5.4 Odd and even functions

Def 5.4.1

$f : \mathbb{R} \to \mathbb{R}$ is called <u>even</u> iff

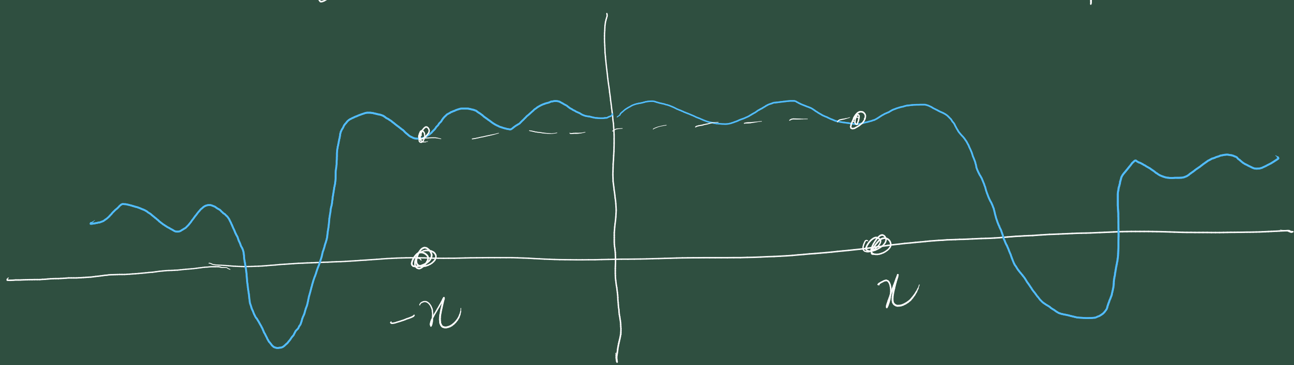
$$\forall x \in \mathbb{R} \quad f(-x) = f(x)$$

$f : \mathbb{R} \to \mathbb{R}$ is called <u>odd</u> iff

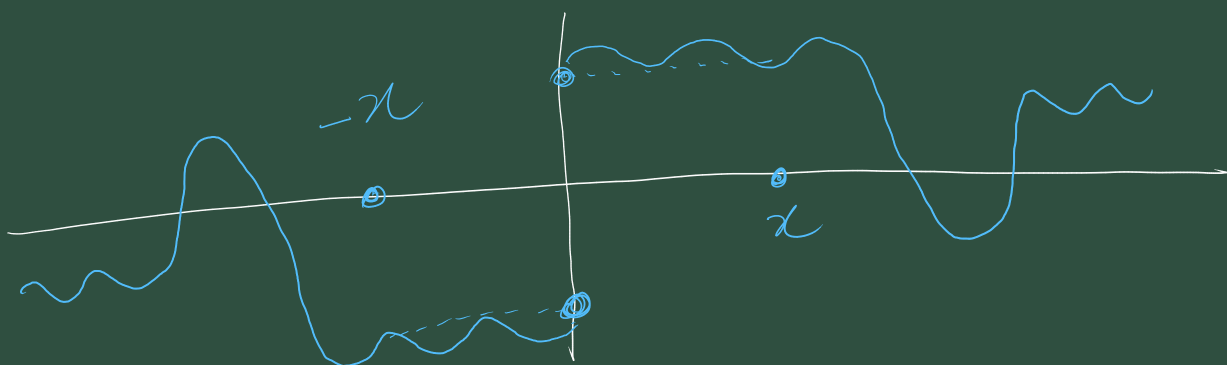
$$\forall x \in \mathbb{R} \quad f(-x) = -f(x)$$

Have very characteristic behaviour in their graphs.

