6G5Z3011 Multi-variable calculus and analytical methods

Tutorial Sheet 9

Qs 1 − 4 on the existence of Fourier series and working with the inner product.
Qs 5 − 13 on finding Fourier series and working with odd and even functions.

1. Consider the inner product $\langle\ ,\ \rangle$ defined by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\ dx.$$

Show that for every positive integer n,

$$\langle 1, p_n\rangle = 0,$$

where p_n is the function defined by $p_n(x) = \sin(nx)$.

Also, using the formula

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

show that if m and n are positive integers then

$$\langle q_m, q_n\rangle = \begin{cases} 0, & \text{if } m \neq n \\ \pi, & \text{if } m = n \end{cases},$$

where q_r is the function defined by $q_r(x) = \cos(rx)$.

2. For each of the following definitions decide whether the function f will have a Fourier series in the interval $(-\pi, \pi)$. Justify your answers.

   (a)
   $$f(x) = \begin{cases} -1, & \text{if } -\pi < x \leq \frac{-\pi}{2} \\ 0, & \text{if } \frac{-\pi}{2} < x \leq \frac{\pi}{2} \\ -1, & \text{if } \frac{\pi}{2} < x \leq \pi \end{cases}$$

   (b) $f(x) = \cos\left(\frac{1}{x}\right)$

   (c) $f(x) = 8x^4 - 8x^2 + 1$

   (d) $f(x) = \tan\left(\frac{1}{x}\right)$

3. Sketch the graph of the function

$$f(x) = \begin{cases} 1 + x, & \text{if } -\pi < x \leq 0 \\ 2 + x, & \text{if } 0 < x \leq \pi \end{cases}.$$

What is the value of the Fourier series of this function when (a) $x = 1$ and (b) $x = 0$?

4. Given that the set of functions

$$\{1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \dots\}$$

is an orthogonal set with respect to the inner product defined in question (1) above, and that the Fourier series for a function f is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

show that

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

and

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx, \quad (m > 0).$$

5. Find the Fourier series for the function f which has period 2π and is defined by

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x \leq 0 \\ 1, & \text{if } 0 < x \leq \pi \end{cases}$$

6. Find the Fourier series for the function f, of period 2π, and defined by $f(x) = x$ for $x \in (-\pi, \pi)$.

7. Find the Fourier series for the function f which has period 2π and is defined by

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \leq 0 \\ x, & \text{if } 0 < x \leq \pi \end{cases}$$

8. For each of the following definitions determine whether the function f is odd or even.

   (a) $f(x) = x^3$

   (b) $f(x) = e^x$

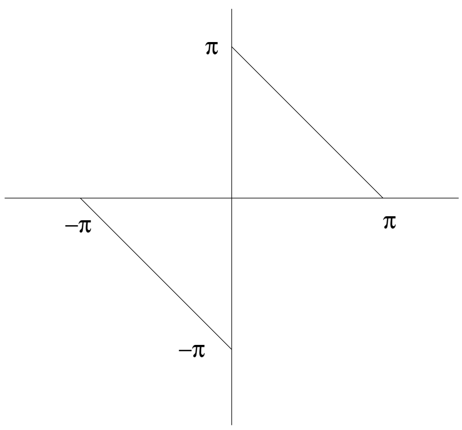
   (c) $f(x) = e^{|x|}$

   (d) $f(x) = x \cos x$

   (e) $f(x) = (\cos x)(\sin^2 x)$

9. Prove that the product of two even functions is even and that the product of two odd functions is also even.

10. Find the Fourier series for the function shown in the diagram below



Verify that the value of the series at $x = 0$ is that predicted by Dirichlet's theorem.

11. Show that if h is an even and integrable function and a is any positive real number then

$$\int_{-a}^{a} h(x)\, dx = 2\int_{0}^{a} h(x)\, dx.$$

12. Find the half range cosine series for the function f defined by $f(x) = x$.

13. Find the half range sine series for the function f defined by $f(x) = x^2$.