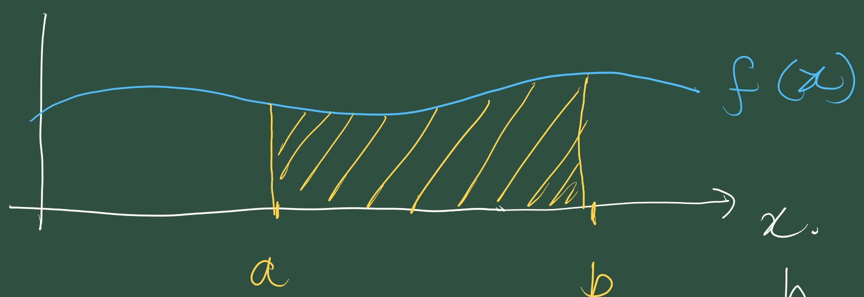
Integration in multi-variable calculus

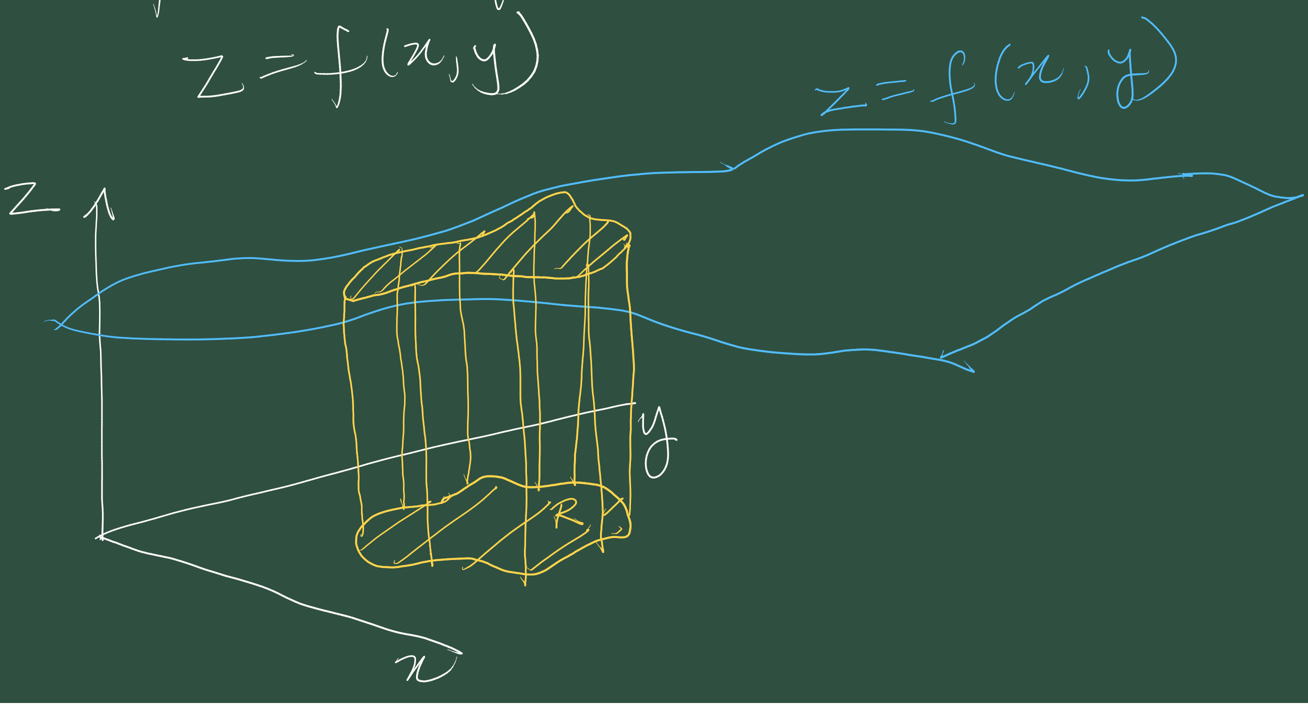
Two types of integral.

Recall single-variable definite integrals



The definite integral $I = \int_a^b f(x)\,dx$ gives the area between the graph and x-axis over the interval $a \leq x \leq b$.

