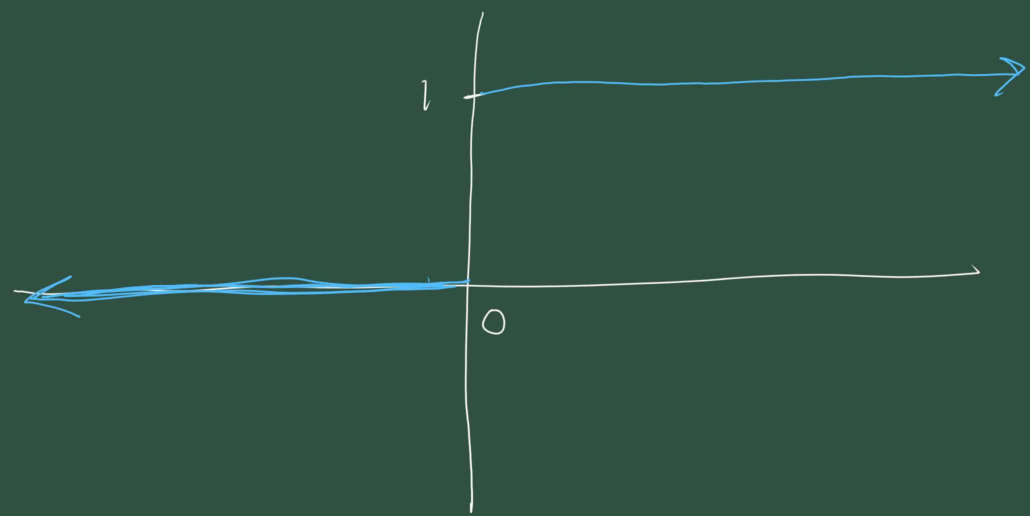
Dirac function, Heaviside step function

<u>Def</u>. The Heaviside step function H is the function defined by

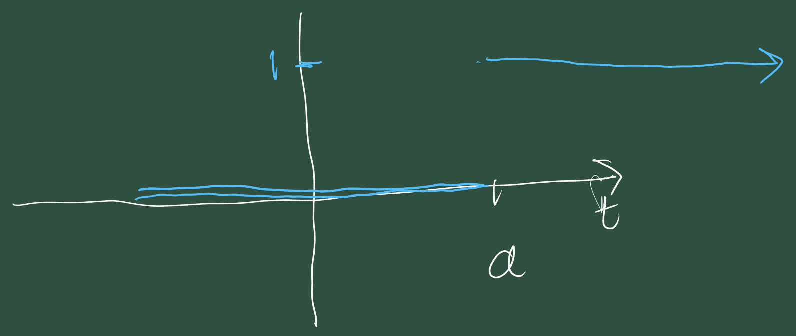
$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0 \end{cases}$$



The location of the step can be moved as in

Let $a > 0$.

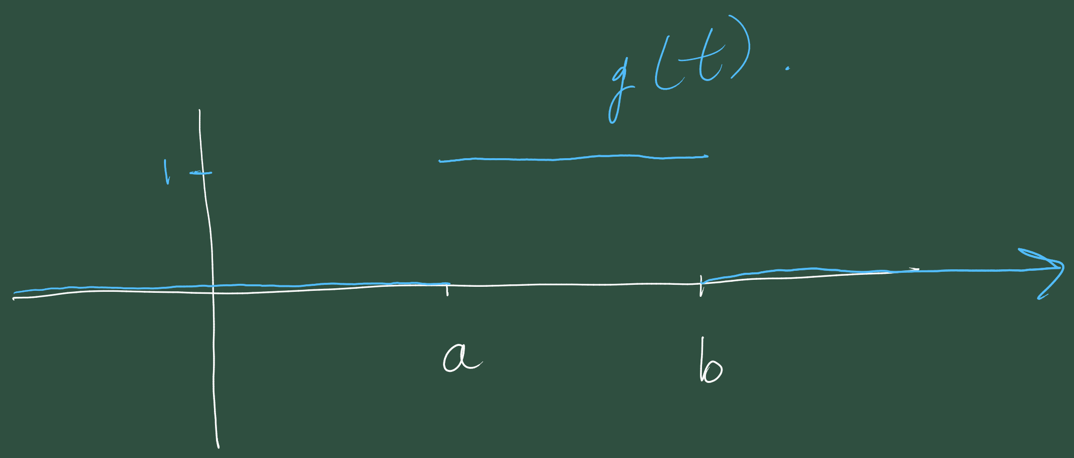
$$H(t-a) = \begin{cases} 0, & t-a < 0 \Leftrightarrow t < a \\ 1, & t-a \geqslant 0 \Leftrightarrow t \geqslant a \end{cases}$$



