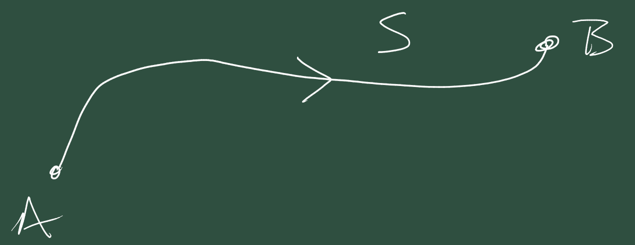
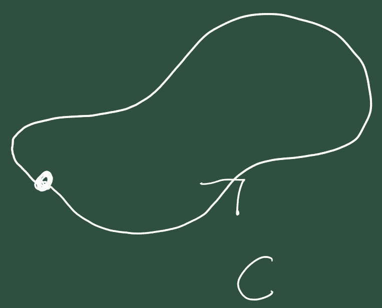
Green's Theorem

Closed path integrals



$$\int_S \sim\sim\sim$$



notation to indicate to the reader the path is closed

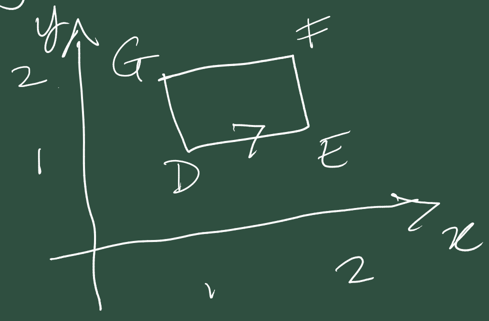
a <u>closed</u> curve C

$$\oint_C \sim\sim\sim$$

Example 3.8.3

$$I = \oint_C 3x^2y^2\,dx + 2xy\,dy$$

where C is the boundary of the square $DEFG$



Use the splitting

$$\oint_C = \int_{DE} + \int_{EF} + \int_{FG} + \int_{GD}.$$

on DE : $y=1,\ dy=0 \qquad x: 1 \longrightarrow 2.$

on EF : $x=2,\ dx=0 \qquad y: 1 \longrightarrow 2$

on FG : $y=2,\ dy=0 \qquad x: 2 \longrightarrow 1$

on GD : $x=1,\ dx=0 \qquad y: 2 \longrightarrow 1.$

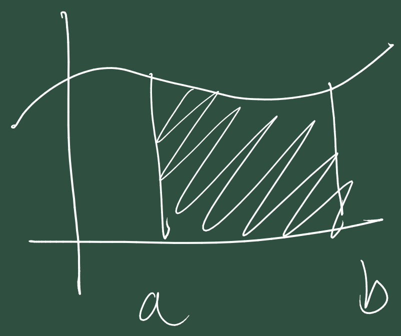
So $\oint_C 3x^2y^2\,dx + 2xy\,dy$

$$= \int_1^2 3x^2\,dx + \int_1^2 4y\,dy + \int_2^1 12x^2\,dx + \int_2^1 2y\,dy$$

$$= \left[x^3\right]_1^2 + \left[2y^2\right]_1^2 + \left[4x^3\right]_2^1 + \left[y^2\right]_2^1$$

$$= 8-1 + 8-2 + 4-32 + 1-4$$

$$= -18$$



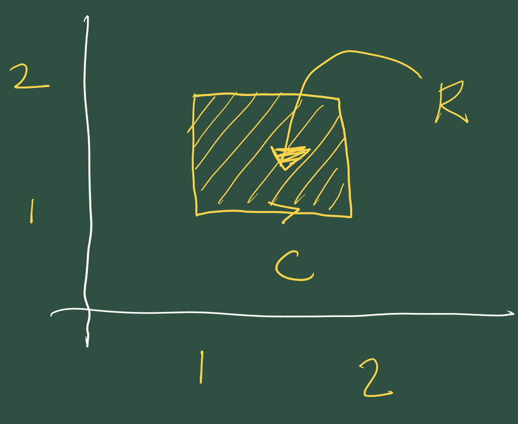
Refer to the notes

Let's verify Green's theorem for the previous example

Ex 3.8.10

$P(x,y) = 3x^2y^2$

$Q(x,y) = 2xy.$



Green's theorem claims that

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx\,dy = \oint_C P\,dx + Q\,dy$$

