6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 10 - SOLUTIONS

Solutions to questions 1 – 4 listed on the following pages under the heading of *Exercise 16*

BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Fourier Series Worked Solutions

Exercise 16

Q1. The given series is even so $b_m = 0$ for all $m > 0$ and when $0 < x < \pi$

$$f(x) = \pi - x$$

Then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\, dx$$

$$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \pi$$

and

$$a_m = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos mx\, dx = \frac{2}{\pi}\left[(\pi - x)\frac{\sin mx}{m} - \frac{\cos mx}{m^2}\right]_0^{\pi} \quad \text{by parts}$$

$$= \frac{2}{\pi}\left[\frac{1 - (-1)^m}{m^2}\right]$$

$$= \frac{4}{\pi m^2} \quad \text{if } m \text{ is odd}$$

$$= 0 \quad \text{if } m \text{ is even.}$$

Hence

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$$

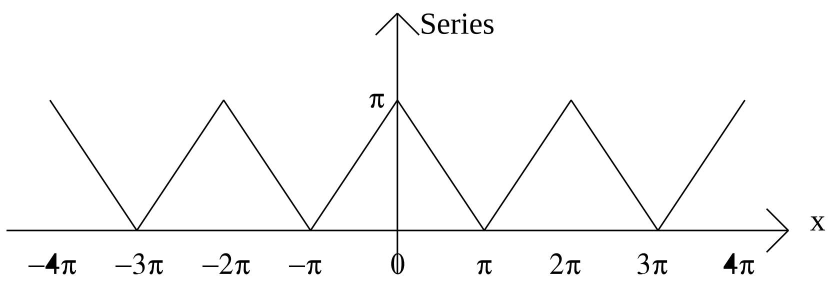
Now putting $x = 0$ gives

$$\pi = \frac{\pi}{2} + \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

so rearranging

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}$$

Since the series has period 2π its graph is



Q2. $f(x) = 0$ if $-\pi < x < 0$ so

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\, dx = \frac{2}{\pi}$$

When m =1

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x\, dx = 0$$

When $m \neq 1$

$$a_m = \frac{1}{\pi}\int_0^{\pi} \sin x \cos mx\, dx$$

Now put $\frac{(A-B)}{2} = x$ and $\frac{(A+B)}{2} = mx$.

Then $A = (m+1)x$ and $B = (m-1)x$

so $\sin x \cos mx = \sin \frac{(A-B)}{2} \cos\frac{(A+B)}{2} = \frac{1}{2}(\sin A - \sin B) = \frac{1}{2}(\sin(m+1)x - \sin (m-1)x)$

Then

$$a_m = \frac{1}{2\pi}\int_0^{\pi} \sin (m+1)x - \sin (m-1)x\, dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos (m+1)x}{m+1} + \frac{\cos (m-1)x}{m-1}\right]_0^{\pi}$$

$$= \frac{-2}{[\pi(m^2 - 1)]} \quad m \text{ even}$$

$$= 0 \quad m \text{ odd}$$

Similarly

$$b_m = \frac{1}{2} \quad \text{when } m = 1$$

$$= 0 \quad \text{when } m \neq 1$$

Hence the series is

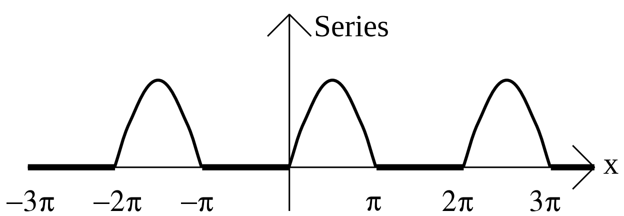
$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} \ldots\ldots.\right) + \frac{1}{2}\sin x$$

Putting $x = 0$ this gives

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} \ldots..\right)$$

so $$\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} \ldots.. = \frac{1}{2}$$

