M202204W Discrete Mathematics
Term 2 Coursework (I)
Deadline for submission: **8th May 2023, at 10:10**

Please print the document **single-sided on A3** paper, **hand write** it and submit on time.

This coursework is worth **20%** of the final mark.

Please answer all questions. You must do this coursework completely by yourself. In this instance, it means you are not even allowed to discuss the coursework questions with other students. Your answer sheet will be checked for plagiarism.

You are allowed to ask the instructor about these questions. But instructor can only clarify the questions or give you suggestions if you get stuck, and instructor will try not to tell you whether you have the correct answer (to be fair to all students).

Name: ____________________

ID: ________________________

Total score: __________________

1. Answer the following questions. (1 mark for total)
   a) Let $A = \{1, 2, 4, 8\}$, and the binary operation $*$ on A is defined as: $a * b = min\,\{a, b\}$. $< A,\ *>$ is a monoid. In $< A,\ *>$, the identity element is ____ and the zero element is ____.
   b) Let $\mathbb{Z}$ be the set of integers，if $\forall a, b \in \mathbb{Z}, a * b = a + b + 5$，the inverse of a will be____.
   c) Define the operations $\cup$ and $\cap$ on negative integer set $\mathbb{N}^-$ as follows: $a \cup b = \max\{a, b\}$, $a \cap b = \min\{a, b\}$. Then ____.
      i. $< \mathbb{N}^-,\ \cup>$ is a semigroup.
      ii. $< \mathbb{N}^-,\ \cap>$ is not a semigroup.
      iii. There is no identity element in $< \mathbb{N}^-,\ \cup>$.
      iv. There is no identity element in $< \mathbb{N}^-,\ \cap>$.
   d) Let the set $A = \{ a + b\sqrt[4]{5} \mid a, b \in \mathbb{Q}\}$, and the operations $+$ and $*$ be the addition and multiplication of real numbers, then ____.
      i. The given set and operations can form a ring.
      ii. The given set and operations can not form a field.

2. Answer the following questions. (1 mark for total)
   a) Suppose that H is a subgroup of finite group G, $|G|$ and $|H|$ represent the number of elements in the G and H respectively. Then $|G|$ must be divided exactly by $|H|$. (Y/N)__________
   b) If a group is finite, the order of the element must be exactly divisible by the order of the finite group. (Y/N)__________
   c) There can only be one generator in a cyclic group. (Y/N)__________
   d) If the order of the group, denoted by $|G|$, is a prime number, then G has non-trivial subgroup. (Y/N)___________

3. Let $*$ be a binary operation on a set A, and suppose that $*$ satisfies the following properties for any a, b and c in A :
   a) $a = a * a$;
   b) $a * b = b * a$;
   c) $a * (b * c) = (a * b) * c$.

   Define a relation $\boldsymbol{R}$ on A by $a\,\boldsymbol{R}\,b$ if and only if $a = a * b$.
   Show that $(A, \boldsymbol{R})$ is a poset, and for all a, b in A, the greatest lower bound $GLB(a, b) = a * b$. (2 marks)

4. Let G be a group. Prove: A sufficient and necessary condition for G to be an Abelian group is that $f(x) = x^{-1}$ is an isomorphism mapping of G. (2 marks)
[Hint: x^{-1} means the inverse element of x]

5. Let $< G, \ast >$ be a finite semigroup, and satisfies the elimination law. Prove that G is a group. Then if $\forall x \in G,\ x \ast x = e$, where e is an identity element, prove that $< G, \ast >$ is an Abelian group. (2 marks)
[Hint: the elimination law
$a \ast b = a \ast c \rightarrow b = c$
$a \ast c = b \ast c \rightarrow a = b$]

6. $< \mathbb{Z}_8, +_8 >$ is an addition module 8 group, and $\mathbb{Z}_8 = \{1, 2, 3, 4, 5, 6, 7, 0\}$, $|\mathbb{Z}_8| = 8$. Find all subgroups of $< \mathbb{Z}_8, +_8 >$ and their corresponding left cosets. (1 mark)
[Hint: $< \mathbb{Z}_n, +_m >: \forall i, j \in \mathbb{Z}_n,\ i +_m j = (i + j)\ mod\ m$]

7. Prove that the $< \mathbb{Z}, + >$ is an infinite cyclic group where $\mathbb{Z}$ is the set of integers.
[Hint: $+$ represent ordinary addition] (2 marks)

8. Show that if G is a group of order n, $\forall a \in G$, $|a|$ is a factor of n, then $a^n = e$. (2 mark)
[Hint: $|a|$ represents the order of an element----the number of times that the element has to perform the group operation to finally get to the identity.]

9. Let $s = (1\,8\,2\,4\,6\,5\,7\,3)$, $t = (1\,3\,2)(5\,6\,4\,8)$. Please find the inverse of them and the result of st and ts. (1 mark)

10. Prove that $< A, \oplus, \otimes >$ is a communitive ring with identity element, where $\forall a, b \in A, a \oplus b = a + b - 1, a \otimes b = a + b - ab$. (1 mark)

11. Please prove that every homomorphism image of ring is also a ring. (1 mark)

12. Let $< L, \mid >$ be a poset, and $\mid$ be a divisible relation in L. Determine whether $< L, \mid >$ is a lattice when
$L = \{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}$
$L = \{1, 2, 3, 4, 5, 6, 8, 9, 10\}$
and draw the Hasse diagrams respectively. (1 mark)

13. Let $<$ L, $\preccurlyeq>$ be a totally ordered set, $|\mathrm{L}| \geq 3$. Prove that $<$ L, $\preccurlyeq>$ is a lattice, but it doesn't have a complement. (1 mark)
[Hint: If in a poset $<$ A, $\preccurlyeq>$, for any $\mathrm{a}, \mathrm{b} \in \mathrm{A}$, have $\mathrm{a} \preccurlyeq \mathrm{b}$ or $\mathrm{b} \preccurlyeq \mathrm{a}$, then A will be a totally ordered set]

14. Let $<$ L, $\vee$, $\wedge>$ be a lattice, $\preccurlyeq$ is the corresponding partial order, a, b, c, d are any member of L. Prove that $(\mathrm{a} \wedge \mathrm{b}) \vee (\mathrm{b} \wedge \mathrm{c}) \vee (\mathrm{c} \wedge \mathrm{a}) = (\mathrm{a} \vee \mathrm{b}) \wedge (\mathrm{b} \vee \mathrm{c}) \wedge (\mathrm{c} \vee \mathrm{a})$. (1 mark)

15. In bounded distributive lattices, prove that the elements with a complement will form a sub-lattice. (1 mark)