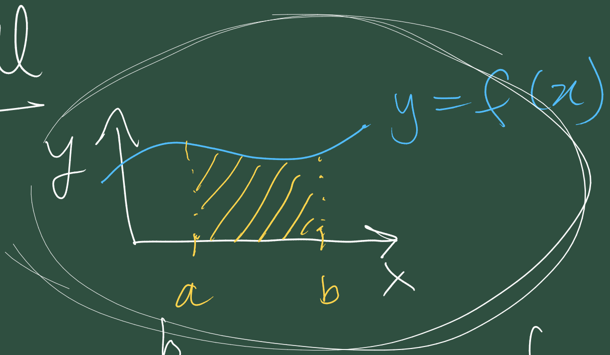
This week is really 3.1 & 3.2 from CLP3.

Multiple Integrals.

Firstly focusing on functions of 2 variables.

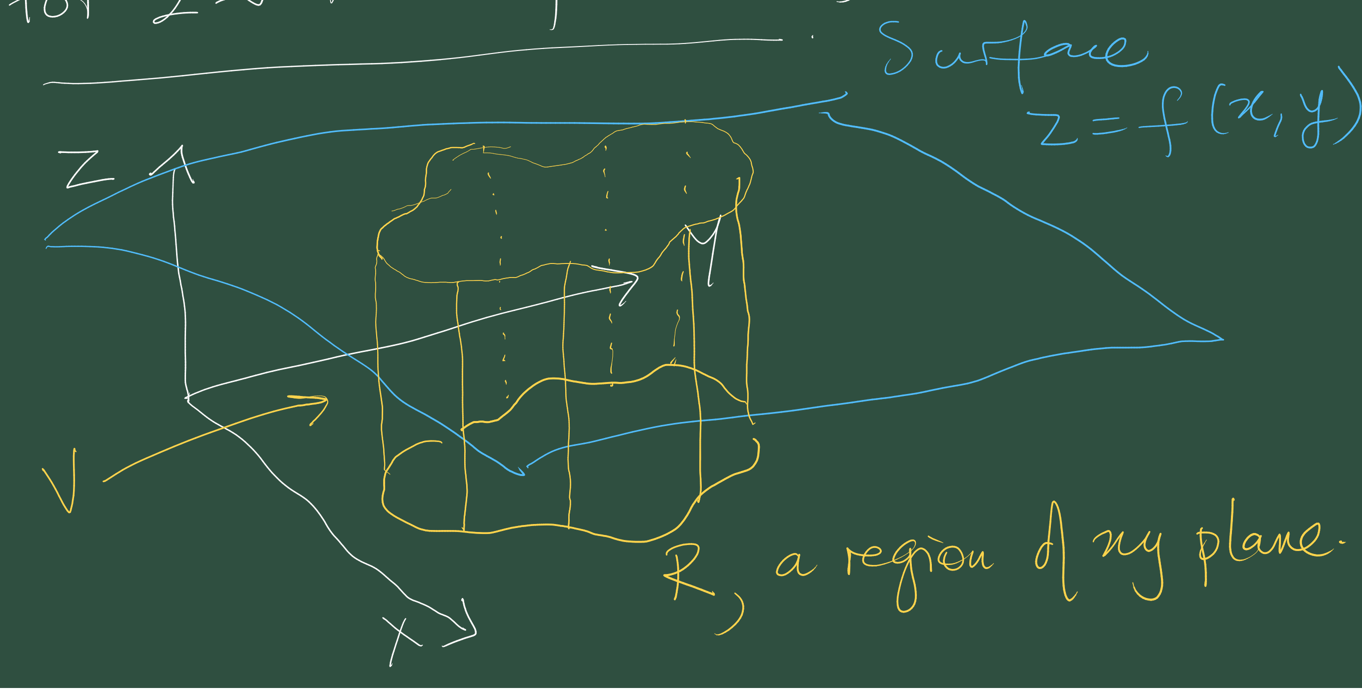
Double integrals are the generalisation to 2-variables of definite integrals.

