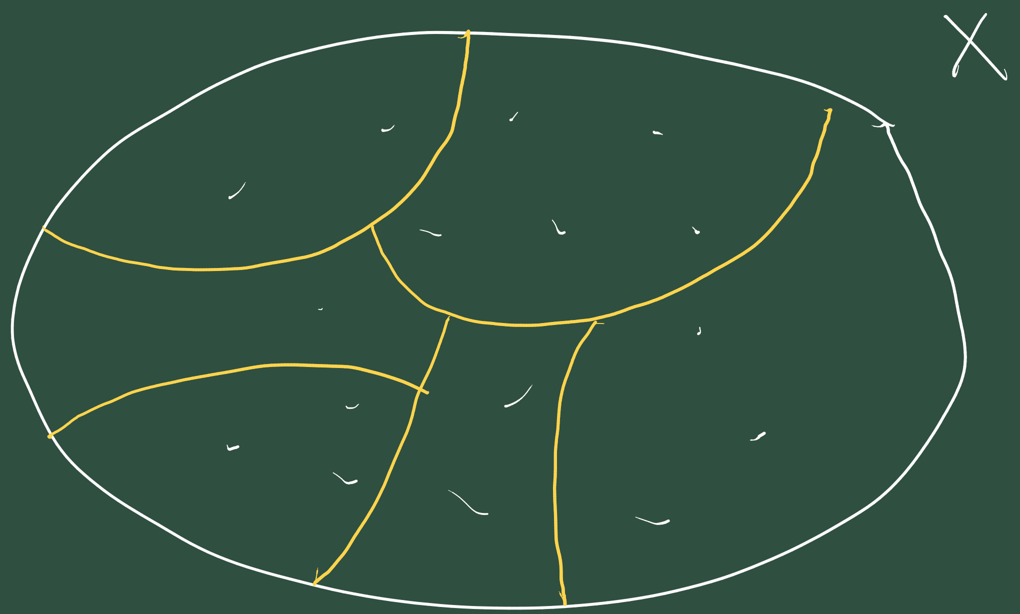
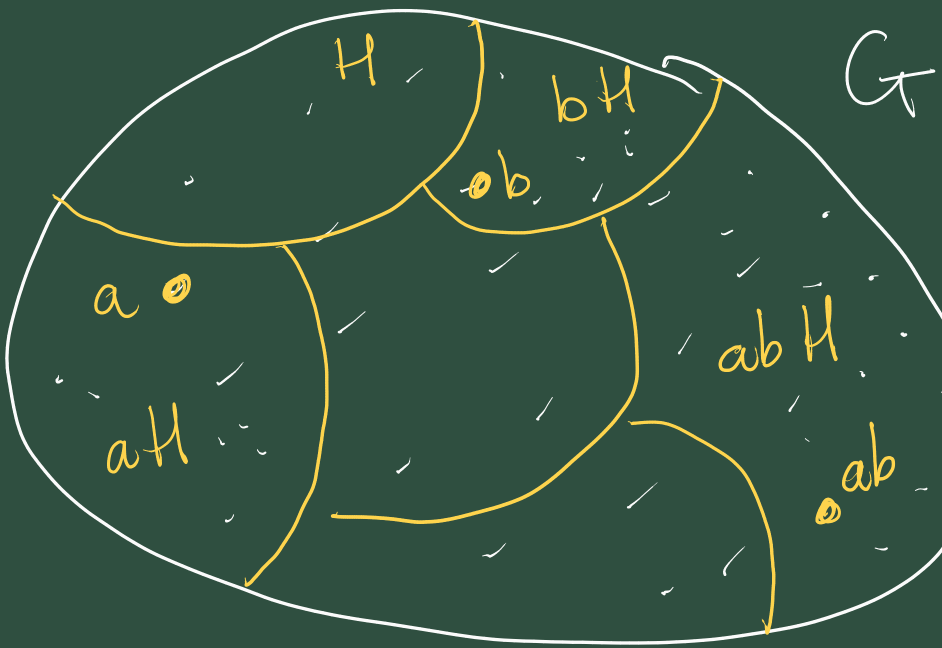
Motivation for factor groups and normal subgroups.

Factor groups are a way getting a simplified view of a particular group with reference to one of its subgroups.

1. Recall the concept of a set partition. View this as a way to simplify a large set.



2. Can also partition a group using cosets of a certain subgroup. eg. a group G with a subgroup H



Initially a set partition of the underlying set G.

This would be improved / be more useful if this partition had a group structure, related to the structure of G.

