Last week we obtained the Fourier series for

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ 1, & \text{if } 0 \leq x < \pi \end{cases}$$

which was

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)x).$$

An interesting fact will come from this when we evaluate it at $x = \pi/2$.

$f(\frac{\pi}{2}) = 1$, from the definition of f.

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \underbrace{\sin\left((2n-1)\frac{\pi}{2}\right)}_{?}$$

"

$$1 = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

This can be rearranged to

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots\right)$$

A remarkable series.

<u>5.4</u> Odd and even functions.

<u>Def</u> Odd and even functions.

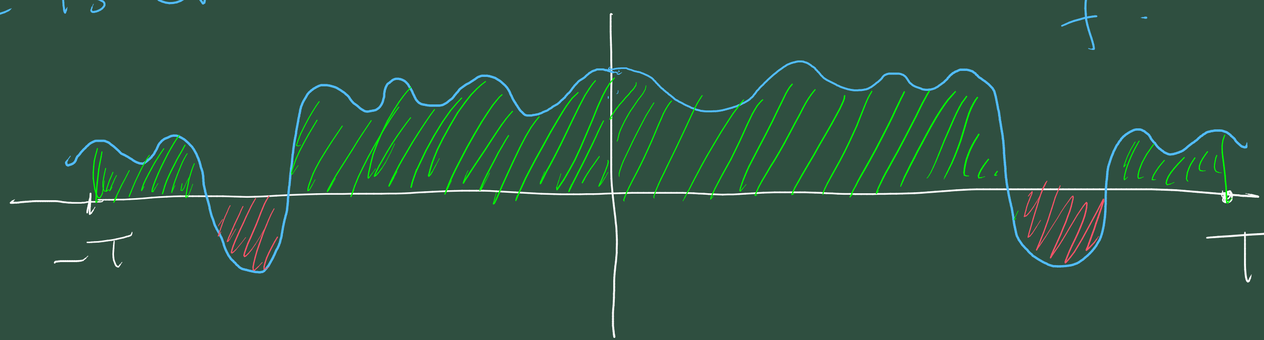
A function $f : \mathbb{R} \to \mathbb{R}$ is <u>even</u> iff $\forall x \in \mathbb{R}$ $f(x) = f(-x)$

and f is <u>odd</u> iff.

$$\forall x \in \mathbb{R} \quad f(x) = -f(-x)$$

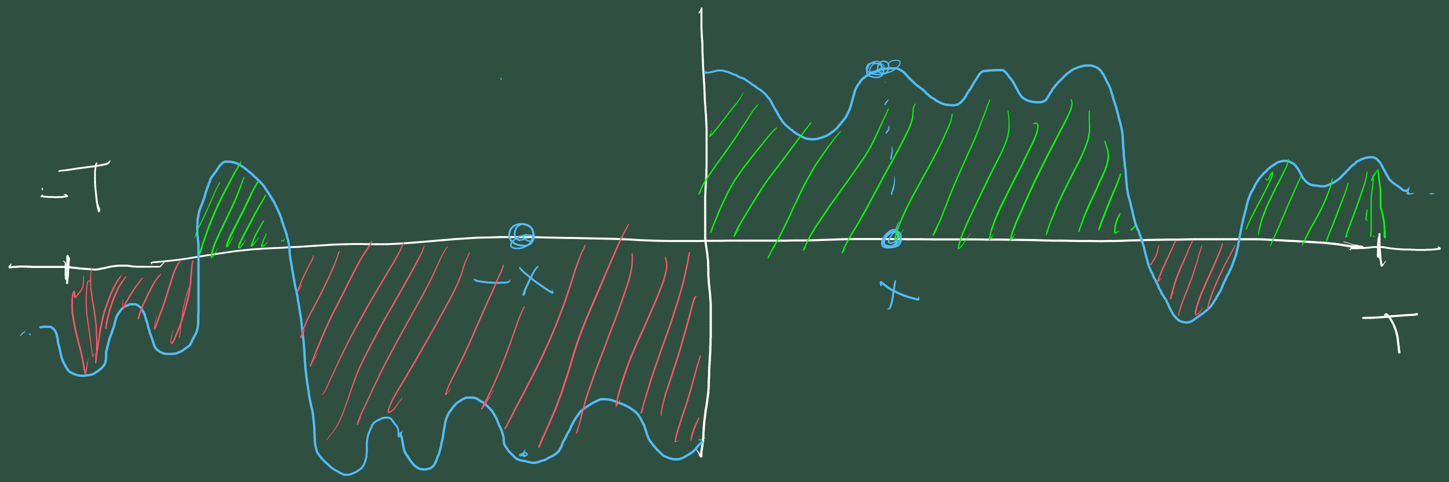
These concepts are very clear in terms geometry of the graphs.

f is even looks like.



reflectional symmetry in y-axis

f is ODD looks like. $f(x) = -f(-x)$

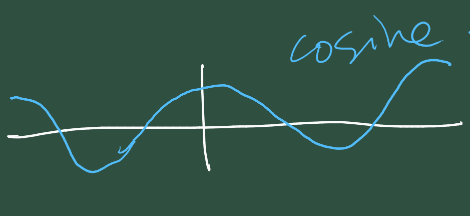


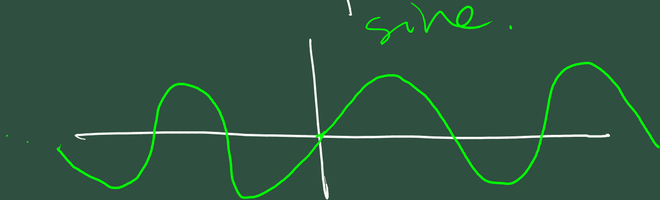
symmetry after reflection in both axes, one after the other.
(or equivalently symmetry in a half-rotation of plane about (0,0)).

odd/even functions, their Fourier series have certain properties, coming from the fact that

cosine is an even. function

sine is an odd function





Theorem 5.4.3.

