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Assignment 2

2.1.1 We want to show that $Kar(\text{prime Vec}) \cong Vec$. It suffices to find a functor, $F: Kar(\text{prime Vec}) \to Vec$ that is full, faithful and essentially surjective on objects by Proposition 1.3.18.

Let us first understand the category $Kar(\text{prime Vec})$. We have $ob(Kar(\text{prime Vec})) = \{(V \in Obj(\text{prime Vec}), p: X \Rightarrow X) \mid p^2 = p\}$ and $Kar(\text{prime Vec})((V,p) \to (W,q)) = \{f \in \text{prime Vec}(V \to W) \mid fp = f = qf\}$. If p is an idempotent on a vector space V, then $V \cong pV \oplus (1-p)V$ as $x \in V$, we can write $x = px + (1-p)x$ and $pV \cap (1-p)V = \{0\}$ since $p(1-p) = 0$. Besides, $f = fp = qf$ implies f is zero on $(1-p)V$ and its image lies entirely in qW since
$$\begin{aligned} qW &= q^2W \oplus q(1-q)W \\ &= qf(pV \oplus (1-p)V) && \text{by construction} \\ &= fp(pV \oplus (1-p)V) && \text{by assumption} \\ &= f(pV)! \end{aligned}$$
So, f is really a map from pV to qW.

Next, we define a functor, $F: Kar(\text{prime Vec}) \to Vec$, $(V,p) \mapsto pV$ on objectwise and $F: Kar(\text{prime Vec})((V,p) \to (W,q)) \to Vec(pV \to qW)$, $f \mapsto f$ on morphism wise, which is well-defined as explained in the previous paragraph.

It is functorial by construction. To see this, if we have $(V,p) \xrightarrow{f} (W,q) \xrightarrow{g} (X,r)$, then $F(g \circ f) = g \circ f = F(g) \circ F(f)$ and for $(V,p) \xrightarrow{I_{(V,p)}} (V,p)$, we get
$$\begin{aligned} F(I_{(V,p)}) &= (pV \xrightarrow{I_{(V,p)}} pV) \\ &= (F(V,p) \xrightarrow{I_{F(V,p)}} F(V,p)) \\ &= I_{F(V,p)}. \end{aligned}$$

Furthermore, it is full because given $(V,p), (W,q) \in Kar(\text{prime Vec})$, $f \in Vec(F(V,p), F(W,q))$, we get $f: pV \to qW$. Since p and q are idempotents, f satisfies $f = fp = qf$ as $fp(pV) = f(p^2V) = f(pV) = qW$ and $qf(pV) = qqW = qW$ and $f \in \text{prime Vec}(V,W)$ as $p: V \to V$ and $q: W \to W$.

To see it is faithful, suppose $f_1 = f_2$ in $Vec(F(V,p), F(W,q))$.

But, by fullness, $f_1, f_2 : (V,p) \to (W,q)$ and $f_1 = f_2$ by assumption.

Finally, it is essentially surjective on objects. If $V \in Vec$ has prime dimension, then we have $F(V, I_V) = I_V V = V$ which is a perfectly fine element in ob Kar(prime Vec) since $I_V I_V = I_V$ and $I_V : V \to V$ with $V \in$ prime Vec. If $0 \in Vec$ has zero dimension, pick any $W \in$ prime Vec and $0_W : W \to 0$, the unique map sending W to zero vector space, we get $F(W, 0_W) = 0_W W = 0$. If $V \in Vec$ has dimension n which is not prime and is not zero, we can pick $W \in$ prime Vec with prime dimension $m > n$ and p, the projection map from m-dimensional vector space to n-dimensional vector space which clearly satisfies $p^2 = p$ such that $F(W,p) = pW = W^n \cong V$.

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2.1.2 We want to construct an equivalence $\iota_{\mathcal{C}} : \mathrm{Kar}(\mathrm{Kar}(\mathcal{C})) \simeq \mathrm{Kar}(\mathcal{C})$. By Proposition 1.3.18, it suffices to find a functor that is full, faithful and essentially surjective on objects.

Let us first understand the category $\mathrm{Kar}(\mathrm{Kar}(\mathcal{C}))$. We have $ob(\mathrm{Kar}(\mathrm{Kar}(\mathcal{C}))) = \{((X,p) \in ob(\mathrm{Kar}(\mathcal{C})), f : (X,p) \to (X,p)) \mid f^2 = f\}$ and $\mathrm{Kar}(\mathrm{Kar}(\mathcal{C}))(((X,p),f) \to ((X',p'),f')) = \{\Phi \in \mathrm{Kar}(\mathcal{C})((X,p) \to (X',p')) \mid \Phi f = \Phi = f'\Phi\}$

