Question 5 = a) We know that $\sum_{x=0}^{N-1} r^x = \frac{1-r^N}{1-r}$

Let $r = e^{-\frac{2\pi i k}{N}}$, then

$$\sum_{x=0}^{N-1} r^x = \sum_{x=0}^{N-1} e^{-\frac{2\pi i k x}{N}} = \frac{1 - e^{-2\pi i k}}{1 - e^{-\frac{2\pi i k}{N}}}$$

b) $\lim_{k \to 0} \frac{1 - e^{-2\pi i k}}{1 - e^{-\frac{2\pi i k}{N}}} = \frac{1-1}{1-1} = \frac{0}{0}$

Use l'Hopital's rule:

$$\lim_{k \to 0} \frac{1 - e^{-2\pi i k}}{1 - e^{-\frac{2\pi i k}{N}}} \overset{H}{=} \lim_{k \to 0} \frac{-2\pi i \, e^{-2\pi i k}}{-\frac{2\pi i}{N} e^{-\frac{2\pi i k}{N}}}$$

$$= \frac{-2\pi i}{-2\pi i} \times N$$

$$= N$$

Assume k is an integer and $k \equiv R \ [N]$, $R > 0$.

Then as k is an integer $e^{-k 2\pi i} = 1$. But as k not an integer multiple of N, we have $\frac{k}{N}$ not an integer $\Rightarrow e^{-\frac{ik 2\pi}{N}} \neq 1$

Consequently, $\sum_{x=0}^{N-1} e^{-\frac{2\pi i k x}{N}} = \frac{1-1}{1 - e^{-\frac{2\pi i k}{N}}} = \frac{0}{1 - e^{-2\pi i k/N}} = 0$

c) Let N/k DFT Eqn/ln, k wave number.

Let $S(x) = \sin\left(\frac{2\pi k x}{N}\right)$

Then $S(q) = F[S(x)] = \sum_{x=0}^{N-1} \sin\left(\frac{2\pi k x}{N}\right) e^{-\frac{2\pi i q x}{N}}$

$$= \frac{1}{2i} \sum_{x=0}^{N-1} \left(e^{\frac{2\pi i k x}{N}} - e^{\frac{-2\pi i k x}{N}}\right) e^{\frac{-2\pi i q x}{N}}$$

$$= \frac{1}{2i} \sum_{x=0}^{N-1} \left(e^{\frac{-2\pi i (q-k) x}{N}} - e^{\frac{-2\pi i}{N}(k+q)x}\right)$$

$$= \frac{1}{2i}\left(\frac{1 - e^{-2\pi i (q-k)}}{1 - e^{\frac{-2\pi i}{N}(q-k)}} - \frac{1 - e^{-2\pi i (k+q)}}{1 - e^{\frac{-2\pi i}{N}(k+q)}}\right)$$

If $k \in \mathbb{N}$, from 5.5, $S(q) = \begin{cases} \frac{1}{2i} N & q = k \\ -\frac{1}{2i} N & q = -k \end{cases}$

& we have no spectral leakage; function is represented by two delta functions!

For $k \notin \mathbb{N}$, no simplification of the sum and we thus get spectral leakage. Our delta function would probably be "less good" has we had a non-integer value of k. Assume in class, we could minimize that effect by increasing N (DFT length), or decreasing the sample frequency. We will however decrease the spectral resolution.