# Fourier Series

## Briefly

- A standard way to express certain functions as infinite sums/series of sine and cosine functions
- Finite versions of these series will approximations to the 'input' functions.
- Applications in signal processing (ie. sound, images, ....)
- Generalises and extends to the Fourier transform.

Some motivation/connections with maths you already know

## Recall Taylor series

A Taylor expansion of a function $f$ at some base point $a$ looks like.

$$f(a+x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}} \underbrace{(x-a)^n}$$

coefficients of the poly. depend on the derivatives of $f$

$$f^{(n)} := \frac{d^n f}{dx^n}.$$

poly. terms in $x$ "standard functions"

approximations to $f$ will be given by

$$f(a+x) \simeq \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

getting better as $k \to \infty$

Fourier series will share some of these properties, using a family of standard functions

$$F = \{ 1, \sin(x), \sin(2x), \sin(3x), \ldots, \cos(x), \cos(2x), \cos(3x), \ldots \}$$

to give a series for $f$ as.

$$f(x) = \frac{1}{2} a_0 1 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Fourier coefficients .... ?

The sine and cosine functions are all periodic, repeating at least every $2\pi$

So in fact the Fourier series will replicate the behaviour of $f$ over the interval $[-\pi, \pi]$ and repeat this across the whole domain of $x$.

Recall some concepts from linear algebra: bases for a vector space, and coefficients for a vector with respect to a basis.

Consider the standard $n$-dim.

Euclidean vector space $V = \mathbb{R}^n$

$V$ has a standard basis

$$B = \{ \underline{e}_1, \underline{e}_2, \underline{e}_3, \dots, \underline{e}_n \}$$

$$\left.\begin{aligned} \underline{e}_1 &= (1, 0, \dots, 0) \\ \underline{e}_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ \underline{e}_n &= (0, \dots, 0, 1) \end{aligned}\right\} \text{all zeros except for a 1 in } i^{th} \text{ position for } \underline{e}_i$$

The basis $B$ is an "orthonormal" basis meaning. $\forall\, i \neq j$ $\underline{e}_i \cdot \underline{e}_j = 0$ ie. $\underline{e}_i, \underline{e}_j$ are orthogonal and $\|\underline{e}_i\| = 1$. (ie $\underline{e}_i$ is unit vector)

Any vector $\underline{x} \in V$ can be expressed as

$$\boxed{\underline{x} = \sum_{i=1}^{n} \lambda_i \underline{e}_i}$$

where $\boxed{\lambda_i = \underline{x} \cdot \underline{e}_i}$

Fourier series will share these properties in that

- We consider an infinite dimensional vector space of "well behaved" function $f$.

- A basis for this space is formed by the family $F$ of standard functions:

$$F = \{ 1, \sin(x), \sin(2x) \ldots \\ \cos(x), \cos(2x), \ldots \}$$

- Any 'vector'/function $f$ from the space can be expressed a linear combination of the basis functions

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

- The coefficients $a_0, a_n, b_n, n \geq 1$

will be given by formulae involving the standard inner product generalises the dot product for functions defined on the interval $(-\pi, \pi)$, so for functions $\phi, \psi : (-\pi, \pi) \to \mathbb{R}$ the inner-product is

$$< \phi, \psi > = \int_{-\pi}^{\pi} \phi(x)\, \psi(x)\, dx$$

So $a_n \sim\!\sim\!\sim \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$

$b_n \sim\!\sim\!\sim \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$

$a_0 \sim\!\sim\!\sim \int_{-\pi}^{\pi} f(x)\, dx.$

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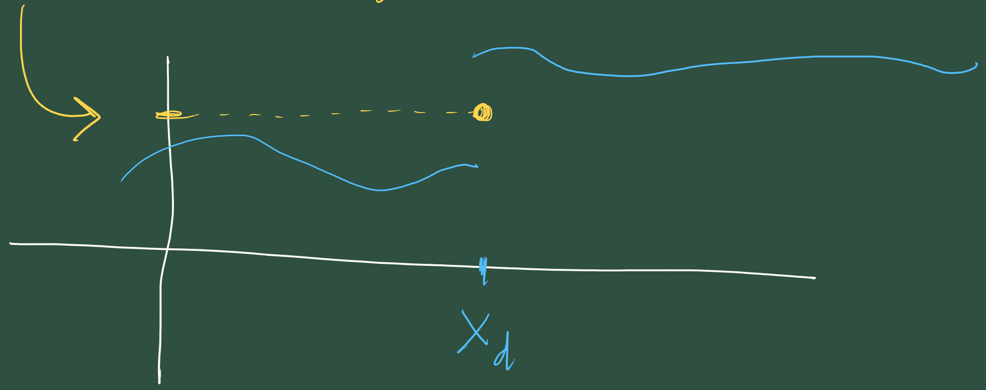
Th. 5.1.1. describes the "well behaved" condition for the input functions $f$.

value taken by the Fourier series at $x_d$.




