$$



$$\oint_C$$

indicates that C is a closed curve

Example 3.8.3

$$I = \oint_C \underbrace{\overbrace{3x^2y^2}^{P(x,y)}dx + \overbrace{2xy}^{Q(x,y)}\,dy}_{L}$$

where C is the closed curve DEFG as shown

We should break I down into the sum of four sub-integrals

$$I = \int_{DE} L + \int_{EF} L + \int_{FG} L + \int_{GD} L$$

On DE: $y=1$, $dy=0$, $x: 1 \to 2$

On EF: $x=2$, $dx=0$, $y: 1 \to 2$

On FG: $y=2$, $dy=0$, $x: 2 \to 1$

On GD: $x=1$, $dx=0$, $y: 2 \to 1$

$$I = \int_1^2 3x^2\,dx + \int_1^2 4y\,dy + \int_2^1 12x^2\,dx + \int_2^1 2y\,dy$$

$$= \left[x^3\right]_1^2 + \left[2y^2\right]_1^2 + \left[4x^3\right]_2^1 + \left[y^2\right]_2^1$$

$$= 8-1 + 8-2 + 4-32 + 1-4 = -18$$

Keep this in mind.

Green's Theorem

First: Verify the theorem for the previous example.

In the previous example

$$P(x,y) = 3x^2y^2$$

$$Q(x,y) = 2xy.$$

Green's theorem says that

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = -18$$

where R is the square region.

$$1 \leq x, y \leq 2.$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

mostly negative for $1 \leq x, y \leq 2$.

$$= \iint_R (\overbrace{2y - 6x^2y}^{f(x,y)})\, dx\, dy$$

$$= \int_1^2 \left(\int_1^2 (2y - 6x^2y)\, dx \right) dy$$

$$= \int_1^2 \Big[2yx - 2x^3y \Big]_{x=1}^{x=2} dy$$

$$= \int_1^2 4y - 16y - (2y - 2y)\, dy$$

$$= \int_1^2 -12y \, dy$$

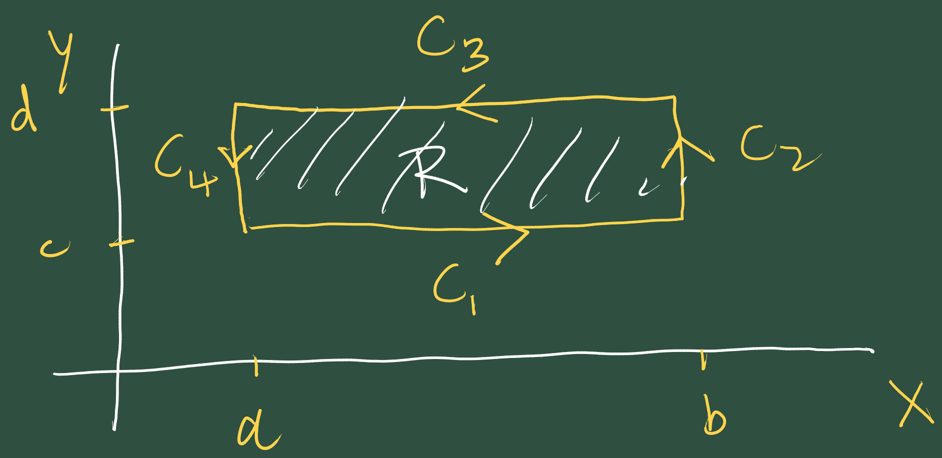
$$= -\left[6y^2 \right]_1^2$$

$$= -\left[24 - 6 \right] = \underline{\underline{-18}},$$

as expected from statement of Green's Theorem.

Sketch of one way to prove G.T.

Consider first, a rectangle aligned with the coord. axes.



We'll prove G.T. for this.

$$C = C_1 + C_2 + C_3 + C_4$$

On C_1: $y = c$, $dy = 0$, $x: a \to b$

" C_2: $x = b$, $dx = 0$, $y: c \mapsto d$

$\therefore\ C_3: \quad y=d,\ dy=0,\quad x: b \to a$

$\therefore\ C_4: \quad x=a,\ dx=0,\ y: d \to c.$

So $\displaystyle\oint_{C} P\,dx + Q\,dy$ L.H.S. of G.T. $P(x,y)$ $Q(x,y)$

$$= \int_a^b P(x,c)\,dx + \int_c^d Q(b,y)\,dy + \int_b^a P(x,d)\,dx + \int_d^c Q(a,y)\,dy$$

↳ as far as we can evaluate it, without knowing P, Q.

