Section 2.1

I talked to Kie Seng about this section.

Definition 1. The 'idempotent completion' $\mathrm{Kar}(\mathscr{C})$ (also called the 'Karoubi envelope') of a category $\mathscr{C}$ is defined as follows:

1. an object of $\mathrm{Kar}(\mathscr{C})$ is a pair (C, p) where C is an object of $\mathscr{C}$ and $C \xrightarrow{p} C$ is an idempotent,
2. a morphism $(C, p) \to (C', p')$ of $\mathrm{Kar}(\mathscr{C})$ is a map $C \xrightarrow{C}{}'$ in $\mathscr{C}$ such that $fp = f = p'f$.

Remark 1. If you have two maps $(C, p) \xrightarrow{f} (C', p') \xrightarrow{g} (C'', p'')$, then their composition gf is also a map $(C, p) \to (C'', p'')$ in $\mathrm{Kar}(\mathscr{C})$ since

$$gfp = g(fp) = gf = (p''g)f = p''gf.$$

Lemma 1. *The identity morphism on (C, p) in $\mathrm{Kar}(\mathscr{C})$ is the morphism p itself:*

$$\mathrm{id}_{(C,p)} = \left(C \xrightarrow{p} C\right).$$

Proof. First, check that p is indeed a morphism $(C, p) \to (C, p)$ in $\mathrm{Kar}(\mathscr{C})$. This amounts to checking that $p^2 = p$, which is guaranteed by the definition of (C, p).

Next, check that p satisfies the identity axioms. Let $(C, p) \xrightarrow{f} (C', p')$ be another morphism in $\mathrm{Kar}(\mathscr{C})$. Then, by definition, $fp = f = p'f$ as maps in $\mathscr{C}$. It follows by the above remark that $fp = f = p'f$ as maps in $\mathrm{Kar}(\mathscr{C})$, as well. $\square$

Part (1)

Let **primeVec** denote the full subcategory of **fdVec** consisting of vector spaces with prime dimensions.

Claim 1. *The idempotent completion of* **primeVec** *is isomorphic to* **fdVec***:*

$$\mathrm{Kar}(\mathbf{primeVec}) \cong \mathbf{fdVec}.$$

Proof. Define a functor $F : \mathrm{Kar}(\mathbf{primeVec}) \to \mathbf{fdVec}$ by

$$(V, p) \longmapsto pV$$

$$\left((V, p) \xrightarrow{f} (V', p')\right) \longmapsto \left(pV \xrightarrow{f|_{pV}} p'V'\right)$$

Why is this well defined on morphisms? We need to check that $\mathrm{Im}(f|_{pV}) \subset p'V'$. But we know that $f(pv) = p'f(v) \subset p'V'$ since f is a morphism in $\mathrm{Kar}(\mathbf{primeVec})$. Thus, F is well defined.

We also need to check that F is indeed a functor. That is, we need to verify that

1. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X in $\mathrm{Kar}(\mathbf{primeVec})$,

2. $F\left((V,p) \xrightarrow{f} (V',p') \xrightarrow{g} (V'',p'')\right) = F(V,p) \xrightarrow{F(f)} F(V',p') \xrightarrow{F(g)} F(V'',p'')$ for all $(V,p),(V',p'),(V'',p'')$ and f,g in Kar(**primeVec**).

To prove 1, notice that $\mathrm{id}_{(V,p)}$ is the morphism $V \xrightarrow{p} V$. Thus, $F(\mathrm{id}_{(V,p)}) = p|_{pV}$. By idempotency, $p(pv) = pv$, so $p|_{pV} = \mathrm{id}_{pV}$ as required. To prove 2, notice that $(g \circ f)|_{pV} = g|_{p'V'} \circ f|_{pV}$.

We want to show that F is an equivalence. To do this, we will prove that F is full, faithful and essentially surjective:

1. Proof that F is full: Take $pV \xrightarrow{g} p'V'$ in **fdVec**. We will construct a morphism $(V,p) \xrightarrow{f} (V',p')$ in Kar(**primeVec**) with $F(f) = g$. Define

$$f(v) = \begin{cases} g(v) & \text{if } v = pw \text{ for some } w \in V, \\ 0 & \text{otherwise.} \end{cases}$$

We need to check that f is a morphism in Kar(**primeVec**). Firstly, we will check that it is linear map from V to V' in **primeVec**. It suffices to check this on the subspace pV. For $v,w \in pV$ and scalars a,b, we have

$$f(av+bw) = g(av+bw) = ag(v) + bg(w) = af(v) + bf(w),$$

since g is linear by assumption. Secondly, we need to check that f satisfies the property $fp = f = p'f$. Again, it suffices to check this on the subspace pV. For $v \in V$, we have $fp(pv) = f(pv) = g(pv)$ since $p^2 = p$. Now, $g(pv) = p'v'$ for some $v' \in V'$. Thus, $fp(pv) = f(pv) = p'v' = p'(p'v') = p'g(pv) = p'f(pv)$ since $(p')^2 = p'$. Therefore, f is a morphism in Kar(**primeVec**).

