6G5Z3011 Multi-variable calculus and analytical methods

Tutorial Sheet 7

Qs 1 – 5 on working with the Laplace transform and its inverse.
Qs 6 – 7 on solving ODEs and systems of ODEs using the Laplace transform method.

1. Find the general solution of the equation

$$\frac{d^3y}{dx^3} = 6.$$

What does the solution become if we are given the conditions

$$y(0) = 0, \quad y(1) = 0, \quad y(2) = 1?$$

2. Using the fact that

$$\sinh t = \frac{1}{2}\left(e^{t} - e^{-t}\right)$$

and the definition of the Laplace transform show that

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}.$$

3. Without using the tables (i.e. by working the integral definition of the transform) show that

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}.$$

4. Using the table of standard transforms and transform properties find the Laplace transforms of the functions defined by

   (a)
   $$1 + 2t + 3t^2,$$

   (b)
   $$5\cos t,$$

   (c)
   $$4e^{2t}\sin 3t.$$

5. Using the table find the inverse Laplace transforms of

   (a)
   $$\frac{7}{s+4},$$

   (b)
   $$\frac{4s+3}{s^2},$$

   (c)
   $$\frac{2s+8}{(s+4)^2+100},$$

   (d)
   $$\frac{3}{(s+1)(s-2)}.$$

6. Use the Laplace transform method to solve the following differential equations subject to the given initial conditions

   (a)
   $$\frac{dy}{dt}+y=1, \quad y(0)=0.$$

(b)

$$\frac{dy}{dt} - 2y = 4e^{-2t}, \quad y(0) = 2.$$

(c)

$$\frac{d^2y}{dt^2} + 2y = 3t, \quad y(0) = 1,\ y'(0) = 7.$$

(d)

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 3, \quad y(0) = 1,\ y'(0) = 4.$$

7. Use the Laplace transform method to solve the following pairs of coupled differential equations.

(a)

$$x + \frac{dy}{dt} = 2$$

$$\frac{dx}{dt} - 6x - 13y = 1$$

with the initial conditions $x(0) = -10$ and $y(0) = 3$.

(b)

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} + y = 0$$
$$\frac{dy}{dt} - \frac{dx}{dt} - 2y + 2x = \sin t.$$

with the initial conditions $x(0) = 0$, $y(0) = 0$ and $y'(0) = 0$.