# 6G5Z3011 Multi-variable calculus and analytical methods

Tutorial Sheet 02

October 7, 2025

## Small increments formula questions

1. Use the small increments formula to estimate the change in the function

$$f(x, y) = \left(x + 2y^2\right)^5$$

when $x$ and $y$ both increase from 1 to 1.01. Check your answer by direct evaluation of the values of $f$.

2. The pressure exerted by a column of gas of density $D$ and length $L$ is $P = \frac{1}{3}DL^2$. Find the percentage change in pressure caused by a 1% increase in density and a 2% increase in length.

3. The surface area $S$ and the volume $V$ of a right circular cone of base radius $r$ and height $h$ are given by

$$S = \pi r^2 + \pi r \left(r^2 + h^2\right)^{\frac{1}{2}}$$
$$V = \frac{1}{3}\pi r^2 h$$

   (a) What percentage increase in $h$ will cancel out a 2% decrease in $r$ if the volume is to required to remain the same.

   (b) In an attempt to estimate the value of $S$ the radius $r$ is measured as $5.00 \pm 0.01$mm and the height $h$ as $12.00 \pm 0.01$mm. Estimate the value of $S$ and quantify the possible error.

   If it is possible to measure just one of the quantities $h$ and $r$ more accurately, which one would you choose? Justify your answer.

4. Consider the function $f$ of $n$ variables defined by

$$f(x_1, x_2, \ldots, x_n) = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}.$$

Suppose that for each $r$ there is an $i_r\%$ increase in the value of $x_r$. Show that the total increase in the value of $f$ is $p_1 i_1 + p_2 i_2 + \cdots + p_n i_n$.

## Chain rule and Jacobian questions

5. Show that if the Cartesian coordinates $(x, y)$ are transformed to the polar coordinates $(r, \theta)$ and $f(x, y)$ is a differentiable function of $x$and $y$ then

$$x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \theta}.$$

6. Consider the transformation from the coordinates $(x, y)$ to a new pair $(s, t)$ defined by

$$s = xy, \quad t = \frac{1}{y}.$$

If $f(x, y)$ is a differentiable function of $x$ and $y$ show that

$$y\frac{\partial f}{\partial y}\left(x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right) = t\frac{\partial f}{\partial t}\left(s\frac{\partial f}{\partial s} - t\frac{\partial f}{\partial t}\right).$$

7. Show that the Jacobian of the transformation from Cartesian coordinates $(x, y)$ to polar coordinates $(r, \theta)$ is given by

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

8. Use the chain rule to prove that if a coordinate transformation from $(x, y)$ to $(s, t)$ is composed with one from $(s, t)$ to $(u, v)$ then

$$\frac{\partial(x, y)}{\partial(s, t)} \frac{\partial(s, t)}{\partial(u, v)} = \frac{\partial(x, y)}{\partial(u, v)}.$$

   Hence establish the reciprocal rule that if a transformation from $(x, y)$ to $(s, t)$ is composed with the inverse transformation from $(s, t)$ to $(x, y)$ then

$$\frac{\partial(x, y)}{\partial(s, t)} \frac{\partial(s, t)}{\partial(x, y)} = 1.$$

9. Show that the Jacobian of the transformation from polar coordinates $(r, \theta)$ to Cartesian coordinates $(x, y)$ is given by

$$\frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}.$$

10. The Cartesian coordinates $(x, y)$ are transformed by the mapping

$$s = y - x, \quad t = (y - x)^2.$$

Complete the following table of coordinate values

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 0 | 1 | 1 | 2 | 0 | 2 | 1 | 2 | 3 | 0 | 3 | 1 | 3 | 2 | 3 |
| $y$ | 0 | 1 | 0 | 1 | 0 | 2 | 1 | 2 | 2 | 0 | 3 | 1 | 3 | 2 | 3 | 3 |
| $s$ | | | | | | | | | | | | | | | | |
| $t$ | | | | | | | | | | | | | | | | |

Plot the resulting points $(s, t)$ from the table and comment on what you notice. Find the Jacobian $\frac{\partial(s,t)}{\partial(x,y)}$. What can be said about the inverse transformation?