F is orthogonal w.r.t. $\langle \, , \rangle$

means:

$$\forall \phi, \psi \in F, \quad \phi \neq \psi \Rightarrow \langle \phi, \psi \rangle = 0$$

There will be a few different cases of this to check.

<u>Ex 5.1.2</u>

eg. consider $\phi(x) = \cos(mx)$

$\psi(x) = \sin(nx)$

Prove $\langle \phi, \psi \rangle = 0$

<u>Proof</u> $\langle \cos(mx), \sin(nx) \rangle$

$$= \int_{-\pi}^{\pi} \cos(mx) \sin(nx) \, dx$$

use the trig. formula.

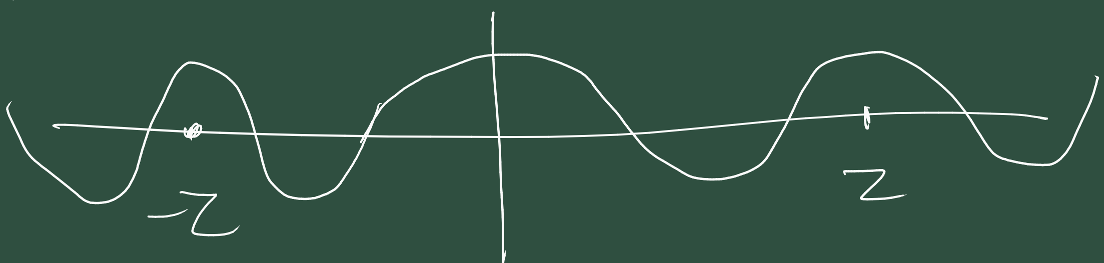
$$\cos(A)\sin(B) = \frac{1}{2}\left(\sin(A+B) + \sin(B-A)\right)$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left(\sin\left((m+n)x\right) + \sin\left((n-m)x\right)\right)dx$$

Assuming $n \neq m$

$$= \left[\frac{-\cos\left((m+n)x\right)}{m+n} - \frac{\cos\left((n-m)x\right)}{n-m}\right]_{-\pi}^{\pi}$$

Remember cosine is an <u>even</u> function, ie. $\forall z \ \cos(-z) = \cos(z)$




$= 0$, as required.

When $n=m$

$$\langle\ ,\ \rangle = \frac{1}{2}\int_{-\pi}^{\pi} \sin(2mx)\,dx$$

$= 0$, for the same reason as above.

Other cases of orthogonality to check are

$\langle \cos(nx), \cos(mx)\rangle$

$= \langle \sin(nx), \sin(mx)\rangle$

$= \langle 1, \cos(nx)\rangle$

$= \langle 1, \sin(nx)\rangle = \underline{\underline{0}}$

Ex 5.1.3

$\langle \sin(nx), \sin(nx)\rangle$

$$= \int_{-\pi}^{\pi} \sin^2(nx)\,dx$$

use $\sin^2(A) = \frac{1}{2}\left(1 - \cos(2A)\right)$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos(2nx)\, dx$$

$$= \frac{1}{2}\left[ x - \frac{\sin(2nx)}{2n} \right]_{-\pi}^{\pi}$$

remember $\sin(a\pi) = 0$
for any $a \in \mathbb{Z}$.

$$= \frac{1}{2} 2\pi = \pi$$

Can also check that

$$\langle \cos(nx), \cos(nx) \rangle = \pi$$

and

$$\langle 1, 1 \rangle = 2\pi.$$

Pause until 5:08.

