Laplace Transforms

ODE will often have more than one solution indeed infinite families of solutions, depending on parameters

Soutions can be made to be unique by supplying extra conditions on y known as initial conditions $y(0) = \ldots$

or boundary conditions, e.g.

$y(a) = \ldots \quad , y(b) = \ldots$

where $[a,b]$ is the domain of the problem.

Example 4.13 Consider

$$\boxed{\frac{d^2y}{dt^2} = 0.}$$

Integrating both sides yield

$$\frac{dy}{dt} = C_1$$

where C_1, C_2 are unknown constants

and again.

$$y(t) = C_1 t + C_2.$$

But if the boundary conditions $y(0) = 1$ and $y(1) = 2$ must be met then:

$$y(0) = C_2 = 1$$

$$y(1) = C_1 + C_2 = 2. \Rightarrow C_1 = 1$$

So $y(t) = t + 1$ is the unique solution to ODE and the boundary conditions.

Motivation: A transform-like process for solving calculation problems.

Consider finding powers or products a^b, $a \cdot b$ for precise decimals a, b

Before computers, these were aided by pre-calculated log tables, vaules of the logarithms function, and properties of log.

$$a^b \xrightarrow{\log} \log(a^b) = b \cdot \log(a)$$

$$a \cdot b \xrightarrow{\log} \log(a \cdot b) = \log(a) + \log(b)$$

harder

easier.

inverse log transform

$= \exp.$

Def 4.2.1.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

other notations $= \bar{f}(s) = F(s)$

a function of the new variable s.
We can picture $\mathcal{L}\{f(t)\}$ as a continuous version of the process of building a power series from a sequence.

$(a_0, a_1, a_2, a_3, \ldots)$ a sequence of $a_i \in \mathbb{R}$

$= f: \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{R}$.

$i \longmapsto a_i$

A power series $A(x)$ is an infinite polynomial function.

$$A(x) = \sum_{i=0}^{\infty} a_i x^i$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt.$$

$$e^{-st} = (e^{-s})^t$$

Examples: Let's find some transforms

<u>Ex 4.2.2.</u> Let f be the constant function. defined by

$$f(t) = 1 \quad , \text{for all } t \geq 0.$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_{t=0}^{t\to\infty}$$

$$= \frac{1}{-s} \left(\lim_{t\to\infty} (e^{-st}) - 1 \right).$$

assuming $s>0$, $\underset{\frac{1}{e^{st}}}{\overset{}{e^{-st}}} \to 0$ as $t \to \infty$

$$= \frac{1}{-s}(0-1) \quad , \text{for } s>0$$

$$= \frac{1}{s}$$

Ex 4.2.3 Consider f defined by

$$f(t) = t, \text{ for } t \geq 0$$

Claim: $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$.

See notes, done by integration by parts

Ex 4.2.4 Consider f defined by

$$f(t) = e^{-at}, \text{ for } t \geq 0$$

then

$$\mathcal{L}\{f(t)\} = \frac{1}{s-a}.$$

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} \, dt$$

$$= \int_0^\infty e^{-(s+a)t} \, dt.$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= \frac{1}{-(s+a)}\left(0 - 1\right) \text{ for } s+a>0$$

i.e. for $s > -a$.

$$= \frac{1}{s+a}.$$

We'll use the table of transforms when solving ODEs together with some important properties of the transform.

Theorem 4.4.1

f, g functions $\alpha, \beta \in \mathbb{R}$.

$\mathcal{L}$ is linear.

i.e. $\mathcal{L}\{\alpha f + \beta g\}$

$$= \alpha \mathcal{L}\{f\} + \beta \mathcal{L}\{g\}$$

Proof: Because integration is linear.

$$\boxed{\mathcal{L}\{\alpha f + \beta g\}}$$

$$= \int_0^\infty e^{-st} (\alpha f + \beta g)(t)\, dt$$

$$= \int_0^\infty e^{-st} (\alpha f(t) + \beta g(t))\, dt$$

$$= \int_0^\infty \left[\alpha e^{-st} f(t) + \beta e^{-st} g(t) \right] dt$$

int. is linear.

$$= \alpha \int_0^\infty e^{-st} f(t)\, dt + \beta \int_0^\infty e^{-st} g(t)\, dt$$

$$\boxed{= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}}.$$

Application.

Ex 4.4.2

$$\mathcal{L}\{3t + 4e^{-2t}\}$$

$$= 3\mathcal{L}\{t\} + 4\mathcal{L}\{e^{-2t}\}.$$

, by linearity

$$= \frac{3}{s^2} + \frac{4}{s+2}$$

, reading transforms from the table.

