BSc Mathematics/BSc Combined Honours

MA2101 Mathematical Methods

Laplace Transform Worked Solutions

Exercise 3

Q1. a) $f(t) = 6\{H(t-2) - H(t-3)\}$

b) $f(t) = (t-2)\{H(t-2) - H(t-6)\} + 4H(t-6)$

$= (t-2)H(t-2) - (t-6)H(t-6)$

c) $f(t) = 3t\{H(t) - H(t-1)\} + (4-t)\{H(t-1) - H(t-4)\}$

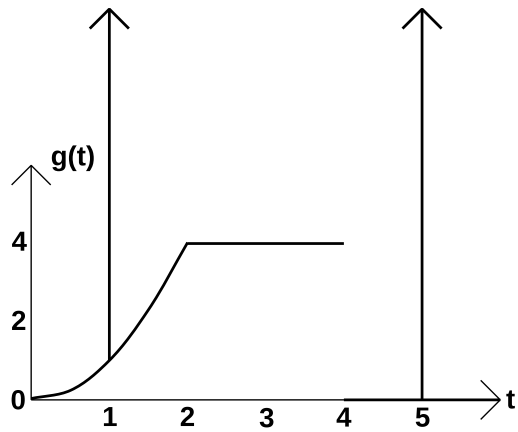
$= 3tH(t) - 4(t-1)H(t-1) + (t-4)H(t-4)$

Q2. In (0,1) the gradient is 2 and the intercept is 0 so $f(t) = 2t$.
In (1,3) the gradient is -2 and the line passes through (2,0) so $f(t) = 4 - 2t$.
In (3,4) the gradient is 2 and the line passes through (4,0) so $f(t) = 2t - 8$.

Hence $f(t) = 2t\{H(t) - H(t-1)\} + (4 - 2t)\{H(t-1) - H(t-3)\} + (2t - 8)\{H(t-3) - H(t-4)\}$

$= 2tH(t) - 4(t-1)H(t-1) - 4(t-3)H(t-3) + 2(t-4)H(t-4)$

Q3.



Q4. $L\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st}f(t-a)H(t-a)dt$

$= \int_a^\infty e^{-st}f(t-a)H(t-a)dt$ since $H(t-a) = 0$ if $t < a$

$= \int_0^\infty e^{-s(u+a)}f(u)H(u)du$ where $u = t - a$

$= e^{-sa}\int_0^\infty e^{-su}f(u)H(u)du = e^{-sa}F(s)$

Exercise 4

Q1.a) $3e^{-s} + \frac{4e^{2s}}{s}$

b) $tH(t-4) = (t-4)H(t-4) + 4H(t-4)$

For first term using notation of tables $a = 4$ and $f(t-4)=t-4$ so $f(t)=t$ and $F(s) = \frac{1}{s^2}$

Then $L\{tH(t-4)\} = \frac{e^{-4s}}{s^2} + \frac{4e^{-4s}}{s}$

c) $a = 3$ and $f(t-3) = \cos(t-3)$ so $f(t) = \cos t$ and $F(s) = \frac{s}{s^2+1}$

Then $L\{\cos(t-3)H(t-3)\} = \frac{s}{s^2+1}e^{-3s}$

d) Let $u = t-1$ so $t = u+1$ and $t^2 = u^2 + 2u + 1$.

Then $t^2H(t-1) = \{(t-1)^2 + 2(t-1) + 1\}H(t-1)$ so $a = 1$ and $f(t-1) = (t-1)^2 + 2(t-1) + 1$.

Hence $f(t) = t^2 + 2t +1$ and so $F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$.

Thus $L\{t^2H(t-1)\} = (\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s})e^{-s}$

2.a) $2\delta(t-4) + 5H(t-5)$

b) Using notation of tables $a = 1$ and if $F(s) = \frac{3}{s+4} + \frac{2}{s^2+4}$, $f(t) = 3e^{-4t} + \sin 2t$

Hence inverse transform $f(t-a)H(t-a) = (3e^{-4(t-1)} + \sin 2(t-1))H(t-1)$

3.a)i) Taking Laplace Transforms

$$s^2L(y) - sy_0 - y_1 - 6(sL(y) - y_o) + 5L(y) = 4e^{-2s}$$

Inserting initial conditions

$$(s^2 - 6s + 5)L(y) = 4e^{-2s}$$

so $$L(y) = \frac{4}{s^2 - 6s + 5}e^{-2s} = \frac{4}{(s-1)(s-5)}e^{-2s}$$

Now by partial fractions $\frac{4}{(s-1)(s-5)} = \frac{1}{s-5} - \frac{1}{s-1}$

so $$L(y) = \frac{1}{s-5}e^{-2s} - \frac{1}{s-1}e^{-2s}$$

For the first term $a = 2$ and $F(s) = \frac{1}{s-5}$ so $f(t) = e^{5t}$ and $f(t-a)H(t-a) = e^{5(t-2)}H(t-2)$