Now, $F(f) = g$ since $f|_{pV} = g$ by definition. Thus, F is full.

2. Proof that F is faithful: Take morphisms $(V,p) \xrightarrow{f,g} (V',p')$ in Kar(**primeVec**) with $F(f) = F(g)$. Then $f|_{pV} = g|_{pV}$. So, for $v \in V$,

$$f(v) = f(pv) = g(pv) = g(v),$$

since $f = fp$ and $g = gp$. Thus, $f = g$.

3. Proof that F is essentially surjective: Take V' in **fdVec**. We will construct (V,p) in Kar(**primeVec**) with $F(V,p) \cong V'$. Let $\{e_1,...,e_n\}$ be a basis for V'. Let ϕ be the smallest prime such that $\phi \geq n$. Let $e_{n+1},...,e_\phi$ be formal symbols. Define V to be the vector space with basis $\{e_1,...,e_\phi\}$. Define $p : V \to V$ to be the projection onto the first n basis vectors. That is, p is the linear map defined by

$$p(e_i) = \begin{cases} e_i & \text{if } i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $p^2 = p$. So $(V,p) \in$ Kar(**primeVec**). Also, pV has dimension n. So $F(V,p) = pV \cong V'$.

□

Part (2)

In this part, we will construct an equivalence $\iota_{\mathscr{C}} : \mathrm{Kar}(\mathrm{Kar}(\mathscr{C})) \xrightarrow{\cong} \mathrm{Kar}(\mathscr{C})$.

Before we do this, lets understand $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$. An object K of $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$ is a pair (X, q), where X is an object of $\mathrm{Kar}(\mathscr{C})$, such that $X \xrightarrow{q} X$ is an idempotent. We can write $X = (C, p)$ where C is an object of $\mathscr{C}$ and $C \xrightarrow{p} C$ is an idempotent. Thus, K is a triple (C, p, q) with p and q idempotents. Notice that $(C, p) \xrightarrow{q} (C, p)$ is a morphism in $\mathrm{Kar}(\mathscr{C})$ and consequently it is a morphism $C \xrightarrow{q} C$ in $\mathscr{C}$ satisfying the property $qp = q = pq$. Summing up, an object K of $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$ is a tuple (C, p, q) with

1. C an object in $\mathscr{C}$,
2. $C \xrightarrow{p,q} C$ idempotents in $\mathscr{C}$, and
3. $qp = q = pq$.

A morphism $(C, p, q) \xrightarrow{f} (C', p', q')$ of $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$ is a map $(C, p) \xrightarrow{f} (C', p')$ in $\mathrm{Kar}(\mathscr{C})$ such that $fq = f = q'f$. Unwrapping definitions further, f is a morphism $C \to C'$ in $\mathscr{C}$ satisfying $fp = f = p'f = fq = q'f$.

Define the 'forgetful' functor $\iota_{\mathscr{C}} : \mathrm{Kar}(\mathrm{Kar}(\mathscr{C})) \to \mathrm{Kar}(\mathscr{C})$ by

$$(C, p, q) \longmapsto (C, q),$$
$$\left((C, p, q) \xrightarrow{f} (C', p', q')\right) \longmapsto \left((C, q) \xrightarrow{f} (C', q')\right).$$

We must check that $\iota_{\mathscr{C}}$ is indeed a functor. Note that both $\mathrm{id}_{(C,p,q)}$ and $\mathrm{id}_{(C,q)}$ are the same morphism $C \xrightarrow{q}$ in $\mathscr{C}$. Thus, $\iota_{\mathscr{C}}(\mathrm{id}_{(C,p,q)}) = \mathrm{id}_{(C,q)}$. The fact that $\iota_{\mathscr{C}}$ preserves composition is trivial.

Claim 2. *The functor $\iota_{\mathscr{C}}$ is an equivalence.*

Proof. We will show that $\iota_{\mathscr{C}}$ is full, faithful and essentially surjective.

1. Proof that $\iota_{\mathscr{C}}$ is full: Fix objects (C, p, q) and (C', p', q') in $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$. Consider $\iota_{\mathscr{C}}(C, p, q) \xrightarrow{f} \iota_{\mathscr{C}}(C', p', q')$. That is, f is a morphism $C \to C'$ in $\mathscr{C}$ satisfying $fq = f = q'f$. We will show that f is also a morphism in $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$. That is, we need to verify $fp = f = p'f$.

