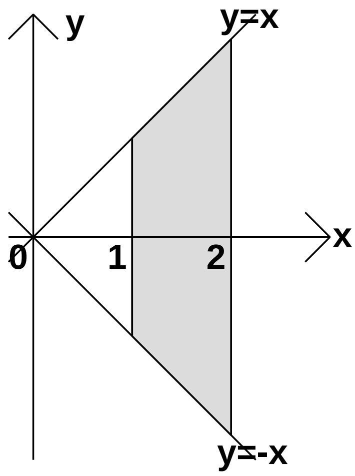
BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Partial Integration Worked Solutions

Exercise 11



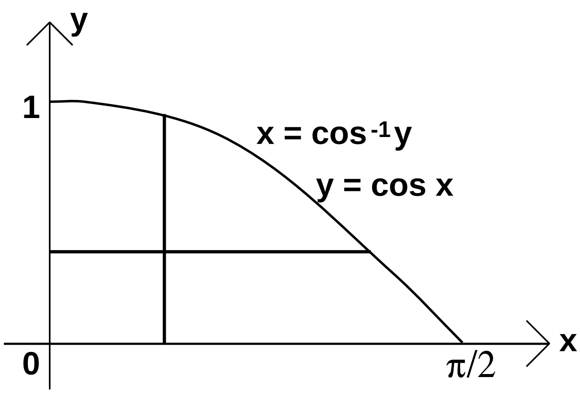
Q1. The region of integration is the shaded area shown in the diagram.

$$\int_1^2\int_{-x}^{x} \left(\frac{y}{x} + 1\right) dydx = \int_1^2 \left[\frac{y^2}{2x} + y\right]_{-x}^{x} dx$$

$$= \int_1^2 \left(\frac{x}{2} + x\right) - \left(\frac{x}{2} - x\right) dx$$

$$= \int_1^2 2x\, dx = \left[x^2\right]_1^2 = 3$$

Q2.a) If $x = \cos^{-1}y$, then $y = \cos x$.
For the given integral x ranges from 0 to $\cos^{-1}y$ and then y ranges from 0 to 1 so the region of integration is that shown in the diagram. Equivalently y ranges from 0 to cos x and then x ranges from 0 to $\pi/2$. Then



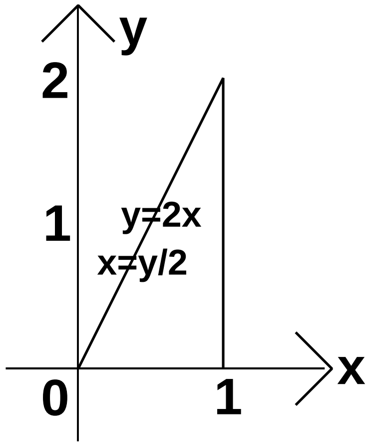
$$\int_0^1\int_0^{\cos^{-1}y} \sec x\, dxdy = \int_1^{\pi/2}\int_0^{\cos x} \sec x\, dydx$$

$$= \int_0^{\pi/2} \left[y \sec x\right]_0^{\cos x} dx$$

$$= \int_0^{\pi/2} (\cos x \sec x - 0)\, dx$$

$$= \int_0^{\pi/2} 1\, dx = \pi/2$$

Q2.b) For the given integral x ranges from y/2 to 1 and then y ranges from 0 to 2 so the region of integration is that shown in the diagram. Equivalently y ranges from 0 to 2x and then x ranges from 0 to 1. Then



