Chain rule and small ~~irerements~~ formula

increments

Small increments

Recall for a function $f(x)$ of a single variable with derivative $f'(x)$

small ~~irerements~~ / small change approximation

$$f(x+\Delta x) \approx f(x) + \Delta x \, f'(x)$$

OR.

$$\boxed{\Delta f} = \underbrace{f(x+\Delta x)}_{\text{new}} - \underbrace{f(x)}_{\text{old}} \approx \boxed{\Delta x \, f'(x)}$$

This will generalise / extend to multi-variable setting

So for a function $f(x,y)$

$$\boxed{\Delta f} = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$\approx \boxed{\Delta x \frac{\partial f}{\partial x} + \Delta y \frac{\partial f}{\partial y}}$$

where both partial derivatives are evaluated at the old / base point.

The general extension to a function

$f(x_1, \ldots, x_n)$

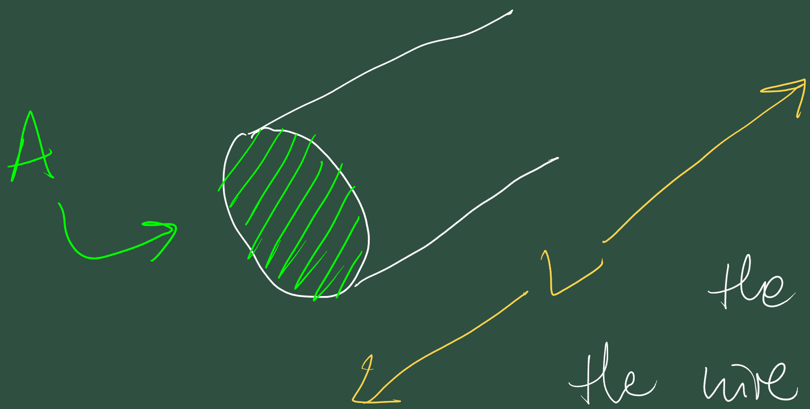
$$\boxed{\Delta f \approx \sum_{i=1}^{n} \Delta x_i \frac{\partial f}{\partial x_i}}$$

<u>Example</u>

The resistance of a length of wire is given by the formula

$$R = \rho \frac{L}{A}, \qquad \rho \text{ "rho"}$$

where L, A are the length and ~~cro~~ cross-section area of the piece of wire



and ρ is the resistivity of the wire material.

Suppose A decreases by 1% and L increases by 2.5%. Use the small increments formula to approximate the corresponding ΔR.

Start with the formula:

$$\Delta R \approx \Delta L \frac{\partial R}{\partial L} + \Delta A \frac{\partial R}{\partial A}.$$

Now $\Delta L = 0.025 L$ and $\Delta A = -0.01 A$

$$\frac{\partial R}{\partial L} = \frac{\partial}{\partial L}\left(\rho \frac{L}{A}\right) = \frac{\rho}{A}$$

$$\frac{\partial R}{\partial A} = \frac{\partial}{\partial A}\left(\rho \frac{L}{A}\right) = -\rho \frac{L}{A^2}$$

$$\boxed{\Delta R} \approx 0.025 L \frac{\rho}{A} - 0.01 A\left(-\rho \frac{L}{A^2}\right)$$

$$= 0.025 \underbrace{\left(\rho \frac{L}{A}\right)}_{R} + 0.01 \underbrace{\left(\rho \frac{L}{A}\right)}_{R}$$

$$= \boxed{0.035 R}$$

So ΔR is approximately an increase of 3.5%

Chain rule

1-variable summary.

Suppose $y = f(x)$ and moreover that $x = g(t)$

So $y = f(x) = f(g(t)) = (f \circ g)(t)$

Chain rule tells us how $\frac{dy}{dt}$

depends on $\frac{df}{dx}$ and $\frac{dg}{dt}$

$$\frac{dy}{dt} = \frac{df}{dx} \cdot \frac{dg}{dt}$$

OR.

$$y'(t) = f'(g(t))\, g'(t)$$

So we need to understand

If $y = f(x_1, \ldots, x_n)$ and the x_i in turn are given by

$$x_i = u_i(z_1, \ldots, z_n) \text{ for each } 1 \le i \le n.$$

Then how does.

$$\frac{\partial y}{\partial z_j} \text{ depend on } \frac{\partial y}{\partial x_i} \text{ and } \frac{\partial u_i}{\partial z_j}?$$

Focus on 2-variable case

Suppose $z = f(x, y)$ and a second coord system s, t and functional relations

$$x = u(s,t) \quad \& \quad y = v(s,t)$$

Start with small Increments formula

$$\Delta z \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

and make it all relative to a change, Δs, in s

$$\frac{\Delta z}{\Delta s} \approx \frac{\partial f}{\partial x} \frac{\Delta x}{\Delta s} + \frac{\partial f}{\partial y} \frac{\Delta y}{\Delta s}$$

Consider the effect of letting $\Delta s \to 0$. This will give.

$$\frac{\Delta z}{\Delta s} \longrightarrow \frac{\partial z}{\partial s}, \quad \frac{\Delta x}{\Delta s} \longrightarrow \frac{\partial x}{\partial s}, \quad \frac{\Delta y}{\Delta s} \longrightarrow \frac{\partial y}{\partial s}$$

and the $\approx$ becomes $=$

and in summary.