Theorem 4.4.3

If $F(s) = \mathcal{L}\{f(t)\}$ then.

$$-\frac{dF}{ds} = \mathcal{L}\{t f(t)\}$$

Proof (see notes) relies on interchanging the operations.

$$\frac{d}{ds}\left(\int_0^\infty \text{———} \, dt \right)$$

$$= \int_0^\infty \frac{d}{ds}(\text{———}) \, dt .$$

Application. Finding transform of polynomial terms. for integers $n \geq 0$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} .$$

Proved by repeating the following

We know $\mathcal{L}\{t\} = \frac{1}{s^2}$

$$\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\}.$$

$$= -\frac{d}{ds}\mathcal{L}\{t\}\text{, by theorem 4.4.3}$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2}\right)$$

$$= -\left(\frac{-2}{s^3}\right)$$

$$= \frac{2}{s^3} = \frac{2!}{s^3}$$

and in general if

$$\mathcal{L}\{t^{n-1}\} = \frac{(n-1)!}{s^n}$$

then

$$\mathcal{L}\{t^n\} = \mathcal{L}\{t \cdot t^{n-1}\}$$

$$= -\frac{d}{ds}\left(\frac{(n-1)!}{s^n}\right)$$

$$= -(n-1)! \left(\frac{-n}{s^{n+1}} \right)$$

$$= \frac{n!}{s^{n+1}}, \text{ since } n(n-1)! = n!$$

So induction proves this for all $n \geq 0$

Theorem 4.4.5 Key for solving ODEs

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = s\,\mathcal{L}\{x(t)\} - x(0)$$

Proof (integration by parts).

(see notes).

$$\frac{d}{dt}\left(e^{-st}\right) = -s\,e^{-st}$$

What about other higher order derivatives? Well they come from repeated applications of this theorem.

eg.

$$\boxed{\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\}} = \mathcal{L}\left\{\frac{d}{dt}\left(\frac{dx}{dt}\right)\right\}$$

$$= s\mathcal{L}\left\{\frac{dx}{dt}\right\} - \left.\frac{dx}{dt}\right|_{t=0}, \text{ from theorem}$$

$$= s\left(s\mathcal{L}\{x\} - x(0)\right) - \left.\frac{dx}{dt}\right|_{t=0}$$

$$\boxed{= s^2\mathcal{L}\{x\} - sx(0) - \left.\frac{dx}{dt}\right|_{t=0}.}$$

Repeating this process (by induction) we prove Corollary 4.4.6.

Ex 4.5.1 Solve (using dot notation)

$$\ddot{y} - 5\dot{y} + 4y = 12$$

with the initial conditions

$$y(0) = \dot{y}(0) = 0$$

Solution:

$$\mathcal{L}\{\ddot{y} - 5\dot{y} + 4y\} = \mathcal{L}\{12\}$$

$$= 12\mathcal{L}\{1\}$$

$$= \frac{12}{s}$$

Use known transform and linearity to get

$$\mathcal{L}\{\ddot{y}\} - 5\mathcal{L}\{\dot{y}\} + 4\mathcal{L}\{y\} = \frac{12}{s}$$

Write $\bar{y} = \mathcal{L}\{y(t)\}$, and use transform of derivatives properties

$$s^2\bar{y} - sy(0) - \dot{y}(0) - 5(s\bar{y} - y(0))$$

$$+ 4\bar{y} = \frac{12}{s}$$

Putting in $y(0) = \dot{y}(0) = 0$

$$s^2\bar{y} - 5s\bar{y} + 4\bar{y} = \frac{12}{s}$$

$$(s^2 - 5s + 4)\bar{y} = \frac{12}{s}$$

$$\Rightarrow \quad \bar{y} = \frac{12}{s(s^2-5s+4)}$$

Now find $y(t)$ as $y(t) = \mathcal{L}^{-1}\{\bar{y}\}$

But before consulting the tables we need the partial fraction expansion of R.H.S.

$$\frac{12}{s(s^2-5s+4)} = \frac{12}{s(s-1)(s-4)}$$

$$= \frac{\alpha}{s} + \frac{\beta}{s-1} + \frac{\gamma}{s-4}$$

for some, as yet undetermined, constants α, β, γ.

$$= \frac{\alpha(s-1)(s-4) + \beta s(s-4) + \gamma s(s-1)}{s(s-1)(s-4)}$$

$$= \frac{\alpha(s^2-5s+4) + \beta(s^2-4s) + \gamma(s^2-s)}{s(s-1)(s-4)}$$

$$= \frac{(\alpha+\beta+\gamma)s^2 + (-5\alpha-4\beta-\gamma)s + 4\alpha}{s(s-1)(s-4)}$$

Comparing numerators above we must have:

$$\alpha + \beta + \gamma = 0$$

$$-5\alpha - 4\beta - \gamma = 0$$

$$4\alpha = 12. \Rightarrow \alpha = 3$$

$$\Rightarrow \begin{cases} \beta + \gamma = -3 \\ -4\beta - \gamma = 15 \end{cases}$$

$$-3\beta = 12$$

$$\Rightarrow \beta = -4.$$

$$\Rightarrow \gamma = -\beta - \alpha$$
$$= 4 - 3 = 1$$

So.

$$\bar{y} = \frac{12}{s(s-4)(s-1)} = \frac{3}{s} - \frac{4}{s-1} + \frac{1}{s-4}.$$

So now we can take $\mathcal{L}^{-1}$, use linearity, and the tables to get:

$$y(t) = \mathcal{L}^{-1}\{\bar{y}\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{3}{s} - \frac{4}{s-1} + \frac{1}{s-4} \right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 4\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$+ \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= 3 - 4e^{t} + e^{4t}$$

This is the solution to the original ODE.