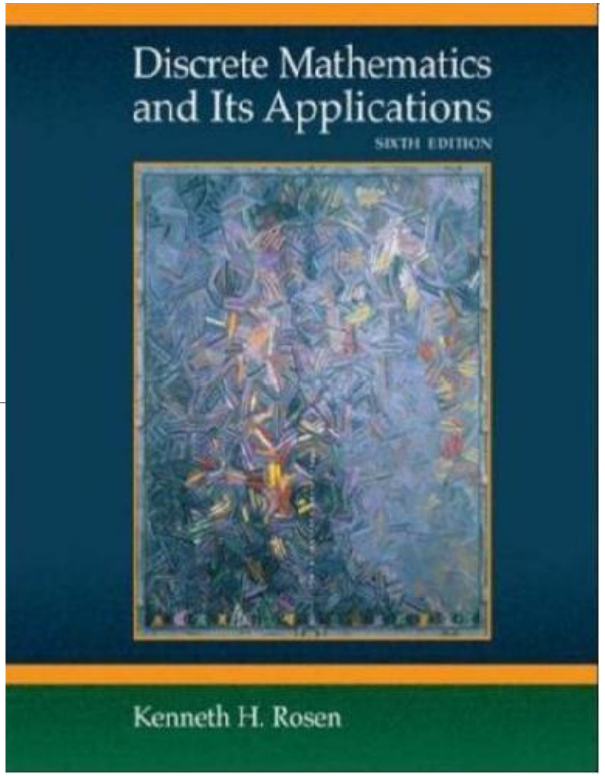
Discrete Mathematics



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SD 404

Algebraic Structure

- **Outline：**
- Introduction to Algebraic Structure
- Semigroup and Monoid
- Group and Subgroup
- Abelian Group, Cyclic Group and Permutation Group
- Ring and Field
- **Lattice**
- Boolean algebra

Review

- Algebraic system <A, ∘>
  or $<S, \triangle, *>$
- 4 properties
  - ☐ Closure
  - ☐ Commutativity
  - ☐ Associativity
  - ☐ Distributivity
- ✓ 3 constants
  - ☐ Identity
  - ☐ Zero
  - ☐ Inverse

- ✓ 9 special algebraic systems
  - ☐ Semigroup
  - ☐ Monoid
  - ☐ Group
  - ☐ Abelian Group, Cyclic Group, Permutation Group
  - ☐ Coset
  - ☐ Ring and Field
- ✓ 2 relations
  - ☐ Homomorphism
  - ☐ Isomorphism

Lattice

- A partially ordered set in which every pair of elements has both a **least upper bound** and a **greatest lower bound** is called a **lattice**.

- Let $(L, \preccurlyeq)$ be a lattice. We denote lub($\{a, b\}$) by $a \vee b$ and call it the join of a and b. Similarly, we denote glb($\{a, b\}$) by $a \wedge b$ and call it the meet of a and b. Then $<L, \vee, \wedge>$ is the corresponding algebraic system of $(L, \preccurlyeq)$.

