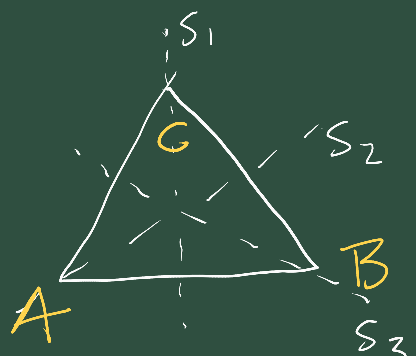
Chap 5 Permutation groups

Recall D_3, the group of symmetries of the Δ

$\begin{pmatrix} \text{start} \\ \text{finish} \end{pmatrix}$



$$e = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}$$

$$r = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$

$$r^2 = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$

$$S_1 = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

$$S_2 = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$$

$$S_3 = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}.$$

S_i are reflections.
r is rot by $\frac{2\pi}{3}$ radians clockwise.

These are examples of "permutation notation" associating to each symmetry of Δ the "permutation" of the vertices of Δ induced by that symmetry.

Def Given a set X, a permutation π of X is a mapping/function

$$\pi : X \longrightarrow X$$

that is bijective, ie. surjective and injective, ie. onto and 1-1.

onto: $\forall y \in X \ \exists x \in X \ \ \pi(x) = y$

1-1 $\forall x_1, x_2 \in X \ (\pi(x_1) = \pi(x_2)) \Rightarrow x_1 = x_2$

For a set $X =$

we have the "symmetric group of X."

$S_X = \{\pi : \pi$ is a permutation of $X\}$.

is a group under the operation of composition.

For finite sets, we use numerals for labels.

i.e. $X = \{1, 2, 3, \dots, n\}$

and write S_n for the symmetric of this set, i.e. group of all permutations of these n objects, under the op. of composition.

Theorem 5.1 S_n is a group, of order $\boxed{|S_n| = n!}$ ✓

Proof From theory of functions,

if $\pi_1, \pi_2 \in S_n$, i.e. two permutations then $\pi_1 \circ \pi_2$ will also be bijective, i.e. 1-1 and onto.

So $\pi_1 \circ \pi_2 \in S_n$

$e \in S_n$ ✓ as the identity mapping is a permutation.

Function composition is always associative. in the usual sense.

If $\pi \in S_n$, then the inverse mapping $\pi^{-1} : X \longrightarrow X$, will also be bijective.

So $\pi^{-1} \in S_n$ and

$$\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = e.$$

We use "two line permutation notation" for elements of S_n, for $\pi \in S_n$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & (n-1) & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & & \pi(n) \end{pmatrix} \begin{array}{l} \text{arguments} \\ \text{values.} \end{array}$$

$\boxed{\text{How many of these are there?}}$

Bijective condition means in the second row we see every element of $X = \{1, \ldots, n\}$ once and only once

n choices for $\pi(1)$
once this is assigned
$n-1$ choices for $\pi(2)$
once both assigned
$n-2$ choices for $\pi(3)$
⋮
2 choices for $\pi(n-1)$
1 choice for $\pi(n)$

$n!$ factorial ways of doing this.

Practice with notation

sigma
tau
mu

Example 5.2 Consider $\sigma, \tau, \mu \in S_5$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$$

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

$$\sigma \circ \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} = \mu.$$

Claim
$\{id, \sigma, \tau, \mu\}$ form a subgroup of S_5.

$$\underline{\underline{\tau \circ \mu}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} = \sigma$$

$$\underline{\underline{\mu^{-1}}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} = \mu$$

Ex 5.4 Composition is not expected to be commutative.

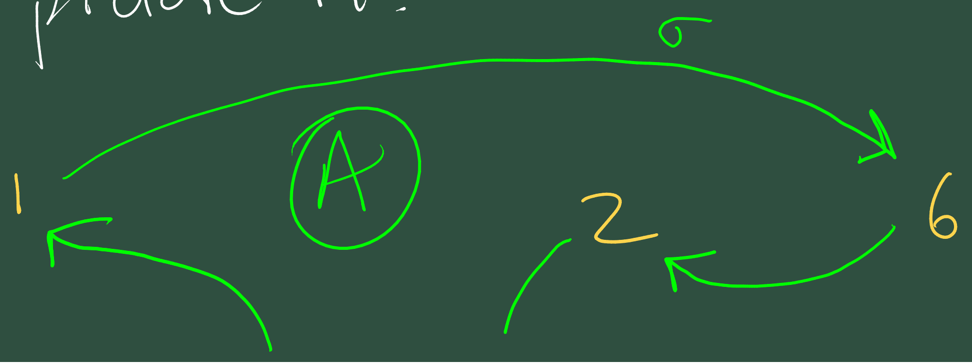
- Two line notation is very wasteful
- It somehow hides/obscures the "structure of the permutation"

A better notation Cycle notation.

