Taylor series in multi-variables

Recall for single variable function $f(x)$, it has a Taylor series, expanded around 0,

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.}$$

Taylor series.
Mclaurin series.

where $f^{(n)}$ is the n^{th} derivative $\frac{d^n f}{dx^n}$

which is valid for some neighborhood of 0, which could be all of $\mathbb{R}$ for certain functions.

For partial finite sums of the series.

$$f(x) \approx \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n$$

we get an approximation to f, getting better with $k \to \infty$.

What about for multi-variable functions?

Consider a function $f(x, y)$ and its behaviour near (a, b)

we'll write $h := \Delta x$ and $k := \Delta y$, for the small variations in x, y. Then the Taylor series for f about (a, b) will be

$$f(a+h, b+k) = \sum_{n=0}^{\infty} \frac{1}{n!} (D^n f)(a, b)$$

where D is the differential operator

$$D = h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}$$

and by $D^n f$ we mean

$$D^n f = \underbrace{D(D(D(\ldots(Df))\ldots)}_{n \text{ copies of } D.}$$

So $D^0 f = f$

$$Df = h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}$$

$$D^2 f = D(Df)$$
$$= \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)\left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\right)$$

$$= h^2 \frac{\partial^2 f}{\partial x^2} + \underbrace{hk \frac{\partial^2 f}{\partial x \partial y} + kh \frac{\partial^2 f}{\partial y \partial x}}$$

$$+ k^2 \frac{\partial^2 f}{\partial y^2}$$

$$= h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}$$

and in general

$$D^n f = \sum_{j=0}^{n} \binom{n}{j} h^j k^{n-j} \frac{\partial^n f}{\partial x^j \partial y^{n-j}}$$

where $\binom{n}{j}$ is the binomial coefficient "n choose j"

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$$

<u>Example</u>

$f(x,y) = \sin(x+3y) + \cos(3x+y)$

Find the Taylor expansion of f

around $\left(\pi/2, 0\right) = (a,b)$ to get an approximation.

$f(\pi/2 + h, k) \approx$ poly in h, k.

$$\boxed{(D^0 f)(\pi/2, 0)} = f(\pi/2, 0)$$

$$\boxed{= 1}$$

$$\boxed{(D^1 f)(\pi/2, 0)} = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)\Bigg|_{(x,y)=\left(\frac{\pi}{2},0\right)}$$

$$= \Big[\left(h\left(\cos(x+3y) - 3\sin(3x+y)\right)\right) + k\left(3\cos(x+3y) - \sin(3x+y)\right)\Big]_{(\pi/2,0)}$$

$$\boxed{= 3h + k}$$

For $D^2 f$, we'll need

at $(\pi/2, 0)$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+3y) - 9\cos(3x+y) = -1$$

$$\frac{\partial^2 f}{\partial y^2} = -9\sin(x+3y) - \cos(3x+y) = -9$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3\sin(x+3y) - 3\cos(3x+y) = -3$$

So $(D^2 f)(\pi/2, 0) = -h^2 - 9k^2 - 6hk$

So putting this together gives the order 2 approximation.

$$f\left(\frac{\pi}{2}+h, k\right) \approx 1 + 3h + k - \frac{1}{2}h^2 - \frac{9}{2}k^2 - 3hk$$

Let's visualize this with computer.

This all extends to 3 or more variables in "obvious" way

say $f(x_1, \ldots, x_n)$

then

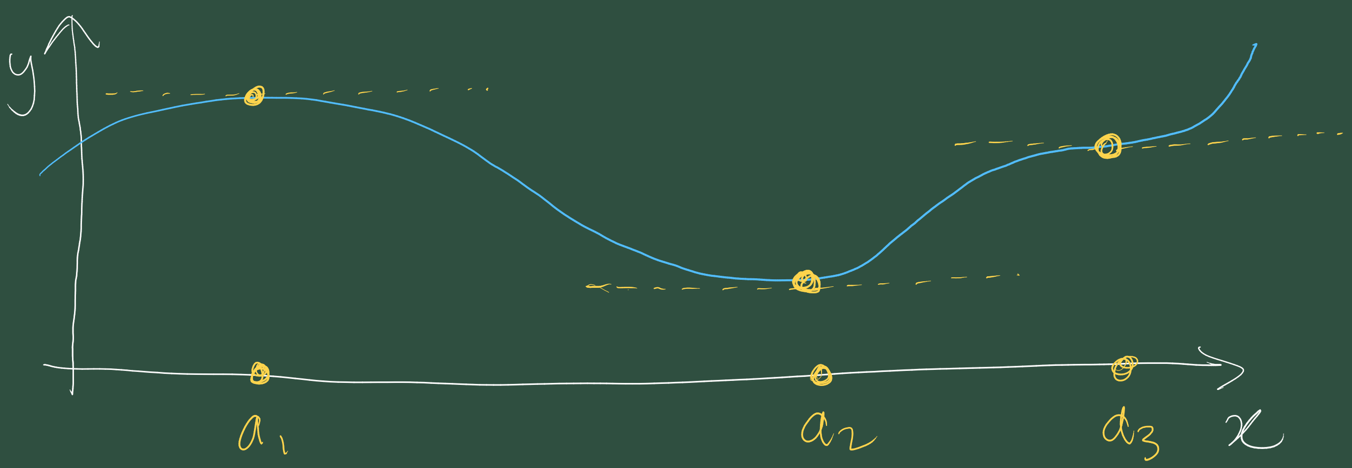
$$f(a_1+h_1, \ldots, a_n+h_n) = \sum_{n=0}^{\infty} \frac{1}{n!}(D^n f)(a_1, \ldots, a_n)$$

where $$D = \sum_{j=1}^{n} h_j \frac{\partial}{\partial x_j}$$

Optimisation in 2 variables.

