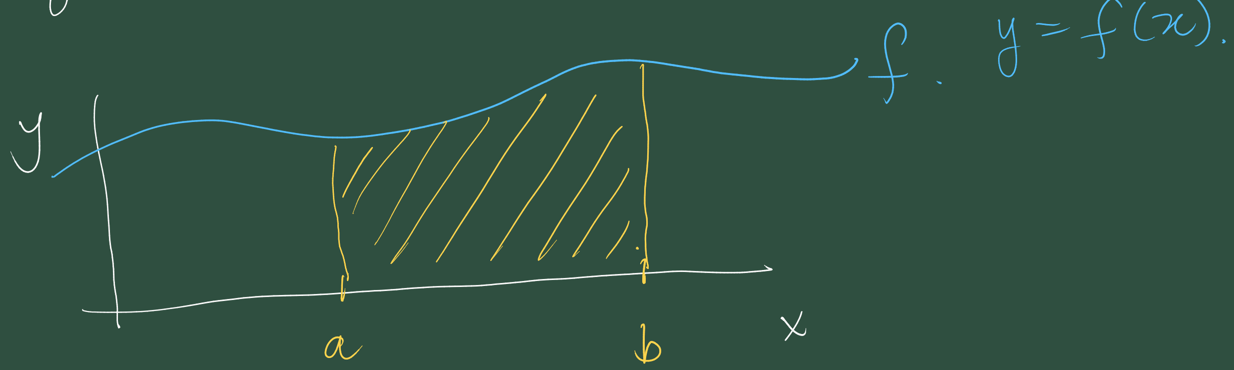
Integration.

We'll looking at two types of multi-variable integration

Single-variable.

Definite integrals.