The graph of the series is:-



Q3. For the half range cosine series of the given function, $b_m = 0$ for every $m>0$,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx = \frac{4}{\pi}$$

and if $m>0$

$$a_m = \frac{2}{\pi}\int_0^{\pi} \sin x \cos mx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin (m+1)x - \sin (m-1)x \, dx$$ using the result of the previous question,

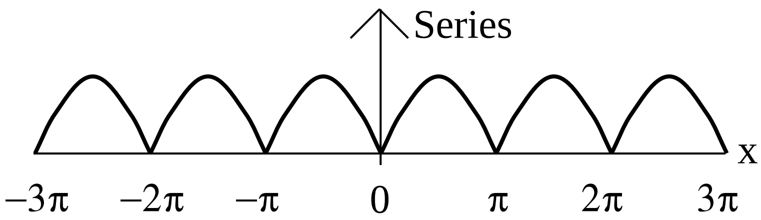
$$= \frac{-4}{[\pi(m^2 - 1)]} \qquad m \text{ even}$$

$$= 0 \qquad m \text{ odd}$$

Hence the cosine series for sin x is

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} \text{.......}\right)$$

Since the series is composed of even functions only, its graph is symmetric about the y-axis so the graph of sin x in $[-\pi,0]$ is the reflection in the y-axis of the graph in $[0,\pi]$. Since the terms in the series have period 2π, the rest of the graph has this period.



Q4. The series obtained for x was

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \text{....} \right)$$

Integrating we get

$$\frac{x^2}{2} = c + 2\left(-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \text{....}\right)$$ where c is the constant of integration

so $$x^2 = 2c - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} - \text{....}\right)$$

To find the constant in the series for x^2 we need to find $\frac{1}{2}a_0$ and since x^2 is even

$$a_0 = \frac{2}{\pi} \quad x^2 \, dx = \frac{2\pi^2}{3}$$

then $$x^2 = \int_0^{\pi} \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} - \text{....}\right)$$

Integrating again

$$\frac{x^3}{3} = c + \frac{x\pi^2}{3} - 4\left(\sin x - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \text{....}\right)$$

so $$x^3 = 3c + x\pi^2 - 12\left(\sin x - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \text{....}\right)$$

Now if we put x = 0 we get c = 0.

Hence

$$x^3 = 2\pi^2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right) - 12\left(\sin x - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \ldots\right)$$

a) We now put $x = \frac{\pi}{2}$ in the Fourier Series for x

$$\frac{\pi}{2} = 2\left(1 - \frac{0}{2} - \frac{1}{3} + \frac{0}{4} + \frac{1}{5} - \frac{0}{6} - \frac{1}{7} + \ldots\right) = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$$

so $$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)} = \frac{\pi}{4}$$

b) Using Parseval's theorem with f(x) = x, since the a_m are all zero and $b_m = 2\frac{(-1)^{m+1}}{m}$ we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2}\sum_{m=1}^{\infty} b_m^2$$

so that $$\frac{\pi^2}{3} = \frac{1}{2}\sum_{m=1}^{\infty}\frac{4}{m^2}$$ and hence $$\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}$$

c) If we put $x = \pi/2$ in the series for x^3 we obtain

$$\frac{\pi^3}{8} = 2\pi^2\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right) - 12\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right)$$

Now using the result in (a)

$$\frac{\pi^3}{8} = 2\pi^2\frac{\pi}{4} - 12\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right)$$

and rearranging $$\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m-1)^3} = \frac{\pi^3}{32}$$

d) When $f(x) = x^2$, all the b_m are zero but $a_0 = \frac{2\pi^2}{3}$ and $a_m = 4\frac{(-1)^m}{m^2}$ so by Parseval's theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{1}{4}\frac{4\pi^4}{9} + \frac{1}{2}\sum_{m=1}^{\infty}\frac{16}{m^4}$$

so that $$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\sum_{m=1}^{\infty}\frac{1}{m^4}$$ and hence $$\sum_{m=1}^{\infty}\frac{1}{m^4} = \frac{\pi^4}{90}$$