Recall



$$\text{Area} = \int_a^b f(x)\,dx,$$ formally defined by means of limiting values of Riemann sums

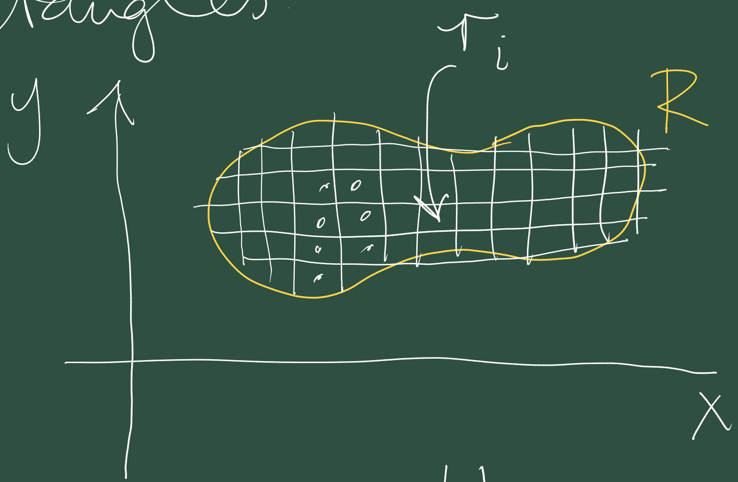
For 2-variable functions

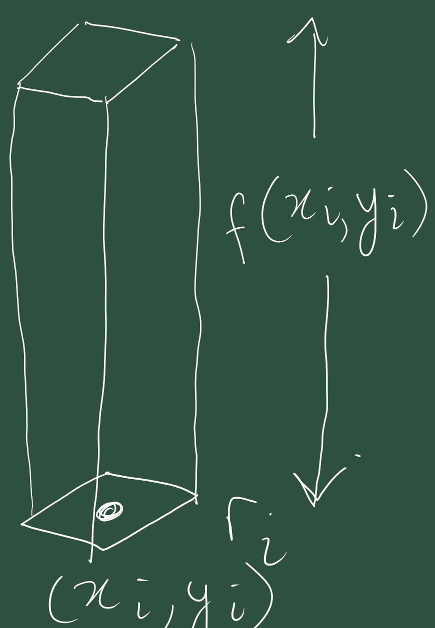


Volume V between xy-plane and surface, over R, will be given by the <u>double integral</u>

$$V = \iint_R f(x,y)\, dx\, dy$$

Can be formally defined using limiting values of sums (like Riemann sums) using a subdivision of R into small rectangles.





$$R \approx \bigcup_i r_i$$

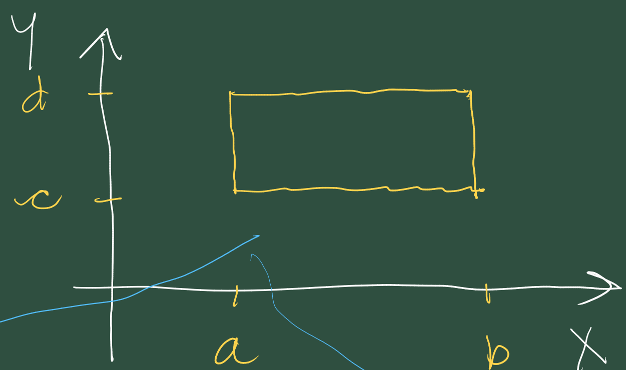
Over each r_i we can take a sample point (x_i, y_i) which produces a sub-volume, the "tower" shown, of height $f(x_i, y_i)$ so that

