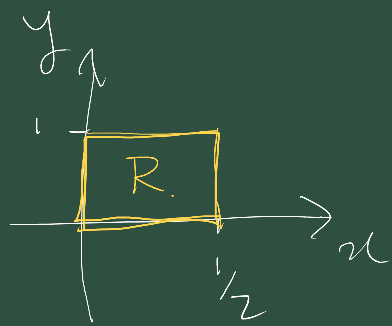
Q4 (a).

$$I = \int_0^1 \left(\int_0^{0.5} y e^{xy} \cdot dx \right) dy$$



$$= \int_0^{0.5} \left(\underbrace{\int_0^1 y e^{xy}\, dy}_{\text{inner integral.}} \right) dx$$

$$\to \int_0^1 y e^{xy}\, dy.$$

$$= \int_0^1 y \frac{\partial}{\partial y}\left(\frac{1}{x} e^{xy} \right) dy.$$, preparing for int. by parts

$$= \left[\frac{y}{x} e^{xy} \right]_{y=0}^{y=1} - \int_0^1 \frac{1}{x} e^{xy}\, dy.$$

$$= \frac{e^x}{x} - \left[\frac{1}{x^2} e^{xy} \right]_{y=0}^{y=1}$$

$$= \frac{e^x}{x} - \left(\frac{e^x}{x^2} - \frac{1}{x^2}\right)$$

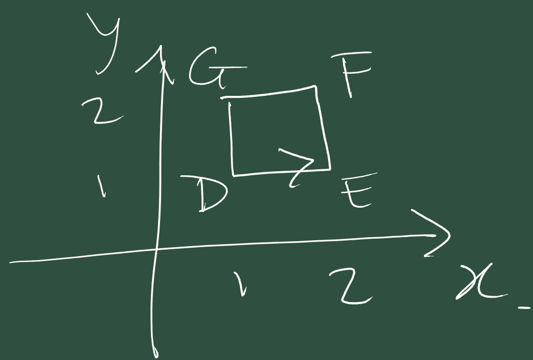
$$= \frac{e^x}{x} - \frac{e^x}{x^2} + \frac{1}{x^2}$$

Now proceed to integrate this ~~from~~ wrt x from 0 to 0.5.

Green's Theorem

Ex 3.8.3 A closed path integral

$$I = \oint_C \underbrace{3x^2y^2}_{P}\,dx + \overbrace{2xy}^{Q}\,dy$$



$$= \int_{DE} P\,dx + Q\,dy + \int_{EF} P\,dx + Q\,dy$$

$$+ \int_{FG} P\,dx + Q\,dy + \int_{GD} P\,dx + Q\,dy$$

on DE $\quad y = 1,\ dy = 0 \quad x: 1 \mapsto 2.$

" $\quad EF \quad x = 2,\ dx = 0 \quad y: 1 \mapsto 2$

" $\quad FG \quad y = 2,\ dy = 0 \quad x: 2 \mapsto 1$

" $\quad GD. \quad x = 1,\ dx = 0,\ y: 2 \mapsto 1$

$$= \int_1^2 3x^2\,dx + \int_1^2 4y\,dy + \int_2^1 12x^2\,dx$$

$$+ \int_2^1 2y\,dy$$

$$= \left[x^3\right]_1^2 + \left[2y^2\right]_1^2 + \left[4x^3\right]_2^1$$

$$+ \left[y^2\right]_2^1$$

$$= (8-1) + (8-2) + (4-32)$$

$$+ (1-4) = 7 + 6 - 28 - 3$$

$$= 13 - 31$$

$= -18$

Ex 3.8.10. Verify G.T. for previous example.

$R =$ interior of C.

