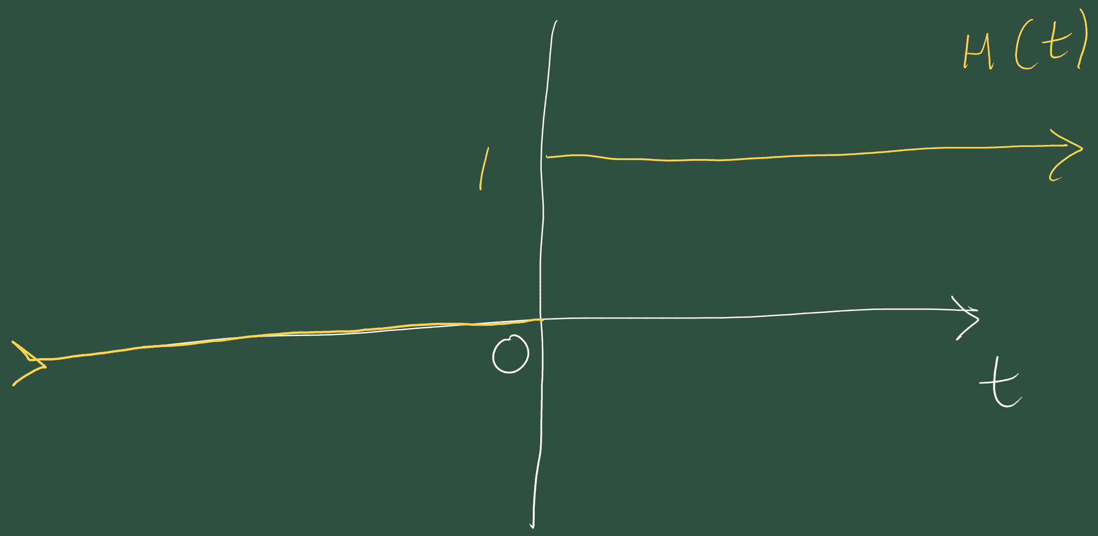
## Heaviside




Heaviside step function $H$ defined by

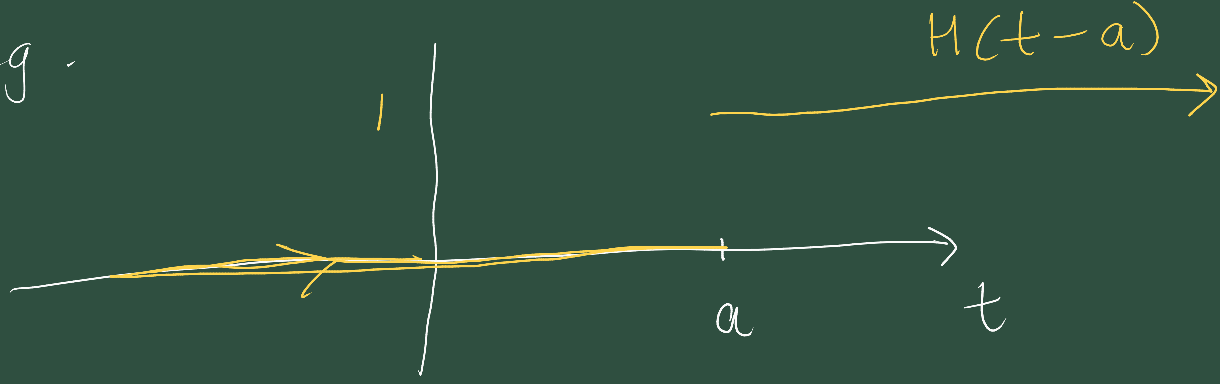
$$H(t) = \begin{cases} 0 & , \ t < 0 \\ 1 & , \ t \geq 0 \end{cases}$$

unit step function

Shift the location/time of the switch by using, for some constant $a$

$$H(t-a) = \begin{cases} 0 & , \ t-a < 0 , \ t < a \\ 1 & , \ t-a \geq 0 , \ t \geq a \end{cases}$$

