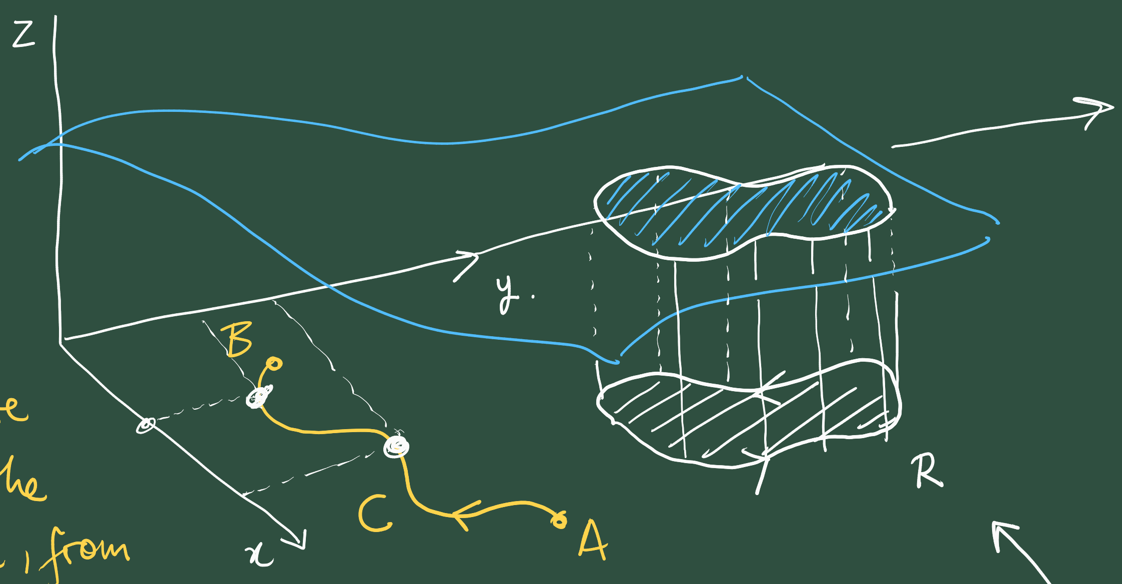
Line integrals

Intro Two types of integrals in this unit. Focus on functions of two variables.

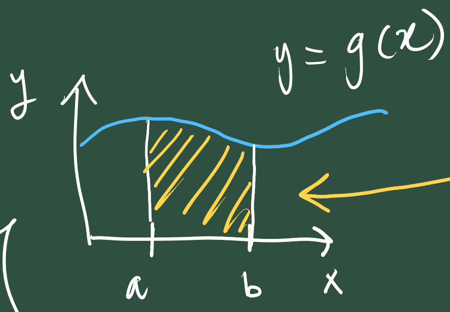
$$z = f(x, y)$$



a curve C in the xy-plane, from A to B.

We will meet so called double integrals.

$$I = \iint_R f(x,y)\, dx\, dy$$ = volume of this cylindrical part of $\mathbb{R}^3$ bounded by R in the xy plane (bottom) and above by the surface $z = f(x,y)$, with straight sides parallel to z-axis



$$\text{Area} = \int_a^b g(x)\, dx$$

Also see path integrals (line integrals)

which are expressed as

$$J = \int_C \underbrace{h(x,y)\, dx + k(x,y)\, dy}$$

which is the path integral of the "linear differential form"

Path integrals have important applications in pure and applied mathematics, and are connected to double integrals through <u>Green's Theorem</u>

<u>Line Integrals</u> The expression L,

$$L = P(x,y)\,dx + Q(x,y)\,dy$$

is called a linear differential form. These appear in path / line integrals over a given curve C in xy-plane. Such integrals can be evaluated / computed by using the specification of C to convert L to be expressed in one of the variables only

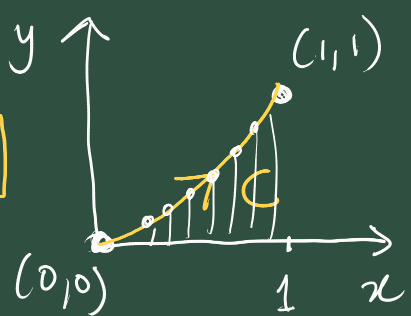
<u>Example</u> Consider $L = 10x^2y\,dx + (3x+2y)\,dy$ and integrate this along curve C which is defined by $C: y = x^2$, from $(0,0)$ to $(1,1)$.

$$I = \int_C L$$

On C, $\boxed{y = x^2}$

$\Rightarrow \boxed{dy = 2x\,dx}$ by chain rule

and $\boxed{x: 0 \longrightarrow 1}$



$$= \int_C 10x^2y\,dx + (3x+2y)\,dy$$

convert everything to x

$$= \int_0^1 10x^4\,dx + (3x+2x^2)\,2x\,dx.$$

$$= \int_0^1 (10x^4 + 4x^3 + 6x^2)\,dx. \quad ,$$

a familiar single variable integral.

$$= \left[2x^5 + x^4 + 2x^3 \right]_0^1$$

$$= 2 + 1 + 2 = 5$$

<u>Theorem</u> (Some basic properties of path integrals).

