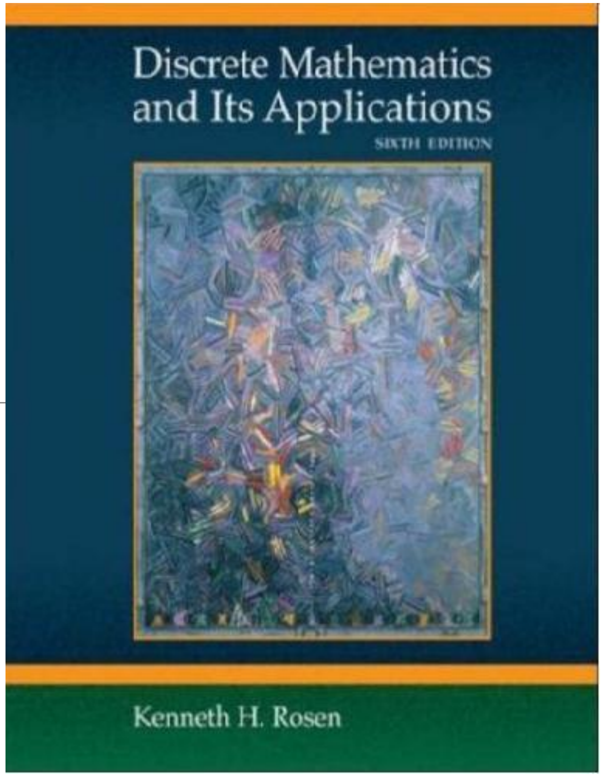
Discrete Mathematics



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Chapter 9: Relations

Outline:

Partial Ordering

- **Definition:**

A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

- The notation $a \preccurlyeq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R), that is $(S, \preccurlyeq)$.

- The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$.

Example 1

- Show that the “greater than or equal” relation (≥) is a partial ordering on the set of integers.

1. R : $\forall a (a \geq a)$ ✓
2. A : $\forall a \forall b (a \geq b \wedge b \geq a \rightarrow a = b)$
3. T : $\forall a \forall b \forall c (a \geq b \wedge b \geq c \rightarrow a \geq c)$.

≥ is a partial ordering on the set of integers and (**Z**, ≥) is a poset.

Example 2

- Show that the divisibility relation | is a partial ordering on the set of positive integers.

1. R: $\forall a (a \mid a)$
2. A: $\forall a \forall b (a \mid b \wedge b \mid a \rightarrow a = b)$.
3. T: $\forall a \forall b \forall c (a \mid b \wedge b \mid c \rightarrow a \mid c)$.

We see that ($\mathbf{Z}^+$, |) is a poset.

Example 3

- Show that the inclusion relation ⊆ is a partial ordering on the power set of a set *S*.

1. R: $\forall A (A \subseteq A)$ ✓

2. A: $\forall A \forall B (A \subseteq B \wedge B \subseteq A \to A = B)$.

3. T: $\forall A \forall B \forall C (A \subseteq B \wedge B \subseteq C \to A \subseteq C)$.

poset: $(P(S), \subseteq)$.

Exercise 9.6

- **1.** Which of these relations on {0, 1, 2, 3} are partial orderings?

Determine the properties of a partial ordering that the others lack.

a) {(0, 0), (1, 1), (2, 2), (3, 3)} ✓

b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)} ✗

c) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)} ✓

d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)} ✓

e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)} ✗

Exercise 9.6

- **5.** Which of these are posets?

a) (Z, =) ✓

b) (Z, ≠) ✗

c) (Z, ≥) ✓

d) (Z, ∤) ✗

- **6.** Which of these are posets?

a) (R, =) ✓

b) (R, <) ✗

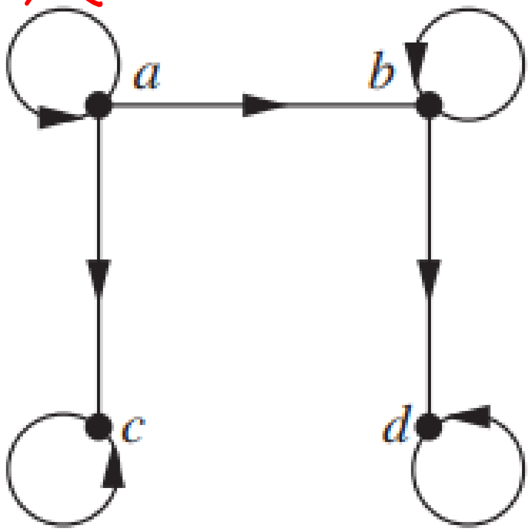
c) (R, ≤) ✓

d) (R, ≠) ✗

Exercise 9.6

- Determine whether the relation with the directed graph shown is a partial order.

9.



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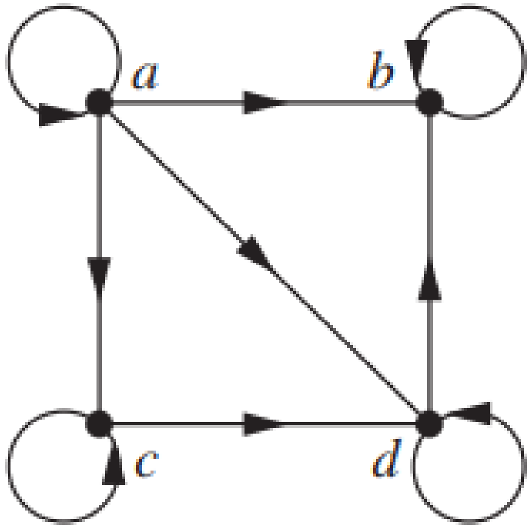


11.

Comparable & Incomparable

- **Definition:**
- The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.
- **Example:**

$3 \preccurlyeq 9$ $\quad (3,9) \in |$

In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Total Ordering

- **Definition:**

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain*.

- **Example:**

$\forall a \in S\ \forall b \in S\ (a \preccurlyeq b \vee b \preccurlyeq a)$

The poset $(\mathbf{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.