For functions of 2-variables
$z = f(x,y)$



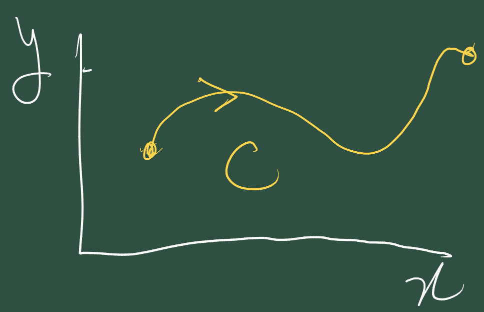
The **volume** between the surface and the xy-plane over the region R is give by what's called the **double integral** of f over R, written as.

$$I = \iint_R f(x,y)\,dx\,dy.$$ (next week).

We also have "line integrals" or "path integrals" of linear different forms L.

$$L = P(x,y)\,dx + Q(x,y)\,dy$$

taken over paths in the xy-plane, expressed as



$$J = \int_C L$$

$$= \int_C P\,dx + Q\,dy.$$

(geometric interpretation not so immediately available). But such integrals arise in maths and physics.

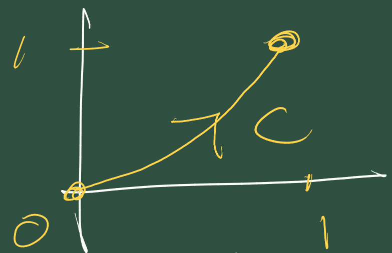
<u>Example 0.1</u>

Consider $L = 10x^2y\,dx + (3x+2y)\,dy$

integrate this along the curve

$$C: \; y = x^2 \quad \text{from } (0,0) \text{ to } (1,1).$$

Idea: Convert the path integral into a standard 1-variable integral using the specification of C.



$$C: \; y = x^2, \quad dy = 2x\,dx$$

Use these substitutions to transform the path integral.

$$I = \int_C 10x^2y\,dx + (3x+2y)\,dy.$$

$$= \int_0^1 10x^4\,dx + (3x + 2x^2)\,2x\,dx$$

$$= \int_0^1 (10x^4 + 4x^3 + 6x^2)\,dx.$$

$$= \left[2x^5 + x^4 + 2x^3\right]_0^1 = 5.$$

<u>Theorem</u> Basic properties of path integrals.
(generalisations of properties we're familiar with for single variable integrals).

1. Linearity. For $\alpha, \beta \in \mathbb{R}$, L_1, L_2 linear differentials forms

$$\int_C \alpha L_1 + \beta L_2$$

$$= \alpha \int_C L_1 + \beta \int_C L_2.$$

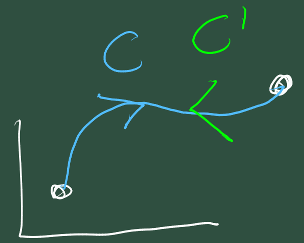
<u>OR</u>

$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

$$= \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy.$$

(cf. $\displaystyle\int_a^b (\alpha f_1(x) + \beta f_2(x))\, dx = \alpha\int_a^b f_1(x)\, dx + \beta\int_a^b f_2(x)\, dx$).

2. Direction of integration.

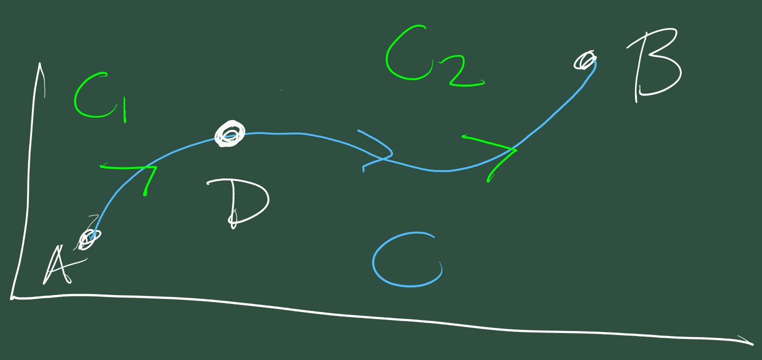


For a lin. diff. form L.

$$\int_C L = -\int_{C'} L$$

cf. $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

3. Subdividing the path / curve.



$C_1 : A \longrightarrow D$

$C_2 : D \longrightarrow B$

$C : A \longrightarrow B$

$$\int_C L = \int_{C_1} L + \int_{C_2} L$$

cf.