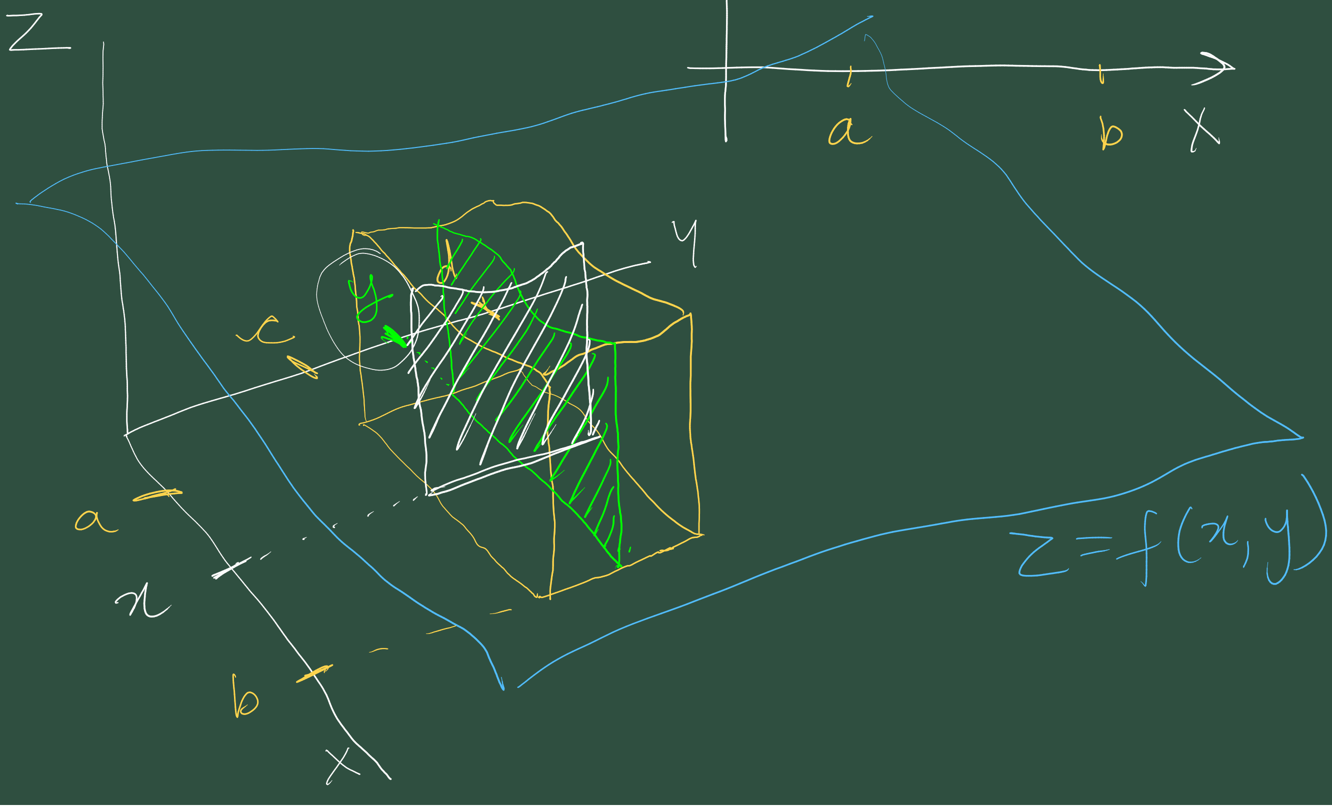
$$V \approx \sum_i (\text{area } r_i)\, f(x_i, y_i)$$

$$\xrightarrow[\text{as areas } r_i \to 0]{} \iint_R f(x,y)\,dx\,dy$$

We shall evaluate these double integrals by interpreting as a pair of nested repeated integrals.

<u>Simplest case</u>: Where R is a rectangle aligned to the x and y axes.





Consider the "curtain" lying parallell to x-axis, at the location y shown whose top edge runs along the surface. Its area is given by $\int_a^b f(x,y)\,dx$.

