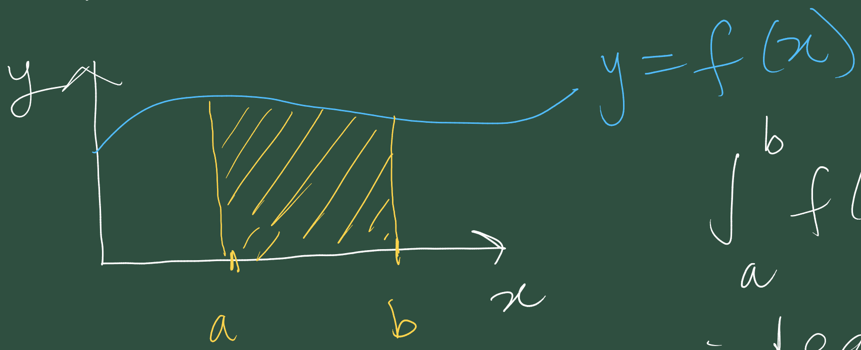
CLP3: Section 3.1.

~~Mult~~

Multiple integrals (Double integrals)

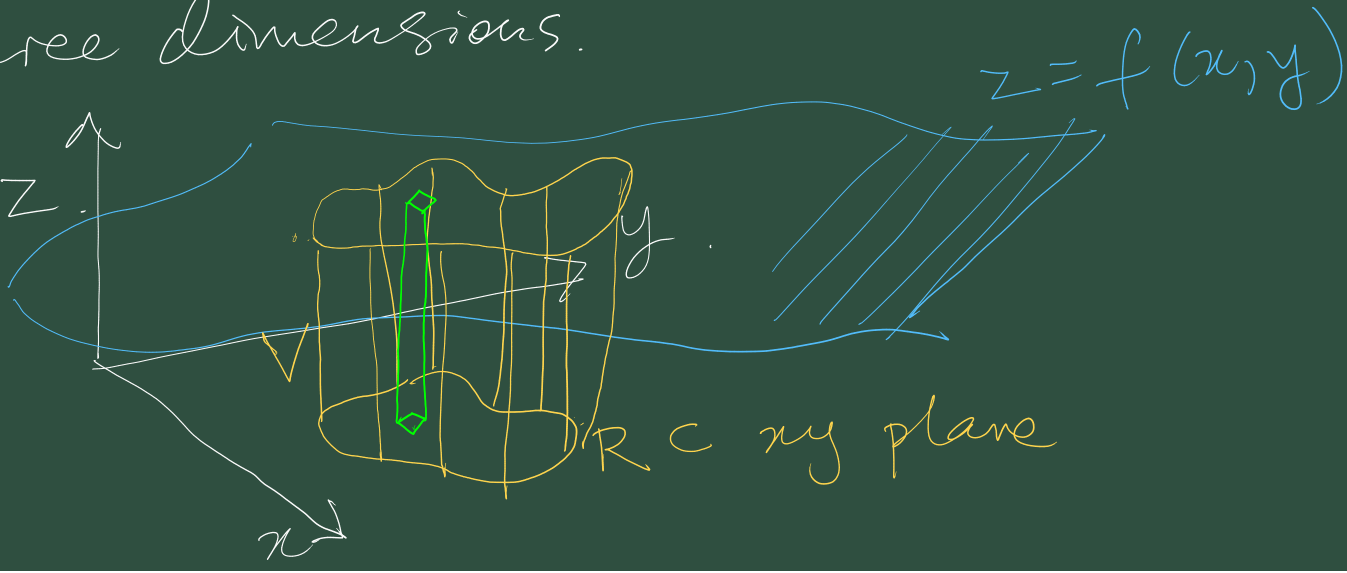
'Double integrals' generalise to two-variable functions, the concept of 'definite integral' of a single-variable function.



