

Q 4.

claim $\forall n \geq 1 \quad \sum_{j=1}^n j^3 = \frac{1}{4} n^2 (n+1)^2$

Base case: $n=1$.

$$\sum_{j=1}^1 j^3 = 1^3 = 1 = \frac{4}{4} = \frac{1}{4} 1^2 (1+1)^2$$

This is clearly true. ✓

Assume result is true for some $k \geq 1$

i.e., $\sum_{j=1}^k j^3 = \frac{1}{4} k^2 (k+1)^2$

The $(k+1)$ th case of the result begins as.

$$\begin{aligned} \sum_{j=1}^{k+1} j^3 &= \left(\sum_{j=1}^k j^3 \right) + (k+1)^3 \\ &= \frac{1}{4} k^2 (k+1)^2 + (k+1)^3, \quad \text{by assumption.} \\ &= \frac{1}{4} (k+1)^2 \left[k^2 + 4(k+1) \right] \end{aligned}$$

Proof: Assume $c|ab$
and that $\gcd(c, a) = 1$

$\Rightarrow ab = \beta c$, from def.
of divisibility.

$\beta \in \mathbb{K}$.

From Theorem 2.3.

we know $\exists x, y \in \mathbb{K}$
such that

$$1 = xc + yab$$

$$1 = xc + ya$$

$$\Rightarrow b = bxc + bya$$

$$\Rightarrow b = bxc + y\beta c$$

$$\Rightarrow b = c(\underbrace{xb + y\beta}_{\in \mathbb{K}})$$

$\Rightarrow c|b$, by definition.

So by the principle of induction

$$\forall n \geq 1 \quad 8 \mid 3^{2n} + 7.$$

Q2. Claim: if $a \in \mathbb{Z}$ then one of $a, a+2, a+4$ is divisible by 3.

We know

$$a = 3q + r$$

for some $q, r \in \mathbb{Z}$ $r = 0, 1, 2$.

Argue separately for cases $r = 0, 1, 2$.

If $r = 0$ then $a = 3q$ so result holds

if $r = 1$ then $a+2 = 3q+1+2 = 3q+3 = 3(q+1)$

a

$$\Rightarrow \frac{a+2}{a+4} = 3 \frac{\quad}{\in \mathbb{Z}}.$$

if $r = 2$ then $a+4 = 3q+2+4 = 3q+6 = 3(q+2)$