

A COMBINATORIAL DESCRIPTION OF KNOT FLOER HOMOLOGY

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ABSTRACT. Given a grid presentation of a knot (or link) K in the three-sphere, we describe a Heegaard diagram for the knot complement in which the Heegaard surface is a torus and all elementary domains are squares. Using this diagram, we obtain a purely combinatorial description of the knot Floer homology of K .

1. INTRODUCTION

Heegaard Floer homology [22] is an invariant for three-manifolds, defined using holomorphic disks and Heegaard diagrams. In [21] and [25], this construction is extended to give an invariant, *knot Floer homology* $\widehat{\mathrm{HFK}}$, for null-homologous knots in a closed, oriented three-manifold. This construction is further generalized in [23] to the case of oriented links. The definition of all these invariants involves counts of holomorphic disks in the symmetric product of a Riemann surface, which makes them rather challenging to calculate.

In its most basic form, knot Floer homology is an invariant for knots $K \subset S^3$, $\widehat{\mathrm{HFK}}(K)$, which is a finite-dimensional bi-graded vector space over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, i.e.

$$\widehat{\mathrm{HFK}}(K) = \bigoplus_{m,s} \widehat{\mathrm{HFK}}_m(K, s).$$

This invariant is related to the symmetric Alexander polynomial $\Delta_K(T)$ by the formula

$$\Delta_K(T) = \sum_{m,s} (-1)^m \mathrm{rank} \widehat{\mathrm{HFK}}_m(K, s)$$

(cf. [21], [25]). The topological significance of this invariant is illustrated by the result that

$$g(K) = \max\{s \in \mathbb{Z} \mid \widehat{\mathrm{HFK}}_*(K, s) \neq 0\},$$

where here $g(K)$ denotes the Seifert genus of K (cf. [20]), and also the fact that $\widehat{\mathrm{HFK}}_*(K, g(K))$ has rank one if and only if K is fibered ([8] in the case where $g(K) = 1$ and [16] in general). The invariant is defined as a version of Lagrangian Floer homology [4] in a suitable symmetric product of a Heegaard surface.

Our aim here is to give a purely combinatorial presentation of knot Floer homology with coefficients in \mathbb{F} for knots in the three-sphere. Our description can be extended to describe link Floer homology, and also it can be extended to describe the “full knot filtration” (and in particular the concordance invariant τ [19]). However, in the interest of exposition, we limit ourselves in the introduction to the case of knot Floer homology, referring the interested reader to Section 3 for more general cases.

To explain our combinatorial description, it will be useful to have the following notions.

A *planar grid diagram* $\tilde{\Gamma}$ consists of a square grid on the the plane with $n \times n$ cells, together with a collection of black and white dots on it, arranged so that:

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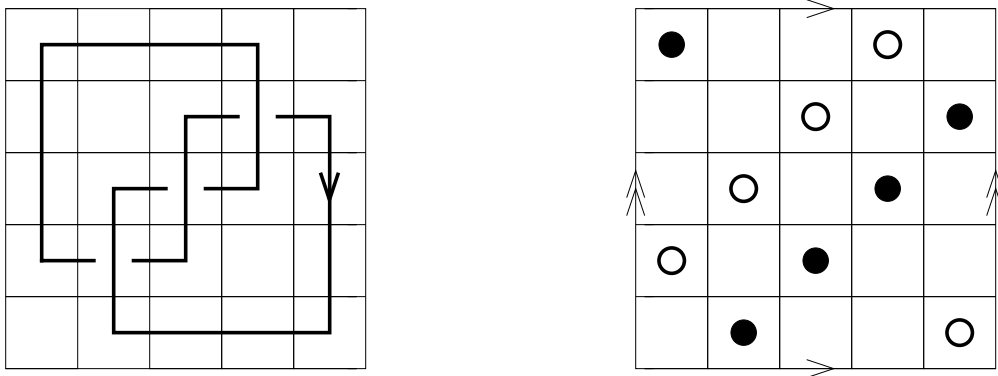


FIGURE 1. **Grid diagram for the trefoil.** We have pictured here a grid diagram for the trefoil, with projection indicated on the left. To pass from a planar to a toroidal grid diagram, we make the identifications suggested by the arrows.

- every row contains exactly one black dot and one white dot;
- every column contains exactly one black dot and one white dot;
- no cell contains more than one dot.

The number n is called the *grid number* of $\tilde{\Gamma}$.

Given a planar grid diagram $\tilde{\Gamma}$, we can place it in a standard position on the plane as follows: the bottom left corner is at the origin, each cell is a square of edge length one, and every dot is in the middle of the respective cell. We then construct a planar knot projection by drawing horizontal segments from the white to the black dot in each row, and vertical segments from the black to the white dot in each column. At every intersection point, we let the horizontal segment be the underpass and the vertical one the overpass. This produces a planar diagram for an oriented link \tilde{L} in S^3 . We say that \tilde{L} has a grid presentation given by $\tilde{\Gamma}$. Figure 1 shows a grid presentation of the trefoil, with $n = 5$.

It is easy to see that every knot (or link) in the three-sphere can be presented by a planar grid diagram. In fact, grid presentations are equivalent to the arc presentations of knots, which first appeared in [1], the square bridge positions of knots of [13], and also to Legendrian realizations of knots, cf. [14]; they have enjoyed a considerable amount of attention over the years. The minimum number n for which a knot $K \subset S^3$ admits a grid presentation of grid number n is called the *arc index* of K .

We find it convenient to transfer our planar grid diagrams to the torus \mathcal{T} obtained by gluing the topmost segment to the bottom-most one, and the leftmost segment to the rightmost one. In the torus, our horizontal and vertical arcs become horizontal and vertical circles. The torus inherits its orientation from the plane. We call the resulting object Γ a *toroidal grid diagram*, or simply a *grid diagram*, for K .

Given a toroidal grid diagram, we associate to it a chain complex $(C(\Gamma), \partial)$ as follows. The generators X of $C(\Gamma)$ are indexed by one-to-one correspondences between the horizontal and vertical circles. More geometrically, we can think of these as n -tuples of intersection points \mathbf{x} between the horizontal and vertical circles, with the property that no intersection point appears on more than one horizontal (or vertical) circle.

We now define functions $A: X \rightarrow \mathbb{Z}$ and $M: X \rightarrow \mathbb{Z}$ (the Alexander and Maslov gradings) as follows.

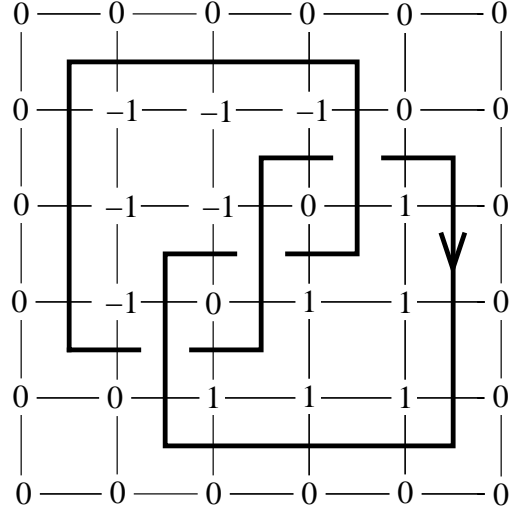


FIGURE 2. **The function a .** Over every lattice point p from Figure 1, we marked minus the winding number of the knot projection around p .

