Quick review of single variable derivative

Consider $f : \mathbb{R} \to \mathbb{R}$

The derivative $f'(a)$ (or written as $\left.\frac{df}{dx}\right|_{x=a}$)

defined as

$$\boxed{f'(a) = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right]}$$

abs change in value of f

change in argument

relative change in value of f.

or

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Let's consider two-variable functions

$$\text{Consider } f : \underset{x}{\mathbb{R}} \times \underset{y}{\mathbb{R}} \longrightarrow \mathbb{R}$$

So values look like $f(x, y) \in \mathbb{R}$

This has two partial derivatives

$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ defined by.

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{x \to a} \left(\frac{f(x,b) - f(a,b)}{x-a} \right)$$

$$\overset{\text{or}}{=} \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

and similarly

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y-b}.$$

$$\overset{\text{or}}{=} \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

Example

Consider f defined by $f(x,y) = x^2 y$. Find its partial derivatives.

$$\frac{\partial f}{\partial x} = \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x-a}.$$

$$= \lim_{x \to a} \left(\frac{x^2 b - a^2 b}{x-a} \right)$$

$$= \lim_{x \to a} \frac{(x^2 - a^2) b}{x-a}$$

$$= \lim_{x \to a} \left(\frac{\cancel{(x-a)}(x+a) b}{\cancel{x-a}} \right)$$

$$= \lim_{x \to a} (x+a) b = \boxed{2ab = \left.\frac{\partial f}{\partial x}\right|_{(a,b)}}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

$$= \lim_{h \to 0} \frac{a^2(b+h) - a^2 b}{h}.$$

$$= \lim_{h \to 0} a^2 = a^2$$

To speak generally about all possible fixed points (a,b) we can just say.

$$\frac{\partial}{\partial x}\left(x^2 y\right) = 2xy$$

$$\frac{\partial}{\partial y}\left(x^2 y\right) = x^2$$

Important principle we see here

To differentiate f with respect to x treat the other variable(s) as constants and use known results and properties of single-variable calculus to differentiate with respect to x.

Example

Consider $f(x,y) = x^2 y^3 \tan(2x)$

Find the two partial derivatives.

$$\boxed{\frac{\partial f}{\partial x}} = y^3 \frac{\partial}{\partial x}\left(x^2 \tan(2x)\right)$$ by linearity, as y^3 is a constant.

$$= y^3\left(2x\tan(2x) + 2x^2\sec^2(2x)\right)$$

using $\frac{d}{dx}\tan(x) = \sec^2(x)$

$\sec = \frac{1}{\cos}$

$$= \boxed{2y^3\left(x\tan(2x) + x^2\sec^2(2x)\right)}$$

$$\boxed{\frac{\partial f}{\partial y}} = \frac{\partial}{\partial y}\left(x^2 y^3 \tan(2x)\right)$$

$$= x^2\tan(2x)\frac{\partial}{\partial y}\left(y^3\right), \text{ by linearity}$$

$$= \boxed{3x^2y^2\tan(2x)}$$

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other notations you might see.

$$f_x = \frac{\partial f}{\partial x}, \qquad f_y = \frac{\partial f}{\partial y}$$

$$\underbrace{\frac{\partial}{\partial x}}_{\text{diff. op.}}\left(f\right)$$

Geometric / graphical interpretation of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

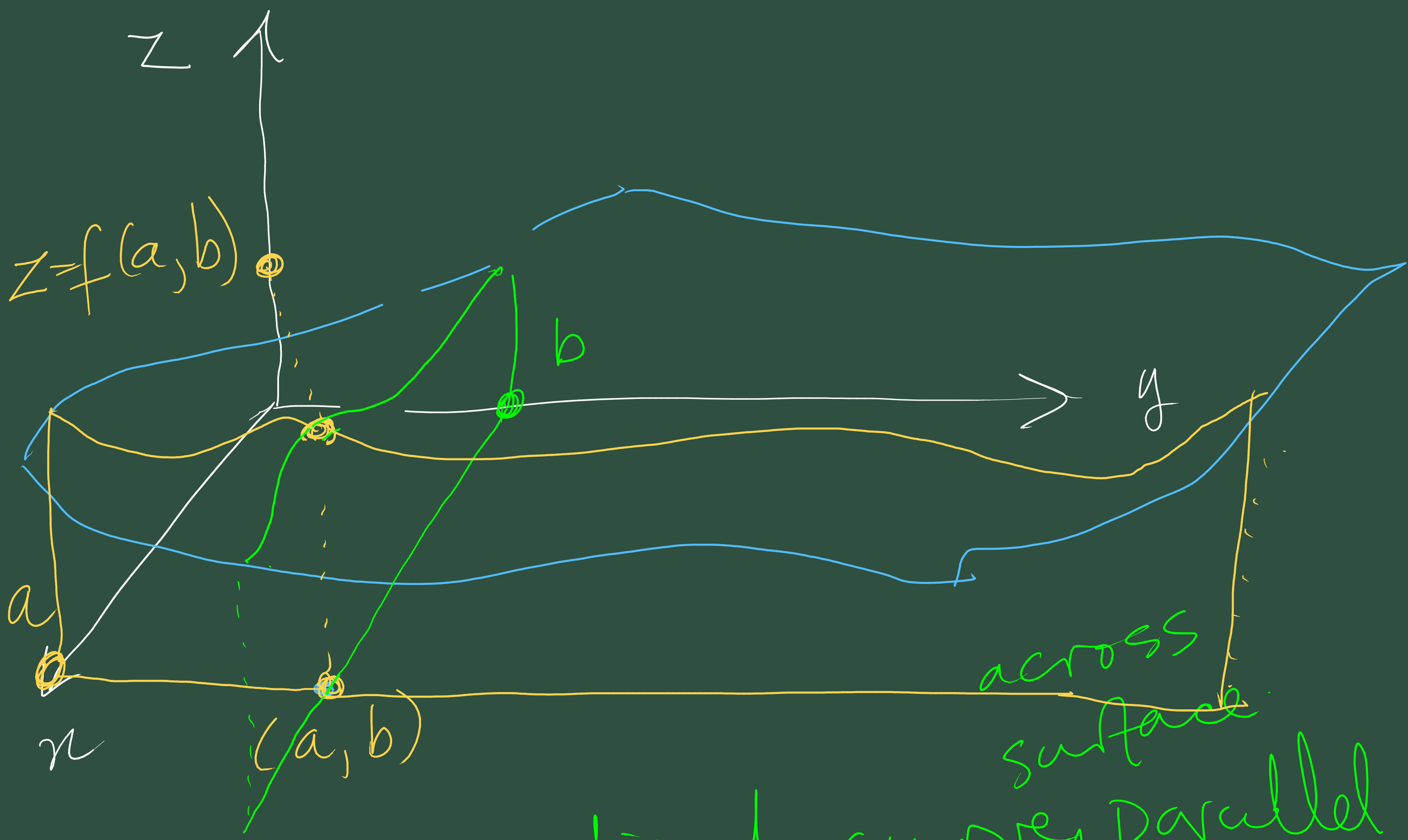
Single-variable case.