The poset $(\mathbf{Z}^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

Exercise 9.6

- **14.** Which of these pairs of elements are comparable in the poset (**Z**$^+$, |)?

a) 5, 15 ✓ **b)** 6, 9 ✗ **c)** 8, 16 ✓ **d)** 7, 7 ✓

- **15.** Find two incomparable elements in these posets.

a) (P({0, 1, 2}), ⊆)

{0}, {1}

b) ({1, 2, 4, 6, 8}, |)

4, 6

6, 8

Well Ordering

- **Definition:**

$(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that is a total ordering and every nonempty subset of S has a least element.

$\forall A \subseteq S\ \exists a \in A\ \forall b \in A\ (A \neq \phi \wedge a \preccurlyeq b)$

Exercise 9.6

- **54.** Determine whether each of these posets is well-ordered.

a) $(S, \leq)$, where $S = \{10, 11, 12, \ldots\}$ ✓

b) $(\mathbf{Q} \cap [0, 1], \leq)$ (the set of rational numbers between 0 and 1 inclusive)

$\{\frac{1}{n} \mid n \in Z^+\}$ ✗

c) $(S, \leq)$, where S is the set of positive rational numbers with denominators not exceeding 3 分母

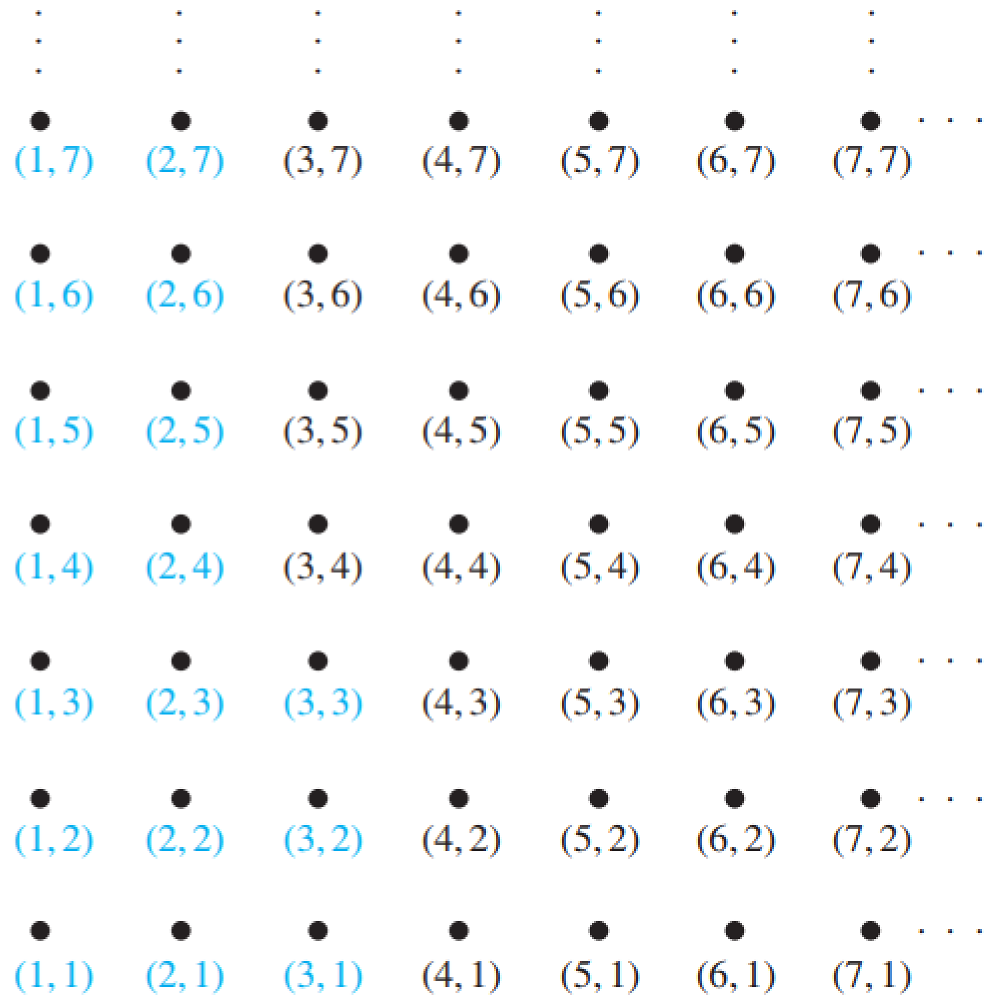
$\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots\ldots\}$ ✓

d) $(\mathbf{Z}^-, \geq)$, where $\mathbf{Z}^-$ is the set of negative integers ✓

Lexicographic Order

Example:

- In the poset $(\mathbf{Z} \times \mathbf{Z}, \preccurlyeq)$, $(3, 5) \prec (4, 8)$, $(3, 8) \prec (4, 5)$, and $(4, 9) \prec (4, 11)$.
- Ordered pairs in $\mathbf{Z}^+ \times \mathbf{Z}^+$ that are less than $(3, 4)$



Lexicographic Order (Cont.)

Example:

- $(1, 2, 3, 5) \prec (1, 2, 4, 3)$
- *discreet* $\prec$ *discrete*
- *discreet* $\prec$ *discreetness*
- *discrete* $\prec$ *discreti*
- *discrete* $\prec$ *discretion*

Exercise 9.6

- **16.** Let $S = \{1, 2, 3, 4\}$. With respect to the lexicographic order based on the usual "less than" relation,

a) find all pairs in $S \times S$ less than (2, 3).

(1,1)(1,2)(1,3)(1,4)(2,1)(2,2)

b) find all pairs in $S \times S$ greater than (3, 1).

(3,2)(3,3)(3,4)(4,1)(4,2)(4,3)(4,4).

