<u>Recall</u> $G \cong H$. "G is isomorphic to H"

means $\exists$ $\phi : G \to H$ a bijection

satisfying the homomorphism property

$$\forall\ a, b \in G \quad \phi(ab) = \phi(a)\phi(b)$$

<u>Theorem 9.10</u> Isomorphism is an <u>equivalence relation</u> on the class of groups

<u>Proof</u>: We need to show $\cong$ is reflexive, symmetric, transitive.

<u>Reflexive</u> If G is a group then the identity map $id : G \to G$ defined by $id(g) = g$ is a bijection and trivially satisfies the homomorphism property

$$id(ab) = ab = id(a)\, id(b)$$

Therefore $G \cong G$.

Symmetric See theorem 9.6 part 1. (yesterday)
we prove if $\phi: G \to H$, is an isomorphism then $\phi^{-1}: H \to G$ is also an isomorphism.

ϕ phi

ψ psi

Transitive property

Assume $G \cong H$ and $H \cong K$.
we need to show that $G \cong K$

Let $\phi: G \to H$, $\psi: H \to K$ be the two isomorphism, then consider the map

$$\psi \circ \phi : G \longrightarrow K$$

From the theory of functions if ϕ and ψ are bijections then so is $\psi \circ \phi$.

and for any $a, b \in G$

$$(\psi \circ \phi)(ab) = \psi(\phi(ab)), \text{ def of comp.}$$

$$= \psi(\phi(a)\phi(b)), \text{ hom. prop. for } \phi$$

$$= \psi(\phi(a))\,\psi(\phi(b)), \text{ hom. prop for } \psi$$

$$= \overbrace{(\psi \circ \phi)(a)}\ \overbrace{(\psi \circ \phi)(b)}$$

, def of comp.

Therefore $G \cong K$.

So $\cong$ is an equivalence relation on groups.

Finally, an important result is Cayley's which in some ways simplifies our view groups.

Idea: think of a Cayley table.

for instance. Cayley table for D_3 shown in figure 3.7.

in each row we see a permutation of the six elements of the group.

This shows a way to associate to each element of $D3$ a permutation from S_6.

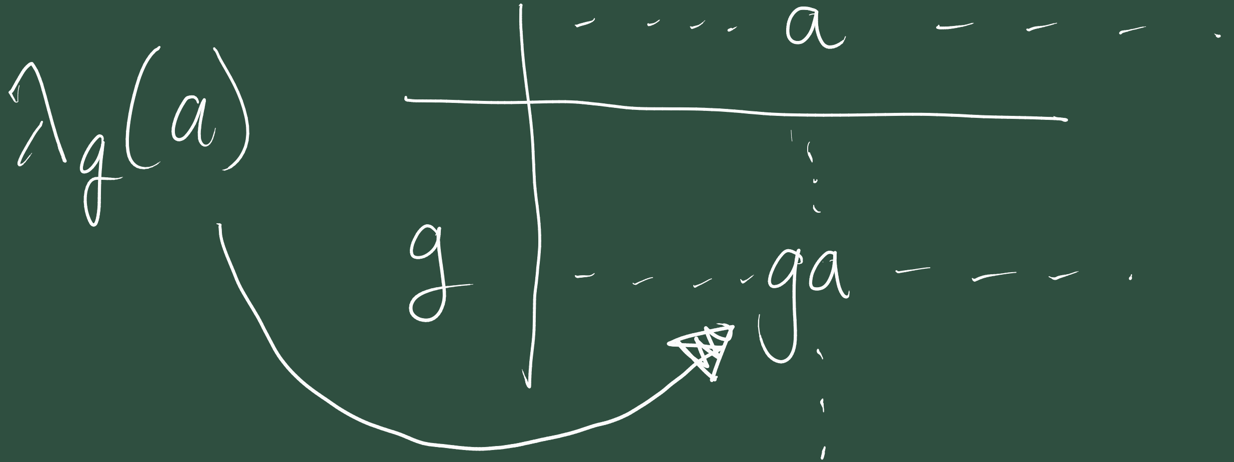
This will turn out to be an isomorphism from D_3 to a subgroup of S_6.

