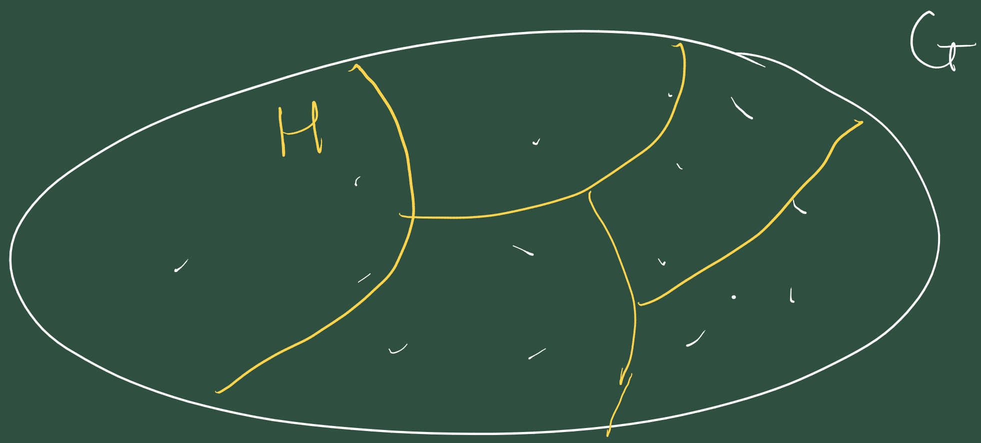
For a group $G$ and one of its subgroups $H$ we can partition $G$ into left cosets of $H$




This at least provides us with a simplified view of the set $G$.

Can we put a group structure on this collection of cosets? Yes, provided $H$ is a normal subgroup of $G$.

$$\Leftrightarrow \forall g \in G \quad gH = Hg$$

When $H$ is normal in $G$ then the collection of cosets, $G/H$, is a group under the product

Theorem 10.4

$$(aH)(bH) = abH.$$

with identity element $eH = H$

and inverse elements

$$(xH)^{-1} = x^{-1}H$$

This group $G/H$ is a simplified version of $G$.

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A group that has no proper non-trivial normal subgroups is called <u>simple</u>

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eg. the prime order cyclic groups $\mathbb{Z}_p$ are all simple.

in fact $\mathbb{Z}_p$ has no proper non-trivial subgroups at all. Its only subgroups are $\mathbb{Z}_p$ and $\{0\}$

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In section 10.2. it's proved the alternating groups $A_n$ for $n \geq 5$ are all simple.

We won't examine the details of 10.2.

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Q 2 of Ex 10.4. $r^4 = e = s^2$, $\underline{sr = r^{-1}s}$

$$D_4 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$$

$\underbrace{\qquad\qquad}_{\text{rotations}}$ $\underbrace{\qquad\qquad}_{\text{reflections}}$

We found the subgroups.

$R_1 = \{e, r, r^2, r^3\}$, $R_2 = \{e, r^2\}$

$S_1 = \{e, s\}$, $S_2 = \{e, rs\}$, $S_3 = \{e, r^2 s\}$,

$S_4 = \{e, r^3 s\}$

Then just two others.

$A = \langle r^2, s\rangle = \{e, r^2, s, r^2 s\}$

$B = \langle r^2, rs\rangle = \{e, r^2, rs, r^3 s\}$.

Then which of these are normal in $D_4$.

Well $|R_1| = 4 = \frac{|D_4|}{2} = \frac{8}{2}$

Such subgroups (of half the size of $G$) are always normal for the two left cosets will be $R_1$, $D_4 \setminus R_1$

and ~~also~~ these are also the two right cosets.

So $D_4/R_1$ will be a group.

$= \{ R_1, sR_1 \}$

$R_1 = \{e, r, r^2, r^3\}$, $sR_1 = \{s, rs, r^2s, r^3s\}$

Such a 2-element group will always have the same Cayley table.

If $K = \{e, x\}$, $x^2 = e$.

then the Cayley table is.

| | $e$ | $x$ |
|---|---|---|
| $e$ | $e$ | $x$ |
| $x$ | $x$ | $e$ |

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Let's consider $R_2$. Is it normal in $D_4$? See theorem 10.3.