   We know that $pq = q = qp$ and $p'q' = q' = q'p'$. Using these properties, along with the relation $fq = f = q'f$, we get
   $$fp = (fq)p = f(qp) = fq = f = q'f = (p'q')f = p'(q'f) = p'f,$$
   as required. Now, it is obvious that $\iota_{\mathscr{C}}(f) = f$. Thus, $\iota_{\mathscr{C}}$ is full.

2. Proof that $\iota_{\mathscr{C}}$ is faithful: This is trivial by expanding definitions. Suppose f, g are morphisms $(C, p, q) \to (C', p', q')$ in $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$ with $\iota_{\mathscr{C}}(f) = \iota_{\mathscr{C}}(g)$. Then f and g considered as morphisms $C \to C'$ in $\mathscr{C}$ must be equal. It follows by definition that f and g are equal, when considered as morphisms $(C, p, q) \to (C', p', q')$.

3. Proof that $\iota_{\mathscr{C}}$ is essentially surjective: Let (C, q) be an object in $\mathrm{Kar}(\mathscr{C})$. Then (C, id_C, q) is an object in $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$ and $\iota_{\mathscr{C}}(C, \mathrm{id}_C, q) = (C, q)$.

$\square$

Part (3)

Claim 3. *There is a fully faithful functor $i_{\mathscr{C}} : \mathscr{C} \to \mathrm{Kar}(\mathscr{C})$ given by $C \mapsto (C, \mathrm{id}_C)$.*

Remark 2. This gives an embedding of $\mathscr{C}$ into $\mathrm{Kar}(\mathscr{C})$.

Proof. Define $i_{\mathscr{C}} : \mathscr{C} \to \mathrm{Kar}(\mathscr{C})$ by

$$C \longmapsto (C, \mathrm{id}_C)$$
$$\left(C \xrightarrow{f} C'\right) \longmapsto \left((C, \mathrm{id}_C) \xrightarrow{f} (C', \mathrm{id}_{C'})\right)$$

We need to check that this is a functor. Note that $\mathrm{id}_{(C,\mathrm{id}_C)}$ is the identity morphism id_C in $\mathscr{C}$. Thus, $i_{\mathscr{C}}(\mathrm{id}_C) = \mathrm{id}_{(C,\mathrm{id})}$ as required. The fact that $i_{\mathscr{C}}$ preserves composition is trivial.

We will show that $i_{\mathscr{C}}$ is full. Fix objects C, C' in $\mathscr{C}$. Consider the morphism $(C, \mathrm{id}_C) \xrightarrow{f} (C', \mathrm{id}_{C'})$ in $\mathrm{Kar}(\mathscr{C})$. By definition, this is a map $C \xrightarrow{f} C'$ in $\mathscr{C}$. Then $i_{\mathscr{C}}(f) = f$ and so $i_{\mathscr{C}}$ is full.

Finally, we prove that $i_{\mathscr{C}}$ is faithful. Again this follows by expanding definitions. Fix C, C'. Suppose we have morphisms $C \xrightarrow{f,g} C'$ in $\mathscr{C}$ with $i_{\mathscr{C}}(f) = i_{\mathscr{C}}(g)$. Then the maps $(C, \mathrm{id}_C) \xrightarrow{f,g} (C', \mathrm{id}_{C'})$ are equal. By definition, it must be the case that f, g are equal as maps $C \to C'$ in $\mathscr{C}$. □

Part (4)

Definition 2. Given a functor $F : \mathscr{C} \to \mathscr{D}$, there is a functor $\mathrm{Kar}(F) : \mathrm{Kar}(\mathscr{C}) \to \mathrm{Kar}(\mathscr{D})$ given by

$$\mathrm{Kar}(F)(C, p) = (F(C), F(p))$$
$$\mathrm{Kar}(F)((C, p) \xrightarrow{f} (C', p')) = (F(C), F(p)) \xrightarrow{F(f)} (F(C'), F(p')).$$

We need to check that this is well defined: is $F(f)$ a map in $\mathrm{Kar}(\mathscr{D})$? It needs to satisfy the property $F(f)F(p) = F(f) = F(p')F(f)$. This follows from the corresponding property of f in $\mathrm{Kar}(\mathscr{C})$ and functorality:

$$F(f)F(p) = F(fp) = F(f) = F(p'f) = F(p')F(f).$$

The fact that $\mathrm{Kar}(F)$ preserves identities and composition is straightforward to verify.

Theorem 1. *A functor $F : \mathrm{Kar}(\mathscr{C}) \to \mathrm{Kar}(\mathscr{D})$ is determined up to natural isomorphism by its restriction $F|_{\mathscr{C}} : \mathscr{C} \to \mathrm{Kar}(\mathscr{D})$:*

$$F \cong (\iota_{\mathscr{D}} \circ \mathrm{Kar}(F|_{\mathscr{C}})).$$

Remark 3. By $F|_{\mathscr{C}}$ we mean the composition $F \circ i_{\mathscr{C}}$ of F and the embedding $i_{\mathscr{C}} : \mathscr{C} \to \mathrm{Kar}(\mathscr{C})$.

