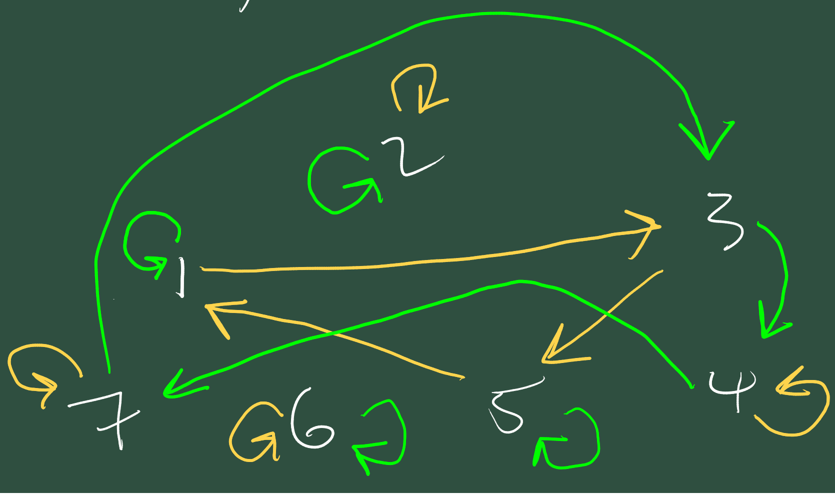
Example 5.7

$$\underbrace{(1\,3\,5)(3\,4\,7)}_{\text{Composition of the two functions } \sigma\circ\mu}=(2)(6)(1\,3\,4\,7\,5)=(1\,3\,4\,7\,5).$$

"σ after μ"

$$(\sigma\circ\mu)(2)=\sigma(\mu(2))=2.$$

$$\underset{\substack{\downarrow\\(1\,3\,5)}}{\sigma},\ \underset{\substack{\downarrow\\(3\,4\,7)}}{\mu}\in S_7.$$



$$\mu\sigma = (3\,4\,7)(1\,3\,5)$$
$$= (1\,4\,7\,3\,5)(2)(6)$$
$$= (1\,4\,7\,3\,5) \neq \sigma\mu.$$
$$= (7\,3\,5\,1\,4)$$

If a group is Abelian then its Cayley table is symmetric about the main diagonal.

$\cdot$	x	y
x		xy
y	yx	

Mock. Q6.

(a). Let G be a group with element g.

The subgroup of G generated by g is written as $\langle g \rangle$ and is defined

$$\langle g \rangle = \{ g^k : k \in \mathbb{Z} \}.$$

Where powers are defined as.

Remember:

For $k > 0$

$$g^k = \underbrace{g * g * \dots * g}_{k \text{ copies of } g.}$$

$$g^{-k} = \underbrace{g^{-1} * g^{-1} * \dots * g^{-1}}_{k \text{ copies of } g^{-1}}$$

$$g^0 = e.$$

Well the order of g, written as $|g|$ is defined as:

the least positive integer exponent

k such that $g^k = e$, if such a k exists, and ∞ otherwise.

(b). No, a counter-example is provided by Klein Viergruppe

$$V = \{e, r, h, v\}$$

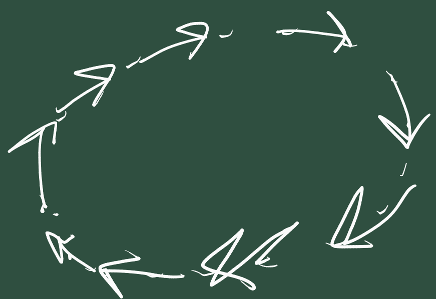
where all non-identity elements, r, h, v, have order 2, i.e.

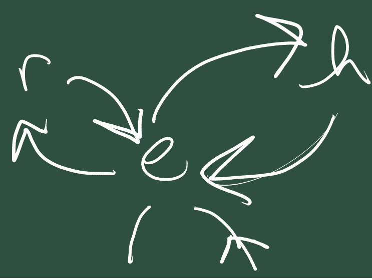
$$r^2 = h^2 = v^2 = e.$$

so $V \neq \langle r \rangle, \neq \langle h \rangle, \neq \langle v \rangle$,

so V is not cyclic.

G is cyclic if $\exists\, g \in G$ such that $G = \langle g \rangle$.





$(\sqrt{})$

(c). See before in our discussion on An coursework.