$= \{ (x,y) : 1 \le x \le 2, \ 1 \le y \le 2 \}$

$P(x,y) = 3x^2y^2, \quad Q(x,y) = 2xy$

The R.H.S. of G.T. in the double integral

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

$$= \iint_R (2y - 6x^2y)\, dx\, dy$$

$$= \int_1^2 \left(\int_1^2 (2y - 6x^2y)\, dx \right) dy.$$

$$= \int_1^2 \Big[2xy - 2x^3y \Big]_{x=1}^{x=2} dy$$

$$= \int_1^2 4y - 16y - (2y - 2y) \ dy$$

$$= \int_1^2 -12y \, dy$$

$$= \left[-6y^2\right]_1^2$$

$$= -24 - (-6)$$

$= -18$ which equals the L.H.S of G.T. we calculated earlier.

<u>Proof of G.T.</u> (Outline).

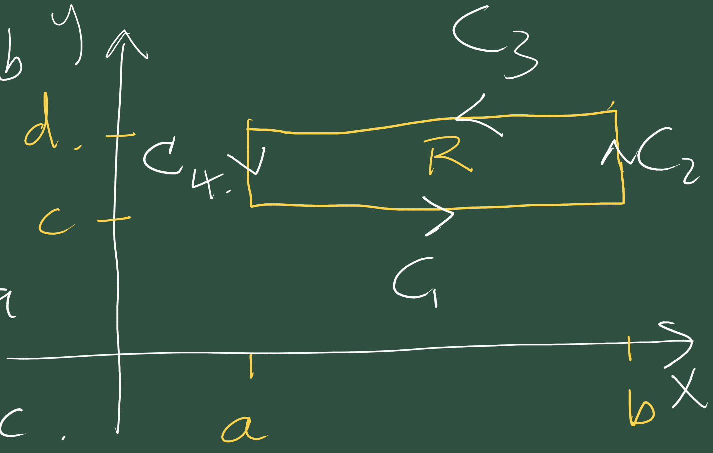
Consider first the simplest type of region R, a rectangle aligned with coordinate axes.

on C_1: $y = c,\ dy = 0\ \ x: a \to b$

C_2, $x = b,\ dx = 0,\ y: c \to d$

C_3: $y = d,\ dy = 0,\ x: b \to a$

C_4: $x = a,\ dx = 0,\ y: d \to c$.



So LHS of GT will be. (P, Q being general)

$$\oint_C P\,dx + Q\,dy$$

$$= \int_{C_1} \_\_\_\_ + \int_{C_2} \_\_\_\_ + \int_{C_3} \_\_\_\_ + \int_{C_4} \_\_\_\_$$

$$= \left[\int_a^b P(x,c)\,dx + \int_c^d Q(b,y)\,dy + \int_b^a P(x,d)\,dx + \int_d^c Q(a,y)\,dy \right]$$

Can't go further without knowledge of P, Q.

Now the RHS of GT will be.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \int_c^d \left(\int_a^b \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \right) dy$$

apply linearity to expand as

$$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x}\,dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y}\,dy \right) dx$$