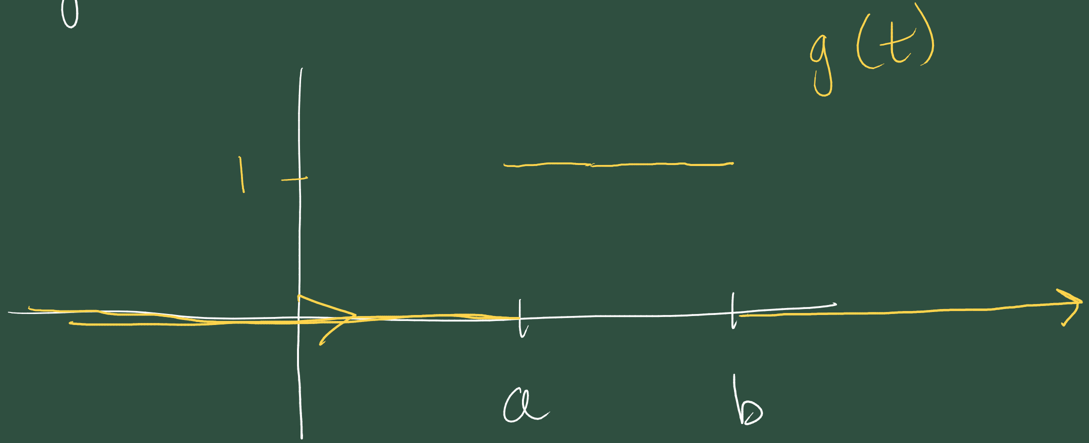
eg.




Ex 4.7.7

1. Let $0 < a < b$. Consider $g$ defined as

$$g(t) = H(t-a) - H(t-b)$$

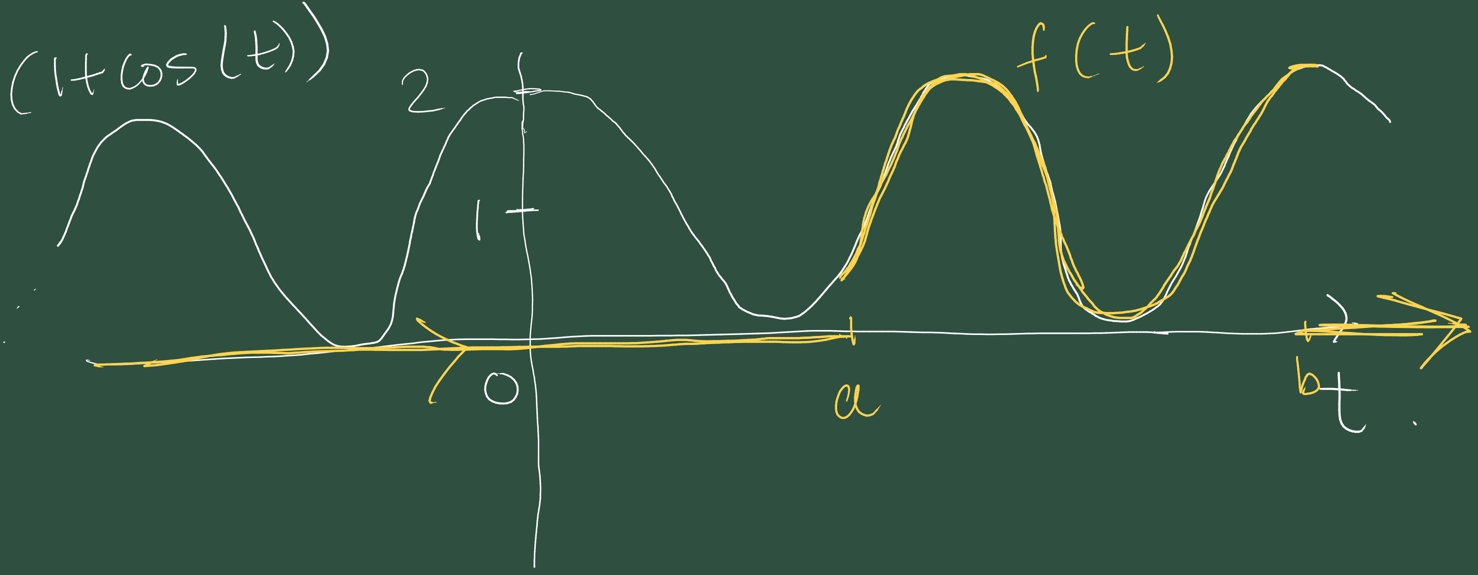



$$g(t) = \begin{cases} 0, & t < a \\ 1, & a \leq t < b \\ 0, & t \geq b \end{cases}$$

