Taylor series

Recall the 1-variable case.

Given a nice function $f(x)$ (continuous and differentiable) it will have a Taylor series, let's say based at point $x=0$, given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)} = \frac{d^n f}{dx^n}$ "n^{th} derivative of f"

valid for some interval around 0

(also sometimes called Mclaurin series in the case of base point $=0$).

Finite initial sums of the series

$$f(x) \approx \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n$$

provide approximations to f, getting better and better as $k \to \infty$.

What about multi-variable functions?

Consider $f(x, y)$ and its behaviour near a base point (a, b)

we'll write $h = \Delta x$, i.e. for small changes in x-variable and $k = \Delta y$. Then the Taylor series for f about (a, b) will be.

$$f(a+h, b+k) = \sum_{n=0}^{\infty} \frac{1}{n!} (D^n f)(a, b)$$

where D is the differential operator

$$D = h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}$$

and by $D^n f = D(D(D(\ldots D(f))))$

$$\underbrace{D(D(D(\ldots D}_{n \text{ copies of } D \text{ applied in composition}}(f))))$$

so $D^0 f = f$

$$Df = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$$

$$D^2 f = D(Df)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x\partial y}$$

$\vdots$

and in general

$$D^n f = \sum_{j=0}^{n}\binom{n}{j} h^j k^{n-j}\frac{\partial^n f}{\partial x^j\,\partial y^{n-j}}$$

where $\binom{n}{j}$ are the binomial coefficients. "n choose j" given by

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$$

<u>Example</u> Let $f(x,y) = \sin(x+3y) + \cos(3x+y)$.

Find the beginning of the Taylor series for f around $(\pi/2, 0) = (a,b)$ to get an approximation for

$f(\pi/2 + h, k) \approx$ poly in h, k.

$$(D^0 f)(\pi/2, 0) = f(\pi/2, 0) = 1.$$

$$(D^1 f)(\pi/2, 0) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(\pi/2, 0)}$$

$$= \big(\cos(x+3y) - 3\sin(3x+y)\big)_{\pi/2, 0}\, h + \big(3\cos(x+3y) - \sin(3x+y)\big)_{(\pi/2, 0)}\, k$$

$$= 3h + k.$$

$$(D^2 f) = h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x\partial y}.$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+3y) - 9\cos(3x+y)$$

at $(\pi/2, 0) = -1$

$$\frac{\partial^2 f}{\partial y^2} = -9\sin(x+3y) - \cos(3x+y)$$

$$\text{at } (\pi/2, 0) = -9$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3\sin(x+3y) - 3\cos(3x+y)$$

$$\text{at } (\pi/2, 0) = -3$$

So $(D^2 f)(\pi/2, 0) = -h^2 - 9k^2 - 6hk$

Stopping here we get the poly. approximation

$$f(\pi/2 + h, k) \approx 1 + 3h + k - \frac{1}{2}h^2 - \frac{9}{2}k^2 - 3hk$$

This will all generalize to 3 or more variables in the "obvious" way.

say $f(x_1, \dots, x_m)$

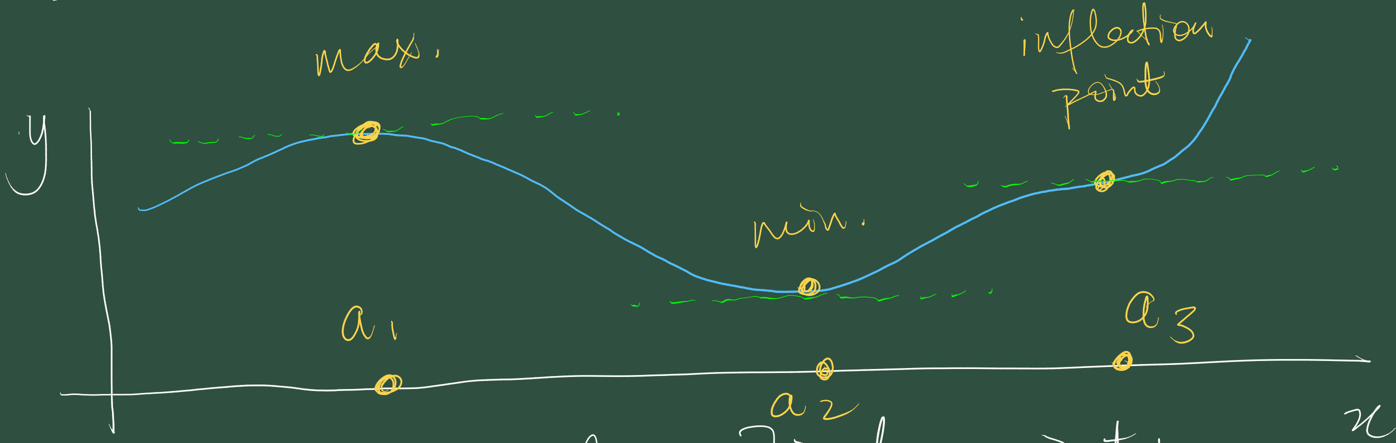
then $f(a_1 + h_1, a_2 + h_2, \dots, a_m + h_m)$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (D^n f)(a_1, \dots, a_m)$$