Hasse Diagram

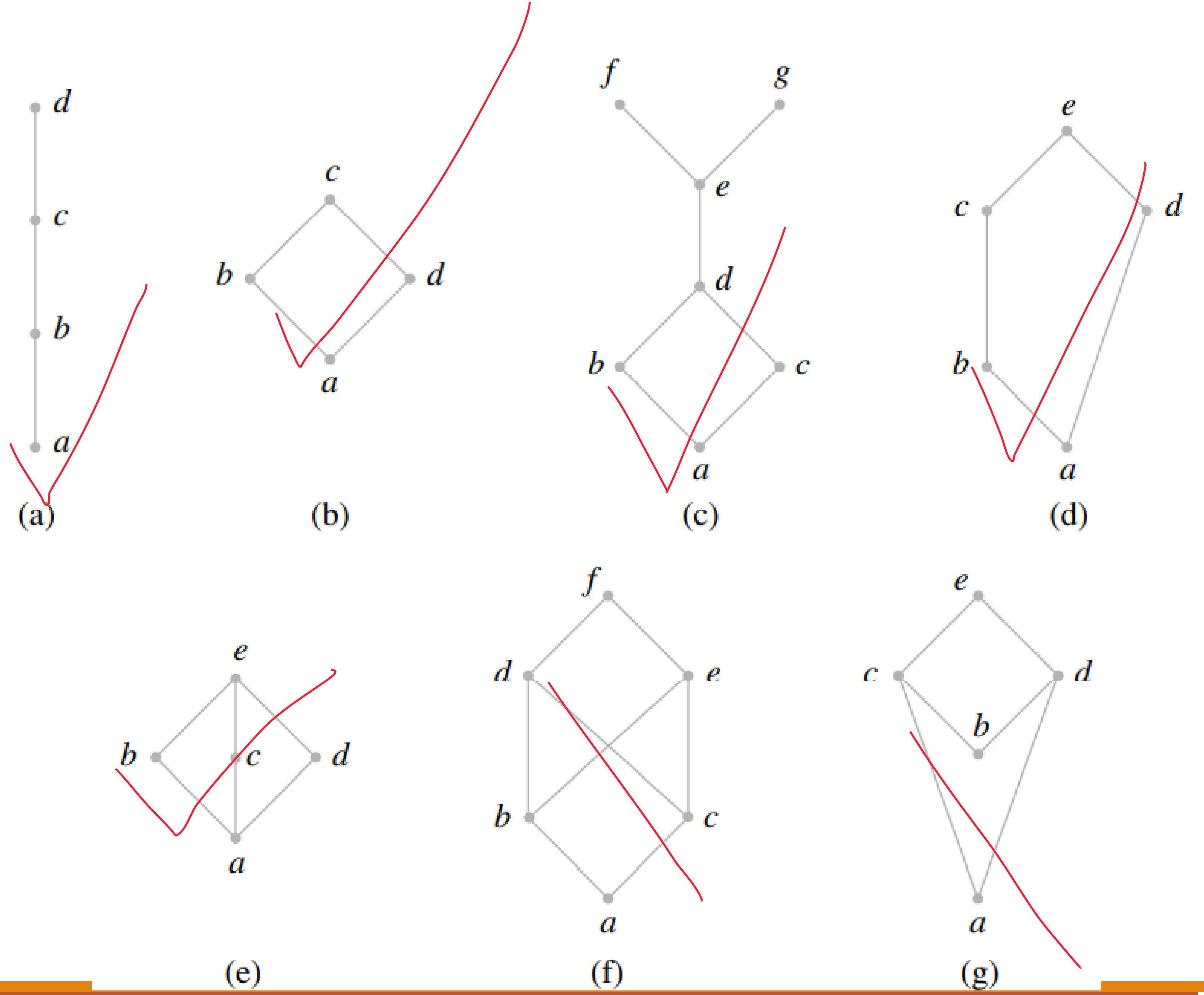
A **Hasse diagram** is a graphical rendering of a partially ordered set displayed via the cover relation of the partially ordered set with an implied upward orientation. A point is drawn for each element of the poset, and line segments are drawn between these points according to the following two rules:

- 1. If $x \preccurlyeq y$ in the poset, then the point corresponding to x appears lower in the drawing than the point corresponding to y.

- 2. The line segment between the points corresponding to any two elements x and y of the poset is included in the drawing iff x covers y or y covers x.

Example 1

- Which of the following Hasse diagrams represent lattices?



Example 2

- Let S={a,b}, draw the Hasse diagram of lattice <P(S), ⊆> and the operation tables of ∨ and ∧.

$\vee$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$

$\wedge$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{a,b\}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$

Sublattice

Definition:

