Q2) In each case we clearly have a closed binary operation $\circ : G \times G \to G$

$$\begin{array}{c|c} & y \\ \hline x & x \circ y \end{array}$$

$(G, \circ)$ forms a group iff.

1. Associativity

   $\forall x, y, z \in G \quad x(yz) = (xy)z$

2. Identity

   $\exists e \in G \quad \forall x \in G \quad xe = ex = x.$

3. Inverses

   $\forall x \in G \quad \exists x^{-1} \in G \quad x x^{-1} = x^{-1} x = e.$

Let's consider existence of identity first

A) No identity. B) a C) a D) a

So A) is not a group

Existence of inverses.

B) yes. all self-inverse.

C) $a^{-1} = a$, $b = d^{-1}$, $c = c^{-1}$

D) d has no inverse.

So D) is not a group.

So now need to check associativity for
B), C).

in A) a is a right-identity

$\forall x \in G \quad xa = x$.

But a is not a left-identity.

as $ab = c \neq b$.

Checking associativity. we don't need
to include the identity

Consider $\{b, c, d\}$.

$b(cd) = (bc)d$? $\quad a = a$ ✓

$c(bd) = (cb)d$? $\quad a = a$ ✓

$b(dc) = (bd)c$? $\quad a = a$ ✓

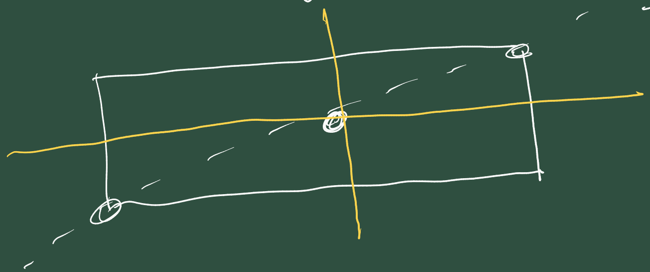
$c(db) = (cd)b$? $\quad a = a$ ✓

$d(bc) = (db)c$? $\quad a = a$ ✓

$d(cb) = (dc)b$? $\quad a = a$ ✓

So B) is associative.

Q3) Consider a non-square rectangle. G is its group of symmetries



this has 4 symmetries

μ_1 = reflection in vertical axis

μ_2 = reflection in horizontal axis.

e = identity.

ρ = rotation by a half-turn.

G

$\circ$	e	ρ	μ_1	μ_2
e	e	ρ	μ_1	μ_2
ρ	ρ	e	μ_2	μ_1
μ_1	μ_1	μ_2	e	ρ
μ_2	μ_2	μ_1	ρ	e

$(\mathbb{Z}_4, +)$

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

In the body of each table, each element of the group appears once and only once in each row and column.

G is called Klein Viergruppe.

Group structures are different as

everything in G is self-inverse but not so in $\mathbb{Z}_4$.

Q6.

$$U(12) := \{ x \in \mathbb{Z}_{12} : \gcd(x,12)=1 \}$$

group of units modulo 12 under multiplication. called $\mathbb{Z}_{12}^{\times}$ in Number Theory.

$$= \{ 1, 5, 7, 11 \}$$

The Cayley table for $U(12)$ is

$\times$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

multiplication modulo 12

This has the same group structure as G. In later chapters we will say G and $U(12)$ are <u>isomorphic</u>.

and write it as $G \cong U(12)$.

$\boxed{Q10}$

$$G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

(cite. L.A.)

Consider associativity, identity, inverses. $I \in G$ Q?

mat. mult. is associative on G. (everywhere)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \checkmark \quad x, y, z = 0.$$

Given $A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in G$

Is A^{-1} in G? Does A^{-1} even exist?

A^{-1} exists in $GL_3(\mathbb{R})$ because $\det A = 1$

and $$A^{-1} = \begin{pmatrix} 1 & -x & -y+xz \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{pmatrix} \in G. \checkmark$$

$x' = -x$

$z' = -z$.

$y' = -y + xz$

and from given product formula we see

$$A A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

So G satisfies the inverses condition.

So G is a group.

also say G is a subgroup $GL_3(\mathbb{R})$

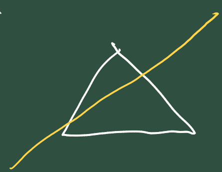
$\boxed{Q35}$ Recall D_3 from the first lecture. Identify all its subgroups

$$D_3 = \{ e, \rho_1, \rho_2, \mu_1, \mu_2, \mu_3 \}.$$

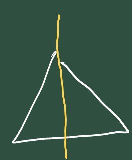
ρ_1 = rot. by $2\pi/3$ radians clockwise

ρ_2 = " " $4\pi/3$ " "

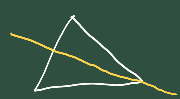
μ_1 = reflection in



μ_2 = " "



μ_3 = " "



Subgroups are D_3 itself, $\{e\}$,

$$H_1 = \{ e, \rho_1, \rho_2 \}$$

$H_2 = \{e, \mu_1, \mu_2, \mu_3\}$?

So H_2 is not a subgroup as it's not closed as $\mu_1 \circ \mu_2 = \rho_1 \notin H_2$.

$H_3 = \{e, \mu_1\}$, $H_4 = \{e, \mu_2\}$

$H_5 = \{e, \mu_3\}$

$H_6 = \{e, \rho_1, \mu, \rho_2, \ldots\} = D_3$

$H_7 = \{e, \mu_1, \mu_2, \rho_1 \ldots\} = D_3$

No other possibilities. All the subgroups are

$\{e\}, D_3, H_1, H_3, H_4, H_5$ ∎

$\boxed{Q48}$ For any group G, its <u>centre</u>.

$$Z(G) := \{ x \in G : \forall g \in G \\ xg = gx \}$$

Zentrum

forms a $\boxed{\text{subgroup of } G.}$

Now if G is abelian, i.e. the op. is commutative on G, then

$$Z(G) = G$$

So the interesting cases are for non-abelian groups G, the centre $Z(G)$ is kind of "the subgroup of its commutative elements".

<u>Proof</u> Use Prop 3.30.

$$H = Z(G) = \{ x \in G : \forall g \in G \; xg = gx \}$$

1. Consider $e \in G$, then for all $g \in G$

$$eg = g = ge$$

So yes, $e \in Z(G)$.

2. Closure

Let $x_1, x_2 \in Z(G)$

Question is $x_1 x_2 \in Z(G)$?

For any $g \in G$

$$(x_1 x_2) g = x_1 g x_2 \quad , \text{ since } x_2 \in Z(G)$$

$$= g x_1 x_2 \quad , \text{ since } x_1 \in Z(G)$$

$$= g(x_1 x_2)$$

$$\Rightarrow \quad x_1 x_2 \in Z(G)$$

3. Let $x \in Z(G)$ i.e. $\forall g \in G$ $xg = gx$

Q Is $x^{-1} \in Z(G)$?

$$a = (a^{-1})^{-1}$$

For any $g \in G$

$$x^{-1} g = ((x^{-1} g)^{-1})^{-1}$$

$$= (g^{-1} x)^{-1}$$

$$= (x g^{-1})^{-1}, \quad x \in Z(G)$$

$$= (g x^{-1})$$

$$= g x^{-1}$$

And so $x^{-1} \in Z(G)$.

So by Prop 3.30 $Z(G)$ forms a subgroup

$$Z(D_3) = \{ \quad e \quad \}$$