Now, we define
$$\iota_{\mathcal{C}} : \mathrm{Kar}(\mathrm{Kar}(\mathcal{C})) \to \mathrm{Kar}(\mathcal{C})$$
$$((X,p),f) \mapsto (X,f)$$
on object wise and
$$\iota_{\mathcal{C}} : \mathrm{Kar}(\mathrm{Kar}(\mathcal{C})) \to \mathrm{Kar}(\mathcal{C})$$
$$\Phi \mapsto \Phi$$
on morphism wise. We see that it is well-defined on object since $X \in ob(\mathcal{C})$, $f \in \mathcal{C}(X \to X)$, and $f^2 = f$ as $f \in \mathrm{Kar}(\mathcal{C})((X,p) \to (X,p))$. It is also well-defined on morphism since $\Phi \in \mathcal{C}(X \to X')$ and $\Phi f = \Phi = f'\Phi$ by the definition of Φ. It is functorial by construction. To see this, if we have $((X,p),f) \xrightarrow{\Phi} ((X',q'),f') \xrightarrow{\Psi} ((X'',q''),f'')$, then $\iota_{\mathcal{C}}(\Psi \circ \Phi) = \Psi \circ \Phi = \iota_{\mathcal{C}}(\Psi) \circ \iota_{\mathcal{C}}(\Phi)$
$= \iota_{\mathcal{C}}(\Phi) \circ \iota_{\mathcal{C}}(\Psi)$

. For $((X,p),f) \xrightarrow{1_{((X,p),f)}} ((X,p),f))$, then $\iota_{\mathcal{C}}(1_{((X,p),f)})$
$$= ((X,f) \xrightarrow{1_{((X,p),f)}} (X,f))$$
$$= \iota_{\mathcal{C}}((X,p),f) \xrightarrow{1_{\iota_{\mathcal{C}}((X,p),f)}} \iota_{\mathcal{C}}((X,p),f)$$
$$= 1_{\iota_{\mathcal{C}}((X,p),f))}$$

Next, we see that it is full because given $((X,p),f), ((X',p'),f') \in \mathrm{Kar}(\mathrm{Kar}(\mathcal{C}))$, $\Phi \in \mathrm{Kar}(\mathcal{C})((X,f) \to (X',f')))$, then $\iota_{\mathcal{C}}(\Phi) = \Phi$ by construction since we can get $\Phi \in \mathrm{Kar}\,\mathrm{Kar}(\mathcal{C})(((X,p),f) \to ((X',p'),f'))$ because $\Phi p = \Phi f p = \Phi f = \Phi = f'\Phi = p'f'\Phi = p'\Phi$ as $\Phi f = f'\Phi = \Phi$, $fp = pf = f$, and $p'f' = f'p' = f'$.

To see it is faithful, consider $((X,p),f), ((X',p'),f') \in \mathrm{Kar}(\mathrm{Kar}(\mathcal{C}))$, $\Phi_1 = \Phi_2$ in $\mathrm{Kar}(\mathcal{C})((X,f) \to (X',f'))$. But, by fullness, Φ_1, Φ_2 are in $\mathrm{Kar}\,\mathrm{Kar}(\mathcal{C})(((X,p),f) \to ((X',p'),f'))$ and $\Phi_1 = \Phi_2$ by assumption.

Finally, it is essentially surjective on object. Given

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$(X, f) \in Kar(\mathcal{C})$, we can find $((X,f), f)$ such that $\mathcal{C}((X,f),f) = (X, f)$ and $((X,f),f)$ is a perfectly fine element in ob $Kar\,Kar(\mathcal{C})$ since $(X,f) \in Kar(\mathcal{C})$, $f \in \mathcal{C}(X \to X)$, $f^2 = f$, and so $f : (X,f) \to (X,f)$.

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Assignment 2

2.1.3 We want to show there is a fully faithful functor, $F: \mathcal{C} \to Kar(\mathcal{C})$.
$X \mapsto (X, I_X)$

First, we see that F is well-defined on object wise since $X \in ob\,\mathcal{C}$, $I_X : X \to X$, and $I_X I_X = I_X$.
We also define, for any $p: X \to X'$ in $\mathcal{C}$, $F(p) = p : (X, I_X) \to (X', I_{X'})$, which is well-defined since $p \in \mathcal{C}(X, X')$ and $pI_X = p = I_{X'}p$.

Next, F is full because given $X, X' \in \mathcal{C}$ and $p : (X, I_X) \to (X', I_{X'})$, we get $F(p) = p$. The reason is such p must satisfy the condition being a morphism in $\mathcal{C}$, that is, $p \in \mathcal{C}(X, X')$

Finally, F is faithfull because given $p_1 = p_2$ in $Kar(\mathcal{C})((X, I_X) \to (X', I_{X'}))$, we automatically get $p_1 = p_2$ in $\mathcal{C}(X \to X')$ by assumption.

