6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 04 - SOLUTIONS

Solutions to questions 1 – 5 listed on the following pages under the heading of *Exercise 10*

Solutions to questions 6 – 8 listed on the following pages under the heading of *Exercise 11*

BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Partial Integration Worked Solutions

Exercise 10

Q1. $$\int_1^2\int_0^3 (x^2y + y^2x)\,dx\,dy = \int_1^2 \left[\frac{x^3y}{3} + \frac{y^2x^2}{2}\right]_0^3 dy$$

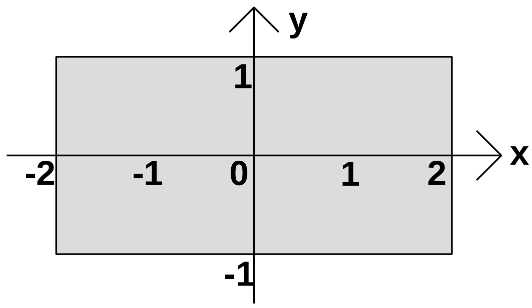
$$= \int_1^2 9y + \frac{9y^2}{2}\,dy = \left[\frac{9y^2}{2} + \frac{3y^3}{2}\right]_1^2$$

$$= (18 + 12) - \left(\frac{9}{2} + \frac{3}{2}\right)$$

$$= 24.$$

This is the volume under the surface, $z = x^2y + y^2x$ above the rectangle $0 \leqslant x \leqslant 1,\ 1 \leqslant y \leqslant 2$.

Q2. Region, R is the shaded area shown in the diagram.



$$\int_{-1}^1\int_{-2}^2 (4xy + \sin x + \cos y)\,dx\,dy = \int_{-1}^1 \Big[(2x^2y - \cos x + x\cos y)\Big]_{-2}^2 dy$$

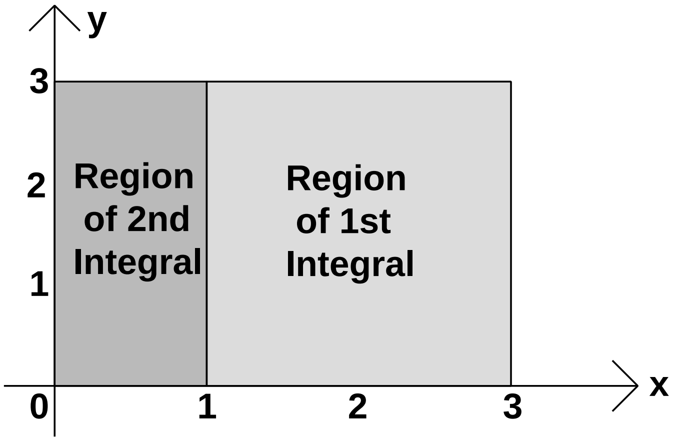
$$= \int_{-1}^1 \Big[(8y - \cos 2 + 2\cos y) - (8y - \cos(-2) - 2\cos y)\Big] dy$$

$$= \int_{-1}^1 4\cos y\,dy$$

$$= \Big[4\sin y\Big]_{-1}^1$$

$$= 8\sin 1$$

Q3.



In the first integral x ranges from 1 to 3 and y from 0 to 3. In the second integral y ranges from 0 to 3 and x from 0 to 1. So for combined region x ranges from 0 to 3 and y ranges from 0 to 3.

Combined integral = $\int_0^3 \int_0^3 (9x^2y^2 + 4xy + 5)dxdy$

$$= \int_0^3 \left[3x^3y^2 + 2x^2y + 5x \right]_0^3 dy$$

$$= \int_0^3 \left[81y^2 + 18y + 15 \right] dy$$

$$= \left[27y^3 + 9y^2 + 15y \right]_0^3$$

= 855

Q4.a) $\int_0^1 \int_0^{0.5} ye^{xy}\, dydx = \int_0^{0.5} \int_0^1 ye^{xy}\, dxdy$

$$= \int_0^{0.5} \left[e^{xy} \right] dy$$

$$= \int_0^{0.5} (e^y - 1)dy$$

$$= \left[e^y - y \right]_0^{0.5} = 0.149$$

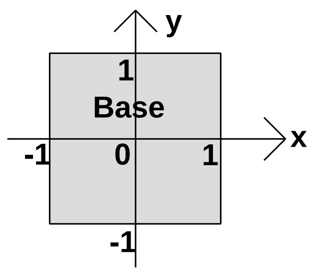
Q4.b) $$\int_0^1 \int_0^{0.5} x\sin xy \, dxdy = \int_0^{0.5} \int_0^1 x\sin xy \, dydx$$

$$= \int_0^{0.5} \Big[-\cos xy \Big]_0^1 dx$$

$$= \int_0^{0.5} (1 - \cos x)dx$$

$$= \Big[x - \sin x \Big]_0^{0.5} = 0.021$$

Q5. Average height of surface = $\dfrac{\text{Volume of Surface}}{\text{Area of Base}}$



