6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 01 - SOLUTIONS

Solutions to questions 1 - 4 list on the following pages under the heading of *Exercise 5*.

Then solutions to questions 5 - 10 listed on the following pages under the heading *Multi-variable calculus Partial differentiation problem solutions*.

BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Partial Differentiation Worked Solutions

Exercise 5

Q1.a) $\dfrac{\partial f}{\partial x} = 6x\ln(y)$, $\dfrac{\partial f}{\partial y} = \dfrac{3x^2}{y}$

b) $\dfrac{\partial g}{\partial x} = 8xy\cos(x^2+2y)$, using the function of a function rule,

$\dfrac{\partial g}{\partial y} = 8y\cos(x^2+2y) + 4\sin(x^2+2y)$ using the function of a function and product rules

c) $\dfrac{\partial h}{\partial x} = 14xy + yz$, $\dfrac{\partial h}{\partial y} = 7x^2 + xz$, $\dfrac{\partial h}{\partial z} = -\dfrac{1}{z^2} + xy$

Q2.a) $\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+y^2}$ so $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{-4xy}{(x^2+y^2)^2}$ using the function of a function (or the quotient) rule.

$\dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2}$ so $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{-4xy}{(x^2+y^2)^2}$ using the function of a function (or the quotient) rule.

Hence $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial x \partial y}$.

Also $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{(x^2+y^2)2 - 2y(2y)}{(x^2+y^2)^2} = \dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$

and $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{(x^2+y^2)2 - 2x(2x)}{(x^2+y^2)^2} = \dfrac{2(x^2-y^2)}{(x^2+y^2)^2}$ using the quotient rule.

Now adding $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

Q3.a) Suppose δf is a small change in $f(x, y)$ caused by a change in δx alone (from x to $x + \delta x$). Then, since y is unchanged,

$$\begin{aligned}\delta f &= f(x + \delta x, y) - f(x, y)\\ &= (x + \delta x)^2y^3 - x^2y^3\\ &= (x^2 + 2x\delta x + (\delta x)^2)y^3 - x^2y^3\\ &= 2xy^3\delta x + (\delta x)^2y^3\end{aligned}$$

Then $\dfrac{\delta f}{\delta x} = 2xy^3 + y^3\delta x$

and letting $\delta x \to 0$ we get

$$\frac{\partial f}{\partial x} = 2xy^3$$

b) Suppose δf is a small change in $f(x, y)$ caused by a change in δy alone (from y to $y + \delta y$). Then, since x is unchanged,

$$\begin{aligned}\delta f &= f(x, y + \delta y) - f(x, y)\\ &= x^2(y + \delta y)^3 - x^2y^3\\ &= x^2(y^3 + 3y^2\delta y + 3y(\delta y)^2 + (\delta y))^3 - x^2y^3\\ &= 3x^2y^2\delta y + 3x^2y(\delta y)^2 + x^2(\delta y)^3\end{aligned}$$

Then $\dfrac{\delta f}{\delta y} = 3x^2y^2 + 3x^2y\delta y + x^2(\delta y)^2$

and letting $\delta y \to 0$ we get

$$\frac{\partial f}{\partial y} = 3x^2y^2$$

Q4. When the height of the hill is 400m,

$$400 = 1000e^{-u} + 110$$

so $e^{-u} = 0.29$

and $u = -\ln(0.29) = 1.238$

Then $x^2+y^2 = 1.24$ which is the equation of a circle centre (0, 0) radius $\sqrt{1.24} = 1.11$

when the height of the hill is 600m, the contour is

$x^2+y^2 = 0.71$ which is the equation of a circle centre (0, 0) radius $\sqrt{0.71} = 0.84$