- **17.** Find the lexicographic ordering of these n-tuples:

a) (1, 1, 2), $<$ (1, 2, 1) **b)** (0, 1, 2, 3), $<$ (0, 1, 3, 2)

c) (1, 0, 1, 0, 1), (0, 1, 1, 1, 0) (0,1,1,1,0) $<$ (1,0,1,0,1)

Exercise 9.6

- **18.** Find the lexicographic ordering of these strings of lowercase English letters:

a) *quack, quick, quicksilver, quicksand, quacking*

1 3 5 4 5

b) *open, opener, opera, operand, opened*

1 3 4 5 2

c) *zoo, zero, zoom, zoology, zoological*

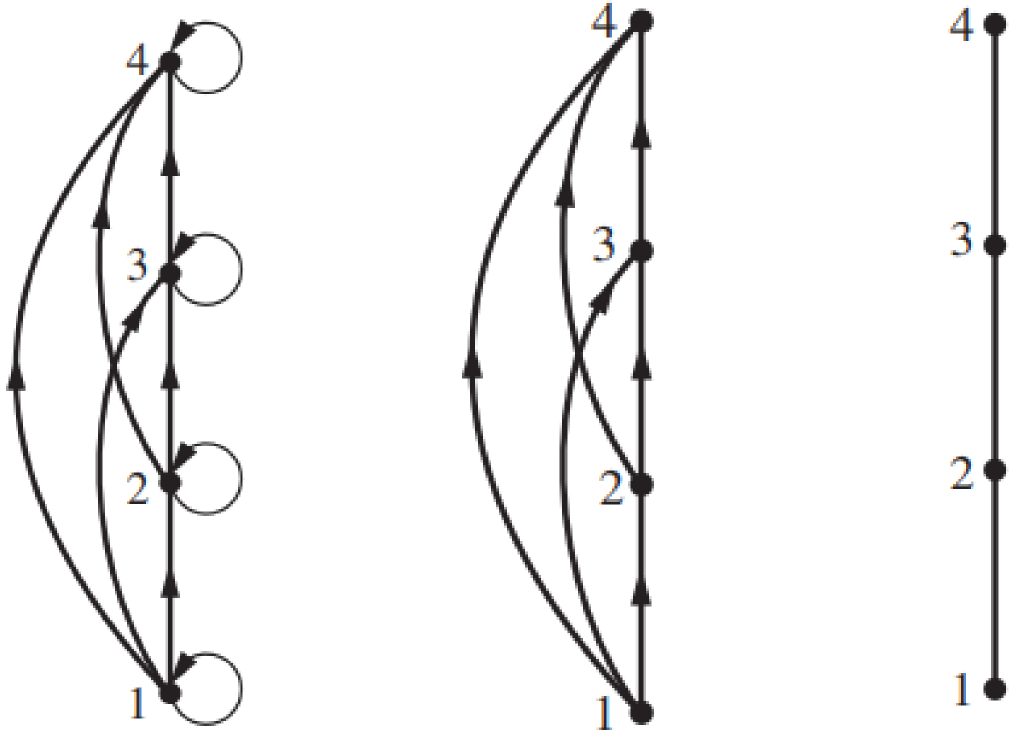
2 1 5 4 3

- **19.** Find the lexicographic ordering of the bit strings 0, 01, 11, 001, 010, 011, 0001, and 0101 based on the ordering $0 \prec 1$.

$0 < 0001 < 001 < 01 < 010 < 0101 < 011 < 11$

Hasse Diagrams

- The directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$.



(a) (b) (c)

Diagram contains sufficient information to find the partial ordering.

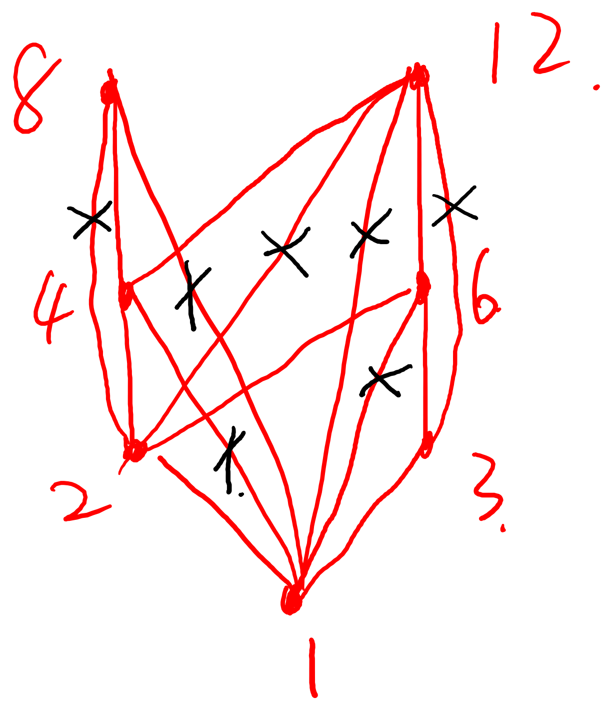
Steps for Drawing Hasse Diagrams

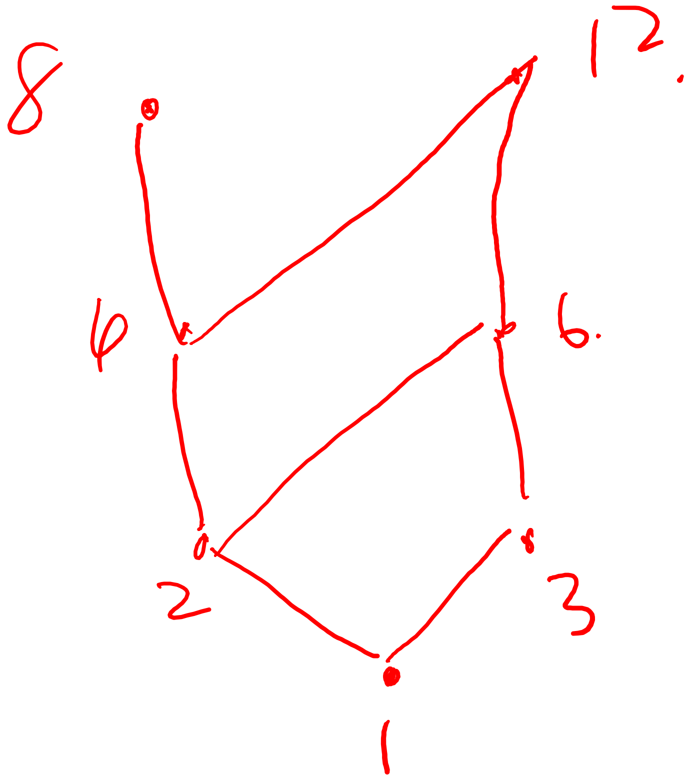
$(S, \preccurlyeq)$

- ✓ Start with the directed graph for this relation.
- ✓ Remove all the loops.
- ✓ Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$.
- ✓ Arrange each edge so that its initial vertex is below its terminal vertex.
- ✓ Remove all the arrows .