$\int_a^b f(x)\,dx$, the definite integral of f over (a,b), gives the area underneath the curve

2-variable function $f(x,y)$,

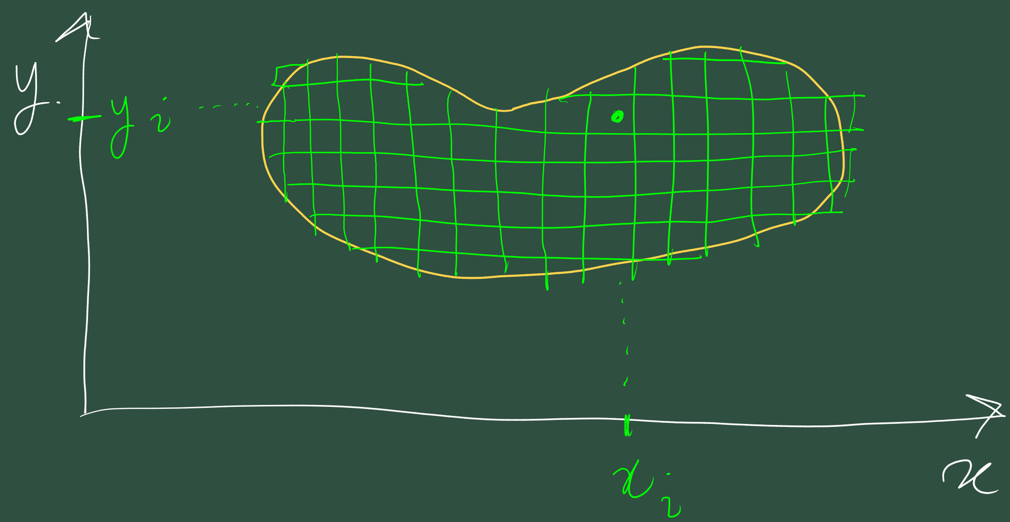
$z=f(x,y)$ defines a surface in three dimensions.



The volume V between the surface and xy plane over the region R will be represented by the "double integral"

$$V = \iint_R f(x,y)\, dx\, dy$$

These can be formally defined in a similar way to Riemann sums for single-variable integrals.



- partition R as $\bigcup_i r_i$, where $\{r_i\}$ is a collection of rectangles approximating R.

- in each r_i, select a representative point (x_i, y_i)
- over each r_i, position a rectangular cylinder of height $f(x_i, y_i)$