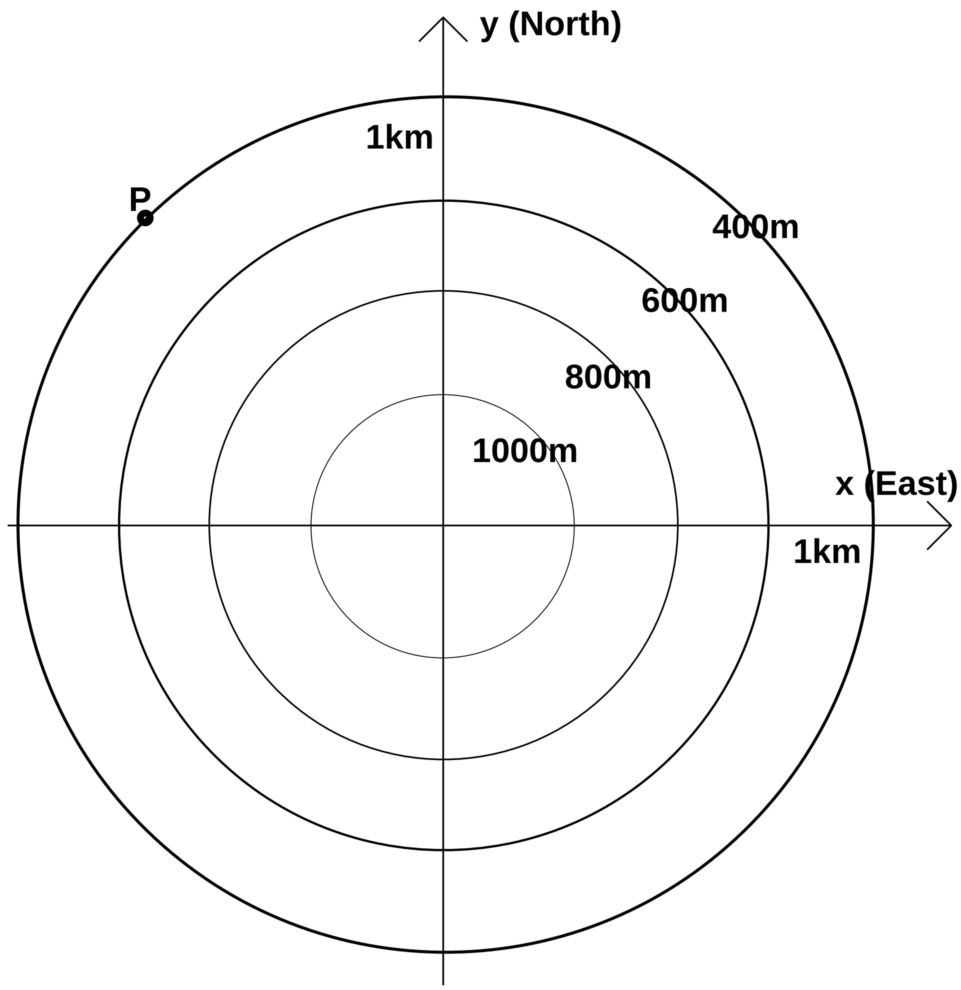
when the height of the hill is 800m, the contour is

$x^2+y^2 = 0.37$ which is the equation of a circle centre (0, 0) radius $\sqrt{0.37} = 0.61$

and when the height of the hill is 1000m, the contour is

$x^2+y^2 = 0.11$ which is the equation of a circle centre (0, 0) radius $\sqrt{0.11} = 0.34$

Contour plot is therefore:-



Due north west of the peak $y= -x$ and since the 400m contour is $x^2+y^2 = 1.24$, we get $2x^2 = 1.24$. so the point, P is (-0.787, 0.787) . Now since

$$z(x,y) = 1000e^{-(x^2+y^2)} + 110$$

$$\frac{\partial z}{\partial x} = (-2x)1000e^{-(x^2+y^2)} = 456.3$$

so the easterly gradient is 456.3 m/km upwards

and $$\frac{\partial z}{\partial y} = (-2y)1000e^{-(x^2+y^2)} = -456.3$$

so the northerly gradient is 456.3 m/km downwards.

Multi-variable calculus \\ Partial differentiation problem solutions

1. The two required second-order partial derivatives are

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^x\sin(y)\right) = e^x\sin(y),$$

and

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(e^x\cos(y)\right) = -e^x\sin(y).$$

So adding them together does indeed result in 0.