$g$ is switched 'on' between $a, b$.
switched 'off' otherwise

2. Consider

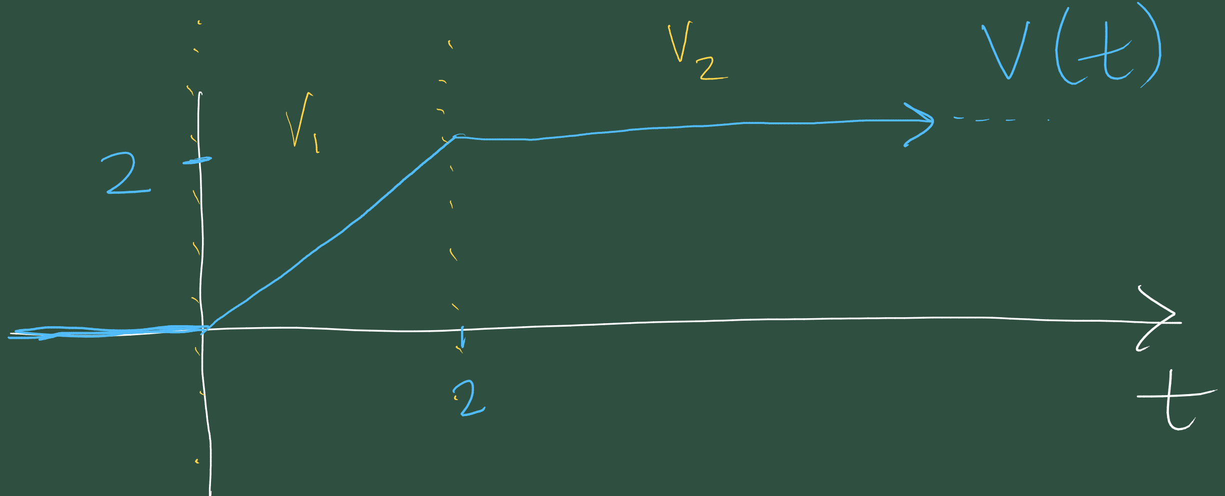
$$f(t) = (1 + \cos(t))\left(H(t-a) - H(t-b)\right)$$




$$f(t) = \begin{cases} 0 & , \quad t < a \\ (1+\cos(t)) & , \quad a \le t < b \\ 0 & , \quad t \geq b \end{cases}$$

3. Consider the case-wise defined function V

$$V(t) = \begin{cases} 0 & , \quad t < 0 \\ t & , \quad 0 \le t < 2 \\ 2 & , \quad t \geq 2. \end{cases}$$



Express $V$ as $V = V_1 + V_2$.

where $V_1 = \begin{cases} t, & 0 \le t < 2 \\ 0, & \text{otherwise.} \end{cases}$

$$V_2 = \begin{cases} 2, & t \geq 2 \\ 0, & t < 2 \end{cases}$$

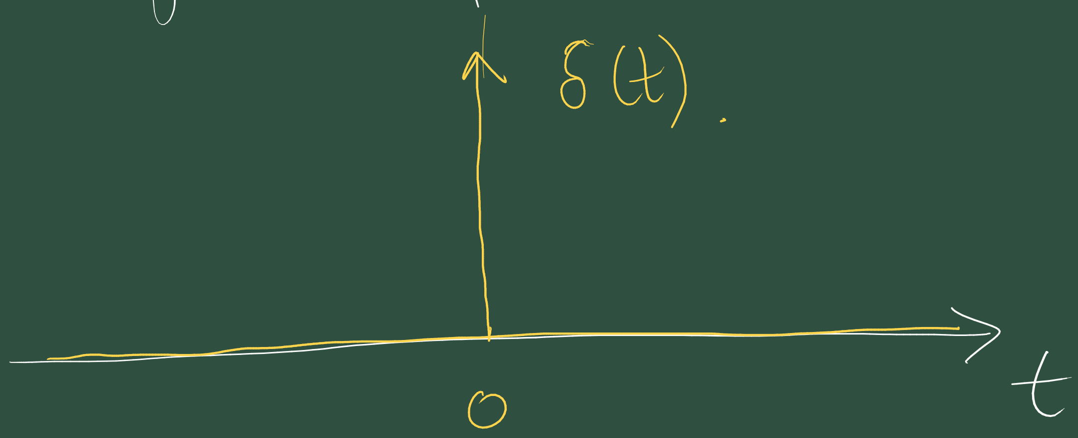
and so
$$V_1(t) = t\big(H(t) - H(t-2)\big)$$
and $V_2(t) = 2H(t-2)$