Area of base = 2×2 = 4

$$\text{Volume of surface} = \int_{-1}^{1} \int_{-1}^{1} (x^2 + y^2) \, dxdy$$

$$= \int_{-1}^{1} \Big[\frac{x^3}{3} + xy^2) \Big]_{-1}^{1} dy$$

$$= \int_{-1}^{1} (\frac{2}{3} + 2y^2) \, dy$$

$$= \Big[\frac{2y}{3} + \frac{2y^3}{3} \Big]_{-1}^{1}$$

$$= \frac{8}{3}$$

Therefore the average height of the surface = $\frac{1}{4} \times \frac{8}{3} = \frac{2}{3}$

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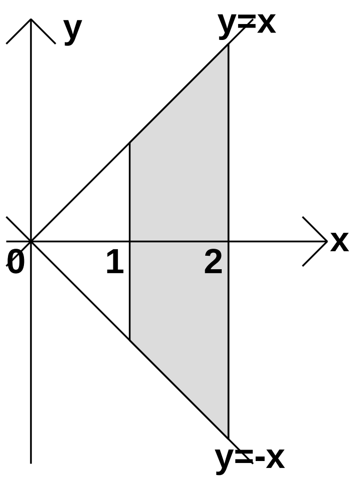
Exercise 11



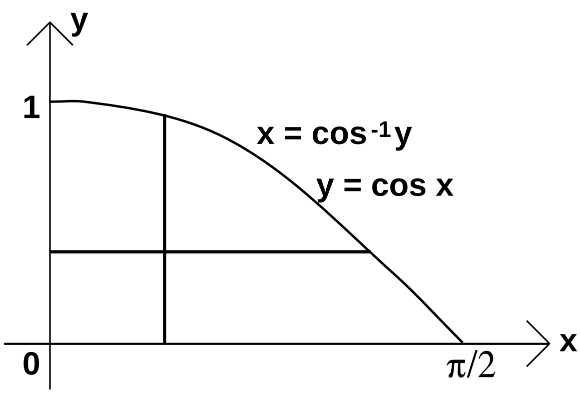
Q1. The region of integration is the shaded area shown in the diagram.

$$\int_1^2\int_{-x}^{x} \left(\frac{y}{x} + 1\right) dydx = \int_1^2 \left[\frac{y^2}{2x} + y\right]_{-x}^{x} dx$$

$$= \int_1^2 \left(\frac{x}{2} + x\right) - \left(\frac{x}{2} - x\right) dx$$

$$= \int_1^2 2x\ dx = \Big[\ x^2\ \Big]_1^2 = 3$$

Q2.a) If x = $\cos^{-1}y$, then y = cos x.
For the given integral x ranges from 0 to $\cos^{-1}y$ and then y ranges from 0 to 1 so the region of integration is that shown in the diagram. Equivalently y ranges from 0 to cos x and then x ranges from 0 to $\pi/2$. Then



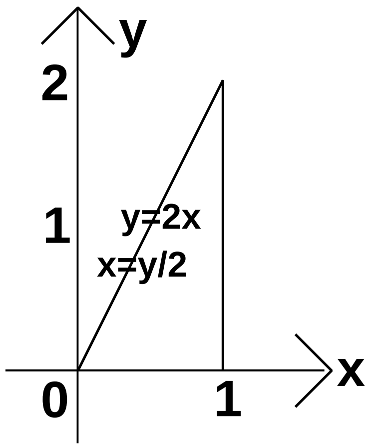
$$\int_0^1\int_0^{\cos^{-1}y} \sec x\ dxdy = \int_0^{\pi/2}\int_0^{\cos x} \sec x\ dydx$$

$$= \int_0^{\pi/2} \Big[\ y \sec x\ \Big]_0^{\cos x} dx$$

$$= \int_0^{\pi/2} (\cos x \sec x - 0)\ dx$$

$$= \int_0^{\pi/2} 1\ dx = \pi/2$$

Q2.b) For the given integral x ranges from y/2 to 1 and then y ranges from 0 to 2 so the region of integration is that shown in the diagram. Equivalently y ranges from 0 to 2x and then x ranges from 0 to 1. Then



