Exercise 1

1. Find out every subgroup of group $<\mathbf{Z}_6, +_6>$, then find out every left coset of each subgroup.

$Z_6 = \{[0], [1], [2], [3], [4], [5]\}$

identity: $[0]$, $[0]^{-1}=[0]$, $[1]^{-1}=[5]$, $[2]^{-1}=[4]$

$[3]^{-1}=[3]$.

① $\langle\{[0]\}, +_6\rangle$, ② $\langle Z_6, +_6\rangle$

③ $\langle\{[0],[3]\}, +_6\rangle$

④ $\langle\{[0],[2],[4]\}, +_6\rangle$

$\cancel{\langle\{[0],[1],[5]\}, +_6\rangle}$

①. $\{[0]\}$ $\{[1]\}$ $\{[2]\}$ $\{[3]\}$ $\{[4]\}$ $\{[5]\}$

②. $\langle Z_6, +_6\rangle$ ③. $\langle\{[0],[3]\}, +_6\rangle$, $\langle\{[1],[4]\}, +_6\rangle$ $\langle\{[2],[5]\}, +_6\rangle$

④. $\langle\{0,2,4\}, +_6\rangle$, $\langle\{1,3,5\}, +_6\rangle$

2. Let $<H, *>$ be a subgroup of $<G, *>$, if $A=\{x|x\in G, x*H*x^{-1}=H\}$, prove that $<A, *>$ is a subgroup of $<G, *>$.

Hint:

Theorem 7

- A necessary and sufficient condition for a nonempty subset H of a group $<G, *>$ to be a subgroup is that for $\forall a, b \in H \rightarrow a * b^{-1} \in H$.

$\forall a,b \in A$ $\quad a*b^{-1} \in A$

$(a*b^{-1})*H*(a*b^{-1})^{-1} = H$

$= (a*b^{-1})*H*(b*a^{-1})$

$= a*(b^{-1}*H*b)*a^{-1}$

$= a*H*a^{-1} = H$

$\boxed{a*H*a^{-1}=H}$

$b*H*b^{-1}=H$.

$(b^{-1}*(b)*H*(b^{-1})*b) = b^{-1}*H*b$

$H = b^{-1}*H*b$

3. Let $<G, *>$ be a group, R is a relation on G such that

$R=\{(a, b) \mid \text{there exists } x\in G \text{ such that } b=x * a * x^{-1}\}$

Show that R is an equivalence relation on G.

(R). $\forall a\in G,\ (a,a)\in R.$ $\quad a = x*a*x^{-1},\ x=e.$

(S). $\forall (a,b)\in R,\ \boxed{(b,a)\in R}$

$b = x*a*x^{-1}$ $\qquad a = y*b*y^{-1}$

$a = x^{-1}*b*x$

$a = x^{-1}*b*(x^{-1})^{-1}$ $\qquad \boxed{(a,c)\in R}$ $\quad c = y*(x*a*x^{-1})*y^{-1}$

$= (y*x)*a*(x^{-1}*y^{-1})$

(T). $(a,b)\in R,\ (b,c)\in R,\ b=x*a*x^{-1},\ c=y*b*y^{-1} = (y*x)*a*(y*x)^{-1}.$

4. Let $<H, *>$ be a subgroup of $<G, *>$, show that among all cosets of H,

there is only one coset A such that $<A, *>$ is a subgroup of $<G, *>$.

H is $<G, *>$ subgroup.

(Assume there is another coset. aH is also a subgroup of $<G,*>$)

$aH = H \Leftarrow aH\subseteq H,\ H\subseteq aH$

$\forall a*h\in aH,\ a*h\in H.$

$a*h_1=e,\ a=h_1^{-1}\in H.$

$\forall h\in H,\ h\in aH.$

$h = (a*a^{-1})*h = a*(a^{-1}*h) = a*(h_1*h)\in aH.$