# Inverse of a Matrix using Elementary Row Operations

*Also called the Gauss-Jordan method.*

This is a fun way to find the Inverse of a Matrix:

Play around with the rows (adding, multiplying or swapping) until we make Matrix **A** into the Identity Matrix **I**

$$\left[\begin{array}{c|c} A & I \end{array}\right] \xrightarrow{\text{"Elementary Row Operations"}} \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

And by ALSO doing the changes to an Identity Matrix it magically turns into the Inverse!

The **"Elementary Row Operations"** are simple things like adding rows, multiplying and swapping ... but let's see with an example:

## Example: find the Inverse of "A":

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

We start with the matrix A, and write it down with an Identity Matrix I next to it:

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

(This is called the "Augmented Matrix")

### Identity Matrix

The "Identity Matrix" is the matrix equivalent of the number "1":

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- It's symbol is the capital letter **I**.

Now we do our best to turn "A" (the Matrix on the left) into an Identity Matrix. The goal is to make Matrix A have **1**s on the diagonal and **0**s elsewhere (an Identity Matrix) ... and the right hand side comes along for the ride, with every operation being done on it as well.

But we can only do these **"Elementary Row Operations"**:

- **swap** rows
- **multiply** or divide each element in a a row by a constant
- replace a row by **adding** or subtracting a multiple of another row to it

And we must do it to the **whole row**, like this:

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Start with **A** next to **I**

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Add}$$

Add row 2 to row 1,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Divide by 5}$$

then divide row 1 by 5,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & -2 & -0.4 & 0.6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Subtract} \times 2$$

Then take 2 times the first row, and subtract it from the second row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Multiply by } -\tfrac{1}{2}$$

Multiply second row by -1/2,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right] \text{ Swap}$$

Now swap the second and third row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & -0.2 & 0.3 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right] \text{ Subtract}$$

Last, subtract the third row from the second row,

And we are done!

And matrix **A** has been made into an Identity Matrix ...

... and at the same time an Identity Matrix got made into $\mathbf{A^{-1}}$

$$A^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}$$

DONE! Like magic, and just as fun as solving any puzzle.

**And note: there is no "right way" to do this, just keep playing around until we succeed!**

(Compare this answer with the one we got on Inverse of a Matrix using Minors, Cofactors and Adjugate. Is it the same? Which method do you prefer?)

## Larger Matrices

We can do this with larger matrices, for example, try this 4x4 matrix:

$$B = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Start Like this:

$$\left[\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

See if you can do it yourself (I would begin by dividing the first row by 4, but you do it your way).

You can check your answer using the Matrix Calculator (use the "inv(A)" button).

## Why it Works

I like to think of it this way:

- when we turn "8" into "1" by dividing by 8,
- and do the same thing to "1", it turns into "1/8"

$$\left[\begin{array}{c|c} 8 & 1 \end{array}\right] \xrightarrow{\text{"Divide by 8"}} \left[\begin{array}{c|c} 1 & \tfrac{1}{8} \end{array}\right]$$

And "1/8" is the (multiplicative) **inverse of 8**

Or, more technically:

The **total effect of all the row operations** is the same as multiplying by $\mathbf{A^{-1}}$

So **A** becomes I (because $\mathbf{A^{-1}A} = \mathbf{I}$)
And I becomes $\mathbf{A^{-1}}$ (because $\mathbf{A^{-1}I} = \mathbf{A^{-1}}$)

$$\left[\begin{array}{c|c} A & I \end{array}\right] \rightarrow \left[\begin{array}{c|c} A^{-1}A & A^{-1}I \end{array}\right] \rightarrow \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

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here's more:

- Multiplying Matrices
- Determinant of a Matrix
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