PROBLEMS FOR THE CATEGORY THEORY READING COURSE, 2017

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1. Assignment 1, due end of Week 4

1.1. Functors.

(1) Leinster 1.2.21 (functors preserve isomorphisms)
(2) Leinster 1.2.27, 1.2.28b (full, faithful)

1.2. Natural transformations.

(1) In fdVec, show that the functors id and $**$ are naturally isomorphic.
(2) Show the the vertical composition of two natural transformations is in fact a natural transformation.
(3) Prove carefully that the horizontal composition of two natural transformations is again a natural transformation.
(4) Show that a functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories if and only if it is *fully faithful* and *essentially surjective*. Clearly state where you are using the axiom of choice, or add hypotheses so it is unnecessary.

1.3. Universal properties.

(1) Prove that two initial objects in a category are isomorphic.
(2) For each of the following categories, decide whether there is an initial, final, and/or zero object, and if so, describe them: FinSet, fdVec, Top, Top_* (pointed topological spaces), field extensions of a fixed field F, Graphs (your answer may depend on which class of graphs you consider), Semigroups, Groups.
(3) Describe the product of two objects as the terminal object in some category.
(4) Describe the tensor product of two vectors spaces as the initial object in some category.
(5) Describe both the product and coproduct in the following categories: FinSet, Top, Top_*, AbGroup, Group, Graphs.

1.4. Adjunctions.

(1) Consider the forgetful functor from abelian groups to groups. What is its left adjoint?
(2) In the category of finite dimensional vector spaces, show that $- \otimes V$ is biadjoint to $- \otimes V^*$.
(3) Prove that the 'hom-set isomorphism' and 'unit/counit' definitions of an adjunction are equivalent.

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2. Assignment 2, due at the end of Week 7, will consist of at least:

2.1. Idempotent completion. The 'idempotent completion' $\mathrm{Kar}(\mathcal{C})$ (also called the 'Karoubi envelope') is defined as follows:

$$\mathrm{Obj}\,\mathrm{Kar}(\mathcal{C}) = \left\{(X \in \mathrm{Obj}(\mathcal{C}), p : X \to X) \mid |p^2 = p\right\}$$
$$\mathrm{Kar}(\mathcal{C})((X,p) \to (X',p')) = \{f \in \mathcal{C}(X \to X') \mid |fp = f = p'f\}.$$

(1) Let primeVec denote the full subcategory of Vec consisting of vector spaces with prime dimensions. Show carefully that $\mathrm{Kar}(\mathrm{primeVec}) \cong \mathrm{Vec}$.
(2) Construct an equivalence $\iota_{\mathcal{C}} : \mathrm{Kar}(\mathrm{Kar}(\mathcal{C})) \cong \mathrm{Kar}(\mathcal{C})$.
(3) Show that there is a fully faithful functor $\mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ given by $X \mapsto (X, 1_X)$.
(4) Note that given a functor $F : \mathcal{C} \to \mathcal{D}$, there is functor $\mathrm{Kar}(F) : \mathrm{Kar}(\mathcal{C}) \to \mathrm{Kar}(\mathcal{D})$ given by

$$\mathrm{Kar}(F)(X,p) = (F(X), F(p))$$
$$\mathrm{Kar}(F)(f : (X,p) \to (X',p')) = F(f).$$

(You might say that Kar is a 2-functor from CAT to itself — what does this mean at the level of natural transformations?) Given a functor $F : \mathrm{Kar}(C) \to \mathrm{Kar}(D)$, show that it is determined up to natural isomorphism by its restriction $F|_C : C \to \mathrm{Kar}(D)$, by showing $F \cong (\iota_{\mathcal{D}} \circ \mathrm{Kar}(F|_C))$.
(5) (Not on the problem set: there is a forgetful 2-functor CAT $\to$ SemiCat, the 2-category of 'semicategories' (categories without identities) and their functors and natural transformations. The idempotent completion gives a 2-functor SemiCAT $\to$ CAT. Are they an adjoint pair?)

2.2. Limits.

(1) Recall that given a functor $F : \mathcal{J} \to \mathcal{C}$, the *limit* of F, written $\lim_{\mathcal{J}} F$ is a terminal object in the category of cones over F. Explicitly, a cone consists of
  (a) an object $X \in \mathrm{Obj}\,\mathcal{C}$,
  (b) for each $j \in \mathrm{Obj}\,\mathcal{C}$, a map $f_j : X \to F(j)$,
  (c) such that for any $g : j \to j'$, $F(g) \circ f_j = f_{j'}$.

Another way of saying this is that the category of cones is the comma category $(J \downarrow F)$, where here we interpret $\mathcal{J}$ as a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{J} \to \mathcal{C})$ by $c \mapsto (j \mapsto c)$ and we interpret F as a functor $1 \to \mathrm{Fun}(\mathcal{J} \to \mathcal{C})$ by $1 \mapsto (j \mapsto F(j))$. Explain carefully why these are talking about the same thing!