This is an exercise in notation. Suppose (C, p) and (D, q) are objects in $\mathrm{Kar}(\mathscr{C})$. Let $C \xrightarrow{f} D$ be a morphism in $\mathscr{C}$ satsifying $fp = f = qf$. Write f_p^q for the corresponding morphism $(C, p) \to (D, q)$ in $\mathrm{Kar}(\mathscr{C})$. Also, given a morphism $(C, p) \xrightarrow{g_p^q} (D, q)$ in $\mathrm{Kar}(\mathscr{C})$, write $\overline{g_p^q}$ for the underlying morphism $C \to D$ in $\mathscr{C}$.

Remark 4. If $\overline{f_p^q} = \overline{g_p^q}$ then $f_p^q = g_p^q$.

Remark 5. We have that $\overline{f \circ g} = \overline{f} \circ \overline{g}$.

Remark 6. If we have $(C,p) \xrightarrow{f_p^{p'}} (C',p')$ and $(C',p') \xrightarrow{g_{p'}^{p''}} (C'',p'')$, then we get a morphism $(g \circ f)_p^{p''}$ from (C,p) to (C'',p'').

Lemma 2. *Given a functor $F : \mathrm{Kar}(\mathscr{C}) \to \mathrm{Kar}(\mathscr{D})$ and (C,p) an object of $\mathrm{Kar}(\mathscr{C})$,*

$$F(C,p) = (\widehat{C_p}, \overline{F(p_p^p)})$$

for some object $\widehat{C_p}$ in $\mathscr{D}$.

Proof. Write $F(C,p) = (\widehat{C_p}, \widehat{p})$ for some object $(\widehat{C_p}, \widehat{p})$ in $\mathrm{Kar}(\mathscr{D})$. By lemma 1, p_p^p is the identity of (C,p). Thus, $F(p_p^p)$ is the identity of $(\widehat{C_p}, \widehat{p})$. Again using lemma 1, it follows that $\overline{F(p_p^p)} = \widehat{p}$. $\square$

Remark 7. We will use the notation $(\widehat{C_p}, \overline{F(p_p^p)})$ to denote the element $F(C,p)$ from herein.

Recall that an element of $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$ is $\Big((C,p),(C,p) \xrightarrow{q} (C,p)\Big)$ where (C,p) is an object and q is a morphism of $\mathrm{Kar}(\mathscr{C})$. A map $\Big((C,p),(C,p) \xrightarrow{q} (C,p)\Big) \to \Big((C',p'),(C',p') \xrightarrow{q'} (C',p')\Big)$ in $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$ is a map $(C,p) \xrightarrow{f_p^{p'}} (C',p')$ in $\mathrm{Kar}(\mathscr{C})$ satisfying $f_p^{p'} q = f_p^{p'} = q' f_p^{p'}$. That is, it is the map $(f_p^{p'})_q^{q'}$.

Lemma 3. *We have that*

$$\iota_{\mathscr{C}}\Big((C,p),(C,p) \xrightarrow{q} (C,p)\Big) = (C, C \xrightarrow{\overline{q}} C) \text{ and } \iota_{\mathscr{C}}\Big((f_p^{p'})_q^{q'}\Big) = f_{\overline{q}}^{\overline{q}'}.$$

Proof. We just need to translate the definition of $\iota_{\mathscr{C}}$ into the new notation. Previously, we wrote an object of $\mathrm{Kar}(\mathrm{Kar}(\mathscr{C}))$ as the triple (C,p,q) and defined

$$\iota_{\mathscr{C}}(C,p,q) = (C,q),$$
$$\iota_{\mathscr{C}}\Big((C,p,q) \xrightarrow{f} (C',p',q')\Big) = (C,q) \xrightarrow{f} (C',q').$$

Now, the morphism q in the triple (C,p,q) is actually the underlying map $C \xrightarrow{\overline{q}} C$ of the morphism $(C,p) \xrightarrow{q} (C,p)$. Then, translating into the new notation, (C,p,q) becomes $\Big((C,p),(C,p) \xrightarrow{q} (C,p)\Big)$ and (C,q) becomes $(C,\overline{q})$, giving us the required result.