Is there a "multiplication"/"product" we can perform on the cosets themselves?

Well there is a strong candidate for one, ie. the natural product of sets arising from the product in G.

given two subsets $X, Y \subseteq G$ we can multiply them to give another subset. by

$$XY = \{ xy \in G : x \in X, y \in Y \}$$

product of elements of G

But when these subsets are cosets, ie. $X = aH$, $Y = bH$ say, in order to make everything work and "hang together" we would require that

$$\boxed{(aH)(bH) = abH.}$$

Q? When might this be true? ie. what property does H have to have?

$$(aH)(bH) = (ab)H.$$

$$\Leftrightarrow \forall h_1, h_2 \in H \; \exists h_3 \in H \quad a h_1 b h_2 = ab h_3$$

$\Leftrightarrow \forall h_1, h_2 \exists h_3 \quad h_1 b h_2 = b h_3$

$\Leftrightarrow \forall h_1, h_2 \exists h_3 \quad h_1 b = b h_3 h_2^{-1}$

$\Leftrightarrow \boxed{\forall h_1 \exists h_4 \quad h_1 b = b h_4}$

$\Leftrightarrow \quad Hb = bH.$

if $\forall b \in G$ $Hb = bH$, i.e. left cosets and right cosets are the same then H is called a normal subgroup.

This allows us to form that "Factor group" of G by H (also called "quotient group")

which is the group of (left) cosets of H in G under the operation

$$aH\, bH = abH$$

this will be a simplified version of

the original group G, it's denoted by

G/H.

Section 10.1.

Def of normal subgroup.

<u>Eg 10.1</u> If G is abelian then all its subgroups are normal since.

$$gH = \{gh : h \in H\}$$
$$= \{hg : h \in H\} \text{ since } gh = hg$$
$$= Hg$$

So the genuinely interesting cases are where G is non-abelian.

<u>Eg. 10.2</u>

$G = S_3$, $H = \{(1), (12)\}$

$(123)(12) = (13)$

H is <u>not</u> normal in S_3

$(123)H = \{(123), (13)\}$

$H(123) = \{(123), (12)\}$

$(13)(123) = (12)$

and so $(123)H \neq H(123)$

But $N = \{(1), (123), (132)\}$

is a normal subgroup of S_3

and S_3/N consists of the cosets

$N, \ (12)N$

the Factor group S_3/N has

Cayley table

	N	$(12)N$
N	N	$(12)N$
$(12)N$	$(12)N$	N

which is the only possible group structure for a group of order 2, i.e.

$\cong \mathbb{Z}_2$

	0	1
0	0	1
1	1	0

Theorem 10.3 gives us alternative views of concept of Normality.

1. N normal in G

$\forall g \in G \quad gN = Ng$

$\iff \forall g \in G \quad \underbrace{gNg^{-1}}_{\text{the conjugate of } N \text{ by } g} = N.$

Theorem 10.4 Prove all these claims.

namely, if N is a normal subgroup of G then set of cosets, G/N, is a group under the operation.

$$aN\, bN = abN$$

with identity $eN = N$

and inverse elements which satisfy.

$$(aN)^{-1} = \underline{a^{-1}}N$$

Eg 10.6 $G = (\mathbb{Z}, +)$

$N = 3\mathbb{Z} = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$

a normal subgroup of G

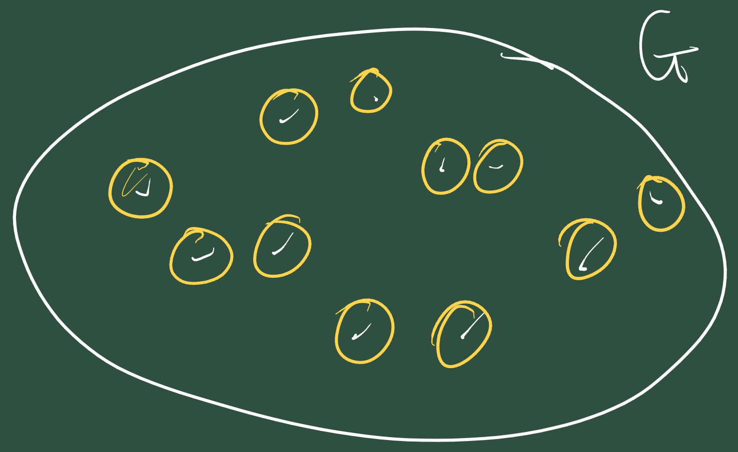
The factor group $\mathbb{Z}/3\mathbb{Z}$ is exactly the group $\mathbb{Z}_3$ of integers modulo 3 under addition.