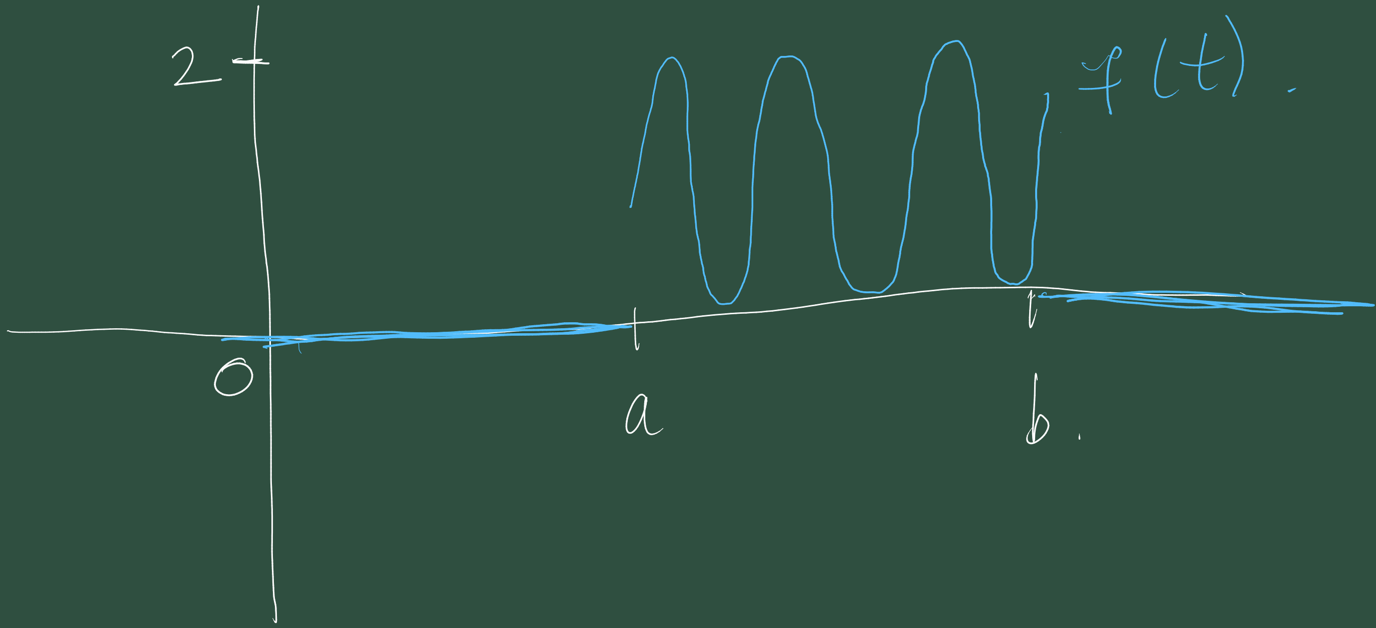
Ex 4.7.7.

1. Assume $0 < a < b$

$$g(t) = H(t-a) - H(t-b) = \begin{cases} 0, & t < a \\ 1, & a \le t < b \\ 0, & t \ge b \end{cases}$$