Let us define a function a on lattice points p to be minus one times the winding number of the knot projection around p . (This is shown in Figure 2 for our trefoil example.) Each black or white dot in the diagram lies in a square. We thus obtain $2n$ distinguished squares, and each of them has four corners. We denote the resulting collection of corners $\{c_{i,j}\}$, $i \in \{1, \dots, 2n\}$, $j \in \{1, \dots, 4\}$. We set

$$(1) \quad A(\mathbf{x}) = \sum_{p \in \mathbf{x}} a(p) - \frac{1}{8} \left(\sum_{i,j} a(c_{i,j}) \right) - \frac{n-1}{2}.$$

Next, given a pair of generators \mathbf{x} and \mathbf{y} , and an embedded rectangle r in \mathcal{T} whose edges are arcs in the horizontal and vertical circles, we say that r connects \mathbf{x} to \mathbf{y} if \mathbf{x} and \mathbf{y} agree along all but two horizontal circles, if all four corners of r are intersection points in $\mathbf{x} \cup \mathbf{y}$, and indeed, if we traverse each horizontal boundary components of r in the direction dictated by the orientation that r inherits from \mathcal{T} , then the arc is oriented so as to go from a point in \mathbf{x} to the point in \mathbf{y} . Let $R_{\mathbf{x},\mathbf{y}}$ denote the collection of rectangles connecting \mathbf{x} to \mathbf{y} .

It is easy to see that if $\mathbf{x}, \mathbf{y} \in X$, then if \mathbf{x} and \mathbf{y} differ along all but two horizontal circles, then there are exactly two rectangles in $R_{\mathbf{x},\mathbf{y}}$; otherwise $R_{\mathbf{x},\mathbf{y}} = \emptyset$, cf. Figure 3.

Given $\mathbf{x}, \mathbf{y} \in X$, it is easy to find an oriented, null-homologous curve $\gamma_{\mathbf{x},\mathbf{y}}$ composed of horizontal and vertical arcs, where each horizontal arc goes from a point in \mathbf{x} to a point in \mathbf{y} (and hence each vertical arc goes from a point in \mathbf{y} to a point in \mathbf{x}). Now, suppose that D is a two-chain whose boundary is a collection of horizontal and vertical arcs, and $\mathbf{x} \in X$. We let $W(D)$ and $B(D)$ denote the number of white and black dots in D respectively. Moreover, at each intersection point x of the horizontal and vertical circles, D has four local multiplicities. We define the local multiplicity of D at x , $p_x(D)$, to be the average of these four local multiplicities. Moreover, given $\mathbf{x} \in X$, let

$$P_{\mathbf{x}}(D) = \sum_{x \in \mathbf{x}} p_x(D).$$

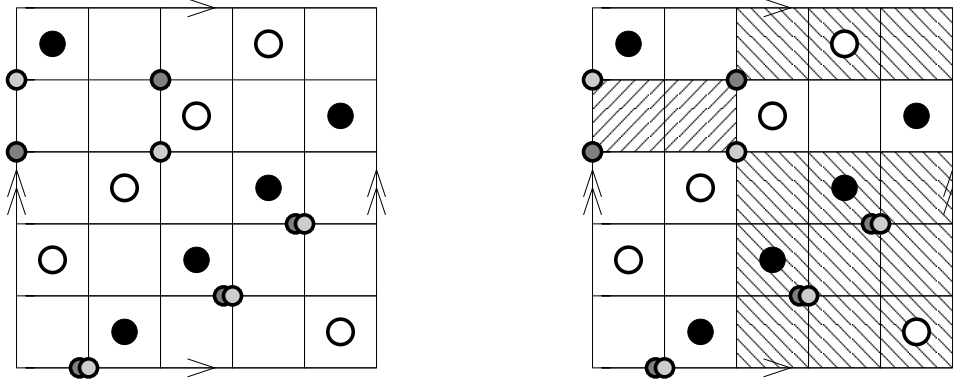


FIGURE 3. **Rectangles.** At the left, we have indicated two generators \mathbf{x} and \mathbf{y} in X for the grid diagram of the trefoil considered earlier. The generators \mathbf{x} and \mathbf{y} are represented by the collections of (smaller) shaded dots centered on the intersection points of the grid, with \mathbf{x} represented by the more darkly shaded circles and \mathbf{y} represented by the more lightly shaded ones. Note that three dots in \mathbf{x} occupy the same locations on the grid as \mathbf{y} -dots, while two do not. At the right, we have indicated the two rectangles in $R_{\mathbf{x},\mathbf{y}}$, which are shaded by (the two types of) diagonal hatchings. One of these rectangles r has $P_{\mathbf{x}}(r) + P_{\mathbf{y}}(r) = 1$ and $W(r) = B(r) = 0$ (and hence it represents a non-trivial differential from \mathbf{x} to \mathbf{y}), while the other rectangle r' has $P_{\mathbf{x}}(r') + P_{\mathbf{y}}(r') = 5$ and $W(r') = B(r') = 2$.

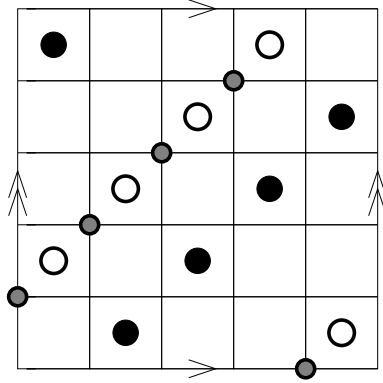


FIGURE 4. **The generator \mathbf{x}_0 .** We have illustrated here the generator with Maslov grading equal to $1 - n$.

Now, M is uniquely characterized up to an additive constant by the property that for each $\mathbf{x}, \mathbf{y} \in X$,

$$(2) \quad M(\mathbf{x}) - M(\mathbf{y}) = P_{\mathbf{x}}(D) + P_{\mathbf{y}}(D) - 2 \cdot W(D),$$

where here D is some two-chain whose boundary is $\gamma_{\mathbf{x},\mathbf{y}}$. (Observe that we have displayed here a simple special case of Lipshitz's formula for the Maslov index of a holomorphic disk in the symmetric product, cf. [12].) The additive indeterminacy is removed by the following convention. Consider the generator \mathbf{x}_0 which occupies the lower left-hand corner of each square which contains a white dot, cf. Figure 4. We declare that $M(\mathbf{x}_0) = 1 - n$.

Consider $C(\Gamma)$, the \mathbb{F} -vector space generated by elements of X . We define a differential

$$\partial: C(\Gamma) \longrightarrow C(\Gamma)$$

by the formula

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in X} \sum_{r \in R_{\mathbf{x}, \mathbf{y}}} \left\{ \begin{array}{ll} 1 & \text{if } P_{\mathbf{x}}(r) + P_{\mathbf{y}}(r) = 1 \text{ and } W(r) = B(r) = 0 \\ 0 & \text{otherwise} \end{array} \right\} \cdot \mathbf{y}.$$

The condition that $P_{\mathbf{x}}(r) + P_{\mathbf{y}}(r) = 1$ and $W(r) = B(r) = 0$ is, of course, equivalent to the condition that the interior of the rectangle r contains no black points, white points, or points amongst the \mathbf{x} and \mathbf{y} .

It is easy to see that ∂ drops Maslov grading by one and preserves Alexander grading. It is also elementary to verify that $\partial^2 = 0$. Thus, we can take the homology of this complex to obtain a bigraded vector space over \mathbb{F} .

Let V be the two-dimensional bigraded vector space spanned by one generator in bigrading $(-1, -1)$ and another in bigrading $(0, 0)$.