Example 5

- Draw the Hasse diagram representing the partial ordering $\{(a, b) | a$ divides $b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.



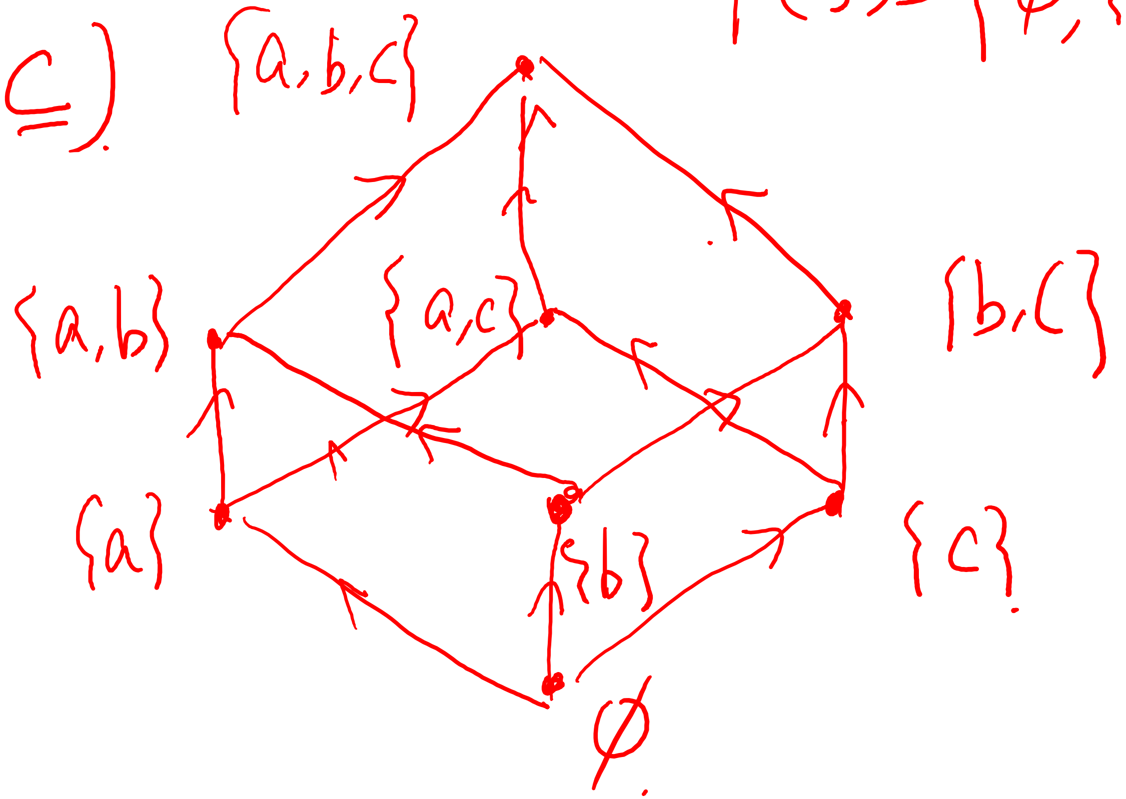


Example 6

- Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$ where $S = \{a, b, c\}$.

$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

$(P(S), \subseteq)$



Exercise 9.6

- **21.** Draw the Hasse diagram for the "less than or equal to" relation on {0, 2, 5, 10, 11, 15}.

- **20.** Draw the Hasse diagram for the "greater than or equal to" relation on {0, 1, 2, 3, 4, 5}.

Exercise 9.6

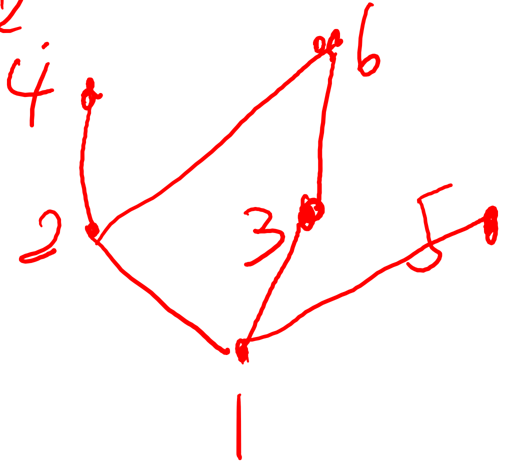
"|"

- **22.** Draw the Hasse diagram for divisibility on the set

a) {1, 2, 3, 4, 5, 6}. **b)** {3, 5, 7, 11, 13, 16, 17}.

c) {2, 3, 5, 10, 11, 15, 25}. **d)** {1, 3, 9, 27, 81, 243}.

a)



c).

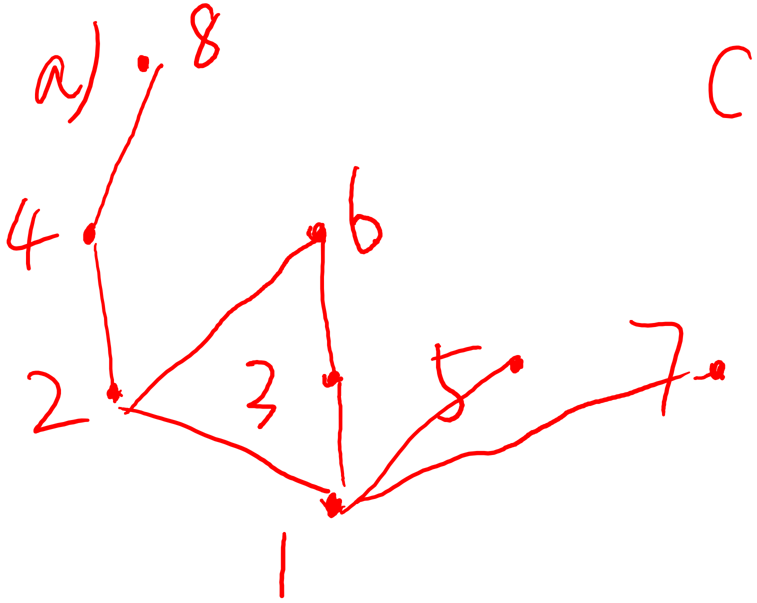
d).

