# Answers for Homework 1

## Textbook 2.

Both (a) and (b) are monoid and **identity** of (a) is *b* and of (b) is *a*. *Association* and *enclosure* should be proved.

## Textbook 8.

It is a *commutative* monoid with **identity** 1.

## Textbook 14.

It is a *commutative* monoid with **identity** 2.

## Textbook 16.

It is a *commutative* semigroup with no **identity**.

## Let $S_k=\{x|x\in\mathbb{Z}, x\geq k\}$, $k\geq 0$, show that $<S_k, +>$ is a semigroup.

Clearly $S_k$ is **not empty** since there are always some integers larger than a given integer $k$.

Moreover, + is a **valid binary operation** on $S_k$ (**closed** & **unique image entry**; prove by yourself).

Finally, since addition is associative for all integers, for any elements $a$, $b$

and $c$ in $S_k$, $(a+b)+c = a+(b+c)$ and operation + is *associative* on $S_k$.

## Show that <P(S), ⊕> is a monoid. (A⊕B=(A∪B)—(A∩B)).

Prove that <P(**S**), ⊕> is a semigroup by yourself (the same steps as exercise above).

For any subset **A** of ***S***, A ⊕ **Ø**= (**A** ∪ **Ø**) – (**A**∩**Ø**) = **A** and **Ø** ⊕ **A** = (**Ø** ∪ **A**) – (**Ø**∩**A**). Hence, **Ø** is the **identity** of P(**S**) and <P(**S**), ⊕> is a monoid.

## Let *f*: ℝ→ℝ, *f*(x)=5ˣ, show that *f* is a homomorphism from <ℝ, +> to <ℝ, ·>.

For any real numbers $a$ and $b$, $f(a+b) = 5^{a+b} = 5^a \cdot 5^b = f(a) \cdot f(b)$. Thus, $f$ is a homomorphism from <ℝ, +> to <ℝ, ·>.

## Let H={*x*|*x*=*dn*}, where *d* is a non-zero certain integer, *n*∈ℤ. Show that <ℤ, +> and <ℍ, +> are isomorphic.

Let $f$ be a function from ℤ to ℍ, which is defined as $f(n) = dn$.

For any element $x$ in ℍ, there must be some integer $n$ in ℤ such that $x = dn$ and therefore, $f$ is onto.

For any integers $x \neq y$, $f(x) - f(y) = d(x\text{-}y) \neq 0$ because neither $d$ nor $x\text{-}y$

equals to 0, suggesting that $f(x) \neq f(y)$ and $f$ is injective.

For any integers $x$ and $y$, $f(x+y) = d(x+y) = dx + dy = f(x) + f(y)$. So, $f$ is a homomorphism from $<\mathbb{Z}, +>$ to $<\mathbb{H}, +>$.

In conclusion, $<\mathbb{Z}, +>$ and $<\mathbb{H}, +>$ are isomorphic.