and changing order in 2nd integral

$$= \int_c^d \Big[Q(x,y) \Big]_{x=a}^{x=b} dy$$

$$- \int_a^b \Big[P(x,y) \Big]_{y=c}^{y=d} \, dx.$$

$$= \int_c^d \Big(Q(b,y) - Q(a,y) \Big) dy$$

$$- \int_a^b \Big(P(x,d) - P(x,c) \Big) dx$$

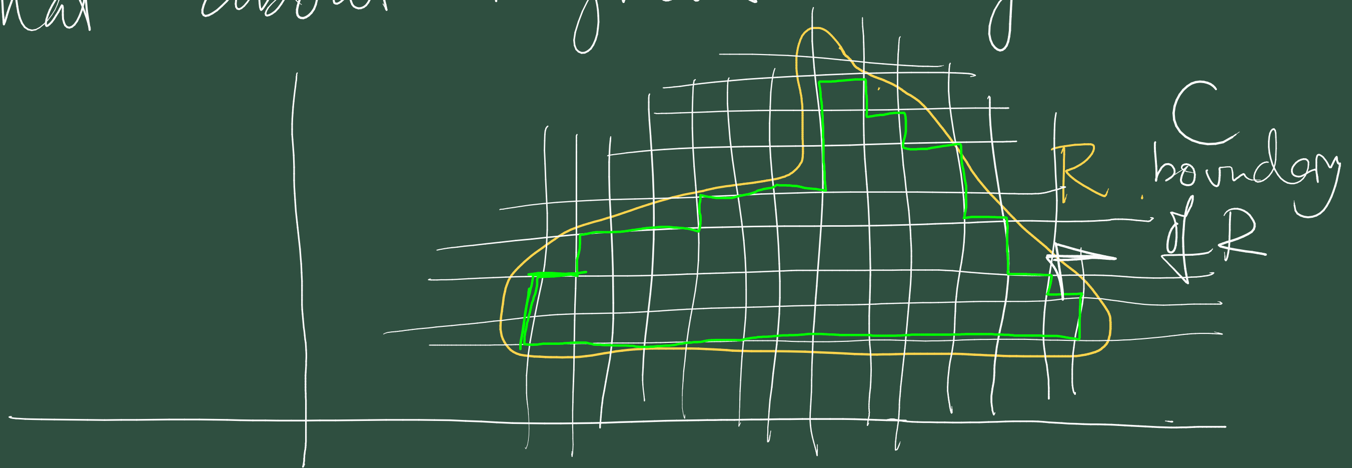
$$= \Big[\int_c^d Q(b,y)\, dy + \int_d^c Q(a,y)\, dy$$

$$+ \int_b^a P(x,d)\, dx + \int_a^b P(x,c)\, dx. \Big]$$

$=$ LHS, calculated above.

So GT is true for rectangles aligned with coordinate axes.

What about regions in general.



Consider a subdivision of xy plane and the approximation

$$R \approx \bigcup_{\square} \square$$

This approx can be made better and better by using smaller rectangles.

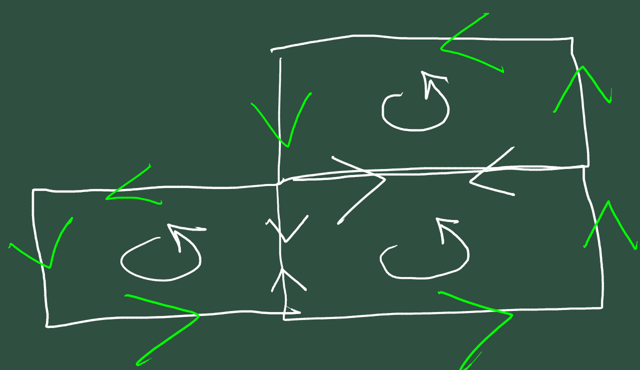
RHS of G.T.

$$\boxed{\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\, dy.}$$

$$\approx \iint_{\substack{\text{union} \\ \text{of } \square}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

$$= \sum_{\square} \iint_{\square} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

$$= \sum_{\square} \oint_{\square} P\, dx + Q\, dy$$



$$= \oint Pdx + Qdy.$$

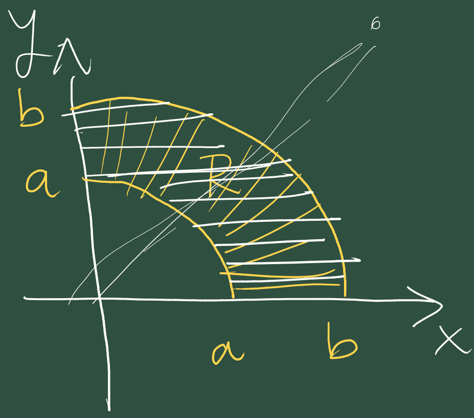
green exterior rectangle edges

$$\approx \boxed{\oint_C Pdx + Qdy.}$$

approx. of G.T

as we take limit as subdivision $\longrightarrow \infty$ we get the exact form of Green's Theorem.

Changing coordinate systems in multiple integrals



Eg. Consider evaluating.

$$I = \iint_R \frac{1}{x^2+y^2}\, dx\, dy$$

over the region R, the annulus region between circles of radius a and b $(a<b)$ in the upper-right quadrant of the plane.

