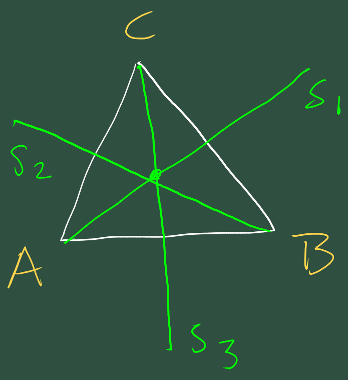
Initial examples

r = rot by $\frac{2\pi}{3}$ radians anti-clockwise.

Consider D_3.



$$D_3 = \{e, r, r^2, s_1, s_2, s_3\}$$

Each element of D_3 can be represented by its action on the vertices A, B, C.

Using a 'two-line permutation symbol'

$$e = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}$$

← arguments/start

← values $e(A), e(B), e(C)$

$$r = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \qquad r^2 = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$

$$s_1 = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \qquad s_2 = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$

$$s_3 = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix} \qquad X = \{A, B, C\}$$

Def: Given a set X, a <u>permutation</u> π of X is a mapping/function

$$\pi : X \longrightarrow X$$

that is bijective, i.e. injective and surjective, i.e. 1-1 and onto.

injective means

$$\forall\, x,y \in X \quad \pi(x)=\pi(y) \Rightarrow x=y$$

surjective means

$$\forall\, y\in X\ \exists\, x\in X \quad \pi(x)=y.$$

In general we can define a group S_X consisting of all the permutations of X under the operation of composition.

$\pi_1, \pi_2 : X \longrightarrow X$

$\pi_1, \pi_2 : X \longrightarrow X$, $\quad \pi_1\pi_2 := \pi_1 \circ \pi_2$

i.e. $(\pi_1 \circ \pi_2)(x) = \pi_1(\pi_2(x))$

Called the "symmetric group of X"

For finite sets X we standardize the notation and write

$$X = \{\underbrace{1, 2, 3, \dots, n}_{\text{serving as a collection of labels for the elements of } X.}\}$$

write S_n for S_X in this case.

Theorem 5.6' S_n is a group and $|S_n| = n!$

Proof From theory of functions. given two bijections/permutations $\pi_1, \pi_2 : X \to X$ then their composition

$$\pi_1 \circ \pi_2 : X \to X$$ is also a bijection.

So $\pi_1, \pi_2 \in S_n \Rightarrow \pi_1 \pi_2 \in S_n$.

The three conditions from defn of a group

1. Associativity. Done. ✓ Composition of functions is always associative.

2. Identity. The identity of S_n is the identity mapping $e : X \to X$ defined by $\forall x \in X \quad e(x) = x$.

which has the property that for all mappings $\pi : X \to X$.

$$e\pi = \pi e = \pi.$$

3. Inverses. Again from theory of functions given a bijection $\pi : X \to X$ ~~condi~~

Consider the inverse function

$\pi^{-1} : X \to X$ has the property

$\pi^{-1} \circ \pi = \pi \circ \pi^{-1} = e.$

and importantly π^{-1} is <u>also</u> a bijection

so $\pi^{-1} \in S_n$.

So S_n is a group.

Claim: $|S_n| = n!$

<u>Proof</u>: Count the number of possible two-line permutation symbols.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$$

There are $\underline{n}$ choices for $\pi(1)$?

Having chosen $\pi(1)$.

Then there are $\underline{n-1}$ choices for $\pi(2)$?

Having chosen $\pi(2)$

Then the are $n-2$ choices for $\pi(3)$

$\vdots$

Having chosen $\pi(n-1)$

There are 1 choices for $\pi(n)$

So in total there are $n \times (n-1) \times (n-2) \times \dots \times 3.2.1$
$= n!$

different possible πs we could construct.

Therefore $|S_n| = n!$ ☒

Some practice with this notation

<u>Example 5.2</u>

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$$

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

Check $\sigma\tau$, by carefully applying the composition to fill out the two-line symbol for $\sigma\tau$

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} = \mu$$

second line is $\sigma(\tau(x))$ for $x = 1, 2, 3, 4, 5$

Continue at 5:00!

