PROBLEMS FOR THE CATEGORY THEORY READING COURSE, 2017

SCOTT MORRISON

1. Assignment 1, due end of Week 4

1.1. Functors.

(1) Leinster 1.2.21 (functors preserve isomorphisms)
(2) Leinster 1.2.27, 1.2.28b (full, faithful)

1.2. Natural transformations.

(1) In fdVec, show that the functors id and $**$ are naturally isomorphic.
(2) Show the the vertical composition of two natural transformations is in fact a natural transformation.
(3) Prove carefully that the horizontal composition of two natural transformations is again a natural transformation.
(4) Show that a functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories if and only if it is *fully faithful* and *essentially surjective*. Clearly state where you are using the axiom of choice, or add hypotheses so it is unnecessary.

1.3. Universal properties.

(1) Prove that two initial objects in a category are isomorphic.
(2) For each of the following categories, decide whether there is an initial, final, and/or zero object, and if so, describe them: FinSet, fdVec, Top, Top_* (pointed topological spaces), field extensions of a fixed field F, Graphs (your answer may depend on which class of graphs you consider), Semigroups, Groups.
(3) Describe the product of two objects as the terminal object in some category.
(4) Describe the tensor product of two vectors spaces as the initial object in some category.
(5) Describe both the product and coproduct in the following categories: FinSet, Top, Top_*, AbGroup, Group, Graphs.

1.4. Adjunctions.

(1) Consider the forgetful functor from abelian groups to groups. What is its left adjoint?
(2) In the category of finite dimensional vector spaces, show that $- \otimes V$ is biadjoint to $- \otimes V^*$.
(3) Prove that the 'hom-set isomorphism' and 'unit/counit' definitions of an adjunction are equivalent.

Date: Version of May 19, 2017.

2. Assignment 2, due at the end of Week 7:

2.1. Idempotent completion. The 'idempotent completion' $\mathrm{Kar}(C)$ (also called the 'Karoubi envelope') is defined as follows:

$$\mathrm{Obj}\,\mathrm{Kar}(C) = \left\{(X \in \mathrm{Obj}(C), p : X \to X) \mid |p^2 = p\right\}$$
$$\mathrm{Kar}(C)((X,p) \to (X',p')) = \{f \in C(X \to X') \mid |fp = f = p'f\}.$$

(1) Let primeVec denote the full subcategory of Vec consisting of vector spaces with prime dimensions. Show carefully that $\mathrm{Kar}(\mathrm{primeVec}) \cong \mathrm{Vec}$.
(2) Construct an equivalence $\iota_C : \mathrm{Kar}(\mathrm{Kar}(C)) \cong \mathrm{Kar}(C)$.
(3) Show that there is a fully faithful functor $C \to \mathrm{Kar}(C)$ given by $X \mapsto (X, 1_X)$.
(4) Note that given a functor $F : C \to \mathcal{D}$, there is functor $\mathrm{Kar}(F) : \mathrm{Kar}(C) \to \mathrm{Kar}(\mathcal{D})$ given by

$$\mathrm{Kar}(F)(X,p) = (F(X), F(p))$$
$$\mathrm{Kar}(F)(f : (X,p) \to (X',p')) = F(f).$$

(You might say that Kar is a 2-functor from CAT to itself — what does this mean at the level of natural transformations?) Given a functor $F : \mathrm{Kar}(C) \to \mathrm{Kar}(D)$, show that it is determined up to natural isomorphism by its restriction $F|_C : C \to \mathrm{Kar}(D)$, by showing $F \cong (\iota_{\mathcal{D}} \circ \mathrm{Kar}(F|_C))$.
(5) (Not on the problem set: there is a forgetful 2-functor CAT $\to$ SemiCat, the 2-category of 'semicategories' (categories without identities) and their functors and natural transformations. The idempotent completion gives a 2-functor SemiCAT $\to$ CAT. Are they an adjoint pair?)

2.2. Limits.

(1) Recall that given a functor $F : \mathcal{J} \to C$, the *limit* of F, written $\lim_{\mathcal{J}} F$ is a terminal object in the category of cones over F. Explicitly, a cone consists of
   (a) an object $X \in \mathrm{Obj}\, C$,
   (b) for each $j \in \mathrm{Obj}\, C$, a map $f_j : X \to F(j)$,
   (c) such that for any $g : j \to j'$, $F(g) \circ f_j = f_{j'}$.

   Another way of saying this is that the category of cones is the comma category $(J \downarrow F)$, where here we interpret $\mathcal{J}$ as a functor $C \to \mathrm{Fun}(\mathcal{J} \to C)$ by $c \mapsto (j \mapsto c)$ and we interpret F as a functor $1 \to \mathrm{Fun}(\mathcal{J} \to C)$ by $1 \mapsto (j \mapsto F(j))$. Explain carefully why these are talking about the same thing!

2.3. The Yoneda embedding.

(1) Prove the Yoneda lemma.
(2) Explain why $\mathrm{Fun}(C^{\mathrm{op}} \to \mathcal{D}) = \mathrm{Fun}(C \to \mathcal{D}^{\mathrm{op}})$. Prove that $\mathrm{Set}^{\mathrm{op}} \not\cong \mathrm{Set}$, but that $\mathrm{fdVec}^{\mathrm{op}} \cong \mathrm{fdVec}$.