To get the second result, translate into the new notation again. We get that $(C,p,q) \xrightarrow{f} (C',p',q')$ becomes

$$\Big((C,p),(C,p) \xrightarrow{q} (C,p)\Big) \xrightarrow{(f_p^{p'})_q^{q'}} \Big((C',p'),(C',p') \xrightarrow{q'} (C',p')\Big)$$

and $(C,q) \xrightarrow{f} (C',q')$ becomes $(C,\overline{q}) \xrightarrow{f_{\overline{q}}^{\overline{q}'}} (C,\overline{q}')$. This gives the required result. $\square$

Proof of theorem 1. The idea will be to find isomorphisms

$$F(C,p) \xrightarrow{\cong} \iota_{\mathscr{D}} \circ \mathrm{Kar}(F|_{\mathscr{C}})(C,p)$$

that are natural in (C,p). Firstly, compute

$$\mathrm{Kar}(F|_{\mathscr{C}})(C,p) = \big(F \circ i_{\mathscr{C}}(C), F \circ i_{\mathscr{C}}(p)\big) = \left(F(C,\mathrm{id}_C), F(C,\mathrm{id}_C) \xrightarrow{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})} F(C,\mathrm{id}_C)\right).$$

Thus, by lemma 3, we calculate

$$\iota_{\mathscr{D}} \circ \mathrm{Kar}(F|_{\mathscr{C}})(C,p) = \left(\widehat{C_{\mathrm{id}_C}}, \widehat{C_{\mathrm{id}_C}} \xrightarrow{\overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})}} \widehat{C_{\mathrm{id}_C}}\right).$$

We have a map $(C,p) \xrightarrow{p_p^{\mathrm{id}_C}} (C,\mathrm{id}_C)$. Applying F we get a map

$$F(C,p) = \left(\widehat{C_p}, \overline{F(p_p^p)}\right) \xrightarrow{F(p_p^{\mathrm{id}_C})} \left(\widehat{C_{\mathrm{id}_C}}, \overline{F\left((\mathrm{id}_C)_{\mathrm{id}_C}^{\mathrm{id}_C}\right)}\right) = F(C,\mathrm{id}_C).$$

We will show that this is also a map

$$\left(\widehat{C_p}, \overline{F(p_p^p)}\right) \xrightarrow{F(p_p^{\mathrm{id}_C})} \left(\widehat{C_{\mathrm{id}_C}}, \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})}\right).$$

To do this, we need $\overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})} \circ \overline{F(p_p^{\mathrm{id}_C})} = \overline{F(p_p^{\mathrm{id}_C})}$. By functoriality, $\overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})} \circ \overline{F(p_p^{\mathrm{id}_C})} = \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C} \circ p_p^{\mathrm{id}_C})}$. Then $p_p^{\mathrm{id}_C} = p_{\mathrm{id}_C}^{\mathrm{id}_C} \circ p_p^{\mathrm{id}_C}$ since $\overline{p_p^{\mathrm{id}_C}} = \overline{p_{\mathrm{id}_C}^{\mathrm{id}_C} \circ p_p^{\mathrm{id}_C}}$, which gives us the required result. We will show that this map $F(p_p^{\mathrm{id}_C})$ is an isomorphism in $\mathrm{Kar}(\mathscr{D})$.

We construct its inverse in a similar way. We have a map $(C,\mathrm{id}_C) \xrightarrow{p_{\mathrm{id}_C}^p} (C,p)$ which gives a map

$$F(C,\mathrm{id}_C) = \left(\widehat{C_{\mathrm{id}_C}}, \overline{F\left((\mathrm{id}_C)_{\mathrm{id}_C}^{\mathrm{id}_C}\right)}\right) \xrightarrow{F(p_{\mathrm{id}_C}^p)} \left(\widehat{C_p}, \overline{F(p_p^p)}\right) = F(C,p).$$

This is also a map

$$\left(\widehat{C_{\mathrm{id}_C}}, \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})}\right) \xrightarrow{F(p_{\mathrm{id}_C}^p)} \left(\widehat{C_p}, \overline{F(p_p^p)}\right).$$

since we have that $\overline{F(p_{\mathrm{id}_C}^p)} \circ \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})} = \overline{F(p_{\mathrm{id}_C}^p)}$ by reasoning analogous to above.

We will show that these maps $F(p_p^{\mathrm{id}_C})$ and $F(p_{\mathrm{id}_C}^p)$ are mutually inverse. We need that

$$\left(\widehat{C_{\mathrm{id}_C}}, \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})}\right) \xrightarrow{F(p_{\mathrm{id}_C}^p)} \left(\widehat{C_p}, \overline{F(p_p^p)}\right) \xrightarrow{F(p_p^{\mathrm{id}_C})} \left(\widehat{C_{\mathrm{id}_C}}, \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})}\right)$$

is the identity. That is, we need that $\overline{F(p_p^{\mathrm{id}_C}) \circ F(p_{\mathrm{id}_C}^p)} = \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})}$. Since $\overline{(p_p^{\mathrm{id}_C} \circ p_{\mathrm{id}_C}^p)} = \overline{p_{\mathrm{id}_C}^{\mathrm{id}_C}}$, we have that $p_p^{\mathrm{id}_C} \circ p_{\mathrm{id}_C}^p = p_{\mathrm{id}_C}^{\mathrm{id}_C}$. Then the required result follows by functoriality.