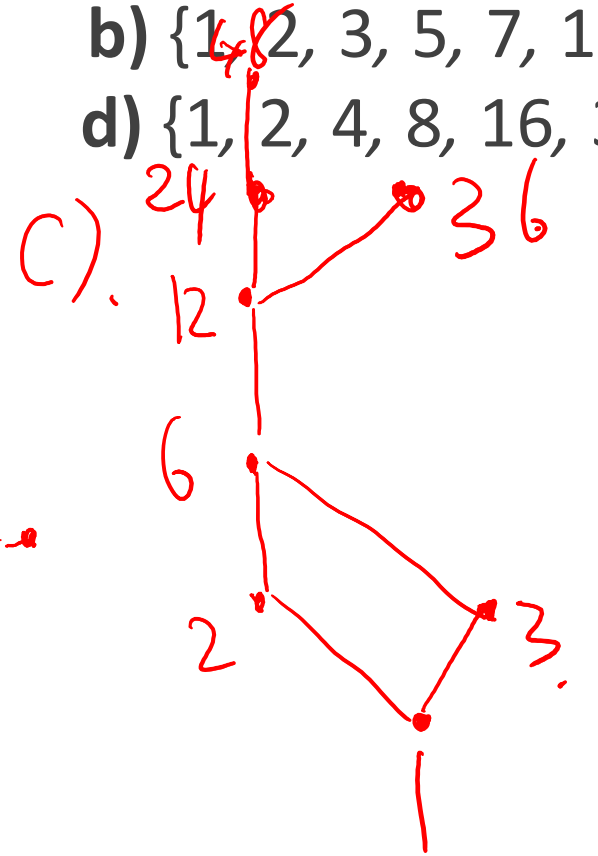
Exercise 9.6

- **23.** Draw the Hasse diagram for divisibility on the set

a) {1, 2, 3, 4, 5, 6, 7, 8}. **b)** {1, 2, 3, 5, 7, 11, 13}.

c) {1, 2, 3, 6, 12, 24, 36, 48}. **d)** {1, 2, 4, 8, 16, 32, 64}.





Exercise 9.6

- List all ordered pairs in the partial ordering with the accompanying Hasse diagram.

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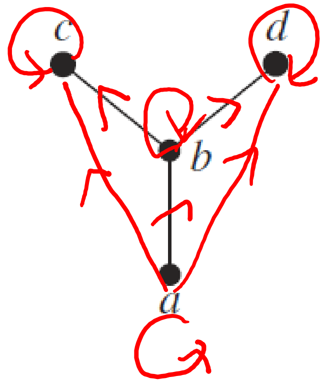

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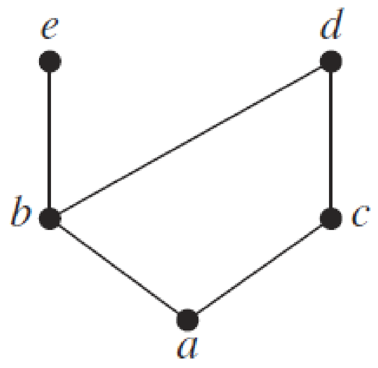

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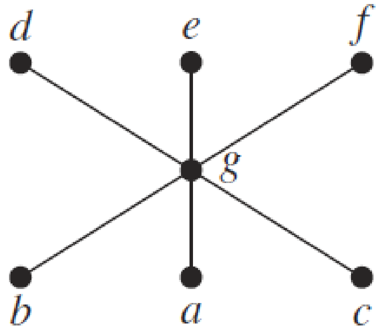

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Maximal and Minimal Elements

- a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$.

  $\neg \exists b \in S\ (a \prec b)$，上方没边的元素

- a is **minimal** if there is no element $b \in S$ such that $b \prec a$.

  $\neg \exists b \in S (b \prec a)$ 下方没边的元素

- Maximal and minimal elements are easy to spot using a Hasse diagram.

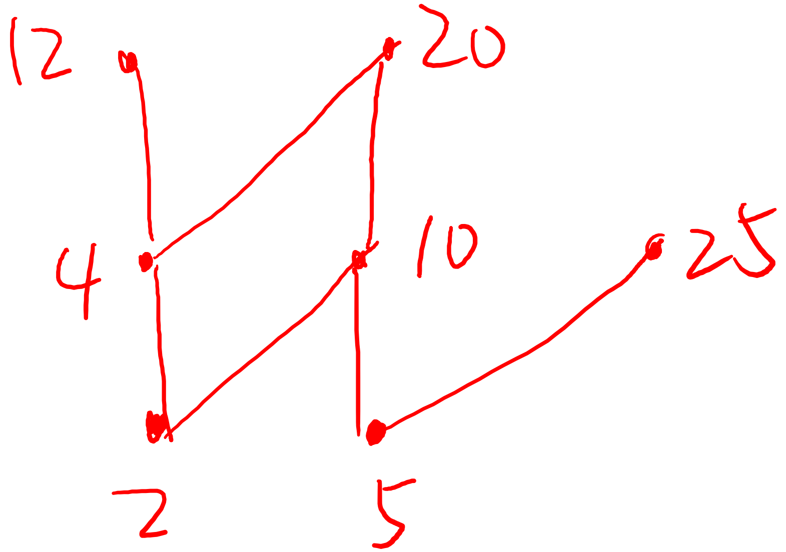
Example 6

- Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}, |) are maximal, and which are minimal?

$4<12$, $2<4$

maximal: 12, 20, 25.

minimal: 2, 5



Greatest and Least Element

- a is the **greatest element** of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$.

$\forall b \in S\ (b \preccurlyeq a)$ 对于其他所有顶点有向下通路

- a is the **least element** of the poset $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.

$\forall b \in S\ (a \preccurlyeq b)$ 对于其他所有顶点有向上通路

- There is exactly one greatest element of a poset, if such an element exists.

- There is exactly one least element of a poset, if such an element exists.

Example 7

- Determine whether the posets represented by each of the Hasse diagrams in Figure 6 have a greatest element and a least element.