Use part (2) of this

Let's consider conjugations of $R_2$, $gR_2g^{-1}$ for $g \in D_4$, and prove that $gR_2g^{-1} \subseteq R_2 = \{e, r^2\}$

if $g = e, r, r^2, r^3$

then $geg^{-1} = e \in R_2$.

$gr^2g^{-1} = gg^{-1}r^2 = r^2 \in R_2.$

If $g = s$ then

$ses^{-1} = ses = ss = s^2 = e \in R_2.$

$$sr^2s = r^{-2}ss = r^{-2}s^2 = r^{-2}e$$
$$= \underbrace{r^{-2} = r^2}_{} \in R_2$$

(they're both just "half turns"

and similarly for $g = rs, r^2s, r^3s$ we will still have.

$$gr^2g^{-1} = r^2 \in R_2.$$

So for all $g \in D_4$ $gR_2g^{-1} = R_2$ by theorem 10.3 $R_2$ is normal in $D_4$.

So $D_4/R_2$ will be a group.

with elements. $aH\,bH = abH.$

$eR_2 = \{e, r^2\} = R_2$

$rR_2 = \{r, r^3\}$

$sR_2 = \{ s, r^2s \}$

$rsR_2 = \{ rs, r^3s \}$

Let's show the Cayley table for $D_4/R_2$.

| | $R_2$ (e) | $rR_2$ (a) | $sR_2$ (b) | $rsR_2$ (c) |
|---|---|---|---|---|
| $R_2$ | $R_2$ | $rR_2$ | $sR_2$ | $rsR_2$ |
| $rR_2$ | $rR_2$ | $R_2$ | $rsR_2$ | $sR_2$ |
| $sR_2$ | $sR_2$ | $rsR_2$ | $R_2$ | $rR_2$ |
| $rsR_2$ | $rsR_2$ | $sR_2$ | $rR_2$ | $R_2$. |

$rR_2\, sR_2 = rsR_2$

$rR_2\, rsR_2 = r^2sR_2 = sR_2.$

$$\begin{aligned} sR_2\, rR_2 &= srR_2 \\ &= r^{-1}sR_2 \\ &= r^3sR_2 \\ &= rsR_2. \end{aligned}$$

| | $y$ |
|---|---|
| $x$ | $xy$ |

which is symbolically equivalent to.

| | e | a | b | c |
|---|---|---|---|---|
| e | e | a | b | c |
| a | a | e | c | b |
| b | b | c | e | a |
| c | c | b | a | e |

$D_4/R_2 \cong \{e, a, b, c\}$

$a^2 = b^2 = c^2 = e$.

which is abelian

and in fact $D_4/R_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

as ~~direct~~ distinct from $\mathbb{Z}_4$.
Isomorphically

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The subgroups $S_1, S_2, S_3, S_4$ are all non-normal.

Consider $S_1 = \langle s \rangle = \{e, s\}$.

Consider the conjugate of $S_1$ by $r$

$$rS_1r^{-1} = \{ rer^{-1}, rsr^{-1} \}$$

now $rer^{-1} = rr^{-1} = e \in S_1$

$$rsr^{-1} = r(r^{-1})^{-1}s$$
$$= rrs = r^2s \notin S_1$$

So $rS_1r^{-1} \neq S_1$

so by theorem 10.3 $S_1$ is not normal in $D_4$. So $D_4$ cannot be simplified with reference to $S_1$.

And similarly $S_2, S_3, S_4$ are all not-normal.

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For subgroups $A, B$ both are normal since $|A| = |B| = 4 = \frac{|D_4|}{2}$

and both factor groups $D_4/A$, $D_4/B$ will be of order 2, and hence isomorphic to $\mathbb{Z}_2$.

Next look at Q4, 13.

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Q3. $a_n = 10^{2^n} + 1 = 10\ldots\ldots01$

Hint: show $a_n \mid a_{n+1} - 2$.

From this one can argue that $\gcd(a_n, a_{n+1}) = 1$

$\boxed{a_{n+1} - 2 = q\, a_n}$

$\Rightarrow a_{n+1} - q\, a_n = 2$

To push further, try and do more with the hint, examine the quotient $q$ carefully.

Q4.

