

# PROBLEMS FOR THE CATEGORY THEORY READING COURSE, 2017

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## 1. ASSIGNMENT 1, DUE END OF WEEK 4

### 1.1. Functors.

- (1) Leinster 1.2.21 (functors preserve isomorphisms)
- (2) Leinster 1.2.27, 1.2.28b (full, faithful)

### 1.2. Natural transformations.

- (1) In  $\mathbf{fdVec}$ , show that the functors  $\text{id}$  and  $**$  are naturally isomorphic.
- (2) Show the the vertical composition of two natural transformations is in fact a natural transformation.
- (3) Prove carefully that the horizontal composition of two natural transformations is again a natural transformation.
- (4) Show that a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is part of an equivalence of categories if and only if it is *fully faithful* and *essentially surjective*. Clearly state where you are using the axiom of choice, or add hypotheses so it is unnecessary.

### 1.3. Universal properties.

- (1) Prove that two initial objects in a category are isomorphic.
- (2) For each of the following categories, decide whether there is an initial, final, and/or zero object, and if so, describe them:  $\mathbf{FinSet}$ ,  $\mathbf{fdVec}$ ,  $\mathbf{Top}$ ,  $\mathbf{Top}_*$  (pointed topological spaces), field extensions of a fixed field  $F$ ,  $\mathbf{Graphs}$  (your answer may depend on which class of graphs you consider),  $\mathbf{Semigroups}$ ,  $\mathbf{Groups}$ .
- (3) Describe the product of two objects as the terminal object in some category.
- (4) Describe the tensor product of two vectors spaces as the initial object in some category.
- (5) Describe both the product and coproduct in the following categories:  $\mathbf{FinSet}$ ,  $\mathbf{Top}$ ,  $\mathbf{Top}_*$ ,  $\mathbf{AbGroup}$ ,  $\mathbf{Group}$ ,  $\mathbf{Graphs}$ .

### 1.4. Adjunctions.

- (1) Consider the forgetful functor from abelian groups to groups. What is its left adjoint?
- (2) In the category of finite dimensional vector spaces, show that  $- \otimes V$  is biadjoint to  $- \otimes V^*$ .
- (3) Prove that the ‘hom-set isomorphism’ and ‘unit/counit’ definitions of an adjunction are equivalent.

## 2. ASSIGNMENT 2, DUE AT THE END OF WEEK 7:

**2.1. Idempotent completion.** The ‘idempotent completion’  $\text{Kar}(C)$  (also called the ‘Karoubi envelope’) is defined as follows:

$$\begin{aligned}\text{Obj Kar}(C) &= \{(X \in \text{Obj}(C), p : X \rightarrow X) \mid p^2 = p\} \\ \text{Kar}(C)((X, p) \rightarrow (X', p')) &= \{f \in C(X \rightarrow X') \mid fp = f = p'f\}.\end{aligned}$$

- (1) Let  $\text{primeVec}$  denote the full subcategory of  $\text{Vec}$  consisting of vector spaces with prime dimensions. Show carefully that  $\text{Kar}(\text{primeVec}) \cong \text{Vec}$ .
- (2) Construct an equivalence  $\iota_C : \text{Kar}(\text{Kar}(C)) \cong \text{Kar}(C)$ .
- (3) Show that there is a fully faithful functor  $C \rightarrow \text{Kar}(C)$  given by  $X \mapsto (X, 1_X)$ .
- (4) Note that given a functor  $F : C \rightarrow \mathcal{D}$ , there is functor  $\text{Kar}(F) : \text{Kar}(C) \rightarrow \text{Kar}(\mathcal{D})$  given by

$$\begin{aligned}\text{Kar}(F)(X, p) &= (F(X), F(p)) \\ \text{Kar}(F)(f : (X, p) \rightarrow (X', p')) &= F(f).\end{aligned}$$

(You might say that  $\text{Kar}$  is a 2-functor from  $\text{CAT}$  to itself – what does this mean at the level of natural transformations?) Given a functor  $F : \text{Kar}(C) \rightarrow \text{Kar}(D)$ , show that it is determined up to natural isomorphism by its restriction  $F|_C : C \rightarrow \text{Kar}(D)$ , by showing  $F \cong (\iota_D \circ \text{Kar}(F|_C))$ .

- (5) (Not on the problem set: there is a forgetful 2-functor  $\text{CAT} \rightarrow \text{SemiCat}$ , the 2-category of ‘semicategories’ (categories without identities) and their functors and natural transformations. The idempotent completion gives a 2-functor  $\text{SemiCat} \rightarrow \text{CAT}$ . Are they an adjoint pair?)