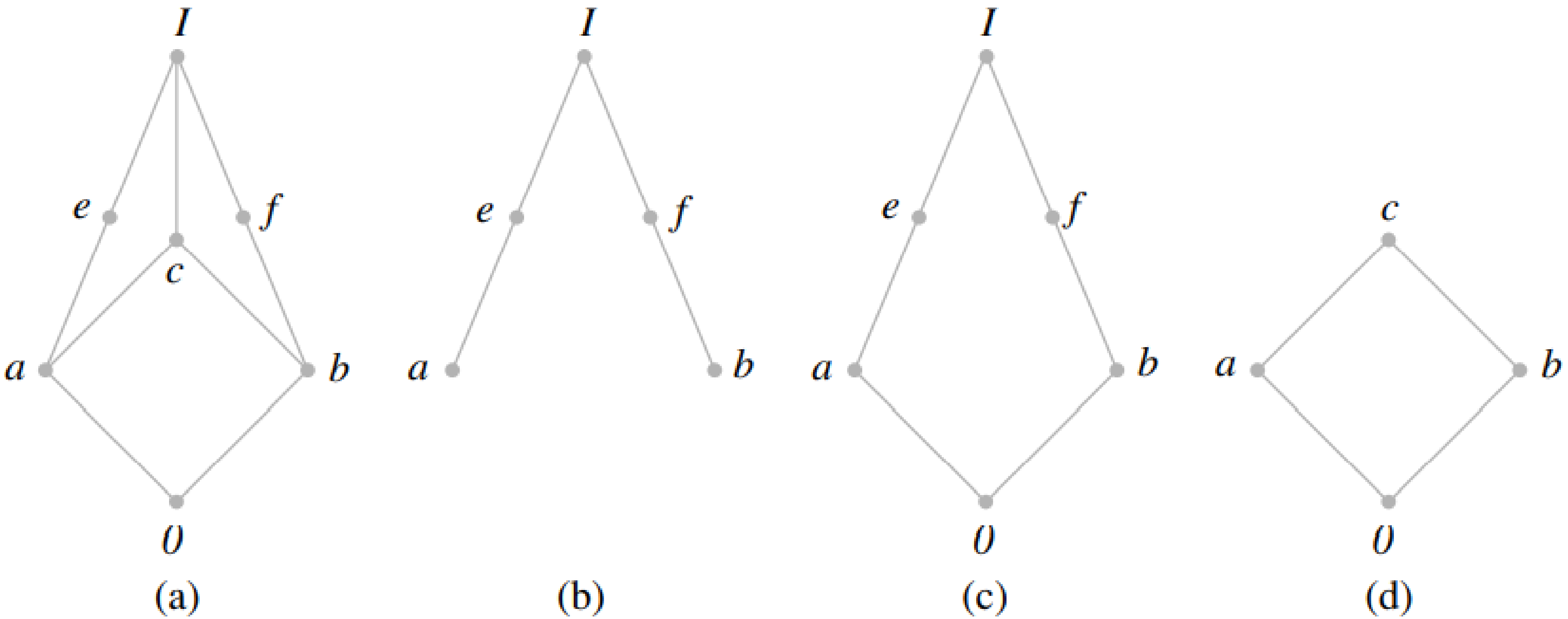
● Let $(L, \preccurlyeq)$ be a lattice. A nonempty subset S of L is called a **sublattice** of L <u>if $a \vee b \in S$ and $a \wedge b \in S$</u> whenever $a \in S$ and $b \in S$.

Example:

● Let E^+ be the set of all positive even integers, then $(E^+, |)$ is a sublattice of $(\mathbf{Z}^+, |)$.

Example 3

- Consider the lattice $(L, \preccurlyeq)$ shown in Figure (a). Which one is its sublattice?



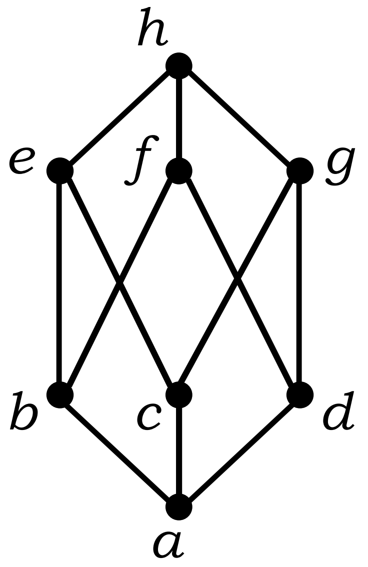
Example 4

- Let $(L, \preccurlyeq)$ be a lattice shown in the figure, $L=\{a, b, c, d, e, f, g, h\}$.

