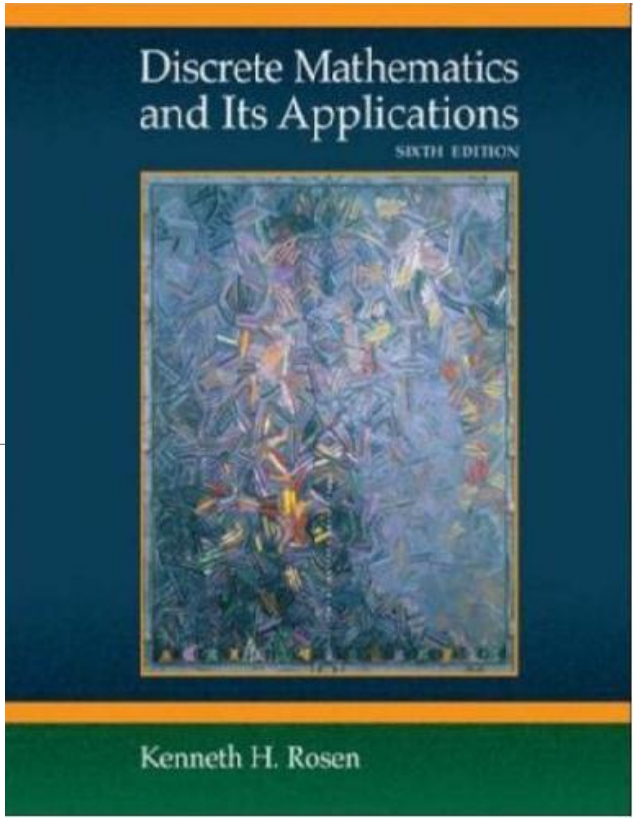
北京交通大學

Discrete Mathematics



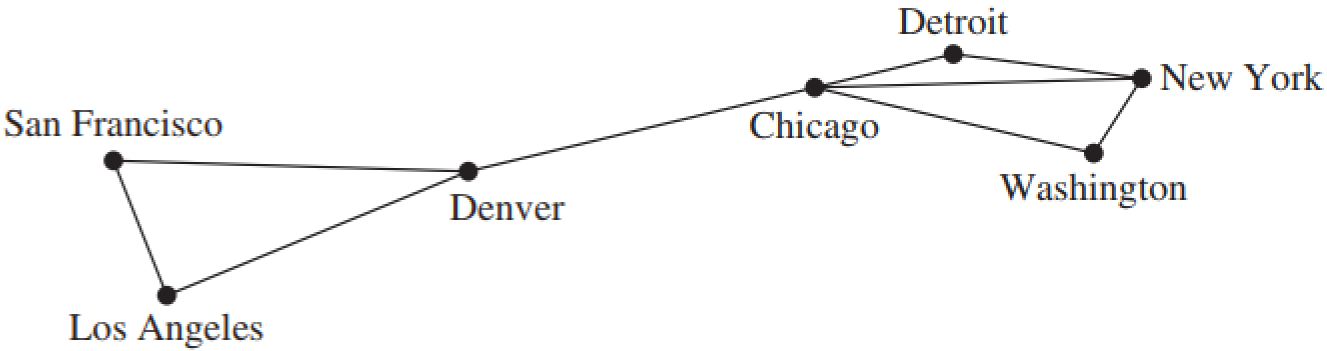
Haiyang Liu
haiyangliu@bjtu.edu.cn
18611249791

Graph

Definition:

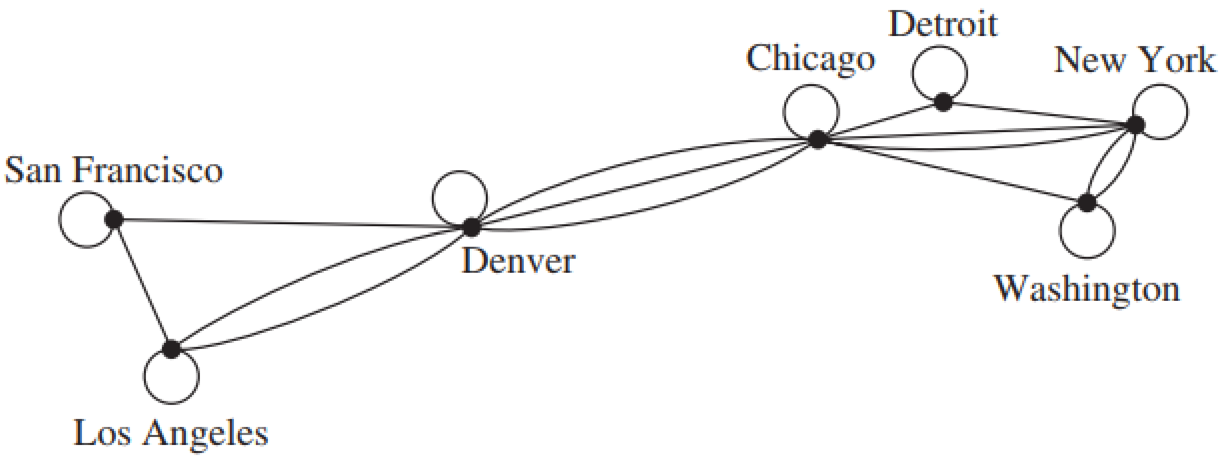
- A graph $G = (V, E)$ consists of V , a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints.

Example:

- 

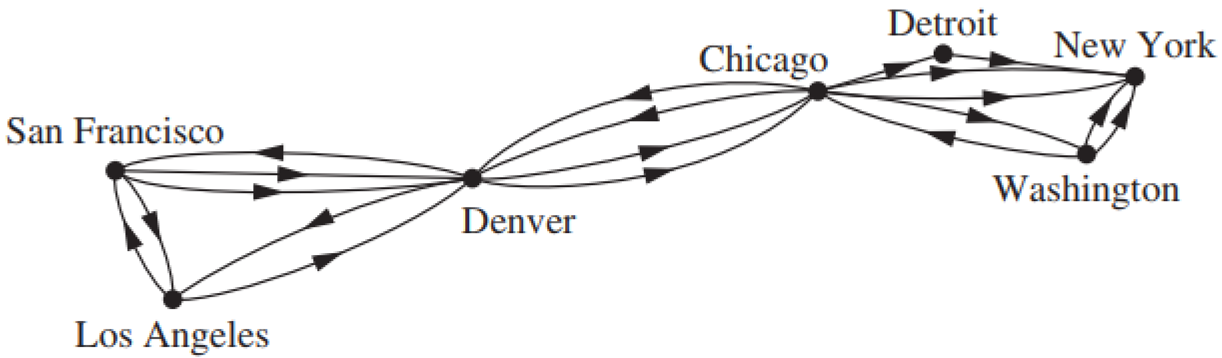
Types of Graphs

- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a **simple graph**.
- Graphs that may have multiple edges connecting the same vertices are called **multigraphs**.
- Graphs that may include loops, are called **pseudographs**.