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Ex 5.2.2.1

Assuming the existence of the Fourier series we can justify these coefficient formulae.

eg.

$$< f(x), \sin(mx) >$$

$$= < \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx), \sin(mx) >$$

Using the fact that the inner-product $< , >$ is linear, we can express this as

$$= \frac{1}{2}a_0 < 1, \sin(mx) > + \sum_{n=1}^{\infty} a_n < \cos(nx), \sin(mx) >$$

$$+ \sum_{n=1}^{\infty} b_n < \sin(nx), \sin(mx) >$$

Almost all these inner products are zero, by orthogonality of $\mathcal{F}$.

$$= b_m < \sin(mx), \sin(mx) >$$

$$= b_m \pi, \text{ from above.}$$

$$\Rightarrow b_m = \frac{1}{\pi} < f(x), \sin(mx) >$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx$$

Using similar argument we can.

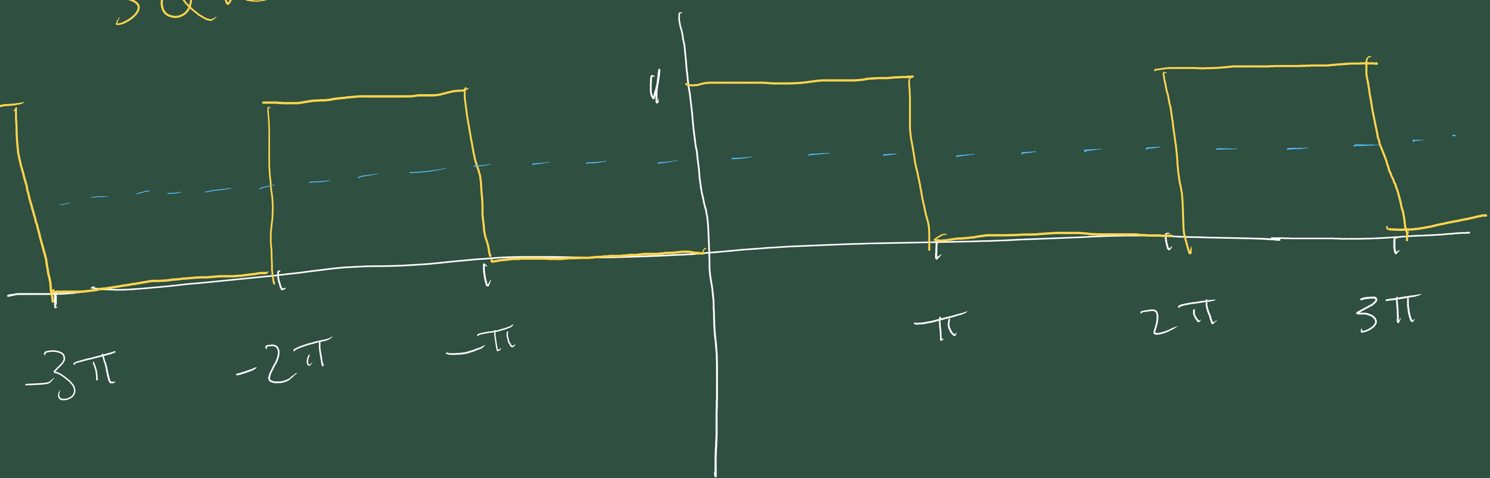
establish formulae for $a_0$, $a_n$.

Can now apply all of this and obtain our first Fourier series.

Consider the square-wave function.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

Square-wave.




Finding the Fourier Series means finding $a_0$, $a_n$, $b_n$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} dx = \frac{1}{\pi}\left[x\right]_0^{\pi}$$

$$= \frac{1}{\pi}\pi = 1.$$

For $n \geq 1$.

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)\,dx. \quad , \text{ due to } f(x)=\begin{cases}0\\1\end{cases}$$

$$= \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} \boxed{= 0}$$

For $n \geq 1$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}$$

$$= \frac{1}{n\pi}\left[-\cos(n\pi) + \cos(0)\right].$$

$$= \frac{1}{n\pi}\left[-\cos(n\pi) + 1\right].$$

$$= \boxed{\begin{cases} 0, & \text{for } n \text{ even} \\ \dfrac{2}{n\pi}, & \text{for } n \text{ odd} \end{cases}}$$

So the Fourier series will be.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n\pi} \sin(nx)$$

$$= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2}{(2m-1)\pi} \sin((2m-1)x)$$

But let's look at it.

Notice the "Gibb's phenomena" at the point $x=0$ of discontinuity.