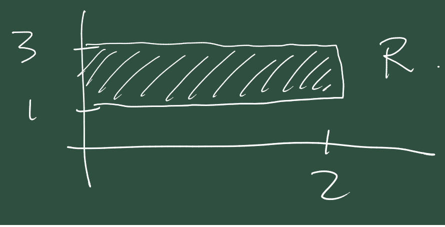
fixed

V can be expressed as continuous sum of areas of these curtains from $y=c$ to $y=d$. This leads to:

$$V = \underbrace{\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy}_{\text{"repeated integral"}}$$

These are typically evaluated from the inside out.

Example. Consider $f(x,y) = 2xy + 4x + 3y + 1$. Integrate this over rectangular region R defined by $R = \{(x,y) : 0 \le x \le 2,\ 1 \le y \le 3\}$



So Vol. between xy-plane and surface, over R, is given by

$$V = \iint_R f(x,y)\,dx\,dy$$

$$= \int_1^3 \left(\int_0^2 (2xy + 4x + 3y + 1)\,dx \right) dy$$

$$= \int_1^3 \Big[x^2 y + 2x^2 + 3xy + x \Big]_{x=0}^{x=2} \, dy$$

$$= \int_1^3 (4y + 8 + 6y + 2)\,dy$$

$$= \int_1^3 \underbrace{(10y + 10)}\,dy$$

area of the curtain parallel to x-axis. as in diagram above.

$$= \Big[5y^2 + 10y \Big]_1^3$$

$$= 45 + 30 - (5 + 10)$$

$$= 60$$

Equally well we could have the inner integral be with respect to y and the outer integral be with respect to x. That would look like.

area of a curtain parallel to y-axis.

$$V = \int_0^2 \left(\int_1^3 f(x,y)\, dy \right) dx$$

$$= (\text{exercise}) = 60.$$

Non-rectangular regions (or non-aligned rectangles).

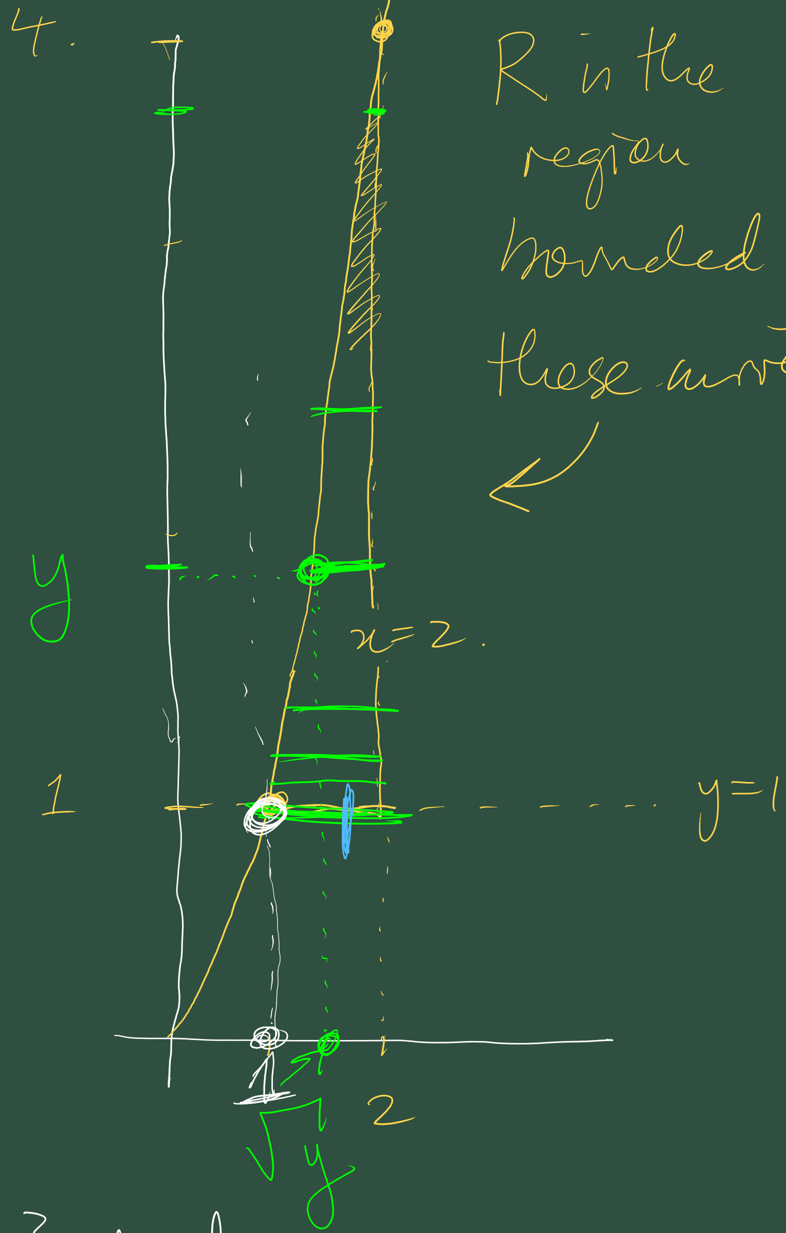
Here, careful consideration of boundary curves is needed.