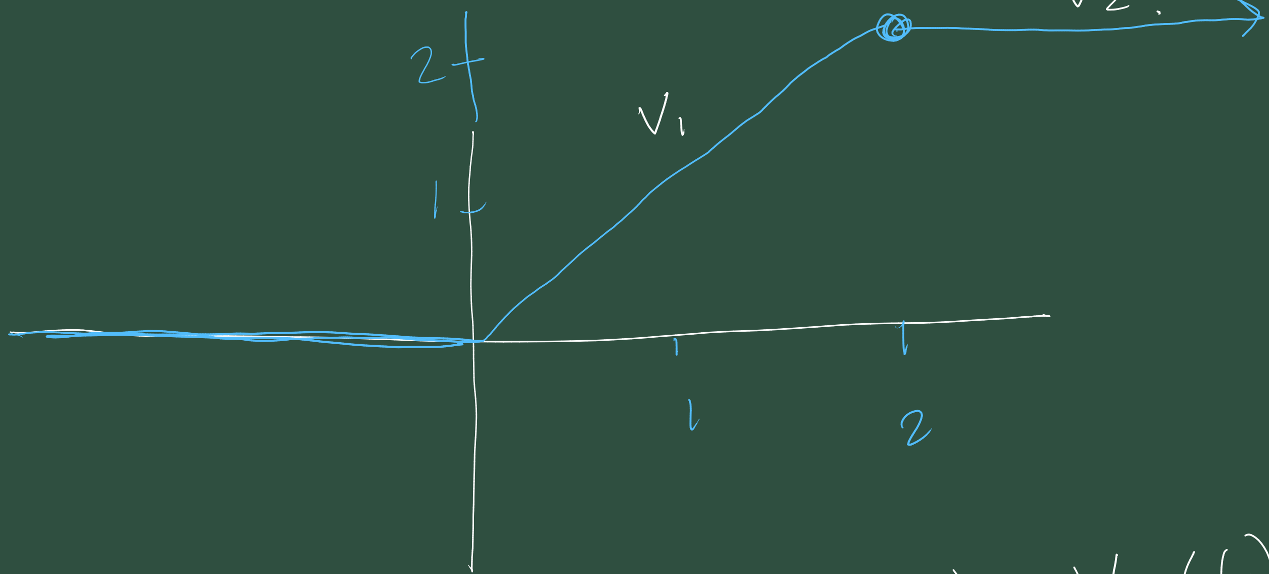


2.
$$f(t) = (1+\cos(t))\big(H(t-a) - H(t-b)\big)$$
$$= \begin{cases} 0, & t < a \\ 1+\cos(t), & a \le t < b \\ 0, & t \ge b \end{cases}$$



3. Consider the function V

$$V(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < 2 \\ 2, & t \ge 2. \end{cases}$$



Write V as $\quad V(t) = V_1(t) + V_2(t)$

$$V_1(t) = \begin{cases} t, & 0 \le t < 2 \\ 0, & \text{otherwise.} \end{cases}$$

$$V_2(t) = \begin{cases} 0, & t < 2. \\ 2, & t \geq 2. \end{cases}$$

From the definitions we can say.

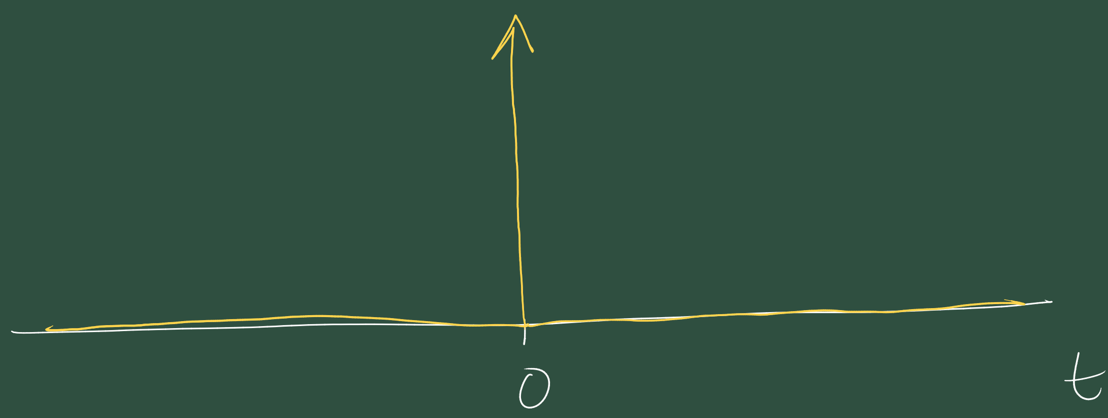
$$V_1(t) = t\left(H(t) - H(t-2)\right)$$

$$V_2(t) = 2H(t-2)$$

So $V(t) = tH(t) - (t-2)H(t-2).$

Dirac delta function.