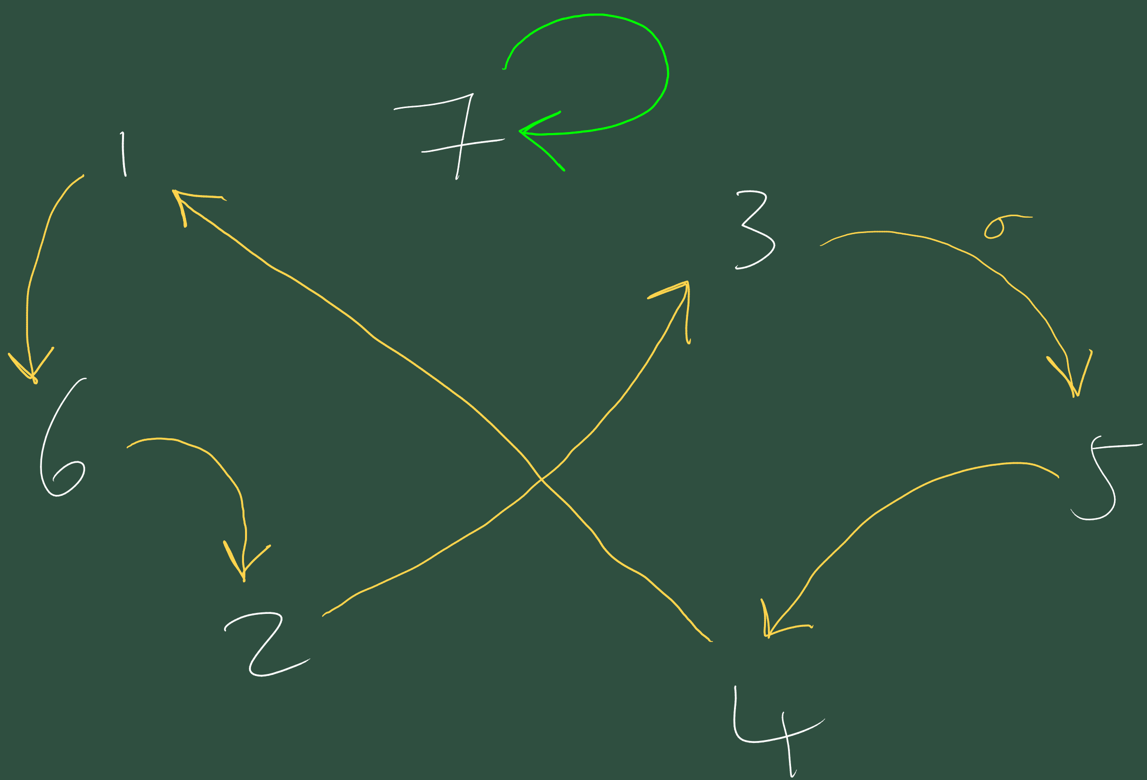
Two line notation is quite wasteful

- The top line is always $1, 2, \dots, n$
- It somehow hides/obscures the 'real structure' of the permutation.

A better notation is 'cycle notation'

<u>Example 5.5</u>.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 5 & 1 & 4 & 2 & 7 \end{pmatrix}$$



Draw the mapping arrows of σ.

This reveals to us that σ is

composed of two cycles.

$$(1\ 6\ 2\ 3\ 5\ 4)$$

$$(7)$$

So this σ from above can be written as $\sigma \in S_7$.

$$\sigma = (1\ 6\ 2\ 3\ 5\ 4)(7)$$
$$= (1\ 6\ 2\ 3\ 5\ 4)$$

and then it is understood that $\sigma(7)=7$ since 7 does not appear in the cycle notation $(1\,6\,2\,3\,5\,4)$

And

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 5 & 6 \end{pmatrix} \in S_6$$

$$\tau = (1)(4\ 3\ 2)(5)(6)$$
$$= (4\ 3\ 2) = (3\ 2\ 4) = (2\ 4\ 3)$$

And

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} \in S_6$$

$$\mu = (4\,3\,1\,2)(5\,6)$$

operation of the composition of the two cycles.

(4312) and (56) are disjoint cycles. Compositions of disjoint cycles are commutative

$$(4312)(56) = (56)(4312)$$

Ex 5.6.

The preferred ~~represe~~ representation of a permutation will be as a product of disjoint cycles.

$\sigma = (1\ 3\ 5\ 2)$, $\tau = (2\ 5\ 6)$

These are not disjoint. S_6

What is the disjoint cycle representation

for ~~yyy~~ $\lambda = \sigma\tau$

$$\lambda = \sigma\tau = (1\ 3\ 5\ 2)(2\ 5\ 6)$$

$$= (5\ 6\ 1\ 3)(2)$$

$$= (5\ 6\ 1\ 3)$$

But

$$\alpha = \tau\sigma = (1\ 3\ 6\ 2) \neq \lambda$$

So τ, σ do not commute as they are not disjoint.

Ex 5.10

$$\sigma\tau = (1\ 6\ 2\ 4)(1\ 3)(4\ 5\ 6)$$

$$= (1\ 3\ 6)(4\ 5\ 2)$$

$$\tau = \tau_1 \circ \tau_2$$

$$\sigma \circ \tau = \sigma \circ \tau_1 \circ \tau_2$$

Transpositions

Are 2-cycles / cycles of length 2.

Any cycle can be expressed as a product/composition of transpositions.

$$(a_1\, a_2 \ldots\ldots a_n) = (a_1\, a_n) \;\ldots\ldots\; (a_1 a_3)(a_1 a_2)$$

Discuss the results relying on transpositions at tutorial.