Integers

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$$

Formally defined / constructed with the specification of a set of axioms

~~Define~~ Consider a set S with two binary operations.

$$+ : S \times S \longrightarrow S$$
$$\cdot : S \times S \longrightarrow S$$

Axioms 1–10 define a commutative ordered ring $(S, +, \cdot, >)$

But there are many of these
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \ldots$

The induction axiom 11a is true of $\mathbb{Z}$, but crucially, not true of $\mathbb{Q}, \mathbb{R}, \ldots$ or indeed any other commutative ordered ring.

eg. Consider $P = \mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$

consider $J = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$
$\subset P$.

$1 \in P$, for sure.

and if $j \in J$ then $j+1 \in J$.

But crucially $J \neq P = \mathbb{Q}^+$.

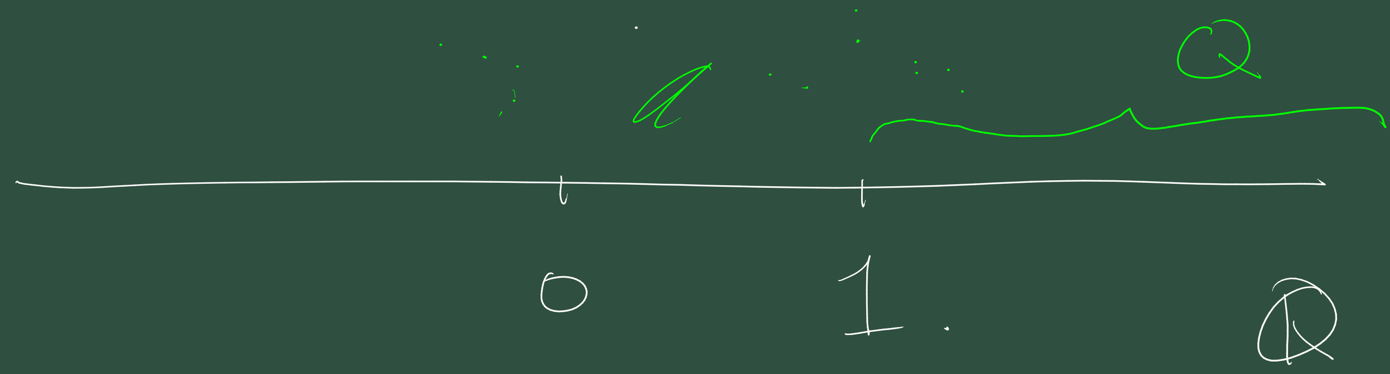
Can also take another version of this axiom 11b, well-ordered axiom.

Surely true of $\mathbb{Z}$, but not true of $\mathbb{Q}$ or $\mathbb{R}$

eg. Consider $Q \subset \mathbb{Q}$

defined $Q = \{q \in \mathbb{Q} \mid q > 1\}$.
$\subset \mathbb{Q}^+$

but $\mathbb{Q}$ has no least element.



This shows $\mathbb{Q}$ do not satisfy IIb.

IIa gives us the powerful proof technique of proof-by-induction for statements about $\mathbb{Z}$.

<u>Ex 1.2</u>

Q1. Here the statement $P(n)$ is

$$\underbrace{\sum_{j=1}^{n} j}_{1+2+3+\dots+n} = \frac{1}{2} n (n+1).$$

To prove this by induction we first observe that

$$\boxed{\sum_{j=1}^{1} j} = 1 = \boxed{\frac{1}{2}\cdot 1\cdot(1+1)}$$

hence $P(1)$ is true.

Let $k \geq 1$, and assume that $P(k)$ is true, i.e. we assume

$$\sum_{j=1}^{k} j = \frac{1}{2}k(k+1).$$

$P(k)$.

From this we try to derive $P(k+1)$

$$\boxed{\sum_{j=1}^{k+1} j} = \left(\sum_{j=1}^{k} j\right) + k+1$$

$$= \frac{1}{2}k(k+1) + (k+1), \text{ by assumption}$$

$$= \frac{1}{2}\left(k(k+1) + 2(k+1)\right).$$

$$= \boxed{\frac{1}{2}(k+1)(k+2)}$$

is $P(k+1)$

So by the Induction axiom.

$$\forall n \geq 1 \quad \sum_{j=1}^{n} j = \frac{1}{2} n(n+1).$$

Q 2. Here $P(n)$ is the statement

$$\sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$1^2 + 2^2 + 3^2 + \dots + n^2$$

Observe that $P(1)$ is true, since.

$$\sum_{j=1}^{1} j^2 = 1 = \frac{1}{6} 1(2)(3) \quad \checkmark$$

Assume that $P(k)$ is true, for some $k \geq 1$

ie. $$\sum_{j=1}^{k} j^2 = \frac{1}{6} k(k+1)(2k+1)$$

$$\boxed{\sum_{j=1}^{k+1} j^2} = \left(\sum_{j=1}^{k} j^2\right) + (k+1)^2$$

$$= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$$

, by assumption

$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$$

$$= \frac{1}{6}(k+1)\left[\underbrace{2k^2 + 7k + 6}\right].$$

factorising this quadratic.

$$\boxed{= \frac{1}{6}(k+1)(k+2)(2k+3).}$$

which is $P(k+1)$

So by the principle of induction.

$$\forall n \geq 1 \quad \sum_{j=1}^{n} j^2 = \frac{1}{6} n(n+1)(2n+1)$$

Q2. What is the result for.

$$\sum_{j=1}^{n} j^5 = ?$$

some order 6. polynomial.

$$= a_6 n^6 + a_5 n^5 + a_4 n^4 + \dots + a_1 n + a_0$$

Use 7 different values of n.
to give me 7 linear equations
in the unknowns $a_0, \dots, a_6$.
Solve these using row-reduction
to identify the polynomial.

Q3 $\forall n \geq 1 \quad \sum_{j=1}^{n} (2j-1) = n^2$

Q4 $\forall n \geq 1 \quad \sum_{j=1}^{n} j^3 = \frac{1}{4} n^2 (n+1)^2$

Bernoulli polynomials.
Faulharber polynomials