Types of Graphs

- **Undirected Graph**
- **Directed Graph**: (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.
- **Simple Directed Graph**
- **Directed Multigraph**
- **Mixed Graph**



Types of Graphs

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

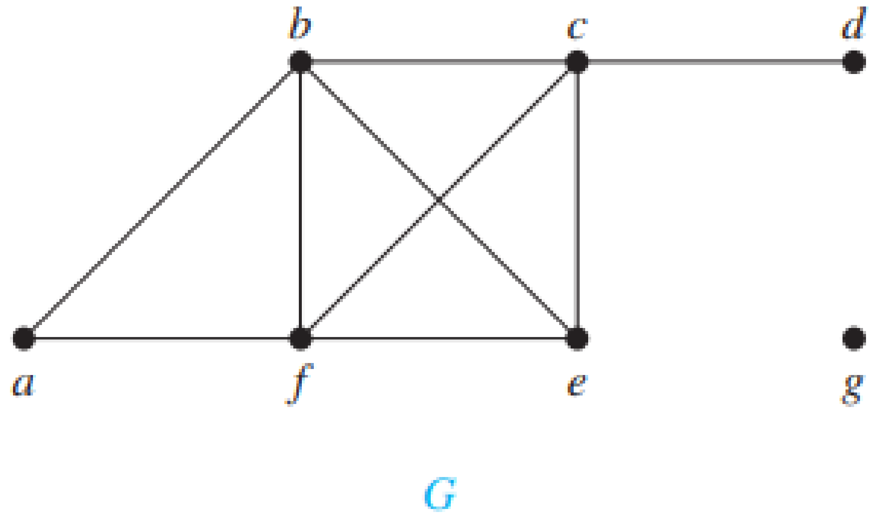
- Are the edges of the graph undirected or directed (or both)?
- If the graph is undirected, are multiple edges present that connect the same pair of vertices? If the graph is directed, are multiple directed edges present?
- Are loops present?

Definitions of Undirected Graphs

- Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.
- The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.
- The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Example 1

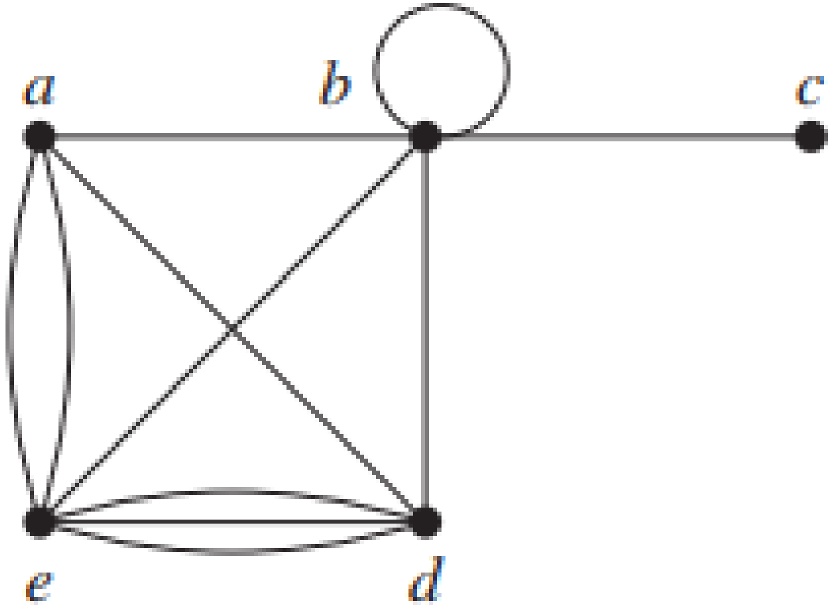
- What are the degrees and what are the neighborhoods of the vertices in the graphs *G*?



- A vertex of degree zero is called **isolated**.
- A vertex of degree one is called **pendant**.

Example 2

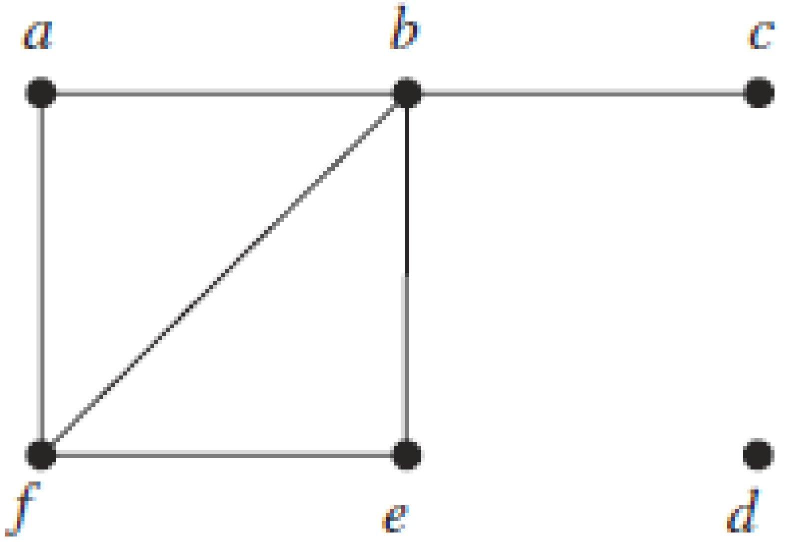
- What are the degrees and what are the neighborhoods of the vertices in the graphs *H*?

