$$\Rightarrow z = \frac{1-xy}{x+y}$$

8 $\quad x, y, z$ related by the equation

$$\boxed{xy + yz + zx = 1}$$

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

This equation implicitly defines z as a function $z(x,y)$.

Take $\frac{\partial}{\partial x}$ of both sides

$$\frac{\partial}{\partial x}(xy + yz + zx) = \frac{\partial}{\partial x}(1) = 0$$

$$\Leftrightarrow \quad y + y\frac{\partial z}{\partial x} + \underbrace{z + x\frac{\partial z}{\partial x}}_{} = 0$$

using linearity and product rule

$$\frac{\partial z}{\partial x} = -\frac{y+z}{x+y}.$$

$$\frac{\partial}{\partial x}(yz^2)$$
$$= y\frac{\partial}{\partial x}(z^2)$$
$$= y\,2z\,\frac{\partial z}{\partial x}.$$

Chain rule and small increments formula.

Recall a use of the single-variable derivative of a function $f(x)$

$f'(x)$

Small increments / small changes approximation.

$$\boxed{f(x+\underbrace{\Delta x}_{\text{small change in } x}) \approx f(x) + \Delta x f'(x)}$$

This will generalize to the 2-variable case

$$\boxed{\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)}$$

$$\boxed{\approx \Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y}}$$

where both partial derivatives evaluated at original point (x, y)

and to n-variables in general

it appears as $\left(f(x_1, x_2, \ldots, x_n) \right)$

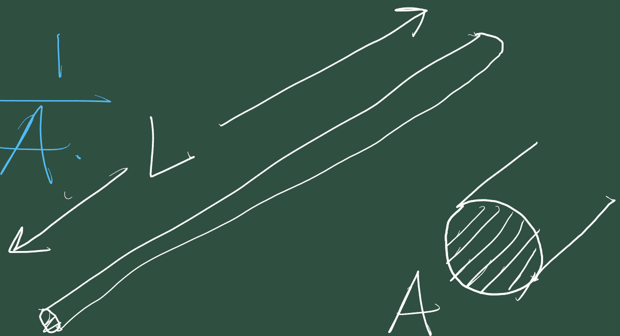
$$\boxed{\Delta f \approx \sum_{i=1}^{n} \Delta x_i \frac{\partial f}{\partial x_i}}$$

$\frac{1}{x} \searrow \frac{-1}{x^2}$

<u>Example</u> The resistance R of a length of wire can be modelled as

$$\boxed{R = \rho \frac{L}{A}}$$

$\frac{1}{A}$



$R(L, A)$

L is length of wire,

A is cross-section area of the wire

ρ is the resistivity of the wire material

ρ = "rho"

Suppose A decreases by 1% and L increases by 2.5%. Use the small increments formula to approximate change in resistance, ΔR, of the wire.

$$\boxed{\Delta R} \approx \Delta L \frac{\partial R}{\partial L} + \Delta A \frac{\partial R}{\partial A}.$$

$$= 0.025L \underbrace{\frac{\rho}{A}} - 0.01A \underbrace{\left(-\frac{\rho L}{A^2}\right)}$$

$$= \boxed{0.035 \underbrace{\frac{\rho L}{A}}_{R.}}$$

= 3.5% increase in R.

$\left(1 + \frac{r}{100}\right)R$. — new value

$$\Delta R = \frac{c}{100} R.$$

Chain rule

First, recall 1-variable concept.

Suppose $y = f(x)$

and in turn $x = g(t)$

So $y = f(g(t)) = (f \circ g)(t)$.

Chain rule tells us about $\frac{dy}{dt}$.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

OR.

$$y'(t) = f'(g(t))\, g'(t)$$

For more than 1 variable.

Suppose $y = f(x_1, \ldots, x_n)$

and the x_i in turn are given

as:
$$x_i = u_i(z_1, \ldots, z_n)$$

We need to know how

$\frac{\partial y}{\partial z_i}$ depends on $\frac{\partial y}{x_j}$ and $\frac{\partial u_j}{\partial z_i}$

Focus on 2-variable case

Start with small increments

formula.
$$z = f(x, y), \quad x = u(s, t)$$
$$y = v(s, t)$$

$$\Delta z \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y.$$

$$\Rightarrow \frac{\Delta z}{\Delta s} \approx \frac{\partial f}{\partial x} \frac{\Delta x}{\Delta s} + \frac{\partial f}{\partial y} \frac{\Delta y}{\Delta s}$$

Consider this as $\Delta s \longrightarrow 0$

then $\frac{\Delta z}{\Delta s} \longrightarrow \frac{\partial z}{\partial s}$, $\frac{\Delta x}{\Delta s} \longrightarrow \frac{\partial x}{\partial s}$

$$\frac{\Delta y}{\Delta s} \longrightarrow \frac{\partial y}{\partial s}$$ and the

approximation becomes equality.

giving us.

$$\boxed{\begin{gathered} \frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ \text{and similarly} \\ \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \end{gathered}}$$

Chain rule equations for the transformation $(x, y) \longrightarrow (s, t)$

And in general case.

$$(x_1, \dots, x_n) \longrightarrow (z_1, \dots, z_n)$$

we would n chain

rule equations for $\dfrac{\partial y}{\partial z_i}$

for $i = 1, \dots, n$

$$\frac{\partial y}{\partial z_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial z_i}$$

Chain rule equations for $(x_1, \dots, x_n) \Rightarrow (z_1, \dots, z_n)$.

Textbook CLP3 Section 2.4

Q6.
$$w = x^2 + y^2 + z^2$$

and $x = st$, $y = s\cos(t)$,
$z = s\sin(t)$.

