Quick review of single variable derivative.

For a function $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$

its derivative, $f'(a)$, evaluated at a, is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

absolute change in f between x, a

change in value of argument

relative change in value of f.

Also write $\left.\frac{df}{dx}\right|_{x=a}$ to denote this.

We want to give an equivalent definition for a function of two variables. $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$$(x, y) \longmapsto f(x, y)$$

function of 2 or more variables

We can imitate single-variable definition to define two 'partial derivatives' of f.

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{x \to a} \left(\frac{f(x, b) - f(a, b)}{x - a} \right)$$

$$\text{or} \quad \lim_{h \to 0} \left(\frac{f(a+h, b) - f(a, b)}{h} \right)$$

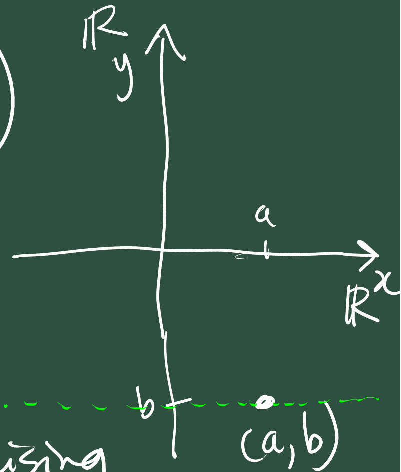
relative change in value of f, when varying the 'x' coordinate

Similarly, the partial derivative of f with respect to y, is defined as

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{y \to b} \left(\frac{f(a, y) - f(a, b)}{y - b} \right).$$

Example 1. Consider f defined by

$f(x,y) = x^2 y$. Find its partial derivatives.



$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{x \to a} \left(\frac{f(x,b) - f(a,b)}{x-a} \right)$$

$$= \lim_{x \to a} \left(\frac{x^2 b - a^2 b}{x-a} \right)$$

$$= \lim_{x \to a} \left(\frac{(x^2 - a^2) b}{x-a} \right) \text{, factorising}$$

$$= b \lim_{x \to a} \left(\frac{\cancel{(x-a)}(x+a)}{\cancel{x-a}} \right) \text{, factorising, } b \text{ does not depend on } x$$

$$= b \lim_{x \to a} (x+a)$$

$$= b\,2a = 2ab$$

And

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{y \to b} \left(\frac{f(a,y) - f(a,b)}{y-b} \right)$$

$$= \lim_{y \to b} \left(\frac{a^2 y - a^2 b}{y-b} \right)$$

$$= a^2 \lim_{y \to b} \underbrace{\left(\frac{y-b}{y-b} \right)}_{=1}$$

$$= a^2$$

Notice $\boxed{f(x,y) = x^2 y.}$

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 2ab$$

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = a^2$$

So we can drop explicit reference to fixed point (a,b) instead declare the derivatives.

$$\boxed{\frac{\partial f}{\partial x} = 2xy}\ , \quad \boxed{\frac{\partial f}{\partial y} = x^2}$$

$$\left(\frac{d}{dx} \tan(x) = \sec^2(x) = \frac{1}{\cos^2(x)} \right)$$

There is a principle at work here:
To differentiate f with respect to a variable x, we treat other variables as fixed constants, and go ahead and differentiate f using our normal single variables rules and techniques for differentiating with respect to x.
Similarly for partial derivatives with respect to other variables.

<u>Example 2</u> Consider f defined by

$$f(x,y) = x^2 y^3 \tan(2x).$$

Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ using above principle.

$$\frac{\partial f}{\partial x} = y^3 \left(2x \tan(2x) + x^2 \sec^2(2x) 2 \right)$$

, by the prod., chain rule, known derivative for tan.

$$\frac{\partial f}{\partial y} = 3y^2 x^2 \tan(2x).$$

<u>Notation</u>: Other notation commonly used is.

$$f_x = \frac{\partial f}{\partial x} \quad , \quad f_y = \frac{\partial f}{\partial y}$$

and $f_x(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$

Maybe f_x is simpler/cleaner.