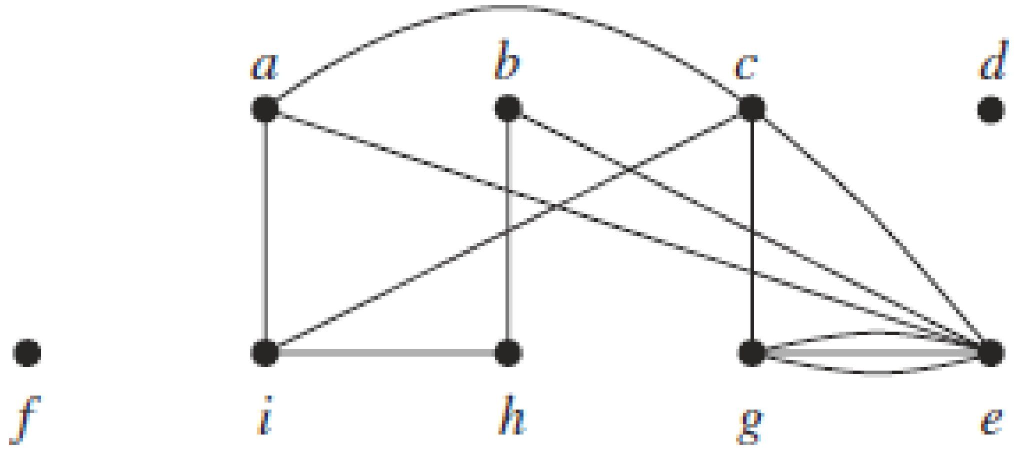


H

Exercise 10.2(1,3)

- Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.





Theorem 1: The Handshaking Theorem

- Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2|E|=2m = \sum_{v\in V}\deg(v)$$

(Note that this applies even if multiple edges and loops are present.)

Example:

- How many edges are there in a graph with 10 vertices each of degree six?
- Can a simple graph exist with 15 vertices each of degree five?

Theroem 2

- An undirected graph has an even number of vertices of odd degree.

Proof:

Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{v\in V} \deg(v) = \sum_{v\in V_1} \deg(v) + \sum_{v\in V_2} \deg(v)$$

Exercise 10.2(52)

- Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that

a) $2e/v \geq m$.

b) $2e/v \leq M$.

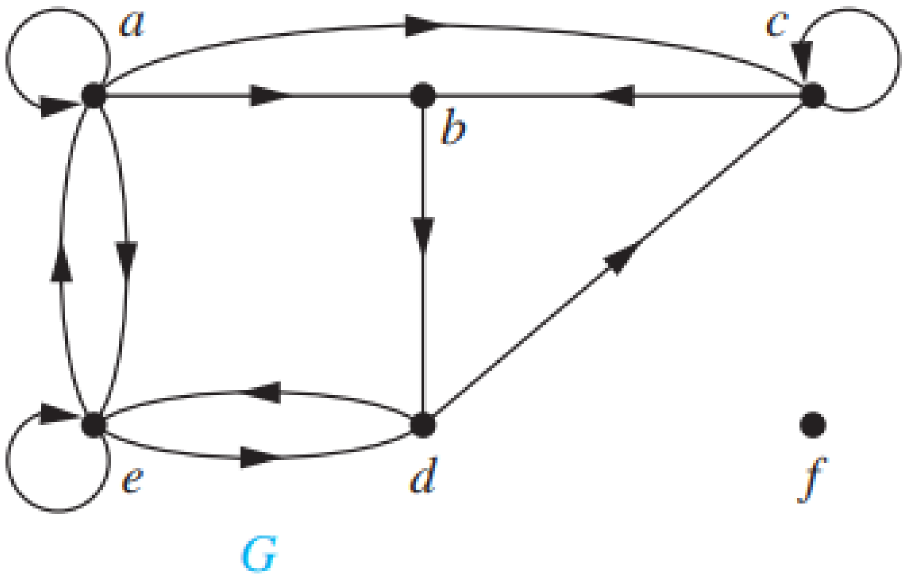
Definitions of Directed Graphs

- When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex of (u, v), and v is called the terminal or end vertex of (u, v). The initial vertex and terminal vertex of a loop are the same.

- In a graph with directed edges the in-degree of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Example 3

- Find the in-degree and out-degree of each vertex in the graph *G* with directed edges?



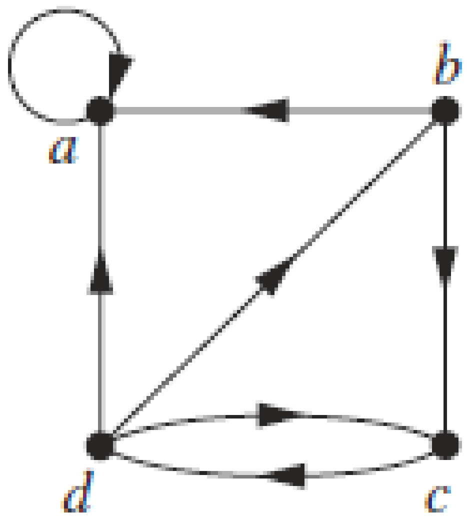
Theorem 3

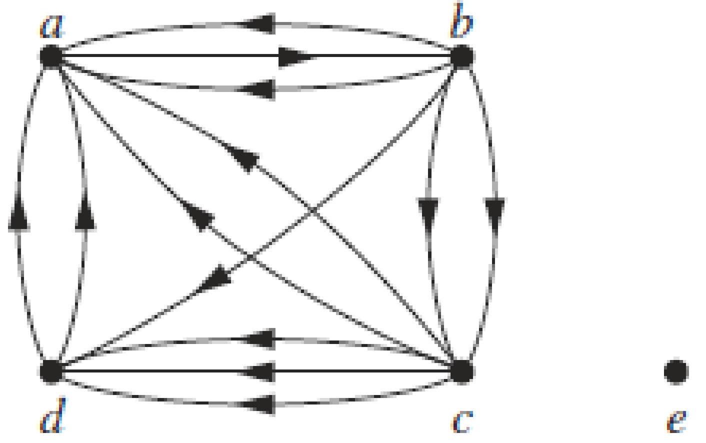
- Let $G = (V, E)$ be a graph with directed edges. Then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$$

Exercise 10.2 (7, 9)

- Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.