Method 1 Direct substitution.

$$\boxed{w} = (st)^2 + (s\cos(t))^2 + (s\sin(t))^2$$

$$= s^2t^2 + s^2 \quad , \quad \cos^2 t + \sin^2 t = 1$$

$$\boxed{= s^2(t^2+1)}$$

Then diff. as usual. to get.

$$\boxed{\frac{\partial w}{\partial s} = 2s(t^2+1)}$$

$$\frac{\partial w}{\partial t} = 2s^2 t$$

Method 2 Use chain rule(s).

$$\boxed{\frac{\partial w}{\partial s}} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= 2xt + 2y\cos(t) + 2z\sin(t)$$

$$= 2st^2 + 2s\cos^2(t) + 2s\sin^2(t)$$

$$= 2st^2 + 2s$$

$$\boxed{= 2s(t^2+1)}$$

AND.

$$\boxed{\frac{\partial w}{\partial t}} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$+ \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= 2xs + 2y(-s\sin(t))$$

$$+ 2zs\cos(t)$$

$$= 2s^2t - 2s^2\cos(t)\sin(t)$$

$$+ 2s^2\sin(t)\cos(t)$$

$$\boxed{= 2s^2 t.}$$

Q23 (a) $f(t), \quad f'(t)$

f is unknown 1-variable function.

$$w(x,y) = \underbrace{e^{-y}}\,\underbrace{f(x-y)}.$$

Claim: w is a solution of

$$w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0.$$

$$\boxed{\frac{\partial w}{\partial x}} = e^{-y} \frac{\partial}{\partial x}\left(f(x-y)\right)$$

$$\boxed{= e^{-y} \frac{df}{dt}\bigg|_{t = x-y}.}$$

$$\boxed{\frac{\partial w}{\partial y}} = \text{(needs product rule)}$$

$$e^{-y} \frac{\partial}{\partial y}\left(f(x-y)\right)$$

$$+ f(x-y)\frac{\partial}{\partial y}\left(e^{-y}\right)$$

$$= e^{-y}\left(-\frac{df}{dt}\right) + f(x-y)\left(-e^{-y}\right)$$

$$\boxed{= -e^{-y}\left.\frac{\partial f}{\partial y}\right|_{x-y} - e^{-y} f(x-y).}$$

So $w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$

$$= e^{-y} f(x-y) + e^{-y}\frac{df}{dt}(x-y)$$

$$- e^{-y}\frac{df}{dt}(x-y) - e^{-y} f(x-y)$$

$$= 0$$

$f(t)_{t=x-y}$

$$w = f(x-y)$$

$$\frac{\partial w}{\partial x} = \frac{df}{dt}\frac{\partial t}{\partial x}.$$

$$\frac{\partial w}{\partial y} = \frac{df}{dt}\frac{\partial t}{\partial y}$$

Recall the chain rule equations.

$$\frac{\partial y}{\partial z_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial z_i} \quad ; i = 1, \dots, n$$

The RHS might look familiar.

$$\overset{\text{vector}}{\square}_i = \sum_{j=1}^{n} \overset{\text{vector}}{\square}_j \; \overset{\text{matrix}}{\boxed{\;i\;}}_j$$

This looks like matrix multiplication with vectors.

Yes, the chain rule equations have a nice expression in matrix form.

2-variable case.

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$$

and similarly

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

$$\begin{pmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \underbrace{\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}}$$

$$|J| = \frac{\partial(x,y)}{\partial(s,t)}$$

Known as the Jacobian matrix of $(x, y) \to (s, t)$

index the rows

Index the columns

In general for a transformation

$(x_1, \dots, x_n) \to (z_1, \dots, z_n)$

J is a $n \times n$ matrix.

$$J_{i,j} = \frac{\partial x_i}{\partial z_j}$$

row, col.

and chain rule eqns. are.

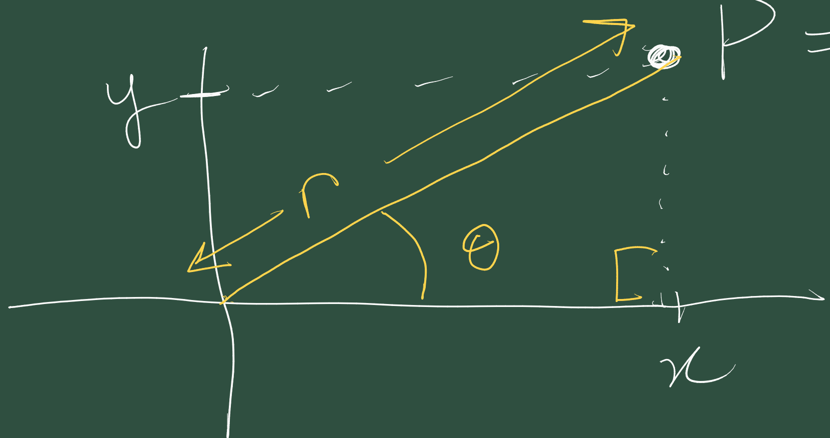
$$\left(\frac{\partial y}{\partial z_1}, \dots, \frac{\partial y}{\partial z_n}\right)$$

$$= \left(\frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n}\right) J.$$

see Q7 on Tut. sheet 02

$$|J| = \frac{\partial(x_1, \dots, x_n)}{\partial(z_1, \dots, z_n)},$$ the Jacobian determinant.

Q7. Change from Cartesian to polar coords.



$P = (x, y) = (r, \theta)$

$$\tan\theta = \frac{y}{x}.$$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}}$$

Jacobians (determinants)

feature importantly in integration

$\dfrac{\partial(x, y)}{\partial(r, \theta)}$ = determinant of Jacobian matrix.