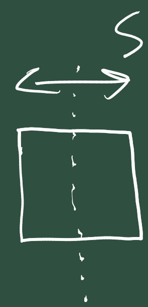
Factor groups are simplifications of the original group.
Which leads to the concept of a "simple group"

A simple group G is one that has no <u>proper</u> <u>non-trivial</u> normal subgroups. So it cannot be simplified using this concept of factor groups.

Remember, every group G has the two normal subgroups G, $\{e\}$
But the factor groups arising from these are not interesting simplifications.
as. $G/G \cong \{e\}$, the trivial group

and $G/\{e\} \cong G$





$\underline{Q2.}$ $D_4 = \langle r, s \rangle$

where $r^4 = e$, $s^2 = e$,

$sr = r^{-1}s$

rotational

$D_4 = \{e, r, r^2, r^3,$
$s, rs, r^2s, r^3s\}$

D_4 has subgroups $\{e\}$, D_4

$R_1 = \langle r \rangle = \{e, r, r^2, r^3\} = \langle r^3 \rangle$

$R_2 = \langle r^2 \rangle = \{e, r^2\}$

$S_1 = \langle s \rangle = \{e, s\}$

$S_2 = \langle rs \rangle = \{e, rs\}$

2 element subgroups

$$S_3 = \langle r^2 s \rangle = \{e, r^2 s\}$$
$$S_4 = \langle r^3 s \rangle = \{e, r^3 s\}$$
reflectional

there is one more subgroup

$$A = \langle r^2, s \rangle = \{e, r^2, s, r^2 s\}$$

$$B = \langle r^2, rs \rangle = \{e, r^2, rs, r^3 s\}$$

mixed

$$C = \langle r^2, r^2 s \rangle = \{e, r^2, r^2 s, s\} = A.$$

$$D = \langle r^2, r^3 s \rangle = B.$$

So we have proper non-trivial subgroups $R_1, R_2, \underbrace{S_1, S_2, S_3, S_4}_{\text{not normal.}}$, A, B.

normal. $|A| = \frac{|D_4|}{2}$

Which are normal?

$\underline{R_1}$ Note $|R_1| = 4 = \underline{|D_4|}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2.$

in such cases $\left(|H| = \frac{|G|}{2}\right)$ are always for ~~these~~ the only cosets are R_1, $D_4 \setminus R_1$, i.e. the complement of R_1.

So factor group exists and

$D_4/R_1 \cong \mathbb{Z}_2$

	R_1	sR_1
R_1	R_1	sR_1
sR_1	sR_1	R_1

$\underline{R_2} = \{e, r^2\}$ Is R_2 normal in D_4?

Is $xR_2 = R_2x$ for all x in D_4?

well $x = r^i s^j$ $\quad i = 1,2,3,0$, $j = 0,1$

we just need to check

if $x = r^i$ then $xr^2 = r^i r^2 = r^{i+2} = r^2 r^i$
$= r^2 x$

but if $x = r^i s$

then $xr^2 = r^i s r^2 = r^i r^{-2} s = r^{i-2} s$

$\boxed{\text{But } r^{-2} = r^2.}$

and $r^2 x = r^2 r^i s = r^{i+2} s$

So xr^2 and r^2x not necessarily the same.

$R_2 = \{e, r^2\}$

eg when $x = s$

$sR_2 = \{s, sr^2 = r^{-2}s = r^2 s\}$

$R_2 s = \{s, r^2 s\} = sR_2.$

try.

$rsR_2 = \{rs, rsr^2 = rr^{-2}s = r^{-1}s = r^3 s\}.$

$R_2 rs = \{rs, r^2 rs = r^3 s\}.$

$\boxed{\text{So it seems } R_2 \text{ is normal in } D_4 \text{ also.} \quad r^{-2} = r^2.}$

and no there in the factor group.

$|D_4 / R_2| = 4.$

$\rho \quad \sigma \quad \rho\sigma$

	R_2,	rR_2,	sR_2,	rsR_2
R_2	R_2	rR_2	sR_2	rsR_2
rR_2	rR_2	R_2	rsR_2.	sR_2.
sR_2	sR_2	rsR_2	R_2	rR_2.
rsR_2	rsR_2	sR_2	rR_2	R_2

$$\cong D_2 = \langle \rho, \sigma \rangle$$

$$\rho^2 = \sigma^2 = e$$

$$\sigma\rho = \rho^{-1}\sigma$$