Circles are given by $x^2+y^2=a^2$

$x^2+y^2=b^2$

$$I = \int_0^b \sim\sim\sim\, dy$$

$$= \int_0^a \int_{\sqrt{a^2-y^2}}^{\sqrt{b^2-y^2}} \frac{1}{x^2+y^2}\, dx\, dy$$

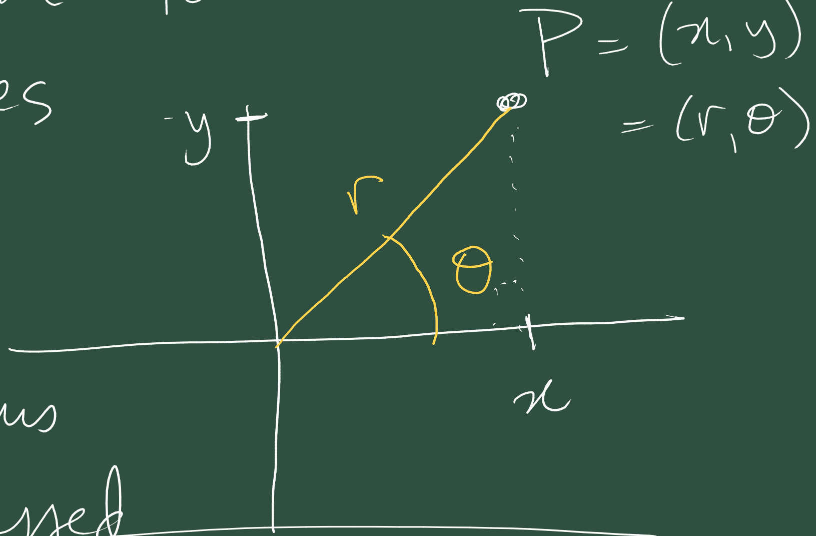
$$+ \int_a^b \int_0^{\sqrt{b^2-y^2}} \frac{1}{x^2+y^2}\, dy$$

= YIKES.....

Can we do better? Yes!!

Region and integrand are very "circular"

A better system to use is polar coordinates



Link between the two systems can be expressed

$$\boxed{x = r\cos\theta\,, \qquad y = r\sin\theta}$$

<u>OR</u>

$$r = \sqrt{x^2+y^2}\,, \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

Q? How to translate integrals from one system to another?

In general. multiple integrals will transform as.

$$\iint_R \cdots \iint f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

$$(x_1, \ldots, x_n) \to (u_1, \ldots, u_n)$$

$$= \int \cdots \int f(\ldots, x_i(u_1, \ldots, u_n), \ldots)$$

$$\underbrace{\left| \frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)} \right|}_{\text{absolute value of Jacobian determinant of transformation } (x_1, \ldots, x_n) \to (u_1, \ldots, u_n)} du_1\, du_2 \cdots du_n$$

$$\frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)} = \left| \begin{matrix} \cdots & & \cdots \\ \vdots & \dfrac{\partial x_i}{\partial u_j} & \vdots \\ \cdots & & \cdots \end{matrix} \right|$$

$n \times n$ determinant

So our example becomes.

$$I = \cdots \iint_R \frac{1}{r^2} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr\, d\theta$$

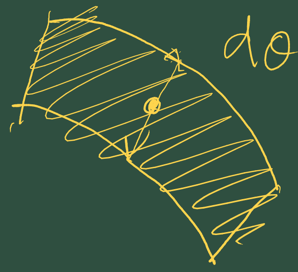
$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

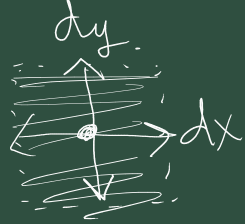
$$= r$$

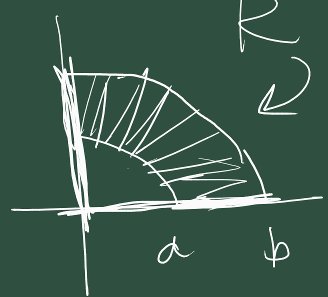


So

$$\boxed{I = \iint_R \frac{1}{r^2} \overbrace{r\,dr\,d\theta}^{=dx\,dy}}$$

So $r\,dr\,d\theta$ is the "area element" in polar coords.