$$= -18.$$

Verify this.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - 6x^2 y$$

$$\iint_R 2y - 6x^2 y = \int_1^2 \left(\int_1^2 2y - 6x^2 y \, dx \right) dy$$

$$= \int_1^2 \left[2xy - 2x^3 y \right]_{x=1}^{x=2} dy.$$

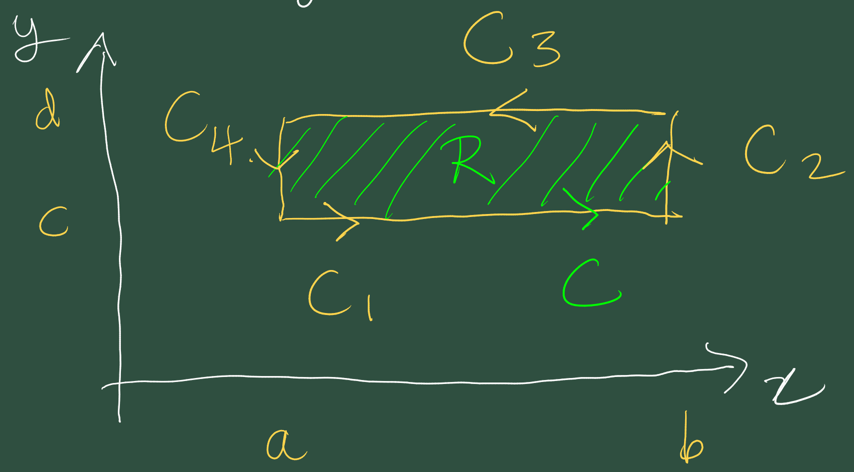
$$= \int_1^2 4y - 16y - (2y - 2y) \, dy.$$

$$= \int_1^2 -12y \, dy$$

$$= \left[-6y^2 \right]_1^2$$

$$= -24 + 6 = \underline{\underline{-18}}, \text{ as expected.}$$

But why IS Green's theorem true.
We'll prove it directly for rectangular regions, and then use a limiting argument for the general case.



$$C = C_1 + C_2 + C_3 + C_4$$

on C_1 : $y = c,\ dy = 0 \quad x : a \to b$

C_2 : $x = b,\ dx = 0 \quad y : c \to d$

C_3 : $y = d,\ dy = 0 \quad x : b \to a$

C_4 : $x = a,\ dx = 0 \quad y : d \to c$.

The L.H.S. of Green's Theorem is

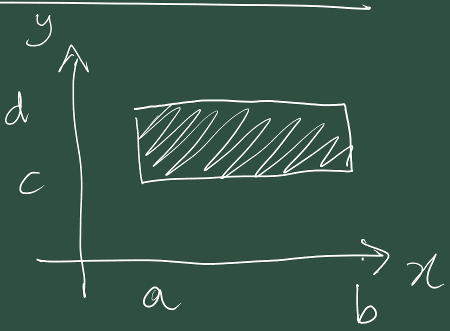
$$\oint_C P(x,y)\,dx + Q(x,y)\,dy$$

$$= \int_{C_1} - + \int_{C_2} - + \int_{C_3} - + \int_{C_4} -$$

$$= \int_a^b P(x,c)\,dx + \int_c^d Q(b,y)\,dy$$

$$\boxed{+\int_b^a P(x,d)\,dx + \int_d^c Q(a,y)\,dy}.$$

Now examine the R.H.S. of Green's theorem.



$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$$= \iint_R \frac{\partial Q}{\partial x}\,dx\,dy - \iint_R \frac{\partial P}{\partial y}\,dy\,dx.$$

$$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x}\,dx\right)dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y}\,dy\right)dx$$