S_3 is non-abelian since

$$(1\ 2)(1\ 3) = (1\ 3\ 2)$$

$$(1\ 3)(1\ 2) = (1\ 2\ 3)$$

and $(1\ 3\ 2) \neq (1\ 2\ 3)$

(d) Well any cycle of length m

$(a_1\ a_2 \ldots\ldots a_m)$

can be written as a product of $\underline{m-1}$ transpositions as:

$$(a_1 a_2 \ldots a_m)$$

$$= (a_1 a_m) \ldots\ldots (a_1 a_4)(a_1 a_3)(a_1 a_2)$$

and if $\sigma \in S_n$ and σ is a cycle then σ has length $\leq n$. and so by above σ can be written as a product of k transpositions where $k \leq n-1$.

$$(a_1 \ldots\ldots a_m)$$

$$= (a_2 a_1)(a_3 a_2) \ldots\ldots (a_{m-1} a_{m-2})(a_m a_{m-1})$$

(e). Let σ, μ be two <u>odd</u> permutations. This means.

$$\sigma = \tau_1 \tau_2 \ldots\ldots \tau_k$$

and

$$\mu = \alpha_1 \alpha_2 \ldots\ldots \alpha_l$$

for transpositions τ_i and α_j where k, l are both <u>odd</u>

The product $\sigma\mu$ of these can be written as

$$\sigma\mu = \tau_1 \dots \tau_k \alpha_1 \dots \alpha_l$$

this is a product of $k+l$ transpositions and clearly $k+l$ is an even integer since k, l are both odd.

Hence $\sigma\mu$ is an even permutation.

(f). Let G be a group with element g.

$\lambda_g : G \longrightarrow G$ defined by.

$$\lambda_g(a) = ga \quad , \text{ for } a \in G.$$

Prove λ_g is a permutation of G.

i.e. that λ_g is a bijection $G \longrightarrow G$.

so 1-1 and onto.

Firstly, since $g, a \in G$ then $ga \in G$.

since groups are closed under their product so λ_g is indeed a function $G \to G$.

To prove 1-1, suppose $a, b \in G$ and $\boxed{\lambda_g(a) = \lambda_g(b)}$

$$\Rightarrow \quad ga = gb$$

$$\Rightarrow \quad g^{-1} g a = g^{-1} g b.$$

$$\boxed{\Rightarrow \quad a = b.}, \text{ as } g^{-1} g = e.$$

Hence λ_g is injective

To prove $\boxed{\lambda_g \text{ is onto}}$, $\boxed{\text{let } y \in G}$ we need to show there exists an $x \in G$ such that

$$\lambda_g(x) = y.$$

This is true for $\boxed{x = g^{-1} y. \in G.}$

$$\boxed{\lambda_g(x) = gx = g^{-1}gy = y.}$$

Hence λ_g is onto, and hence λ_g is a permutation of G.

Q7.

(a). If G is a finite group with a subgroup H. then $|H|$ divides $|G|$.

(b) (i). A <u>left coset</u> of H in G with representative $a \in G$ is the subset

$$aH = \{ah : h \in H\} \subseteq G.$$

A <u>right coset</u> of H in G with rep. $a \in G$ is the subset.

$$Ha = \{ha : h \in H\} \subseteq G$$

(ii) Remember a disjunction.

$$(A \text{ OR } B) \equiv \boxed{((\neg B) \Rightarrow A)}$$

We will prove that

$$g_1 H \cap g_2 H \neq \phi \Rightarrow g_1 H = g_2 H.$$

So let $g_1, g_2 \in G$, H be a subgroup of G. and assume that

$$\boxed{g_1 H \cap g_2 H \neq \phi.}$$

So let $x \in g_1 H \cap g_2 H$.

$\Rightarrow x = g_1 h_1 = g_2 h_2$ for some elements $h_1, h_2 \in H$.

$\Rightarrow g_1 = g_2 h_2 h_1^{-1}$

and $g_2 = g_1 h_1 h_2^{-1}$

In order to prove that $g_1 H = g_2 H$ we will prove $g_1 H \subseteq g_2 H$ and $g_2 H \subseteq g_1 H$.

Let $y \in g_1 H$

$\Rightarrow \quad y = g_1 h$, for some $h \in H$

$$= g_2 \underbrace{h_2 h_1^{-1}}_{\in H} h$$

$$\in g_2 H.$$

Therefore $g_1 H \subseteq g_2 H$.

Secondly, let $y \in g_2 H$.

$\Rightarrow \quad y = g_2 h$, for some $h \in H$.

$$= g_1 \underbrace{h_1 h_2^{-1}}_{\in H} h.$$

$$\in g_1 H.$$

Therefore $g_2 H \subseteq g_1 H$.