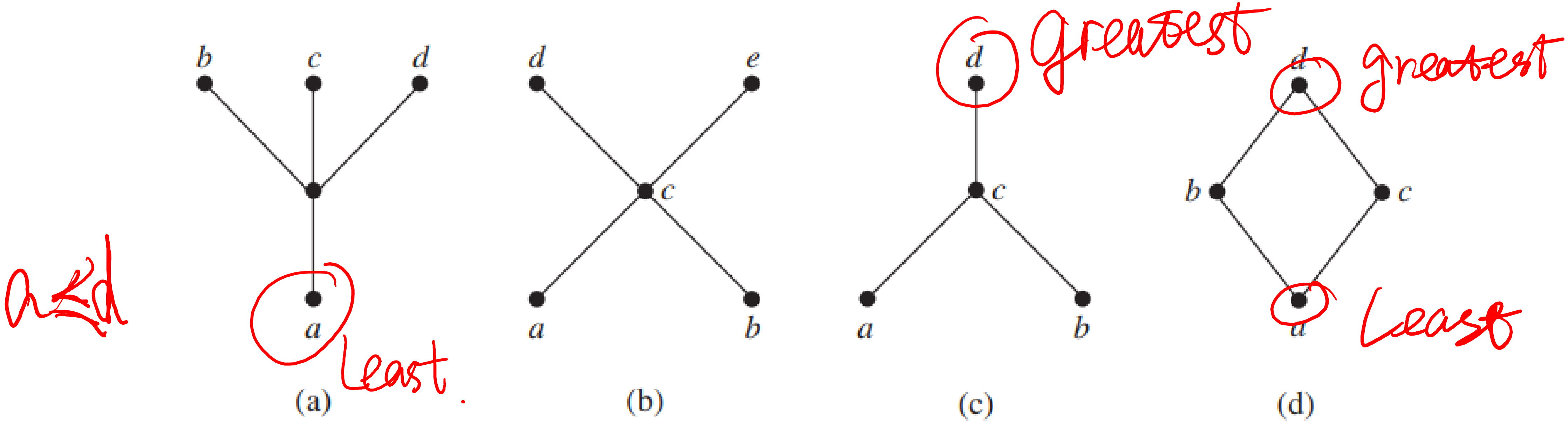


FIGURE 6 **Hasse Diagrams of Four Posets.**

Example 8

- Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

Least: ϕ

greatest: S.

Example 9

- Is there a greatest element and a least element in the poset $(\mathbf{Z}^+, |)$?

no greatest

Least : 1

Upper and Lower Bound

- Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** of A. Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

Example 10

- Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

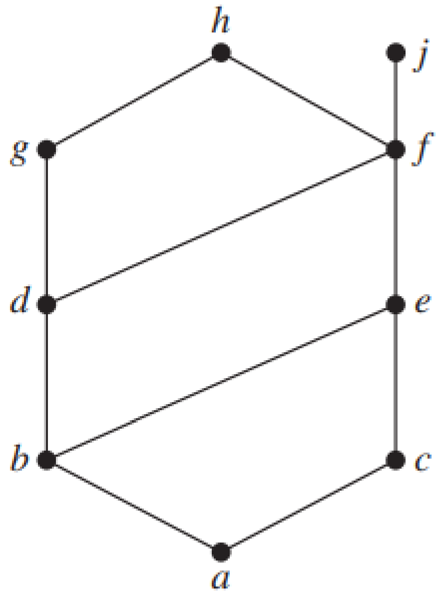


FIGURE 7 The Hasse Diagram of a Poset.

$\{a,b,c\}$ upper: e, f, j, h.
lower: a.

$\{j,h\}$ upper: ×
lower: a, b, c, d, e, f.

$\{a,c,d,f\}$ upper: f, h, j.
lower: a.

Least Upper and Greatest Lower bound

- The element x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A, denoted by lub(A).
- The element y is called the **greatest lower bound** of the subset A if y is a lower bound that is greater than every other lower bound of A, denoted by glb(A).
- The least upper bound of a set in a poset is unique if it exists.
- The greatest lower bound of a set in a poset is unique if it exists.

Example 11

- Find the greatest lower bound and the least upper bound of {*b, d, g*}, if they exist, in the poset shown in Figure 7.

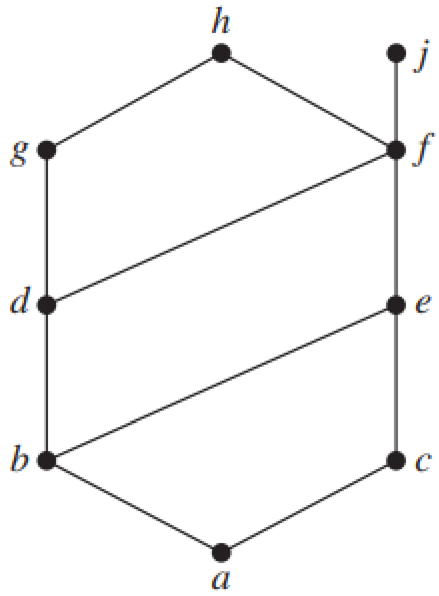


FIGURE 7 **The Hasse Diagram of a Poset.**

$\{b, d, g\}$ lower: a, b.

upper: h, g.

glb = b

lub = g

Example 12

- Find the greatest lower bound and the least upper bound of the sets {3, 9, 12} and {1, 2, 4, 5, 10}, if they exist, in the poset ($\mathbf{Z}^+$, |).

整除 { 下界：公约数.
　　　 上界：公倍数

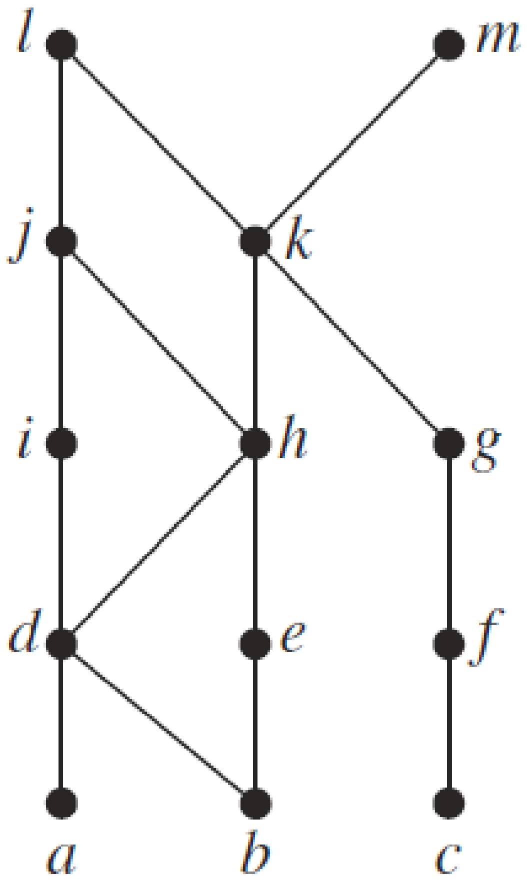
{ glb = 最大公约数
　 lub = 最小公倍数.