$$= \int_c^d \Big[Q(x,y)\Big]_{x=a}^{x=b}\,dy$$

$$- \int_a^b \Big[P(x,y)\Big]_{y=c}^{y=d}\,dx.$$

$$= \int_c^d Q(b,y) - Q(a,y)\,dy$$

$$- \int_a^b P(x,d) - P(x,c)\,dx.$$

$$= \int_c^d Q(b,y)\,dy - \int_c^d Q(a,y)\,dy$$

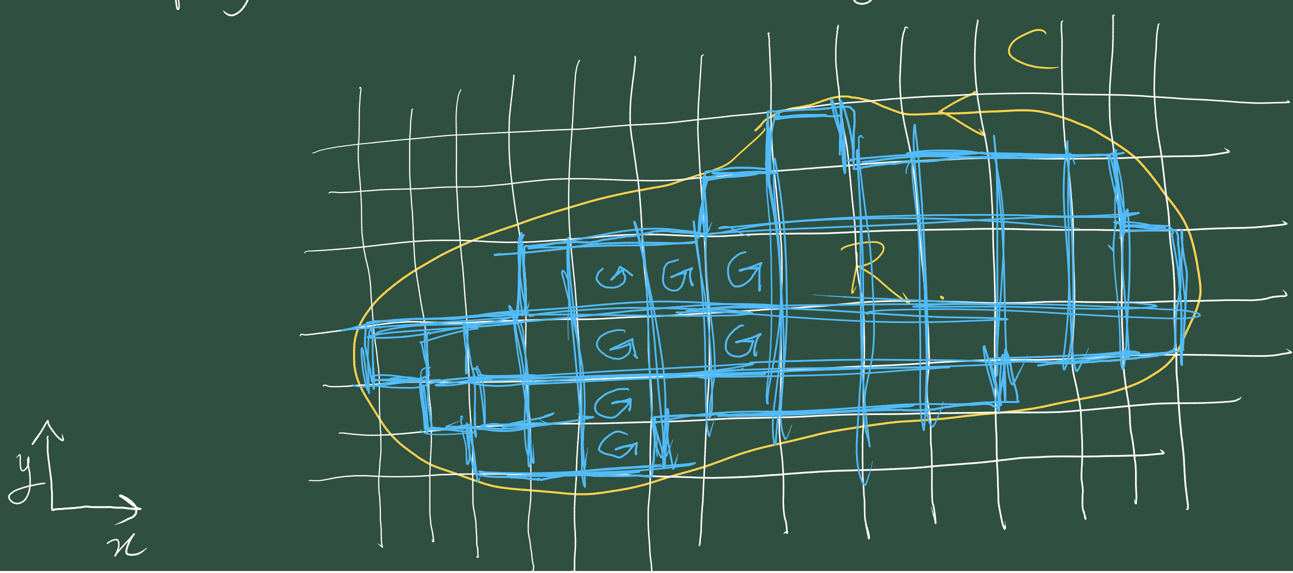
$$- \int_a^b P(x,d)\,dx + \int_a^b P(x,c)\,dx.$$

$$= \boxed{\int_c^d Q(b,y)\,dy + \int_d^c Q(a,y)\,dy + \int_b^a P(x,d)\,dx + \int_a^b P(x,c)\,dx.}$$

= LHS of Green's theorem.

So Green's Theorem is true for such 'nice' regions

So now consider a general simply connected region.



Consider a sub-division of the plane into small rectangles (parallel to the x, y axes).

We can approximate R as the union of rectangles within R.

and as size of rectangles $\longrightarrow 0$ their union $\longrightarrow R$.

RHS of GT

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$\simeq \iint_{\text{union of rectangles}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$$

$$= \sum_{\square} \iint_{\square} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$$

$$= \sum_{\square} \oint_{\square} P\,dx + Q\,dy$$, by special case of G.Ts.

$$= \oint_\gamma P\,dx + Q\,dy.$$

, where γ is the boundary of the union of rectangles on interior of R.

as the integrals over all ~~e~~shared edges of interior rectangles cancel out as each interior edge is traversed once in each direction

$$\approx \oint_C P\,dx + Q\,dy. = \text{LHS of G.T.}$$

So RHS $\approx$ LHS.

but taking limit as size of partitions $\to 0$ will give equality here.

Changing coordinate systems in multi-integrals.

Example

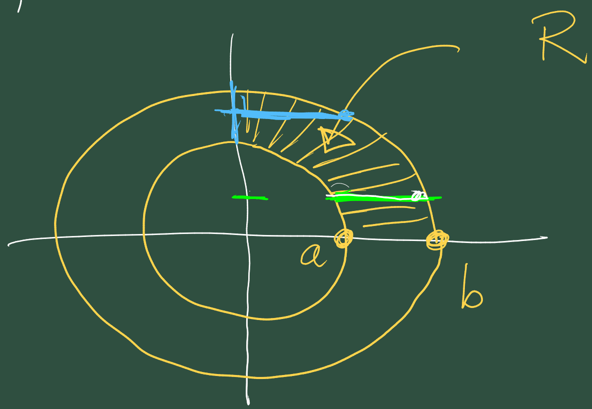
Consider this integral.

$$I = \iint_R \frac{1}{x^2+y^2}\,dx\,dy$$

where R is the 'annular' region in the upper-right quadrant between circles of radius a, b.

