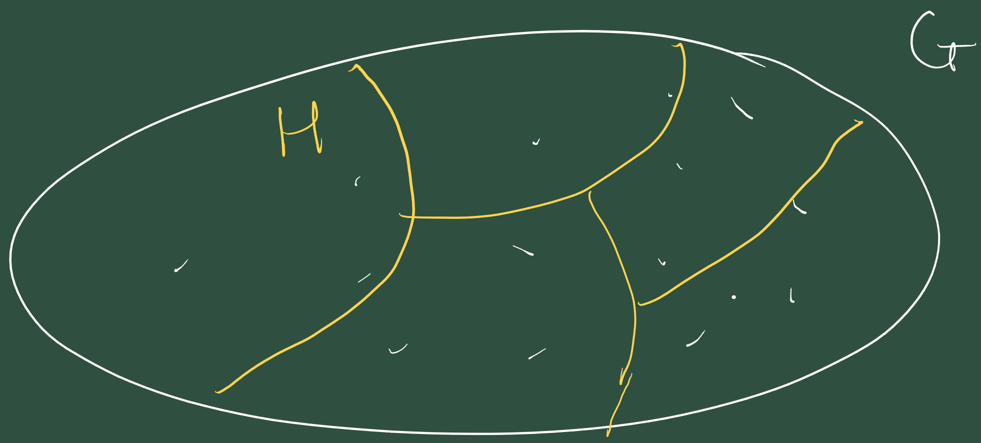
For a group G and one of its subgroups H we can partition G into left cosets of H



This at least provides us with a simplified view of the set G.

Can we put a group structure on this collection of cosets? Yes, provided H is a normal subgroup of G.

$$\Leftrightarrow \forall g \in G \quad gH = Hg$$

When H is normal in G then the collection of cosets, G/H, is a group under the product

Theorem 10-4

$$(aH)(bH) = abH.$$

with identity element $eH = H$

and inverse elements

$$(xH)^{-1} = x^{-1}H$$

This group G/H is a simplified version of G.

A group that has no proper non-trivial normal subgroups is called <u>simple</u>

---

eg. the prime order cyclic groups $\mathbb{Z}_p$ are all simple.

in fact $\mathbb{Z}_p$ has no proper non-trivial subgroups at all. Its only subgroups are $\mathbb{Z}_p$ and $\{0\}$

In section 10.2. it's proved the alternating groups A_n for $n \geq 5$ are all simple.

<u>We won't examine the details of 10.2.</u>

Q2 of Ex 10.4. $r^4 = e = s^2$, $\underline{sr = r^{-1}s}$

$$D_4 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$$

rotations reflections

We found the subgroups

$R_1 = \{e, r, r^2, r^3\}$, $R_2 = \{e, r^2\}$

$S_1 = \{e, s\}$, $S_2 = \{e, rs\}$, $S_3 = \{e, r^2 s\}$,

$S_4 = \{e, r^3 s\}$

Then just two others.

$A = \langle r^2, s \rangle = \{e, r^2, s, r^2 s\}$

$B = \langle r^2, rs \rangle = \{e, r^2, rs, r^3 s\}$.

Then which of these are normal in D_4.

Well $|R_1| = 4 = \dfrac{|D_4|}{2} = \dfrac{8}{2}$

Such subgroups (of half the size of G) are always normal for the two <u>left cosets</u> will be R_1, $D_4 \setminus R_1$ and ~~also~~ these are also the two right cosets.

So D_4/R_1 will be a group.

$= \{ R_1, sR_1 \}$

$R_1 = \{e, r, r^2, r^3\}$, $sR_1 = \{s, rs, r^2s, r^3s\}$

Such a 2-element group will always have the same Cayley table.

If $K = \{e, x\}$, $x^2 = e$.

then the Cayley table is.

	e	x
e	e	x
x	x	e

Let's consider R_2. Is it normal in D_4? See theorem 10.3.

Use part (2) of this

Let's consider conjugations of R_2, gR_2g^{-1} for $g \in D_4$, and prove that $gR_2g^{-1} \subseteq R_2 = \{e, r^2\}$

if $g = e, r, r^2, r^3$

then $geg^{-1} = e \in R_2$.

