Laplace Transforms

Following chapter 4 in CLP3.

Ex 4.1.3.

$$\frac{d^2y}{dt^2} = 0.$$

Solve this subject to the boundary conditions $y(0)=1, \; y(1)=2$.

Solution

$$\frac{d^2y}{dt^2} = 0$$

$$\Rightarrow \frac{dy}{dt} = c, \text{ a constant.}$$

$$\Rightarrow y(t) = ct + d, \quad d \text{ a constant.}$$

The parameters c, d are determined by given boundary conditions

$y(0) = 1 \Rightarrow \quad c \cdot 0 + d = 1 \Rightarrow d = 1$

$y(1) = 2 \Rightarrow \quad c \cdot 1 + 1 = 2 \Rightarrow c = 1$

So the unique solution

is $\quad y(t) = t + 1.$

Some general remarks about the concept of "transforming" problems.

<u>Hard</u> problems in one domain might be <u>transformed</u> to <u>easier</u> problems in another domain.

Solve the problem there

then apply an <u>inverse transform</u> to recover the solution to the original.

Eg. Arithmetic computations

Hard $\xrightarrow{\log}$ Easier

a^b $\qquad$ $\log(a^b) = b \cdot \log(a)$

ab $\qquad$ $\log(ab) = \log(a) + \log(b)$

exp.

Def Laplace transform

$$\mathcal{L}\{f(t)\} = \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\text{function of the new variable } s}.$$

other notations.

$$\mathcal{L}\{f(t)\} = F(s) = \bar{f}(s)$$

Another 'transform' you might recall

Power series / Taylor series.

↳ takes an infinite sequence of coefficients $a_0, a_1, a_2, \ldots$

and builds an associated function.

$$A(x) = \sum_{i=0}^{\infty} a_i x^i$$

Can view a sequence $a_0, a_1, a_2, \ldots$

as a function $f: \mathbb{Z}^{\geq 0} \to \mathbb{R}$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

$$(e^{-s})^t$$

Can regard the Laplace transform as "kind of" a continuous version of the power series transform.

Ex 4.2.2 Consider the constant function f, defined by

$$f(t) = 1, \quad \text{for } t \geq 0.$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\}$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty.$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^{t\to\infty}.$$

$$= -\frac{1}{s}\left[\lim_{t\to\infty}\left(e^{-st}\right) - 1\right].$$

$$= -\frac{1}{s}(0-1), \text{ provided } s>0$$

$$= \frac{1}{s}, \text{ provided } s>0.$$

In notes.

Ex 4.2.3.

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \text{ provided } s>0$$

Ex 4.2.4

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}.$$

for $s-a>0$

ie. $s>a$.

Note: misprint in statement of 4.2.4.

Properties of the transform

Theorem 4.4.1

$\mathcal{L}\{\alpha f + \beta g\}$

$= \int_0^\infty e^{-st} (\alpha f + \beta g)(t)\, dt.$

$= \int_0^\infty e^{-st} (\alpha f(t) + \beta g(t))\, dt.$

$= \alpha \int_0^\infty e^{-st} f(t)\, dt$

$+ \beta \int_0^\infty e^{-st} g(t)\, dt.$

$= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}.$

Ex 4.4.4.

We know $\mathcal{L}\{t\} = \frac{1}{s^2}$

So $\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\}$

$= -\frac{d}{ds}\left(\frac{1}{s^2}\right)$

$= -\left(-2/s^3\right)$

$$= 2/s^3$$

Keep repeating this for $\mathcal{L}\{t^3\}, \mathcal{L}\{t^4\}, \ldots$ to get

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Corollary 4.4.6.

$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} = \mathcal{L}\left\{\frac{d}{dt}\left(\frac{dx}{dt}\right)\right\}$$

$$= s\mathcal{L}\left\{\frac{dx}{dt}\right\} - \left.\frac{dx}{dt}\right|_{t=0}$$

$$= s\left(s\mathcal{L}\{x(t)\} - x(0)\right) - \left.\frac{dx}{dt}\right|_{t=0}$$

$$= s^2\mathcal{L}\{x(t)\} - sx(0) - \left.\frac{dx}{dt}\right|_{t=0}.$$

Keep applying this to get Cor 4.4.6.

We'll use the table to apply the transform and it's inverse. noting that $\mathcal{L}^{-1}$ will be linear also.

<u>Ex 4.5.1</u> In simpler notation. $\ddot{y} = \frac{d^2y}{dt^2}$

$\dot{y} = \frac{dy}{dt}$.

$$\ddot{y} - 5\dot{y} + 4y = 12$$

subject to
$y(0) = \dot{y}(0) = 0$

$$\mathcal{L}\{\ddot{y} - 5\dot{y} + 4y\} = \mathcal{L}\{12\}$$

$$\mathcal{L}\{\ddot{y}\} - 5\mathcal{L}\{\dot{y}\} + 4\mathcal{L}\{y\} = 12\mathcal{L}\{1\}$$

, by linarity.

Use transform properties and write

$\bar{y} = \mathcal{L}\{y\}$

$$s^2\bar{y} - sy(0) - \dot{y}(0) - 5\left(s\bar{y} - y(0)\right)$$

$$+ 4\bar{y} = \frac{12}{s}$$

Put in initial conditions.

$$s^2\bar{y} - 5s\bar{y} + 4\bar{y} = \frac{12}{s}$$

$$(s^2 - 5s + 4)\,\bar{y} = \frac{12}{s}$$

$$\Rightarrow \quad \bar{y}(s) = \frac{12}{s(s^2-5s+4)} \qquad (*)$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{\bar{y}\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{12}{s(s^2-5s+4)} \right\}$$

$$= \, ?$$

In order to use results from table we first need to expand $\bar{y}$ into its partial fraction expansion.

$$\bar{y} = \frac{12}{s(s-4)(s-1)}$$

$$= \frac{\alpha}{s} + \frac{\beta}{s-4} + \frac{\gamma}{s-1} \quad , \text{ by partial fractions}$$

$$= \frac{\alpha(s^2-5s+4)+\beta s(s-1)+\gamma s(s-4)}{s(s-4)(s-1)}$$

$$= \frac{(\alpha+\beta+\gamma)s^2+(-5\alpha-\beta-4\gamma)s+4\alpha}{s(s-4)(s-1)}$$

Comparing numerators from first and last expressions for $\bar{y}$

$$\Rightarrow \begin{cases} 4\alpha = 12 \\ -5\alpha-\beta-4\gamma = 0 \\ \alpha+\beta+\gamma = 0 \end{cases}$$

$$\Rightarrow \quad \alpha = 3$$

$$\beta+\gamma = -3$$

$$-(\beta+4\gamma) = 15$$

$$-3\gamma = 12$$

$$\Rightarrow \gamma = -4$$

$$\Rightarrow \beta = 1.$$

So $\quad \bar{y} = \frac{3}{s} + \frac{1}{s-4} - \frac{4}{s-1}.$

So $\boxed{y(t)} = \mathcal{L}^{-1}\{\bar{y}\}$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$-4\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

, by linearity of $\mathcal{L}^{-1}$

$$\boxed{= 3 + e^{4t} - 4e^{t}}$$

the solution to the original ODE.