We can now state the following:

Theorem 1.1. *Fix a grid presentation Γ of a knot K , with grid number n . Then, the homology of the above chain complex $H_*(C(\Gamma), \partial)$ is isomorphic to the bigraded group $\widehat{\text{HFK}}(K) \otimes V^{\otimes(n-1)}$.*

The key point of the above theorem is to find a suitable Heegaard diagram for S^3 compatible with the knot K . Indeed, the diagram we use has genus one, with Heegaard torus \mathcal{T} , and K is represented as a collection of horizontal and vertical arcs. This Heegaard diagram has the property that the knot pierces \mathcal{T} in several pairs of points, and the very interesting property that the complement in \mathcal{T} of the attaching circles is a collection of squares. In this case, properties of the Maslov index ensure that the only holomorphic disks are rectangles. The chain complex $C(\Gamma)$ we have described above, then, agrees with the Heegaard Floer complex for this diagram.

There are several other variants of Theorem 1.1. There is, for example, a version which calculates $\widehat{\text{HFK}}(K)$ directly (though, of course, it is uniquely determined by the above result), except there one needs to consider a variant of the above the chain complex defined over a suitable polynomial algebra.

We also consider in this paper several other versions of Theorem 1.1. We discuss how to calculate the other variants of knot Floer homology, and also a variant for links.

This paper is organized as follows. In Section 2, we describe the construction of link Floer homology using Heegaard diagrams with the property that the link crosses the Heegaard surface in many points. This construction is then identified with the usual construction using methods from [23]. In Section 3, we identify chain complex $C(\Gamma)$ with the link Floer homology complex using the toroidal grid diagram of L , interpreted as a Heegaard diagram for L , and state some more general consequences. Finally, in Section 4, we describe some simple examples to illustrate our results.

Further remarks. Whereas the constructions in this paper give a purely combinatorial chain complex for knot Floer homology, Theorem 1.1 is still somewhat impractical, as the chain complex $C(\Gamma)$ typically has far too many generators: for a knot with arc index n , the procedure gives a chain complex with $n!$ generators. It remains a very interesting challenge to come up with more efficient methods for calculating the homology of the complexes we describe here.

In a different direction, the relationship between our combinatorial description and Legendrian knots seems tantalizing: one wonders whether this is perhaps the hint of a connection with the holomorphic invariants of those objects, compare [2], [3], [15].

We would like to remind the reader that we have kept the introduction as elementary as possible. The more general results of Section 3 actually lead to a calculation of link Floer homology for links in S^3 . Also, the extra data about the “knot filtration” allows one to calculate the concordance invariant τ for knots. It is also the input needed to determine the ranks of Heegaard Floer homology groups of Dehn surgeries on a given knot K , see [24].

Acknowledgements. This paper grew out of attempts at understanding an earlier preprint by the third author, who made the revolutionary observation that for Heegaard diagrams of a certain special form, the corresponding Heegaard Floer homology groups can be calculated combinatorially.

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2. LINK FLOER HOMOLOGY WITH MULTIPLE BASEPOINTS

We review here the construction of knot and link Floer homology, considering the case where the link meets the Heegaard surface in extra intersection points. The fact that Heegaard Floer homology can be extracted from this picture follows essentially from [23].

Let $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be a Heegaard diagram, where Σ is a surface of genus g , k is some positive integer, $\alpha = \{\alpha_1, \dots, \alpha_{g+k-1}\}$ are pairwise disjoint, embedded curves in Σ which span a half-dimensional subspace of $H_1(\Sigma; \mathbb{Z})$ (and hence specify a handlebody U_α with boundary equal to Σ), $\beta = \{\beta_1, \dots, \beta_{g+k-1}\}$ is another collection of attaching circles specifying U_β , and $\mathbf{w} = \{w_1, \dots, w_k\}$ and $\mathbf{z} = \{z_1, \dots, z_k\}$ are distinct marked points with

$$\mathbf{w}, \mathbf{z} \subset \Sigma - \alpha_1 - \dots - \alpha_{g+k-1} - \beta_1 - \dots - \beta_{g+k-1}.$$

The data (Σ, α, β) specifies a Heegaard splitting for some oriented three-manifold Y . In the present applications, we will be interested in the case where the ambient three-manifold is the three-sphere, and hence, we make this assumption hereafter.

Let $\{A_i\}_{i=1}^k$ resp. $\{B_i\}_{i=1}^k$ be the connected components of $\Sigma - \alpha_1 - \dots - \alpha_{g+k-1}$ resp. $\Sigma - \beta_1 - \dots - \beta_{g+k-1}$.

We suppose that our basepoints are placed in such a manner that each component A_i or B_i contains exactly one basepoint amongst the \mathbf{w} and exactly one basepoint amongst the \mathbf{z} . We can label our basepoints so that A_i contains z_i and w_i , and then B_i contains w_i and $z_{\nu(i)}$, for some permutation ν of $\{1, \dots, k\}$.

In this case, the basepoints uniquely specify an oriented link L in $S^3 = U_\alpha \cup U_\beta$, by the following conventions. For each $i = 1, \dots, k$, let ξ_i denote the arc in A_i from z_i to w_i and let η_i denote the arc in B_i from w_i to $z_{\nu(i)}$. Let $\xi_i \subset U_\alpha$ be an arc obtained by pushing the interior of ξ_i into U_α , and $\tilde{\eta}_i$ be the arc obtained by pushing the interior of η_i into U_β . Now, we can let L be the oriented link obtained as the difference

$$\bigcup_{i=1}^k (\tilde{\xi}_i - \tilde{\eta}_i).$$

Definition 2.1. *In the above case, we say that $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is a $2n$ -pointed Heegaard diagram compatible with the oriented link L in S^3 .*

Let ℓ denote the number of components of L . Clearly, $k \geq \ell$. In the case where $k = \ell$, these are the Heegaard diagrams used in the definition of link Floer homology [23], see also [21], [25]. In the case where $k > \ell$, these Heegaard diagrams can still be used to calculate link Floer homology, in a suitable sense.

Definition 2.2. *A periodic domain is a two-chain of the form*

$$P = \sum_{i=1}^k (a_i \cdot A_i + b_i \cdot B_i)$$

which has zero local multiplicity at all of the $\{w_i\}_{i=1}^k$. A Heegaard diagram is said to be admissible if every non-trivial periodic domain has some positive and negative local multiplicities at some other points.

For expository simplicity, we consider now the case where our link is in fact a knot. The case of links can be handled with a few minor notational changes, as outlined in Subsection 2.1 below.

Let $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be a Heegaard diagram compatible with an oriented knot K .

The *relative Alexander grading* of two intersection points \mathbf{x} and \mathbf{y} is defined by the formula

$$A(\mathbf{x}) - A(\mathbf{y}) = \left(\sum_{i=1}^n n_{z_i}(\phi) \right) - \left(\sum_{i=1}^n n_{w_i}(\phi) \right),$$

where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is any homotopy class from \mathbf{x} to \mathbf{y} . We find it convenient to remove the additive indeterminacy in A : there is a unique choice with the property that

$$\sum_{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} T^{A(\mathbf{x})} \equiv \Delta_K(T) \cdot (1 - T^{-1})^{n-1} \pmod{2},$$

where $\Delta_K(T)$ is the symmetrized Alexander polynomial which could be made to work over \mathbb{Z} by introducing signs. (These conventions are chosen to be consistent with those made in Proposition 2.3 below.)