3, 36.

1, 20

Exercise 9.6

- **32.** Answer these questions for the partial order represented by this Hasse diagram.



a) Find the maximal elements. l, m

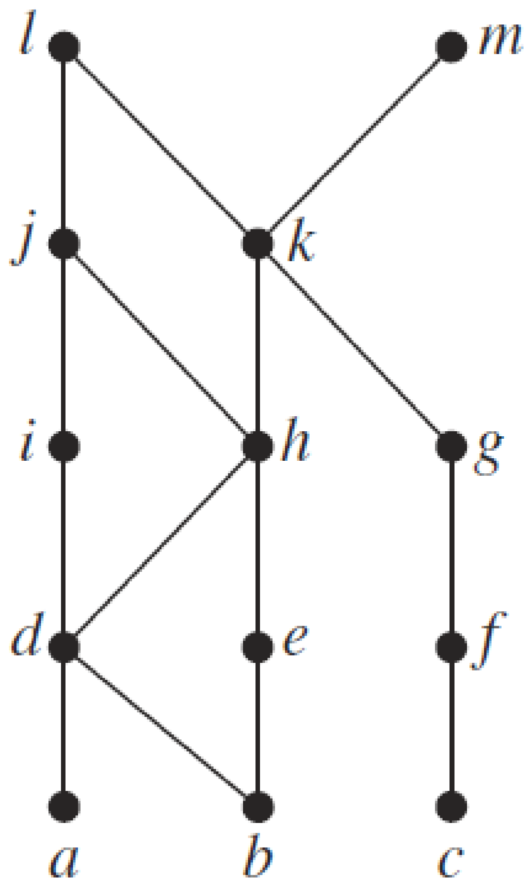
b) Find the minimal elements. a, b, c

c) Is there a greatest element? no.

d) Is there a least element? no

Exercise 9.6

- **32.** Answer these questions for the partial order represented by this Hasse diagram.



e) Find all upper bounds of {*a, b, c*}.

k, l, m

f) Find the least upper bound of {*a, b, c*}, if it exists. k.

g) Find all lower bounds of {*f, g, h*}.

no

h) Find the greatest lower bound of {*f, g, h*}, if it exists.

no

Exercise 9.6

- **33.** Answer these questions for the poset ({3, 5, 9, 15, 24, 45}, |).

a) Find the maximal elements. 24, 45

b) Find the minimal elements. 3, 5.

c) Is there a greatest element? no

d) Is there a least element? no

e) Find all upper bounds of {3, 5}. 15, 45.

f) Find the least upper bound of {3, 5}, if it exists. 15.

g) Find all lower bounds of {15, 45}. 3, 5, 15.

h) Find the greatest lower bound of {15, 45}, if it exists. 15.

Exercise 9.6

- **34.** Answer these questions for the poset ({2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, |).

a) Find the maximal elements. 27, 48, 60, 72.

b) Find the minimal elements. 2, 9

c) Is there a greatest element? no

d) Is there a least element? no

e) Find all upper bounds of {2, 9}. 18, 36, 72.

f) Find the least upper bound of {2, 9}, if it exists. 18

g) Find all lower bounds of {60, 72}. 2, 4, 6, 12

h) Find the greatest lower bound of {60, 72}, if it exists. 12

Exercise 9.6

- **35.** Answer these questions for the poset ({{1}, {2}, {4}, {1, 2}, {1, 4}, {2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}}, ⊆).

a) Find the maximal elements. {1,2} {1,3,4}, {2,3,4}

b) Find the minimal elements. {1} {2} {4}.

c) Is there a greatest element? no

d) Is there a least element? no

Exercise 9.6

- **35.** Answer these questions for the poset ({{1}, {2}, {4}, {1, 2}, {1, 4}, {2, 4}, {3, 4}, {1, 3, 4}, {2, 3, 4}}, ⊆).