Claim: $T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \begin{array}{l} a, b, c \in \mathbb{R}. \\ ac \neq 0 \end{array} \right\}$

is a group under matrix multiplication

i.e. $T$ is a subgroup of $GL(2, \mathbb{R})$

<u>Prove</u> $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in T$ ✓

Given $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} \in T.$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}$$

$\in T.$ ✓

note $adcf = acdf \neq 0$ since $ac \neq 0$ and $df \neq 0$

and $$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \in T. \checkmark$$

So by Proposition 3.30 $T$ is a subgroup of $GL(2, \mathbb{R})$.

a). $U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in T \ : \ x \in \mathbb{R} \right\}$

U is a subgroup thanks to Prop 3.30 and the following observations

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U.$ ✓

$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} \in U$ ✓

$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \in U$ ✓

b). $\boxed{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix}$

$\boxed{= \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}$

So U is abelian as all its matrices commute with each other.

c). To prove $U$ is normal in $T$.

Consider conjugation of $U$ by matrices from $T$.

i.e.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -b/ac \\ 0 & \frac{1}{c} \end{pmatrix}$$

$$= \begin{pmatrix} a & ax+b \\ 0 & c \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -b/ac \\ 0 & \frac{1}{c} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{b}{c} + \frac{ax+b}{c} \\ 0 & 1 \end{pmatrix} \in U$$

Therefore $tUt^{-1} \subset U$ for all $t \in T$.

So by theorem 10.3 $U$ is normal in $T$.

d) Claim $T/U$ is abelian.

$T/U$ consists of cosets of $U$

such as.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} U \quad , \quad \begin{pmatrix} d & f \\ 0 & e \end{pmatrix} U$$

Let's try multiplying these.

$$\underbrace{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} U}\ \underbrace{\begin{pmatrix} d & f \\ 0 & e \end{pmatrix} U}$$

$$= \begin{pmatrix} \cdot \end{pmatrix}\begin{pmatrix} \cdot \end{pmatrix} U$$

$$= \begin{pmatrix} ad & af+be \\ 0 & ce \end{pmatrix} U \qquad (*)$$

AND.

$$\begin{pmatrix} d & f \\ 0 & e \end{pmatrix} U \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} U$$

$$= \begin{pmatrix} ad & db+fc. \\ 0 & ce \end{pmatrix} U \qquad (**)$$

If. $\underbrace{(*) = (**)}$ then $T/U$ is abelian. $\llcorner$ i.e. are these two left cosets the same.

Using parts 1, 5 of Lemma 6.3 we'll compute the product.

$$\begin{pmatrix} ad & af+be \\ 0 & ce \end{pmatrix}^{-1} \begin{pmatrix} ad & db+fc \\ 0 & ce \end{pmatrix}$$

and hopefully observe it's in $U$.

$$= \frac{1}{adce} \begin{pmatrix} ce & -af-be \\ 0 & ad \end{pmatrix} \begin{pmatrix} ad & db+fc \\ 0 & ce \end{pmatrix}$$

$$= \frac{1}{adce} \begin{pmatrix} cead & \sim\sim\sim \\ 0 & adce \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cdots \\ 0 & 1 \end{pmatrix} \in U,$$ from the 1s and 0

So as described above $(*) = (**)$ and $T/U$ is abelian as required.

(e) Is $T$ normal in $GL(2,\mathbb{R})$?

Use theorem 10.3 to judge this.

So we'll conjugate $T$ with a typical matrix from $GL(2,\mathbb{R})$ and check we still remain in $T$

So consider products like

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}^{-1}$$

$$= \frac{1}{wz-xy}\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} z & -x \\ -y & w \end{pmatrix}$$

$$= \frac{1}{wz-xy}\begin{pmatrix} wa & wb+xc \\ ya & yb+zc \end{pmatrix}\begin{pmatrix} z & -x \\ -y & w \end{pmatrix}$$

$$= \frac{1}{wz-xy}\begin{pmatrix} \cdot & \cdot \\ yaz-y^2b-yzc & \cdot \end{pmatrix}$$

Claim this bottom left entry is not always zero, hence $\notin T$.

such as when $y=a=z=b=c=1$

So conjugation of $T$ does not stay in $T$. so by theorem 10.3 $T$ is not normal in $GL(2,\mathbb{R})$.

$\boxed{Q5}$ Recall from chapter 3.

we know if $H_1, H_2$ are subgroups of a group $G$ then so is $H_1 \cap H_2$.

Seek to use part (2) of theorem 10.3

Assume $H_1, H_2$ are both normal

Consider conjugates of intersection.