We consider now the chain complex $CFK^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ over the polynomial algebra $\mathbb{F}[U_1, \dots, U_k]$ which is freely generated by intersection points between the tori $\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_{g+k-1}$ and $\mathbb{T}_\beta = \beta_1 \times \dots \times \beta_{g+k-1}$ in $\text{Sym}^{g+k-1}(\Sigma)$. This module is endowed with the differential

$$(3) \quad \partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi)=1\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) U_1^{n_{w_1}(\phi_1)} \cdot \dots \cdot U_k^{n_{w_k}(\phi_k)} \cdot \mathbf{y},$$

where, as usual, $\pi_2(\mathbf{x}, \mathbf{y})$ denotes the space of homology classes of Whitney disks connecting \mathbf{x} to \mathbf{y} , $\mathcal{M}(\phi)$ denotes the moduli space of pseudo-holomorphic representatives of ϕ , $\mu(\phi)$ denotes its formal dimension (Maslov index), $n_p(\phi)$ denotes the local multiplicity of ϕ at the reference point p (i.e. the algebraic intersection number of ϕ with the subvariety $\{p\} \times \text{Sym}^{g+k-2}(\Sigma)$), and $\#()$ denotes a count modulo two. As usual, in the definition of pseudo-holomorphic disks, one uses a suitable perturbation of the condition on the disk that it be holomorphic with respect to the complex structure on $\text{Sym}^{g+k-1}(\Sigma)$ induced from some complex structure on Σ , as explained in [22, Section 3]; see also [5], [17], [6], [7] for more general discussions. We use here a sufficiently small perturbation to retain the property that if u is pseudo-holomorphic, then for all $p \in \Sigma - \alpha_1 - \dots - \alpha_{g+k-1} - \beta_1 - \dots - \beta_{g+k-1}$,

$n_p(\phi) \geq 0$, cf. [22, Lemma 3.2]. For the case where the Heegaard diagram is admissible, it is easy to see that Equation (3) gives a finite sum, compare [22].

Moreover, there is a *relative Maslov grading*, defined by

$$(4) \quad M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2 \sum_{i=1}^n n_{w_i}(\phi).$$

The relative Maslov grading can be lifted to an absolute grading using the observation that $(\Sigma, \alpha, \beta, \mathbf{w})$ is a multiply-pointed Heegaard diagram for S^3 (a *balanced n -pointed Heegaard diagram* in the terminology of [23]), and consequently, if one sets all the $U_i = 0$, the homology groups of the resulting complex, one obtains a relatively graded group which is isomorphic to $H_*(T^{k-1}; \mathbb{F})$ (compare [23, Theorem 4.5]). The Maslov grading is fixed by the requirement that

$$(5) \quad H_*(CFK^- / \{U_i = 0\}) \cong H_{*+k-1}(T^{k-1}; \mathbb{F}).$$

So far, we have made no reference to the basepoints \mathbf{z} , and indeed, the complex $CF^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ so far is the chain complex for $HF^-(S^3)$ for a multi-pointed Heegaard diagram, in the sense of [23, Section 4.5].

This complex admits an *Alexander filtration* defined by the convention the conventions that any element $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ has Alexander filtration level $A(\mathbf{x})$, and multiplication by the variables U_i drops Alexander filtration by one, i.e.

$$A(U_1^{a_1} \cdots U_k^{a_k} \cdot \mathbf{x}) = A(\mathbf{x}) - a_1 - \dots - a_k.$$

Non-negativity of local multiplicities of pseudo-holomorphic disks ensures that this function indeed defines a filtration on the complex; i.e. we have an increasing sequence of subcomplexes $\mathcal{F}^-(K, m) \subset CFK^-(K)$ indexed by integers m , which are generated over $\mathbb{F}[U_1, \dots, U_k]$ by intersection points \mathbf{x} with $A(\mathbf{x}) \leq m$.

In the case where $k = 1$, the above construction gives the chain homotopy type of the “knot filtration” on $CF^-(S^3)$, called $CFK^-(K)$, which is a chain complex over the polynomial algebra $\mathbb{F}[U]$. In this case, it was shown in [21] and [25] that the filtered chain homotopy type of the complex is a knot invariant. Our goal here is to show that this filtered chain homotopy type is also independent of k .

In practice, it is often more convenient to consider the simpler complex $C(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ generated by intersection points of \mathbb{T}_α and \mathbb{T}_β with coefficients in \mathbb{F} , endowed with the differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\left\{ \phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \begin{array}{l} \mu(\phi) = 1, \\ n_{w_i}(\phi) = n_{z_i}(\phi) = 0 \ \forall i = 1, \dots, n \end{array} \right\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot \mathbf{y}.$$

For this complex, the function A defines an Alexander grading which is preserved by the differential. One can think of $C(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ as obtained from $CFK^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ by first setting all the $U_i = 0$, and then taking the graded object associated to the Alexander filtration.

The following proposition shows how to extract the usual knot Floer homology from the above variants using multiple basepoints. The result is an adaptation of the results from [23, Section 6.1], but we sketch the proof here for the reader’s convenience.

Proposition 2.3. *Let $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be a $2k$ -pointed admissible Heegaard diagram compatible with a knot K . Then, the filtered chain homotopy type of $CFK^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ agrees with the filtered chain homotopy type of $CFK^-(K)$, where the latter is thought of as a chain*

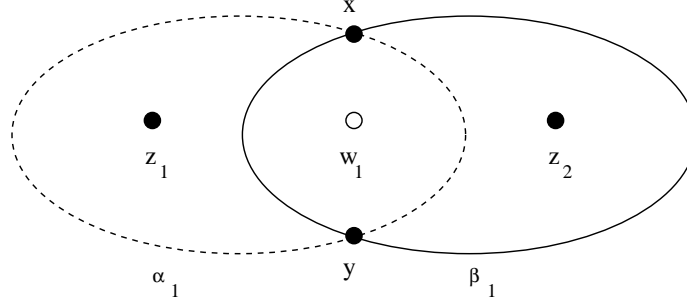


FIGURE 5. **Local picture near z_1 .** We stabilize $(\Sigma, \{\alpha_2, \dots, \alpha_{g-k+1}\}, \{\beta_2, \dots, \beta_{g-k+1}\}, \{p, w_2, \dots, w_k\}, \{z_2, \dots, z_k\})$ by introducing this local picture at p .

complex over $\mathbb{F}[U_1, \dots, U_k]$ by the convention that each U_i acts as U . Moreover, we have an identification

$$(6) \quad H_*(C(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}), \partial) \cong \widehat{HFK}(K) \otimes V^{\otimes(k-1)},$$

where V is the two-dimensional vector space spanned by two generators, one in bigrading $(-1, -1)$, another in bigrading $(0, 0)$.

Proof. Observe that if we fix the number of basepoints k , then the homology, and even the filtered chain homotopy type, of an admissible Heegaard diagram $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ with $2k$ basepoints is independent of the particular choice of diagram, depending only on k . Indeed, we can connect any two such diagrams by a sequence of index one/two stabilizations (and their inverses), handleslides, and isotopies, such that at every step of this process we only encounter admissible Heegaard diagrams. This follows from the same arguments as in [22], [21], [25].

We now claim that the Floer homology transforms as above when the number of basepoints is decreased by two. It suffices to consider a single (admissible) pointed Heegaard diagram with $2k - 2$ basepoints and show that the filtered chain homotopy type of its complex is identified with the filtered chain homotopy type of another (admissible) pointed Heegaard diagram with $2k$ basepoints.