<u>f even</u>



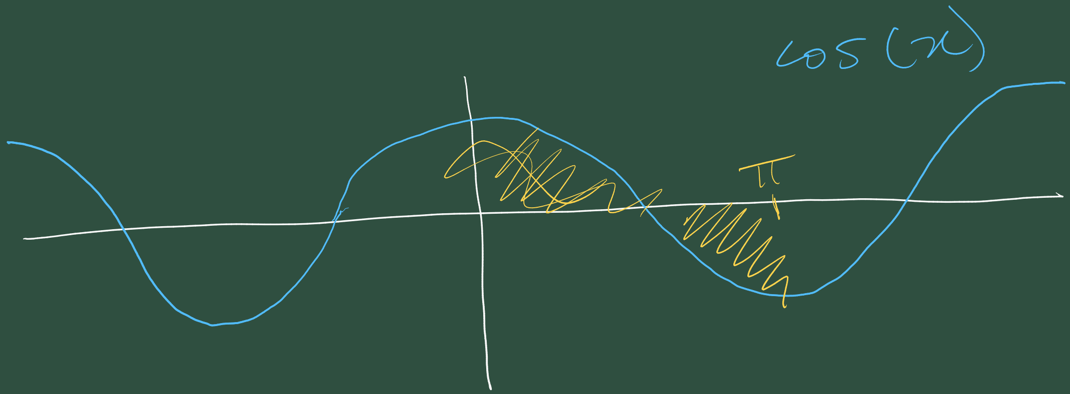
<u>f odd</u>

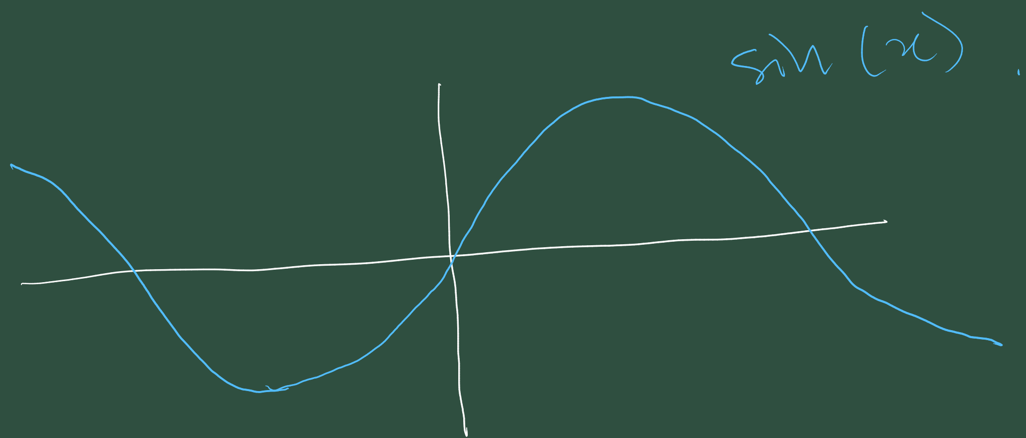


odd/even properties will affect Fourier series because

- sine is an odd function
  cosine is an even function

- integrals of odd/even functions have characteristic behaviour over intervals centred on origin 0.





odd/even functions have a certain multiplicative behaviour.

Theorem 5.4.3

1. even × even = even

2. odd × odd = even

3. odd × even = odd.

<u>Proofs</u> Just keep careful track of minus signs.

1. Let g_1, g_2 be even functions.

$\forall x \in \mathbb{R}$.

$$(g_1 g_2)(-x) = g_1(-x)\, g_2(-x),$$

$$= g_1(x)\, g_2(x)$$

$$= (g_1 g_2)(x).$$

Hence $g_1 g_2$ is even.

<u>3.</u> Let f be odd and g be even

For any $x \in \mathbb{R}$

$$(fg)(-x) = f(-x)\, g(-x).$$

$$= -f(x)\, g(x),$$ since f is odd g is even.

$$= -(fg)(x)$$

Hence fg is odd. ∎

Theorem 5.4.4

Let g be even and h odd.

Then:

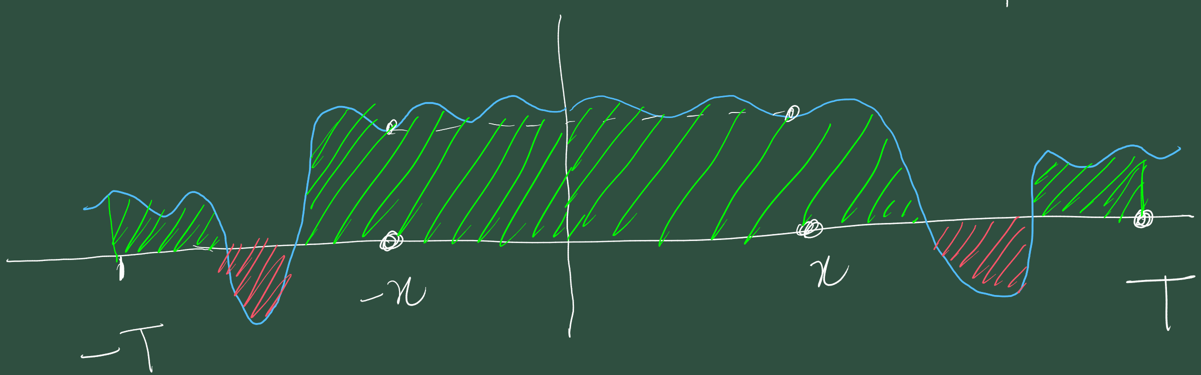
- $$\int_{-T}^{T} g(x)\,dx = 2\int_0^T g(x)\,dx.$$