The definite integral I is

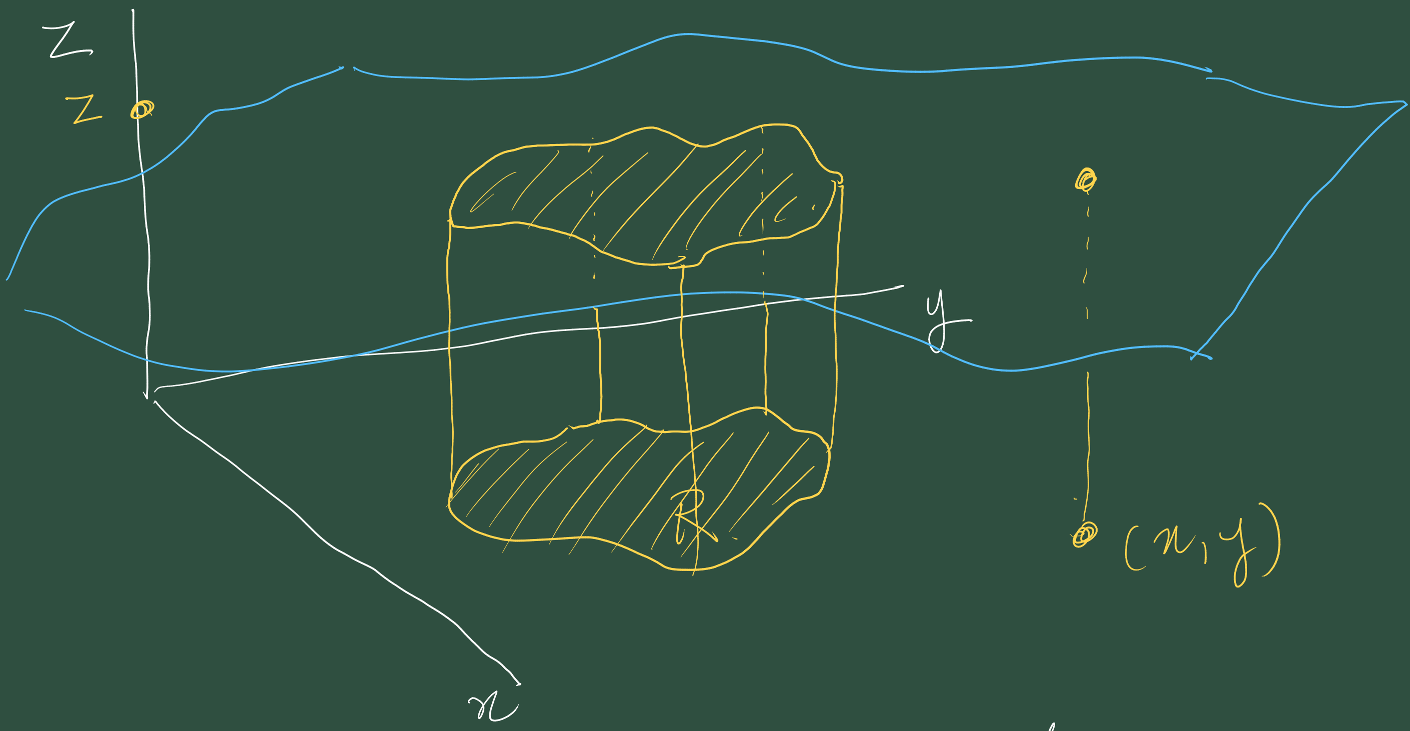
$$I = \int_a^b f(x)\,dx.$$

and gives the area between graph of f and x-axis. between a and b.

For function of two-variables

$$z = f(x, y)$$

this defines a surface over the xy plane of height z.



Given a subset R of xy plane we can think about the volume lying over R and 'underneath' the surface $z = f(x,y)$. This will be given by the double integral of f over R.

$$I = \iint_R f(x,y)\,dx\,dy$$

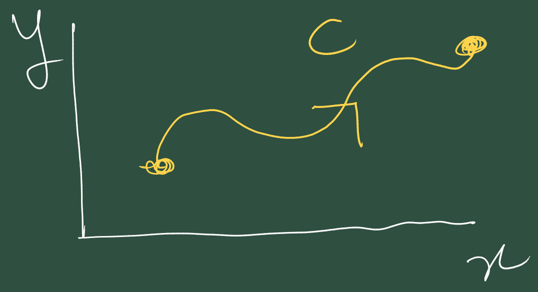
Discuss these in detail next week.

A second type of integration of two variable functions is so called "path integrals" or "line integrals"

of linear differential forms. L,

$$L = P(x,y)\,dx + Q(x,y)\,dy,$$

taken over paths in the xy plane



and expressed as.

$$\int_C L = \int_C (P\,dx + Q\,dy)$$

Example 1

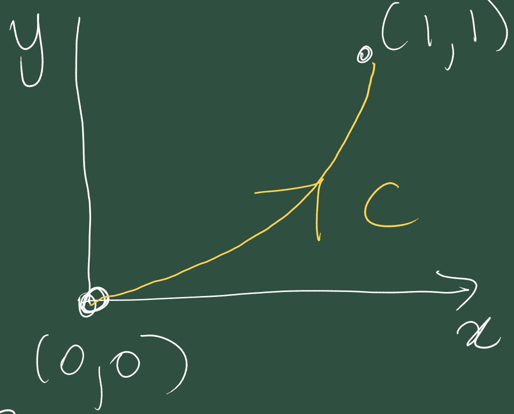
Consider the lin. diff. form

$$L = 10x^2y\,dx + (3x+2y)\,dy$$

integrate this along the curve C.

C: $y = x^2$ from $(0,0)$ to $(1,1)$.

To evaluate.



$$I=\int_C 10x^2y\,dx + (3x+2y)\,dy$$

the idea is to paramatrise the curve and convert everything into terms of a single-variable.

$\boxed{y=x^2}$ then $\boxed{dy = 2x\,dx}$.

