Answer for Homework 3

P426, 10:

$G = \mathbb{Z}_4 = \{[0], [1], [2], [3]\}$ and obviously H is a nonempty subset of G.

$$[0]H = \{[0], [2]\},$$

$$[1]H = \{[1], [3]\},$$

$$[2]H = \{[2], [0]\}, \text{ and}$$

$$[3]H = \{[3], [1]\}.$$

P431, 6:

First prove that <S, +> is an **Abelian** by showing the **closure**, *association*, **identity** (0), **inverse** ($-a-b\sqrt{5}$) and *commutation*.

Then, show that <S-{0}, ∗> is a *commutative* **monoid** with **closure**, *association* and **identity** (1) and *commutation*.

Finally, ∗ is distributive on + and therefore <S, +, ∗> is a *commutative* **ring** with **identity** (1).

1.

(1) $\{p_1, p_4, p_5\}$, $\{p_2, p_3, p_6\}$, $\{p_3, p_6, p_2\}$, $\{p_4, p_5, p_1\}$, $\{p_5, p_1, p_4\}$, $\{p_6, p_2, p_3\}$.

(2) $\{p_1\}$, $\{p_2\}$, $\{p_3\}$, $\{p_4\}$, $\{p_5\}$, $\{p_6\}$.

2.

(1) It is a **ring** but not a **field** since there is no **identity** in <**A**-{0}, ·>.

(2) It is not a **ring** since + in not **closed** on **A**.

(3) It is a **field**. Obviously <**A**, +> is an **Abelian**, and to show that <**A**-{0}, ·> is an Abelian, the following properties need to be proved.

Since **A** is a subset of $\mathbb{R}$, the *commutative law* and *associative law* are satisfied on the normal multiplication.

For any $x=a+b\sqrt{5}$ and $y=c+d\sqrt{5}$ in **A**-{0}, $x \cdot y = (a+b\sqrt{5})\cdot(c+d\sqrt{5}) = (ac+5bd) + (ad+bc)\sqrt{5}$.

Since 1 is in **A**-{0} and for any x in **A**-{0}, $x\cdot 1 = 1\cdot x = x$, 1 is the **identity** of **A**-{0}.

Since for any $x=a+b\sqrt{5}$ in **A-{0}**, $x\cdot\frac{1}{a+b\sqrt{5}} = \frac{a-b\sqrt{5}}{a^2-5b^2}\cdot x = 1 = x\cdot\frac{a-b\sqrt{5}}{a^2-5b^2}$, the **inverse** of x is $\frac{a}{a^2-5b^2} + \frac{b}{5b^2-a^2}\sqrt{5}$.

In conclusion, <**A**, +, ·> is a **field**.

3.

$(a+b)^2 = (a+b)*(a+b) = (a+b)*a + (a+b)*b = a*a + b*a + a*b + b*b = a^2 + a*b + b*a + b^2$.

4.

(1) $(a+a) = (a+a)*(a+a) = (a+a)*a + (a+a)*a = a*a + a*a + a*a + a*a =$ $a + a + a + a$. Add $(a^{-1}+a^{-1})$ to both sides and the result will be $a+a = e$.

(2) For any a and b in G, $(a+b) = (a+b)*(a+b) = a + a*b + b*a + b$. Therefore, $a*b = b*a$ and <A, +, $*$> is a *commutative* ring.