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2.1.4 Given a functor $F: \mathcal{C} \to D$, there is a functor $Kar(F): Kar(\mathcal{C}) \to Kar(D)$ given by
$$Kar(F)(X,p) = (F(X), F(p))$$
$$Kar(F)(f:(X,p) \to (X',p')) = F(f).$$
We see it is well-defined on object wise because $F(X) \in ob\, D$, $F(p): F(X) \to F(X)$ as $p: X \to X$ and F is a functor, as well as
$$\begin{aligned}(F(p))^2 &= F(p)F(p) \\ &= F(p^2) && \text{as } F \text{ is a functor,} \\ &= F(p), && \text{as } p \text{ is an idempotent,}\end{aligned}$$
so $F(p)$ is an idempotent. Moreover, we see that $F(f) \in Kar(\mathcal{C})((F(X), F(p)) \to (F(X'), F(p')))$ since F is a functor, $F(f) \in D(F(X) \to F(X'))$, and
$$\begin{aligned}F(f)F(p) &= F(fp) && \text{as } F \text{ is a functor,} \\ &= F(f) && \text{by assumption,} \\ &= F(p'f) && \text{by assumption,} \\ &= F(p')F(f) && \text{as } F \text{ is a functor,}\end{aligned}$$
as desired.

Now, given a functor $F: Kar(\mathcal{C}) \to Kar(D)$, we want to show that it is determined up to natural isomorphism by its restriction $F|_{\mathcal{C}}: \mathcal{C} \to Kar(D)$, by proving $F \cong (\iota_D \circ Kar(F|_{\mathcal{C}}))$.

Note that $\mathcal{C}$ is a full subcategory of $Kar(\mathcal{C})$ by $X \mapsto (X, id_X)$ as it is deemed as an embedding, by Corollary 1.3.19.

By lemma 1.3.11, it suffices to find an isomorphism $\alpha_{(X,p)}: F(X,p) \to (\iota_D \circ Kar(F|_{\mathcal{C}}))(X,p)$ for all $(X,p) \in$ ~~$Kar(\mathcal{C})$~~ $Kar(\mathcal{C})$ and show the naturality axiom holds.

Note that
$$\begin{aligned}(\iota_D \circ Kar(F|_{\mathcal{C}}))(X,p) &= \iota_D(F|_{\mathcal{C}}(X), F|_{\mathcal{C}}(p)) \\ &= \iota_D(F(X, 1_X), F(p)) \\ &= (\pi_1(F(X, 1_X)), F(p))\end{aligned}$$
where $\pi_1(F(X, 1_X))$ is the underlying object of $F(X, 1_X)$ and $F|_{\mathcal{C}}(p) = F(p)$ because $p: X \to X$ defines $p: (X, 1_X) \to (X, 1_X)$ as $p 1_X = 1_X p = p$.

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Assignment 2

2.1.4 We claim that $\alpha^{-1}_{(X,p)} := F((X, 1_X) \xrightarrow{p} (X, p))$. This is because $F(p) : F(X, 1_X) \to F(X, p)$, that is, $F(p) : \pi_1(F(X, 1_X)) \to \pi_1 F(X, p)$. Similarly, $\alpha_{(X,p)} := F((X,p) \xrightarrow{p} (X, 1_X))$ as $F(p) : \pi_1(F(X,p)) \to \pi_1 F(X, 1)$. Note that, on morphism wise, $F((X, 1_X) \xrightarrow{p} (X, 1_X) \xrightarrow{p} (X, p)) = F((X, 1_X) \xrightarrow{p^2=p} (X,p))$ $= F((X, 1_X) \xrightarrow{p} (X, p) \xrightarrow{p} (X, p))$ and $F((X,p) \xrightarrow{p} (X,p) \xrightarrow{p} (X, 1_X))$ $= F((X,p) \xrightarrow{p^2=p} (X, 1_X)) = F((X, p) \xrightarrow{p} (X, 1_X) \xrightarrow{p} (X, 1_X))$ which implies $\alpha^{-1}_{(X,p)}$ and $\alpha_{(X,p)}$ are indeed morphisms in $Kar(D)$.

Then, $\alpha^{-1}_{(X,p)} \circ \alpha_{(X,p)} = F((X, 1_X) \xrightarrow{p} (X, p)) \circ F((X, p) \xrightarrow{p} (X, 1_X))$

$= F((X, 1_X) \xrightarrow{p} (X, p) \xrightarrow{p} (X, 1_X))$ as F is a functor, functor,

$= F((X, 1_X) \xrightarrow{p^2=p} (X, 1_X))$

$= id_{F(X, 1_X)}$

and $\alpha_{(X,p)} \circ \alpha^{-1}_{(X,p)} = F((X,p) \xrightarrow{p} (X, 1_X)) \circ F((X, 1_X) \xrightarrow{p} (X, p))$

$= F((X,p) \xrightarrow{p} (X, 1_X) \xrightarrow{p} (X, p))$

$= F((X,p) \xrightarrow{p^2=p} (X, p))$

$= id_{F(X,p)}$

which show that they are inverses to each other.

