$$\frac{\partial^3 f}{\partial x^2 \partial y}$$

$$D(f) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$D^2 f = h^2 \frac{\partial^2 f}{\partial x^2} + k^2 \frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x \partial y}$$

$$= D(D(f))$$

$$D^3 f = h^3 \frac{\partial^3 f}{\partial x^3} + \ldots\ldots = D(D^2(f))$$

Chap 2. Q9.

$$T : V_4(\mathbb{R}) \longrightarrow V_3(\mathbb{R})$$

T is the linear transformation defined by

$$T((a, b, c, d))$$
$$= (a-b+c+d,\ a+2b-c+d,\ 3a-2c).$$

Finding a basis for $\ker(T)$.

Given a vector $\underline{x} = (a, b, c, d)$.

$\underline{x} \in \ker(T) \iff T(\underline{x}) = \underline{0} \in V_3(\mathbb{R})$

$\iff (a-b+c+d,\ a+2b-c+d,\ 3a-2c) = (0,0,0)$

$$\iff \left\{ \begin{array}{l} a-b+c+d = 0 \\ a+2b-c+d = 0 \\ 3a \quad -2c \quad = 0 \end{array} \right\}$$

Find sols to these by Gaussian Elimination.

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 & 0 \\ 3 & 0 & -2 & 0 & 0 \end{pmatrix}$$

$r_2' = r_2 - r_1$
$r_3' = r_3 - 3r_1$

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & \underline{\underline{3}} & -2 & 0 & 0 \\ 0 & 3 & -5 & -3 & 0 \end{pmatrix}$$

$r_2' = \frac{1}{3} r_2$

$r_1' = r_1 + r_2'$

$r_3' = r_3 - r_2$

$$\begin{pmatrix} 1 & 0 & 1/3 & 1 & 0 \\ 0 & 1 & -2/3 & 0 & 0 \\ 0 & 0 & \underline{\underline{-3}} & -3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2/3 & 0 \\ 0 & 1 & 0 & 2/3 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$r_1' = r_1 - 1/3\, r_3'$
$r_2' = r_2 + 2/3\, r_3'$
$r_3' = \frac{1}{-3} r_3$

This is now in reduced row echelon form so I can read off the solutions, treating d as a free parameter, we get

$$\boxed{\begin{aligned} c &= -d \\ b &= -2/3\, d \\ a &= -2/3\, d \end{aligned}}$$

eg. $d = 3$
$c = -3$
$b = -2$
$a = -2$
✓

So $\underline{x} \in \ker T \iff x = \left(-\tfrac{2}{3}d, -\tfrac{2}{3}d, -d, d\right)$

$= d\left(-\tfrac{2}{3}, -\tfrac{2}{3}, -1, 1\right)$

for any $d \in \mathbb{R}$.

So $\ker(T)$ is a 1-dimensional subspace with basis $\left(-\tfrac{2}{3}, -\tfrac{2}{3}, 1, 1\right)$

Now $\operatorname{Im} T$. From the dimension sum formula.

$\dim \operatorname{domain} T = \dim \ker T + \dim \operatorname{Im} T$.
$4 = 1 + \boxed{3}$

$\operatorname{Im} T$ will be the span of

$$T(\underline{e}_1), T(\underline{e}_2), T(\underline{e}_3), T(\underline{e}_4)$$

where $\{\underline{e}_i\}$ is the standard basis for $V_4(\mathbb{R})$

$e_1 = (1,0,0,0)$, $\underline{e}_2 = (0,1,0,0)$ etc.

So $\operatorname{Im} T$ is the span of.

$$\left.\begin{array}{l} T(\underline{e}_1) = (1,1,3) \\ T(\underline{e}_2) = (-1,2,0) \\ T(\underline{e}_3) = (1,-1,-2) \\ T(\underline{e}_4) = (1,1,0) \end{array}\right\} \subseteq V_3(\mathbb{R})$$

Question? How lin. dependent are they?

Can discover this by row-reducing the vectors. Because elementary row operations preserve the 'row-space' of a matrix.

$$\begin{pmatrix} \underline{\underline{1}} & 1 & 3 \\ -1 & 2 & 0 \\ 1 & -1 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$

$r_2' = r_2 + r_1$

$r_3' = r_3 - r_1$

$r_4' = r_4 - r_1$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & \underline{\underline{3}} & 3 \\ 0 & -2 & -5 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix}$$

$r_2' = \frac{1}{3} r_2$
$r_1' = r_1 - r_2'$
$r_3' = r_3 + 2r_2'$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

now in reduced row echelon form.

So $\operatorname{Im} T$ is spanned by these 4 rows

So $\operatorname{Im} T$ has the basis $\{ (1,0,0), (0,1,0), (0,0,1) \}$

so i.e $\operatorname{Im} T = V_3(\mathbb{R})$

Mock Exam 2

Q2 (a).

Find eigenvalues and eigenvectors of $A = \begin{pmatrix} -1 & 1 \\ 2 & 4 \end{pmatrix}$.

Def: A non-zero vector x is an eigenvector of A with associated eigen-value λ iff.

$$A \underline{x} = \lambda \underline{x}$$

$$\Leftrightarrow (A - \lambda I)\underline{x} = \underline{0} \quad (*)$$

Such a matrix-vector equation $B\underline{y} = \underline{0}$ has the unique solution $\underline{y} = \underline{0}$ whenever B is invertible, ie.

$\underline{y} = B^{-1}\underline{0} = \underline{0}$, ie whenever $\det(B) \neq 0$.

So non-zero solutions to $(*)$ exist only when $\det(A - \lambda I) = 0$. This allows us to identify the e-values first.

eigenvalues are solutions to

$$\det(A - \lambda I) = 0$$

$$\Leftrightarrow \left| \begin{pmatrix} -1-\lambda & 1 \\ 2 & 4-\lambda \end{pmatrix} \right| = 0$$

$$\Leftrightarrow (-1-\lambda)(4-\lambda) - 2 = 0$$

$$\Leftrightarrow \lambda^2 - 3\lambda - 6 = 0.$$

$$\Leftrightarrow \lambda = \frac{3 \pm \sqrt{9+24}}{2}$$

$$\Leftrightarrow \lambda = \frac{3 \pm \sqrt{33}}{2}.$$

So we have two eigenvalues λ_+, λ_-

So taking $\underline{\underline{\lambda_+}}$ first. It's $\underline{\text{eigenvectors } \underline{x}}$ will be solutions to.

$$(A - \lambda_+ I)\underline{x} = \underline{0}$$

$$\Leftrightarrow \begin{pmatrix} -1 - \frac{3+\sqrt{33}}{2} & 1 \\ 2 & 4 - \frac{3+\sqrt{33}}{2} \end{pmatrix} \underline{x} = \underline{0}$$

$$\Leftrightarrow \begin{pmatrix} \frac{-5-\sqrt{33}}{2} & 1 \\ 2 & \frac{5-\sqrt{33}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -5-\sqrt{33} & 2 \\ 4 & 5-\sqrt{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note that the second row is a multiple of the first.

$$\frac{4}{-5-\sqrt{33}} = \frac{5-\sqrt{33}}{2}$$

$$8 = -25 + 33 \checkmark$$

So

$\iff (-5-\sqrt{33})x_1 + 2x_2 = 0$

So an eigenvector (i.e. a basis for this eigenspace) is given by setting

$x_1 = 1$ and so $x_2 = \dfrac{5+\sqrt{33}}{2}$

So the eigenvector for λ_+ is

$$\underline{x} = \begin{pmatrix} 1 \\ \frac{5+\sqrt{33}}{2} \end{pmatrix} \checkmark$$