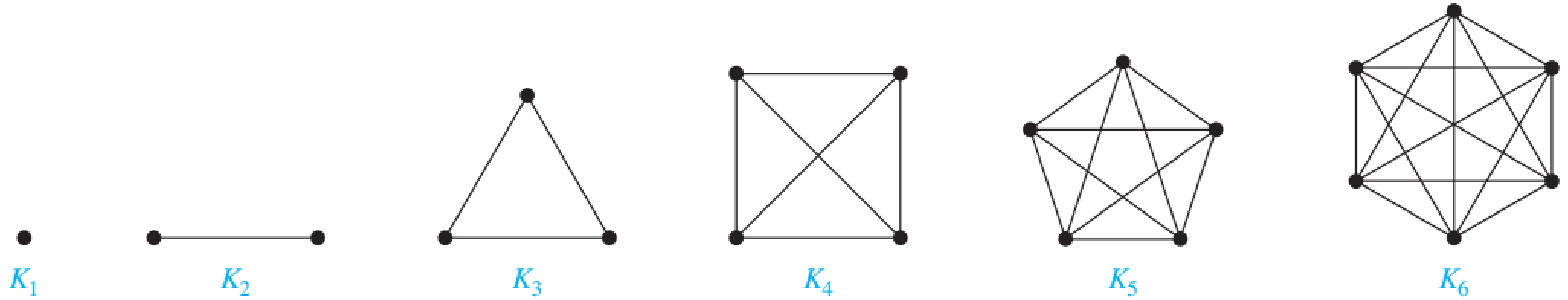


Special Simple Graphs

- Complete Graphs
- Cycles
- Wheels
- *n*-Cubes

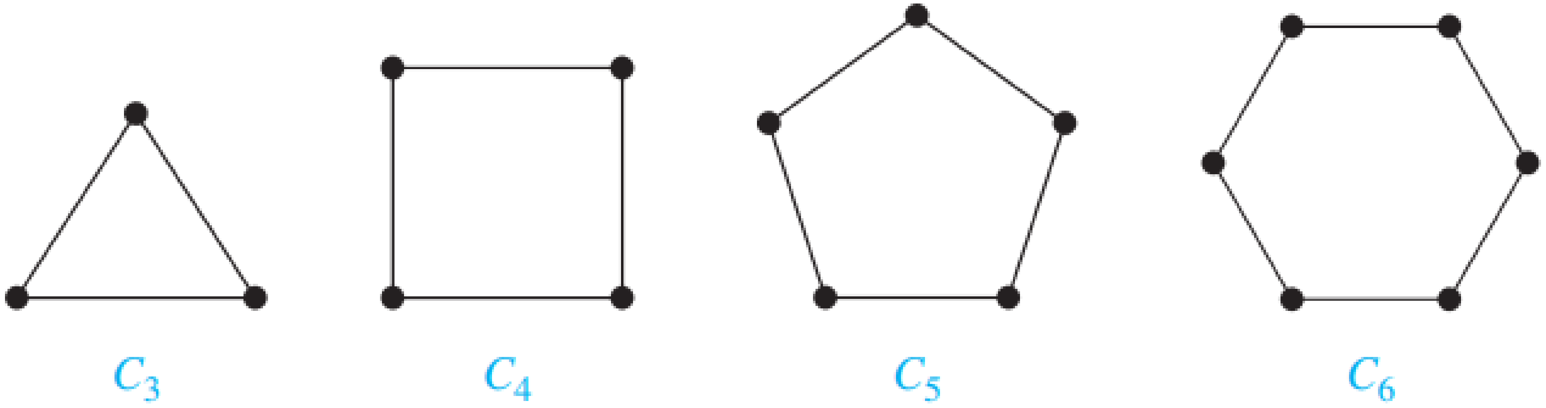
Complete Graphs

- **Complete Graphs:** A **complete graph on *n* vertices**, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.



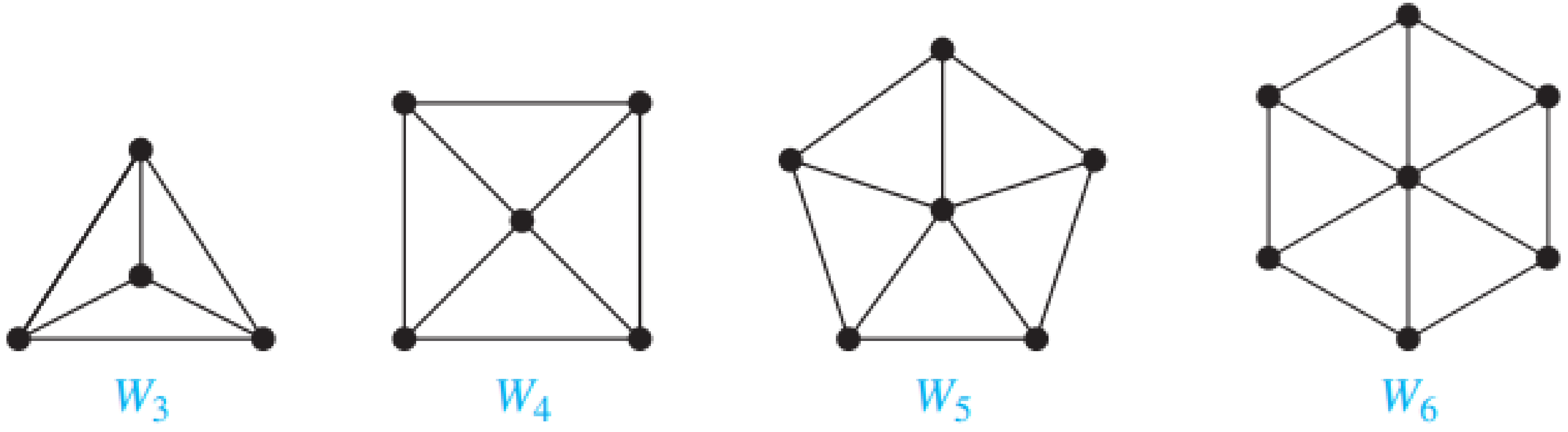
Cycles

- A **cycle** $\boldsymbol{C_n}$, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.