1. Path integrals are linear.

L_1, L_2. lin. diff. forms. , $\alpha, \beta \in \mathbb{R}$.

$$\int_C \alpha L_1 + \beta L_2$$

$$= \alpha \left(\int_C L_1 \right) + \beta \left(\int_C L_2 \right).$$

<u>OR</u>

$$\int_C P\,dx + Q\,dy$$

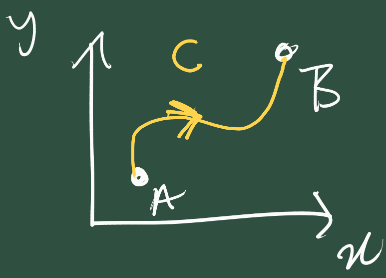
$$= \left(\int_C P(x,y)\,dx \right) + \left(\int_C Q(x,y)\,dy \right)$$

Recall: $\displaystyle\int_a^b (\alpha f_1(x) + \beta f_2(x))\,dx$

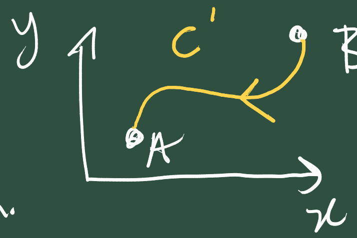
$$= \alpha \left(\int_a^b f_1(x)\,dx \right) + \beta \left(\int_a^b f_2(x)\,dx \right)$$

2. Reversing direction.

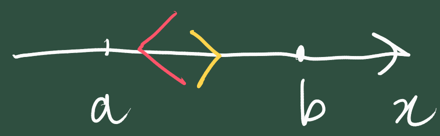
$$\int_C L = -\int_{C'} L$$



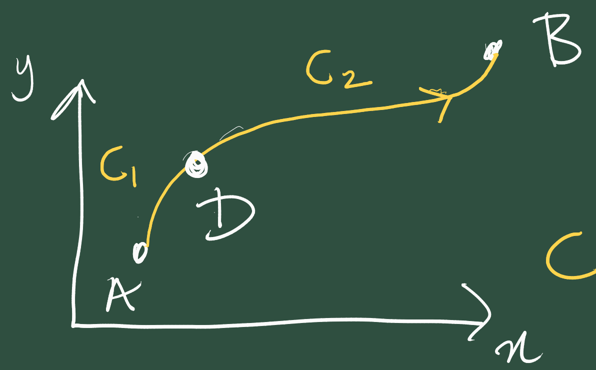
where C' is the curve C, traversed in the opposite direction.



$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$



3. Subdividing the curve.



concatenation of the curves.

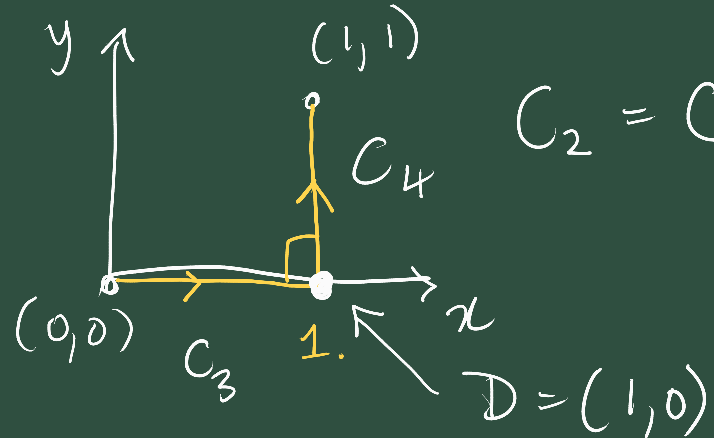
$$C = C_1 + C_2$$

$$\int_C L = \int_{C_1} L + \int_{C_2} L.$$

<u>Recall</u>

$$\left(\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx \right)$$

Example Let's integrate L from prev. example.
from $(0,0)$ to $(1,1)$, this time along the curve C_2



$C_2 = C_3 + C_4$

$$\int_{C_2} L = \int_{C_3} L + \int_{C_4} L \quad \text{, by subdividing } C_2$$

On C_3: $y=0$, $x: 0 \longrightarrow 1$,
$\Rightarrow dy = 0$

On C_4: $x=1$, $dx=0$, $y: 0 \longrightarrow 1$.

$L = 10x^2 y\,dx + (3x+2y)\,dy$.

$$\text{So } \int_{C_2} L = \int_0^1 10 \cdot x^2 \cdot 0 \cdot dx + (3x + 2 \cdot 0) \cdot 0 \text{ , on } C_3$$

$$+ \int_0^1 10y \cdot 0 + (3+2y)\,dy \text{ , on } C_4.$$

$$= \int_0^1 (3+2y)\,dy,$$ all other contributions are zero.

$$= \left[3y + y^2\right]_0^1$$

$$\int_{C_2} L = 4.$$

And recall from previous example.

