Discrete Mathematics

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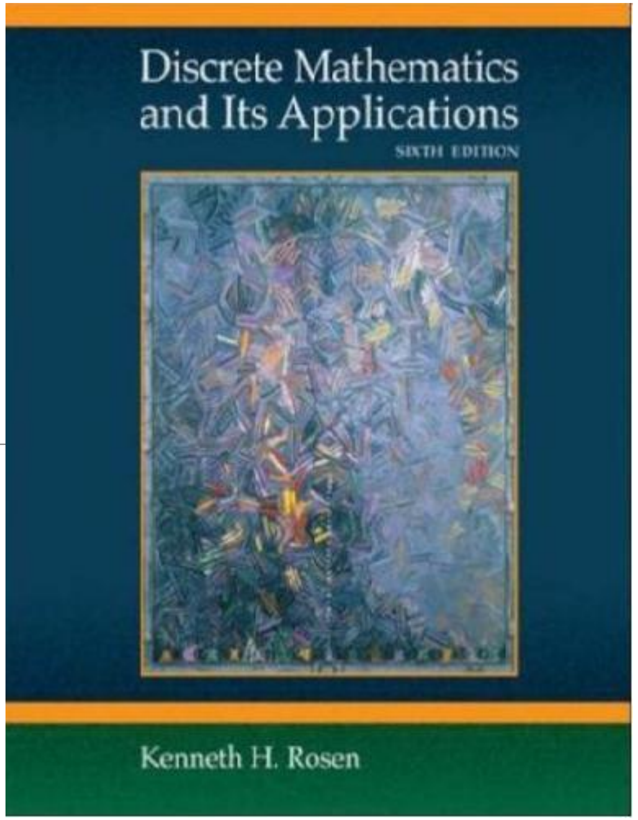


Algebraic Structure

- **Outline：**
- Introduction to Algebraic Structure
- Semigroup and Monoid
- Group and Subgroup
- Abelian Group, Cyclic Group and Permutation Group
- Ring and Field
- Lattice
- **Boolean Algebra**

Review

- Algebraic system <A, ∘> or $<S, \triangle, *>$
  - ✓4 properties
    - ☐Closure
    - ☐Commutativity
    - ☐Associativity
    - ☐Distributivity
  - ✓3 constants
    - ☐Identity
    - ☐Zero
    - ☐Inverse
  - ✓10 special algebraic systems
    - ☐Semigroup
    - ☐Monoid
    - ☐Group
    - ☐Abelian Group, Cyclic Group, Permutation Group
    - ☐Coset
    - ☐Ring and Field
    - ☐The corresponding algebraic system of Lattice
  - ✓2 relations
    - ☐Homomorphism
    - ☐Isomorphism

Boolean Algebra

- **Boolean Lattice**: A distributive complemented lattice is called a Boolean lattice.
- Every element in a Boolean lattice has a unique complement.
- Define a unary operation "$\bar{\ }$" where $\bar{a}$ denotes the complement of a.
- Let $(L, \preccurlyeq)$ be a Boolean lattice, the corresponding algebraic system $<L, \vee, \wedge, \bar{\ }>$ is called a **Boolean algebra**.
- A Boolean algebra with finite elements is called a **finite Boolean algebra**.
- Example: Boolean lattice $(P(S), \subseteq)$

  Boolean algebra $<P(S), \cup, \cap, \bar{\ }>$

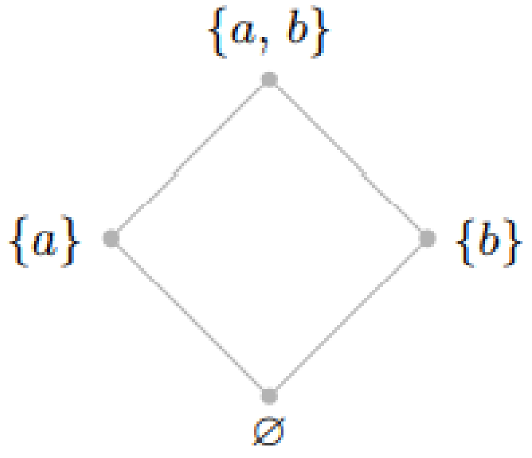
Example 1

- $S = \{a, b\}$, $<P(S), \cup, \cap, \overline{\ }>$

$\cup$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$

$\cap$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{a,b\}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$

	$\overline{\ \ }$
$\emptyset$	$\{a,b\}$
$\{a\}$	$\{b\}$
$\{b\}$	$\{a\}$
$\{a,b\}$	$\emptyset$



Theorem 1

- For any two elements *a*, *b* in a Boolean algebra
- $\overline{(\overline{a})} = a$
- $\overline{a \vee b} = \overline{a} \wedge \overline{b}$
- $\overline{a \wedge b} = \overline{a} \vee \overline{b}$

Proof:

Exercise 1

- Show that in a Boolean algebra, $b \wedge \overline{c}$ =0 if and only if $b \preccurlyeq c$.