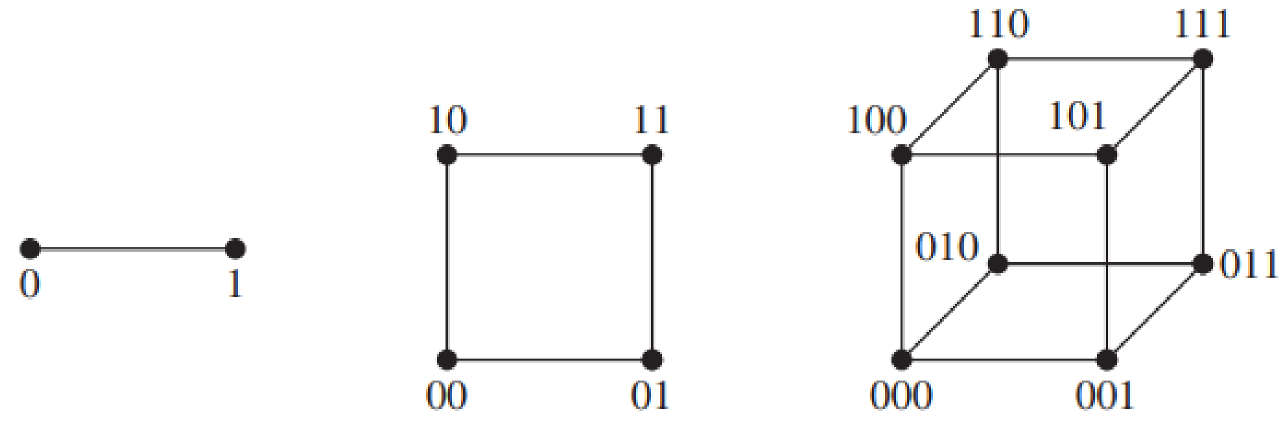
Wheels

- We obtain a **wheel** W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.



n-Cubes

- A ***n*-cube**, denoted by Q_n, is a graph that has vertices representing the 2^n bit strings of length n.



Q_1 Q_2 Q_3

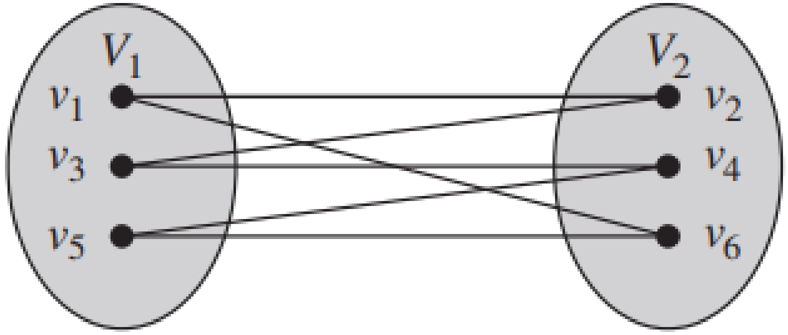
Bipartite Graphs

Definition:

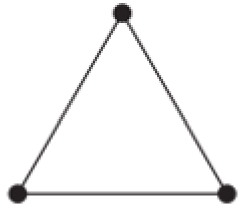
- A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint nonempty sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

Example:

- C_6



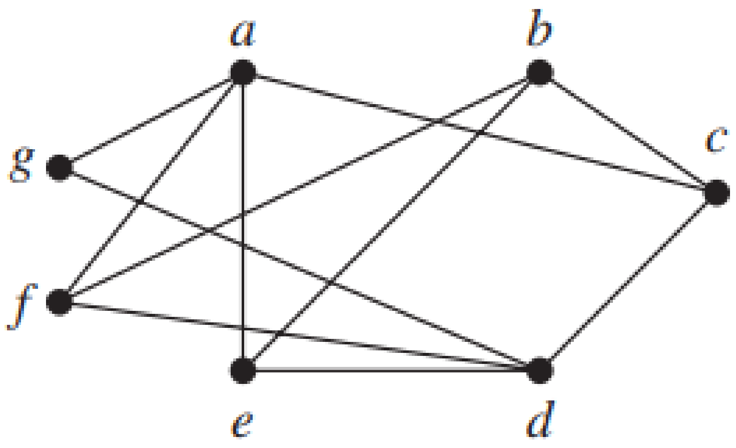
- K_3



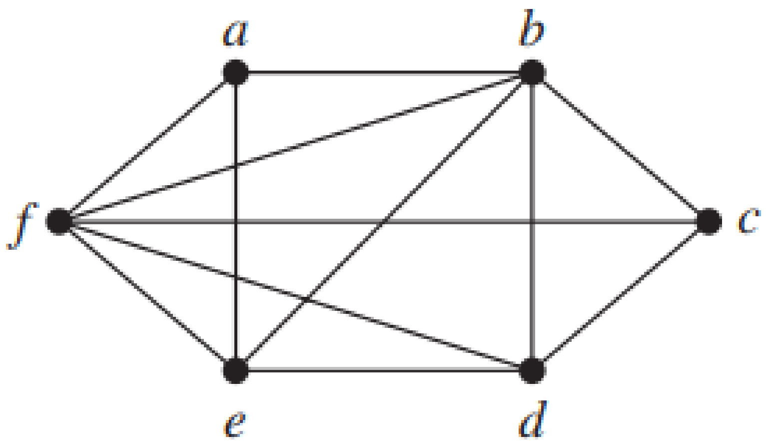
K_3

Example 4

- Are the graphs *G* and *H* bipartite?



