

6G5Z3011 Multi-variable calculus and analytical methods

Tutorial Sheet 01

September 30, 2025

1. Find the first derivatives of the following functions

(a) $f(x, y) = 3x^2 \ln(y)$

(b) $g(x, y) = 4y \sin(x^2 + 2y)$

(c) $h(x, y, z) = 7x^2y + \frac{1}{z} + xyz + 2$

2. Given that $f(x, y) = \ln(x^2 + y^2)$, show that

(a)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

(b)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

3. Consider the function f defined by

$$f(x, y) = x^2 y^3.$$

Show using the limit definition of the partial derivative that $\frac{\partial f}{\partial x} = 2xy^3$ and $\frac{\partial f}{\partial y} = 3x^2y^2$.

4. Consider a hill whose height above sea level at a point x km east and y km north of its peak is given by the value $z(x, y)$ of the function z defined by

$$z(x, y) = 1000e^{-u} + 110 \text{ metres,}$$

where $u = x^2 + y^2$. Plot the 400, 600, ... , 1000 metre contours of this hill on a map.

Find the coordinates of the point P , due north west of the peak, which is exactly 400 metres above sea level, and mark it on your plot. What is the slope of the hill at this point P , in a (i) northerly direction, and in an (ii) easterly direction.

5. Demonstrate that the function ϕ , defined by

$$\phi(x, y) = e^x \sin(y)$$

is a solution of Laplace's differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

6. Demonstrate that the function ψ , defined by

$$\psi(x, y, t) = e^{-t}(\sin(x) + \cos(y)),$$

is a solution of the partial differential equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial \psi}{\partial t}.$$

7. Find all the locations (x, y) where the two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are simultaneously zero, where f is the function defined by

$$f(x, y) = \cos(x^2 + y^2).$$

8. Use the technique of implicit partial differentiation to find expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where x, y, z are related by the condition

$$xy + yz + zx = 1.$$

9. Consider the 1-dimensional heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

which describes the distribution of heat in a region at time t . Show that the function u defined by

$$u(x, t) = e^{-\beta t} \sin(\alpha x)$$

is a solution of the heat equation when a certain relationship holds between the parameters α and β .

10. Consider a general triangle with angles A, B, C whose opposite sides have lengths a, b, c respectively.

Find an expression that gives the rate of change of angle A as side length a is varied, but b and c are kept fixed. To do this make use of implicit differentiation and the cosine formula

$$a^2 = b^2 + c^2 - 2bc \cos(A).$$

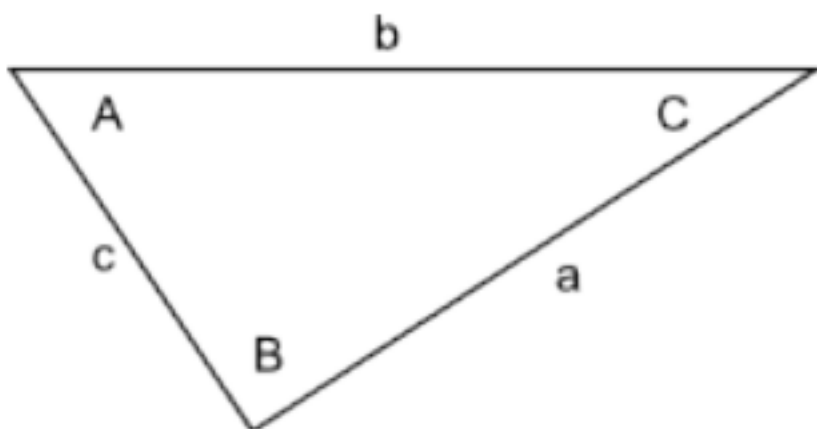


Figure 1: Triangle