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$$

and similarly can show

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

These are the chain rule equations for the transformation $(x, y) \longrightarrow (s, t)$

In general for a transformation from $(x_1, \dots, x_n) \longrightarrow (s_1, \dots, s_n)$ we get a set of chain rule equations:

$$\boxed{\frac{\partial z}{\partial s_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial s_i} \quad \text{for each } 1 \le i \le n}$$

End of section exercises 2.4 CLP 3

Q6.

~~Fin~~ $w = x^2 + y^2 + z^2$

and $x = st$, $y = s\cos(t)$, $z = s\sin(t)$

obtain $\frac{\partial w}{\partial s}$, $\frac{\partial w}{\partial t}$.

Method 1 Direct substitution

$$w = (st)^2 + (s\cos(t))^2 + (s\sin(t))^2$$

$$= s^2t^2 + s^2 \underbrace{(\cos^2(t) + \sin^2(t))}_{=1}.$$

$$w = s^2(t^2+1).$$

Then take partial derivatives as normal

$$\boxed{\frac{\partial w}{\partial s} = 2s(t^2+1)}$$

$$\frac{\partial w}{\partial t} = s^2\,2t = 2s^2t$$

Method 2. Employ chain rule.

$$\boxed{\frac{\partial w}{\partial s}} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= 2xt + 2y\cos(t) + 2z\sin(t)$$

$$= 2stt + 2s\cos(t)\cos(t) + 2s\sin(t)\sin(t)$$

$$= 2st^2 + 2s$$

$$= \boxed{2s(t^2+1)}, \text{ as before}$$

Similarly can also confirm expression for $\frac{\partial w}{\partial t}$.

Q23 (a).

f is a single variable function.

define $w(x,y)$ as

$$w(x,y) = e^{-y} f(x-y)$$

Show $w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$.

Think of f as f(t). Then a new coord system v is brought in

(x,y)

under transformation $t = x - y$.

so that.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial x}$$

&

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial y}$$

$$\frac{\partial t}{\partial x} = 1$$

$$\frac{\partial t}{\partial y} = -1$$

So $\boxed{\frac{\partial w}{\partial x}} = \frac{\partial}{\partial x}\left(e^{-y} f(x-y)\right)$

$$= e^{-y}\frac{\partial f}{\partial x}$$

$$= e^{-y} \frac{\partial f}{\partial t} \frac{\partial t}{\partial x}$$

$$\boxed{= e^{-y} \frac{\partial f}{\partial t}}$$

AND

$$\boxed{\frac{\partial w}{\partial y}} = \frac{\partial}{\partial y}\left(e^{-y} f(x-y) \right)$$

$$= e^{-y} \frac{\partial}{\partial y}\left(f(x-y) \right)$$

$$+ \; -e^{-y} f(x-y), \quad \text{prod. rule.}$$

$$= e^{-y} \cdot \frac{\partial f}{\partial t} \frac{\partial t}{\partial y} - e^{-y} f(x-y)$$

$$\boxed{= -e^{-y} \frac{\partial f}{\partial t} - e^{-y} f(x-y)}$$

So back to the PDE

$$w + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

$$= e^{-y} f(x-y) + e^{-y}\frac{\partial f}{\partial t}$$

$$- e^{-y}\frac{\partial f}{\partial t} - e^{-y} f(x-y)$$

$= 0$, as expected.

Look again at general chain rule.

$$\frac{\partial z}{\partial s_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\,\frac{\partial x_j}{\partial s_i}$$

This R.H.S. looks like something from Linear Algebra.

$$\underbrace{\Box_i}_{\text{vector}} = \sum_{j=1}^{n} \underbrace{\Box_j}_{\text{vector.}} \underbrace{\Box_{i,j}}_{\text{matrix}}$$

This looks likes obtaining a (row) vector as a product of a (row) vector with a matrix

In the 2-variable case can express the pair of chain rule equations as:

$$\left(\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\underbrace{\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}}$$

Known as the Jacobian matrix of the transformation from $(x, y) \to (s, t)$

In general "the Jacobian" refers to the determinant of this

So for

$$(x_1, \dots, x_n) \longrightarrow (s_1, \dots, s_n)$$

$$|J| = \frac{\partial(x_1, \dots, x_n)}{\partial(s_1, \dots, s_n)}$$

"notation for the Jacobian determinant"

$$= \left| \left(\frac{\partial x_i}{\partial s_j} \right)_{i,j} \right|$$

the $n \times n$ matrix determinant.

We'll see meaning/relevance/interpretation of this when we're changing coordinate systems in future integrals.