To see that it satisfies the naturality axiom, consider a morphism $g : (X, p) \to (Y, q)$ and the commutativity square

$$\begin{tikzcd}
(\pi_1 F(X, 1_X), F((X, 1_X) \xrightarrow{p} (X, 1_X))) \arrow[r, "L_D Kar(F|_{\mathcal{C}})(g) := F(g)"] \arrow[d, "F((X, 1_X) \xrightarrow{p} (X, p))"'] & (\pi_1 F(Y, 1_Y), F((Y, 1_Y) \xrightarrow{g} (Y, 1_Y))) \arrow[d, "F((Y, 1_Y) \xrightarrow{q} (Y, q))"] \\
F(X, p) \arrow[r, "F(g)"] & F(Y, q)
\end{tikzcd}$$

commutes since $F((X, 1_X) \xrightarrow{p} (X, p) \xrightarrow{g} (Y, q))$

$= F((X, 1_X) \xrightarrow{g} (Y, 1_Y) \xrightarrow{q} (Y, q))$

which is true by Karoubi envelope relation, that is, $gp = qg = g$ and $g \in \mathcal{C}(X, Y)$ implies $g \in Kar(\mathcal{C})((X, 1_X) \to (X, 1_Y))$ as $g 1_X = 1_Y g = g$.

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Assignment 2

2.2.1 Given a functor $F: J \to \mathcal{C}$, the limit of F $\lim_J F$ is a terminal object in the category of cones over F. Explicitly, a cone consists of

(a) an object $X \in \mathrm{Obj}\,\mathcal{C}$

(b) for each $j \in \mathrm{Obj}\,J$, a map $f_j : X \to F(j)$,

(c) such that for any $g: j \to j'$, $F(g) \circ f_j = f_{j'}$

We claim that the category of cones is the comma category $(J \downarrow F)$, where we interpret $J \in [\mathcal{C}, [J, \mathcal{C}]]$ and $F \in [\mathcal{I}, [J, \mathcal{C}]]$, so that it makes sense in the comma category.

To do this, we define $\tilde{F}(\cdot) := F$ and $\tilde{F}(\overset{id}{\circlearrowright}) := id_F$. We also define, for all $C \in \mathcal{C}$, $\tilde{J}(C) := (j \mapsto C)$ and for all $g \in J(j, j')$, $\tilde{J}(C)(g) := (g: j \to j' \mapsto id_C)$

$$\begin{tikzcd} & \mathcal{I} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "J"'] & {[J,\mathcal{C}]} \end{tikzcd}$$

The objects are pairs $(C \in \mathcal{C},\ f: \tilde{J}(C) \to F)$

Then, a map $(C, f) \to (C', f')$ in $J \downarrow F$ is a map $q: C \to C'$ in $\mathcal{C}$ making the triangle

$$\begin{tikzcd} \tilde{J}(C) \arrow[rr, "\tilde{J}(q)"] \arrow[dr, "f"'] & & \tilde{J}(C') \arrow[dl, "f'"] \\ & F & \end{tikzcd}$$

commutes.

Since f is a natural transformation, then for every $g: j \to j'$ in J, the square

$$\begin{tikzcd} C = \tilde{J}(C)(j) \arrow[r, "\tilde{J}(C)(g) = id_C"] \arrow[d, "f_j"'] & \tilde{J}(C)(j') = C \arrow[d, "f_{j'}"] \\ F(j) \arrow[r, "F(g)"'] & F(j') \end{tikzcd}$$

commutes. Then, this corresponds to the components in the cone since for any $g: j \to j'$, $F(g) \circ f_j = f_{j'}$, and for all $j \in \mathrm{Obj}\,J$ and an $X \in \mathrm{Obj}\,\mathcal{C}$, a map $f_j : X \to F(j)$.

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To make $\tilde{J}(q)$ a natural transformation, we define for all $j \in J$, $\tilde{J}(q)_j = q$. Then, for every map $g: j \to j'$ in J, the square

$$\begin{tikzcd}
c = \tilde{J}(c)(j) \arrow[r, "\tilde{J}(c)(g) = id_c"] \arrow[d, "\tilde{J}(q)_j = q"'] & \tilde{J}(c)(j') = c \arrow[d, "\tilde{J}(q)_{j'} = q"] \\
c' = \tilde{J}(c')(j) \arrow[r, "\tilde{J}(c')(g) = id_{c'}"'] & \tilde{J}(c')(j') = c'
\end{tikzcd}$$

commutes because $q \circ id_c = q = id_{c'} \circ q$.

