BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Partial Integration Worked Solutions

Exercise 10

Q1. $$\int_1^2\int_0^3 (x^2y + y^2x)\,dx\,dy = \int_1^2 \left[\frac{x^3}{3}y + y^2\frac{x^2}{2} \right]_0^3 dy$$

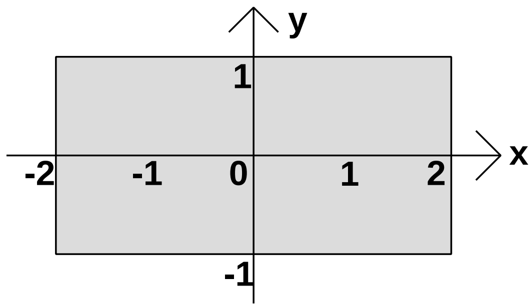
$$= \int_1^2 9y + \frac{9y^2}{2}\,dy \quad = \quad \left[\frac{9y^2}{2} + \frac{3y^3}{2} \right]_1^2$$

$$= (18 + 12) - \left(\frac{9}{2} + \frac{3}{2}\right)$$

$$= 24.$$

This is the volume under the surface, $z = x^2y + y^2x$ above the rectangle $0 \leqslant x \leqslant 1$, $1 \leqslant y \leqslant 2$.

Q2. Region, R is the shaded area shown in the diagram.



$$\int_{-1}^1\int_{-2}^2 (4xy + \sin x + \cos y)\,dx\,dy = \int_{-1}^1 \left[(2x^2y - \cos x + x\cos y) \right]_{-2}^2 dy$$

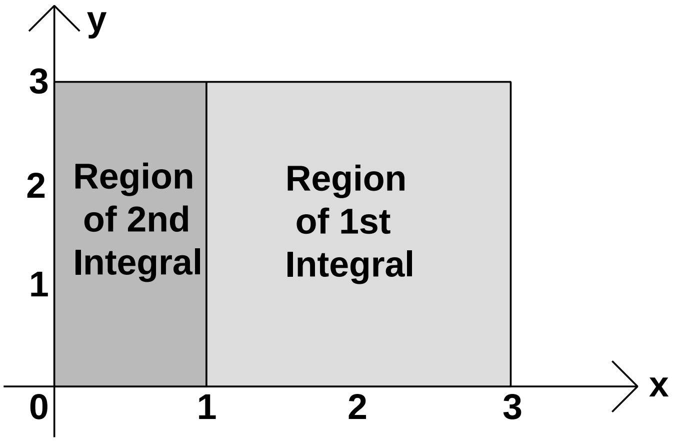
$$= \int_{-1}^1 \left[(8y - \cos 2 + 2\cos y) - (8y - \cos(-2) - 2\cos y) \right] dy$$

$$= \int_{-1}^1 4\cos y\,dy$$

$$= \left[4\sin y \right]_{-1}^1$$

$$= 8\sin 1$$

Q3.



In the first integral x ranges from 1 to 3 and y from 0 to 3. In the second integral y ranges from 0 to 3 and x from 0 to 1. So for combined region x ranges from 0 to 3 and y ranges from 0 to 3.

$$\text{Combined integral} = \int_0^3 \int_0^3 (9x^2y^2 + 4xy + 5)dxdy$$

$$= \int_0^3 \Big[3x^3y^2 + 2x^2y + 5x \Big]_0^3 dy$$

$$= \int_0^3 \Big[81y^2 + 18y + 15 \Big] dy$$

$$= \Big[27y^3 + 9y^2 + 15y \Big]_0^3$$

$$= 855$$

Q4.a) $$\int_0^1 \int_0^{0.5} ye^{xy}\, dydx = \int_0^{0.5} \int_0^1 ye^{xy}\, dxdy$$

$$= \int_0^{0.5} \Big[e^{xy} \Big] dy$$

$$= \int_0^{0.5} (e^y - 1)dy$$

$$= \Big[e^y - y \Big]_0^{0.5} = 0.149$$

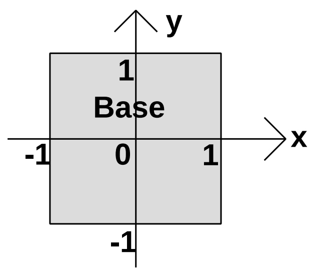
Q4.b) $$\int_0^1 \int_0^{0.5} x\sin xy \, dxdy = \int_0^{0.5} \int_0^1 x\sin xy \, dydx$$

$$= \int_0^{0.5} \Big[-\cos xy \Big]_0^1 dx$$

$$= \int_0^{0.5} (1 - \cos x)dx$$

$$= \Big[x - \sin x \Big]_0^{0.5} = 0.021$$

Q5. Average height of surface = $\dfrac{\text{Volume of Surface}}{\text{Area of Base}}$



Area of base = $2 \times 2 = 4$

Volume of surface = $$\int_{-1}^1 \int_{-1}^1 (x^2 + y^2) \, dxdy$$

$$= \int_{-1}^1 \Big[\frac{x^3}{3} + xy^2) \Big]_{-1}^1 dy$$

$$= \int_{-1}^1 (\frac{2}{3} + 2y^2) \, dy$$

$$= \Big[\frac{2y}{3} + \frac{2y^3}{3} \Big]_{-1}^1$$

$$= \frac{8}{3}$$

Therefore the average height of the surface = $\frac{1}{4} \times \frac{8}{3} = \frac{2}{3}$