Now for the RHS of G.T.

$$\boxed{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

$$= \iint_R \frac{\partial Q}{\partial x}\,dx\,dy - \iint_R \frac{\partial P}{\partial y}\,dx\,dy,$$

by linearity

$$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x}\,dx\right) dy$$

$$- \int_a^b \left(\int_c^d \frac{\partial P}{\partial y} \, dy \right) dx$$

, forming into repeated integrals

$$= \int_c^d \left[Q(x,y) \right]_a^b dy - \int_a^b \left[P(x,y) \right]_c^d dx$$

$$= \int_c^d \left(Q(b,y) - Q(a,y) \right) dy - \int_a^b \left(P(x,d) - P(x,c) \right) dx$$

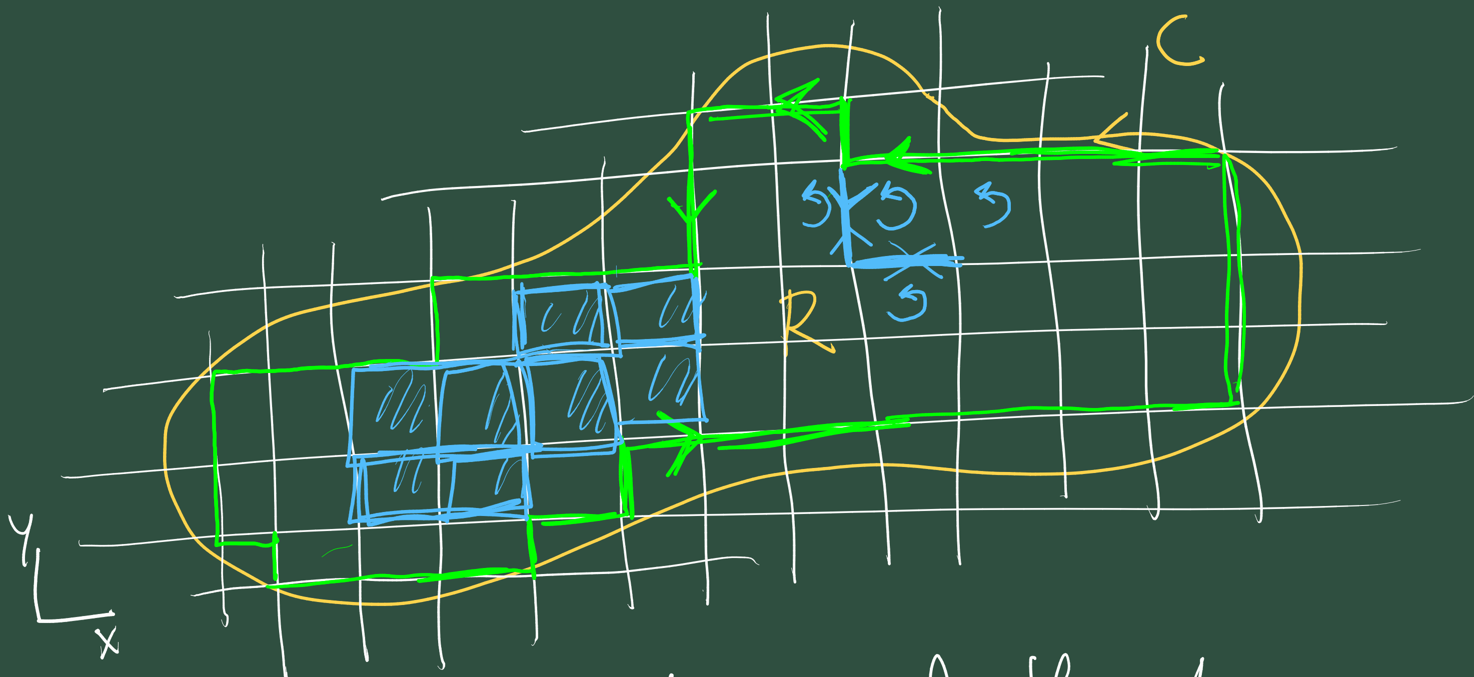
$$\boxed{= \int_c^d Q(b,y)\, dy + \int_d^c Q(a,y)\, dy + \int_b^a P(x,d)\, dx + \int_a^b P(x,c)\, dx.}$$

(note change of direction in 2nd and 3rd integrals)

$=$ L.H.S. of GT above.

So G.T. is true for these rectangles.