Let $g \in G$

consider $\underline{g(H_1 \cap H_2)g^{-1}}$

$= \{ghg^{-1} : h \in H_1 \cap H_2\}$.

$\subseteq gH_1g^{-1} \underline{\underline{\subseteq H_1}}$, by (2) of theorem 10.3.

and also.

$\underline{\underline{g(H_1 \cap H_2)g^{-1}}}$

$\subseteq gH_2g^{-1} \underline{\underline{\subseteq H_2}}$ by (2) of theorem 10.3.

Therefore $g(H_1 \cap H_2)g^{-1} \subseteq H_1 \cap H_2$

and so by (2) of theorem 10.3

$H_1 \cap H_2$ is normal in $G$.

$\boxed{\text{Q6}}$ Claim: If $G$ is abelian then all of it's factor groups $G/H$ are also abelian.

Proof $(aH)(bH) = abH$

$= baH$, since $ab=ba$ as $G$ is abelian

$= (bH)(aH)$

Q7 for $G = D_4$, we found the normal subgroup $R_1 = \{e, r, r^2, r^3\} = \langle r \rangle$

$R_1$ is abelian, and $|D_4/R_1| = 2$.

and so $\cong \mathbb{Z}_2$ an abelian.

but $D_4$ is not abelian.

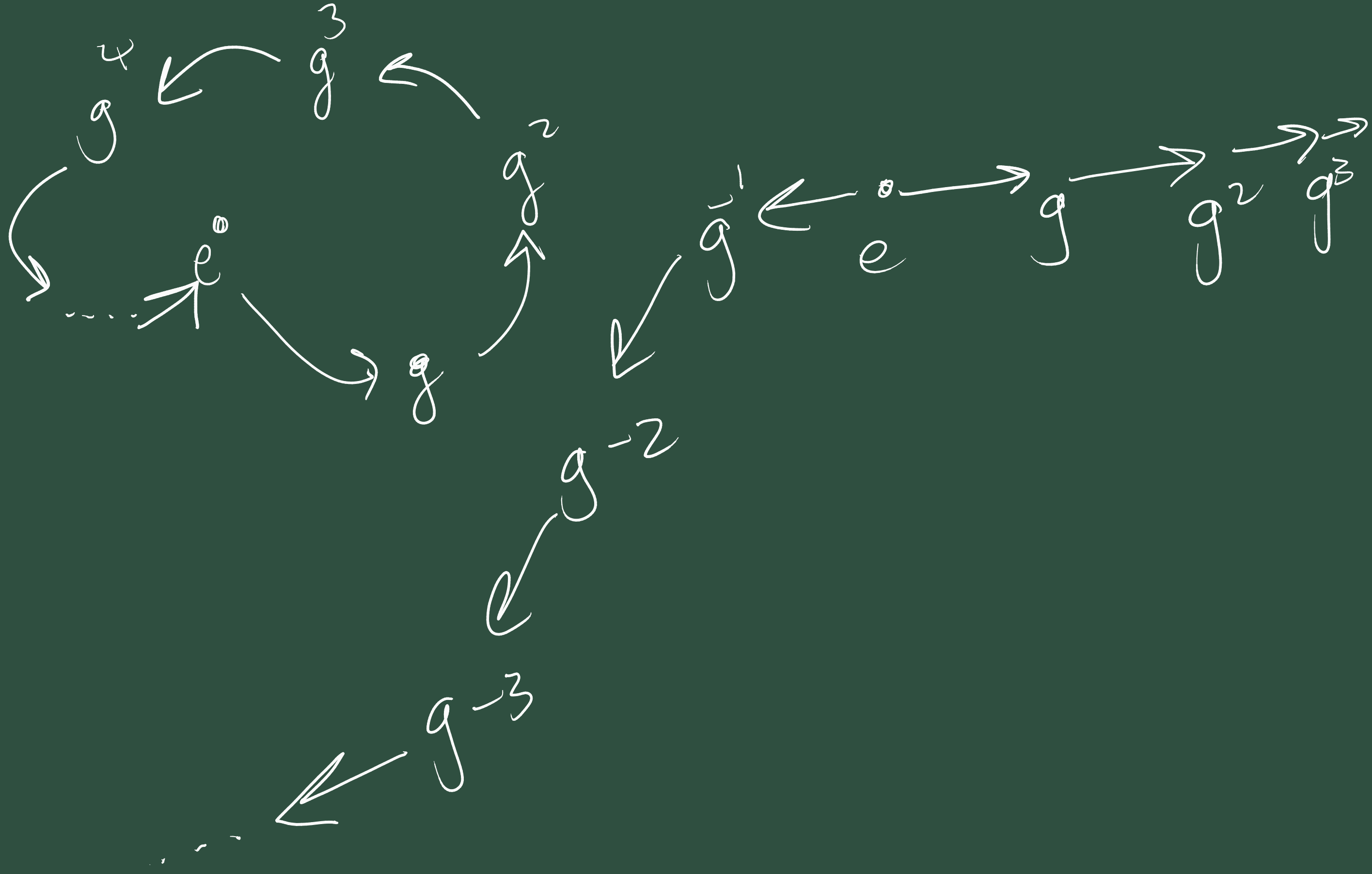
$$sr = r^{-1}s = r^3 s \neq rs$$

so the conjecture in 7 is false.

Qs 8, 9.

Remember a group $G$ is cyclic if it has a generator $g \in G$

$$G = \langle g \rangle = \{ g^k ; k \in \mathbb{Z} \}$$




$\boxed{Q8}$ Assume $G$ is cyclic.
i.e. $G = \langle g \rangle$ for some $g \in G$.

Consider a normal subgroup $H$ and consider $G/H$.

Claim: $G/H$ is cyclic

$$G/H = \langle \underline{gH} \rangle$$

<u>Proof</u> A typical element of $G/H$ takes the form $xH$ for some $x \in G$. But $x = g^k$, for some $k \in \mathbb{Z}$

So $xH = g^k H$.

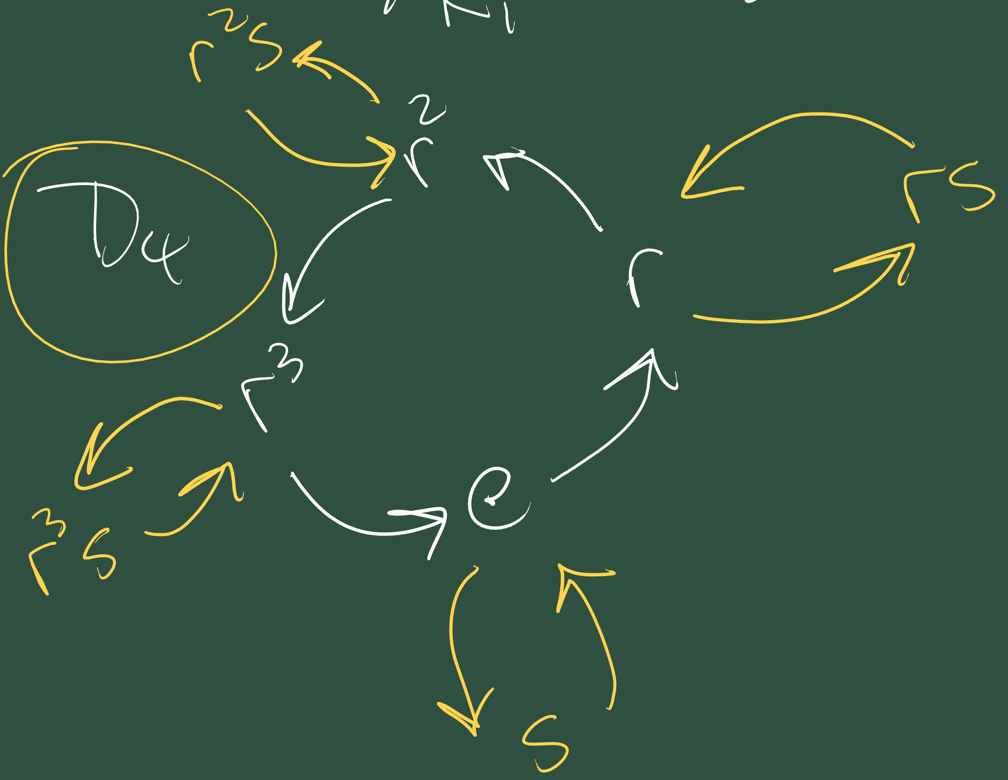
$$= (gH)^k$$

This shows that $G/H = \langle gH \rangle$ and so $G/H$ is cyclic.

$\boxed{\text{Q9}}$ Conjecture is false.

$D_4$ is not cyclic.