3. Assignment 3, due May 12:

(1) Prove that every monoidal category is monoidally equivalent to a strict monoidal category.
   (Hint: given a monoidal category C, define a new monoidal category $\mathsf{List}C$, whose objects are lists of objects in C. In $\mathsf{List}C$, tensor product of objects is concatenation of lists, and the tensor unit is the empty list. There should be a functor $\mathsf{List}C$ to C defined by sending a list $[x_1, x_2, \ldots, x_n]$ to $(((1 \otimes x_1) \otimes x_2) \otimes \cdots \otimes x_n$. Your job is to describe what happens at the level of morphisms, and check that everything works.)

(2) Let C be a monoidal category. We say a 'monoid object' in C (or, as we gain confidence, just a monoid in C) is a tuple $(A \in \operatorname{Obj} C, \iota : 1 \to A, m : A \otimes A \to A)$ satisfying some conditions. Look up, or work out, what these conditions should be. (Hint: look at chapter 7 of Etingof, Gelaki, Nikshych, and Ostrik's book Tensor Categories.) You should be able to show that a monoid object in Vec is what is usually called an associative unital algebra.
   (a) A 'module object' for a monoid object $A \in C$ is a tuple $(M, \triangleright : A \otimes M \to M)$ satisfying an appropriate condition (what is it?). A morphism f between module objects M and M' is a morphism between the underlying objects, such that $f \circ \triangleright_M = \triangleright_{M'} \circ (1_A \otimes f)$. Draw the string diagram corresponding to this axiom. Define composition of module morphisms, by imitating the definition for modules over a ring. Show that modules for a fixed monoid object form a category.

(3) Show that $\mathrm{Rep}G$, for G a finite group, forms a monoidal category.

(4) (For this part, you may assume we are looking at representations over the complex numbers.)
   - If $\mathrm{Rep}G \cong \mathrm{Rep}H$, as categories, are G and H isomorphic?
     (Hint: no, give a counterexample — any pair of non-isomorphic groups with the same number of irreducible representations (equivalently, the same number of conjugacy classes) will do.)
   - What about if $\mathrm{Rep}G \cong \mathrm{Rep}H$ as monoidal categories, and moreover this equivalence is compatible with the forgetful functors to Vec?
     (Hint: think about the monoidal automorphisms of the forgetful functor. You should prove that (not necessarily monoidal) automorphisms of the forgetful functor $\mathrm{Rep}G \to \mathrm{Vec}$ is a group isomorphic to $\mathbb{C}[G]^\times$, and then identify the subgroup of monoidal automorphisms is G is itself. To do the first part, show that any such automorphism is determined by its component on the regular representation $\mathbb{C}[G]$; for this you may like to use that the regular representation is faithful, or (more or less equivalently) that there is a surjective G-linear map from the regular representation to any irreducible representation of G.)

4. Assignment 4, attempt 2, due May 26:

4.1. Braided monoidal categories.

(1) Show that Temperley-Lieb is a braided monoidal category, with braiding satisfying

$$\times = A \;) \; (\; + A^{-1} \asymp ,$$

for some A so that $\delta = -A^2 - A^{-2}$.

(2) We can take a link diagram, and interpret it as a morphism $0 \to 0$ in Temperley-Lieb, by interpreting crossings according to the above formula. (Explain carefully what we're meant to do with $\times$.) The morphism space $0 \to 0$ consists of just scalar multiples of the empty diagram, so this associates a number to a link diagram. Explain why this number is a *framed* link invariant, i.e. that it doesn't change when we modify the link diagram by Reidemeister moves II and III, and a modified version of the usual Reidemeister move I.

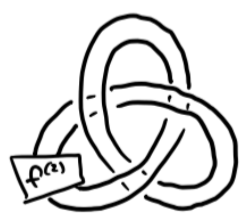
(3) Calculate the invariant of the trefoil, according to this recipe. Show that the trefoil is not isotopic to the unknot, by showing this invariant takes different values on the two knots.

(4) What is this invariant usually called?

(5) Show that in $\mathrm{Kar}(TL)$, we have $(2, 1_2) \cong (2, f^{(2)}) \oplus (0, 1_0)$. Here $f^{(2)}$ denotes the second Jones-Wenzl idempotent:

$$f^{(2)} = \;) \; (\; - \frac{1}{\delta} \asymp .$$

(6) We can make another knot invariant by replacing each string of a knot diagram by two parallel strings, and somewhere on the knot inserting a copy of $f^{(2)}$.



Explain why this doesn't depend on where we insert the $f^{(2)}$. Calculate this invariant for the unknot.

(7) We say a monoid (A, m) in a braided monoidal category is commutative if $m \circ \beta = m$, where β denotes the braiding. Define a monoidal structure on the category of modules for a commutative monoid.

(Scott Morrison) Mathematical Sciences Institute, Australian National University
E-mail address: scott.morrison@anu.edu.au