Consider now a general simply connected region R.



Consider a sub-division of the plane into rectangles. The union of the rectangles inside R, approximates R

As the size of the rectangles $\longrightarrow 0$ this approx $\longrightarrow R$.

RHS of G.T.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$\approx \iint_{\substack{\text{union}\\ \text{of rectangles}}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \sum_{\square} \iint_{\square} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \sum_{\square} \oint_{\square} P\,dx + Q\,dy, \quad \text{by the simple case of G.T. proved above}$$

$$\approx \oint_C P\,dx + Q\,dy, \quad \text{LHS of G.T.}$$

note that integrals along all edges of interior rectangles cancel out with the integrals of neighboring retangles except for those near the boundary.

So we have an approx. form of G.T. which becomes = in the limit as size of rectangles $\to 0$ (i.e. their number $\to \infty$).

Changing coordinate system in a double integral

Consider this integral

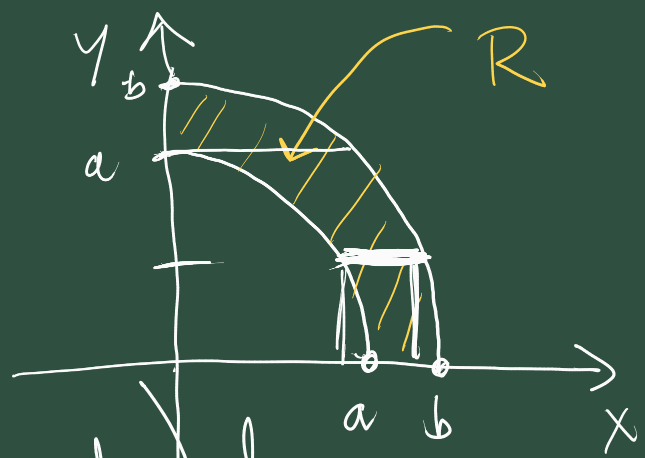
$$I = \iint_R \frac{1}{x^2+y^2}\,dx\,dy$$

"between circles"

over the region R, the annular region in the upper-right quadrant between circles of radius a and b.

$x^2+y^2=a^2$

$x^2+y^2=b^2$



$$I = \int_0^a \left(\int_{\sqrt{a^2-y^2}}^{\sqrt{b^2-y^2}} \frac{1}{x^2+y^2}\,dx \right) dy + \int_a^b \left(\int_0^{\sqrt{b^2-y^2}} \frac{1}{x^2+y^2}\,dx \right) dy$$

= my heart is sinks.... looks awkward. Is there a better way?

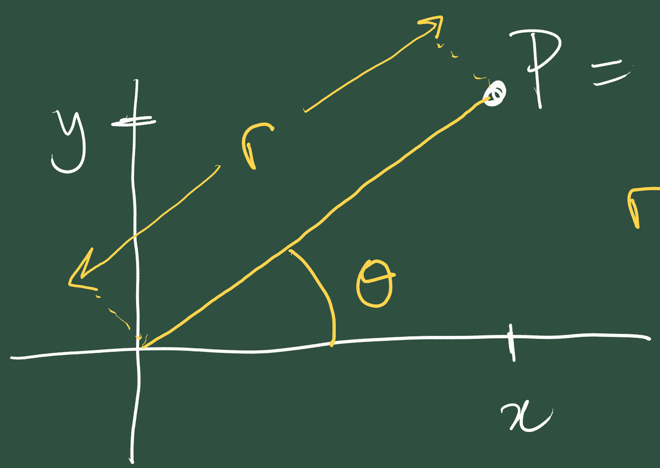
Observation: everything about this integral look "circular"

The region is "circular" and x^2+y^2 is the (radial distance)^2 of (x,y) from $(0,0)$

We should adopt the coordinate best suited to describing circles.

The polar coordinates (r,θ).

$P = (x,y) = (r,\theta)$



r = length of radius defined by P

θ = angle between radius and x-axis.

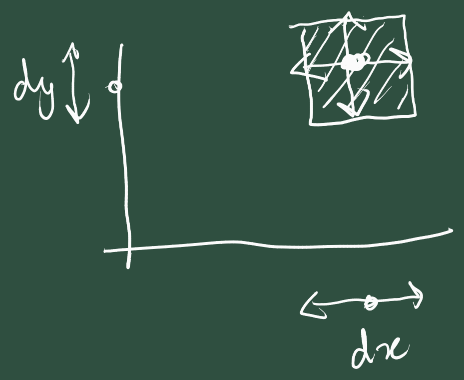
Transformation equations

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

We'd like to "lower" the integral into r, θ.