<u>Example</u>. Integrate $f(x,y) = 4x^3 + 4y^3$ over the region R, defined as the region bounded by the curves $x = 2$, $y = 1$, and $y = x^2$

① - Sketch the region R.

$$x = \sqrt{y} \iff \boxed{y = x^2}$$

R is the region bounded by these curves.



$$V = \iint_R 4x^3 + 4y^3 \, dx\, dy.$$

$$= \int_1^4 \left(\int_{\sqrt{y}}^{2} (4x^3 + 4y^3)\,dx \right) dy$$

non-rectangular regions produce non-constant limits of integration

$$=$$

$$= \int_1^4 \left[x^4 + 4xy^3 \right]_{x=\sqrt{y}}^{x=2} dy$$

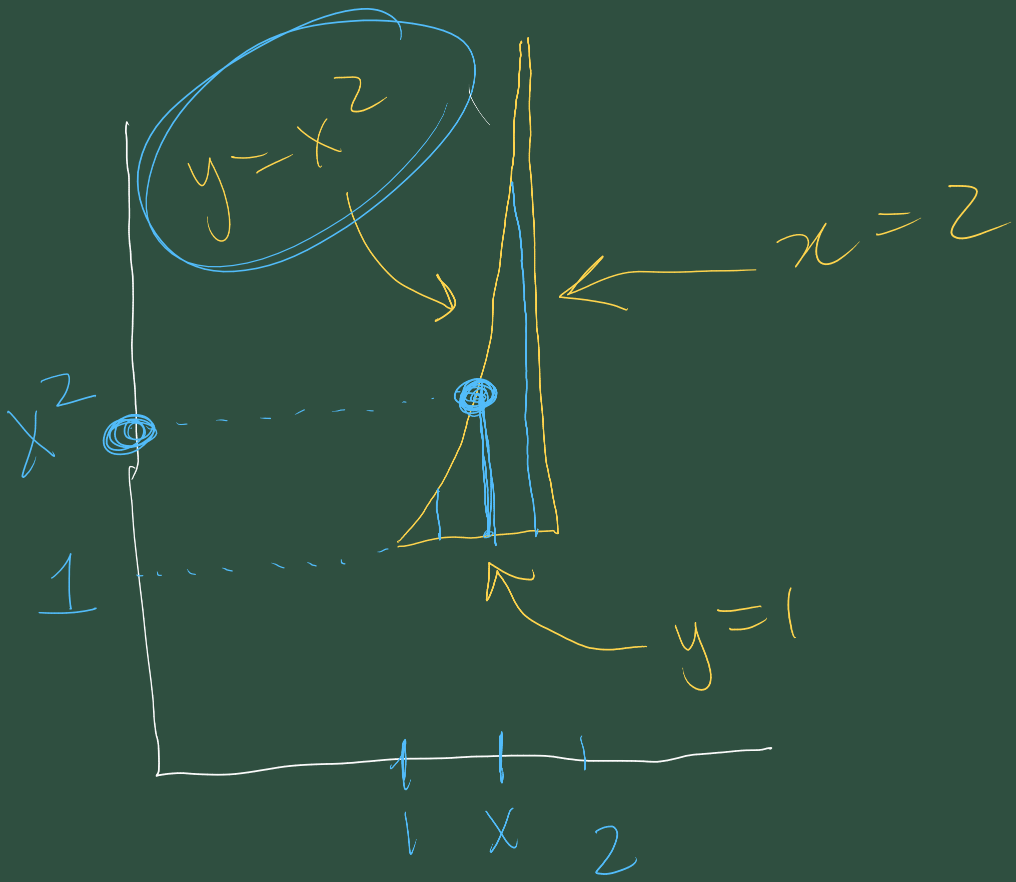
$$= \int_1^4 \left(16 + 8y^3 - \left(y^2 + 4y^{7/2}\right)\right) dy$$