and so
$$\begin{aligned} V(t) &= t\big(H(t) - H(t-2)\big) \\ &\quad + 2H(t-2) \\ &= tH(t) - (t-2)H(t-2). \end{aligned}$$

Another new function that we can incorporate in the Dirac delta function defined by.

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

an infinite spike at $t=0$




Note location/time of the spike can be shifted as.

$$\delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & t\neq a \end{cases}$$

Aim: Be able to solve, using the Laplace transform method, ODEs featuring these functions.

So we need to understand their transforms.

Th. 4.7.9 $\mathcal{L}\{H(t-a)\} = \dfrac{e^{-as}}{s}$

Assume $a>0$,

Proof: $\mathcal{L}\{H(t-a)\}$

$$= \int_0^\infty e^{-st} \boxed{H(t-a)}\, dt.$$

$\begin{cases} 0, & t<a \\ 1, & t \geq a \end{cases}$

$$= \int_a^\infty e^{-st}\, dt.$$ , implementing def of $H(t-a)$ and using $\int_0^\infty = \int_0^a + \int_a^\infty$

$$= \left[ \frac{e^{-st}}{-s} \right]_a^\infty$$

$$= 0 - \frac{e^{-sa}}{-s},$$ for $s>0$

$$= \frac{e^{-as}}{s},$$ as required.

<u>Th. 4.7.10</u> How to transform products of Heavisides and other functions. We require that both

factors are evaluated at the same shifted argument of $t$.

$$\mathcal{L}\{f(t-a)H(t-a)\}$$

$$= e^{-as}F(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$

<u>Proof</u> $\mathcal{L}\{f(t-a)H(t-a)\}$

$$= \int_0^\infty e^{-st} f(t-a)H(t-a)\,dt.$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt,$$ by the def. of $H(t-a)$

and

$$\int_0^\infty = \int_0^a + \int_a^\infty$$

"looks like $\mathcal{L}\{?\}$"

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt$$

use the substitution

$u = t-a.$

$du = dt$

$$= \int_0^\infty e^{-s(u+a)} f(u) \qquad du$$

$$= \int_0^\infty e^{-sa} e^{-su} f(u)\,du.$$

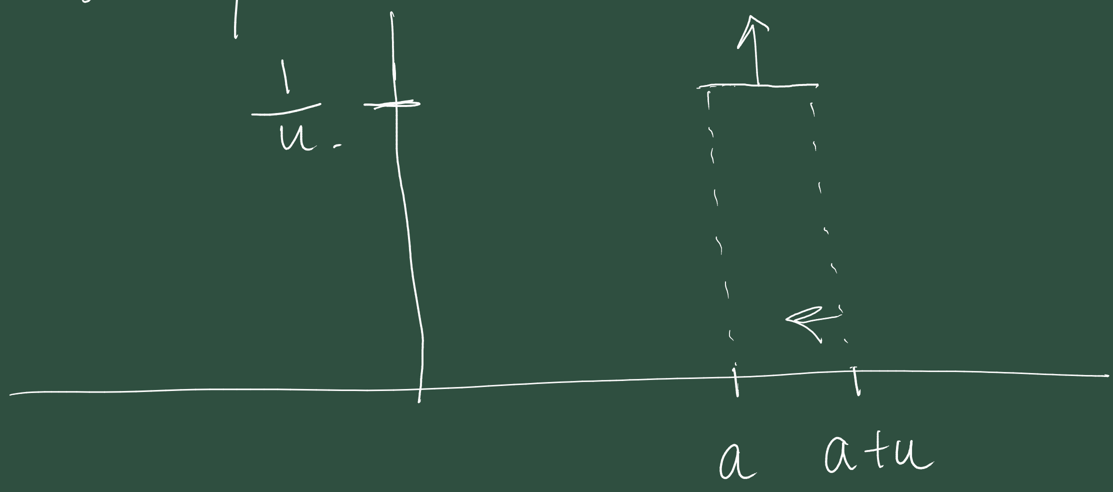
$$= e^{-sa} \boxed{\int_0^\infty e^{-su} f(u)\,du}, \text{ by linearity}$$