Finding and classifying critical/stationary points on surface defined by $f(x, y)$

1-variable case $\quad y = f(x)$



Three types of critical point local maximums (a_1), local minimums (a_2), inflection point (a_3), all characterised by $\frac{df}{dx} = 0$ and were classified by $\frac{d^2f}{dx^2}$ as

$$\left.\frac{d^2f}{dx^2}\right|_{a_1} < 0 \implies \text{local max.}$$

$$\left.\frac{d^2f}{dx^2}\right|_{a_2} > 0 \implies \text{local min.}$$

$$\left.\frac{d^2f}{dx^2}\right|_{a_3} = 0 \implies \text{inflection point}$$

For the surface $z = f(x,y)$

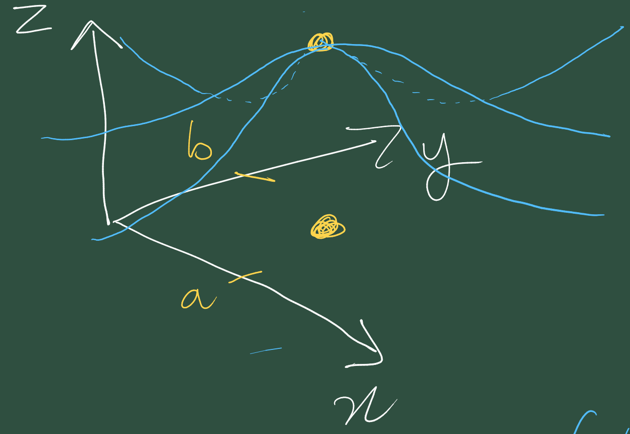
<u>Define</u> a critical point is a point (a,b) where both partial derivatives vanish

i.e. $\frac{\partial f}{\partial x}(a,b) = 0$ & $\frac{\partial f}{\partial y}(a,b) = 0$.

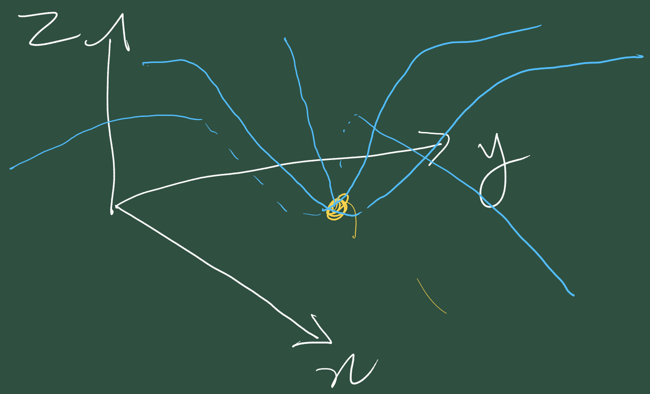
So geometrically the tangent plane to the surface at (a,b) is horizontal.

Again they come in three types

local max. ("top of a hill")



local min. ("bottom of a crater")



saddle point (mixture of a min/max)

Again these are classified using second-order derivatives.
The Hessian determinant is

$$D = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\[2ex] \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

$$= \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$

And if

$D(a,b) < 0$ then (a,b) is a saddle

If $D(a,b) > 0$

then $\begin{cases} \dfrac{\partial^2 f}{\partial x^2}(a,b) > 0 \Rightarrow \text{local min.} \\[2ex] \dfrac{\partial^2 f}{\partial x^2}(a,b) < 0 \Rightarrow \text{local max.} \end{cases}$

we can justify by analysing the 2^{nd} order Taylor poly.

<u>Example</u>

Consider

$f(x,y) = x^3 + 3xy^2 - 15x - 12y$.

Find and classify its critical points.

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 15$$

$$\frac{\partial f}{\partial y} = 6xy - 12$$

Tip: Look at simpler one first.

$$\frac{\partial f}{\partial y} = 0 \iff 6xy - 12 = 0.$$

$$\iff y = \frac{2}{x}.$$

Now examine $\frac{\partial f}{\partial x} = 0$ under this assumption.

$$\frac{\partial f}{\partial x} = 0 \iff 3x^2 + 3\left(\frac{2}{x}\right)^2 - 15 = 0$$

$$\iff 3x^2 + \frac{12}{x^2} - 15 = 0$$

$$\iff 3x^4 + 12 - 15x^2 = 0$$

$$\iff x^4 - 5x^2 + 4 = 0$$

$\Leftrightarrow (x^2-1)(x^2-4)=0$

$\Leftrightarrow x=-1,+1,-2,+2.$

Combining these with $y=\frac{2}{x}$

we get four critical points.

$(-1,-2),(1,2),(-2,-1),(2,1)$

Hessian determinant

$$D=\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2.$$

$$=6x\,6x-(6y)^2$$

$$=36x^2-36y^2$$

$$=36(x^2-y^2).$$

$$D(-2,-1)=D(2,1)=36(3)$$
$$=108>0.$$

$$\frac{\partial^2 f}{\partial x^2}=\begin{matrix}-12, \text{ at } x=-2\\ 12, \text{ at } x=2\end{matrix}$$

So $(-2,-1)$ is a local max.

and $(2,1)$ is a local min.

$$D(-1,-2) = D(1,2) = 36(-3)$$

$$= -108 < 0$$

So $(-1,-2)$ and $(1,2)$ are both saddles.

Let's confirm all this with Matlab.