Proofs come from the defs.

eg. If g_1, g_2 are both even.

For $x \in \mathbb{R}$

$$(g_1 g_2)(-x) = g_1(-x)\, g_2(-x)$$

$$= g_1(x)\, g_2(x), \text{ both } g_1, g_2 \text{ are even.}$$

$$= (g_1 g_2)(x), \text{ def of prod. functions.}$$

Therefore $g_1 g_2$ is even.

Other two cases can be proved in a similar manner.

Theorem 5.4.4. Integrating odd/even functions across intervals centred on origin gives systematic results

If h is odd and g is even then.

$$\int_{-T}^{T} g(x)\,dx = 2\int_{0}^{T} g(x)\,dx.$$

AND.

$$\int_{-T}^{T} h(x)\,dx = 0.$$

Proofs ① Can be "seen" immediatley from the symmetry propeties of their graphs.

② But they can also be proved algebraically. $\forall x \in \mathbb{R}\ h(-x) = -h(x)$

$$\int_{-T}^{T} h(x)\,dx = \int_{-T}^{0} h(x)\,dx + \int_{0}^{T} h(x)\,dx$$

On the first use the substitution.

$y = -x,\ dy = -dx.$

$$= -\int_{T}^{0} h(-y)\,dy + \int_{0}^{T} h(x)\,dx.$$

$$= -\int_T^0 -h(y)\,dy + \int_0^T h(x)\,dx.$$

$$= \int_T^0 h(y)\,dy + \int_0^T h(x)\,dx.$$

$$= -\int_0^T h(y)\,dy + \int_0^T h(x)\,dx$$

$= 0$, as the two integrals are the same.

Other result proved with a similar approach.

Bringing this all together into Theorems 5.4.5 and 5.4.6.

For instance, proof of 5.4.6.

If h is odd then the

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\,dx = 0$$

also for every $n \geq 1$ the product.

function.

$$h(x)\cos(nx)$$

is again odd.

So $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{h(x)\cos(nx)}_{\text{odd}}\,dx = 0$$

and for any $n \geq 1$.

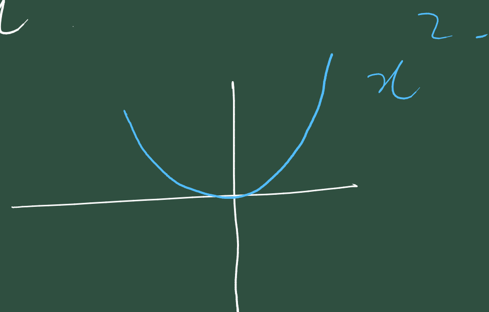
$$h(x)\sin(nx)$$

is an <u>even</u> function, and so

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} h(x)\sin(nx)\,dx.$$

Next: let's apply all this to help us obtain the Fourier series for f defined by $f(x) = x^2$.



The function f is an even function.

So by theorem 5.4.5 all the sine coefficients b_n satisfy.

$$b_n = 0 \quad \text{for } n \geq 1.$$

and f will have a series of the form

$$x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

where.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

and for $n \geq 1$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)\,dx.$$

This requires integration by parts twice to resolve into simpler integrals.

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \frac{d}{dx}\left(\frac{\sin(nx)}{n}\right)dx.$$

$$= \frac{2}{\pi}\left(\underbrace{\left[\frac{x^2\sin(nx)}{n}\right]_0^{\pi}}_{=0} - \int_0^{\pi} 2x\frac{\sin(nx)}{n}dx\right)$$

$$= \frac{-4}{n\pi} \int_0^{\pi} x \sin(nx)\, dx.$$

$$= \frac{-4}{n\pi} \int_0^{\pi} x \frac{d}{dx}\left(\frac{-\cos(nx)}{n}\right) dx.$$

$$= \frac{-4}{n\pi} \left(\left[\frac{-x\cos(nx)}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{-\cos(nx)}{n}\, dx. \right)$$

, by int. by parts.

$$= \frac{-4}{n\pi} \left(\frac{-\pi \cos(n\pi)}{n} + \frac{1}{n} \int_0^{\pi} \cos(nx)\, dx \right)$$

$$= \frac{-4}{n\pi} \left(\frac{-\pi \cos(n\pi)}{n} + \frac{1}{n} \underbrace{\left[\frac{\sin(nx)}{n} \right]_0^{\pi}}_{=0} \right)$$

$$= \frac{4}{n^2} \cos(n\pi)$$

$$= (-1)^n \, 4/n^2 = a_n.$$

So the F.S. for x^2 will be.

$$x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

$$\boxed{x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}\cos(nx)}$$

Another interesting series comes from an evaluation of this at $x=0$.

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

$$\Rightarrow \left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots\cdots\right) = \frac{\pi^2}{12}.$$