By analogous reasoning we can show that

$$\left(\widehat{C_p}, \overline{F(p_p^p)}\right) \xrightarrow{F(p_p^{\mathrm{id}_C})} \left(\widehat{C_{\mathrm{id}_C}}, \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})}\right) \xrightarrow{F(p_{\mathrm{id}_C}^p)} \left(\widehat{C_p}, \overline{F(p_p^p)}\right)$$

is the identity. Therefore, $F(p_p^{\mathrm{id}_C})$ is an isomorphism from $F(C,p)$ to $\iota_{\mathscr{D}} \circ \mathrm{Kar}(F|_{\mathscr{C}})(C,p)$.

It remains to prove that this isomorphism is natural in (C,p). For every $(C,p) \xrightarrow{f_p^q} (D,q)$, we need to check that the square

$$\begin{tikzcd}
F(C,p) \arrow[r, "F(f_p^q)"] \arrow[d, "F(p_p^{\mathrm{id}_C})"'] & F(D,q) \arrow[d, "F(q_q^{\mathrm{id}_D})"] \\
\left(\widehat{C_{\mathrm{id}_C}}, \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})}\right) \arrow[r, "\iota_{\mathscr{D}} \circ \mathrm{Kar}(F \circ i_{\mathscr{C}})(f_p^q)"] & \left(\widehat{D_{\mathrm{id}_D}}, \overline{F(p_{\mathrm{id}_D}^{\mathrm{id}_D})}\right)
\end{tikzcd}$$

commutes. Firstly, let us compute

$$\mathrm{Kar}(F \circ i_{\mathscr{C}})(f_p^q) = \left(F \circ i_{\mathscr{C}}(\overline{f_p^q})\right)_{F\circ i_{\mathscr{C}}(p)}^{F\circ i_{\mathscr{C}}(q)} = \left(\left(\widehat{C_{\mathrm{id}_C}}, \overline{F\left((\mathrm{id}_C)_{\mathrm{id}_C}^{\mathrm{id}_C}\right)}\right) \xrightarrow{F\left((\overline{f_p^q})_{\mathrm{id}_C}^{\mathrm{id}_D}\right)} \left(\widehat{D_{\mathrm{id}_D}}, \overline{F\left((\mathrm{id}_D)_{\mathrm{id}_D}^{\mathrm{id}_D}\right)}\right)\right)_{F\circ i_{\mathscr{C}}(p)}^{F\circ i_{\mathscr{C}}(q)}$$

Notice that $(\overline{f_p^q})_{\mathrm{id}_C}^{\mathrm{id}_D} = f_{\mathrm{id}_C}^{\mathrm{id}_D}$. Then, applying lemma 3,

$$\begin{aligned}
\iota_{\mathscr{D}} \circ \mathrm{Kar}(F \circ i_{\mathscr{C}})(f_p^q) &= \left(\left(\widehat{C_{\mathrm{id}_C}}, \overline{F \circ i_{\mathscr{C}}(p)}\right) \xrightarrow{F\left(f_{\mathrm{id}_C}^{\mathrm{id}_D}\right)} \left(\widehat{D_{\mathrm{id}_D}}, \overline{F \circ i_{\mathscr{C}}(q)}\right)\right) \\
&= \left(\left(\widehat{C_{\mathrm{id}_C}}, \overline{F(p_{\mathrm{id}_C}^{\mathrm{id}_C})}\right) \xrightarrow{F\left(f_{\mathrm{id}_C}^{\mathrm{id}_D}\right)} \left(\widehat{D_{\mathrm{id}_D}}, \overline{F(q_{\mathrm{id}_D}^{\mathrm{id}_D})}\right)\right).
\end{aligned}$$

Using this, we get the naturality condition easily by functoriality:

$$F(f_{\mathrm{id}_C}^{\mathrm{id}_D}) \circ F(p_p^{\mathrm{id}_C}) = F\left((f \circ p)_p^{\mathrm{id}_D}\right) = F(f_p^{\mathrm{id}_D}) = F\left((q \circ f)_p^{\mathrm{id}_D}\right) = F(q_q^{\mathrm{id}_D}) \circ F(f_p^q).$$

□

Section 2.2

Part (1)

Recall the following definitions.

Definition 3. Given a functor $F : \mathcal{J} \to \mathscr{C}$, the category $\mathbf{C}_F$ of cones is the category where:

1. objects are pairs $(C, (C \xrightarrow{f_j} Fj)_{j \in \mathcal{J}})$ of objects $C \in \mathscr{C}$ and families of morphisms $(C \xrightarrow{f_j} Fj)_{j\in\mathcal{J}})$ in $\mathscr{C}$ such that the triangle

$$\begin{tikzcd}
C \arrow[d, "f_j"'] \arrow[dr, "f_{j'}"] & \\
Fj \arrow[r, "Fg"'] & Fj'
\end{tikzcd}$$

commutes for all $j \xrightarrow{g} j'$ in $\mathcal{J}$,

2. morphisms $(C, (f_j)_{j \in \mathcal{J}}) \to (C', (f'_j)_{j \in \mathcal{J}})$ are maps $C \xrightarrow{f} C'$ in $\mathscr{C}$ such that the triangle

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "f_j"'] & C' \arrow[d, "f'_j"] \\ & Fj \end{tikzcd}$$

commutes for all $j \in \mathcal{J}$.

