6G5Z3011 MULTI-VARIABLE CALCULUS AND ANALYTICAL METHODS

TUTORIAL SHEET 01

(1) Find the first derivatives of the following functions
  (a) $f(x,y) = 3x^2 \ln(y)$
  (b) $g(x,y) = 4y\sin(x^2+2y)$
  (c) $h(x,y,z) = 7x^2y + \frac{1}{z} + xyz + 2$

(2) Given that $f(x,y) = \ln(x^2+y^2)$, show that
  (a)
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$
  (b)
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

(3) Consider the function f defined by
$$f(x,y) = x^2y^3.$$
Show using the limit definition of the partial derivative that $\frac{\partial f}{\partial x} = 2xy^3$ and $\frac{\partial f}{\partial y} = 3x^2y^2$.

(4) Consider a hill whose height above sea level at a point $x$km east and $y$km north of its peak is given by the value $z(x,y)$ of the function z defined by
$$z(x,y) = 1000e^{-u} + 110 \text{ metres},$$
where $u = x^2+y^2$. Plot the 400, 600, ... , 1000 metre contours of this hill on a map.

Find the coordinates of the point P, due noth west of the peak, which is exactly 400 metres above sea level, and mark it on your plot. What is the slope of the hill at this point P, in a (i) northerly direction, and in an (ii) easterly direction.

(5) Demonstrate that the dunction ϕ, defined by
$$\phi(x,y) = e^x \sin(y)$$
is a solution of Laplace's differential equation
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

(6) Demonstrate that the function ψ, defined by
$$\psi(x,y,t) = e^{-t}(\sin(x) + \cos(y)),$$
is a solution of the partial differentiatial equation
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial \psi}{\partial t}.$$

(7) Find all the locations (x,y) where the two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are simultaneously zero, where f is the function defined by
$$f(x,y) = \cos(x^2+y^2).$$

(8) Use the technique of implicit partial differentiation to find expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where x, y, z are related by the condition
$$xy + yz + zx = 1.$$

(9) Consider the 1-dimensional heat equation
$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$
which describes the distribution of heat in a region at time t. Show that the function u defined by
$$u(x,t) = e^{-\beta t}\sin(\alpha x)$$
is a solution of the heat equation when a certain relationship holds between the parameters α and β.

(10) Consider a general triangle with angles A, B, C whose opposite sides have lengths a, b, c respectively.

Find an expression that gives the rate of change of angle A as side length a is varied, but b and c are kept fixed. To do this make use of implicit differentiation and the cosine formula
$$a^2 = b^2 + c^2 - 2bc\cos(A).$$

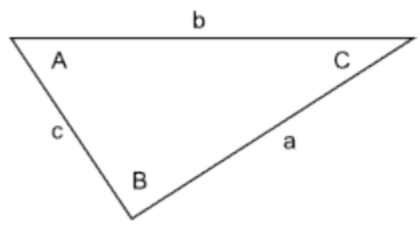

Figure 1: Triangle