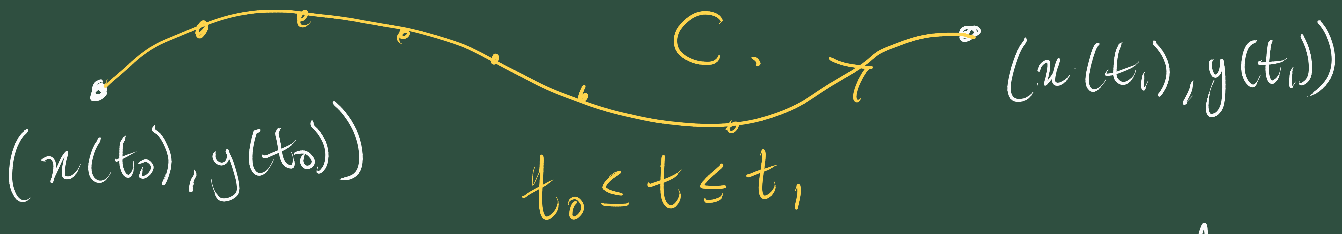
$$\int_C L = 5$$

Although both integrals went between the same endpoints $A=(0,0)$, $B=(1,1)$, they went along different curves, and so produced different values. This is typical behaviour. This is known as "path dependence" of L.

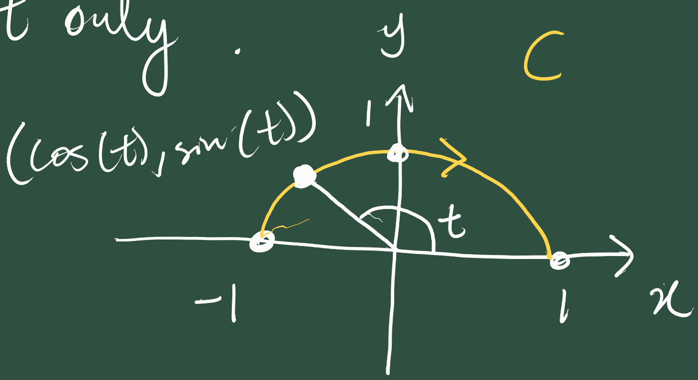
But in some special cases the form L can be "path independent", where integrals along different curves, but between same endpoints, will always give same value. See later.

<u>Parametrised paths</u>. Sometimes a curve C is parametrised by a third variable, t say.

$$C : \{(x(t), y(t)) : t_0 \le t \le t_1\}$$



Integrals along such curves can be evaluated by converting them to "t only".



__Example:__

C: Upper semi-circle from $(-1,0)$ to $(1,0)$

C has the parametrisation as

$$C : \{ (\cos(t), \sin(t)) : , \quad t: \pi \to 0 \}$$

t measured in radians.

Evaluate the integral $I = \int_C x^2 \, dy - yx \, dx$

__Transform using:__

$x = \cos(t)$, $y = \sin(t)$

$dx = -\sin(t)\, dt$, $dy = \cos(t)\, dt$, by chain rule.

$t: \pi \to 0$.

$$I = \int_\pi^0 \cos^2(t) \cos(t)\, dt - \sin(t)\cos(t)(-\sin(t)\, dt)$$

$$= \int_\pi^0 \cos(t) \big(\overbrace{\cos^2(t) + \sin^2(t)}^{=1} \big)\, dt.$$

$$= \int_{\pi}^{0} \cos(t)\, dt = \Big[\sin(t)\Big]_{\pi}^{0} = \sin(0) - \sin(\pi)$$
$$= 0$$

Tut sheet 03

Q1

$$f(x,y) = \ln(x+y^2).$$

Find Taylor series expansion of f around the $(1,0)$.

$$f(1+h, k) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(D^n f\right)(1,0)$$

$$\approx \sum_{n=0}^{2} \frac{1}{n!}\left(D^n f\right)(1,0)$$

for $n=0$

$$D^0 f = f$$

So $(D^0 f)(1,0) = f(1,0) = \ln(1+0^2)$
$= \ln(1) = 0$

$n=1$

$$Df = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$= h\frac{1}{x+y^2} + k\frac{1}{x+y^2}2y$$

So $(Df)(1,0) = h + 0 = h$

$\underline{n=2}$

$$D^2 f = D(D(f))$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x\partial y}$$

$$= h^2\left(\frac{-1}{(x+y^2)^2}\right) + 2k^2\left(\frac{-y\cdot 2y}{(x+y^2)^2} + \frac{1}{x+y^2}\right)$$

$$+ 2hk\left(\frac{-2y}{(x+y^2)^2}\right)$$

So $(D^2 f)(1,0)$

$$= -h^2 + 2k^2$$

So

$$f(1+h,k) \approx 0 + h + \frac{1}{2}\left(-h^2+2k^2\right)$$

$$= h + \frac{1}{2}(-h^2 + 2k^2)$$

as required.