Similarly for the second term $f(t-a)H(t-a) = e^{(t-2)}H(t-2)$

Hence $y = e^{5(t-2)}H(t-2) - e^{(t-2)}H(t-2)$.

ii) Taking Laplace Transforms

$$s^2L(y) - sy_0 - y_1 + 4(sL(y) - y_o) + 3L(y) = \frac{e^{-s}}{s^2}$$

Inserting initial conditions

$$(s^2 + 4s + 3)L(y) = \frac{e^{-s}}{s^2}$$

so factorising $$L(y) = \frac{1}{(s+3)(s+4)s^2}e^{-s}$$

Using partial fractions $\frac{1}{(s+3)(s+4)s^2} = \frac{1}{12s^2} - \frac{7}{144s} + \frac{1}{9(s+3)} - \frac{1}{16(s+4)}$

In this case a = 1 and $F(s) = \frac{1}{12s^2} - \frac{7}{144s} + \frac{1}{9(s+3)} - \frac{1}{16(s+4)}$ so $f(t) = \frac{t}{12} - \frac{7}{144} + \frac{1}{9}e^{-9t} - \frac{1}{16}e^{-4t}$

and $f(t-a)H(t-a) = [\frac{t-1}{12} - \frac{7}{144} + \frac{1}{9}e^{-3(t-1)} - \frac{1}{16}e^{-4(t-1)}]H(t-1)$.

b) $f(t) = t[H(t)-H(t-1)] + [H(t-1)-H(t-2)] + (3-t)[H(t-2)-H(t-3)]$

$= tH(t) - (t-1)H(t-1) - (t-2)H(-2) + (t-3)H(t-3)$

Taking Laplace Transforms

$$sL(y) - y_0 - L(y) = \frac{1}{s^2}e^{0s} - \frac{1}{s^2}e^{-1s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s^2}e^{-3s}$$

so, since $y_0 = 0$, $\quad (s-1)L(y) = \frac{1}{s^2}(1 - e^{-s} - e^{-2s} - e^{-3s})$

and $\quad L(y) = \frac{1}{(s-1)s^2}(1 - e^{-s} - e^{-2s} - e^{-3s})$

Now by partial fractions $\frac{1}{(s-1)s^2} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}$

and, if $F(s) = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}$ then $f(t) = e^t - 1 - t$

Hence $\quad y = e^t - 1 - t - (e^{t-1} - 1 - (t-1))H(t-1) - (e^{t-2} - 1 - (t-2))H(t-2) - (e^{t-3} - 1 - (t-3))H(t-3)$

Q4.a) $4H(t-1) + 7e^{-2t}$ $\quad$ $4H(t-1)$ is steady state, $7e^{-2t}$ is transient

b) $\frac{36 - 12s}{(s^2+9)(s+3)} = \frac{4}{s+3} - \frac{4s}{s^2 + 9}$. Inverse is $4e^{-3t} - 4\cos 3t$. $4e^{-3t}$ is transient, $4\cos 3t$ is steady state.

c) $\frac{(5s + 17)}{s^2 + 6s + 10} = \{\frac{5(s+3)}{(s+3)^2+1} + \frac{2}{(s+3)^2+1}\}$. Inverse is $5e^{-3t}\cos t + 2e^{-3t}\sin t$, which is transient.

Q5. Taking Laplace Transforms

$$s^2L(y) - sy_0 - y_1 + 6(L(y) - y_0) + 8L(y) = \frac{16}{s}$$

Inserting initial conditions and reorganising equation

$$(s^2 + 6s + 8)L(y) = \frac{16 + 22s + 3s^2}{s(s + 2)(s + 4)}$$

$$= \frac{2}{s} + \frac{4}{s + 2} - \frac{3}{s + 4} \quad \text{by partial fractions}$$

so $\quad y = 2 + 4e^{-2t} - 3e^{-4t}$

$4e^{-2t} - 3e^{-4t}$ is transient, 2 is steady state.

$\lim_{t \to \infty} y(t) = 2$ because $4e^{-2t}$ and $3e^{-4t}$ both tend to 0 as $t \to \infty$.

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} s\left(\frac{2}{s} + \frac{4}{s + 2} - \frac{3}{s + 4}\right) = \lim_{s \to 0} \left(2 + \frac{4s}{s + 2} - \frac{3s}{s + 4}\right) = 2$$

so the initial value theorem holds.

$\lim_{t \to 0} y(t) = y(0) = 3$ $\quad s \to \infty$

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\left(\frac{2}{s} + \frac{4}{s + 2} - \frac{3}{s + 4}\right) = \lim_{s \to 0} \left(2 + \frac{4s}{s + 2} - \frac{3s}{s + 4}\right) = 2 + 4 - 3 = 3$$

so the final value theorem holds.