6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 03

Qs 1,2 on **Taylor Series**, Qs 3 – 6 on **Finding and classifying stationary points**

(1) Use Taylor's theorem to expand the function

$$f(x,y) = \ln(x + y^2)$$

about the point $(1,0)$. Show that if h and k are small then

$$f(1+h,k) \approx h + \frac{1}{2}(-h^2 + 2k^2).$$

(2) Use Taylor's theorem to expand the function

$$f(x,y) = e^{xy}$$

about the point $(0,0)$. Show that if h and k are small then

$$f(h,k) \approx 1 + hk.$$

(3) Locate and classify the stationary points of the following functions.
   (a) $f(x,y) = (x^3 + 3x)(y^2 - 6y)$
   (b) $g(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1$

(4) What is the maximum number of stationary points that the function

$$q(x,y) = ax^2 + 2hxy + by^2 + 2gy + 2fx + c$$

can have? What conditions on the coefficients a, b, c, f, g, h ensure that at the point $(1,1)$ there is (a) a maximum, (b) a minimum or (c) a saddle point?

(5) (a) Use a software package to plot the following surfaces of the following functions in the region $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.
   (a) $f(x,y) = (x^2 - 4)(y^2 - 4)$
   (b) $g(x,y) = (x^2 - 4)(4 - y^2)$
   (c) $h(x,y) = (4 - x^2)(y^2 - 4)$
   (d) $i(x,y) = (4 - x^2)(4 - y^2)$

   (b) By inspecting the surface plot identify the location of any stationary points in the region and state what kind of points they are. Verify that the appropriate differential conditions hold, i.e.
   (i) the first partial derivatives vanish at the stationary points,
   (ii) at any saddle points we have $D < 0$,
   (iii) at any maxima we have $D > 0$ and the second partial derivative with respect to x is negative
   (iv) at any minima we have $D > 0$ and the second partial derivative with respect to x is positive.

   Note that D is the Hessian determinant given by

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

(6) Locate and classify the stationary points of the following functions and evaluate the function value at any extremum.
   (a) $g(x,y) = e^{x+y}(x^2 - xy + y^2)$
   (b) $h(x,y) = 6\ln(x+y) - 2xy - 4x - 6y + x^3 + 7$