2.3. The Yoneda embedding.

(1) Prove the Yoneda lemma.
(2) Explain why $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathcal{D}) = \mathrm{Fun}(\mathcal{C} \to \mathcal{D}^{\mathrm{op}})$. Prove that $\mathrm{Set}^{\mathrm{op}} \ncong \mathrm{Set}$, but that $\mathrm{fdVec}^{\mathrm{op}} \cong \mathrm{fdVec}$.

3. Possible problems for later:

3.1. More on adjoints.

(1) Recall the $C - \mathrm{Set}$ is the category of functors from C to Set. Given a functor $F : C \to \mathcal{D}$, we have the pull-back functor $F^* : \mathcal{D} - \mathrm{Set} \to C - \mathrm{Set}$ given by precomposition by F. If F^* has adjoints, we call them the right push-forward F_* and the left pushforward $F_!$ (pronounced usually 'F-shriek').

(You may like to read https://arxiv.org/abs/1009.1166.)

  (a) [[Calculate some examples.]]

  (b) Show that the polynomial functors, namely those of the form $F_! G^* H_* : \mathcal{E} - \mathrm{Set} \to \mathcal{B} - \mathrm{Set}$ for some diagram

$$\mathcal{B} \xleftarrow{F} C \xrightarrow{G} \mathcal{D} \xleftarrow{H} \mathcal{E},$$

  are closed under composition. (You may assume that all categories are finitely presented, i.e. the path category of some finite graph modulo finitely many relations. You may like to look at https://ncatlab.org/nlab/show/polynomial+functor, although as is often the case at the *nLab*, the presentation there is more general than we need.)

3.2. Abelian categories.

3.3. Monoidal categories.

(1) Prove that every monoidal category is monoidally equivalent to a strict monoidal category.

(2) Find an example of a monoidal functor which is not naturally isomorphic to any strict monoidal functor.

(3) Let C be a monoidal category. We say a 'monoid object' in C (or, as we gain confidence, just a monoid in C) is a tuple $(A \in \mathrm{Obj}\, C, \iota : 1 \to A, m : A \otimes A \to A)$ satisfying some conditions. Look up, or work out, what these conditions should be. You should be able to show that a monoid object in Vec is what is usually called an associative unital algebra.

  (a) A 'module object' for a monoid object $A \in C$ is a tuple $(M, \triangleright : A \otimes M \to M)$ satisfying an appropriate condition (what is it?). A morphism f between module objects M and M' is a morphism between the underlying objects, such that $f \circ \triangleright_M = \triangleright_{M'} \circ (1_A \otimes f)$. Draw the string diagram corresponding to this axiom. Define composition of module morphisms, by imitating the definition for modules over a ring. Show that modules for a fixed monoid object form a category.

(4) Show that $\mathrm{Rep}G$, for G a finite group, forms a monoidal category.

(5) If $\mathrm{Rep}G \cong \mathrm{Rep}H$, as categories, are G and H isomorphic? What about if $\mathrm{Rep}G \cong \mathrm{Rep}H$ as monoidal categories? (Hint: think about the forgetful functor vector spaces, and its automorphisms.)

3.4. Enriched categories.

(1)

3.5. A problem involving Yoneda, and rigid monoidal categories.

(1) Show if $F : C \to \mathcal{D}$ is a fully faithful functor, then $f \in C(X \to Y)$ is a monomorphism if and only if $F(f)$ is. (Similarly for epimorphisms.)

(2) Use this, and the Yoneda embedding, so show that in a rigid abelian category, the functor $- \otimes X$ is exact. (See Proposition 2.1.8 of Bakalov-Kirillov if you need some help; they don't explain how they are using Yoneda, however!)

3.6. **Braided monoidal categories.**

(1) Explain how any object X in a pivotal braided category gives an oriented link invariant.
(2) Describe how the Temperley-Lieb category has the structure of a pivotal braided category.
(3) Calculate the invariant of the trefoil corresponding to the object 1 in Temperley-Lieb. (What is this invariant usually called?)
(4) Show that in $\mathrm{Kar}(TL)$, we have $(2, 1_2) \cong (2, f^{(2)}) \oplus (0, 1_0)$. (Here $f^{(2)}$ denotes the second Jones-Wenzl idempotent.)
(5) Calculate the invariant of the unknot corresponding to the object $(2, f^{(2)})$ in (the idempotent completion of) Temperley-Lieb.
(6) We say a monoid (A, m) in a braided monoidal category is commutative if $m \circ \beta = m$. Define a monoidal structure on the category of modules for a commutative monoid.

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