So, the first triangle gives us for all $j \in J$ and a $\cancel{q: c \to}$ $q: c \to c'$,

$$\begin{tikzcd}
c = \tilde{J}(c)(j) \arrow[rr, "\tilde{J}(q)_j = q"] \arrow[dr, "f_j"'] & & \tilde{J}(c')(j) = c' \arrow[dl, "f_j'"] \\
& F(j) &
\end{tikzcd}$$

the relation $q \circ f_j' = f_j$ which is the condition for a morphism $(c, (f_j)_{j \in J}) \to (c', (f_j')_{j \in J})$ in cone.

In short, all the condition in comma category $(J \downarrow F)$ corresponds to the criteria satisfied by a cone.

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Assignment 2.

2.3.1 We want to prove the Yoneda lemma.

The Yoneda lemma states if $\mathscr{A}$ is a locally small category, then $[\mathscr{A}^{op}, Set](H_A, X) \cong X(A)$ naturaly in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{op}, Set]$.

The idea is to define a bijection between the sets $[\mathscr{A}^{op}, Set](H_A, X)$ and $X(A)$ for all A and X. Then, we need to show that the bijection is natural in A and X.

First, pick $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{op}, Set]$. We then define ~~functions~~ functions $[\mathscr{A}^{op}, Set](H_A, X) \overset{(\hat{\ })}{\underset{(\tilde{\ })}{\rightleftarrows}} X(A)$ and show that they are mutually inverse.

Given $f: H_A \to X$, we define $\hat{f} \in X(A)$ by $\hat{f} := f_A(1_A)$. Let $\xi \in X(A)$. We have to define a natural transformation $g^{\xi}: H_A \to X$ such that for each $C \in \mathscr{A}$, we get $g^{\xi}_C : H_A(C) = \mathscr{A}(C, A) \to X(C)$ where the family $g^{\xi} = (g^{\xi}_C)_{C \in \mathscr{A}}$ satisfies naturality.

Pick $C \in \mathscr{A}$ and $\varphi \in \mathscr{A}(C, A)$, define $\underbrace{g^{\xi}_C(\varphi)}_{\tilde{\xi}_C(\varphi)} := (X(\varphi))(\xi) \in X(C)$ since $X(\varphi)$ is a map from $X(A) \to X(C)$. To see the naturality holds, consider any map $\theta : C_2 \to C_1$ in $\mathscr{A}$, $\varphi : C_1 \to A$, the ~~square~~ square

$$\begin{tikzcd}
\mathscr{A}(C_1, A) \arrow[r, "H_A(\theta) = -\circ\theta"] \arrow[d, "g^{\xi}_{C_1}"] & \mathscr{A}(C_2, A) \arrow[d, "g^{\xi}_{C_2}"] \\
X(C_1) \arrow[r, "X(\theta)"] & X(C_2)
\end{tikzcd}$$

$$\begin{tikzcd}
\varphi \arrow[r, mapsto] \arrow[d, mapsto] & H_A(\theta)(\varphi) = \varphi\circ\theta \arrow[d, mapsto] \\
(X(\varphi))(\xi) \arrow[r, mapsto] & {[X(\varphi\circ\theta)](\xi)} \quad X(\theta)((X(\varphi)(\xi)))
\end{tikzcd}$$

commutes since $X(\varphi\circ\theta)(\xi) = (X\theta \circ X\varphi)(\xi) = X\theta(X\varphi(\xi))$ by the ~~functorialit~~ functoriality.

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of contravariant functore X.

Now, given $\xi \in X(A)$, we have to show $\hat{\tilde{\xi}} = \xi$.
Indeed, $\xi \mapsto \tilde{\xi} = g^{\xi} \mapsto \hat{g^{\xi}} = g^{\xi}_A(1_A) = X(1_A)(\xi)$
$= 1_{X(A)}(\xi)$ by functoriality of X
$= \xi$ since $\xi \in$ ~~$X A$~~ $X(A)$.

Next, given any $f: H_A \to X$, we have to show that $\tilde{\hat{f}} = f$. It suffices to show that $(\tilde{\hat{f}})_C = f_C$ for all $C \in \mathscr{A}$ because two natural transformations are equal if and only if all their components are equal. But, it all boils down to show that $(\tilde{\hat{f}})_C(\varphi) = f_C(\varphi)$ for every $C \in \mathscr{A}$ and $\varphi: C \to A$ in $\mathscr{A}$ since two functions from $H_A(C) = \mathscr{A}(C, A)$ to $X(C)$ are equal if and only if they take same values at every element of the domain.