$$\int_0^2\int_{y/2}^1 e^{x^2}\,dxdy = \int_0^1\int_0^{2x} e^{x^2}\,dydx$$

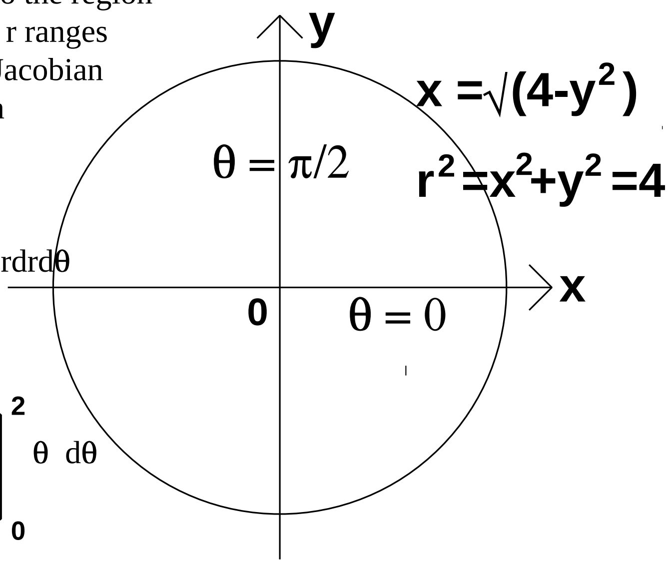
$$= \int_0^1 \Big[\, ye^{x^2} \Big]_0^{2x}\,dx$$

$$= \int_0^1 2xe^{x^2}\,dx$$

$$= \int_0^1 e^u du \quad \text{where } u = x^2 \text{ so that } du = 2x\,dx$$

$$= e - 1$$

Q3.a) Since $r^2 = x^2 + y^2$, $(x^2 + y^2)^{5/2} = r^5$. Also $\tan^{-1}(y/x) = \theta$. If $x = \sqrt{(4-y^2)}$, $r^2 = x^2 + y^2 = 4$ so $r = 2$. For the given integral x ranges from 0 to $\sqrt{(4-y^2)}$ and then y ranges from 0 to 2 so the region of integration is that shown in the diagram. Equivalently r ranges from 0 to 2 and then θ ranges from 0 to $\pi/2$. Finally the Jacobian from (x,y) to (r,θ) is r so dxdy is replaced by $rdrd\theta$. Then

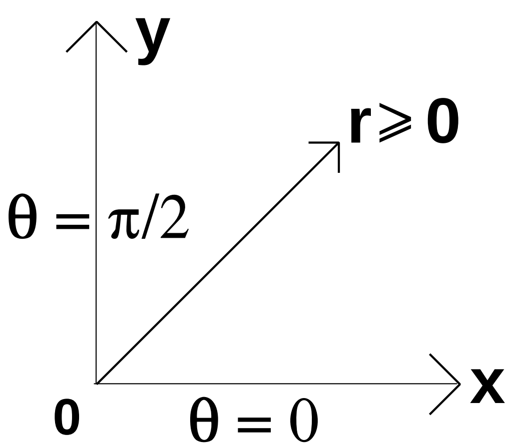


$$\int_0^2\int_0^{\sqrt{(4-y^2)}} (x^2 + y^2)^{5/2}\tan^{-1}(y/x)\,dxdy = \int_0^{\pi/2}\int_0^2 r^5\theta\; rdrd\theta$$

$$= \int_0^{\pi/2} \Big[\, \frac{r^7}{7} \Big]_0^2\, \theta\; d\theta$$

$$= \frac{128}{7}\int_0^{\pi/2} \theta\, d\theta = \frac{16\pi^2}{7}$$

Q3.b) For the given integral x and y both range from 0 to ∞.
The region of integration is therefore the first quadrant where
$0 \leqslant \theta \leqslant \pi/2$ and r ranges from 0 to ∞.
Also $r^2 = x^2 + y^2$ and the Jacobian from (x,y) to (r,θ) is r.
Then



$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta$$

$$= \int_0^{\pi/2} \int_0^\infty \frac{e^{-u}}{2}\, du\, d\theta$$ using the substitution $u = r^2$.

$$= \int_0^{\pi/2} \left[\frac{-e^{-u}}{2} \right]_0^\infty d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\, d\theta = \frac{\pi}{4}$$