where $$D = \sum_{j=1}^{m} h_j \frac{\partial}{\partial x_j}$$

Optimisation. Finding and classifying critical points on a surface defined by $f(x,y)$.

1-variable case $\quad y = f(x)$



Three types of critical point: local maximum, local minimum and inflection point, all characterised

$$\frac{df}{dx} = 0 \quad \text{at } x = a_1, a_2, a_3.$$

and were classified by $\frac{d^2 f}{dx^2}$ according to.

$$\left.\frac{d^2f}{dx^2}\right|_{a_1} < 0 \qquad \text{local maximum}$$

$$\left.\frac{d^2f}{dx^2}\right|_{a_2} > 0 \qquad \text{local minimum.}$$

$$\left.\frac{d^2f}{dx^2}\right|_{a_3} = 0 \Leftarrow \text{at an inflection pt.}$$

3rd one is not an equivalence.

For 2-variable functions $z = f(x,y)$.

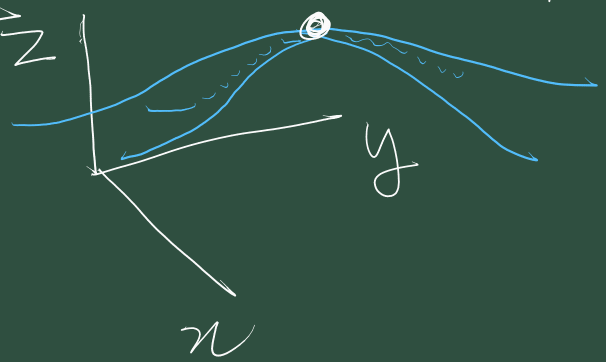
<u>Def</u> a critical point of f is a point (a,b) where both partial derivatives vanish, ie.

$$\frac{\partial f}{\partial x}(a,b) = 0 \quad \& \quad \frac{\partial f}{\partial y}(a,b) = 0$$

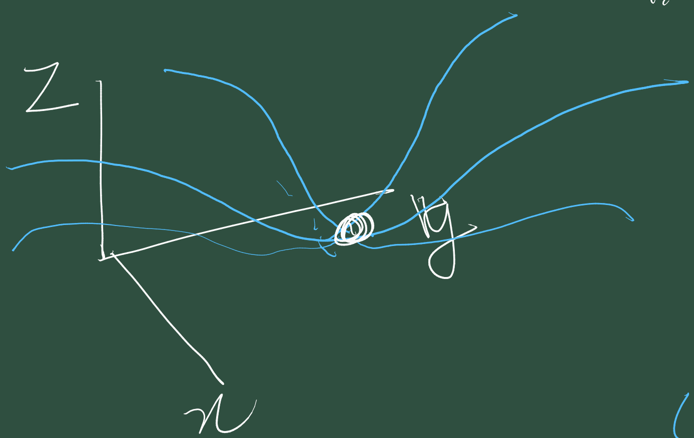
So geometrically the tangent plane to surface defined by $z = f(x, y)$ is horizontal at a critical point.

