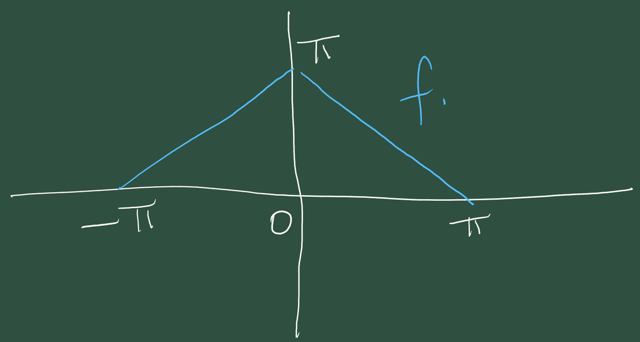
Q1 Tut sheet 10



The function f has the case-wise definition

$$f(x) = \begin{cases} \pi + x, & \text{if } -\pi < x < 0 \\ \pi - x, & \text{if } 0 < x < \pi \end{cases}$$

We observe that f is an even function. i.e. $f(-x) = f(x)$

So the sine coefficients $b_n = 0$, for $n \geq 1$.

and f will have a Fourier series

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

for coefficients a_n, $n \geq 0$. which are given by

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (\pi - x)\, dx.$$

$$= \frac{2}{\pi}\left(\pi x - \frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\frac{\pi^2}{2}$$

$$= \pi$$

For $n \geq 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx.$$

requires integration by parts, ...
or using Matlab we get.

$$= \frac{2}{\pi}\,\frac{2}{n^2}\sin^2\left(\frac{\pi n}{2}\right)$$

$$= \begin{cases} 0, & \text{if } n \text{ even. as } \sin(m\pi) = 0 \text{ for } m \in \mathbb{Z}. \\ \dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ & \text{as } \sin\left(m\pi + \frac{\pi}{2}\right) = \pm 1 \text{ for } m \in \mathbb{Z}. \end{cases}$$

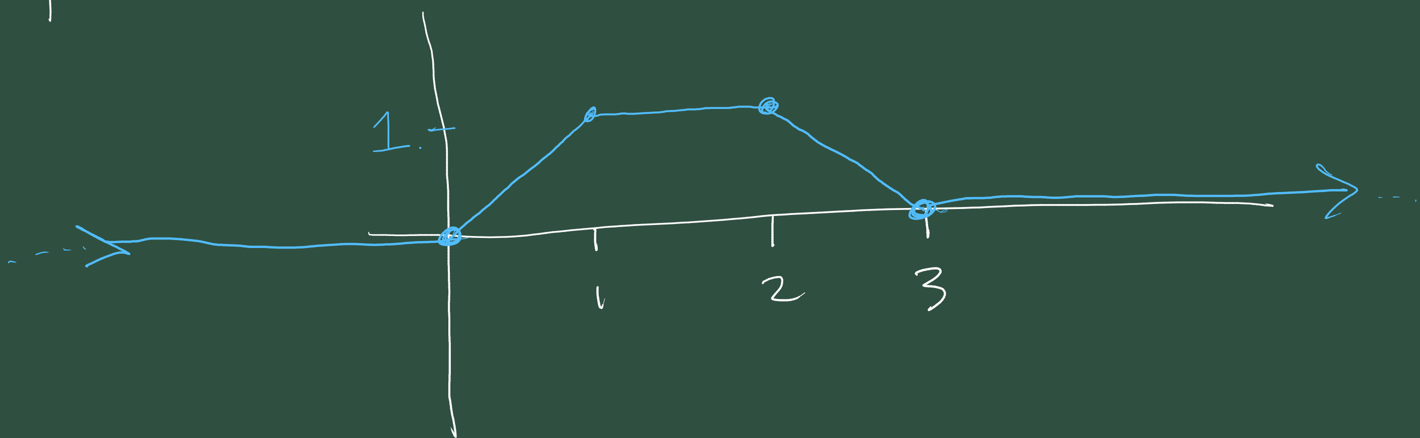
So the Fourier series will be

$$f(x) = \frac{\pi}{2} + \sum_{\substack{n=1 \\ n \text{ odd.}}}^{\infty} \frac{4}{\pi n^2} \cos(nx)$$

$$= \frac{\pi}{2} + \sum_{m=1}^{\infty} \frac{4}{\pi (2m-1)^2} \cos((2m-1)x)$$

Tut Sheet 8

Q 7 (b). First we'll want to express f in terms of Heaviside step functions



Build an expression for f using differences of Heavisides to represent each interval.

For $t \geqslant 0$

$$f(t) = t\left(H(t) - H(t-1)\right)$$
$$+ \ 1\left(H(t-1) - H(t-2)\right)$$
$$+ \ (3-t)\left(H(t-2) - H(t-3)\right).$$

$$= tH(t) - (t-1)H(t-1) - (t-2)H(t-2)$$
$$- (t-3)H(t-3).$$

To solve

$$\dot{y} - y = f(t) \quad \& \quad y(0)=0.$$

first transform the whole ODE using linearity, transform of derivative property, and $\mathcal{L}\{f(t-a)H(t-a)\}$ property (all summarised on table) to get.

$$s\bar{y} - y(0) - \bar{y} = \frac{1}{s^2}\left(1 - e^{-s} - e^{-2s} - e^{-3s}\right)$$

using $y(0)=0$ this becomes

$$s\bar{y} - \bar{y} = \frac{1}{s^2}\left(1 - e^{-3} - e^{-2s} - e^{-3s}\right)$$

$$\Rightarrow \bar{y} = \frac{1}{(s-1)s^2}\left(1 - e^{-s} - e^{-2s} - e^{-3s}\right).$$

In principle we just take inverse transform now to get the solution. $y(t) = \mathcal{L}^{-1}\{\bar{y}(s)\}$.

To do this we'll need

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)s^2}\right\},$$

we need partial fraction expansion of $\frac{1}{(s-1)s^2}$, from MATLAB.

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right\}.$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

, using linearity

$$= e^t - 1 - t.$$

in our $f(t)$ to use with the.

$$\mathcal{L}^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a)H(t-a)$$

So now

$$y(t) = \mathcal{L}^{-1}\{\bar{y}(s)\}$$

$$= e^{t} - 1 - t$$
$$- \left[e^{t-1} - 1 - (t-1)\right] H(t-1)$$
$$- \left[e^{t-2} - 1 - (t-2)\right] H(t-2)$$
$$- \left[e^{t-3} - 1 - (t-3)\right] H(t-3)$$