Therefore $\boxed{g_1 H = g_2 H.}$

This proves the required result.

(iii) Let $g \in G$.

$H = eH$ is one of the left-cosets of H in G.

we can define a mapping ϕ.

$$\phi : H \longrightarrow gH$$

and prove that ϕ is a bijection.

This will prove that $|H| = |gH|$

and hence all left cosets are the same size.

Define $\phi : H \longrightarrow gH$

by $\phi(h) = gh$

ϕ is injective: since

$$\left.\begin{aligned} &\phi(h_1) = \phi(h_2) \\ \Rightarrow\ &gh_1 = gh_2 \\ \Rightarrow\ &h_1 = h_2 . \end{aligned}\right\} \text{ similar to Q6 (f).}$$

ϕ is surjective since. for $gh \in gH$.

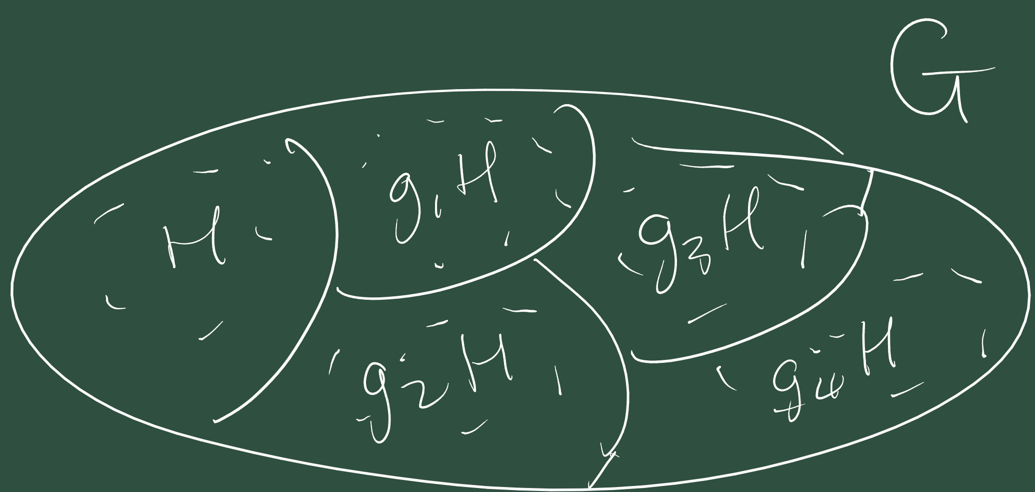
$$\phi(h) = gh.$$

So ϕ is clearly surjective.

So $\phi : H \longrightarrow gH$ is a bijection and hence $|H| = |gH|$.

A very similar argument with g on the right proves that $|H| = |Hg|$

(iv) Well the left cosets of H in G, these partition G.
(justified by the fact $g \in gH$ as $g = ge$, and the fact that different cosets are disjoint (ii)).



and all cosets have the same size, namely $|H|$ (part (iii)).

so $|G| = [G : H]\, |H|$

where $[G:H]$ is the number of cosets of H in G.

(c) Consider D_6.

$D_6 = \{e, r, r^2, r^3, r^4, r^5$
$s, rs, r^2s, r^3s, r^4s, r^5s\}$.

and these satisfy $r^6 = e$, $s^2 = e$.

$sr = r^{-1}s$

Consider the subgroup $H = \langle r^2 \rangle = \{e, r^2, r^4\}$

(i). $|D_6| = 12$, $|H| = 3$

So ~~the~~ there will be four left cosets of H in D_6, all of size 3.

1. $H = eH = \boxed{\{e, r^2, r^4\}} = r^2H = r^4H$.
2. $rH = \boxed{\{r, r^3, r^5\}} = r^3H = r^5H$.
3. $sH = \{s, sr^2, sr^4\}$

$= \{s, r^{-2}s, r^{-4}s\}$

$$= \boxed{\{s, r^4 s, r^2 s\}} = \underbrace{r^4 s H}_{} = \underline{r^2 s H}.$$

$$4: \quad rsH = \{rs, rsr^2, rsr^4\}$$

$$= \{rs, rr^{-2}s, rr^{-4}s\}$$

$$= \boxed{\{rs, r^5 s, r^3 s\}} = r^5 s H = r^3 s H.$$

(ii). Looking for non-normality here.

Consider $K = \langle s \rangle = \{e, s\}$.

Consider $rK = \{r, rs\}$.

but $Kr = \{r, sr\}$

$$= \{r, r^{-1}s\}$$

$$= \{r, r^5 s\} \neq rK.$$

since $rs \neq r^5 s$.