Q4. $\boxed{P(n) \equiv \underbrace{\sum_{j=1}^{n} j^3}_{1+8+27+\dots+n^3} = \frac{1}{4} n^2 (n+1)^2}$ for all $n \geq 1$

The base case is when $n=1$.

$$\boxed{\sum_{j=1}^{1} j^3} = 1^3 = 1 = \boxed{\frac{1}{4} 1^2 (1+1)^2}$$

and thus is true.

We assume that $\boxed{\sum_{j=1}^{k} j^3 = \frac{1}{4} k^2 (k+1)^2}$

and try and prove the next case.

$$\boxed{\sum_{j=1}^{k+1} j^3} = \left(\sum_{j=1}^{k} j^3\right) + (k+1)^3$$

$= \frac{1}{4} k^2 (k+1)^2 + (k+1)^3$, by the above assumption

$= (k+1)^2 \left(\frac{1}{4} k^2 + k+1\right)$, factoring

$= \frac{1}{4} (k+1)^2 \left(k^2 + 4k + 4\right)$ factoring

$=$

$$= \boxed{\frac{1}{4}(k+1)^2(k+2)^2},$$ factoring.

So this proves that $P(k) \Rightarrow P(k+1)$

So by the principle of induction $P(n)$ is true for all $n \geq 1$.

Ex 2.3

Q4 Use induction to prove

a) $\forall n \geq 1 \quad 7 \mid (2^{3n} - 1)$

b) $\forall n \geq 1 \quad 8 \mid (3^{2n} + 7)$

c) $\forall n \geq 1 \quad 3 \mid (2^n + (-1)^{n+1})$

Q6 $\quad \forall n \geq 1 \quad \boxed{(1+\frac{1}{2})^n \geq 1 + \frac{n}{2}} \equiv P(n)$

Base case is $n=1$

$(1+1/2)^1 = 1 + 1/2$. So yes $P(1)$ is true.

Assume $P(k)$ is true for some $k \geq 1$, ie. $\boxed{(1+\frac{1}{2})^k \geq 1 + \frac{k}{2}}$ $P(k)$

Now try and derive $P(k+1)$

$$\left(1+\tfrac{1}{2}\right)^{k+1} = \left(1+\tfrac{1}{2}\right)\left(1+\tfrac{1}{2}\right)^{k}$$

$$\geqslant \left(1+\tfrac{1}{2}\right)\left(1+k/2\right), \text{ by assumption}$$

$$= 1 + \frac{k}{2} + \frac{1}{2} + \frac{k}{4}$$

$$\geqslant \quad = 1 + \frac{2k+2+k}{4}$$

$$= 1 + \frac{3k+2}{4}$$

$$> 1 + \frac{2k+2}{4} \qquad -\frac{k}{4}$$

$$= 1 + \frac{k+1}{2}$$

which is saying

$$\left(1+\tfrac{1}{2}\right)^{k+1} > 1 + \frac{k+1}{2}.$$

$$\Rightarrow P(k+1).$$

ie. $P(k) \Rightarrow P(k+1)$.

So $P(n)$ for all $n \geqslant 1$, by induction.

~~Sec~~ Ex 2.3

Q8. b) $\forall n \geq 1$ $\boxed{8 \mid (3^{2n} + 7)}$ $P(n)$.

Base cases

$n=1$ $3^{2n}+7 = 3^2+7 = 16 = 2\cdot 8$

$n=2$ $3^{2n}+7 = 3^4+7 = 81+7 = 88 = 8\cdot 11$

Let's assume that $\boxed{8 \mid 3^{2k}+7}$, for some $k \geq 1$

$$3^{2k+2}+7 = 3^2\, 3^{2k} + 7.$$

$$= 3^2(3^{2k}+7) - 8\cdot 7.$$

since this is a linear combination of integers divisible by 8

$$\Rightarrow 8 \mid 3^{2k+2}+7.,$$

$$\Rightarrow 8 \mid 3^{2(k+1)}+7.$$

So we've shown that $P(k) \Rightarrow P(k+1)$.

So by induction, $8 \mid 3^{2n}+7$.

$\boxed{Q7}$ If a is odd then

$$24 \mid a(a^2-1)$$

eg. $a=5$, $a(a^2-1) = 5\cdot(25-1)$
$= 5\cdot 24$

$a=7$, $7(7^2-1) = 7\cdot 48 = 7\cdot 2\cdot 24$

$a=3$, $3(3^2-1) = 3\cdot 8 = 24$

If a is odd we can write

$a = 2n+1$, for some $n\in\mathbb{Z}$.

$$a(a^2-1) = (2n+1)\left((2n+1)^2-1\right)$$
$$= (2n+1)(4n^2+4n)$$
$$= 4(2n+1)(n^2+n)$$
$$= 4\cdot\underline{(2n+1)\,n\,(n+1)}$$

If we can show that

$$6 \mid (2n+1)\,n(n+1)$$

then we will know $24 \mid a(a^2-1)$

$$(2n+1)\,n(n+1) = 2\left[(2n+1)\,\frac{n(n+1)}{2}\right]$$

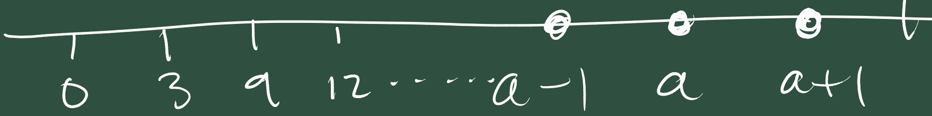
So far we've shown that

$$8 \mid a(a^2-1) \qquad a \text{ is odd.}$$

But we also need a factor of 3.

$$a(a^2-1) = a(a+1)(a-1)$$

three consecutive integers. one of which will be divisible by 3.



So $\Rightarrow$ $3 \mid a(a^2-1)$

$8 \mid a(a^2-1)$ AND $3 \mid a(a^2-1)$

Is it true that if

$x \mid z$ AND $y \mid z$

is it true $xy \mid z$.

$50 \mid 100$ $\qquad$ $25 \mid 100$

$50 \cdot 25 \nmid 100$

<u>Theorem</u> if $\boxed{\gcd(x, y) = 1}$

then $\left\{ \text{if } (x \mid z \text{ AND } y \mid z) \right.$

$\left. \quad \text{then } xy \mid z \right\}$

<u>Proof</u> Assume $\gcd(x, y) = 1$

$1 = mx + ny$, for some $m, n \in \mathbb{Z}$.

$\Rightarrow z = mxz + nyz$

we also assume that $x|z$ AND $y|z$

ie. $z = ax$, $z = by$, for some $a, b \in \mathbb{Z}$

$$z = mxz + nyz$$
$$= mxby + nyax$$
$$= (mb + na)xy$$

Therefore $xy|z$.

QED

$\gcd(8,3) = 1$

$\Rightarrow 24 \mid a(a^2-1)$.