To this end, let $(\Sigma, \{\alpha_2, \dots, \alpha_{g+k-1}\}, \{\beta_2, \dots, \beta_{g+k-1}\}, \{w_2, \dots, w_k\}, \{p, z_2, \dots, z_k\})$ be a $2k - 2$ -pointed Heegaard diagram which represents K . Let C' denote its corresponding filtered Heegaard Floer complex $CFK^-(\Sigma, \{\alpha_2, \dots, \alpha_{g+k-1}\}, \{\beta_1, \dots, \beta_{g+k-1}\}, \{p, w_3, \dots, w_k\}, \{z_2, \dots, z_k\})$, thought of as a polynomial algebra $\mathbb{F}[U_2, \dots, U_k]$. It is generated by the corresponding intersection points X' of the tori \mathbb{T}'_α and \mathbb{T}'_β in $\text{Sym}^{g+k-2}(\Sigma)$.

We can stabilize this by introducing a new pair of attaching circles α_1 and β_1 , replacing the point p by the three basepoints z_1, w_1 , and z_2 as illustrated in Figure 5, to obtain a new $2k$ -pointed Heegaard diagram

$$(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}) = (\Sigma, \{\alpha_1, \dots, \alpha_{g+k-1}\}, \{\beta_1, \dots, \beta_{g+k-1}\}, \{w_1, \dots, w_k\}, \{z_1, \dots, z_k\}).$$

Admissibility of the original diagram clearly implies admissibility of this stabilization.

The set of generators X of the complex $CFK^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ has the form $X' \times \{x, y\}$, where x and y are the two points of intersection of α_1 and β_1 . Let C_x be the subgroup of C generated by intersection points of type $X' \times \{x\}$ and C_y be the subgroup generated by those of type $X' \times \{y\}$.

It is shown in the proof of [23, Proposition 6.5] that for a suitable choice of complex structure on Σ , the chain complex $CFK^-(\Sigma, \alpha, \mathbf{w}, \mathbf{z})$ is identified with the mapping cone of the chain map

$$U_1 - U_2: C'[U_1] \longrightarrow C'[U_1],$$

where the domain is identified with C_x and the range with C_y . More concretely, under the natural identifications of groups $C_x \cong C'[U_1]$, $C_y \cong C'[U_1]$, we have that the differential of C is identified with the matrix

$$\begin{pmatrix} \partial' & U_1 - U_2 \\ 0 & \partial' \end{pmatrix}$$

the Alexander filtration of a generator in $C'[U_1]$ thought of as supported in C_x is one higher than the Alexander filtration of the corresponding element, thought of as supported in C_y .

For the second assertion, view all the U_i as being set to zero, and the above argument shows that

$$C(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}) \cong C(\Sigma', \alpha', \beta' \mathbf{w}', \mathbf{z}') \otimes V;$$

and hence a corresponding identification holds on the level of homology. Iterating this until we remain with two basepoints, we obtain the stated identification. \square

It is perhaps more traditional to consider the filtration of $\widehat{CF}(S^3)$ (rather than CF^-). This filtration is induced from the filtration of $CF^-(S^3)$ by setting $U = 0$. According to Proposition 2.3, this filtration is obtained as the induced filtration of $CFK^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})/U_1$.

2.1. Modifications for links. Recall that knot Floer homology has a generalization to the case of oriented links \vec{L} . For an ℓ -component, oriented link \vec{L} in the three-sphere, this takes the form of a multi-graded theory

$$\widehat{\text{HFL}}(\vec{L}) = \bigoplus_{d \in \mathbb{Z}, h \in \mathbb{H}} \widehat{\text{HFL}}_d(\vec{L}, h),$$

where $\mathbb{H} \cong H_1(S^3 - \vec{L}) \cong \mathbb{Z}^\ell$, with the latter isomorphism induced by an ordering of the link components. We sketch now the changes to be made to the above discussion for the purposes understanding link Floer homology for Heegaard diagrams with extra basepoints, where by “extra” here we mean more than twice ℓ .

Suppose now that $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is a Heegaard diagram compatible with an oriented link \vec{L} in the sense of Definition 2.1.

We find it convenient to label the basepoints keeping track of which link component they belong to. Specifically, suppose L is a link with ℓ components, and for $i = 1, \dots, \ell$, we choose n_i basepoints to lie on the i^{th} component. Letting S be the index set of pairs (i, j) with $i = 1, \dots, \ell$ and $j = 1, \dots, n_i$. We now have basepoints $\{z_{i,j}\}_{(i,j) \in S}$ and $\{w_{i,j}\}_{(i,j) \in S}$.

We can now form the chain complex $CFL^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ defined over $\mathbb{F}[\{U_{i,j}\}_{(i,j) \in S}]$ analogous to the version before, generated by intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, with differential

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi)=1\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot \left(\prod_{(i,j) \in S} U_{i,j}^{n_{w_{i,j}}(\phi)} \right).$$

This complex has a relative Maslov grading as before. It also has a relative Alexander grading which in this case is an ℓ -tuple of integers,

$$A: \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathbb{Z}^\ell,$$

determined up to an overall additive constant by the formula

$$A(\mathbf{x}) - A(\mathbf{y}) = \left(\sum_{j=1}^{n_1} (n_{z_{1,j}}(\phi) - n_{w_{1,j}}(\phi)), \dots, \sum_{j=1}^{n_\ell} (n_{z_{\ell,j}}(\phi) - n_{w_{\ell,j}}(\phi)) \right).$$

The indeterminacy in this case is a little more unpleasant to pin down (i.e. one must go beyond the multi-variable Alexander polynomial, which can be identically zero), but one can do this with the help of Proposition 2.4.

The complex $CFL^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ inherits an Alexander filtration induced by the Alexander multi-grading of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and the convention that $U_{i,j}$ drops the multi-grading by the i^{th} basis vector. In the case where $k = \ell$, the filtered chain homotopy type of $CFL^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ was shown to be a link invariant in [23]; it is the link filtration $CFL^-(\vec{L})$.

The analogue of $C(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ can be defined as well: it is generated by intersection points of \mathbb{T}_α and \mathbb{T}_β over \mathbb{F} , endowed with the differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\left\{ \phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \begin{array}{l} \mu(\phi) = 1, \\ n_{w_{i,j}}(\phi) = n_{z_{i,j}}(\phi) = 0 \ \forall (i,j) \in S \end{array} \right\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot \mathbf{y}.$$

This differential drops Maslov grading by one and preserves the Alexander multi-grading, and hence the homology groups $H_*(C(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}))$ inherit a Maslov grading and an Alexander multi-grading.

Proposition 2.4. *Let $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be a $2k$ -pointed admissible Heegaard diagram compatible with an oriented link \vec{L} , with n_i pairs of basepoints corresponding to the i^{th} component of \vec{L} . Then, there is a filtered chain homotopy equivalence $CFL^-(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ with the usual link filtration $CFL^-(\vec{L})$, viewed as a chain complex over $\mathbb{F}[\{U_{i,j}\}_{(i,j) \in S}]$. Moreover, there are (relative) multi-graded identifications*

$$H(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}) \cong \widehat{HFL}(\vec{L}) \otimes \bigotimes_{i=1}^{\ell} V_i^{\otimes (n_i - 1)},$$

where V_i is the two-dimensional vector space spanned by one generator in Maslov and Alexander gradings zero, and another in Maslov grading -1 and Alexander grading corresponding to minus the i^{th} basis vector.