✓ Let $L_1=\{h, e, c, g\}$

✓ Let $L_2=\{a, b, f, d\}$

✓ Let $L_3=\{a, b, d, e, f, g, h\}$



- Let $(L, \preccurlyeq)$ be a lattice, S be a nonempty subset of L. Then $(S, \preccurlyeq)$ must be a poset, but not necessarily a lattice.
- Even if $(S, \preccurlyeq)$ is lattice, it is not necessarily a sublattice of $(L, \preccurlyeq)$

Theorems of Lattice (1)

- Let $(L, \preccurlyeq)$ be a lattice. $<L, \vee, \wedge>$ is the corresponding algebraic system of $(L, \preccurlyeq)$. For $\forall a, b \in L$,

✓ $a \preccurlyeq a \vee b$, $b \preccurlyeq a \vee b$.

✓ $a \wedge b \preccurlyeq b$, $a \wedge b \preccurlyeq a$.

✓ $a \vee b = b$ if and only if $a \preccurlyeq b$.

✓ $a \wedge b = a$ if and only if $a \preccurlyeq b$.

✓ $a \wedge b = a$ if and only if $a \vee b = b$.

Cont.

- $a \vee b = b$ if and only if $a \preccurlyeq b$.
- $a \wedge b = a$ if and only if $a \preccurlyeq b$.
- $a \wedge b = a$ if and only if $a \vee b = b$.
- Proof:

Suppose that $a \vee b = b$. Since a $\preccurlyeq a \vee b = b$, we have $a \preccurlyeq b$.

Conversely, if $a \preccurlyeq b$, then, since $b \preccurlyeq b$, b is an upper bound of a and b;

so by definition of least upper bound we have $a \vee b \preccurlyeq b$. Since $a \vee b$ is an upper bound, $b \preccurlyeq a \vee b$, so $a \vee b = b$.

Theorems of Lattice (2)

Let $(L, \preccurlyeq)$ be a lattice. $<L, \vee, \wedge>$ is the corresponding algebraic system of $(L, \preccurlyeq)$. For $\forall a, b, c, d \in L$,

- 1. If $a \preccurlyeq b$, then

(a) $a \vee c \preccurlyeq b \vee c$. (b) $a \wedge c \preccurlyeq b \wedge c$.

- 2. $a \preccurlyeq c$ and $b \preccurlyeq c$ if and only if $a \vee b \preccurlyeq c$.
- 3. $c \preccurlyeq a$ and $c \preccurlyeq b$ if and only if $c \preccurlyeq a \wedge b$.
- 4. If $a \preccurlyeq b$ and $c \preccurlyeq d$, then

(a) $a \vee c \preccurlyeq b \vee d$. (b) $a \wedge c \preccurlyeq b \wedge d$.

Cont.

- 4. If $a \preccurlyeq b$ and $c \preccurlyeq d$, then

(a) $a \vee c \preccurlyeq b \vee d$. (b) $a \wedge c \preccurlyeq b \wedge d$.

- Proof:

$b \preccurlyeq b \vee d$, $a \preccurlyeq b$, so $a \preccurlyeq b \vee d$.

$d \preccurlyeq b \vee d$, $c \preccurlyeq d$, so $c \preccurlyeq b \vee d$.

So $b \vee d$ is an upper bound of a and c.

By the definition of lub, we have $a \vee c \preccurlyeq b \vee d$.

Cont.

- 1. If $a \preccurlyeq b$, then

(a) $a \vee c \preccurlyeq b \vee c$.   (b) $a \wedge c \preccurlyeq b \wedge c$.

- 4. If $a \preccurlyeq b$ and $c \preccurlyeq d$, then

(a) $a \vee c \preccurlyeq b \vee d$.   (b) $a \wedge c \preccurlyeq b \wedge d$.

- Proof:

Replace d in 4(a)(b) with c.

Theorems of Lattice (3)

Let $(L, \preccurlyeq)$ be a lattice. $<L, \vee, \wedge>$ is the corresponding algebraic system of $(L, \preccurlyeq)$. For $\forall a, b, c \in L$,

- **1. Idempotent Properties:** (a) $a \vee a = a$ (b) $a \wedge a = a$
- **2. Commutative Properties:** (a) $a \vee b = b \vee a$ (b) $a \wedge b = b \wedge a$
- **3. Associative Properties:**

(a) $a \vee (b \vee c) = (a \vee b) \vee c$ (b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

- **4. Absorption Properties:**

(a) $a \vee (a \wedge b) = a$ (b) $a \wedge (a \vee b) = a$

Cont.

