Integers

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, \overbrace{1, 2, 3, 4, \dots}^{J}\}$$

so $J = P$

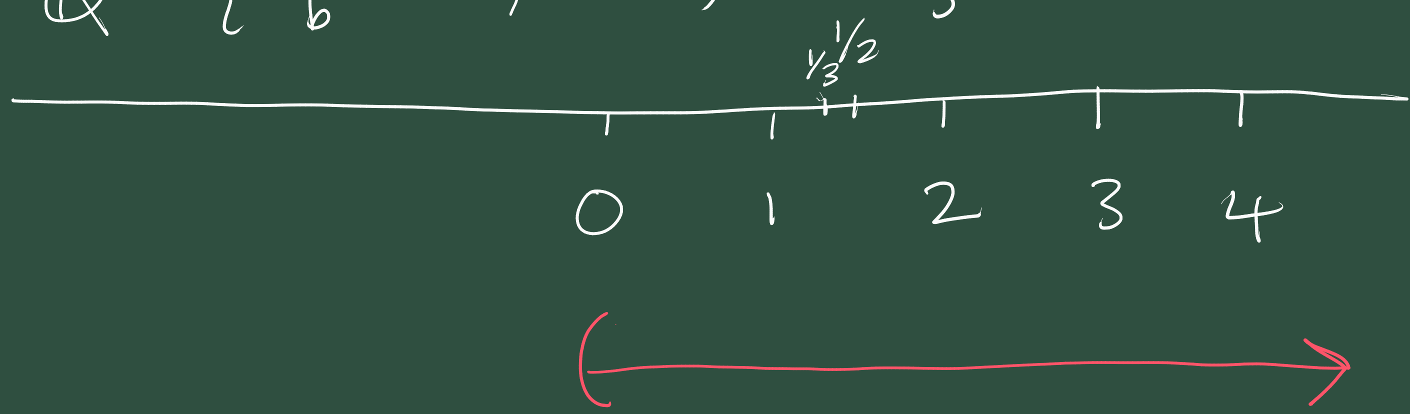
$\mathbb{Z}$ can be formally defined and its features described by a list of axioms.

$\boxed{\text{Axioms 1–11 uniquely define } \mathbb{Z}.}$

Consider the rationals $\mathbb{Q}$.

Is 11 true of $\mathbb{Q}$?

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}$$



$P \subset \mathbb{Q}$

Consider $\mathbb{Z}^{+} = \{1, 2, 3, 4, \dots\} = J$.

J satisfies $1 \in J$, ✓ $j \in J \Rightarrow j+1 \in J$. ✓

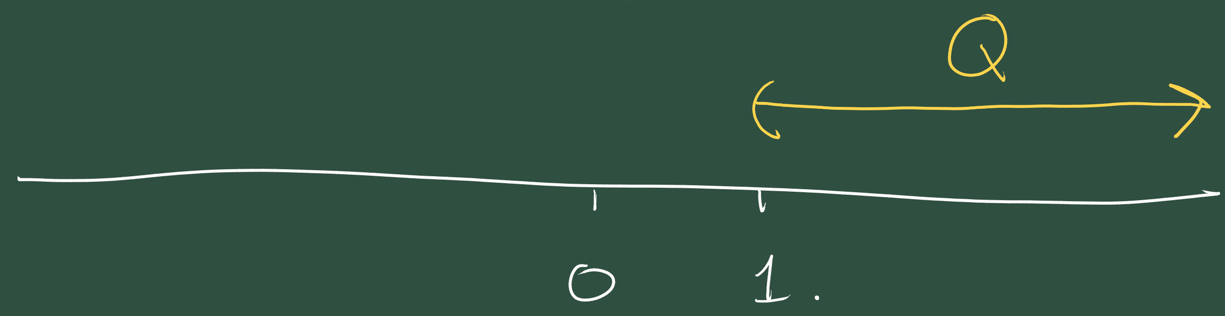
But $J \neq P$. since $\frac{1}{2} \notin J$

and so on.