$$= \left[16y + 2y^4 - \frac{y^3}{3} - 4 \cdot \frac{2y^{9/2}}{9} \right]_1^4$$

$$= 745/9$$

How would it appear expressed in the opposite order?

$$V = \int_1^2 \left(\int_1^{x^2} (4x^3 + 4y^3)\, dy \right) dx$$

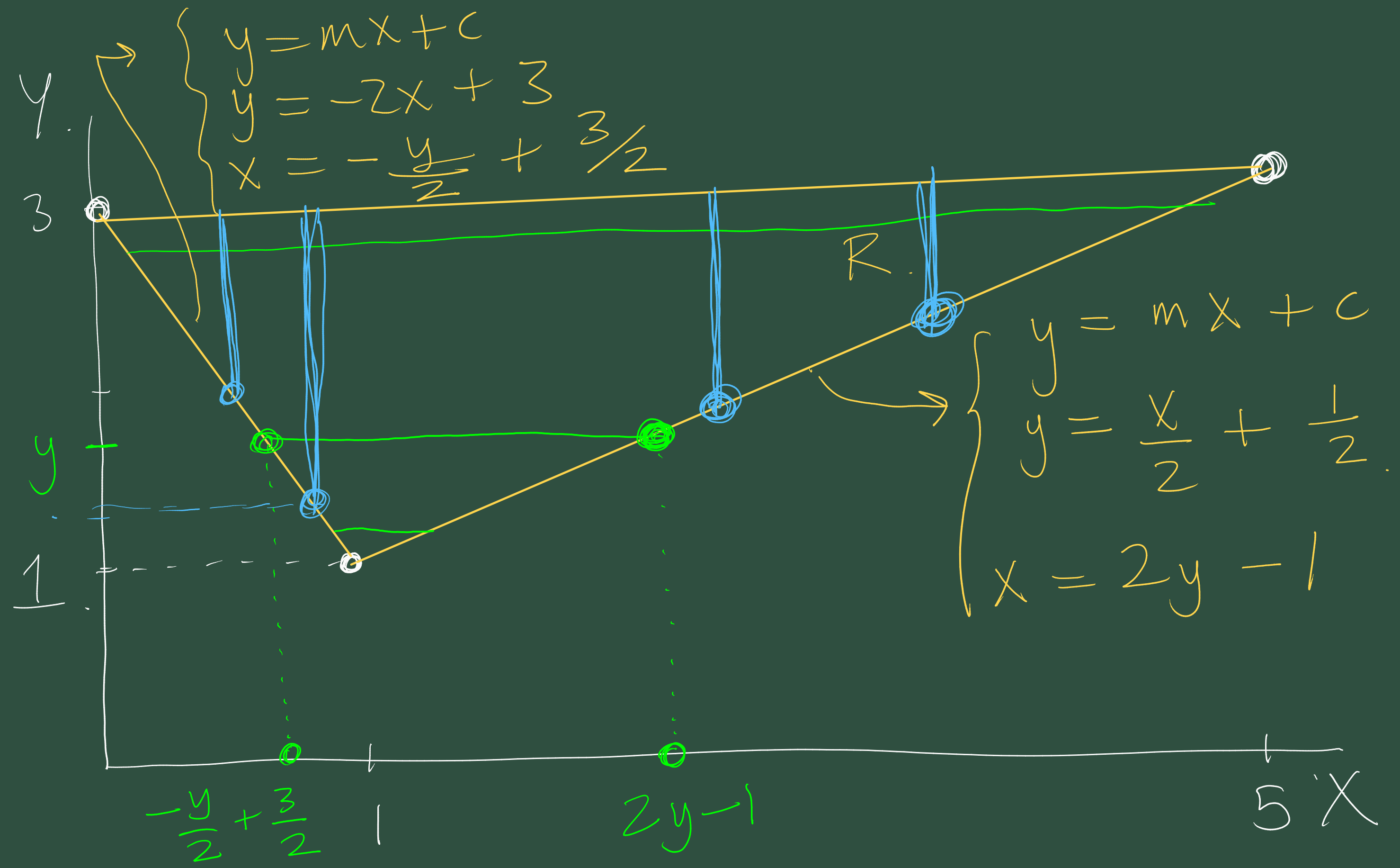


$$= \text{exercise} = 745/9$$

Example Shows the need to sometimes use two or more double integrals.

