Quick review of the def. of the derivative. of a single variable function.

Consider $f : \mathbb{R} \to \mathbb{R}$.

Its derivative $f'(a)$ evaluated at a (or written as $\left.\frac{df}{dx}\right|_{x=a}$) was defined as.

abs. change in f

$$f'(a) = \lim_{x \to a} \left(\boxed{\frac{f(x) - f(a)}{x - a}} \right)$$

relative change in f

change in argument

$$\underline{\underline{\text{or}}} \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Let's see how to extend this to functions of two variables.

Consider $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$.

$$(x, y) \longmapsto f(x, y)$$

Actually there are two 'partial' derivatives we can define

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ → "...... y" 等

↳ "partial derivative of f with respect to x"

x is varying

same y value

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{x \to a} \left(\frac{f(x, b) - f(a, b)}{x - a} \right)$$

$$\overset{\text{or}}{=} \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

And similarly.

same x value

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{h \to 0} \left(\frac{f(a, b+h) - f(a, b)}{h} \right)$$

y is varying.

Lets work with the example function.

$$\boxed{f(x, y) = x^2 y}$$

and let's find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_{(a,b)}} = \lim_{x\to a}\left(\frac{f(x,b)-f(a,b)}{x-a}\right)$$

$$= \lim_{x\to a}\left(\frac{x^2 b - a^2 b}{x-a}\right)$$

$$= \lim_{x\to a}\left(\frac{(x^2-a^2)\,b}{x-a}\right)$$

$$= \lim_{x\to a}\left(\frac{\cancel{(x-a)}(x+a)\,b}{\cancel{(x-a)}}\right)$$

$$= \lim_{x\to a}\left[(x+a)\,b\right]$$

$$= b\lim_{x\to a}(x+a)$$

$$= b\,2a = \boxed{2ab}\,),$$ linearity of limits

And.

$$\boxed{\left.\frac{\partial f}{\partial y}\right|_{(a,b)}} = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}.$$

$$= \lim_{h\to 0}\left(\frac{a^2(b+h)-a^2 b}{h}\right)$$

$$= \lim_{h \to 0} \left(\frac{a^2 \cancel{h}}{\cancel{h}} \right)$$

$$\boxed{= a^2}$$

$f(x, y) = x^2 y$, $\dfrac{\partial f}{\partial x} = 2xy$.

$$\frac{\partial f}{\partial y} = x^2.$$

Important principle at work here.

To differentiate f with respect to <u>one variable, x</u> say, treat the <u>other variable(s) as constants</u>, and go ahead and differentiate using your single variable calculus concepts.

Another example.

Consider f defined by

$$f(x, y) = x^2 y^3 \tan(2x).$$

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

$$\boxed{\frac{\partial f}{\partial x}} = \underbrace{\frac{\partial}{\partial x}}_{\text{operator}} \left(\underbrace{x^2 y^3 \tan(2x)}_{\text{argument.}} \right)$$

$$= y^3 \frac{\partial}{\partial x} \left(x^2 \tan(2x) \right)$$

$$= y^3 \left(2x \tan(2x) + x^2 \underbrace{\sec^2}_{= \tan^2(z)+1} (2x) 2 \right)$$

using $\frac{d}{dz} \tan(z) = \sec^2(z) = \frac{1}{\cos^2(z)}$

and product and chain rules.

$$\boxed{= 2y^3 \left(x \tan(2x) + x^2 \sec^2(2x) \right)}$$

$$\boxed{\frac{\partial f}{\partial y}} = \frac{\partial}{\partial y} \left(x^2 y^3 \tan(2x) \right)$$

$$= x^2 \tan(2x) \frac{\partial}{\partial y} \left(y^3 \right)$$

$$\boxed{= 3 y^2 x^2 \tan(2x)}$$

Notation.