Proof. This follows readily from the proof of Proposition 2.3. □

3. PROOF OF THEOREM 1.1 AND ITS GENERALIZATIONS

The alert reader will have noticed by now that the toroidal grid diagrams from the introduction are a special cases of the multiply-pointed Heegaard diagrams from Section 2. The Heegaard surface is the torus \mathcal{T} , the α -circles are the horizontal circles, and the β -circles are the vertical ones. The basepoints $\{w_i\}_{i=1}^n$ are the white circles, and $\{z_i\}_{i=1}^n$ are the black ones. Since for each i and j , α_i and β_j in the single point (i, j) , we see that the generators X are, of course, the intersection points of \mathbb{T}_α with \mathbb{T}_β in $\text{Sym}^n(\mathcal{T})$. In our coordinate system, these generators can be thought of as graphs of permutations on n letters. To apply the results from Section 2 apply, we must verify the following:

Lemma 3.1. *The diagram $(\mathcal{T}, \{\alpha_1, \dots, \alpha_n\}, \{\beta_1, \dots, \beta_n\}, \{w_1, \dots, w_n\}, \{z_1, \dots, z_n\})$ is admissible.*

Proof. To this end, we describe the periodic domains. The alpha curves cut out the sphere into n annuli $\{A_i\}_{i=1}^n$, where A_i denotes the region containing z_i . Similarly the beta curves cut it into annuli $\{B_i\}_{i=1}^n$ where B_i contains w_i . The formal differences $A_i - B_i$ span the space of periodic domains. Moreover, in an arbitrary periodic domain

$$D = \sum_{i=1}^{n-1} a_i(A_i - B_i),$$

the local multiplicity in the intersection z_i is $a_i - a_{\nu(i)}$, for some permutation ν of $\{1, \dots, n\}$. It follows at once that if a_i is not identically zero, then at least one of these local multiplicities must be positive, and another must be negative. \square

We can now turn to the following:

Proof of Theorem 1.1. The circles $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n cut up \mathcal{T} into n^2 squares $D_{i,j}$ with $1 \leq i, j \leq n$. According to [22, Proposition 2.15], homology classes of Whitney disks $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ are determined by their underlying two-chain

$$\mathcal{D}(\phi) = \sum_{i,j} a_{ij} D_{i,j},$$

where here $a_{i,j} = n_{p_{i,j}}(\phi)$ for some point $p_{i,j} \in D_{i,j}$. Indeed, if \mathbf{x} and \mathbf{y} correspond to permutations σ and τ , then these induced two-chains are the ones that satisfy the property for all $i = 1, \dots, n$ that

$$\partial(\partial\mathcal{D}(\phi) \cap \alpha_i) = (i, \tau(i)) - (i, \sigma(i)).$$

Given $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, Lipshitz gives a formula for its Maslov index $\mu(\phi)$ in terms of combinatorial data on the Heegaard surface [12]. His formula involves an *Euler measure* for the two-chain D underlying ϕ , and also $P_{\mathbf{x}}(\mathcal{D}(\phi)) + P_{\mathbf{y}}(\mathcal{D}(\phi))$. We do not explain the Euler measure here observing only that it vanishes for regions which can be partitioned onto squares, and hence it vanishes for all homology classes of Whitney disks in our Heegaard diagram. Thus,

$$\mu(\phi) = P_{\mathbf{x}}(\mathcal{D}(\phi)) + P_{\mathbf{y}}(\mathcal{D}(\phi)).$$

In particular the relative Maslov grading from Equation (4) specializes to Equation (2).

We can lift from the relative to the absolute Maslov grading by performing handleslides on the β -circles in our diagram which now are allowed to cross the z_i , to reduce to a diagram which has 2^{n-1} intersection points in $\mathbb{T}_{\alpha} \cap \mathbb{T}'_{\beta}$, i.e. in this case, the homology groups of the chain complex are trivial [18]. The handleslides are performed by successively handlesliding β_i over β_{i+1} for $i = 1, \dots, n-1$, as pictured in Figure 6. It is easy to see now that the generator \mathbf{x}_0 from the introduction can be connected to the bottom-most generator of the new chain complex by a collection of (Maslov index zero) triangles. According to (5), the grading of \mathbf{x}_0 should be $1 - n$.

For the Alexander grading, we proceed as follows. Recall that in Section 2, it is determined up to an overall additive constant by the formula

$$A(\mathbf{x}) - A(\mathbf{y}) = \left(\sum_{i=1}^n n_{z_i}(\phi) \right) - \left(\sum_{i=1}^n n_{w_i}(\phi) \right),$$

where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is any homology class connecting \mathbf{x} to \mathbf{y} . Let $A_0: X \rightarrow \mathbb{Z}$ be defined as minus the sum over $x \in \mathbf{x}$ of the winding number of K around x . It is straightforward to

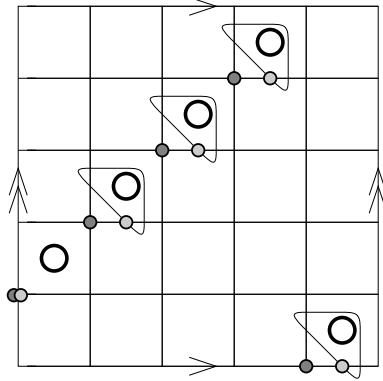


FIGURE 6. **Fixing the Maslov grading.** Handleslide the vertical circles from left to right, to obtain the smaller null-homotopic circles β'_i encircling the various w_i . There is a collection of triangles connecting the generator \mathbf{x}_0 , indicated here with the darkly shaded circles, with the bottom-most generator of $\mathbb{T}_\alpha \cap \mathbb{T}'_\beta$, indicated here with the lightly shaded circles.

see that $A(\mathbf{x}) - A_0(\mathbf{x})$ is independent of the choice of $\mathbf{x} \in X$. Our normalization conventions on A are chosen now so that

$$\sum_{\mathbf{x} \in X} T^{A(\mathbf{x})} \equiv \Delta_K(T) \cdot (1 - T^{-1})^{n-1} \pmod{2},$$

where $\Delta_K(T)$ is a symmetric Laurent polynomial. (Indeed, it is the symmetrized Alexander polynomial.)

To understand the differential, we must count holomorphic disks in $\mathcal{M}(\phi)$ with $\mu(\phi) = 1$. First, we classify all non-negative homology classes ϕ with Maslov index one.

Let $D = \mathcal{D}(\phi)$. First observe that if ∂D is 0 on $(n-1)\alpha$ circles, then it is in fact 0 on all the α circles, and D is generated by the annular regions cut out by the β circles. Now, if such a thing happens then $\mathbf{x} = \mathbf{y}$, and its Maslov index is even.

Thus we can assume that ∂D is non-zero on at least two α circles (say α_{j_1} and α_{j_2}) and similarly non-zero on at least two β circles (β_{i_1} and β_{i_2}). But if ∂D is non-zero on β_i , then $P_{\mathbf{x}}(D) \geq p_{(i, \sigma(i))}(D) \geq \frac{1}{4}$ and $P_{\mathbf{y}}(D) \geq p_{(i, \tau(i))}(D) \geq \frac{1}{4}$. For equality to hold throughout, $\partial D|_{\beta_i}$ has to be a single arc joining $(i, \sigma(i))$ and $(i, \tau(i))$. In fact, since $Mas(\phi) = 1$, equality must hold throughout. So ∂D is non-zero precisely on $\alpha_{j_1}, \alpha_{j_2}, \beta_{i_1}$ and β_{i_2} , and D is one of the two rectangles with four vertices $(i_1, j_1), (i_1, j_2), (i_2, j_1)$, and (i_2, j_2) . Without loss of generality, assume $\sigma(i_1) = j_1$ and $\sigma(i_2) = j_2$. Then $\tau(i_1) = j_2$ and $\tau(i_2) = j_1$, and it agrees with σ on the rest of the values. Also for the Maslov index requirement, $(i, \sigma(i)) = (i, \tau(i))$ does not lie in the interior of D for any other i .