Consider integrating a function $f(x,y)$ over the triangle with vertices at $(1,1)$, $(5,3)$, $(0,3)$

Investigate the repeated integrals and their limits in both orders.



First try x as the inner, y as the outer integral.

$$V = \int_1^3 \left(\int_{-\frac{y}{2}+\frac{3}{2}}^{2y-1} f(x,y)\, dx \right) dy$$

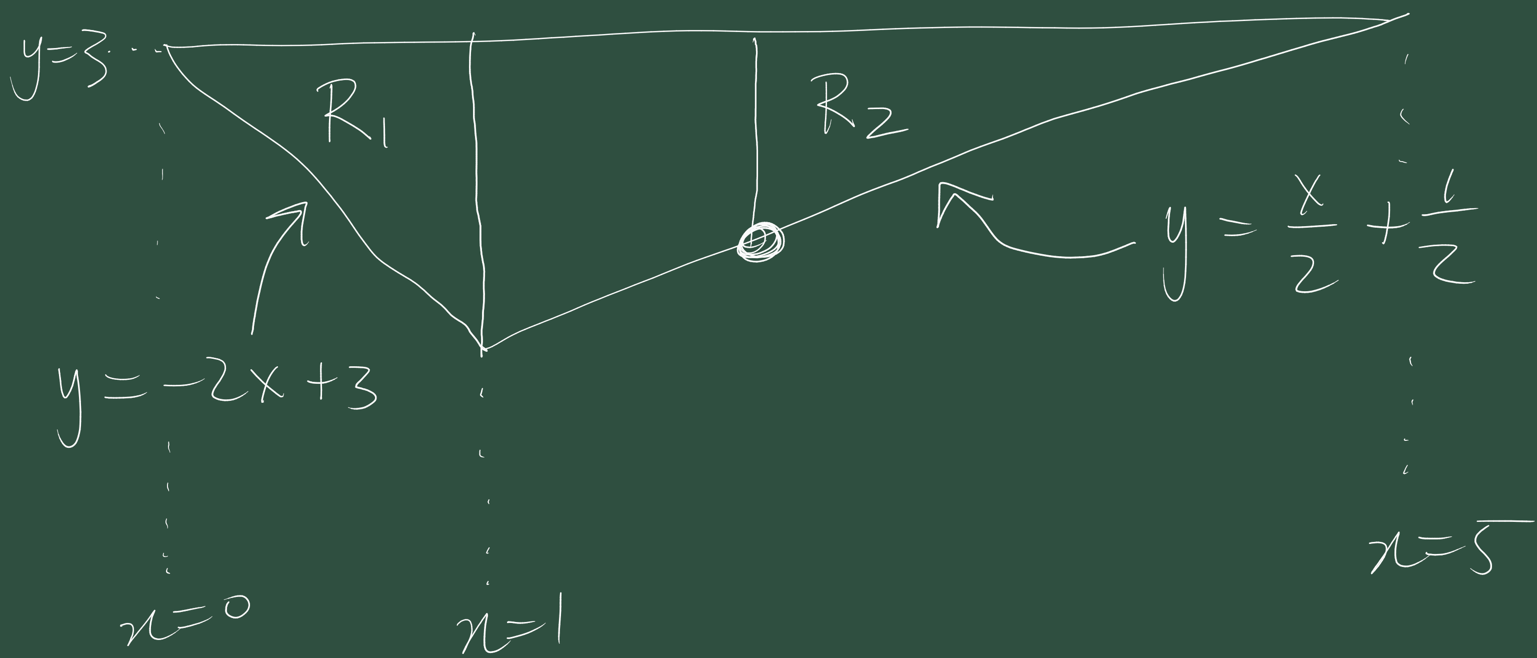
in the other order.