$f'(a) = \left.\frac{df}{dx}\right|_{x=a}$ was the gradient of this tangent line

i.e. the instantaneous rate of change of f at $x = a$.

Partial derivatives of $f(x,y)$

$z = f(x,y)$ defines a surface of height $z = f(x,y)$ above the (x,y) plane.



green cross-sectional curve, parallel to x-axis, at the location $y=b$

yellow " " " " " " "

" y-axis " " " $x=a$

$\left.\dfrac{\partial f}{\partial x}\right|_{(a,b)}$ is the gradient of tangent to $z=f(x,b)$ at $x=a$.

$\left.\dfrac{\partial f}{\partial y}\right|_{(a,b)}$ " " " "

" " $z=f(a,y)$ at $y=b$.

Higher-order derivatives

Just like in single-variable case we have higher order derivatives such as

$$\frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Clairaut's Theorem say.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

E.g. $f(x,y) = x^2 y$.

$\dfrac{\partial f}{\partial x} = 2xy$ from before.

So $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}(2xy) = 2y$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(x^2\right) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2\right) = 2x$$

$$\|$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy) = 2x$$

And higher-order still.

$\dfrac{\partial^3 f}{\partial x^3} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 f}{\partial x^2}\right)$ and so on

CLP3 Sec 2.3. Exercises

Q6.

$$f(r,\theta) = r^m \cos(m\theta)$$

(a) $$f_r = \frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left(r^m \cos(m\theta)\right)$$

$$= \cos(m\theta)\frac{\partial}{\partial r}\left(r^m\right)$$

$$= \cos(m\theta)\, m r^{m-1}$$

$$f_{rr} = \frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right)$$

$$= \frac{\partial}{\partial r}\left(\cos(m\theta)\, m r^{m-1}\right)$$

$$= m\cos(m\theta)\frac{\partial}{\partial r}\left(r^{m-1}\right)$$

$$= m\cos(m\theta)\,(m-1)\,r^{m-2}$$, providing $m \geq 2$.

or $= 0$ if $m = 0, 1$.

$$f_\theta = \frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta}\left(r^m \cos(m\theta)\right)$$

$$= r^m \frac{\partial}{\partial \theta}(\cos(m\theta))$$

$$= r^m(-\sin(m\theta))\, m$$

$$= -m r^m \sin(m\theta)$$

$$f_{\theta\theta} = \frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial \theta}\left(-m r^m \sin(m\theta)\right)$$

$$= -m r^m \frac{\partial}{\partial \theta}(\sin(m\theta))$$

$$= -m r^m \cos(m\theta)\, m$$

$$= -m^2 r^m \cos(m\theta).$$

$$f_{r\theta} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial r}\left(-m r^m \sin(m\theta)\right)$$

$$= -m^2 r^{m-1} \sin(m\theta)$$

(b) The P.D.E. is.

$$f_{rr} + \frac{\lambda}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0$$

$$\Leftrightarrow m(m-1) r^{m-2} \cos(m\theta)$$

$$+ \frac{\lambda}{r} m r^{m-1} \cos(m\theta)$$

$$+ \frac{1}{r^2} \left(-m^2 r^m \cos(m\theta) \right) = 0$$

$$\Leftrightarrow r^{m-2} \cos(m\theta) \left(m(m-1) + m\lambda - m^2 \right) = 0$$

$$\Leftrightarrow r^{m-2} \cos(m\theta) \left(m(-1+\lambda) \right) = 0$$

True iff $\lambda = 1$

So $f(r,\theta)$ is a solution to

$$f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0 .$$