By definition, $(\tilde{\hat{f}})_C(\varphi) = g^{\hat{f}}_C(\varphi) = X(\varphi)(\hat{f}) = X(\varphi)(f_A(1_A))$.
To see $X(\varphi)(f_A(1_A)) = f_C(\varphi)$, the only tool at our disposal is the naturality of f. Consider the commutative square

$$\begin{tikzcd}
\mathscr{A}(A, A) \arrow[r, "H_A(\varphi) = _\circ\varphi"] \arrow[d, "f_A"'] & \mathscr{A}(C, A) \arrow[d, "f_C"] \\
X(A) \arrow[r, "X(\varphi)"'] & X(C)
\end{tikzcd}$$

$$1_A \longmapsto 1_A \circ \varphi = \varphi$$

$$f_A(1_A) \longmapsto X(\varphi)(f_A(1_A)) = f_C(\varphi)$$

and it gives $X(\varphi)(f_A(1_A)) = f_C(\varphi)$, as desired.

Finally, we want to show that the bijection is natural in A and X. By Exercise 1.3.29, it is natural in the pair (A, X) if and only if it is natural in A for each fixed X and natural in X for each fixed A. By Lemma 1.3.11, it is enough to check the naturality of $(\hat{\ })$.

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Assignment 2

2.3.1 Naturality in A states that for each $X \in [\mathscr{A}^{op}, \mathbf{Set}]$ and $\varphi : C \to A$ in $\mathscr{A}$, the squares

$$
\begin{tikzcd}
{[\mathscr{A}^{op}, \mathbf{Set}](H_A, X)} \arrow[r, "- \circ H_\varphi"] \arrow[d, "(\hat{\ })"] & {[\mathscr{A}^{op}, \mathbf{Set}](H_C, X)} \arrow[d, "(\hat{\ })"] \\
X(A) \arrow[r, "X(\varphi)"] & X(C)
\end{tikzcd}
$$

$$f \longmapsto f \circ H_\varphi$$

$$f_A(1_A) \longmapsto X(\varphi)(f_A(1_A)) = (f \circ H_\varphi)_C(1_C)$$

has to commute which is true because

$$
\begin{aligned}
(f \circ H_\varphi)_C(1_C) &= f_C((H_\varphi)_C(1_C)) && \text{by definition of composition in } [\mathscr{A}^{op}, \mathbf{Set}], \\
&= f_C(\varphi \circ 1_C) && \text{by definition of } H_\varphi, \\
&= f_C(\varphi) \\
&= X(\varphi)(f) && \text{by naturality of } f.
\end{aligned}
$$

Furthermore, naturality in X states that for each $A \in \mathscr{A}$ and map
$\begin{tikzcd} \mathscr{A}^{op} \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "X'"'{name=D}] & \mathbf{Set} \arrow[from=U, to=D, Rightarrow, "\theta"] \end{tikzcd}$
in $[\mathscr{A}^{op}, \mathbf{Set}]$, the square

$$
\begin{tikzcd}
{[\mathscr{A}^{op}, \mathbf{Set}](H_A, X)} \arrow[r, "\theta \circ -"] \arrow[d, "(\hat{\ })"] & {[\mathscr{A}^{op}, \mathbf{Set}](H_A, X')} \arrow[d, "(\hat{\ })"] \\
X(A) \arrow[r, "\theta_A"] & X'(A)
\end{tikzcd}
$$

$$f \longmapsto \theta \circ f$$

$$f_A(1_A) \longmapsto \theta_A(\alpha_A(1_A)) = (\theta \circ f)_A(1_A)$$

has to commute which is simply true as $(\theta \circ f)_A = \theta_A \circ f_A$ by definition of composition in $[\mathscr{A}^{op}, \mathbf{Set}]$, so the square does commute.

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In addition, we can check naturality of $(\tilde{\ })$ in A. For each $X \in [\mathscr{A}^{op}, Set]$ and $\varphi : C \to A$ in $\mathscr{A}$, we need to show the square below ~~commut~~ commutes,

$$\begin{tikzcd}
{[\mathscr{A}^{op}, Set](H_A, X)} \arrow[r, "- \circ H_\varphi"] & {[\mathscr{A}^{op}, Set](H_C, X)} \\
X(A) \arrow[u, "(\tilde{\ })"] \arrow[r, "X\varphi"] & X(C) \arrow[u, "(\tilde{\ })"']
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{\xi} = g^{\xi} \arrow[r, mapsto] & g^{\xi} \circ H_\varphi \\
 & g^{X(\varphi)\xi} \\
\xi \arrow[uu, mapsto] \arrow[r, mapsto] & X(\varphi)\xi \arrow[u, mapsto]
\end{tikzcd}$$

To see the two natural transformation agree, use the same argument before, for all $B \in \mathscr{A}$, $f : B \to C$, we get

$$\begin{aligned}
(g^{\xi} \circ H_\varphi)_B(f) &= (g^{\xi}_B)(H_\varphi)_B(f) && \text{by definition of composition in } [\mathscr{A}^{op}, Set] \\
&= g^{\xi}_B(\varphi \circ f) && \text{by definition of } H_\varphi \\
&= X(\varphi \circ f)(\xi) && \text{by definition} \\
&= X(f) \circ X(\varphi)\xi && \text{since } X \text{ is a presheaf} \\
&= g_B^{X(\varphi)\xi}(f) && \text{by definition of } g_B^{X(\varphi)\xi}
\end{aligned}$$

