Question n° 1 =

a) By Taylor expanding in the variable δ, we have

$$f(x \pm \delta) = f(x) \pm f'(x)\delta + \frac{1}{2!} f''(x)\delta^2 \pm \frac{1}{3!} f'''(x)\delta^3 + \dots$$

$$f(x \pm 2\delta) = f(x) \pm 2f'(x)\delta + 2f''(x)\delta^2 \pm \frac{8}{6} f'''(x)\delta^3 + \dots$$

Since we are looking for $f'(x)$ in terms of $f(x \pm \delta)$ and $f(x \pm 2\delta)$,

let solve the system (and neglect higher order terms):

$$\alpha f(x+\delta) + \beta f(x-\delta) + \gamma f(x+2\delta) + \rho f(x-2\delta) = (1) f'(x)$$

$$\Rightarrow f(x)[\alpha + \beta + \gamma + \rho] + f'(x)\delta[\alpha - \beta + 2\gamma - 2\rho]$$
$$+ f''(x)\delta^2\left(\frac{\alpha}{2} + \frac{\beta}{2} + 2\gamma + 2\rho\right)$$
$$+ f'''(x)\delta^3\left(\frac{\alpha}{6} - \frac{\beta}{6} + \frac{4}{3}\gamma - \frac{4}{3}\rho\right) = \delta f'(x)$$

This implies that:

$$\begin{cases} \alpha + \beta + \gamma + \rho = 0 \\ \alpha - \beta + 2\gamma - 2\rho) = 1 \\ \frac{\alpha}{2} + \frac{\beta}{2} + 2\gamma + 2\rho) = 0 \\ \frac{\alpha}{6} - \frac{\beta}{6} + \frac{4}{3}\gamma - \frac{4}{3}\rho) = 0 \end{cases}$$

as we can just multiply

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & -2 & 1 \\ \frac{1}{2} & \frac{1}{2} & 2 & 2 & 0 \\ \frac{1}{6} & -1/6 & \frac{4}{3} & -\frac{4}{3} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & -3 & 1 \\ 0 & 0 & 3/2 & 3/2 & 0 \\ \frac{1}{6} & -1/6 & \frac{4}{3} & \frac{4}{3} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 3/2 & -1/2 \\ 0 & 0 & 3/2 & 3/2 & 0 \\ 0 & -1/3 & 7/6 & -3/2 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1/6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1/12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 0 & -5/8 \\ 0 & 0 & 1 & 0 & -1/12 \\ 0 & 0 & 0 & 1 & 1/12 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & -1/12 \\ 0 & 1 & 0 & 0 & -2/3 \\ 0 & 0 & 1 & 0 & -1/12 \\ 0 & 0 & 0 & 1 & 1/12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 & -2/3 \\ 0 & 0 & 1 & 0 & -1/12 \\ 0 & 0 & 0 & 1 & 1/12 \end{pmatrix}$$

So $\alpha = \frac{2}{3}$, $B = -\frac{2}{3}$, $\gamma = -1/12$ $\rho = \frac{1}{12}$

Thus $\delta f'(x) = \frac{2}{3}\left[f(x+\delta) - f(x-\delta)\right] + \frac{1}{12}\left[f(x-2\delta) - f(x+2\delta)\right]$

$$\Rightarrow f'(x) = \frac{\frac{2}{3}\left(f(x+\delta) - f(x-\delta)\right) + \frac{1}{12}\left(f(x-2\delta) - f(x+2\delta)\right)}{\delta}$$

5) Need to find the best δ to have the best precision.
In fact, machine precision will affect error in our derivative as well have the truncation of the taylor series.

As seen in class :

$$\tilde{f}(x) \to f(x)(1+\varepsilon_g)$$

What the machine gets — machne precision — ↳ order of unity

For consiseness, let $f_{\pm n} = f(x \pm n \times \delta)$.

The total error $E(\delta) = E_{machine}(\delta) + E_{truncation}(\delta)$

let determine $E_t(\delta) = E_{truncation}(\delta) =$

$$\delta f'(x) = \frac{2}{3}(f_+ - f_-) + \frac{1}{12}(f_{2-} - f_{2+})$$

The error on $f_{n\pm}$ is $E_{f_{n\pm}} \le \frac{f^{(4+1)}(\pm\delta)^{4+1}}{(4+1)!}$ as even power cancels out.