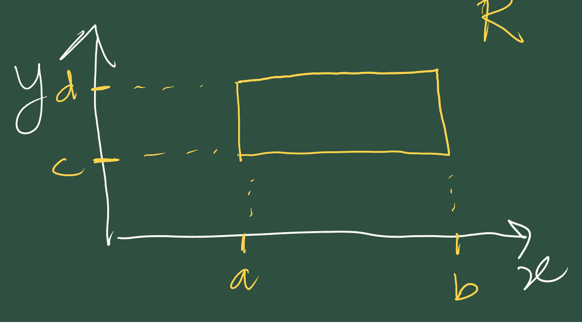
Can define V as

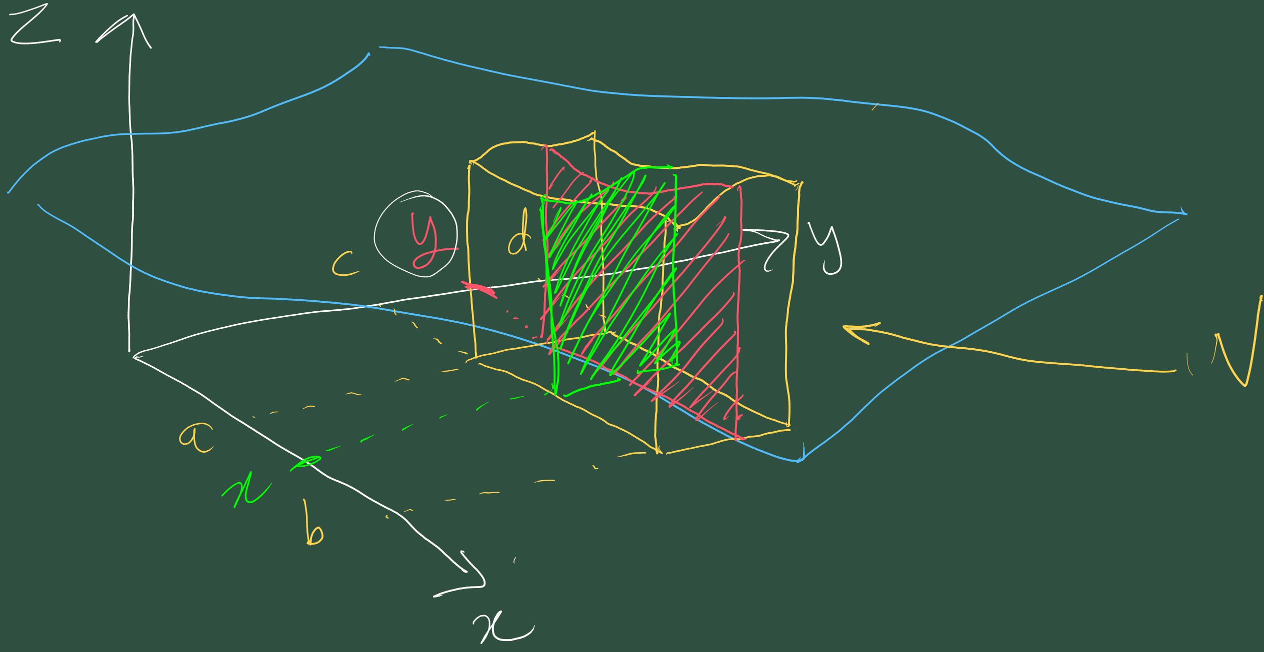
$$V = \iint_R f(x,y)\,dx\,dy$$

$$:= \lim_{\text{size } r_i \to 0} \sum_i (\text{area } r_i)\, f(x_i, y_i)$$

We won't actually calculate such integrals like this, rather we'll use repeated single-variable integration.

<u>Simplest case</u> When R is a rectangle aligned with x, y axes.





Consider the cross-section (red curtain) shown which is located at a particular y value $c \le y \le d$

This has area

$$\int_a^b f(x, y)\, dx$$

this y value held constant

Idea is to express V as the continuous sum of such 'curtain' areas as y varies from c to d

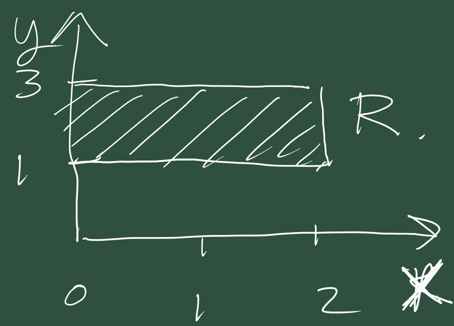
$$V = \int_c^d \left(\underbrace{\int_a^b f(x,y)\, dx}_{} \right) dy$$

y is treated as a constant.

These will typically be evaluated from the inside out.

Example. Consider $f(x,y) = 2xy + 4x + 3y + 1$

Integrate this over the rectangular region R over $0 \le x \le 2$, $1 \le y \le 3$.



V is the volume under surface $z = f(x,y)$, over R,

$$V = \iint_R f(x,y)\,dx\,dy$$

$$= \int_1^3 \left(\int_0^2 (2xy + 4x + 3y + 1)\,dx \right) dy.$$

$$= \int_1^3 \left[x^2 y + 2x^2 + 3xy + x \right]_0^2 dy.$$

$$= \int_1^3 4y + 8 + 6y + 2 \; dy$$

$$= \int_1^3 (10y + 10)\,dy$$

$$= \left[5y^2 + 10y \right]_1^3$$

$$= 45 + 30 - (5 + 10)$$

$$= 60.$$

We could equally well calculate this double integral as

$$V = \int_a^b \Big(\underbrace{\int_c^d f(x,y)\, dy}_{\text{area of 'green curtain' where } x \text{ is held constant}} \Big)\, dx .$$

Exercise: Evaluate this way and confirm $V = 60$ value.