$$= e^{-sa}\mathcal{L}\{f\}, \text{ as required.}$$

$$= e^{-sa} F(s).$$ ∎

<u>Th. 4.7.11</u>

Let's express $\delta(t-a)$ in terms of Heavisides.



Consider $g_u(t) = \frac{1}{u}\left[H(t-a) - H(t-(a+u))\right]$

Note that

$$\delta(t-a) = \lim_{u \to 0} g_u(t)$$

$$\mathcal{L}\{g_u(t)\} = \frac{1}{u}\mathcal{L}\{H(t-a)\} - \frac{1}{u}\mathcal{L}\{H(t-(a+u))\}$$

$$= \frac{1}{u}\frac{1}{s}e^{-as} - \frac{1}{u}\frac{1}{s}e^{-(a+u)s}$$

$$= \frac{1}{us}e^{-as} - \frac{1}{us}\left(e^{-as}e^{-us}\right)$$

$$= e^{-as}\left(\frac{1}{us}\left(1 - e^{-us}\right)\right)$$

claim. $\xrightarrow[u \to 0]{} e^{-as}$, with the use of l'Hôpitals rule.

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Pause.

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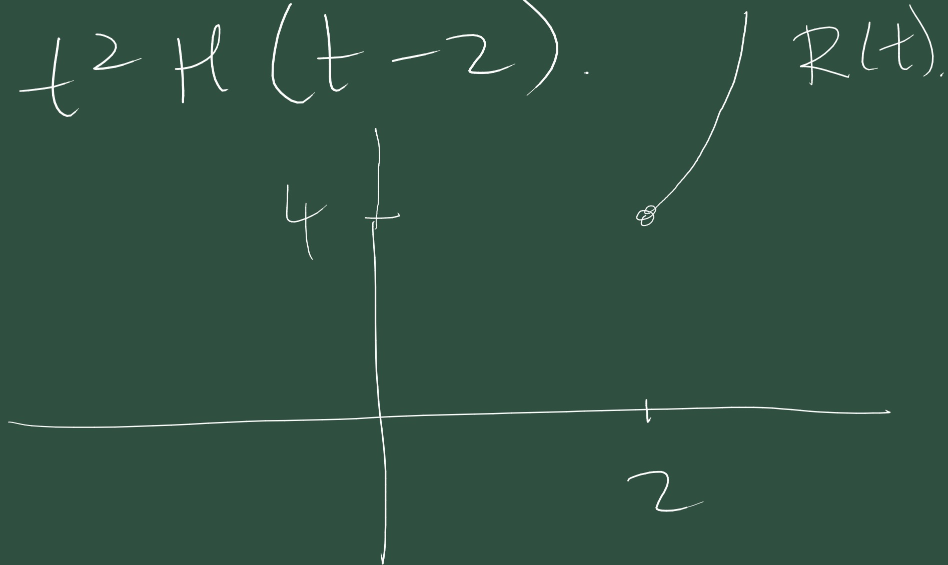
<u>Ex 4.7.12</u>

Consider $R$ defined by

$$R(t) = t^2 H(t-2).$$




Q? $\mathcal{L}\{R(t)\} = ?$

Before using Th. 4.7.10 for this
we would need to express.

$R(t) = g(t-2)\,H(t-2)$.

where $\boxed{g(t-2) = t^2}$ (*).

applying the theorem.

$\mathcal{L}\{R(t)\} = \mathcal{L}\{g(t-2)\,H(t-2)\}$

$= e^{-2s}\,G(s)$

where $G(s) = \mathcal{L}\{g(t)\}$

So what is $g(t) = ?$

from * we can see that.