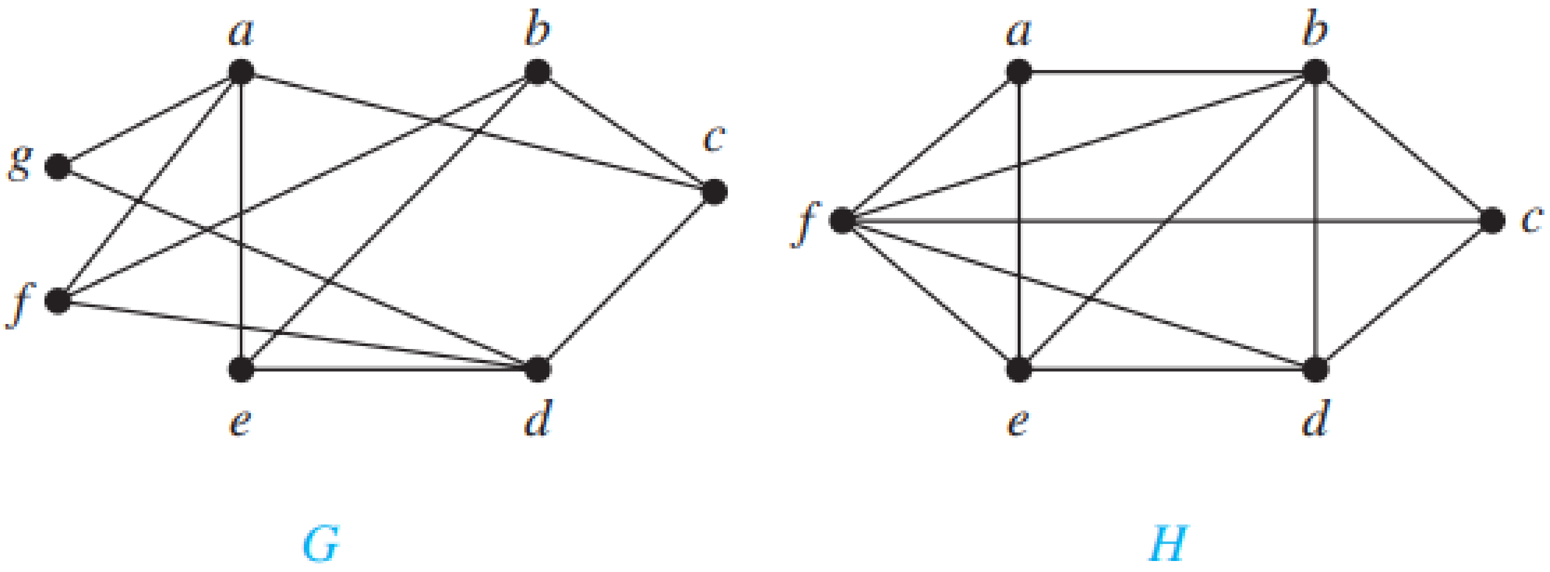
G



H

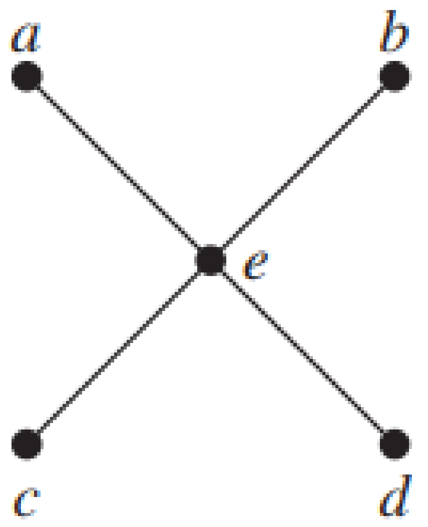
Theorem 4

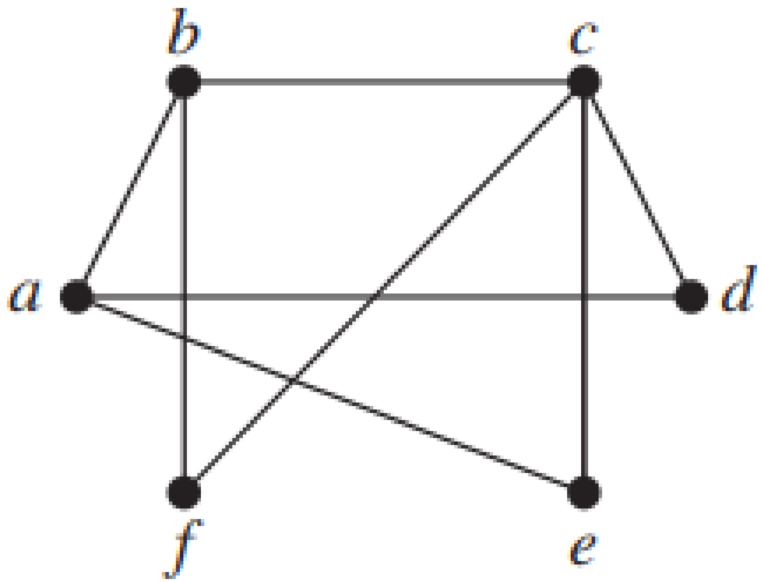
- A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

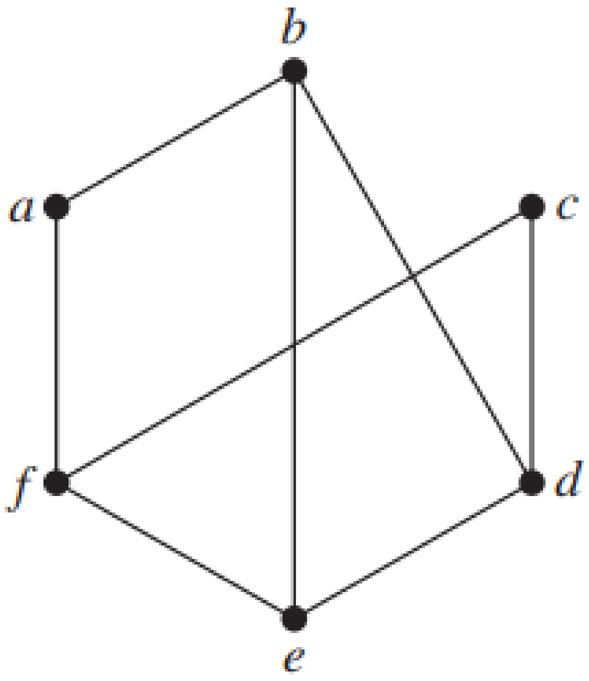


Exercise 10.2(21, 23, 25)

- Determine whether the graph is bipartite





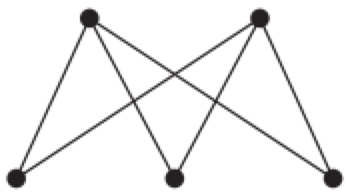


Exercies 10.2 (26)

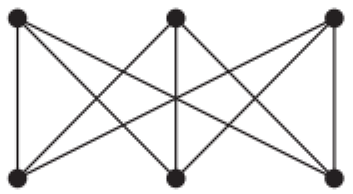
- For which values of n are these graphs bipartite?
- a) K_n
- b) C_n
- c) W_n
- d) Q_n

Complete Bipartite Graphs

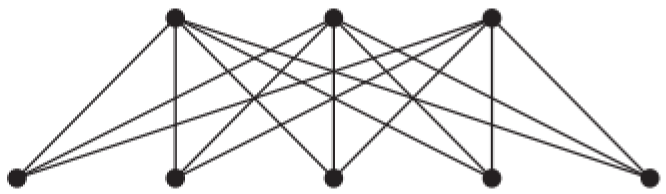
- A **complete bipartite graph** $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of *m* and *n* vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.