But $\frac{\partial f}{\partial x}$ allows us to show $\frac{\partial}{\partial x}(f)$

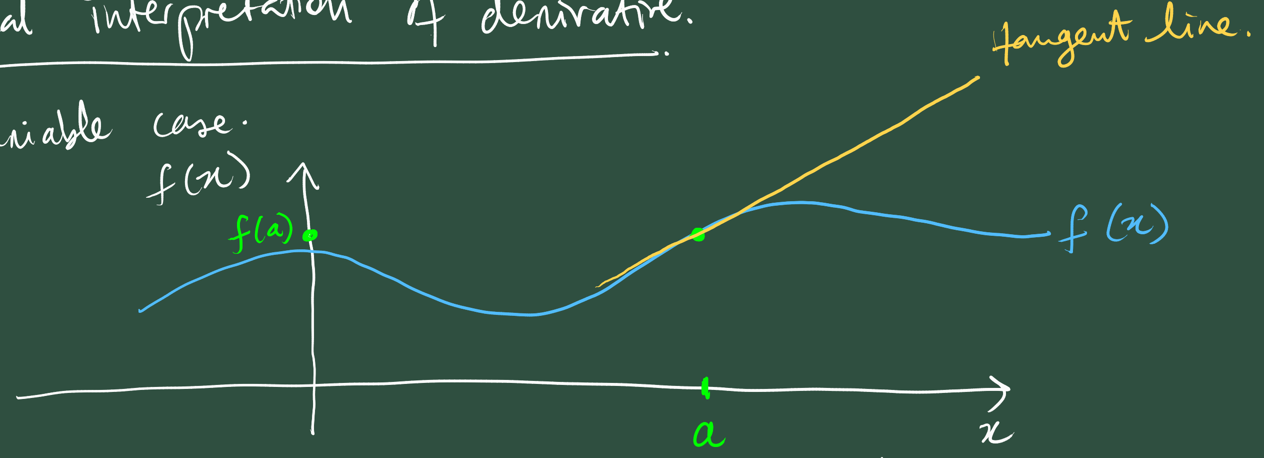
object function

operator

applied to.

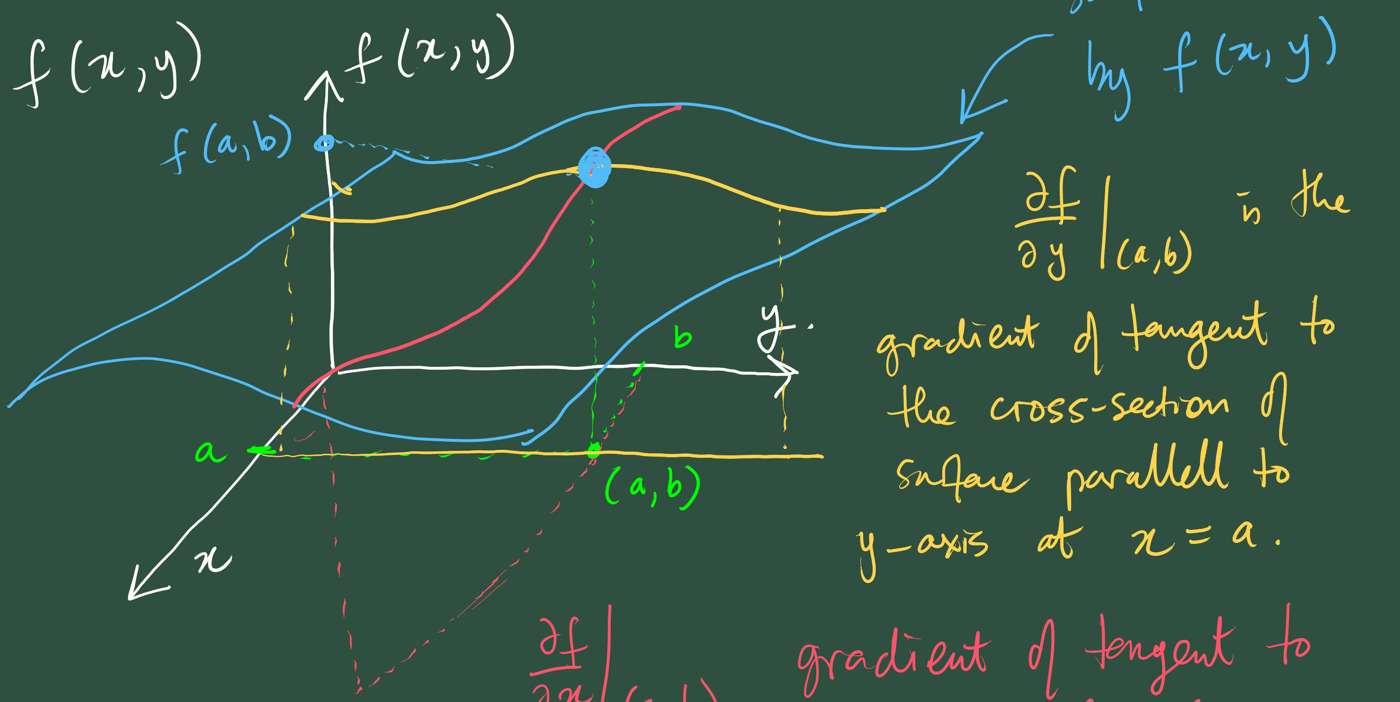
Graphical interpretation of derivative.