$x^2+y^2=b^2$.

$x=\sqrt{b^2-y^2}$



$$= \int_0^b \left(\int_?^? \frac{1}{x^2+y^2}\,dx \right) dy$$

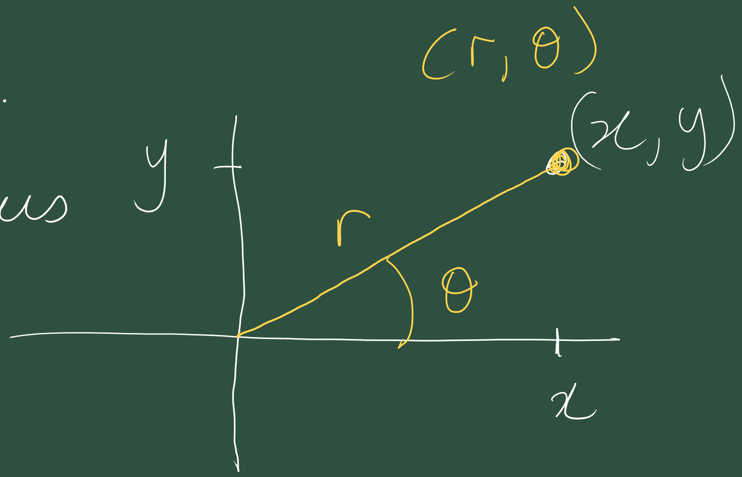
$$= \int_0^a \left(\int_{\sqrt{a^2-y^2}}^{\sqrt{b^2-y^2}} \frac{1}{x^2+y^2}\,dx \right) dy + \int_a^b \left(\int_0^{\sqrt{b^2-y^2}} \frac{1}{x^2+y^2}\,dx \right) dy.$$

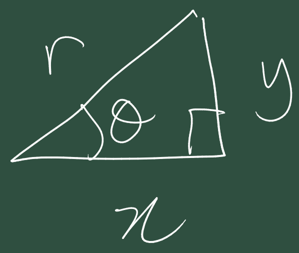
= "my heart is sinking"... save me!

There is a better way.

Both the integrand and region are kind of circular. So let's use polar coordinates.

There are equations relating cartesian and polar systems.





$$x = r\cos\theta$$
$$y = r\sin\theta$$

We'd ~~to~~ like to convert the integral I and express it in polar coordinates.

Rule for converting the differentials, or area element $dx\,dy$

For two coordinate systems (x, y) & (s, t) we have.

$$dx\,dy = \left| \begin{matrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{matrix} \right| ds\,dt$$

using the Jacobian determinant of the transformation from (s,t) to (x,y)

For polar coords.

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

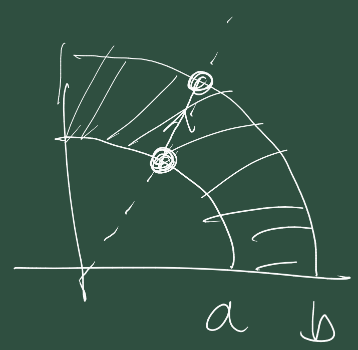
$$= r.$$

So $dx\,dy = r\,dr\,d\theta$

Returning to I.

$$I = \iint_R \frac{1}{x^2+y^2}\,dx\,dy$$

$$= \iint_R \frac{1}{r^2}\,r\,dr\,d\theta$$



$$= \iint_R \frac{1}{r}\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\int_a^b \frac{1}{r} \, dr \right) d\theta$$

$$= \int_0^{\pi/2} \Big[\ln(r) \Big]_a^b \, d\theta$$

$$= \int_0^{\pi/2} \ln(b) - \ln(a) \, d\theta$$

$$= \int_0^{\pi/2} \ln(b/a) \, d\theta$$

$$= \pi/2 \ln(b/a).$$