### 2.2. Limits.

- (1) Recall that given a functor  $F : \mathcal{J} \rightarrow C$ , the *limit* of  $F$ , written  $\lim_{\mathcal{J}} F$  is a terminal object in the category of cones over  $F$ . Explicitly, a cone consists of
  - (a) an object  $X \in \text{Obj } C$ ,
  - (b) for each  $j \in \text{Obj } \mathcal{J}$ , a map  $f_j : X \rightarrow F(j)$ ,
  - (c) such that for any  $g : j \rightarrow j'$ ,  $F(g) \circ f_j = f_{j'}$ .

Another way of saying this is that the category of cones is the comma category  $(J \downarrow F)$ , where here we interpret  $\mathcal{J}$  as a functor  $C \rightarrow \text{Fun}(\mathcal{J} \rightarrow C)$  by  $c \mapsto (j \mapsto c)$  and we interpret  $F$  as a functor  $1 \rightarrow \text{Fun}(\mathcal{J} \rightarrow C)$  by  $1 \mapsto (j \mapsto F(j))$ . Explain carefully why these are talking about the same thing!

### 2.3. The Yoneda embedding.

- (1) Prove the Yoneda lemma.
- (2) Explain why  $\text{Fun}(C^{\text{op}} \rightarrow \mathcal{D}) = \text{Fun}(C \rightarrow \mathcal{D}^{\text{op}})$ . Prove that  $\text{Set}^{\text{op}} \not\cong \text{Set}$ , but that  $\text{fdVec}^{\text{op}} \cong \text{fdVec}$ .

### 3. ASSIGNMENT 3, DUE MAY 12:

- (1) Prove that every monoidal category is monoidally equivalent to a strict monoidal category.  
(Hint: given a monoidal category  $C$ , define a new monoidal category  $\text{List}C$ , whose objects are lists of objects in  $C$ . In  $\text{List}C$ , tensor product of objects is concatenation of lists, and the tensor unit is the empty list. There should be a functor  $\text{List}C \rightarrow C$  defined by sending a list  $[x_1, x_2, \dots, x_n]$  to  $((1 \otimes x_1) \otimes x_2) \otimes \dots \otimes x_n$ . Your job is to describe what happens at the level of morphisms, and check that everything works.)
- (2) Let  $C$  be a monoidal category. We say a ‘monoid object’ in  $C$  (or, as we gain confidence, just a monoid in  $C$ ) is a tuple  $(A \in \text{Obj } C, \iota : 1 \rightarrow A, m : A \otimes A \rightarrow A)$  satisfying some conditions. Look up, or work out, what these conditions should be. (Hint: look at chapter 7 of Etingof, Gelaki, Nikshych, and Ostrik’s book [Tensor Categories](#).) You should be able to show that a monoid object in  $\text{Vec}$  is what is usually called an associative unital algebra.
  - (a) A ‘module object’ for a monoid object  $A \in C$  is a tuple  $(M, \triangleright : A \otimes M \rightarrow M)$  satisfying an appropriate condition (what is it?). A morphism  $f$  between module objects  $M$  and  $M'$  is a morphism between the underlying objects, such that  $f \circ \triangleright_M = \triangleright_{M'} \circ (1_A \otimes f)$ . Draw the string diagram corresponding to this axiom. Define composition of module morphisms, by imitating the definition for modules over a ring. Show that modules for a fixed monoid object form a category.
- (3) Show that  $\text{Rep}G$ , for  $G$  a finite group, forms a monoidal category.
- (4) (For this part, you may assume we are looking at representations over the complex numbers.)
  - If  $\text{Rep}G \cong \text{Rep}H$ , as categories, are  $G$  and  $H$  isomorphic?  
(Hint: no, give a counterexample — any pair of non-isomorphic groups with the same number of irreducible representations (equivalently, the same number of conjugacy classes) will do.)
  - What about if  $\text{Rep}G \cong \text{Rep}H$  as monoidal categories, and moreover this equivalence is compatible with the forgetful functors to  $\text{Vec}$ ?  
(Hint: think about the monoidal automorphisms of the forgetful functor. You should prove that (not necessarily monoidal) automorphisms of the forgetful functor  $\text{Rep}G \rightarrow \text{Vec}$  is a group isomorphic to  $\mathbb{C}[G]^\times$ , and then identify the subgroup of monoidal automorphisms is  $G$  is itself. To do the first part, show that any such automorphism is determined by its component on the regular representation  $\mathbb{C}[G]$ ; for this you may like to use that the regular representation is faithful, or (more or less equivalently) that there is a surjective  $G$ -linear map from the regular representation to any irreducible representation of  $G$ .)