Can also write. $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$

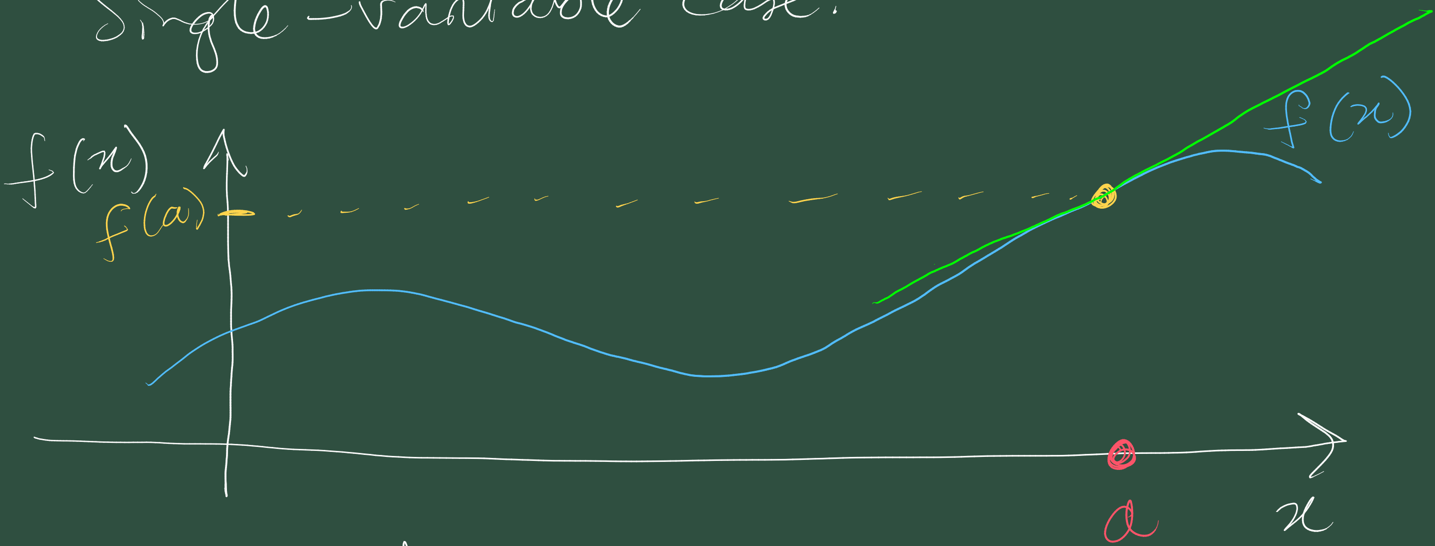
and $f_x(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}$

But $\frac{\partial f}{\partial x}$ allows us to see it as.

$$\underbrace{\frac{\partial}{\partial x}}_{\text{operator}}\overbrace{(f)}^{\text{applied to } f.}$$

Graphical interpretation of partial derivatives.

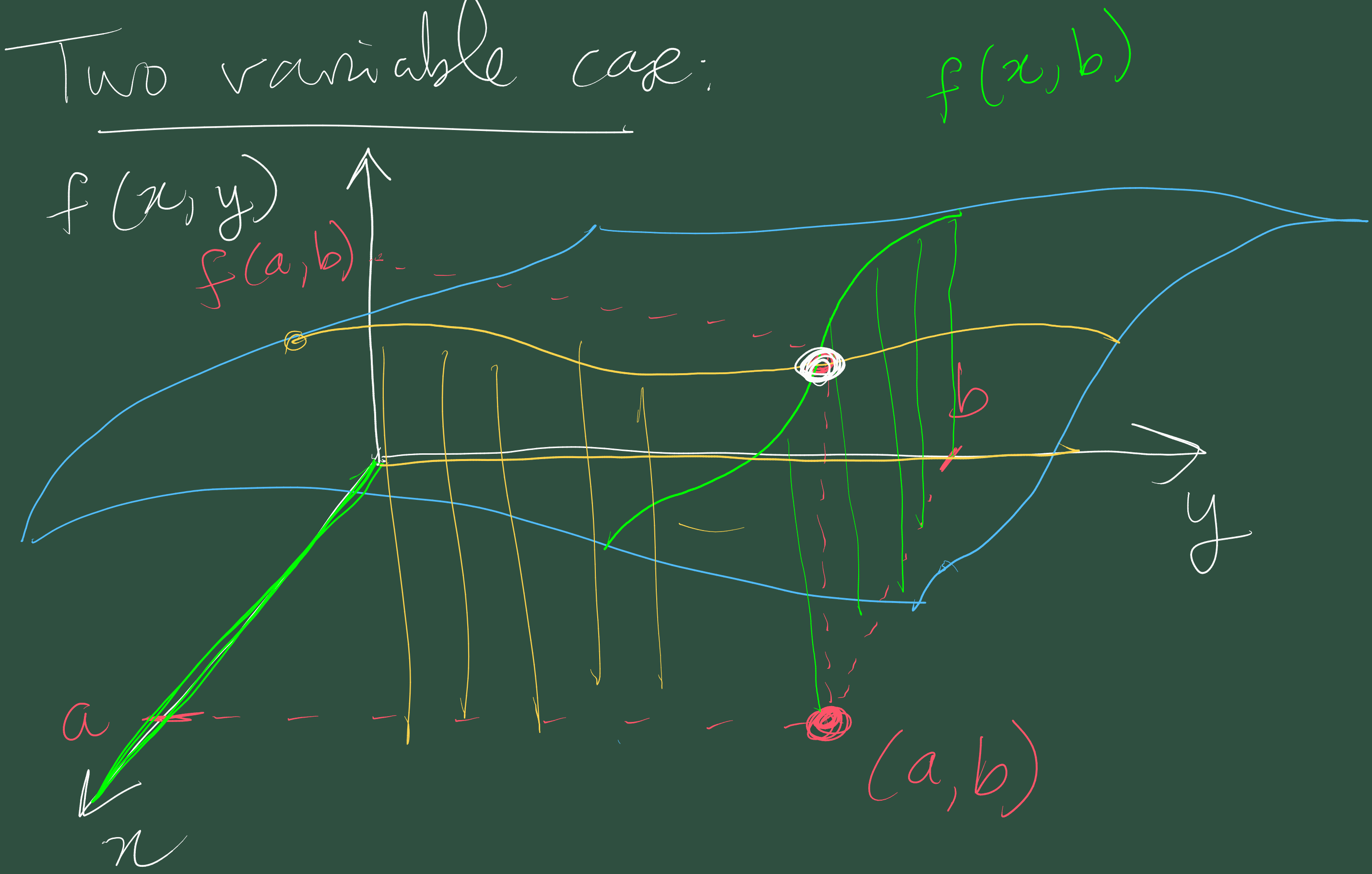
Single-variable case.