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$



Shifting the argument to δ will shift the location of the spike.

so $\delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & t\neq a. \end{cases}$

Our aim is to be able to solve ODEs featuring these functions.

Th. 4.7.9

$$\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s}.$$

Proof By definition (assume $a>0$)

$$\mathcal{L}\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\,dt.$$

$$= \int_0^a \quad + \int_a^\infty$$

$$= \underbrace{\int_0^a e^{-st}\cdot 0\,dt}_{=0} + \int_a^\infty e^{-st}\,dt$$

$$= \int_a^\infty e^{-st}\, dt.$$

$$= \left[\frac{e^{-st}}{-s} \right]_{t=a}^{t \to \infty}$$

$$= 0 - \frac{e^{-sa}}{-s} \qquad , \text{ if } s > 0$$

$$= \frac{e^{-as}}{s}. \qquad \blacksquare$$

Theorem 4.7.10

If $\mathcal{L}\{f(t)\} = F(s)$. then (for $a > 0$).

$$\mathcal{L}\{f(t-a)\, H(t-a)\} = e^{-as} F(s).$$

<u>Proof</u>

$\mathcal{L}\{f(t-a)\, H(t-a)\}$

$$= \int_0^\infty e^{-st} f(t-a)\, H(t-a)\, dt.$$

$$= \int_0^a \cdots + \int_a^\infty \cdots.$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\, dt.$$

use the substitution.

$$t - a = u.$$

so $du = dt$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-sa} e^{-su} f(u)\,du$$

$$= e^{-sa} \underbrace{\int_0^\infty e^{-su} f(u)\,du}_{\text{def of } \mathcal{L}\{f\}}.$$

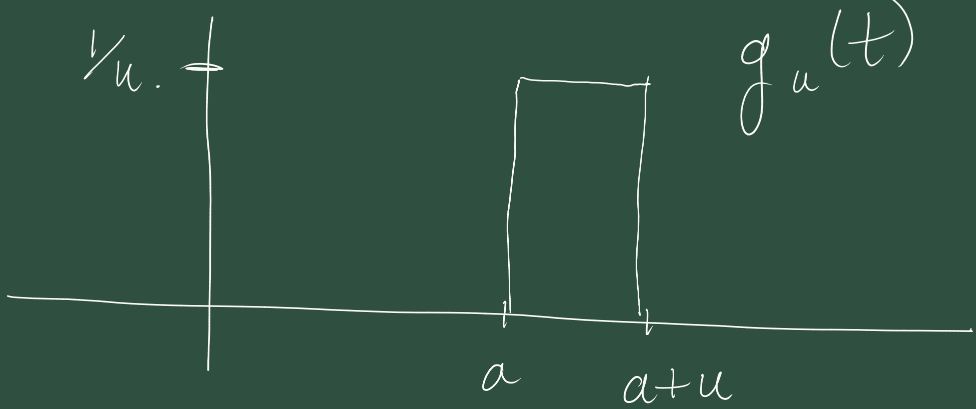
$$= e^{-sa} F(s)$$ ∎

<u>Th. 4.7.11</u>

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

<u>Proof</u> (Assume $a > 0$).

Idea



and "let $u \to 0$" $g_u(t) \to \delta(t-a)$

$$g_u(t) = \frac{1}{u}\left[H(t-a) - H(t-(a+u))\right]$$

So

$$\mathcal{L}\{g_u(t)\} = \frac{1}{u}\mathcal{L}\{H(t-a)\}$$

$$- \frac{1}{u}\mathcal{L}\{H(t-(a+u))\}$$

$$= \frac{1}{u}\frac{e^{-as}}{s} - \frac{1}{u}\frac{e^{-(a+u)s}}{s}$$

$$= \frac{e^{-as}}{us} - \frac{e^{-as}e^{-us}}{us}$$

$$= e^{-as}\left(\frac{1}{us}\left(1 - e^{-us}\right)\right)$$

$$\longrightarrow e^{-as}, \text{ as } u \longrightarrow 0$$

can see this by applying l'Hôpitals rule.