$$I = \iint_R \frac{1}{r}\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_a^b \frac{1}{r}\,dr\,d\theta$$

$$= \int_0^{\pi/2} \Big[\ln(r)\Big]_a^b \,d\theta$$

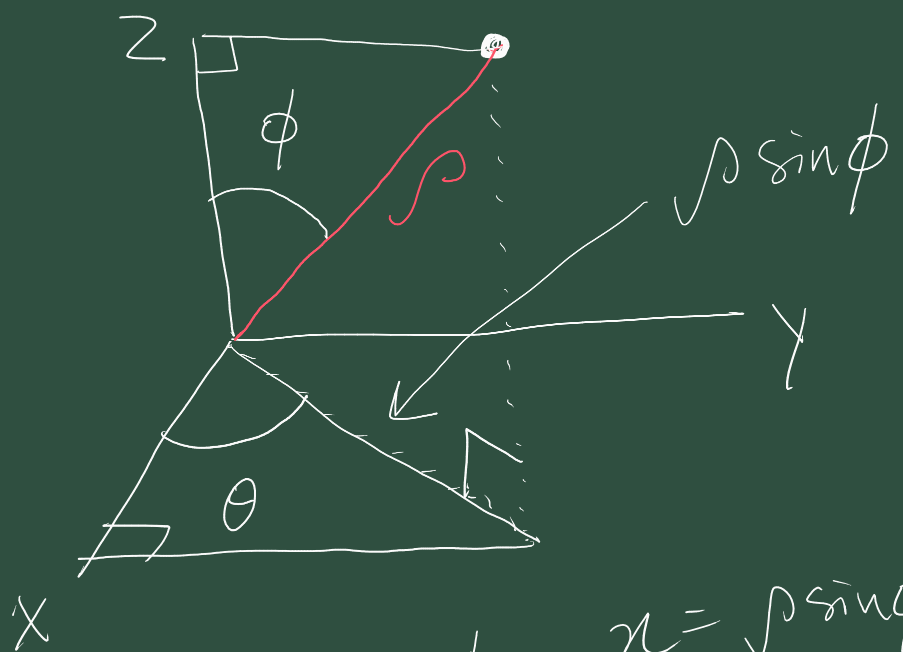
$$= \int_0^{\pi/2} \left(\ln(b) - \ln(a) \right) d\theta$$

$$= \frac{\pi}{2} \left(\ln(b) - \ln(a) \right)$$

$$= \frac{\pi}{2} \ln\left(\frac{b}{a} \right).$$

A 3-d example using spherical coords. $= (\rho, \theta, \phi)$

Greek rho ρ



$z = \rho \cos\phi$, $x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

Let's calculate $\dfrac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix}$$

$$\begin{vmatrix} \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \rho^2\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta \\ \cos\phi & 0 & -\sin\phi \end{vmatrix}$$

(factorising the determinant)

$$= \rho^2\sin\phi\left[\cos\phi\begin{vmatrix} -\sin\theta & \cos\phi\cos\theta \\ \cos\theta & \cos\phi\sin\theta \end{vmatrix} - \sin\phi\begin{vmatrix} \sin\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\theta \end{vmatrix}\right]$$

$$= \rho^2\sin\phi\left[\cos^2\phi\begin{vmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} - \sin^2\phi\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}\right]$$

$$= \rho^2 \sin\phi \, [\underbrace{\cos^2\phi(-1) - \sin^2\phi(1)}_{=-1}]$$

$$= -\rho^2 \sin\phi .$$

Can now use this as the ~~area~~ volume element for translating such integrals.

$$dx\,dy\,dz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$