[Taylor Reminder Thm.]

so $E_t(\delta) = \left| \frac{1}{\delta}\left[\left(\frac{2 \times f^{(5)}\delta^5}{5!}\right) \times \frac{2}{3} + \frac{1}{12}\left(-\frac{\delta^5 f^{(5)} \times 2 \times 2^5}{5!}\right)\right]\right|$

$$= \frac{\delta^4 |f^{(5)}|}{30}$$

Now let's account for machine precision $E_m(\delta) = E_{machine}(\delta)$

The computer receive $\bar{f}'(x) = \frac{1}{\delta}\left[\frac{2}{3}\left((1+\varepsilon_+)f_+ - (1+\varepsilon_-)f_-\right) + \frac{1}{12}\left((1+\varepsilon_{2-})f_{2-} - (1+\varepsilon_{2+})f_{2+}\right)\right]$

so $\delta E_m(\delta) = \delta|\bar{f}'(x) - f'(x)| = \frac{2}{3}(\varepsilon_+ f_+ - \varepsilon_- f_-) + \frac{1}{12}(\varepsilon_{2-}f_{2-} - \varepsilon_{2+}f_{2+})$

let $\varepsilon_+, \varepsilon_-, \varepsilon_{2+}, \varepsilon_{2-} < \varepsilon$ and let's consider the case in which this error is the biggest by setting the $\varepsilon_{n\pm}$ to $\pm\varepsilon$.

so $E_m \le \left| \left[\frac{2}{3}\varepsilon\,(f_+ + f_-) + \frac{\varepsilon}{12}(f_{2-} + f_{2+}) \right] \times \frac{1}{\delta} \right|$

Note: It will of course depend on the sign of $f_{n\pm}$; just make sure you chose $\pm\varepsilon$ st it comes a sum of positive terms

Since δ small, assume $x \pm \delta = x$, $x \pm 2\delta = x$

so $E_m(\delta) \le \left| \frac{\varepsilon}{\delta} \left(\frac{2\times 2}{3} f(x) + \frac{2}{12} f(x) \right) \right|$

$= \left| \frac{\varepsilon}{\delta} f(x) \times \left(\frac{4}{3} + \frac{1}{6} \right) \right|$

$= \frac{3}{2} \frac{\varepsilon}{\delta} |f(x)|$

Thus $E(\delta) = E_m + E_t = \frac{3}{2}\frac{\varepsilon}{\delta}|f(x)| + \frac{\delta^4 |f^{(5)}(x)|}{30}$

Take derivatives wrt δ and set it to 0:

$\frac{\partial E}{\partial \delta} = 0 \Rightarrow -\frac{3}{2}\frac{\varepsilon}{\delta^2}|f(x)| + \frac{2}{15}\delta^3 |f^{(5)}(x)| = 0$

$\Rightarrow \frac{3}{2}\frac{\varepsilon}{\delta^2}|f(x)| = \frac{2}{15}\delta^3 |f^{(5)}(x)|$

$\Rightarrow \delta^5 = \frac{45}{4}\varepsilon \left| \frac{f(x)}{f^{(5)}(x)} \right|$

$\Rightarrow \delta = \left[\frac{45}{4}\varepsilon \left| \frac{f(x)}{f^{(5)}(x)} \right| \right]^{1/5}$

let $f_\alpha(x) = e^{x}$, $f_\beta(x) = e^{0.01x}$

then $\delta_\alpha = \left[\frac{45}{4}\varepsilon\right]^{1/5}$

$$\delta_\beta = \left[\frac{45\varepsilon}{4} \times \frac{1}{(10^{-2})^5}\right]^{1/5} = \left[\frac{45}{4} \times \frac{\varepsilon}{10^{-10}}\right]^{1/5} \sim$$

As seen in class, for 64 bits $\varepsilon \sim 10^{-16}$

so $\delta_\alpha \sim \left[10 \times 10^{-16}\right]^{1/5} = \left[10^{-\frac{15}{5}}\right] = 10^{-3}$

$$\delta_\beta \sim \left[10 \times \frac{\varepsilon}{10^{-10}}\right]^{1/5} = \left[10^{11}\varepsilon\right]^{1/5} = \left[10^{-5}\right]^{1/5} = 10^{-1}$$

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