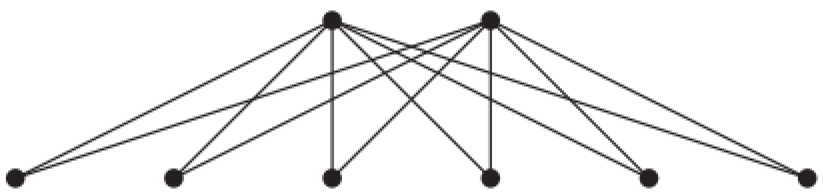
$K_{2,3}$



$K_{3,3}$

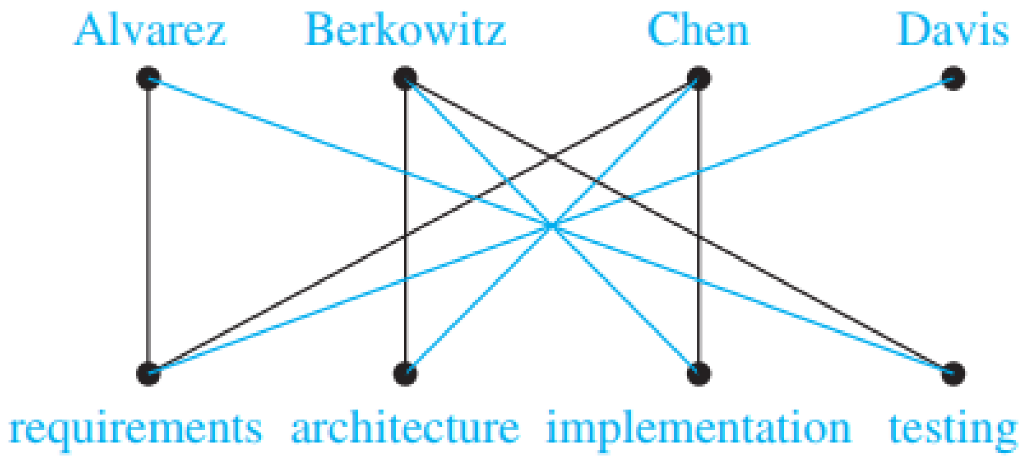


$K_{3,5}$

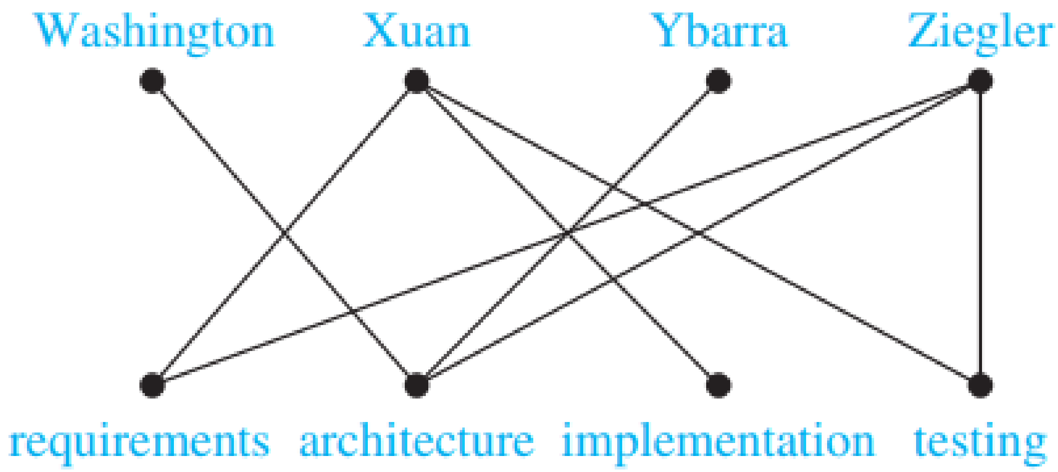


$K_{2,6}$

Job Assignments



(a)



Matching

- A **matching** M in a simple graph $G = (V, E)$ is a subset of the set E of edges of the graph such that no two edges are incident with the same vertex.
- In other words, a matching is a subset of edges such that if $\{s, t\}$ and $\{u, v\}$ are distinct edges of the matching, then s, t, u, and v are distinct.
- A **maximum matching** is a matching with the largest number of edges.
- We say that a matching M in a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) is a **complete matching** from V_1 to V_2 if every vertex in V_1 is the endpoint of an edge in the matching, or equivalently, if $|M| = |V_1|$.

Theorem 5: Hall's Theorem

- The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Exercise 10.2(27)

- Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

a) Use a bipartite graph to model the four employees and their qualifications.

b) Use Hall's theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support.

c) If an assignment of employees to support areas so that each employee is assigned to one support area exists, find one.

Exercise 10.2(27)

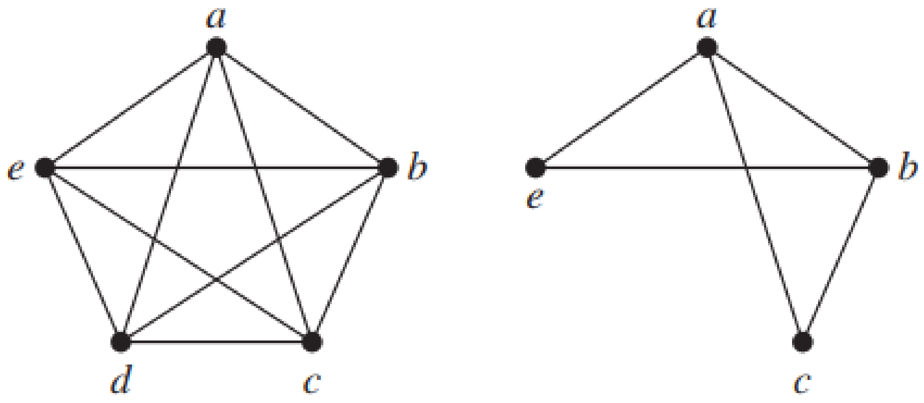
- Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

Exercise 10.2(29)

- Suppose that there are five young women and five young men on an island. Each man is willing to marry some of the women on the island and each woman is willing to marry any man who is willing to marry her. Suppose that Sandeep is willing to marry Tina and Vandana; Barry is willing to marry Tina, Xia, and Uma; Teja is willing to marry Tina and Zelda; Anil is willing to marry Vandana and Zelda; and Emilio is willing to marry Tina and Zelda. Use Hall's theorem to show there is no matching of the young men and young women on the island such that each young man is matched with a young woman he is willing to marry.