So IIa is not true of $\mathbb{Q}$.

And IIb is not true of $\mathbb{Q}$ either.

consider $Q = \{ q \in \mathbb{Q} : q > 1 \}$



Q has no smallest element element, as it contains things arbitrarily close to 1.

Proofs by induction

Axiom IIa provides a powerful proof technique for proving statements about the (positive) integers.

Ex 1.2

Q1. Prove for all $n \geq 1$, $\boxed{\sum_{j=1}^{n} j = \frac{n}{2}(n+1)}$ — P(n)

$1+2+3+\dots+n$.

Let's use induction to prove this.

First, investigate the case where $n=1$

$$\boxed{\sum_{j=1}^{1} j} = 1 = \boxed{\frac{1}{2}(1+1)} \checkmark$$

We observe that it's true.

Now we want to prove, for all $k \geq 1$, that $P(k) \Rightarrow P(k+1)$.

So let's assume $P(k)$, ie we assume that

$$\boxed{\sum_{j=1}^{k} j = \frac{k}{2}(k+1)}, \text{ for some } k \geq 1$$

Consider

$$\sum_{j=1}^{k+1} j = \left(\sum_{j=1}^{k} j\right) + (k+1)$$

, by assumption $= \frac{k}{2}(k+1) + (k+1)$

by normal algebra.

$$
\begin{aligned}
&= (k+1)\left(\frac{k}{2} + 1\right) \\
&= (k+1)\left(\frac{k}{2} + \frac{2}{2}\right) \\
&= (k+1)\,\frac{k+2}{2} \\
&= \boxed{\frac{(k+1)}{2}(k+2)}
\end{aligned}
$$

L is $P(k+1)$

So this proves, that for any $k \geq 1$,

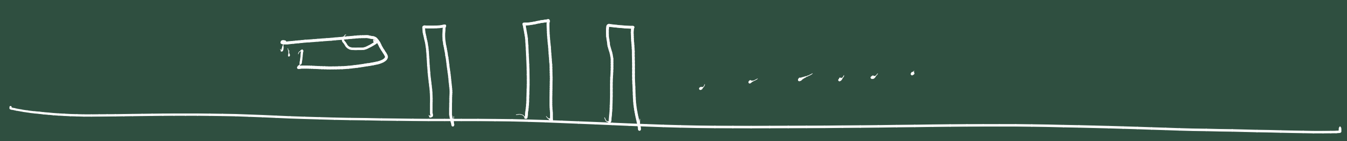
$$P(k) \Rightarrow P(k+1)$$

So by the principle of induction.

$\sum_{j=1}^{n} j = \frac{n}{2}(n+1)$ is true for all $n \geq 1$.

QED.

- Analogy Falling dominoes



Q2. $P(n) \equiv \sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$.

$$\sum_{j=1}^{n} j^2 = 1^2+2^2+3^2+\ldots+n^2$$

Consider $n=1$

$$\boxed{\sum_{j=1}^{1} j^2 = 1^2 = 1} = \boxed{1 = \frac{1}{6} 1(2)(3)} \checkmark$$

Assume that

$$\boxed{\sum_{j=1}^{k} j^2 = \frac{1}{6} k(k+1)(2k+1)} \quad P(k)$$

We consider

$$\boxed{\sum_{j=1}^{k+1} j^2} = \left(\sum_{j=1}^{k} j^2\right) + (k+1)^2$$

$$= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2, \text{ by assumption}$$

$$= (k+1)\left[\frac{1}{6} k(2k+1) + k+1\right]$$

$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6k+6\right]$$

$$= \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right]$$

$$\Updownarrow \quad ''$$

$$= \boxed{\frac{1}{6}(k+1)(k+2)(2k+3)}$$

$P(k+1)$

So we've proved that

$$P(k) \Rightarrow P(k+1)$$

So by the principle of induction

$$\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$$

QED