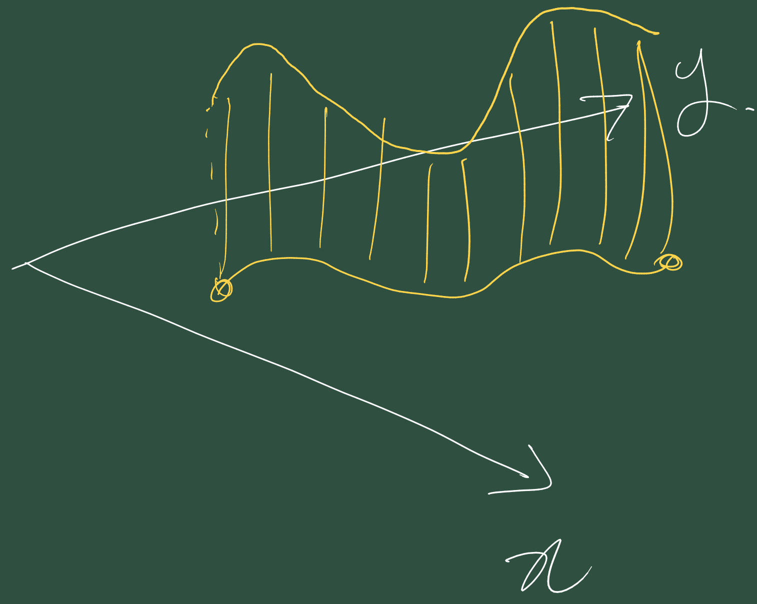
use these to convert it all into "x"

$$I=\int_{x=0}^{x=1} 10x^2x^2\,dx + (3x+2x^2)2x\,dx.$$

$$=\int_0^1 (10x^4+4x^3+6x^2)\,dx$$

$$=\left[2x^5+x^4+2x^3\right]_0^1$$

$$=5$$



"area of curtain"

<u>Theorem</u> Basic properties of path integrals

1. Linearity For two linear diff. forms L_1, L_2 and $\alpha, \beta \in \mathbb{R}$

$$\int_C \alpha L_1 + \beta L_2 = \alpha \int_C L_1 + \beta \int_C L_2$$

OR

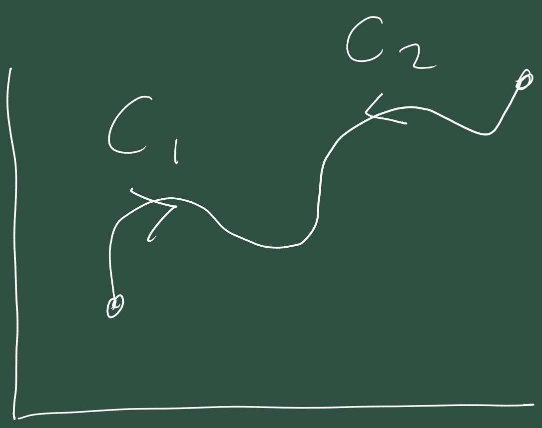
$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

$$= \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy.$$

2. <u>Direction of integration</u>.

Consider the curves C_1, C_2 which go along the same path but in opposite directions

then.



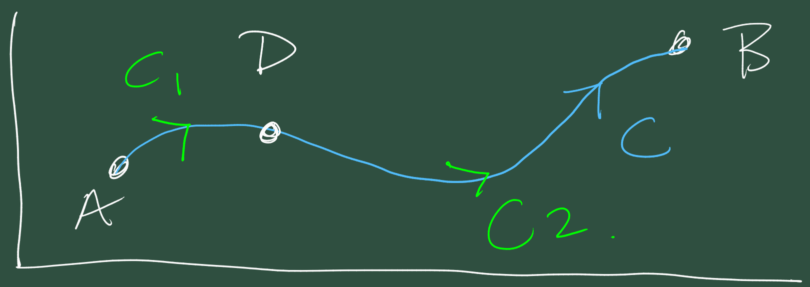
$$\int_{C_1} L = -\int_{C_2} L$$

compare with. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx.$

from 1-variable case.

indeed we can write "$C_1 = -C_2$".