Moreover, naturality in X states that for each $A \in \mathscr{A}$ and map $\begin{tikzcd} \mathscr{A}^{op} \arrow[r, bend left, "X"] \arrow[r, bend right, "X'"'] \arrow[r, phantom, "\theta \Downarrow"] & Set \end{tikzcd}$ in $[\mathscr{A}^{op}, Set]$, the square

$$\begin{tikzcd}
{[\mathscr{A}^{op}, Set](H_A, X)} \arrow[r, "\theta \circ -"] & {[\mathscr{A}^{op}, Set](H_A, X')} \\
X(A) \arrow[u, "(\hat{\ })"] \arrow[r, "\theta_A"] & X'(A) \arrow[u, "(\hat{\ })"']
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{\xi} = g^{\xi} \arrow[r, mapsto] & \theta \circ g^{\xi} \\
 & g^{\theta_A(\xi)} \\
\xi \arrow[uu, mapsto] \arrow[r, mapsto] & \theta_A(\xi) \arrow[u, mapsto]
\end{tikzcd}$$

commutes. To see $\theta \circ g^{\xi} = g^{\theta_A(\xi)}$, take any $B \in \mathscr{A}$, $f : B \to A$,

$$\begin{aligned}
(\theta \circ g^{\xi})_B(f) &= \theta_B \circ g^{\xi}_B f && \text{by definition} \\
&= \theta_B \circ X(f)(\xi) && \text{by definition} \\
&= X'(f)\theta_A(\xi) && \text{since } \theta \text{ is natural,} \\
&= g_B^{\theta_A(\xi)}(f).
\end{aligned}$$

Note that the following square commutes.

$$\begin{tikzcd}
X(B) \arrow[r, "\theta_B"] & X'(B) \\
X(A) \arrow[u, "X(f)"] \arrow[r, "\theta_A"] & X(A') \arrow[u, "X'(f)"']
\end{tikzcd}$$

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Assignment 2

2.3.2 We want to show $\mathrm{Fun}(\mathscr{C}^{op} \to \mathscr{D}) = \mathrm{Fun}(\mathscr{C} \to \mathscr{D}^{op})$.

Suppose $C, C' \in \mathscr{C}^{op}$ and $f : C' \to C$ in $\mathscr{C}$. Then, if $F \in \mathrm{Fun}(\mathscr{C}^{op} \to \mathscr{D})$, then $F(f) : F(C) \to F(C')$ in $\mathscr{D}$. On the other hand if $G \in \mathrm{Fun}(\mathscr{C} \to \mathscr{D}^{op})$, then $G(f) : G(C') \to G(C)$ in $\mathscr{D}^{op}$, and so $G(f) : G(C) \to G(C')$ in $\mathscr{D}$. We see they "behave" the ~~same~~ the same.

Now, we define $\Phi : \mathrm{Fun}(\mathscr{C}^{op} \to \mathscr{D}) \to \mathrm{Fun}(\mathscr{C} \to \mathscr{D}^{op})$
$$F \mapsto F$$
and $\Psi : \mathrm{Fun}(\mathscr{C} \to \mathscr{D}^{op}) \to \mathrm{Fun}(\mathscr{C}^{op} \to \mathscr{D})$.
$$\overline{F} \mapsto \overline{F}$$
If they are well-defined, clearly they are inverses of each other since $\Phi(\Psi)(\overline{F}) = \Phi(\overline{F}) = \overline{F} = I_{\mathrm{Fun}(\mathscr{C} \to \mathscr{D}^{op})}$ and
$$\Psi(\Phi)(F) = \Psi(F) = F = I_{\mathrm{Fun}(\mathscr{C}^{op} \to \mathscr{D})}.$$

First, we easily "see that" both Φ and Ψ are well-defined on object as $\mathrm{ob}(\mathscr{B}^{op}) = \mathrm{ob}(\mathscr{B})$ for any category $\mathscr{B}$ by definition. To see that if $F \in \mathrm{Fun}(\mathscr{C}^{op} \to \mathscr{D})$, then $F \in \mathrm{Fun}(\mathscr{C} \to \mathscr{D}^{op})$, given $f : C' \to C$ in $\mathscr{C}$, we get $F(f) : F(C) \to F(C')$ in $\mathscr{D}$, by construction, which is a functor in $\mathrm{Fun}(\mathscr{C} \to \mathscr{D}^{op})$ since, by definition, $\mathscr{D}^{op}(F(C), F(C')) = \mathscr{D}(F(C'), F(C))$. Similarly, given $f : C' \to C$ in $\mathscr{C}$, $\overline{F}(f) : \overline{F}(C) \to \overline{F}(C')$ which corresponds to a functor in $\mathrm{Fun}(\mathscr{C}^{op} \to \mathscr{D})$. The functoriality follows from the construction itself.