2. The three required partial derivatives are

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^{-t}\cos(x)\right) = -e^{-t}\sin(x),$$

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right) = \frac{\partial}{\partial y}\left(-e^{-t}\sin(y)\right) = -e^{-t}\cos(y)$$

and

$$\frac{\partial\psi}{\partial t} = -e^{-t}\left(\sin(x)+\cos(y)\right).$$

So adding the first and the second, does indeed result in the third.

3. We need to find the simultaneous soutions (x, y) of the equations

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\cos(x^2+y^2)\right) = -2x\sin(x^2+y^2) = 0$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\cos(x^2+y^2)\right) = -2y\sin(x^2+y^2) = 0.$$

The first equation is satisfied whenever $x = 0$ or $x^2 + y^2 = n\pi$, for some $n \in \mathbb{Z}$. The second equation is satisfied whenever $y = 0$ or $x^2 + y^2 = n\pi$.

For a simultaneous solution both equations must be satisfied so the simultaneous solutions are the points (x, y) satisfying $x^2 + y^2 = n\pi$, for any $n \in \mathbb{Z}$. (Note that the $x = y = 0$ solution is included here in the case where $n = 0$). Geometrically, a simultaneous solution is given by any point (x, y) on any of the circles of radius $\sqrt{n\pi}$, for some $n \in \mathbb{Z}$, and centred on the origin.

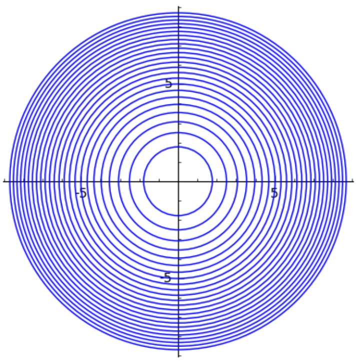


Figure 1: Some of the simultaneous solutions

4. We treat the variable z as a function of the two independent variables x and y. The equation defines z implicitly. To obtain the derivatives we differentiate both sides and use the product rule as appropriate.

   First we obtain the x derivative

$$\begin{aligned}
&\frac{\partial}{\partial x}(xy + yz + zx) = \frac{\partial}{\partial x}(1) \\
\Rightarrow\, & y + y\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x} = 0 \\
\Rightarrow\, & \frac{\partial z}{\partial x} = \frac{-y - z}{x + y}.
\end{aligned}$$

   Secondly, the y derivative

$$\begin{aligned}
&\frac{\partial}{\partial y}(xy + yz + zx) = \frac{\partial}{\partial y}(1) \\
\Rightarrow\, & x + z + y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial y} = 0 \\
\Rightarrow\, & \frac{\partial z}{\partial y} = \frac{-x - z}{x + y}.
\end{aligned}$$

   Both expressions are of course only valid off the line defined by $x = -y$. In fact the original defining equation can be rearranged to make z the subject and then the partial derivatives obtained in the usual way. However in other cases the defining equation will not be so easily rearranged and the technique of implicit differentiation will come in useful.

5. The given function u is a solution if and only if

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -\beta e^{-\beta t}\sin(\alpha x) + \alpha^2 e^{-\beta t}\sin(\alpha x) = 0,$$

   i.e.

$$(\alpha^2 - \beta)\sin(\alpha x) = 0.$$

   Since this equation must be true for all x, the given u is a solution if and only if $\beta = \alpha^2$. So the heat equation has a solution

$$u(x,t) = e^{-\alpha^2 t}\sin(\alpha x),$$

   for any $\alpha \in \mathbb{R}$.

6. Following the guidance in the question we take the cosine formula

$$a^2 = b^2 + c^2 - 2bc\cos(A),$$

   to implicitly define the angle A in terms of the independent variables a, b, c. So assuming b and c are held constant, we differentiate both sides with respect to a to get

$$2a = 2bc\sin(A)\frac{\partial A}{\partial a},$$

   and so

$$\frac{\partial A}{\partial a} = \frac{a}{bc\sin(A)}.$$