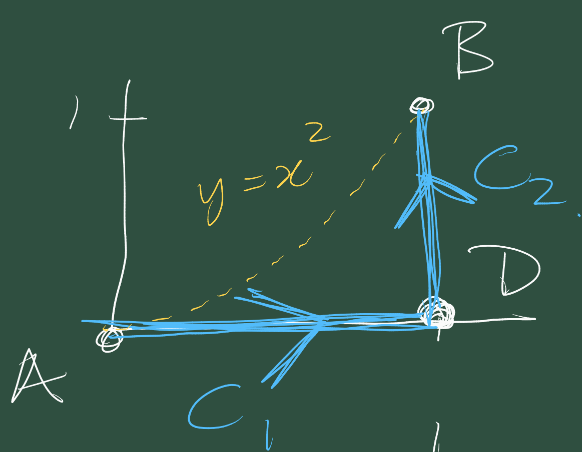
$$\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx$$

Example 02 Same L as before.

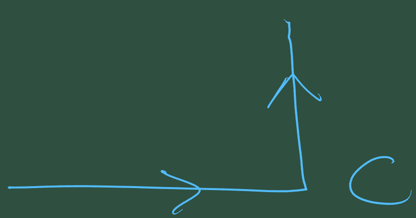
$$L = 10x^2y\,dx + (3x+2y)\,dy$$

Integrate this from $A=(0,0)$ to $B=(1,1)$ along the two straight line segments AD, DB where $D=(1,0)$.

$C = C_1 + C_2$



$$\int_C L = \int_{C_1} L + \int_{C_2} L$$



On C_1: $y=0$, $dy=0$, $x: 0 \longmapsto 1$

On C_2: $x = 1$, $dx = 0$

$y : 0 \mapsto 1$.

So $\int_{C_1} L = \int_0^1 0 \, dx = 0$

$$\int_{C_2} L = \int_0^1 (3 + 2y) \, dy.$$

$$= \left[3y + y^2 \right]_0^1 = 4.$$

And so $\int_C L = 0 + 4 = 4$.

which is different to previous result integrating along $y = x^2$ from A to B.

This is known as path dependence and is the typical expected behaviour of such integrals.

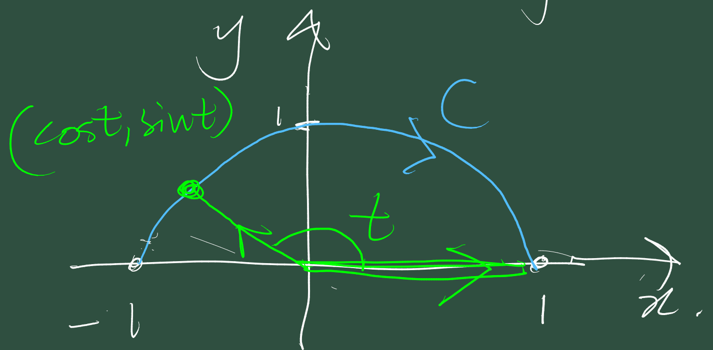
But there are special "path independent" linear differential forms which we'll meet later.

Parametrised paths

<u>Example 0.3</u> Evaluate I.

$$I = \int_C x^2\,dy - yx\,dx.$$

where C is the semi-circular path of radius 1 traversed clockwise from $(-1,0)$ to $(1,0)$



The curve C is parametrised as

$(x,y) = (\cos(t), \sin(t))$

for $\boxed{t : \pi \to 0}$

Use this to translate I into a single-variable t-integral

$$\boxed{\begin{aligned} x &= \cos(t), & dx &= -\sin(t)\,dt \\ y &= \sin(t), & dy &= \cos(t)\,dt. \end{aligned}}$$

So

$$I = \int_{\pi}^{0} \cos^3(t)\,dt + \sin^2(t)\cos(t)\,dt.$$

$$= \int_{\pi}^{0} \cos(t)\Big(\underbrace{\cos^2(t) + \sin^2(t)}_{=1}\Big)\,dt.$$

$$= \int_{\pi}^{0} \cos(t)\,dt$$

$$= \Big[\sin(t)\Big]_{\pi}^{0} = \sin(0) - \sin(\pi)$$

$$= 0 - 0 = 0$$

Question

Compute the integral J.