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2.3.2 We want to prove that $Set^{op} \not\simeq Set$. Suppose, on the contrary, Set^{op} is equivalent to Set. Then, the corresponding functor, $F: Set \to Set^{op}$ is full, faithful and essentially surjective on objects, by Proposition 1.3.18. ~~Proposition 1.3.18~~

Let's divert for a moment and look at the lemma below

Lemma: A full and essentially surjective functor takes initial objects to initial objects and takes terminal objects to terminal objects

Proof. Suppose $F: \mathscr{C} \to \mathscr{D}$ is the described functor and $C \in \mathscr{C}$ is initial. Since F is essentially surjective, for any $D \in \mathscr{D}$, we can write $D = F(C')$ for some $C' \in \mathscr{C}$. Then, consider the map $\varphi: F(C) \to F(C')$ for some C' such that $F(C') \cong D$, for all D. Sine F is full, $\varphi = F(g)$ for some $g: C \to C'$. But, since C is initial, g is unique. If φ' is another map $\varphi': F(C) \to F(C')$ with $\varphi' = F(g')$ for some $g': C \to C'$, then $g = g'$ since C is initial. If ϕ is the isomorphism from $F(C')$ to D, then it remains to show $\phi \circ F(g)$ is unique. But, if $\phi_1 \neq \phi_2$ are such isomorphism, then $\phi_1^{-1}(\phi_2(F(g))) = F(g)$ since $\phi_1^{-1}(\phi_2(F(g)))$ is the unique map from $F(C)$ to $F(C')$. Hence, this implies $\phi_2(F(g)) = \phi_1(F(g))$ as desired.

By duality, the proof works for the terminal object.

Now, consider $F: Set \to Set^{op}$ where $F(\phi) \cong 1$ and $F(1) \cong \phi$. If $S \neq \phi$, then we have $1 \xrightarrow{f} S$ in Set and $F(1) \xrightarrow{F(f)} F(S)$ in Set^{op} which is equal to $\phi = F(1) \xleftarrow{F(f)} F(S)$ in Set. But, this implies $F(S) = \phi$ since there is no map $F(S) \to \phi$ in Set unless $F(S) = \phi$. Hence, we get $F(S) \cong \begin{cases} 1 & \text{if } S = \phi \\ \phi & \text{otherwise} \end{cases}$ which shows that F is not essentially surjective.

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Assignment 2

2.3.2 We want to show that $\mathrm{fdVec}^{op} \simeq \mathrm{fdVec}$. It suffices to find a pair of functor that shows the equivalence.

From Example 1.2.12, we have a contravariant functor $(\ )^* = \mathrm{Hom}(-, k) : \mathrm{fdVec}^{op} \to \mathrm{fdVec}$ where k is the underlying field.

Now, consider
$$\begin{tikzcd} \mathrm{fdVec}^{op} \arrow[r, bend left, "(\ )^*"] & \mathrm{fdVec} \arrow[l, bend left, "(\ )^*"] \end{tikzcd}$$

Then, $(\ )^* \circ (\ )^* = (\ )^{**} \cong 1_{\mathrm{fdVec}}$ as we show in assignment 1.

But, from the first part, $\mathrm{Func}(\mathrm{fdVec}, \mathrm{fdVec}) = \mathrm{Func}(\mathrm{fdVec}^{op}, \mathrm{fdVec}^{op})$. If α is the natural isomorphism from $(\ )^{**}$ to 1_{fdVec}, then it is also a natural isomorphism from $(\ )^{**}$ to $1_{\mathrm{fdVec}^{op}}$. This is because $\mathrm{ob}(\mathrm{fdVec}) = \mathrm{ob}(\mathrm{fdVec}^{op})$ and the naturality square in fdVec is exactly the same as in fdVec. For $f: V \to W$ in fdVec, we get
$$\begin{tikzcd} V^{**} \arrow[r, "f^{**}"] \arrow[d, "\alpha_V"'] & W^{**} \arrow[d, "\alpha_W"] \\ 1(V) \arrow[r, "f"] & 1(W) \end{tikzcd}$$
in fdVec, but we get $f^{op}: W \to V$ in fdVec^{op} and $(f^{op})^{**}: W^{**} \to V^{**}$ in fdVec^{op} which is equal to $(f^{op})^{**}: V^{**} \to W^{**}$ in fdVec. Similary, it works for $1_{\mathrm{fdVec}^{op}}$.

Hence, we get $(\ )^* \circ (\ )^* = (\ )^{**} \cong 1_{\mathrm{fdVec}^{op}}$ as well.

NO: .... DATE: ....