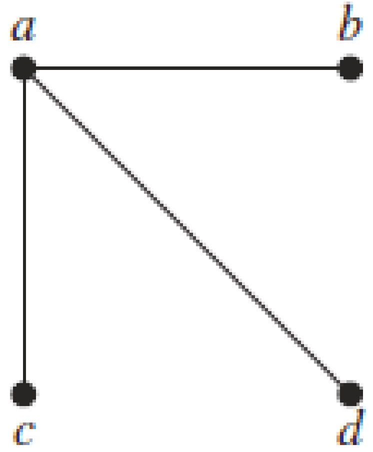
Subgraph

- A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a **proper subgraph** of G if $H \neq G$.
- Let $G = (V, E)$ be a simple graph. The **subgraph induced** by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.
- **Example:**



Exercise 10.2(51)

- Draw all subgraphs of this graph.



Exercise 10.2(48, 49)

- How many subgraphs with at least one vertex does K_2 have?

- How many subgraphs with at least one vertex does K_3 have?

Removing or Adding Edges of a Graph

- Given a graph $G = (V, E)$ and an edge $e \in E$, we can produce a subgraph of G by removing the edge e. The resulting subgraph, denoted by $G - e$, has the same vertex set V as G. Its edge set is $E - e$. Hence, $G - e = (V, E - \{e\})$.

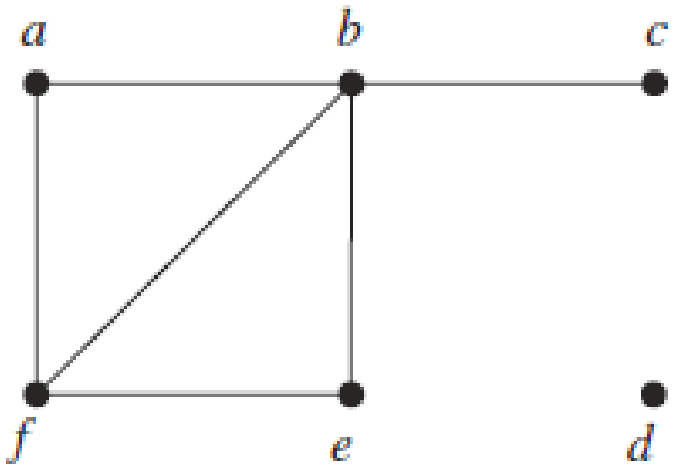
- We denote by $G + e$ the new graph produced by adding a new edge e, connecting two previously nonincident vertices, to the graph G Hence, $G + e = (V, E \cup \{e\})$.

Edge Contractions

- Removes an edge e with endpoints u and v and merges u and w into a new single vertex w, and for each edge with u or v as an endpoint replaces the edge with one with w as endpoint in place of u or v and with the same second endpoint.
- The contraction of the edge e with endpoints u and v in the graph $G = (V, E)$ produces a new graph $G' = (V', E')$, where $V' = V - \{u, v\} \cup \{w\}$ and E' contains the edges in E which do not have either u or v as endpoints and an edge connecting w to every neighbor of either u or v in V.

Exercis 10.2 (33)

- For the graph G find
- **a)** the subgraph induced by the vertices a, b, c, and f .
- **b)** the new graph G_1 obtained from G by contracting the edge connecting b and f .
- **c)** is G_1 a subgraph of G?

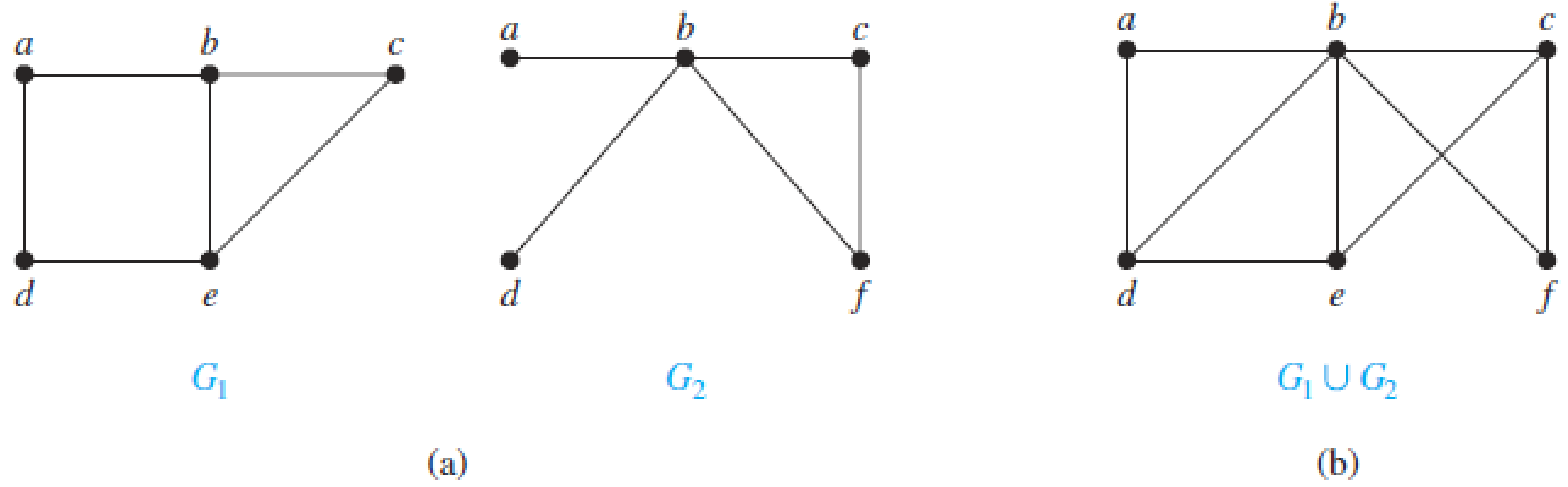


Removing Vertices of a Graph

- When we remove a vertex v and all edges incident to it from $G = (V, E)$, we produce a subgraph, denoted by $G - v$. Observe that $G - v = (V - v, E)$, where E is the set of edges of G not incident to v.

Graph Unions

- The *union* of two simple graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.



Complementary Graph

- The **complementary graph** $\bar{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

Exercis 10.2 (59)

- Describe each of these graphs.
- **a)** $\overline{K_n}$
- **b)** $\overline{K_{m,n}}$
- **c)** $\overline{C_n}$
- **d)** $\overline{Q_n}$