Non-rectangular regions (and non-aligned rectangles)

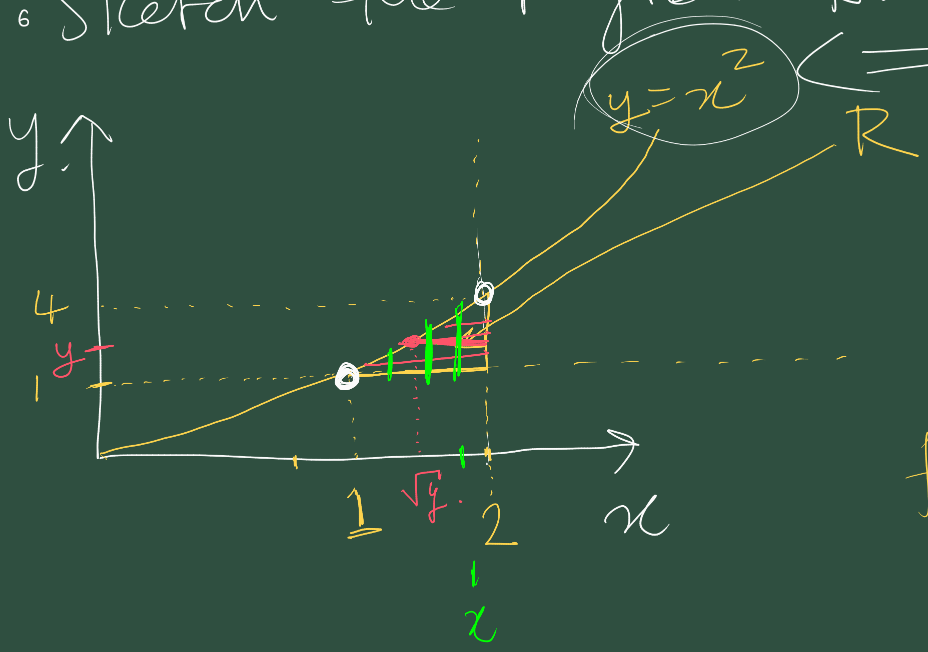
In such cases careful consideration of boundary curves is needed.

Example Integrate $f(x,y) = 4x^3 + 4y^3$

over R, where R is the finite region bounded by the curves $x=2$, $y=1$ and $y=x^2$

. Sketch the region R.

$y=x^2 \Longleftrightarrow x=\sqrt{y}$.



when filling in limits work from outside-in.