Exercise 2

- Let <A, $\vee$, $\wedge$, $\bar{\ }$> be a Boolean algebra, + is a binary operation defined on A as

$a + b = (a \wedge \bar{b}) \vee (\bar{a} \wedge b)$

show that <A, +> is an abelian group.

Exercise 3

- Let <A, $\vee$, $\wedge$, $\bar{\ }$> be a Boolean algebra, + and · are two binary operations defined on A as

$$a + b = (a \wedge \bar{b}) \vee (\bar{a} \wedge b)$$

$$a \cdot b = a \wedge b$$

show that <A, +, ·> is a ring with identity I.

Isomorphism of Boolean Algebra

- Let $<A, \vee, \wedge, ^{-}>$ and $<B, \vee, \wedge, ^{-}>$ be two Boolean algebras, if there exists a bijection f: $A \to B$ such that for $\forall a, b \in A$,

$$f(a \vee b) = f(a) \vee f(b)$$

$$f(a \wedge b) = f(a) \wedge f(b)$$

$$f(\overline{a}) = \overline{f(a)}$$

then we say $<A, \vee, \wedge, ^{-}>$ and $<B, \vee, \wedge, ^{-}>$ are isomorphic.

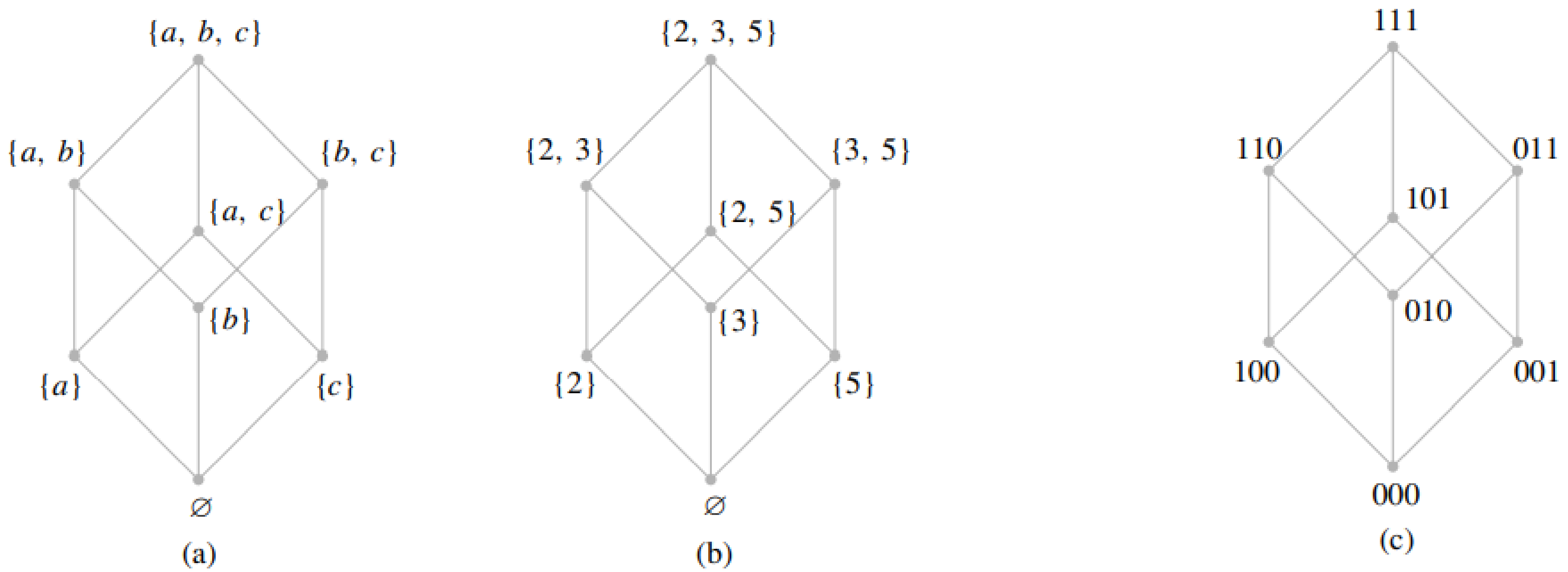
Theorem 2

- If $S_1 = \{x_1, x_2,\ldots,x_n\}$ and $S_2 = \{y_1, y_2,\ldots,y_n\}$ are any two finite sets with n elements, then the Boolean algebras $<P(S_1), \vee, \wedge, \bar{}>$ and $<P(S_2), \vee, \wedge, \bar{}>$ are isomorphic. Consequently, the Hasse diagrams of lattices $(P(S_1), \subseteq)$ and $(P(S_2), \subseteq)$ may be drawn identically.

- Note: $<P(S), \vee, \wedge, \bar{}>$ is completely determined by the number $|S|$ and does not depend in any way on the nature of the elements in S.