$f'(a) = \left.\frac{df}{dx}\right|_a$ is the gradient or slope of this tangent line.

which is the instantaneous rate of change of f with respect to x at a.

Two variable case:



Note the green cross-section curve across the surface parallel to x-axis, at $y=b$..

and the yellow cross-section curve across surface parallel to y-axis, at $x=a$.

$\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$ is the gradient of the curve $z = f(x, b)$

$\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$ is the gradient of the curve $z = f(a, y)$

Higher-order derivatives

Taking partial derivatives of derivatives and so on.....

There are four possible second-order derivatives of $f(x,y)$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Clairaut's Theorem

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\Bigg\}$ the same provided $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous.

eg. If $f(x, y) = x^2 y$

$$\frac{\partial f}{\partial y} = x^2$$

so $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2\right) = 0$$

But $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x^2\right) = 2x.$$

Can extend to higher-orders third, fourth and so on.

Clairaut's theorem generalises to say. $$\frac{\partial^3 g}{\partial x \partial y \partial z} = \frac{\partial^3 g}{\partial y \partial z \partial x}$$ and so on.

the order in which the derivatives are taken doesn't matter

Ex. 2.3.3 from Section 2.3.

Q6.

$$f(r,\theta) = r^m \cos(m\theta).$$

where m is a positive integer.

a) $f_{rr} = \dfrac{\partial^2 f}{\partial r^2} = \dfrac{\partial}{\partial r}\left(\dfrac{\partial f}{\partial r}\right)$

$$\boxed{f_r} = \frac{\partial}{\partial r}\left(r^m \cos(m\theta)\right) = \cos(m\theta)\frac{\partial}{\partial r}(r^m)$$

$$= \boxed{\cos(m\theta)\, m r^{m-1}}$$

$$\boxed{f_\theta} = \frac{\partial}{\partial \theta}\left(r^m \cos(m\theta)\right)$$

$$= r^m \frac{\partial}{\partial \theta}\left(\cos(m\theta)\right)$$

$$= -r^m \sin(m\theta)\, m$$

$$= \boxed{-m r^m \sin(m\theta)}$$

$$f_{rr} = \frac{\partial}{\partial r}\left(m r^{m-1} \cos(m\theta)\right)$$

$$= m(m-1) r^{m-2} \cos(m\theta)$$

, provided $m \geq 2$.

$$f_{r\theta} = \frac{\partial}{\partial r}\left(-m r^{m} \sin(m\theta)\right)$$

$$= -m^2 r^{m-1} \sin(m\theta)$$

$$f_{\theta\theta} = \frac{\partial}{\partial \theta}\left(-m r^{m} \sin(m\theta)\right)$$

$$= -m^2 r^{m} \cos(m\theta).$$

(b) Consider L.H.S.

L.H.S.

$$= m(m-1) r^{m-2} \cos(m\theta)$$

$$+ \frac{\lambda}{r} m r^{m-1} \cos(m\theta)$$

$$-\frac{1}{r^2} m^2 r^m \cos(m\theta)$$

$$= \cos(m\theta) r^{m-2} \left(\cancel{m^2} - m + \lambda m - \cancel{m^2}\right)$$

$$= \underbrace{(\lambda m - m)}\ \underline{\underline{\cos(m\theta) r^{m-2}}}$$

$$= 0.$$

This equation is satisfied for all r, θ if and only if $\lambda = 1$