Single-variable case.



$\left.\frac{df}{dx}\right|_{x=a}$ is the slope/gradient of tangent line to graph of $f(x)$ at $x=a$.

For $f(x,y)$



$\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$ is the gradient of tangent to the cross-section of surface parallell to y-axis at $x=a$.

$\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$ gradient of tangent to cross section of surface parallell to x-axis at $y=b$.

Higher-order derivatives

Just like the single variable case, we can take derivatives of derivatives, ...

For instance there are four possible 2^{nd} order partial derivatives of a function $f(x,y)$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

compact notation

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$=$ by Clairaut's Th.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Some of the 3rd order partial derivatives would be

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right)$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right)$$

Example 3. (continue example 2) $f(x,y) = x^2 y^3 \tan(2x)$ has 2nd order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2y^3\left(x\tan(2x) + x^2\sec^2(2x)\right)\right)$$

$$= 2y^3 \frac{\partial}{\partial x}\left(x\tan(2x) + x^2\sec^2(2x)\right)$$

$$= 2y^3\Big(\tan(2x) + x\,2\sec^2(2x)$$

$$+ 2x\sec^2(2x) + x^2\left(4\sec^2(2x)\tan(2x)\right)\Big)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(3y^2x^2\tan(2x)\right)$$

$$= 6y\,x^2\tan(2x)$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y}} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3y^2 x^2 \tan(2x)\right)$$

$$= 3y^2 \frac{\partial}{\partial x}\left(x^2 \tan(2x)\right)$$

$$\boxed{= 3y^2\left(2x\tan(2x) + 2x^2\sec^2(2x)\right)}$$

$$= 6xy^2\left(\tan(2x) + x\sec^2(2x)\right)$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x}} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(y^3\left(2x\tan(2x) + 2x^2\sec^2(2x)\right)\right)$$

$$\boxed{= 3y^2\left(2x\tan(2x) + 2x\sec^2(2x)\right)}$$

Notice In this example $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

This will always hold for function f with continuous partial derivatives. Clairaut's Theorem

All of this will extend to functions of more than two variables.

Eg. $g(x,y,z) = x^2 y^4 z^3 + y^2$

has three first order partial derivatives

$$\frac{\partial g}{\partial x} = 2x y^4 z^3 + 0$$

$$\frac{\partial g}{\partial y} = 4x^2y^3z^3 + 2y$$

$$\frac{\partial g}{\partial z} = 3x^2y^4z^2 + 0$$