

# 6G5Z3011 Multi-variable calculus and analytical methods

## Tutorial Sheet 01

September 30, 2025

1. Find the first derivatives of the following functions

(a)  $f(x, y) = 3x^2 \ln(y)$

(b)  $g(x, y) = 4y \sin(x^2 + 2y)$

(c)  $h(x, y, z) = 7x^2y + \frac{1}{z} + xyz + 2$

2. Given that  $f(x, y) = \ln(x^2 + y^2)$ , show that

(a)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

(b)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

3. Consider the function  $f$  defined by

$$f(x, y) = x^2 y^3.$$

Show using the limit definition of the partial derivative that  $\frac{\partial f}{\partial x} = 2xy^3$  and  $\frac{\partial f}{\partial y} = 3x^2 y^2$ .

4. Consider a hill whose height above sea level at a point  $x$ km east and  $y$ km north of its peak is given by the value  $z(x, y)$  of the function  $z$  defined by

$$z(x, y) = 1000e^{-u} + 110 \text{ metres,}$$

where  $u = x^2 + y^2$ . Plot the 400, 600, ... , 1000 metre contours of this hill on a map.

Find the coordinates of the point  $P$ , due north west of the peak, which is exactly 400 metres above sea level, and mark it on your plot. What is the slope of the hill at this point  $P$ , in a (i) northerly direction, and in an (ii) easterly direction.

5. Demonstrate that the dunction  $\phi$ , defined by

$$\phi(x, y) = e^x \sin(y)$$

is a solution of Laplace's differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

6. Demonstrate that the function  $\psi$ , defined by

$$\psi(x, y, t) = e^{-t}(\sin(x) + \cos(y)),$$

is a solution of the partial differentiatial equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial \psi}{\partial t}.$$

7. Find all the locations  $(x, y)$  where the two partial derivatives  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  are simultaneously zero, where  $f$  is the function defined by

$$f(x, y) = \cos(x^2 + y^2).$$

8. Use the technique of implicit partial differentiation to find expressions for  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  where  $x, y, z$  are related by the condition

$$xy + yz + zx = 1.$$

9. Consider the 1-dimensional heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

which describes the distribution of heat in a region at time  $t$ . Show that the function  $u$  defined by

$$u(x, t) = e^{-\beta t} \sin(\alpha x)$$

is a solution of the heat equation when a certain relationship holds between the parameters  $\alpha$  and  $\beta$ .

10. Consider a general triangle with angles  $A, B, C$  whose opposite sides have lengths  $a, b, c$  respectively.

Find an expression that gives the rate of change of angle  $A$  as side length  $a$  is varied, but  $b$  and

$c$  are kept fixed. To do this make use of implicit differentiation and the cosine formula

$$a^2 = b^2 + c^2 - 2bc \cos(A).$$

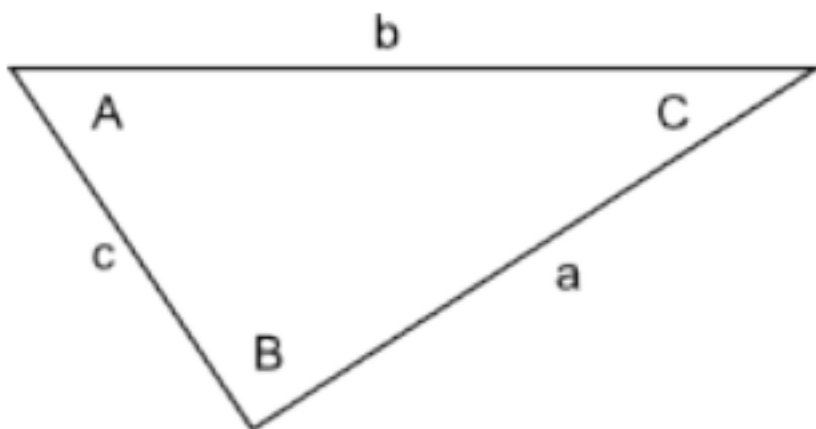


Figure 1: Triangle