3. Subdividing paths.



$$C = C_1 + C_2$$

$$\int_C L = \int_{C_1} L + \int_{C_2} L$$

compare this with

$$\int_a^b f(x)dx$$

$$= \int_a^d f(x)dx$$

$$+ \int_d^b f(x)dx$$

Example 2

Same L as before

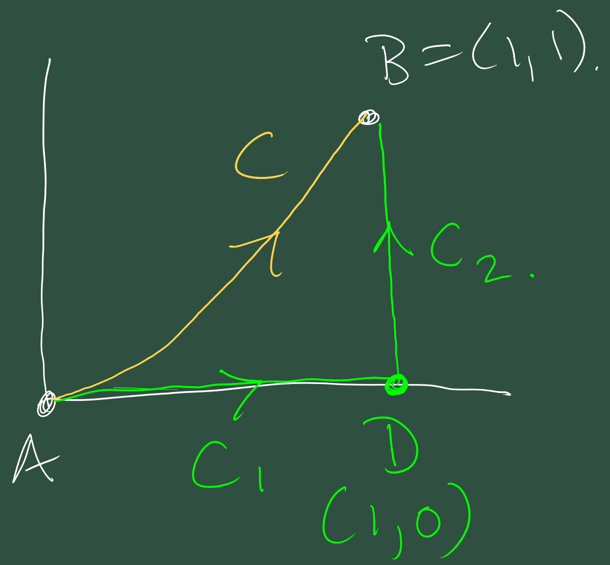
$$L = 10x^2y\,dx + (3x+2y)\,dy$$

and same start and end points.

$$A=(0,0)\ ,\ B=(1,1)$$

Now consider C_1: $A \longrightarrow D=(1,0)$ along x-axis and C_2: $D \longrightarrow B$ along the line $x=1$

So let's evaluate.



$$\int_{C_1+C_2} L$$

$$= \int_{C_1} L + \int_{C_2} L$$

on C_1: $y=0$, $dy=0$
$x: 0 \longmapsto 1$.

on C_2: $x=1$, $dx=0$
$y: 0 \longmapsto 1$.

$$= \int_{x=0}^{x=1} 10x^2 \cdot 0 \cdot dx + 3x \cdot 0$$

$$+ \int_{y=0}^{y=1} 10 \cdot y \cdot 0 + (3+2y)\,dy$$

$$= 0 + \int_0^1 3 + 2y \, dy$$

$$= \left[3y + y^2 \right]_0^1 = 4.$$

So $\int_{C_1 + C_2} L = 4.$, which is different

to $\int_C L = 5$, from example 1.

This shows the path dependence of L, which is the typical expected behaviour of integrals of L.

There are special forms that show "path independence"

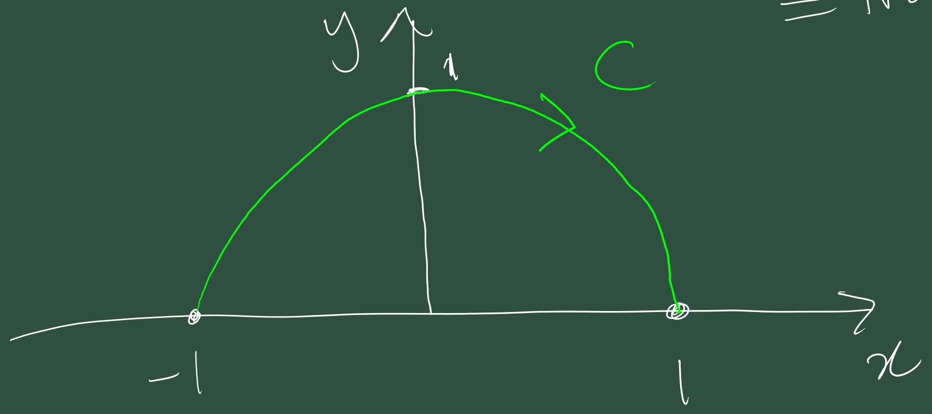
Another example of parametrising a path.

Evaluate I.

$$I = \int_C x^2 \, dy - yx \, dx$$

where C is the semi-circular path of radius 1, traversed clockwise from

$(-1,0)$ to $(1,0)$ "traverse" = travel along"



The curve C is parametrised by the variable t as $x=\cos(t)$, $y=\sin(t)$.

$$C: (\cos(t), \sin(t))$$

with $t: \pi \longrightarrow 0$

So we can expresse I as.

$$I = \int_{t=\pi}^{t=0} \cos^2(t) \underline{\cos(t)}\, dt - \sin(t)\underline{\cos(t)}(-\sin(t)\, dt)$$

$$= \int_{\pi}^{0} \cos(t)\left(\cos^2(t) + \sin^2(t)\right) dt.$$

$$= \int_{\pi}^{0} \cos(t)\, dt = \left[\sin(t)\right]_{\pi}^{0} = 0$$

A further example on stationary points.