$$\int_0^2\int_{y/2}^1 e^{x^2}\,dxdy = \int_0^1\int_0^{2x} e^{x^2}\,dydx$$

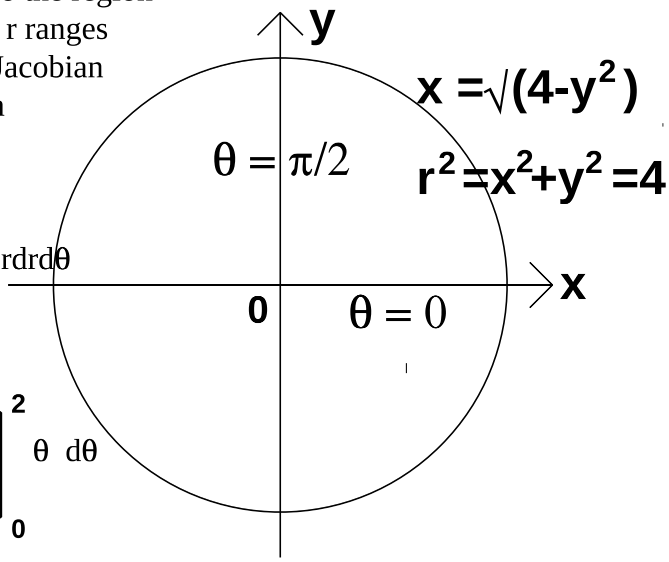
$$= \int_0^1 \Big[ye^{x^2} \Big]_0^{2x} dx$$

$$= \int_0^1 2xe^{x^2}\,dx$$

$$= \int_0^1 e^u du \quad \text{where } u = x^2 \text{ so that } du = 2x\,dx$$

$$= e - 1$$

Q3.a) Since $r^2 = x^2 + y^2$, $(x^2 + y^2)^{5/2} = r^5$. Also $\tan^{-1}(y/x) = \theta$. If $x = \sqrt{(4-y^2)}$, $r^2 = x^2 + y^2 = 4$ so $r = 2$. For the given integral x ranges from 0 to $\sqrt{(4-y^2)}$ and then y ranges from 0 to 2 so the region of integration is that shown in the diagram. Equivalently r ranges from 0 to 2 and then θ ranges from 0 to π/2. Finally the Jacobian from (x,y) to (r,θ) is r so dxdy is replaced by rdrdθ. Then

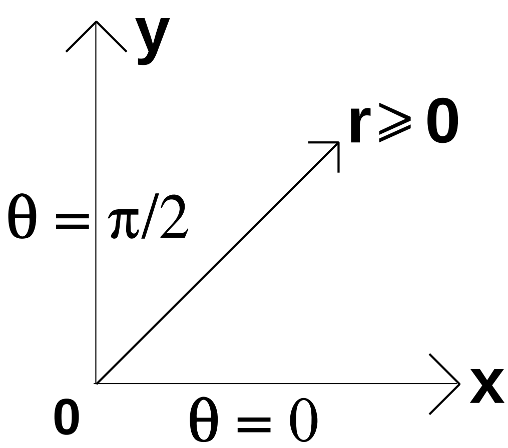


$$\int_0^2\int_0^{\sqrt{(4-y^2)}} (x^2 + y^2)^{5/2}\tan^{-1}(y/x)\,dxdy = \int_0^{\pi/2}\int_0^2 r^5\theta\, rdrd\theta$$

$$= \int_0^{\pi/2} \Big[\frac{r^7}{7} \Big]_0^2 \theta\, d\theta$$

$$= \frac{128}{7}\int_0^{\pi/2} \theta\, d\theta = \frac{16\pi^2}{7}$$

Q3.b) For the given integral x and y both range from 0 to ∞. The region of integration is therefore the first quadrant where $0 \leqslant \theta \leqslant \pi/2$ and r ranges from 0 to ∞.
Also $r^2 = x^2 + y^2$ and the Jacobian from (x,y) to (r,θ) is r.
Then



$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\infty} \frac{e^{-u}}{2}\,du\,d\theta$$ using the substitution $u = r^2$.

$$= \int_0^{\pi/2} \left[\frac{-e^{-u}}{2} \right]_0^{\infty} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\,d\theta = \frac{\pi}{4}$$