Problem Types and Reductions

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SCC 120: Fundamentals of Computer Science

Outline

Problems

Decision problems

Versus optimization problems

- Traveling salesman (TSP): Find shortest tour
  - *Optimization* problem
  - Answer: a tour of length k such that there exists no tour of length less than k
- Traveling salesman decision (TSD): Is there a tour of length less than k?
  - *Decision* problem
  - The answer is either a *yes* or *no*

Decision Problem or Not?

- Sorting an array of integers
- Searching for an integer in an array
- Find minimum integer in an array
- Checking if an array is sorted
- Hamiltonian cycle
- Subset sum
- Propositional satisfiability
- Propositional entailment

Using TSD to Solve TSP

- Can we solve TSP by solving TSD?

Using TSD to Solve TSP

- Suppose for some instance of TSP, the shortest tour is 50
  - Is there a tour of length less than 0? No!
  - Is there a tour of length less than 1? No! And so on, until
  - Is there a tour of length at most 50? Yes!
- Eventually you will find the length of the shortest tour
- However, something is missing …

Using TSD cannot solve TSP

- You won't get the tour (the route), which is what we are interested in

Using TSP to Solve TSD

- Suppose for some instance of TSP, the shortest tour is 50
- We ask Is there a tour of length less than 42?
- Can we apply TSP to answer this?

Using TSP to Solve TSD

- Solve TSP, which tells you the shortest tour is 50.
  - Therefore, no tour of length less than 42!
- TSP can be used to solve TSD

Kinds of Problems

- Decision
  - Yes or no
- Optimization
  - least cost, minimum, maximum, shortest …
- Witness, a variant of decision
  - If yes, provide *witness* (also *certificate* or *proof*)
  - Traveling Salesman Witness: Is there a tour of length less than k? If yes, give the tour as well.
- Function: broad general category
  - Map input to output

Relations Among Problems

- Witness at least as hard as decision
  - TSW at least as hard as decision
- Optimization at least as hard as witness
  - TSP at least as hard as TSW
- By transitivity, optimization at least as hard as decision
- Why study decision problems?
  - Many interesting intractable ones (and undecidable ones!)
  - Simpler than optimization and witness
  - A separation of concerns
    - Understand one class of problems, say decision
    - Understand its relationships with other classes

Reduction from Problem P to Problem P'

- *Rephrasing* of P into P' such that the solution to P' provides the solution to P
- *Rephrasing* is an algorithm

```
// ρ is an instance of P, σ is a solution of ρ
solve(ρ) {

    //ρ' is an instance of P'
    ρ' = transformPToP'(ρ)

    //σ' is a solution of ρ'
    σ' = solveP'(ρ')

    σ = transformSolP'ToSolP(σ')

    return σ
}
```

Example: TSD to TSP

```
//Returns true iff there is a tour of boolean length less than k
solveTSD(Graph g, int k) {

    //k' is length of shortest tour t
    <k',t> = solveTSP(g)

    if(k'< k)
        return true
    else
        return false
}
```

Why Reduce from P to P'?

- To take advantage of solved problems instead of building things from scratch
- To show that solving P is no harder than solving P', that is, P is no more complex than P'

Polynomial-Time Reductions

```
// ρ is an instance of P, σ is a solution of ρ
solve(ρ) {

    //ρ' is an instance of P'
    ρ' = transformPToP'(ρ)

    //σ' is a solution of ρ'
    σ' = solveP'(ρ')

    σ = transformSolP'ToSolP(σ')

    return σ
}
```

- If the *transformPToP'* and *transformSolP'ToSolP* steps can be done in polynomial time
  - If ρ' can be solved in polynomial time, this means that ρ can be solved in polynomial time!