$$J = \int_C x e^y\,dx + x^2 y\,dy$$

where the curve C is parametrised by

$(x, y) = (e^t, e^t)$, $\boxed{-1 \le t \le 1}$,

$$\boxed{\begin{aligned} x &= e^t, & dx &= e^t \cdot dt \\ y &= e^t, & dy &= e^t\, dt \end{aligned}}$$

So

$$J = \int_{-1}^{1} e^t e^{e^t} e^t\, dt + e^{2t} e^t e^t\, dt$$

$$= \int_{-1}^{1} (\underbrace{e^{2t} e^{e^t}}_{e^{2t + e^t}} + e^{4t})\, dt.$$

$$= \underbrace{\int_{-1}^{1} e^{2t} e^{e^t}\, dt}_{} + \underbrace{\int_{-1}^{1} e^{4t}\, dt.}_{}$$

Could try a substitution.
$u = e^t$, $du = e^t\, dt.$

$$\left[\frac{1}{4} e^{4t}\right]_{-1}^{1}$$

$$\boxed{\frac{1}{4}\left(e^{4}-e^{-4}\right)}$$

or integration by parts.

$$= \int_{\frac{1}{e}}^{e} u^2 e^u \cdot \frac{du}{u} = \int_{1/e}^{e} u e^u \, du.$$

$$= \int_{1/e}^{e} u \frac{d}{du}(e^u) \, du.$$

$$= \left[u e^u \right]_{1/e}^{e} - \int_{1/e}^{e} e^u \, du$$

$$= \boxed{\left[u e^u \right]_{1/e}^{e} - \left[e^u \right]_{1/e}^{e}}$$

A further critical points example.

Q: Find + classify any crit. points of

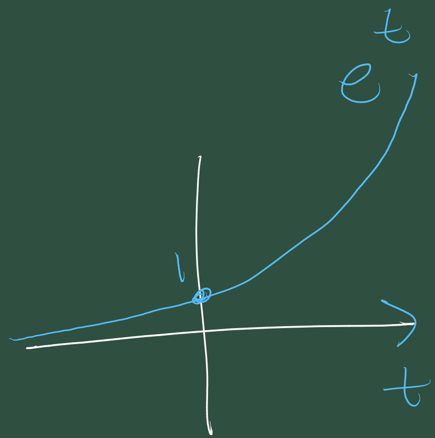
$$g(x,y) = -2x^4 + 4x^2e^y - 4e^{4y}.$$

A: Crit. points (a,b) are the simultaneous solutions to $\frac{\partial g}{\partial x} = 0$ & $\frac{\partial g}{\partial y} = 0$

$$\frac{\partial g}{\partial x} = -8x^3 + 8xe^y = 0$$

AND

$$\frac{\partial g}{\partial y} = 4x^2e^y - 16e^{4y} = 0.$$



Consider $\frac{\partial g}{\partial x} = 0$.

$$\iff x(-x^2 + e^y) = 0.$$

$$\iff x = 0 \quad \underline{\text{OR}} \quad x^2 = e^y$$

Impose these conditions on $\frac{\partial g}{\partial y} = 0$

Well If $x=0$ $\frac{\partial g}{\partial y} = 0$

$$\iff -16e^{4y} = 0$$

which is false, has no solutions

But if $\boxed{x^2 = e^y}$ then ② ensures $\frac{\partial g}{\partial x} = 0$

$$\frac{\partial g}{\partial y} = 0$$

$$\iff e^{2y} - 4e^{4y} = 0.$$

$$\iff e^{2y}(1 - 4e^{2y}) = 0$$

Now $e^{2y} \neq 0$

$$\iff 1 - 4e^{2y} = 0$$

$$\iff e^{2y} = \frac{1}{4}$$

$$\iff e^{y} = \frac{1}{2}$$

$$\iff \boxed{y = \ln(1/2)}. \quad ①$$

ensures $\frac{\partial g}{\partial y} = 0$

So critical points (a,b) are found by combining ① and ②.

So $x^2 = 1/2$

so $x = \pm 1/\sqrt{2} = \pm \frac{1}{\sqrt{2}}$

So there are two critical points.

$$(a, b) = \left(-\frac{1}{\sqrt{2}}, \ln\left(\frac{1}{2}\right)\right)$$

and $\left(\frac{1}{\sqrt{2}}, \ln\left(1/2\right)\right)$