Thus, we have established that the only $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with non-negative local multiplicities and Maslov index equal to one are those whose underlying domain r is a rectangle of the form $r \in R_{\mathbf{x}, \mathbf{y}}$ with $P_{\mathbf{x}}(r) + P_{\mathbf{y}}(r) = 1$. Moreover, we claim that in this case, the number of pseudo-holomorphic representatives of r is odd. In fact, this can be seen by elementary complex analysis, using a (classical) complex structure on the symmetric product of \mathcal{T} , where one shows that in fact the moduli space consists of a single representative. Indeed, for this choice, the moduli space $\mathcal{M}(r)/\mathbb{R}$ can be seen to correspond to involutions of r (with the complex structure it inherits from \mathcal{T}) which switch opposite sides of the rectangle. It

is a simple exercise in conformal geometry that for any rectangle, there is a unique such involution.

We have thus completed the verification that the complex $C(\Gamma)$ from the introduction coincides with the Heegaard Floer complex $C(\mathcal{T}, \alpha, \beta, \mathbf{w}, \mathbf{z})$ in the notation of Section 2. Theorem 1.1 now follows directly from Proposition 2.3 (Equation (6)). \square

3.1. Other variants. There are other variants of Theorem 1.1, which should be clear from the constructions thus far. We state several of them for completeness.

Label the white dots $\{w_1, \dots, w_n\}$, and let $n_{w_i}(r)$ denote the local multiplicity of r at w_i . Consider the chain complex $C^-(\Gamma)$ over the algebra $\mathbb{F}[U_1, \dots, U_n]$ also generated by X , endowed with the differential

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in X} \sum_{r \in R_{\mathbf{x}, \mathbf{y}}} \left\{ \begin{array}{ll} 1 & \text{if } P_{\mathbf{x}}(r) + P_{\mathbf{y}}(r) = 1 \\ 0 & \text{otherwise} \end{array} \right\} U_1^{n_{w_1}(r)} \cdot \dots \cdot U_n^{n_{w_n}(r)} \cdot \mathbf{y},$$

thought of as a filtered chain complex where the filtration level of each generator $\mathbf{x} \in X$ is its Alexander grading, and multiplication by the variable U_i drops filtration level by one.

Theorem 3.2. *Fix a grid presentation Γ of a knot K , with grid number n . The filtered chain homotopy type of K coincides with the filtered chain homotopy type of the knot filtration $CF^-(S^3, K)$.*

Proof. The proof of Theorem 1.1 identifies the filtered chain complex $C^-(\Gamma)$ with the complex denoted $CFK^-(\mathcal{T}, \alpha, \beta, \mathbf{w}, \mathbf{z})$ in Section 2 which, by Proposition 2.3, is identified with $CFK^-(K)$. \square

Of course, the other filtrations $CFK^\infty(S^3, K)$ and $CFK^+(S^3, K)$ from [21] can be extracted from this information.

We call attention to another other construction, which gives a concordance invariant $\tau(K)$ for knots [19], [25]. This is a homomorphism from the smooth concordance group of knots to the integers, which can be used to bound the four-ball genus of knots, giving an alternate proof of the theorem of Kronheimer and Mrowka [11] confirming Milnor's conjecture for the unknotting numbers of torus knots. This feature underscores its similarity with Rasmussen's concordance invariant $s(K)$ [26] from Khovanov homology [10]. However, these two invariants are known to be linearly independent [9].

Recall that the filtration $CFK^-(K)$ of $CF^-(S^3)$ induces also a filtration $\{\widehat{\mathcal{F}}_m(S^3)\}_{m \in \mathbb{Z}}$ of $\widehat{CF}(S^3) = CF^-(S^3)/U \cdot CF^-(S^3)$. The concordance invariant $\tau(K)$ is by definition the minimal $m \in \mathbb{Z}$ with the property that the map

$$H_*(\widehat{\mathcal{F}}_m(S^3)) \longrightarrow \widehat{HF}(S^3) \cong \mathbb{F}$$

is non-trivial.

We have a corresponding chain complex $\widehat{C}(\Gamma) = CF^-(\Gamma)/U_1 \cdot CF^-(\Gamma)$; i.e. whose differential is given by

$$\widehat{\partial} \mathbf{x} = \sum_{\mathbf{y} \in X} \sum_{r \in R_{\mathbf{x}, \mathbf{y}}} \left\{ \begin{array}{ll} 1 & \text{if } P_{\mathbf{x}}(r) + P_{\mathbf{y}}(r) = 1 \text{ and } n_{w_1}(r) = 0 \\ 0 & \text{otherwise} \end{array} \right\} U_2^{n_{w_2}(r)} \cdot \dots \cdot U_n^{n_{w_n}(r)} \cdot \mathbf{y},$$

This is equipped with subcomplexes $\widehat{F}(K, m) \subset \widehat{C}(\Gamma)$, generated by elements $U_2^{a_2} \cdot \dots \cdot U_n^{a_n} \cdot \mathbf{x}$ with integral $a_i \geq 0$, and

$$A(\mathbf{x}) - a_2 - \dots - a_n \leq m.$$

Corollary 3.3. *The concordance invariant $\tau(K)$ is the minimal m for which the map induced on homology*

$$i_*: H_*(\widehat{\mathcal{F}}(K, m)) \longrightarrow H_*(\widehat{C}(\Gamma))$$

is non-trivial.

Proof. Theorem 3.2 actually gives an identification of the filtered chain homotopy type of $\widehat{CFK}(K)$ with $\widehat{C}(\Gamma)$. The result then follows from the definition of $\tau(K)$. \square

For link Floer homology, we have another version. For this, we must number our dots $\{w_{i,j}\}_{(i,j) \in S}$ and $\{z_{i,j}\}_{(i,j) \in S}$ where S is the index set consisting of (i, j) with $i = 1, \dots, \ell$ and $j = 1, \dots, n_i$, and the dots $w_{i,j}$ and $z_{i,j}$ lie on the i^{th} component of \vec{L} .

In this case, the Alexander grading is an ℓ -tuple of integers. It is uniquely characterized by the property that for $\mathbf{x}, \mathbf{y} \in X$, the i^{th} component of the Alexander grading is the winding number of $\gamma_{\mathbf{x}, \mathbf{y}}$ about the sum of the black dots in $\{z_{i,j}\}_{(i,j) \in S}$ minus its winding number around $\{w_{i,j}\}_{(i,j) \in S}$. Again, this can be more succinctly recorded by placing (minus one times) a vector of winding numbers at each vertex, and defining $A(\mathbf{x})$ as the sum of these local contributions at each intersection point in \mathbf{x} . This can then be renormalized to be symmetric.

Once again, we have the chain complex $C(\Gamma)$ as defined in the introduction, which now inherits an ℓ -tuple Alexander gradings and a single Maslov grading.

There is also a refinement, $C^-(\Gamma)$, which is freely generated by X over $\mathbb{F}[\{U_{i,j}\}_{(i,j) \in S}]$, endowed with the differential

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in X} \sum_{r \in R_{\mathbf{x}, \mathbf{y}}} \left\{ \begin{array}{ll} 1 & \text{if } P_{\mathbf{x}}(r) + P_{\mathbf{y}}(r) = 1 \\ 0 & \text{otherwise} \end{array} \right\} \cdot \left(\prod_{(i,j) \in S} U_{i,j}^{n_{w_{i,j}}(r)} \right).$$

Theorem 3.4. *There are multi-graded identifications*

$$H_*(C(\Gamma), \partial) \cong \widehat{HFL}(\vec{L}) \otimes \bigotimes_{i=1}^{\ell} V_i^{\otimes (n_i-1)},$$

where V_i is the two-dimensional vector space spanned by two generators, one in zero Maslov and Alexander multigradings, and the other in Maslov grading one and Alexander multigrading corresponding to minus the i^{th} basis vector. More generally, the multi-filtered chain homotopy type of $CFL^-(S^3, \vec{L})$ is identified with the multi-filtered chain homotopy type of $C^-(\Gamma)$.

