Integers

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots \}$$

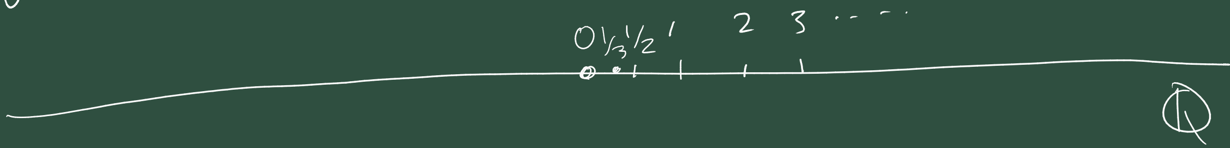
$\mathbb{Z}$ can be formally defined with a system of axioms (see notes).

Axioms 1–10 are true for $\mathbb{Z}$. But also true for $(\mathbb{Q}, +, \times)$, $(\mathbb{R}, +, \times)$, $(\mathbb{C}, +, \times)$, ...

The 11^{th} axiom will only be true for $\mathbb{Z}$.

It is crucial.

Why is 11 not true for $\mathbb{Q}$?



$$P = \mathbb{Q}^+ = \{ q \in \mathbb{Q} : q > 0 \}$$

Consider $J = \{ 1, 2, 3, 4, \ldots \} = \mathbb{Z}^+ \subset P$

$1 \in J$ ✓

if $j \in J$ then $j + 1 \in J$ ✓

But $J \neq P$, for instance $\frac{1}{3} \notin J$

For similar reasons 11 not true for $\mathbb{R}$ and others.

Ex 1.2.

Q1. $P(n)$ is the statement

$$\underbrace{\sum_{j=1}^{n} j}_{1+2+3+\dots+n} = \frac{1}{2}n(n+1)$$

Claim: For all $n \geq 1$ $P(n)$ is true.

Is $P(1)$ true? Yes.

$$\sum_{j=1}^{1} j = 1 = \frac{1}{2}1(1+1) \checkmark$$

Let $k \geq 1$.

Assume that $\boxed{\sum_{j=1}^{k} j = \frac{1}{2}k(k+1) \quad P(k)}$

From this we want to deduce $P(k+1)$.

$$\sum_{j=1}^{k+1} j = \underbrace{1+2+3+\dots+k}+(k+1).$$

$$= \left(\sum_{j=1}^{k} j\right) + k+1$$

$$= \frac{1}{2}k(k+1) + k+1, \text{ by assumption}$$

$$= (k+1)\left(\frac{1}{2}k + 1\right)$$

$$= \underbrace{\frac{1}{2}(k+1)(k+2)}_{P(k+1)}$$

So this is a proof of $P(k) \Rightarrow P(k+1)$

Therefore, by induction,

$$\sum_{j=1}^{n} j = \frac{1}{2}n(n+1)$$ is true for all $n \geq 1$.

Q2. $P(n)$ is $\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$

Base case: Is $P(1)$ true?

$$\sum_{j=1}^{1} j^2 = 1 \overset{\checkmark}{=} \frac{1}{6} 1(2)(3)$$

So yes, $P(1)$ is true.

Assume $P(k)$, i.e.

$$\boxed{\sum_{j=1}^{k} j^2 = \frac{1}{6} k(k+1)(2k+1)}$$

From this, let's deduce $P(k+1)$

$$\boxed{\sum_{j=1}^{k+1} j^2} = \left(\sum_{j=1}^{k} j^2\right) + (k+1)^2$$

$$= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$$

from assumption

$$= (k+1)\left(\frac{1}{6} k(2k+1) + (k+1)\right)$$

$$= \frac{1}{6}(k+1)\left(k(2k+1) + 6k+6\right)$$

$$= \frac{1}{6}(k+1)\left(2k^2+7k+6\right)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

$$\boxed{= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)} \; P(k+1)$$

So by induction, $P(n)$ is true for all $n \geq 1$

Q3 $\displaystyle\sum_{j=1}^{n} (2j-1) = n^2$

Q7. $P(n)$ is the statement

$$2^n \leq \frac{(2n)!}{n!\,n!}$$

Check the base case of $n=1$

$$2^1 = 2 \leq 2 = \frac{2!}{1!\,1!} \quad \checkmark$$

Now we assume that

$$\boxed{2^k \leq \frac{(2k)!}{k!\,k!}}$$

Now let's try to deduce from this

$P(k+1)$

$$\boxed{2^{k+1}} = 2 \cdot 2^k$$

$$(k+1)\,k! = (k+1)!$$

$$< 2 \cdot \frac{(2k)!}{k!\,k!} \quad \text{, from assumption.}$$

$$= 2 \cdot \frac{(2k)!\,(k+1)^2}{(k+1)!\,(k+1)!}$$

$<$

hope!!

$$= \frac{(2k)!\,(2k+2)(k+1)}{(k+1)!\,(k+1)!}$$

$$< \frac{(2k)!\,(2k+2)(2k+1)}{(k+1)!\,(k+1)!}$$

$<$

$$= \boxed{\frac{(2k+2)!}{(k+1)!\,(k+1)!}}$$