Definition 4. Given categories and functors

$$\begin{tikzcd} & \mathscr{B} \arrow[d, "Q"] \\ \mathscr{A} \arrow[r, "P"] & \mathscr{C} \end{tikzcd}$$

the comma category $(P \downarrow Q)$ is the category where

1. objects are tuples (A, h, B) with $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $PA \xrightarrow{h} QB$ in $\mathscr{C}$,
2. morphisms $(A, h, B) \to (A', h', B')$ are pairs $(A \xrightarrow{f} A', B \xrightarrow{g} B')$ such that the square

$$\begin{tikzcd} PA \arrow[r, "Pf"] \arrow[d, "h"] & PA' \arrow[d, "h'"] \\ QB \arrow[r, "Qg"] & QB' \end{tikzcd}$$

commutes.

Consider categories $\mathcal{J}$ and $\mathscr{C}$ and functor $F : \mathcal{J} \to \mathscr{C}$. We can interpret $\mathcal{J}$ as a functor $\mathscr{C} \to \mathrm{Fun}(\mathcal{J} \to \mathscr{C})$ defined on objects by

$$C \longmapsto \left(\begin{array}{c} j \mapsto C \\ (j \xrightarrow{f} j') \longmapsto \mathrm{id}_C \end{array} \right).$$

The functor $\mathcal{J}$ sends a morphism $C \xrightarrow{g} C'$ in $\mathscr{C}$ to the natural transformation $\mathcal{J}(C) \to \mathcal{J}(C')$ with components $J(C)(j) = C \xrightarrow{g} C' = J(C)(j')$. It is easy to verify that this is indeed a natural transformation (i.e. it satisfies the naturality square). To verify that $\mathcal{J}$ is a functor, we need to check that it preserves identities and composition. This is trivial.

Moreover, we can interpret F as a functor $\mathbf{1} \to \mathrm{Fun}(\mathcal{J} \to \mathscr{C})$ defined on objects by

$$1 \longmapsto \left(\begin{array}{c} j \mapsto Fj \\ (j \xrightarrow{f} j') \longmapsto (Fj \xrightarrow{Ff} Fj') \end{array} \right).$$

The functor $F : \mathbf{1} \to \mathrm{Fun}(\mathcal{J} \to \mathscr{C})$ sends the identity morphism in $\mathbf{1}$ to the identity natural transformation on $F(1)$. It follows F preserves identities and composition. Thus, we have the following categories and functors:

$$\begin{tikzcd} & \mathbf{1} \arrow[d, "F"] \\ \mathscr{C} \arrow[r, "\mathcal{J}"] & \mathrm{Fun}(\mathcal{J} \to \mathscr{C}) \end{tikzcd}$$

Using these interpretations of $\mathcal{J}$ and F, we get a comma category $(\mathcal{J} \downarrow F)$. Its objects are tuples $(C, h, 1)$ with $C \in \mathscr{C}$ and $\mathcal{J}(C) \xrightarrow{h} F(1)$. Note that h is a natural transformation, so we get components h_j for all $j \in \mathcal{J}$ satisfying the commutative square:

$$\begin{tikzcd} C \arrow[r, "\mathrm{id}_C"] \arrow[d, "h_j"] & C \arrow[d, "h_{j'}"] \\ Fj \arrow[r, "Fg"'] & Fj' \end{tikzcd}$$

Notice that the above square is exactly saying that $(C, (h_j)_{j\in\mathcal{J}})$ is an object in the category $\mathbf{C}_F$ of cones of F. Thus, every object in $(\mathcal{J} \downarrow F)$ can be considered as an object in $\mathbf{C}_F$. Similarly, an object $(C, (h_j)_{j\in\mathcal{J}})$ in $\mathbf{C}_F$ can be considered as an object $(C, h, 1)$ in $(\mathcal{J} \downarrow F)$ where h is that natural transformation with components h_j.