Proof. Both follow from the proof of Theorem 1.1, combined with Proposition 2.4. \square

4. EXAMPLES

We give a few elementary illustrations of our results.

4.1. Hopf link. Consider the grid presentation Γ for the Hopf link with grid number $n = 4$, shown in Figure 7.

The generators of the chain complex $C(\Gamma)$ are in one-to-one correspondence with permutations σ of the set $\{1, 2, 3, 4\}$. For conciseness, we write the generator consisting of the intersections of the i th horizontal circle with the $\sigma(i)$ th vertical circle as $(\sigma(1)\sigma(2)\sigma(3)\sigma(4))$.

There are eight empty squares in the grid. Each of them produces differentials between generators that differ by a transposition, according to the recipe:

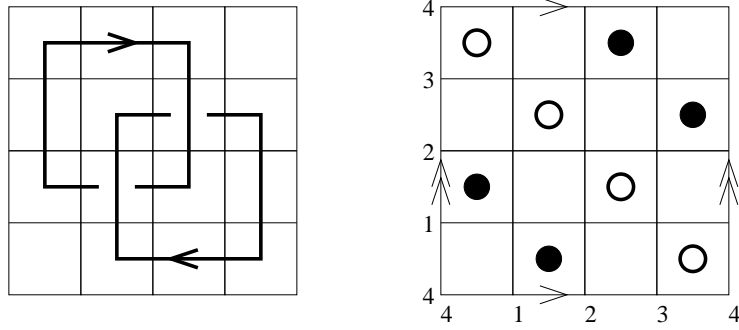
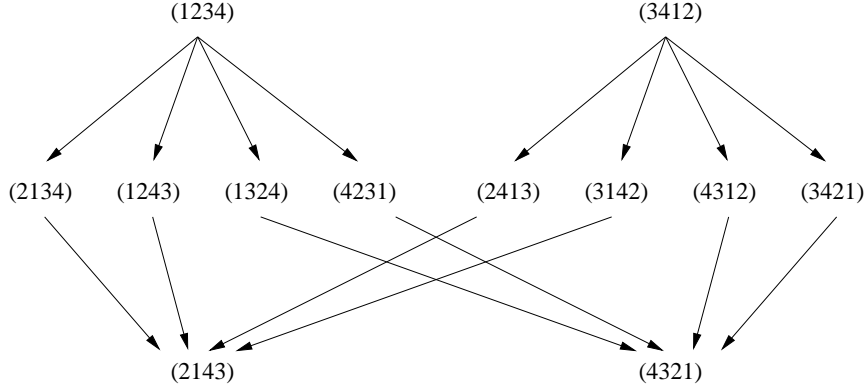


FIGURE 7. Grid presentation of the Hopf link.

FIGURE 8. Part of the chain complex for the Hopf link. This complex appears in Alexander bigrading $(-\frac{1}{2}, -\frac{1}{2})$. Its homology has rank four.

$$\begin{array}{cccc}
 (1 \ 2 \ * \ *) & (* \ 2 \ 3 \ *) & (* \ * \ 3 \ 4) & (1 \ * \ * \ 4) \\
 \downarrow & \downarrow & \downarrow & \downarrow \\
 (2 \ 1 \ * \ *) & (* \ 3 \ 2 \ *) & (* \ * \ 4 \ 3) & (4 \ * \ * \ 1) \\
 \\
 (3 \ 4 \ * \ *) & (* \ 4 \ 1 \ *) & (* \ * \ 1 \ 2) & (3 \ * \ * \ 2) \\
 \downarrow & \downarrow & \downarrow & \downarrow \\
 (4 \ 3 \ * \ *) & (* \ 1 \ 4 \ *) & (* \ * \ 2 \ 1) & (2 \ * \ * \ 3).
 \end{array}$$

The result is that there sixteen differentials in $(C(\Gamma), \partial)$. They connect twelve of the 24 generators, as shown schematically in Figure 8. Each of the other twelve generators is not connected by differentials to any other generators. Therefore, the homology of our complex has total rank $4 + 12 = 16$.

The Alexander bigrading of the generators is computed using the adaptation for links of Equation (1). For example,

$$A(3214) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad A(2143) = \left(-\frac{1}{2}, -\frac{1}{2}\right).$$

To compute the Maslov gradings, we start with the canonical generator (2143) , which has $M = -3$. Then, we relate each of the other generators to the canonical one by a sequence of transpositions. Whenever two generators \mathbf{x} and \mathbf{y} differ by a transposition, if a two-chain D

has boundary $\gamma_{\mathbf{x},\mathbf{y}}$, then D consists of two points and a rectangle, and it is straightforward to apply Equation (2). For example,

$$M(2134) = M(2143) + 1 = -2, \quad M(2314) = M(2134) + 1 = -1, \text{ etc.}$$

The result is that

$$H_*(C(\Gamma), \partial) = \left(V_1^{\otimes 2} \otimes V_2^{\otimes 2} \right) \left[\frac{1}{2}, \frac{1}{2} \right].$$

Here, the notation $[i, j]$ denotes an upward shift in Alexander bigrading, i.e. if V is a bigraded vector space, then $(V[i, j])_{x,y} = V_{x-i, y-j}$.

The link Floer homology of \vec{H} is $(V_1 \otimes V_2)[\frac{1}{2}, \frac{1}{2}]$, cf. [23]. This confirms that

$$H_*(C(\Gamma), \partial) = \widehat{\text{HFL}}(\vec{H}) \otimes V_1 \otimes V_2.$$

4.2. The trefoil. Consider the grid presentation of the trefoil knot shown in Section 1. There are, of course, 120 generators of the chain complex. A quick glance at Figure 1 reveals 15 rectangles containing no black or white dot: fifteen 1×1 , five 2×1 , and five 1×2 . Each rectangle gives rise to $3! = 6$ different differentials. With a little computer assistance or a great deal of patience, one finds that the homology of this complex has rank 48. Indeed, with the conventions used in Subsection 4.1, one finds that the generators correspond to permutations (listed in increasing Alexander grading):

$$\begin{array}{cccccccc} (23451) & (13452) & (23415) & (23541) & (24351) & (32451) & (13542) & (14352) \\ (24153) & (24315) & (25413) & (32415) & (32541) & (35421) & (42351) & (43152) \\ (43521) & (15342) & (15423) & (25143) & (31542) & (32514) & (35241) & (41352) \\ (42153) & (42315) & (43125) & (45321) & (52341) & (52413) & (54312) & (15243) \\ (15324) & (31524) & (41325) & (42135) & (51342) & (51423) & (52143) & (52314) \\ (54132) & (54213) & (15234) & (41235) & (51243) & (51324) & (52134) & (51234). \end{array}$$

These generators have Alexander gradings between -5 and 1 ; there are $1, 5, 11, 14, 11, 5, 1$ generators in gradings $-5, -4, -3, -2, -1, 0, 1$, respectively. The knot Floer homology group for the left-handed trefoil T is non-trivial in only three Alexander-Maslov bigradings $(-1, 0)$, $(0, 1)$, and $(1, 2)$, and it has rank one in these three bigradings. Considering Maslov gradings as well, one immediately verifies that

$$H_*(C(\Gamma)) \cong \widehat{\text{HFK}}(T) \otimes V^{\otimes 4}.$$

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