- **3. Associative Properties**

(a) $a \vee (b \vee c) = (a \vee b) \vee c$ (b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

- Proof:

From the definition of lub, we have $a \preccurlyeq a \vee (b \vee c)$ and $b \vee c \preccurlyeq a \vee (b \vee c)$.

Moreover, $b \preccurlyeq b \vee c$ and $c \preccurlyeq b \vee c$, so, by transitivity, $b \preccurlyeq a \vee (b \vee c)$ and $c \preccurlyeq a \vee (b \vee c)$.

Thus $a \vee (b \vee c)$ is an upper bound of a and b, so $a \vee b \preccurlyeq a \vee (b \vee c)$

Since $a \vee (b \vee c)$ is an upper bound of $a \vee b$ and c, we obtain $(a \vee b) \vee c \preccurlyeq a \vee (b \vee c)$.

Similarly, $a \vee (b \vee c) \preccurlyeq (a \vee b) \vee c$. By the antisymmetry of $\preccurlyeq$, $a \vee (b \vee c) = (a \vee b) \vee c$.

Cont.

- **4. Absorption Properties**

(a) $a \vee (a \wedge b) = a$ (b) $a \wedge (a \vee b) = a$

- Proof:

Since $a \wedge b \preccurlyeq a$ and $a \preccurlyeq a$, we see that a is an upper bound of $a \wedge b$ and a.

So $a \vee (a \wedge b) \preccurlyeq a$.

By the definition of lub, we have $a \preccurlyeq a \vee (a \wedge b)$.

So $a \vee (a \wedge b) = a$.

Example 5

Let $<A, \vee, \wedge>$ be an algebraic system. $\vee$ and $\wedge$ are binary operations with absorption properties. Show that $\vee$ and $\wedge$ have idempotent properties.

- Proof:

By the definition of absorption property, for $\forall a, b \in A$,

$a \vee (a \wedge b) = a \quad (1),$

$a \wedge (a \vee b) = a \quad (2).$

Replace b in (1) with $a \vee b$, we have $a \vee (a \wedge (a \vee b)) = a$.

According to (2) $a \vee (a \wedge (a \vee b)) = a \vee a = a$.

Similarly, $a \wedge a = a$.

Exercise 1

- Let $(L, \preccurlyeq)$ be a lattice. For $\forall a, b, c \in L$, show that

$a \vee (b \wedge c) \preccurlyeq (a \vee b) \wedge (a \vee c)$.

$(a \wedge b) \vee (a \wedge c) \preccurlyeq a \wedge (b \vee c)$.

Isomorphism of Lattices

- Let $(L_1, \preccurlyeq_1)$ and $(L_2, \preccurlyeq_2)$ be two lattices, the corresponding algebraic systems are $< L_1, \vee_1, \wedge_1 >$ and $< L_2, \vee_2, \wedge_2 >$ respectively. If there is a bijection $f: L_1 \to L_2$, such that for $\forall a, b \in L_1$,

$f(a \vee_1 b) = f(a) \vee_2 f(b)$

$f(a \wedge_1 b) = f(a) \wedge_2 f(b)$,

then we say f is a isomorphism from $< L_1, \vee_1, \wedge_1 >$ to $< L_2, \vee_2, \wedge_2 >$.

$(L_1, \preccurlyeq_1)$ and $(L_2, \preccurlyeq_2)$ isomorphic lattices.

Example 6

- Let E^+ be the set of positive even integers, show that $(\mathbf{Z}^+, \leq)$ and $(E^+, \leq)$ are isomorphic lattices.

Exercise 2

- Let A = {1, 2, 3, 6}, S={a, b}, show that (A, |) and (P(S), ⊆) are isomorphic lattice.

Define f : A → P(S) as:

f(1) = ∅, f(2) = {a}, f(3) = {b}, f(6) = {a, b}.

then it is easily seen that f is a one-to-one correspondence.