Proof of Cayley's table

For $g \in G$, we define the map

$$\lambda_g : G \longrightarrow G.$$
$$a \longmapsto ga.$$

So in other words, λ_g is the permutation of G represented by the g^{th} row of the Cayley Table.



We can show that $\lambda_g : G \to G$ is indeed a permutation, ie. a bijection.

- 1-1 since $\lambda_g(a) = \lambda_g(b)$

$$\iff ga = gb$$
$$\iff a = b$$

- onto: since $\forall\ y \in G$ then note that

$$\lambda_g(\underline{g^{-1}y}) = g\,\underline{g^{-1}y} = y$$

hence λ_g is onto.

So $\lambda_g : G \to G$ is a permutation and can be regarded as an element of S_n, where $n = |G|$

Claim: $\overline{G} = \{\lambda_g : g \in G\}$

is a group of permutations (operation is composition of the permutations)

(i.e. a subgroup of the corresponding S_n)

Lastly the isomorphism required is the map $\phi : G \longrightarrow \overline{G}$ defined by $\phi(g) = \lambda_g$

"known as the "left regular representation" of G".

External Direct Products

Given two groups $(G, \cdot)$, $(H, \circ)$ we can build / define a group by "sticking G and H together" as the Cartesian Product of sets

$$G \times H = \{ (g, h) : g \in G, h \in H \}$$

and the operation on $G \times H$ defined as

$$(g_1, h_1)(g_2, h_2) := (g_1 \cdot g_2, h_1 \circ h_2)$$

Prop 9.13 $G \times H$ is a group

Proof: Clearly.

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

$$\in G \times H$$

since $g_1 g_2 \in G$, $h_1 h_2 \in H$.

So this is a closed binary operation on $G \times H$.

The identity

$$e_{G \times H} = (e_G, e_H) \checkmark$$

Observe.

$$(g, h)(e_G, e_H) = (g e_G, h e_H)$$

$$= (g, h)$$

Associativity comes from the associativity of G and H.

$$(g_1, h_1)\left[(g_2, h_2)(g_3, h_3)\right]$$

$= (g_1, h_1)(g_2 g_3, h_2 h_3)$

$= (g_1(g_2 g_3), h_1(h_2 h_3))$

$= ((g_1 g_2) g_3, (h_1 h_2) h_3)$

, by the associativity in G & H.

$= \cdots$

$= [(g_1, h_1)(g_2, h_2)](g_3, h_3)$

Finally the inverse elements are

$$(g, h)^{-1} = (g^{-1}, h^{-1})$$

So this shows that $G \times H$ is a group.

Lots of little results we

Can prove as exercises.

- $|G \times H| = |G||H|$,

  if G, H are finite.

- If G and H are Abelian then so will $G \times H$ be.

Theorem 9.17

$|(g, h)| = \text{lcm}(|g|, |h|)$.

Theorem 9.21

$\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$

iff $\gcd(m, n) = 1$.

<u>Corollary</u> If $m = \prod_{i=1}^{k} p_i^{e_i}$ in

the prime factorisation of m. (p_i are distinct primes)

then $\mathbb{Z}_m \cong \prod_{i=1}^{k} \mathbb{Z}_{p_i^{e_i}}$

since $p_i^{e_i}$ are pairwise co-prime.

Internal direct product

The concept of recognising when an existing group G as being (isomorphic to) the direct product of two of its subgroups.

For a group G we must see it having two subgroups H, K

Satisfying:

- $G = HK = \{ hk : h \in H, k \in K \}$.
- $H \cap K = \{e\}$
- $\forall h \in H \ \forall k \in K \quad hk = kh$

technical conditions that achieve two things.

- the representation $g \in G$

$$g = hk \ , \ h \in H, k \in K$$

is unique.

- closure condition.

$$(h_1 k_1)(h_2 k_2) = h_1 k_1 h_2 k_2$$

$$= \underbrace{h_1 h_2}\underbrace{k_1 k_2}$$

$$\in HK$$

Theorem 9.27

G, H, K satisfying 3 conditions above means.