Ex 5.5 Consider $\sigma \in S_7$

$$\boxed{\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 5 & 1 & 4 & 2 & 7 \end{pmatrix}}$$

Let's picture it.



We see σ consists of two <u>cycles</u> which we can write with cycle notation.

A $\quad (1\ 6\ 2\ 3\ 5\ 4)$

(B) $\quad (7)$

could also write.

A $\quad (3\ 5\ 4\ 1\ 6\ 2)$

For a cycle $\mu = (1\ 6\ 2\ 3\ 5\ 4)$ by convention we can apply μ to any element x

$\mu(x) = x$, for any element x not mentioned in the cycle notation μ.

This allows us to write σ as

composition operator

$$\boxed{\sigma = (1\,6\,2\,3\,5\,4)(7)}$$

$$= \boxed{(1\,6\,2\,3\,5\,4)}$$

Return to Ex 5.2

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} = (4\,5) \in S_5.$$

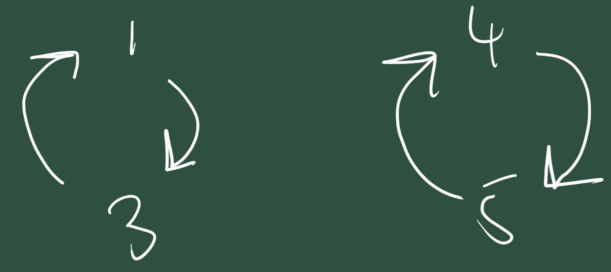
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix} = (1\,3)$$

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} = (1\,3)(4\,5)$$

$$= (4\,5)(1\,3)$$

$$= (5\,4)(3\,1)$$

μ:



Ex 5.6 Composing cycles can be done fairly efficiently "on the page" "in your mind"

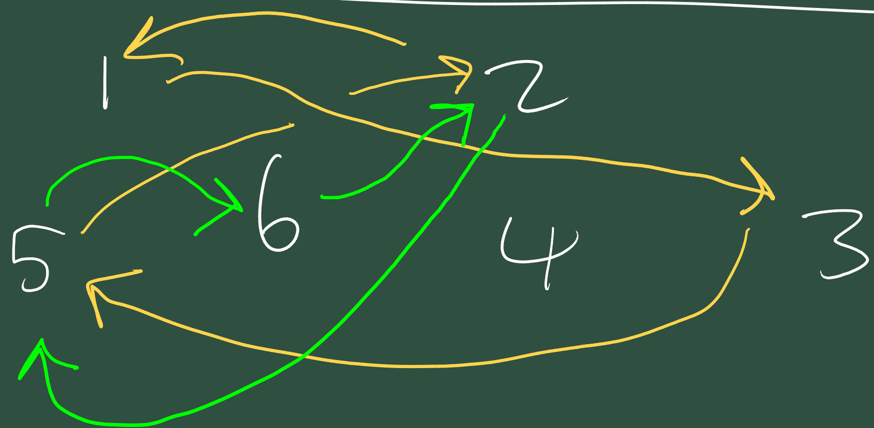
Consider $\sigma = (1\ 3\ 5\ 2)$, $\tau = (2\ 5\ 6)$

$$\sigma\tau = (3\ 5\ 6\ 1)(2)$$
$$= (3\ 5\ 6\ 1)$$
$$= (1\ 3\ 5\ 6)$$

Composition in other order

$$\tau\sigma = (2\ 1\ 3\ 6)(5) = (2\ 1\ 3\ 6)$$
$$\neq \sigma\tau$$

σ, τ

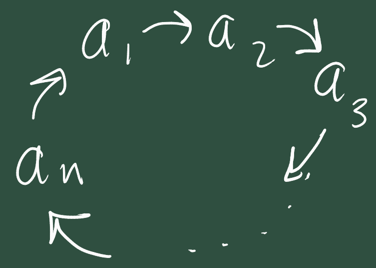


If two cycles π_1, π_2 are non-trivially disjoint, i.e. they operate on disjoint sets of objects, then they will commute.

$$\pi_1 \pi_2 = \pi_2 \pi_1$$

Expressing permutations as products of disjoint cycles is the preferred notation.

