Finding and classifying critical points.

Consider the function g.

$$g(x,y) = (y^2 - 5y)\sin(2x)$$

Critical points are the simultaneous solutions

$$\frac{\partial g}{\partial x} = 0 \quad \& \quad \frac{\partial g}{\partial y} = 0.$$

$$\Leftrightarrow \left\{ \begin{array}{l} 2(y^2 - 5y)\cos(2x) = 0 \\ \& \\ (2y-5)\sin(2x) = 0 \end{array} \right\}$$

The second is satisfied if and only if

$$\left[y = 5/2 \quad \text{or} \quad 2x = m\pi, \ m \in \mathbb{Z} \right]$$

$$\Leftrightarrow \left[y = 5/2 \ \text{or} \ x = m\frac{\pi}{2}, \ m \in \mathbb{Z} \right]$$

Take each part of these conditions, impose it on first equation and find solutions that remain.

If $y = 5/2$, then $\dfrac{\partial g}{\partial x} = 0$

$$\iff 2\left(\left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right)\right)\cos(2x) = 0$$

$$\iff \cos(2x) = 0.$$

$$\iff 2x = m\pi + \frac{\pi}{2}, \text{ for any } m \in \mathbb{Z}$$

$$\iff x = m\frac{\pi}{2} + \frac{\pi}{4}, \text{ for any } m \in \mathbb{Z}.$$

Hence any point of the form

$$(x, y) = \left(m\frac{\pi}{2} + \frac{\pi}{4}, \ 5/2\right), \text{ for } m \in \mathbb{Z}$$

is a critical point.

Secondly, assume that $\boxed{x = m\frac{\pi}{2}}$, for $m \in \mathbb{Z}$. Under this,

$$\frac{\partial g}{\partial x} = 0$$

$$\iff 2(y^2 - 5y)\overbrace{\cos(m\pi)}^{\pm 1} = 0.$$

$$\iff y^2 - 5y = 0$$

$$\iff y(y-5) = 0$$

$$\iff \boxed{y = 0 \text{ or } 5}$$

So this gives us two infinite families of critical points, namely

$(x, y) = \left(m\frac{\pi}{2}, 0\right)$ AND $\left(m\frac{\pi}{2}, 5\right)$

for any $m \in \mathbb{Z}$.

These critical points are classified (max, min, saddle) by considering the second order partial derivatives

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(2(y^2 - 5y)\cos(2x)\right)$$

$$= -4(y^2 - 5y)\sin(2x)$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial y}\left((2y - 5)\sin(2x)\right)$$

$$= 2\sin(2x)$$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left((2y - 5)\sin(2x)\right)$$

$$= 2(2y - 5)\cos(2x)$$

We need to evaluate the Hessian determinant $D = \frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial y^2} - \left(\frac{\partial^2 g}{\partial x \partial y}\right)^2$

at each of the critical points.

Using matlab

$$D\left(m\frac{\pi}{2}+\frac{\pi}{4}, \frac{5}{2}\right)$$

$$= 50\sin^2\left(m\pi+\frac{\pi}{2}\right)$$

$$= 50(\pm 1)^2$$

$= 50 > 0$ So these minor

max accordingly as $\frac{\partial^2 g}{\partial x^2} > 0$ or < 0

$$\frac{\partial^2 g}{\partial x^2}\left(m\frac{\pi}{2}+\frac{\pi}{4}, \frac{5}{2}\right)$$

$$= 25\sin\left(m\pi+\frac{\pi}{2}\right)$$

$$= \begin{cases} 25, & m \text{ even.} \\ -25, & m \text{ odd} \end{cases}$$

So $\left(m\frac{\pi}{2}+\frac{\pi}{4}, \frac{5}{2}\right)$

is a minimum for m even and a maximum for m odd.

Next.

$$D\left(m\frac{\pi}{2}, 0\right) = -100\cos^2(m\pi)$$
$$= -100(\pm 1)^2$$
$$= -100 < 0.$$

So these $\left(m\frac{\pi}{2}, 0\right)$ are saddles.

$$D\left(m\frac{\pi}{2}, 5\right) = -100 < 0$$

So these $\left(m\frac{\pi}{2}, 5\right)$ are saddles.

Let's confirm some of these with some surface plots.

Looking at a selection the surface has the expected appearance.

$$g(x,y) = (y^2 - 5y)\sin(2x).$$

What does Taylor series of this look like.

From MATLAB, series based at $(0,0)$, for g appears as.

$$g(x,y) = -10xy + 2xy^2 + \frac{20}{3}x^3y + \dots$$

Is this the series generated by the defining formula?

$$g(h,k) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n g(0,0).$$

$$= g(0,0) + Dg(0,0) + \frac{1}{2!} D^2 g(0,0) + \dots$$

Where $D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$

Where $D^n g = \underbrace{D(D(D(\ldots\ldots(Dg))\ldots))}_{\text{n operators D applied in composition.}}$

$D^2 g = D(D(g))$