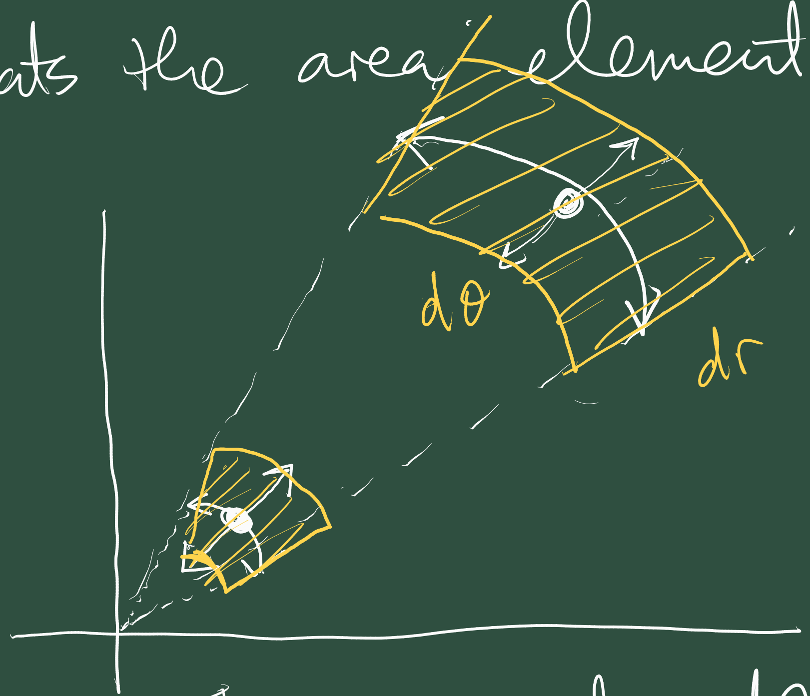
What happens to the differentials $dx\,dy$?

$dx\,dy$ is the "area element" in Cartesian coordinates.



area $= dx\,dy = dA$.

What's the area element in (r,θ)?



The same $dr, d\theta$ produce a larger area, the further we are from the origin.

It turns out that the correct

area element for varying r, θ is

$$\boxed{dA = r\,dr\,d\theta}$$

In general the relation ~~betwe~~ between area elements for coord systems (x, y) and (s, t) is given by the following formula featuring the Jacobian determinant

$$dx\,dy = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}}_{\text{Jacobian determinant of } (x,y) \to (s,t)} ds\,dt$$

$x = r\cos\theta$
$y = r\sin\theta$

For polar coords.

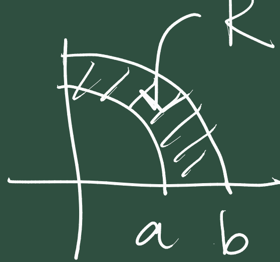
$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

Let's return to the initial integral



$$I = \iint_R \frac{1}{x^2+y^2}\,\underline{dx\,dy}$$

$$= \iint_R \frac{1}{r^2}\, r\,dr\,d\theta$$

$$= \iint_R \frac{1}{r}\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_a^b \frac{1}{r}\,dr \right) d\theta,$$

looks much nicer

$$= \int_0^{\pi/2} \left[\ln(r) \right]_a^b d\theta$$

$$= \int_0^{\pi/2} (\ln(b) - \ln(a))\, d\theta$$

$$= \int_0^{\pi/2} \ln\left(\frac{b}{a}\right) d\theta$$

$$= \ln\left(\frac{b}{a}\right) \int_0^{\pi/2} d\theta \,, \quad \text{linearity.}$$

$$= \frac{\pi}{2} \ln\left(\frac{b}{a}\right).$$

This concept of translating integrals into other coord systems extends to higher dimensions too.

$$\int \cdots \int_R f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

$$= \int \cdots \int_R f(\ldots, x_i(u_1, \ldots, u_n), \ldots) \ldots \left| \frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)} \right| du_1 \ldots du_n$$

for the transformation $(x_1, \ldots, x_n) \longrightarrow (u_1, \ldots, u_n)$

and

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)} = \left| \underbrace{\frac{\partial x_i}{\partial u_j}} \right|$$

entry in i^{th} row and j^{th} col.

Look out for videos on the 3-d coord systems

- spherical coordinates
- cylindrical coordinates.