$$= \int_0^5 \left(\int_{?}^{3} f(x,y)\, dy \right) dx.$$

No single expression in x that will

work here as it's different for $x<1$ and $x>1$.

The way to resolve this is to split R into two regions, for $x<1$ and $x>1$.



$$\iint_R f(x,y)\,dy\,dx = \iint_{R_1} f(x,y)\,dy\,dx$$

$$+ \iint_{R_2} f(x,y)\,dy\,dx.$$

$$= \int_0^1 \left(\int_{-2x+3}^{3} f(x,y)\,dy \right) dx$$

$$+\int_1^5\left(\int_{\frac{x}{2}+\frac{1}{2}}^{3} f(x,y)\,dy\right)dx$$

Textbook Exercises.

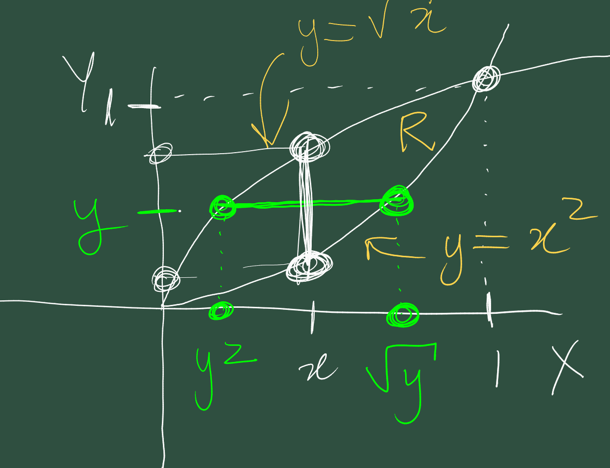
Section 3.1.

Q3 (c).

$$V=\iint_R xy^2\,dx\,dy$$

where R is the finite region in 1st quadrant bounded by $y=x^2$ and $x=y^2 \Leftrightarrow y=\sqrt{x}$.

$$=\int_0^1\left(\int_{y^2}^{\sqrt{y}} xy^2\,dx\right)dy.$$



$$=\int_0^1\left[\frac{x^2y^2}{2}\right]_{x=y^2}^{x=\sqrt{y}}dy$$

$$=\int_0^1\left(\frac{y^3}{2}-\frac{y^6}{2}\right)dy.$$

$$=\left[\frac{y^4}{8}-\frac{y^7}{14}\right]_0^1=\frac{1}{8}-\frac{1}{14}=\frac{7-4}{56}=\frac{3}{56}$$

In the opposite order it would be.

$$V = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} x y^2 \, dy \right) dx$$

$= \text{exercise} = \frac{3}{56}.$

Tutorial sheet 5 (except Q8).