$G \cong H \times K$.

the isomorphism being

$$\phi : G \longrightarrow H \times K$$

$$hk = g \longmapsto \underline{\underline{(h, k)}}$$

Exercises 9.4

Q2. Claim $\mathbb{C}^* \cong H \subset GL_2(\mathbb{R})$.

$$H = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R}, \ a^2+b^2 \neq 0 \right\}.$$

Our task is to define a bijective map $\phi : \mathbb{C}^* \to H$ that satisfy the homomorphism property.

$$\underset{\in \mathbb{C}^*}{\phi(x+iy)} = \underset{\in H.}{\begin{pmatrix} x & y \\ -y & x \end{pmatrix}}$$

and the homomorphism property will be.

$$\phi\left((x_1+iy_1)(x_2+iy_2)\right)$$

$$= \phi\left((x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\right)$$

$$= \begin{pmatrix} x_1x_2 - y_1y_2 & x_1y_2 + x_2y_1 \\ -x_1y_2 - x_2y_1 & x_1x_2 - y_1y_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix} \quad \checkmark$$

$$= \phi(x_1 + iy_1)\,\phi(x_2 + iy_2).$$

This seems like the correct way to define the isomorphism ϕ as the hom. prop. proof works out.

Note $x + iy \neq 0$

$\iff x, y$ not both zero

$\iff x^2 + y^2 \neq 0$

so the map is indeed a bijection between $\mathbb{C}^*$ and H.

Q3. $U(8) \cong \mathbb{Z}_4$?

$\cdot$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$$\overset{?}{\underset{?}{\cong}}$$

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Note that $\forall x \in U(8)$ $x^{-1} = x$

But this is <u>not</u> true in $\mathbb{Z}_4$.

If $\phi : U(8) \to \mathbb{Z}_4$

were an isomorphism.

then take any $y \in \mathbb{Z}_4$

write it as $y = \phi(x)$

then $y^{-1} = \phi(x)^{-1}$

$= \phi(x^{-1})$

$$\Rightarrow \phi(x) = \boxed{y}.$$

So there can't be any such isomorphisms ϕ.

Q5. $U(5)$, $U(10)$, $U(12)$.

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

- in $U(12)$

$x = x^{-1}$ for all x.

So $U(5) \not\cong U(12) \not\cong U(10)$
for same reason as
previous example.

An isomorphism is

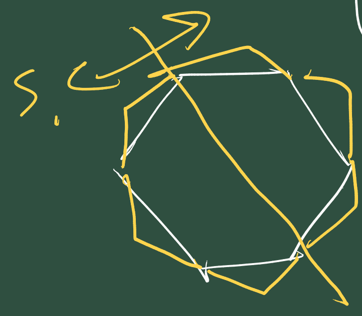
$$\phi : U(5) \longrightarrow U(10)$$

defined by

$$1 \longmapsto 1$$
$$2 \longmapsto 3$$
$$4 \longmapsto 9$$
$$3 \longmapsto 7.$$

this does indeed map
cayley table of $U(5)$ to
that of $U(10)$, ie ϕ.

satisfies the homomorphism property.



Q12

Claim: $S_4 \not\cong D_{12}$.

$$|S_4| = 4! = 24$$

$$|D_{12}| = 2 \cdot 12 = 24.$$

$$D_{12} = \{ \overbrace{e, r, r^2, r^3, \dots, r^{11}}^{\text{rotations}}, \underbrace{s, rs, r^2s, r^3s, \dots, r^{11}s}_{\text{reflections.}} \}$$

Suppose $\phi : D_{12} \to S_4$ were an isomorphism.

D_{12} has a cyclic subgroup of order 6 $H = \{e, r, r^2, r^3, \dots, r^{11}\}$

Then $\phi(H)$ would be a cyclic subgroup of S_4 with generator $\phi(r)$

But does S_4 have such a subgroup?

S_4 has elements like

$$|(a\ b\ c\ d)| = 4$$

$$|(a\ b\ c),| = 3$$

$$|(a\ b),| = 2$$

$$|(a\ b)(c\ d)| = 2$$

So we can have cyclic subgroups of S_4 of orders 4, 3, or 2. But not a subgroup of order 12.

So the conclusion is that no such isomorphism ϕ can exist.

So $D_{12} \not\cong S_4$.