Q. Find and classify the critical points

of $g(x,y) = -2x^4 + 4x^2e^y - 4e^{4y}$.

Need the first and second order partial derivatives

$$\frac{\partial g}{\partial x} = -8x^3 + 8xe^y$$

$$\frac{\partial g}{\partial y} = 4x^2e^y - 16e^{4y}$$

$$\frac{\partial^2 g}{\partial x^2} = -24x^2 + 8e^y. \leftarrow$$

$$\frac{\partial^2 g}{\partial y^2} = 4x^2e^y - 64e^{4y} \leftarrow$$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = 8xe^y$$

Critical points are the simultaneous solutions (x,y) to $\frac{\partial g}{\partial x} = 0$ & $\frac{\partial g}{\partial y} = 0$.

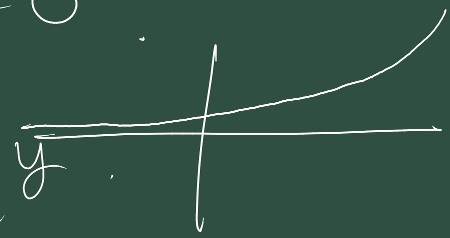
First consider $\frac{\partial g}{\partial x} = 0$

$$\Leftrightarrow -8x^3 + 8xe^y = 0.$$

$$\Leftrightarrow \quad -x^3 + xe^y = 0$$

$$\Leftrightarrow \quad x(-x^2 + e^y) = 0.$$

$$\Leftrightarrow \quad x = 0 \quad \underline{\text{OR}} \quad x^2 = e^y.$$



Now impose these conditions on $\frac{\partial g}{\partial y} = 0$ and solve if possible.

<u>When $x=0$</u>

$$\frac{\partial g}{\partial y} = 0 \Leftrightarrow -16e^{4y} = 0$$

which is never true as exponentials are always > 0.

<u>When $x^2 = e^y$.</u>

$$\frac{\partial g}{\partial y} = 0 \Leftrightarrow 4e^y e^y - 16e^{4y} = 0$$

$$\Leftrightarrow e^{2y} - 4e^{4y} = 0.$$

$$\Leftrightarrow \quad e^{2y}(1-4e^{2y})=0$$

$$\Leftrightarrow \quad 1-4e^{2y}=0, \text{ as } e^{z}>0.$$

$$\Leftrightarrow \quad e^{2y}=\frac{1}{4}$$

$$\Leftrightarrow \quad e^{y}=\sqrt{1/4}=\frac{1}{2}, \qquad -\frac{1}{2} \text{ gives no solutions.}$$

$$\Leftrightarrow \quad y=\ln(1/2).$$

And here $x^2=e^{y}=1/2.$

$$x=\pm\sqrt{1/2}$$

So there are <u>two</u> critical points

$$\left(-\sqrt{1/2}, \ln(1/2)\right), \left(\sqrt{1/2}, \ln(1/2)\right)$$

These are classified using.

$$D=\frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial y^2}-\left(\frac{\partial^2 g}{\partial x \partial y}\right)^2.$$

$$D\left(-\frac{1}{\sqrt{2}}, \ln\left(\frac{1}{2}\right)\right)$$

$$= D\left(\frac{1}{\sqrt{2}}, \ln\left(1/2\right)\right)$$

$$= (-8)(-3) - \left(\frac{4}{\sqrt{2}}\right)^2$$

$$= 24 - 8 = 16 > 0$$

and $\dfrac{\partial^2 g}{\partial x^2} = -8 < 0$

So $\left(\pm\frac{1}{\sqrt{2}}, \ln\left(\frac{1}{2}\right)\right)$

are both local maximums.