$gr^2g^{-1} = gg^{-1}r^2 = r^2 \in R_2.$

If $g = s$ then

$$ses^{-1} = ses = ss = s^2 = e \in R_2.$$

$$sr^2s = r^{-2}ss = r^{-2}s^2 = r^{-2}e$$
$$= \underbrace{r^{-2} = r^2}_{\text{(they're both just "half turns")}} \in R_2$$

and similarly for $g = rs, r^2s, r^3s$ we will still have.

$$gr^2g^{-1} = r^2 \in R_2.$$

So for all $g \in D_4$ $gR_2g^{-1} = R_2$

by theorem 10.3 R_2 is normal in D_4.

So D_4/R_2 will be a group.

with elements. $\quad aH\,bH = abH.$

$eR_2 = \{e, r^2\} = R_2$

$rR_2 = \{r, r^3\}$

$sR_2 = \{s, \underline{r^2 s}\}$

$rsR_2 = \{rs, \underline{r^3 s}\}$

Let's show the Cayley table for D_4/R_2.

	R_2 (e)	rR_2 (a)	sR_2 (b)	rsR_2 (c)
R_2	R_2	rR_2	sR_2	rsR_2
rR_2	rR_2	R_2	rsR_2.	sR_2
sR_2	sR_2	rsR_2.	R_2	rR_2
rsR_2	rsR_2	sR_2	rR_2	R_2.

$rR_2 sR_2 = rsR_2$

$\underline{r}R_2\underline{rs}R_2 = r^2 sR_2 = sR_2.$

$$\begin{aligned} sR_2 rR_2 &= srR_2 \\ &= r^{-1}sR_2 \\ &= \underline{r^3}sR_2 \\ &= rsR_2. \end{aligned}$$

	y
x	xy

which is symbolically equivalent to.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$D_4/R_2 \cong \{e, a, b, c\}$$

$$a^2 = b^2 = c^2 = e.$$

which is abelian

and in fact $D_4/R_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

as ~~direct~~ distinct from $\mathbb{Z}_4$.
isomorphically

The subgroups S_1, S_2, S_3, S_4 are all non-normal.

Consider $S_1 = \langle s \rangle = \{e, s\}$.

Consider the conjugate of S_1 by r

$$rS_1r^{-1} = \{ rer^{-1}, rsr^{-1} \}$$

now $rer^{-1} = rr^{-1} = e \in S_1$

$$rsr^{-1} = r(r^{-1})^{-1}s$$
$$= rrs = r^2s \notin S_1$$

So $rS_1r^{-1} \neq S_1$

so by theorem 10.3 S_1 is not normal in D_4. So D_4 cannot be simplified with reference to S_1.

And similarly S_2, S_3, S_4 are all not-normal.

For subgroups A, B both are normal since $|A| = |B| = 4 = \frac{|D_4|}{2}$

and both factor groups D_4/A, D_4/B will be of order 2, and hence isomorphic to $\mathbb{Z}_2$.

Next look at Q4, 13.

Q3. $a_n = 10^{2^n} + 1 = 10\ldots\ldots01$

Hint: show $a_n \mid a_{n+1} - 2$.

From this one can argue that $\gcd(a_n, a_{n+1}) = 1$

$$\boxed{a_{n+1} - 2 = q\,a_n} \implies a_{n+1} - q\,a_n = 2$$

To push further, try and do more with the hint, examine the quotient q carefully.

Q4.

Claim: $T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \begin{array}{l} a, b, c \in \mathbb{R}. \\ ac \neq 0 \end{array} \right\}$

is a group under matrix multiplication

i.e. T is a subgroup of $GL(2, \mathbb{R})$

Prove $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in T$ ✓

Given $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \in T.$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}$$

$\in T.$ ✓

note $adcf = ac\,df \neq 0$ since $ac \neq 0$ and $df \neq 0$

and $$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \in T.$$ ✓

So by Proposition 3.30 T is a subgroup of $GL(2, \mathbb{R})$.

a). $U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in T \; : \; x \in \mathbb{R} \right\}$

U is a subgroup thanks to Prop 3.30 and the following observations

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U. \checkmark$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \in U \checkmark$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \in U \checkmark$$

b). $\boxed{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}$$

So U is abelian as all its matrices commute with each other.

c). To prove U is normal in T.

Consider conjugation of U by matrices from T.

i.e.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -b/ac \\ 0 & \frac{1}{c} \end{pmatrix}$$

$$= \begin{pmatrix} a & ax+b \\ 0 & c \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -b/ac \\ 0 & \frac{1}{c} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{b}{c} + \frac{ax+b}{c} \\ 0 & 1 \end{pmatrix} \in U$$

Therefore $tUt^{-1} \subset U$ for all $t \in T$.

So by theorem 10.3 U is normal in T.