- $$\int_{-T}^{T} h(x)\,dx = 0$$

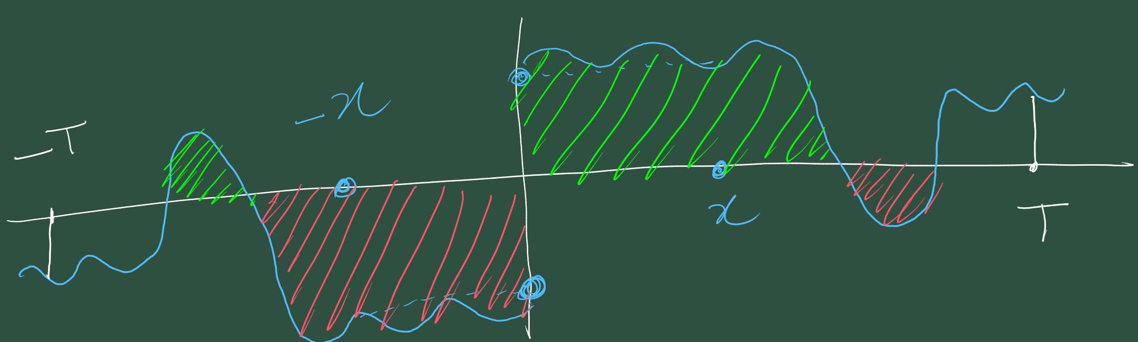
Proof (two diagrams)

In their

f even



f odd



More formally "just keep track of minus signs"

When h is odd.

$$\int_{-T}^{T} h(x)\,dx = \int_{-T}^{0} h(x)\,dx + \int_{0}^{T} h(x)\,dx.$$

use $x = -t$, $dx = -dt$.

$$= \int_{T}^{0} -h(-t)\,dt + \int_{0}^{T} h(x)\,dx.$$

$$= \int_{T}^{0} -(-h(t))\,dt + \int_{0}^{T} h(x)\,dx, \text{ as } h \text{ is odd}$$

$$= \int_{T}^{0} h(t)\,dt + \int_{0}^{T} h(x)\,dx.$$

$$= -\int_{0}^{T} h(t)\,dt + \int_{0}^{T} h(x)\,dx.$$

$= 0$, as both integrals are the same.

For the Fourier series.

Th 5.4.5. g even $\Rightarrow$ $\forall n \geq 1$ $b_n = 0$

Th 5.4.6. h odd $\Rightarrow$ $\forall n \geq 0$ $a_n = 0$

Proof of 5.4.5.

If g is even then for all $n \geq 1$.

$g(x)\sin(nx)$ will be an odd function, as even $\times$ odd $=$ odd

$$\Rightarrow b_n = \int_{-\pi}^{\pi} \underbrace{g(x)\sin(nx)}_{\text{odd}}\,dx = 0$$

and $\forall\, n \geq 1$
$g(x)\cos(nx)$ is an even function.

For $n \geq 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{g(x)\cos(nx)}_{\text{even.}}\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} g(x)\cos(nx)\,dx.$$

For Th 5.4.6. Apply similar reasoning in the case that h is odd.

Let's apply all this now to finding Fourier series for function

$$f(x) = x^2.$$

over $(-\pi, \pi)$.

Example 5.47.

The function f is even, since

$$f(-x) = (-x)^2 = x^2 = f(x)$$

So it's Fourier series will have no sine terms as $\forall n \geq 1$ $b_n = 0$.

So $$x^2 = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

for some $a_0, a_1, a_2, \dots$

$$\boxed{a_0} = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2\, dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \frac{\pi^3}{3}$$

$$\boxed{= \frac{2}{3} \pi^2.}$$

For $n \geq 1$.

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx)\, dx.$$

need integration by parts.