Classified into three types

<u>local max</u> "top of a hill"

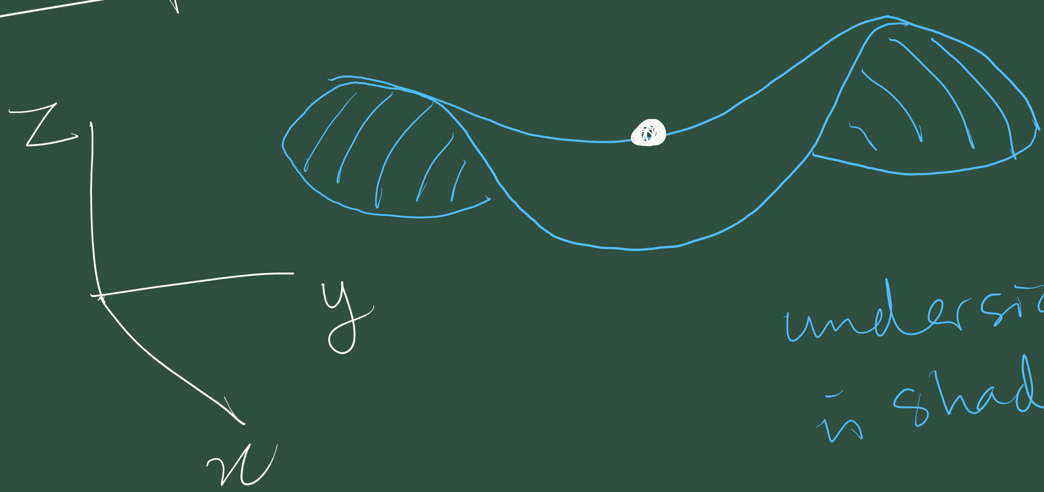


<u>local minimum</u> "bottom of a bowl"



<u>saddle point</u>

centre of a horse's back.



underside is shaded.

An algebraic classification is provided by the second order partial derivatives.

The Hessian determinant D is the det. of the 2×2 matrix of 2^{nd} order partial derivs.

$$D = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

$$= \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

Classification is:

If $D(a,b) < 0$ then (a,b) is a saddle.

If $D(a,b) > 0$ then $\begin{cases} \left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(a,b)} > 0 & \text{local min.} \\ \left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(a,b)} < 0 & \text{local max.} \end{cases}$

This classification can be justified by analysing the 2nd order Taylor poly.

Example

Consider the poly. function $f(x,y) = x^3 + 3xy^2 - 15x - 12y$. Find & classify its critical points.

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 15$$

$$\frac{\partial f}{\partial y} = 6xy - 12$$

Consider sols to $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ simultaneous.

General tip: Solve the simpler equation first and impose its conditions on the other to find simultaneous solutions.

$$\frac{\partial f}{\partial y}=0 \iff 6xy-12=0.$$

$$\iff xy-2=0$$

$$\iff \boxed{y=2/x}$$

(provided $x\neq 0$, but if $x=0$ does not lead to critical points since $\left.\frac{\partial f}{\partial y}\right|_{x=0}=-12\neq 0$)

Now examine $\frac{\partial f}{\partial x}=0$ under this condition.

$$\frac{\partial f}{\partial x}=0 \iff 3x^2+3y^2-15=0$$

$$\iff 3x^2+\frac{12}{x^2}-15=0$$

$$\iff x^4-5x^2+4=0$$

$$\iff (x^2-4)(x^2-1)=0$$

$$\iff x^2-4=0 \ \underline{\text{OR}}\ x^2-1=0$$

$$\iff x=-2 \text{ or } 2 \text{ or } -1 \text{ or } 1.$$

Combining the above will generate four critical points. $y = 2/x$

$(a,b) = (-2,-1), (2,1), (-1,-2),$

$(1,2)$

Classified by D.

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$= 6x \cdot 6x - (6y)^2$$

$$= 36x^2 - 36y^2.$$

$$= 36(x^2 - y^2).$$

$$D(-2,-1) = D(2,1) = 108 > 0$$

$$\frac{\partial^2 f}{\partial x^2} = 6x = \begin{cases} -12 \text{ at } (-2,-1) \text{ a } \underline{\text{max.}} \\ 12 \text{ at } (2,1) \text{ a } \underline{\text{min}} \end{cases}$$

$$D(1,2) = D(-1,-2) = -108 < 0$$

So $(1,2)$ and $(-1,2)$ are both saddles.

Let's "see this" on Matlab.