$$V = \iint_R 4x^3 + 4y^3 \, dx\, dy.$$

$$= \int_1^4 \left(\int_{\sqrt{y}}^2 4x^3 + 4y^3 \, dx \right) dy.$$

$$= \int_1^4 \left[x^4 + 4xy^3 \right]_{x=\sqrt{y}}^{x=2} dy$$

$$= \int_1^4 16 + 8y^3 - (y^2 + 4y^{7/2}) \, dy.$$

$$= \left[16y + 2y^4 - \frac{1}{3}y^3 - \frac{8}{9}y^{9/2} \right]_1^4$$

$$= \left[64 + 512 - \frac{64}{3} - \frac{8}{9}2^9 \right.$$

$$- \left(16 + 2 - 1/3 - 8/9\right)$$

$$= \frac{745}{9}$$

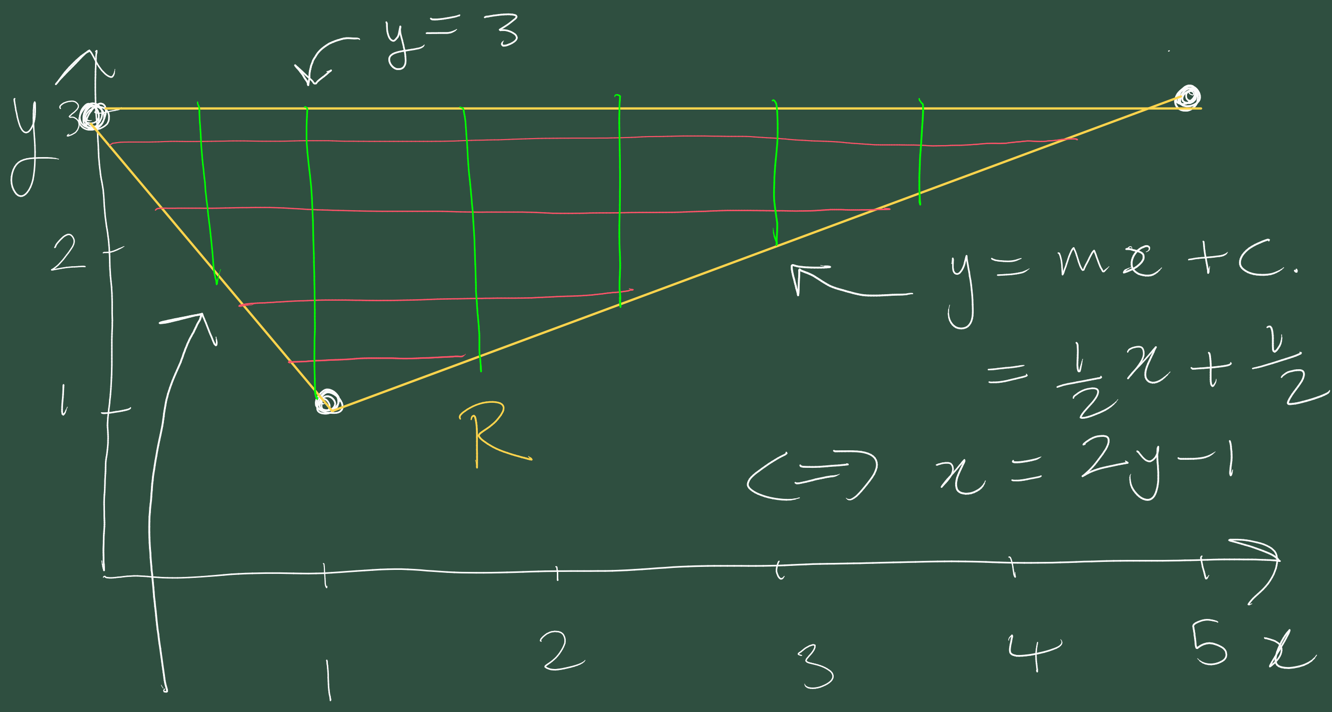
How would this appear if x, y were taken in opposite order

$$V = \int_1^2 \left(\int_1^{x^2} 4x^3 + 4y^3 \, dy \right) dx.$$

Exercise: Confirm this evaluates to $\frac{745}{9}$.

Example (Shows the need sometimes for splitting the region of integration).

Consider integrating a function $f(x,y)$ over the triangular region R with vertices at $(1,1)$, $(5,3)$, $(0,3)$



$$y=mx+c$$
$$=-2x+3$$
$$\iff x=-\frac{y}{2}+\frac{3}{2}$$

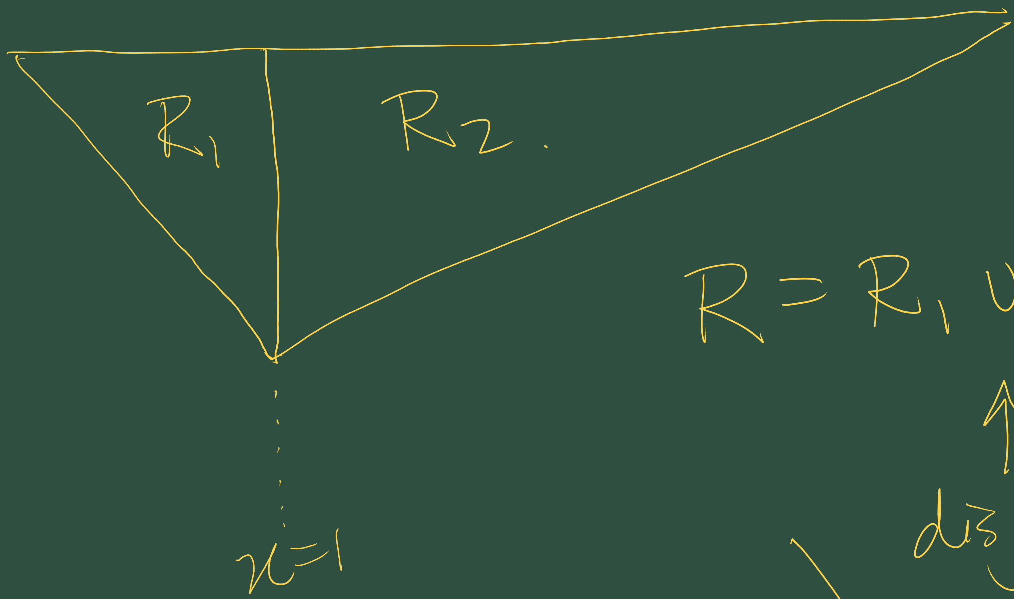
$$V=\iint_R f(x,y)\,dx\,dy$$

$$=\int_1^3\left(\int_{-\frac{y}{2}+3/2}^{2y-1} f(x,y)\,dx\right)dy$$

(let's try in opposite order)