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \frac{d}{dx}\left(\frac{\sin(nx)}{n} \right) dx.$$

$$= \frac{2}{\pi}\left[\left[\frac{x^2 \sin(nx)}{n} \right]_0^{\pi} - \int_0^{\pi} 2x \frac{\sin(nx)}{n}\, dx \right]$$

$$= \frac{-4}{n\pi} \int_0^{\pi} x \sin(nx)\, dx.$$

int. by parts again.

$$= \frac{-4}{n\pi} \int_0^{\pi} x \frac{d}{dx}\left(-\frac{\cos(nx)}{n} \right) dx.$$

$$= \frac{-4}{n\pi}\left\{ \left[-\frac{x\cos(nx)}{n} \right]_0^{\pi} - \underbrace{\int_0^{\pi} -\frac{\cos(nx)}{n}\, dx}_{=0.} \right\}$$

$$= \frac{-4}{n^2\pi}\left(-\pi \cos(n\pi) \right)$$

$$= \boxed{\frac{4}{n^2}(-1)^n = a_n.}$$

$b_n = 0$

$a_0 = \frac{2}{3}\pi^2.$

So the Fourier series will be.

$$\boxed{x^2 = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx).}$$

An interesting result comes from this by evaluating it at $x=0$.

$$0 = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}, \quad \text{since } \cos(0)=1.$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$$= 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots + \frac{(-1)^{n+1}}{n^2}$$

Gives $\frac{\pi^2}{12}$ as the alternating sum of the reciprocals of the squares

Also, recall the Fourier series for square wave from last week.

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left((2n-1)x\right)$$

where $f(x) = \begin{cases} 0 , & -\pi < x \le 0 \\ 1 , & 0 < x < \pi \end{cases}$

If we evaluate this at $x = \frac{\pi}{2}$ we get

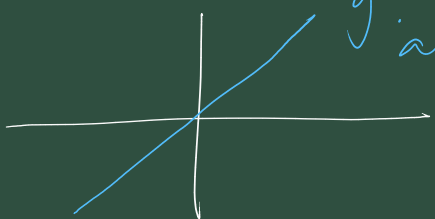
$$f\left(\tfrac{\pi}{2}\right) = 1 = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} (-1)^{n+1}$$

$$\Rightarrow \quad \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

the alternating sum of the reciprocals of the odd numbers.

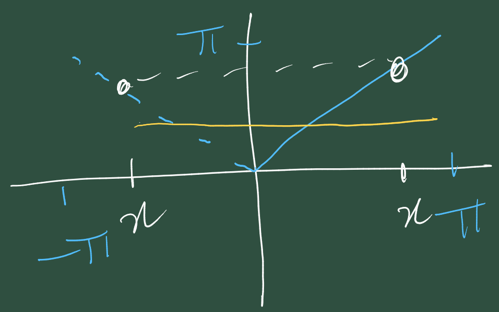
<u>Example 5.4.8</u> Consider $g(x) = x$.



g an odd function.

We want a cosine series for $g(x)$ over $(0, \pi)$.

We do this by altering the definition of g, on $(-\pi, 0)$.



$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

on $(0, \pi)$

for $x \in (-\pi, 0)$ define $g(x) := g(-x)$.

So in other words $g(x) = |x|$ on $(-\pi, \pi)$

which is an <u>even function</u> with a cosine series. (i.e. $b_n = 0$).

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx.$$

$$= \frac{2}{\pi} \left. \frac{x^2}{2} \right|_0^{\pi} = \frac{2}{\pi} \frac{\pi^2}{2} = \pi.$$

For $n \geq 1$.

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) \, dx.$$

int. by parts.

$$= \frac{2}{\pi} \int_0^{\pi} x \frac{d}{dx} \left(\frac{\sin(nx)}{n} \right) dx.$$

$$= \frac{2}{\pi}\left[\underbrace{\left[\frac{x\sin(nx)}{n}\right]_0^\pi}_{=0} - \int_0^\pi \frac{\sin(nx)}{n}\,dx\right]$$

$$= \frac{-2}{n\pi}\int_0^\pi \sin(nx)\,dx$$

$$= \frac{-2}{n\pi}\left[\frac{-\cos(nx)}{n}\right]_0^\pi.$$

$$= \frac{-2}{n^2\pi}\left(-\cos(n\pi) + 1\right)$$

$$= \frac{-2}{n^2\pi}\left(-(-1)^n + 1\right).$$

$$= \frac{-2}{n^2\pi}\left((-1)^{n+1} + 1\right)$$

$$= \begin{cases} 0, & n \text{ even.} \\ \dfrac{-4}{n^2\pi}, & n \text{ odd.} \end{cases}$$

So the Fourier cosine series for $g(x) = x$, on $(0, \pi)$ is

$$x = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$= \frac{\pi}{2} + \sum_{m=1}^{\infty} \frac{-4}{(2m-1)^2 \pi} \cos\big((2m-1)x\big)$$

When evaluated at $x = 0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}$$

$$\Rightarrow \quad \frac{\pi^2}{8} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}.$$

$$= 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$$

Another very interesting infinite series formula involving π.