e) Find all upper bounds of {{2}, {4}}. {2, 4} {2, 3, 4}

f) Find the least upper bound of {{2}, {4}}, if it exists. {2, 4}

g) Find all lower bounds of {{1, 3, 4}, {2, 3, 4}}. {4}, {3, 4}

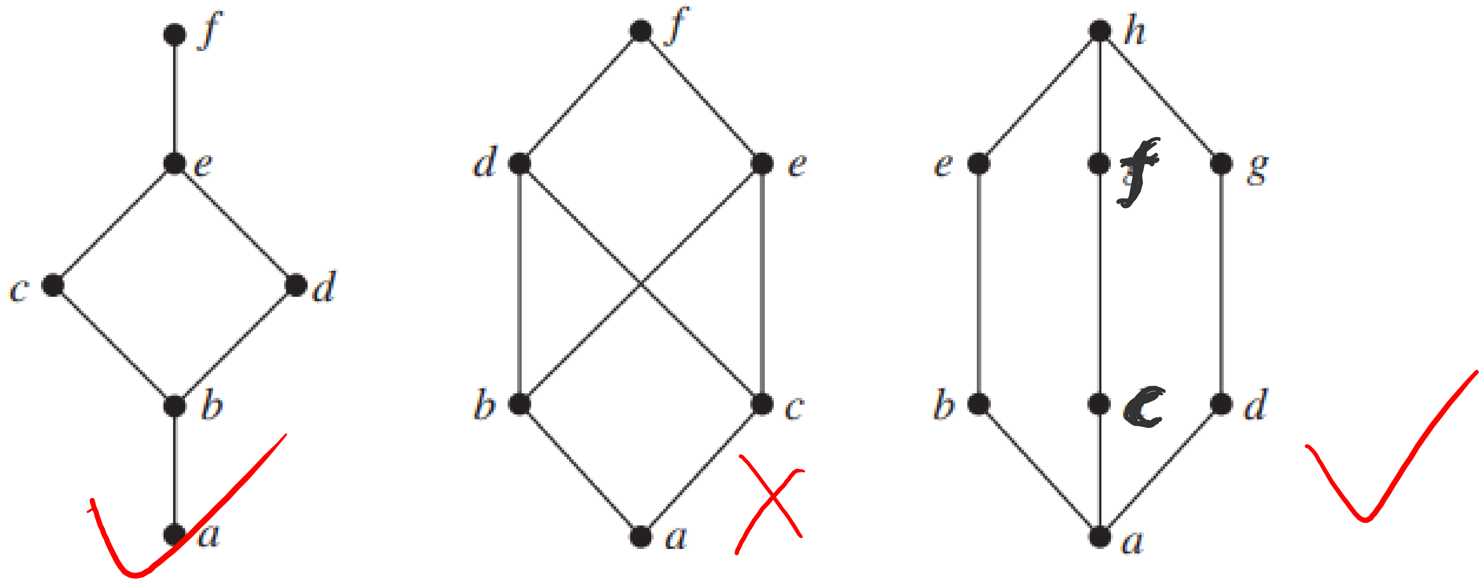
h) Find the greatest lower bound of {{1, 3, 4}, {2, 3, 4}}, if it exists. {3, 4}.

Lattice 格

Definition:

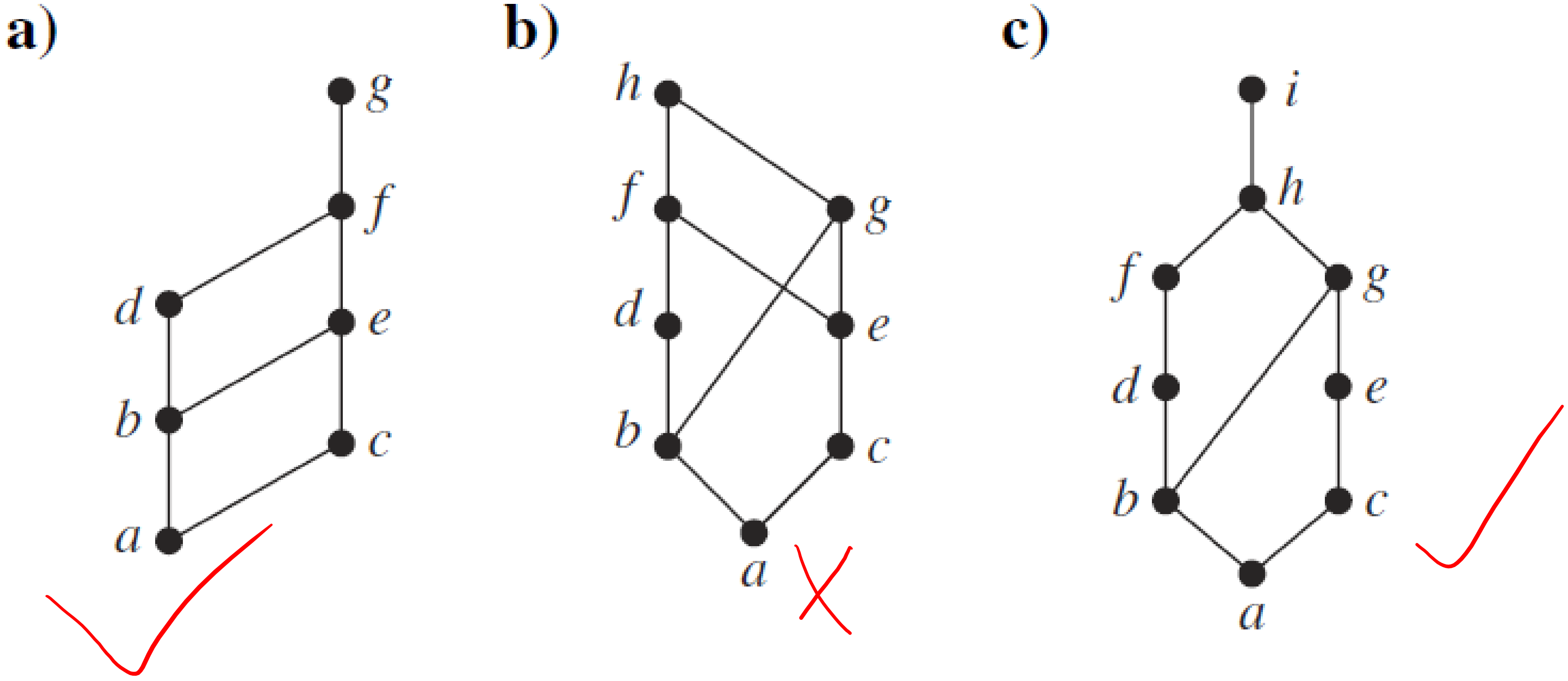
A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

Example:



Exercise 9.6

- **43.** Determine whether the posets with these Hasse diagrams are lattices.



Example 13

- Is the poset ($\mathbf{Z}^+$, |) a lattice?

Yes. Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively.

- Determine whether the posets ({1, 2, 3, 4, 5}, |) and ({1, 2, 4, 8, 16}, |) are lattices.

Example 14

- Determine whether ($P(S)$, $\subseteq$) is a lattice where S is a set.

Yes.

Let A, B be two subsets of S.

$lub = A \cup B$.

$glb = A \cap B$.

Exercise 9.6

- **44.** Determine whether these posets are lattices.

a) $(\{1, 3, 6, 9, 12\}, |)$

b) $(\{1, 5, 25, 125\}, |)$

c) $(\mathbf{Z}, \geq)$

d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S

Partial Orderding	$(\mathbf{Z}, \geq)$ $(\mathbf{Z}, \leq)$	$(\mathbf{Z}^+, \mid)$	$(P(S), \subseteq)$ $(P(S), \supseteq)$
Total Ordering	yes	no	no
Well Ordering	no	no	no
Lattice	yes	yes	yes
Greatest/Least	None/None	None/1	S/∅