A transposition is a 2-cycle and interestingly any cycle can be expressed as a product of transpositions



<u>Proof</u> $(a_1, a_2, \ldots, a_n)$

$$= (a_1, a_n)(a_1, a_{n-1}), \ldots (a_1, a_3)(a_1, a_2)$$

∎

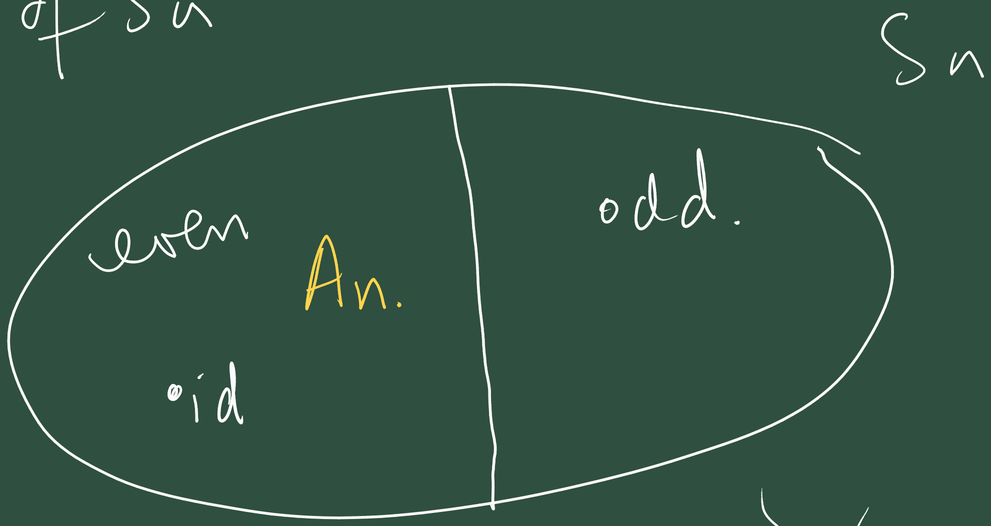
In S_n, each permutation is

inherently odd or even accordingly as ~~the~~ it requires an odd/even number of transpositions to express it.

An n-cycle can be written as a composition of n-1 transpositions.

so an n-cycle is $\begin{cases} \text{odd if } n \text{ is even} \\ \text{even if } n \text{ is odd} \end{cases}$

Leads to an important subgroup of S_n



$id = (1\,2)(2\,1) =$ empty

The Alternating group A_n is the subgroup of S_n consisting of all the even permutations of S_n.

Theorem 5.16 A_n is a subgroup of S_n.

Pf: $id \in A_n$ ✓ since $id = (1\ 2)(1\ 2)$

If $\pi_1, \pi_2 \in A_n$

$$\pi_1 = \boxed{\underbrace{(\cdot\ \cdot) \cdots\cdots (\cdot\ \cdot)}_{\text{even \# of transpositions } m_1}}$$

m_1, m_2 are even.

$$\pi_2 = \boxed{\underbrace{(\ ) \cdots\cdots (\ )}_{m_2 \text{ transpositions}}}$$

$$\pi_1\pi_2 = \underbrace{\boxed{(\ ) \cdots\cdots (\ )}\ \boxed{(\ ) \cdots\cdots (\ )}}_{\underbrace{m_1+m_2}_{\text{even}} \text{ transpositions}}$$

$\Rightarrow \pi_1\pi_2 \in A_n$.

If $\pi_1 = \tau_1\tau_2 \cdots \tau_{m_1}$, τ_i are transpositions

$$(\pi_1)^{-1} = \tau_{m_1}^{-1}\tau_{m_1-1}^{-1} \cdots \tau_3^{-1}\tau_2^{-1}\tau_1^{-1}$$

$$= \underbrace{\tau_{m_1} \tau_{m_1 - 1} \cdots \tau_3 \tau_2 \tau_1}_{m_1 \text{ transpositions}}$$

so $\pi_1^{-1} \in A_n$.

So by Prop 3.30 A_n is a subgroup of S_n.

<u>Prop 5.17</u> $|A_n| = \frac{n!}{2} = \frac{|S_n|}{2}$

$\lambda_\sigma : A_n \to B_n$ is onto

<u>Pf</u> Let $\mu \in B_n$, ie. μ is odd,

so $\mu = \tau_1 \cdots \tau_m$, τ_i are transpositions, m is odd.

Consider $\sigma^{-1}\mu = \sigma \underbrace{\tau_1 \cdots \tau_m}_{\substack{m+1 \text{ transpositions} \\ m+1 \text{ is even}}}$

so $\sigma^{-1}\mu \in A_n$.

$$\lambda_\sigma(\sigma^{-1}\mu) = \sigma\sigma^{-1}\mu$$

$$= \mu.$$

Hence λ_σ is surjective.

$X \xrightarrow{f} Y.$