

6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 9

Qs 1 – 4 on the existence of Fourier series and working with the inner product.

Qs 5 – 13 on finding Fourier series and working with odd and even functions.

- (1) Consider the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx.$$

Show that for every positive integer n ,

$$\langle 1, p_n \rangle = 0,$$

where p_n is the function defined by $p_n(x) = \sin(nx)$.

Also, using the formula

$$\cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

show that if m and n are positive integers then

$$\langle q_m, q_n \rangle = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases},$$

where q_r is the function defined by $q_r(x) = \cos(rx)$.

- (2) For each of the following definitions decide whether the function f will have a Fourier series in the interval $(-\pi, \pi)$. Justify your answers.

(a)

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x \leq \frac{-\pi}{2} \\ 0, & \text{if } \frac{-\pi}{2} < x \leq \frac{\pi}{2} \\ -1, & \text{if } \frac{\pi}{2} < x \leq \pi \end{cases}$$

(b) $f(x) = \cos\left(\frac{1}{x}\right)$

(c) $f(x) = 8x^4 - 8x^2 + 1$

(d) $f(x) = \tan\left(\frac{1}{x}\right)$

- (3) Sketch the graph of the function

$$f(x) = \begin{cases} 1+x, & \text{if } -\pi < x \leq 0 \\ 2+x, & \text{if } 0 < x \leq \pi \end{cases}.$$

What is the value of the Fourier series of this function when (a) $x = 1$ and (b) $x = 0$?

- (4) Given that the set of functions

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \dots\}$$

is an orthogonal set with respect to the inner product defined in question (1) above, and that the Fourier series for a function f is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

show that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

and

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx, \quad (m > 0).$$

- (5) Find the Fourier series for the function f which has period 2π and is defined by

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x \leq 0 \\ 1, & \text{if } 0 < x \leq \pi \end{cases}$$

- (6) Find the Fourier series for the function f , of period 2π , and defined by $f(x) = x$ for $x \in (-\pi, \pi)$.

- (7) Find the Fourier series for the function f which has period 2π and is defined by

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \leq 0 \\ x, & \text{if } 0 < x \leq \pi \end{cases}$$

- (8) For each of the following definitions determine whether the function f is odd or even.

(a) $f(x) = x^3$

(b) $f(x) = e^x$

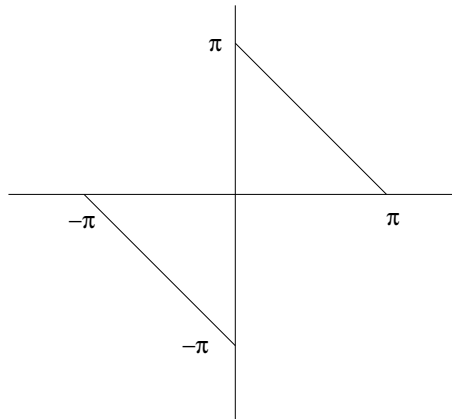
(c) $f(x) = e^{|x|}$

(d) $f(x) = x \cos x$

(e) $f(x) = (\cos x)(\sin^2 x)$

- (9) Prove that the product of two even functions is even and that the product of two odd functions is also even.

- (10) Find the Fourier series for the function shown in the diagram below



Verify that the value of the series at $x = 0$ is that predicted by Dirichlet's theorem.

- (11) Show that if h is an even and integrable function and a is any positive real number then

$$\int_{-a}^a h(x) \, dx = 2 \int_0^a h(x) \, dx.$$

- (12) Find the half range cosine series for the function f defined by $f(x) = x$.

- (13) Find the half range sine series for the function f defined by $f(x) = x^2$.