#### 4. ASSIGNMENT 4, DUE MAY 26:

##### 4.1. Abelian categories.

- (1) Show if  $F : C \rightarrow D$  is a fully faithful functor, then  $f \in C(X \rightarrow Y)$  is a monomorphism if and only if  $F(f)$  is. (Similarly for epimorphisms.)
- (2) Use this, and the Yoneda embedding, to show that in a rigid abelian category, the functor  $- \otimes X$  is exact. (See Proposition 2.1.8 of Bakalov-Kirillov, attached, if you need some help; they don't explain how they are using Yoneda, however! You may cite the literature for any lemmas you like.)

##### 4.2. Braided monoidal categories.

- (1) Explain how any object  $X$  in a pivotal braided category gives an oriented link invariant.
- (2) Describe how the Temperley-Lieb category has the structure of a pivotal braided category.
- (3) Calculate the invariant of the trefoil corresponding to the object 1 in Temperley-Lieb. (What is this invariant usually called?)
- (4) Show that in  $\text{Kar}(TL)$ , we have  $(2, 1_2) \cong (2, f^{(2)}) \oplus (0, 1_0)$ . (Here  $f^{(2)}$  denotes the second Jones-Wenzl idempotent.)
- (5) Calculate the invariant of the unknot corresponding to the object  $(2, f^{(2)})$  in (the idempotent completion of) Temperley-Lieb.
- (6) We say a monoid  $(A, m)$  in a braided monoidal category is commutative if  $m \circ \beta = m$ . Define a monoidal structure on the category of modules for a commutative monoid.

LEMMA 2.1.6. *Suppose that  $V$  has a dual  $V^*$ . Then there exist canonical isomorphisms*

$$(2.1.13) \quad \text{Hom}(U \otimes V, W) = \text{Hom}(U, W \otimes V^*),$$

$$(2.1.14) \quad \text{Hom}(U, V \otimes W) = \text{Hom}(V^* \otimes U, W).$$

PROOF. To  $\psi \in \text{Hom}(U \otimes V, W)$  we associate the composition

$$U \xrightarrow{\text{id} \otimes i_V} U \otimes V \otimes V^* \xrightarrow{\psi \otimes \text{id}} W \otimes V^*$$

which is an element of  $\text{Hom}(U, W \otimes V^*)$ .

Similarly, to  $\varphi \in \text{Hom}(U, W \otimes V^*)$  we assign

$$U \otimes V \xrightarrow{\varphi \otimes \text{id}} W \otimes V^* \otimes V \xrightarrow{\text{id} \otimes e_V} W.$$

One can easily check that these two maps are inverse to each other, establishing (2.1.13). The proof of (2.1.14) is similar.  $\square$

In particular, if both  $U$  and  $V$  have duals, then by Lemma 2.1.6

$$(2.1.15) \quad \text{Hom}(U, V) = \text{Hom}(V^*, U^*) = \text{Hom}(\mathbf{1}, V \otimes U^*).$$

(In the language of abstract nonsense, this means that the category  $\mathcal{C}$  has internal Hom's when it has duals.) For  $f \in \text{Hom}(U, V)$  its image in  $\text{Hom}(V^*, U^*)$  via the isomorphism (2.1.15) will be denoted by  $f^*$ .

If the right dual  $*$  exists for all objects in  $\mathcal{C}$ , then by (2.1.15) it is a contravariant functor, i.e., a functor  $\mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$  where  $\mathcal{C}^{\text{op}}$  is the opposite (or dual) category to  $\mathcal{C}$ . (Recall that  $\mathcal{C}^{\text{op}}$  has the same objects as  $\mathcal{C}$  but with all arrows reversed.)

EXERCISE 2.1.7. Show that, in a rigid category,  $*$  is an equivalence of categories  $\mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ .

Rigidity is a very restrictive requirement. As an illustration, let us prove the following proposition.

PROPOSITION 2.1.8. *In an abelian rigid monoidal category the tensor product functor  $\otimes$  is exact, i.e., for any short exact sequence  $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$  and an object  $X$ , the sequences*

$$0 \rightarrow U \otimes X \rightarrow V \otimes X \rightarrow W \otimes X \rightarrow 0$$

and

$$0 \rightarrow X \otimes U \rightarrow X \otimes V \rightarrow X \otimes W \rightarrow 0$$

are exact.

PROOF. The sequence

$$0 \rightarrow U \otimes X \rightarrow V \otimes X \rightarrow W \otimes X$$

is exact iff

$$0 \rightarrow \text{Hom}(Y, U \otimes X) \rightarrow \text{Hom}(Y, V \otimes X) \rightarrow \text{Hom}(Y, W \otimes X)$$

is exact for any object  $Y$ . But by (2.1.12, 2.1.13),  $\text{Hom}(Y, U \otimes X) = \text{Hom}(Y \otimes {}^*X, U)$ . Since the functor  $\text{Hom}(Y, -)$  is left exact, it follows that  $- \otimes X$  is left exact. Using Exercise 2.1.7 (or repeating the same argument with duals), we see that it is also right exact.  $\square$

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