$$=\int_0^5\left(\int_?^3 f(x,y)\,dy\right)dx$$

no single formula for y will work here



$R = R_1 \cup R_2$

disjoint union.

and use the integration principle.

$$\iint_R f(x,y)\,dy\,dx = \iint_{R_1} f\,dy\,dx + \iint_{R_2} f\,dy\,dx$$

So

$$V = \int_0^1 \left(\int_{-2x+3}^{3} f(x,y)\,dy \right) dx$$

$$+ \int_1^5 \left(\int_{\frac{1}{2}x+\frac{1}{2}}^{3} f(x,y)\,dy \right) dx$$

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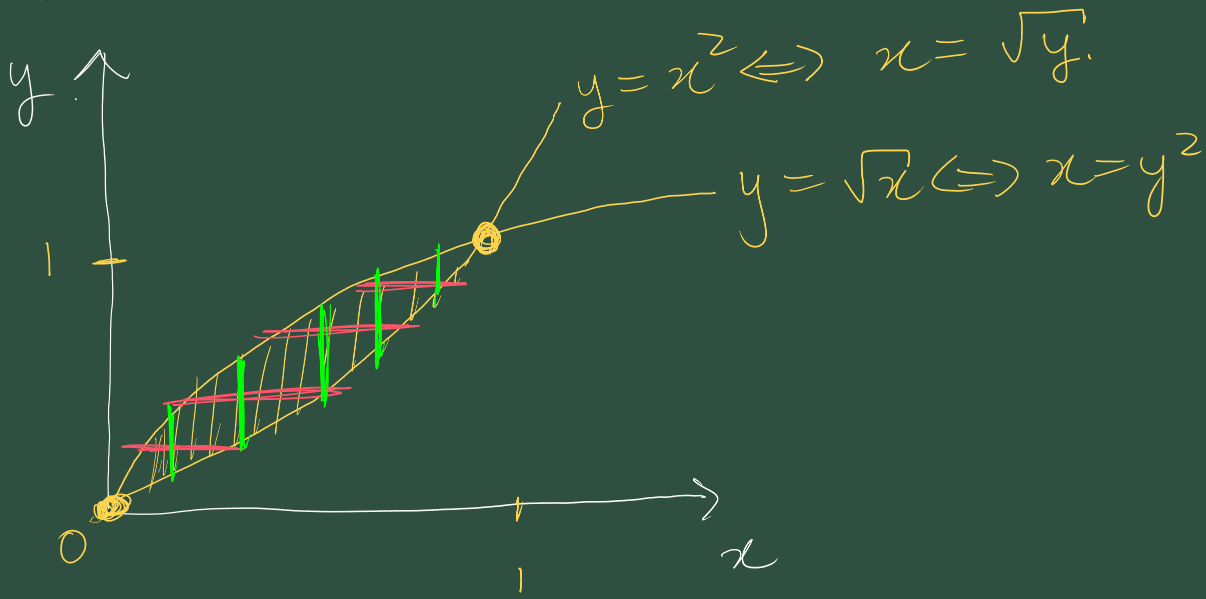
Q3 (c).

$$V = \iint_R x y^2 \, dx\, dy$$

where R is the finite region bounded by $y = x^2$ and $x = y^2 \iff y = \sqrt{x}$.



$$V = \int_0^1 \left(\int_{y^2}^{\sqrt{y}} x y^2 \, dx \right) dy$$

$$= \ldots\ldots = 3/56$$

$$= \int_0^1 \left(\int_{x^2}^{\sqrt{x}} x y^2 \, dy \right) dx = \ldots = 3/56$$