But has the normal cyclic subgroup $R_1 = \langle r \rangle$ and $D_4/R_1 \cong \mathbb{Z}_2$ and hence cyclic



Q13 A special subgroup $Z(G)$
"centre of $G$"

$$Z(G) = \{ x \in G : \forall g \in G \;\; gx = xg \}$$

i.e. the elements $x$ from $G$ that commute with all of $G$.

if $G$ is abelian $G = Z(G)$

a) $Z(S_3) = ?$

$|S_3| = 3! = 6$

$S_3$ the group of all permutations of three objects 1, 2, 3

$$S_3 = \{ (1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2) \}$$

Do any of these elements commute with all elements of $S_3$?

Well trivially $(1)$ does. so $(1) \in Z(S_3)$

Remember if $e$ is the identity of a group $G$

$\forall g \in G \quad eg = g = ge$

Any others? No.

$(1\ 2)(1\ 3) = (1\ 3\ 2)$

$(1\ 3)(1\ 2) = (1\ 2\ 3) \neq (1\ 2)(1\ 3)$

so $(1\ 2) \notin Z(G)$

and $(1\ 3) \notin Z(G)$

Similarly $(2\ 3) \notin Z(G)$

Also

$(1\ 2\ 3)(1\ 2) = (1\ 3)$

$(1\ 2)(1\ 2\ 3) = (2\ 3) \neq (1\ 2\ 3)(1\ 2)$

so $(1\ 2\ 3) \notin Z(G)$

similarly $(1\ 3\ 2) \notin Z(G)$.

so $Z(S_3) = \{(1)\}$, ie the group $S_3$ is very non-abelian.

b). What about $Z(GL(2,\mathbb{R}))$

Are there special matrices $X$ that commute with all matrices of $GL(2,\mathbb{R})$ under multiplication.

Clearly $I \in Z(GL(2,\mathbb{R}))$

Consider.

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} xa & xb \\ yc & yd \end{pmatrix} \qquad rs \neq sr$$

$\forall A \in GL_2(\mathbb{R}) \qquad \alpha I A = A \alpha I$

$$\neq \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

$$= \begin{pmatrix} ax & by \\ cx & dy \end{pmatrix}$$

But if $x=y$ we do get commutativity here

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = xI.$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(xI)A = xAI$$
$$= A(xI)$$

So for any $x \in \mathbb{R}$ $\quad xI \in Z(G)$.

A bit more detail needed to fully prove but I think we can appreciate that

$$Z(GL(2,\mathbb{R})) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \mathbb{R} \right\}.$$

(c)

Claim: For all $G$, $Z(G)$ is a normal subgroup.

<u>Proof</u> Let $g \in G$

$$gZ(G)g^{-1} = \{ gxg^{-1} : x \in Z(G) \}$$
$$= \{ xgg^{-1} : x \in Z(G) \}$$

$= \{x : x \in Z(G)\}$

$= Z(G)$

So by theorem 10.3 $Z(G)$ is normal in $G$.

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d). Claim: If $G/Z(G)$ is cyclic then $G$ is abelian.

or it's contrapositive: If $G$ is non-abelian then $G/Z(G)$ cannot be cyclic.

Proof Assume $G/Z(G)$ is cyclic.

let's $G/Z(G) = \langle gZ(G) \rangle$

Let $x, y \in G$

$xy = \ldots\ldots = yx.$

Consider $xZ(G)$, $yZ(G)$

$\exists\, k_1, k_2$ $\quad xZ(G) = g^{k_1}Z(G)$

$yZ(G) = g^{k_2}Z(G)$

By $1 \Leftrightarrow 4$ of Lemma 6.3.

we can say. $x \in g^{k_1} Z(G)$

and $y \in g^{k_2} Z(G)$.

$\Rightarrow$ $x = g^{k_1} z_1$, for some $z_1 \in Z(G)$

$y = g^{k_2} z_2$, for some $z_2 \in Z(G)$

$$\boxed{xy} = g^{k_1} z_1 g^{k_2} z_2$$

$= z_1 z_2 g^{k_1} g^{k_2}$, $z_i \in Z(G)$

$= z_1 z_2 g^{k_2} g^{k_1}$, powers of $g$ commute with each other.

$= g^{k_2} z_2 g^{k_1} z_1$, $z_i \in Z(G)$

$= \boxed{yx.}$

So $G$ is abelian.