Example 2

- $S = \{a, b, c\}$ and $T = \{2, 3, 5\}$. $(P(S), \subseteq)$ and $(P(T), \subseteq)$.



Boolean Algebra

- If the Hasse diagram of the lattice corresponding to a set with n elements is labeled by sequences of 0's and 1's of length n, then the resulting lattice is named B_n. If $x = a_1a_2 \cdots a_n$ and $y = b_1b_2 \cdots b_n$ are two elements of B_n, then

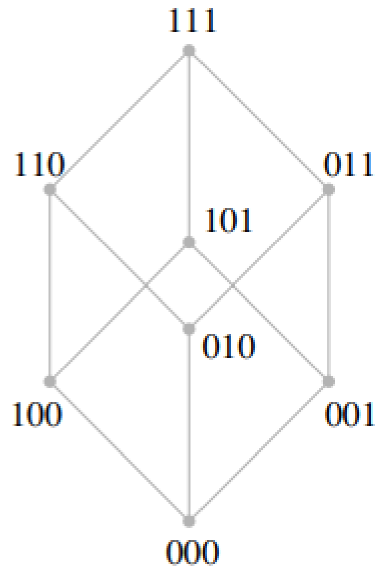
✓ $x \le y$ if and only if $a_k \le b_k$ for $k = 1, 2,..., n$.

✓ $x \wedge y = c_1c_2 \cdots c_n$, where $c_k = \min\{a_k, b_k\} = a_k \wedge b_k$

✓ $x \vee y = d_1d_2 \cdots d_n$, where $d_k = \max\{a_k, b_k\} = a_k \vee b_k$

✓ x has a complement $x' = z_1z_2 \cdots z_n$,

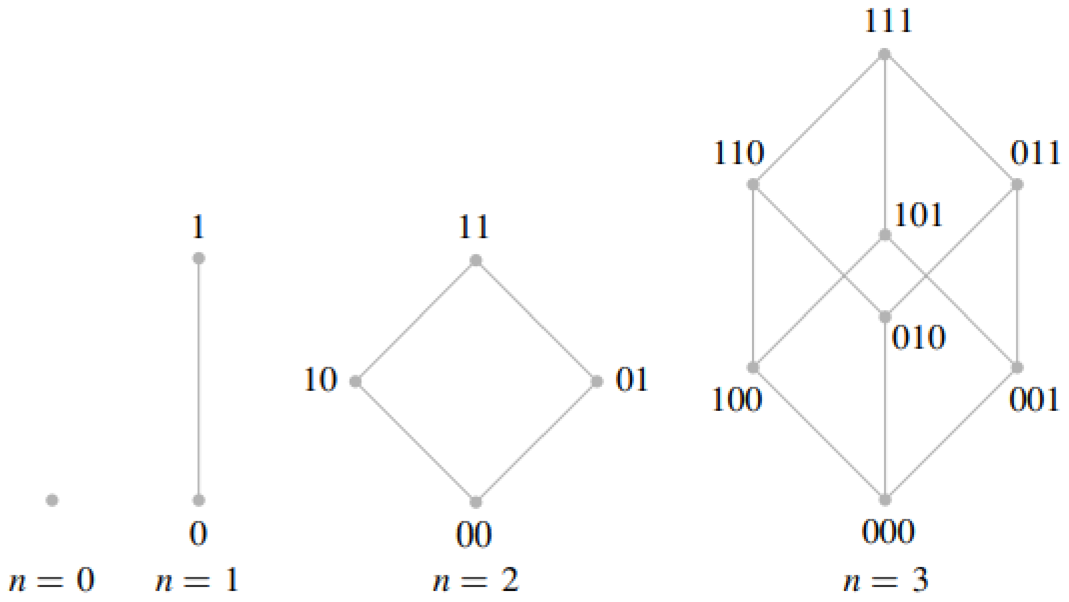
where $z_k = 1$ if $x_k = 0$, and $z_k = 0$ if $x_k = 1$.



(c)

Boolean Algebra

- A finite lattice is called a Boolean algebra if it is isomorphic with B_n for some nonnegative integer n.
- For every positive integer n, there must be a Boolean algebra with 2^n elements; For every Boolean algebra, the member of its elements must be positive integer power of 2.
- Hasse diagrams of the lattices B_n for n = 0, 1, 2, 3



Example 3

- Determine whether $(D_6, |)$, $(D_{20}, |)$ and $(D_{30}, |)$ is a Boolean algebra or not, respectively.

Homework 4

- Textbook p272: 24, 34
- Textbook p278-279: 1-10(判断即可), 12, 20
- 1. Show that in a lattice if $a \preccurlyeq b \preccurlyeq c$, then

(1) $a \vee b = b \wedge c$

(2) $(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c)$

- 2. A bounded lattice is shown in Figure 1, answer the following questions.

(1) Find the complements of a and f.

(2) Is the lattice a complemented lattice? Why?

- 3. Show that in a bounded lattice, the only complement of 0 is I.