$g(t) = g((t+2)-2)$

$= (t+2)^2$ , from (*)

$= t^2 + 4t + 4$,

So $G(s) = \mathcal{L}\{g(t)\}$

$$= \mathcal{L}\{t^2 + 4t + 4\}$$

$$= \mathcal{L}\{t^2\} + 4\mathcal{L}\{t\} + 4\mathcal{L}\{1\}$$

$$= \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}.$$

So

$$\mathcal{L}\{R(t)\} = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

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$$\ddot{y} = \frac{d^2y}{dt^2}$$

Ex. 4.7.13.

$$\dot{y} = \frac{dy}{dt}.$$

Solve.

$$\ddot{y} - 4\dot{y} + 3y = 6H(t-5)$$

subject to initial conditions.

$$y(0) = \dot{y}(0) = 0.$$

Solution 1. Transform whole ODE ✓

$$\mathcal{L}\{\ddot{y} - 4\dot{y} + 3y\} = \mathcal{L}\{6H(t-5)\}$$

apply linearity, write $\bar{y} = \mathcal{L}\{y\}$, use tables to get.

$$s^2\bar{y} - sy(0) - \dot{y}(0) - 4(s\bar{y} - y(0))$$

$$+3\bar{y} = \frac{6}{s}e^{-5s}$$

2. Introduce the initial conditions

$$s^2\bar{y} - 4s\bar{y} + 3\bar{y} = \frac{6}{s}e^{-5s}$$

3. Solve this transformed equation.

$$(s^2 - 4s + 3)\bar{y} = \frac{6}{s}e^{-5s}$$

So its solution

$$\bar{y} = e^{-5s}\left(\frac{6}{s(s^2-4s+3)}\right)$$

4. Obtain $y(t) = \mathcal{L}^{-1}\{\bar{y}\}$.

By theorem 4.7.10, (or from table).

$$\boxed{y(t) = \mathcal{L}^{-1}\{\bar{y}\} = f(t-5)\,H(t-5)}$$

where $f$ is function defined.

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$$

where $\bar{f}(s) = \dfrac{6}{s(s^2-4s+3)}$

So we need.

$$\mathcal{L}^{-1}\left\{\frac{6}{s(s^2-4s+3)}\right\}$$

For this we need the partial fraction expansion of

$$\boxed{\frac{6}{s(s-3)(s-1)}} = \frac{\alpha}{s} + \frac{\beta}{s-3} + \frac{\gamma}{s-1}$$

(for some values $\alpha, \beta, \gamma = ?$).

$$= \frac{\alpha(s^2-4s+3) + \beta s(s-1) + \gamma s(s-3)}{s(s-3)(s-1)}$$

$$= \boxed{\frac{(\alpha+\beta+\gamma)s^2 + (-4\alpha-\beta-3\gamma)s + 3\alpha}{s(s-3)(s-1)}}$$

Comparing numerators we get the linear equations:

$$3\alpha = 6. \Rightarrow \alpha = 2.$$

$$\begin{cases} -4\alpha - \beta - 3\gamma = 0 \\ \alpha + \beta + \gamma = 0 \end{cases}$$

$$-\beta - 3\gamma = 8$$

$$+(\beta + \gamma = -2)$$

$$-2\gamma = 6$$

$$\Rightarrow \gamma = -3$$

$$\beta = -\alpha - \gamma = -2 + 3 = 1.$$

So $\bar{f}(s) = \dfrac{6}{s(s^2-4s+3)}$

$$= \frac{2}{s} + \frac{1}{s-3} + \frac{-3}{s-1}$$

So
$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$

$$= 2 + e^{3t} - 3e^{t}.$$

So finally our solution is

$y(t) = f(t-5)H(t-5)$

$$= \left(2 + e^{3(t-5)} - 3e^{t-5}\right)H(t-5)$$

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Tut 07.

Q2.

$$\sinh(t) = \frac{1}{2}\left(e^{t} - e^{-t}\right)$$

So

$$\mathcal{L}\{\sinh(t)\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{t} - e^{-t}\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{t}\} - \frac{1}{2}\mathcal{L}\{e^{-t}\}$$

$$= \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$= \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

$$= \frac{1}{2}\frac{2}{s^2-1}$$

$$= \frac{1}{s^2-1}$$