Ex 4.7.12

arguments are not shifted the same

1. $R(t) = t^2 H(t-2)$

$\mathcal{L}\{R(t)\} = ?$ In order to apply

Therem 4.7.10 we'll need to express R as

$R(t) = g(t-2)H(t-2)$

where $g(t-2) = t^2$ $(*)$

and then.

$\mathcal{L}\{R(t)\} = e^{-2s} G(s)$, where

$G(s) = \mathcal{L}\{g(t)\}$.

So what is $g(t) = ?$

From $*$ we see.

$$g(t) = g((t+2)-2)$$
$$= (t+2)^2, \text{ from } (*).$$
$$= t^2 + 4t + 4.$$

$$\text{So } G(s) = \mathcal{L}\{t^2 + 4t + 4\}$$
$$= \mathcal{L}\{t^2\} + 4\mathcal{L}\{t\} + 4\mathcal{L}\{1\}$$
$$= \frac{2}{s^3} + 4\frac{1}{s^2} + 4\frac{1}{s}.$$

So

$$\mathcal{L}\{R(t)\} = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right).$$

Ex 4.7.13.

Solve. $\ddot{y} - 4\dot{y} + 3y = 6H(t-5)$,

subject to $y(0) = \dot{y}(0) = 0$

Step 1. Transform the whole equation.

We write $\bar{y} = \mathcal{L}\{y(t)\}$

$$s^2\bar{y} - sy(0) - \dot{y}(0) - 4(s\bar{y} - y(0))$$
$$+ 3\bar{y} = 6\frac{e^{-5s}}{s}$$

2 Insert initial conditions $y(0) = \dot{y}(0) = 0$

$$s^2\bar{y} - 4s\bar{y} + 3\bar{y} = 6\frac{e^{-5s}}{s}$$

3. Solve this equation for $\bar{y}$.

$$(s^2 - 4s + 3)\bar{y} = 6\frac{e^{-5s}}{s}$$

$$\bar{y} = 6\frac{e^{-5s}}{s(s^2 - 4s + 3)}$$

$$= e^{-5s}\frac{6}{s(s-3)(s-1)}$$

So by recent theorem.

$$\boxed{y(t) = \mathcal{L}^{-1}\{\bar{y}\}}$$

$$\boxed{= f(t-5)\,H(t-5)}$$

where. $f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$

where $\bar{f}(s) = \dfrac{6}{s(s-3)(s-1)}$

find partial fraction expansion.

$$\bar{f}(s) = \frac{\alpha}{s} + \frac{\beta}{s-3} + \frac{\gamma}{s-1} \quad (**)$$

$$= \frac{\alpha(s^2-4s+3) + \beta s(s-1) + \gamma s(s-3)}{s(s-3)(s-1)}$$

$$= \frac{(\alpha+\beta+\gamma)s^2 + (-4\alpha-\beta-3\gamma)s + 3\alpha.}{s(s-3)(s-1)}$$

Comparing numerators we get.

$$3\alpha = 6 \Rightarrow \alpha = 2.$$

$$-4\alpha - \beta - 3\gamma = 0$$

$$\alpha + \beta + \gamma = 0$$

$$\Rightarrow \quad -\beta - 3\gamma = 8$$

$$\beta + \gamma = -2$$

Add: $-2\gamma = 6$

$$\gamma = -3$$

So $\beta = 1$.

Returning to $**$.

$$\bar{f}(s) = \frac{2}{s} + \frac{1}{s-3} + \frac{-3}{s-1}$$

So

$$\boxed{f(t)} = \mathcal{L}^{-1}\{\bar{f}(s)\}$$

$$= \boxed{2 + e^{3t} - 3e^{t}}.$$

So our solution to the ODE is.

$$y(t) = f(t-5)H(t-5)$$

$$= \left(2 + e^{3(t-5)} - 3e^{t-5}\right)H(t-5).$$

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2. Find the inverse transform of.

$$u(s) = e^{-3s} + \frac{e^{-3s}}{s^2+1}.$$

$$u(t) = \mathcal{L}^{-1}\{U(s)\}$$

$$= \delta(t-3) + f(t-3)\,H(t-3)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$$= \sin(t)$$

So

$$u(t) = \delta(t-3) + \sin(t-3)\,H(t-3).$$