Now consider a morphism $(C, h, 1) \to (C', h', 1)$ in $(\mathcal{J} \downarrow F)$. By definition, it is a pair $(C \xrightarrow{g} C', 1 \xrightarrow{\mathrm{id}_1} 1)$ of morphisms in $\mathscr{C}$ and $\mathbf{1}$ such that the square of functors and natural transformations in $\mathrm{Fun}(\mathcal{J} \to \mathscr{C})$

$$\begin{tikzcd} \mathcal{J}(C) \arrow[r, "\mathcal{J}(g)"] \arrow[d, "h"'] & \mathcal{J}(C') \arrow[d, "h'"] \\ F(1) \arrow[r, "F(\mathrm{id}_1)"] & F(1) \end{tikzcd}$$

commutes. This is exactly saying that, for all $j \in \mathcal{J}$, the square

$$\begin{tikzcd} \mathcal{J}(C)(j) = C \arrow[r, "\mathcal{J}(g)_j = g"] \arrow[d, "h_j"'] & \mathcal{J}(C')(j) = C' \arrow[d, "h'_j"] \\ F(1)(j) = F(j) \arrow[r, "\mathrm{id}_{F_j}"] & F(1)(j) = F(j) \end{tikzcd}$$

commutes. This is exactly the property that $C \xrightarrow{g} C'$ is a map $(C, (h_j)_{j\in\mathcal{J}}) \to (C', (h'_j)_{j\in\mathcal{J}})$ in $\mathbf{C}_F$. Thus, a morphism $(C \xrightarrow{g} C', 1 \xrightarrow{\mathrm{id}_1} 1)$ from $(C, h, 1)$ to $(C', h', 1)$ in $(\mathcal{J} \downarrow F)$ can be considered as a morphism $C \xrightarrow{g} C'$ from $(C, (h_j)_{j\in\mathcal{J}})$ to $(C', (h'_j)_{j\in\mathcal{J}})$ in $\mathbf{C}_F$. Similarly, a morphism $C \xrightarrow{g} C'$ from $(C, (h_j)_{j\in\mathcal{J}})$ to $(C', (h'_j)_{j\in\mathcal{J}})$ in $\mathbf{C}_F$ can be considered as a morphism $(C \xrightarrow{g} C', 1 \xrightarrow{\mathrm{id}_1} 1)$ from $(C, h, 1)$ to $(C', h', 1)$ in $(\mathcal{J} \downarrow F)$.

Therefore, the comma category $(\mathcal{J} \downarrow F)$ is the same as the category $\mathbf{C}_F$ of cones of F.

Section 2.3

Part (1)

In this section we will prove the Yoneda lemma. Recall the following definitions.

Definition 5. Let $\mathscr{A}$ be a locally small category and A an object in $\mathscr{A}$. Define the functor $H_A : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ by

$$H_A(B) = \mathrm{Hom}_{\mathscr{A}}(B \to A),$$

for objects B in $\mathscr{A}$ and

$$\begin{aligned} H_A(B \xrightarrow{g} B') : \mathrm{Hom}_{\mathscr{A}}(B' \to A) &\longrightarrow \mathrm{Hom}_{\mathscr{A}}(B \to A) \\ p &\longmapsto p \circ g, \end{aligned}$$

for morphisms $B \xrightarrow{g} B'$ in $\mathscr{A}$. Moreover, given a map $A \xrightarrow{f} A'$ in $\mathscr{A}$, define the induced natural transformation $H_f : H_A \to H_{A'}$ by its components

$$\begin{aligned}(H_f)_B : H_A(B) = \mathrm{Hom}_{\mathscr{A}}(B \to A) &\longrightarrow H_{A'}(B) = \mathrm{Hom}_{\mathscr{A}}(B \to A') \\ p &\longmapsto f \circ p,\end{aligned}$$

for all B in $\mathscr{A}$.

Given a functor $X : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$, consider the morphisms $\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]}(H_A \to X)$. These are natural transformations. from H_A to X.

Theorem 2 (Yoneda). *For $\mathscr{A}$ locally small, $\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]}(H_A \to X)$ is a set*[1]*, and*

$$\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]}(H_A \to X) \cong X(A)$$

naturally in A in $\mathscr{A}$ and X in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$.

This is theorem 4.2.1 in Leinster.

Proof. We will define mutually invertible maps

$$\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]}(H_A \to X) \underset{(\tilde{\ })}{\overset{(\hat{\ })}{\rightleftarrows}} X(A)$$

for each A and X. Then we will show that the bijection $(\hat{\ })$ is natural in A and X.

For a natural transformation $\alpha : H_A \Rightarrow X$, defined $\hat{\alpha} = \alpha_A(\mathrm{id}_A)$. This is an element of $X(A)$. Why? By definition of a natural transformation, $H_A(A) = \mathrm{Hom}_{\mathscr{A}}(A \to A) \xrightarrow{\alpha_A} X(A)$ is a map in $\mathbf{Set}$. So $\alpha_A(\mathrm{id}_A)$ is an element of $X(A)$.

For $x \in X(A)$, define a natural transformation $\tilde{x} : H_A \to X$ as follows. Given B in $\mathscr{A}$, we need to define the component $\tilde{x}_B : H_A(B) = \mathrm{Hom}_{\mathscr{A}}(B \to A) \to X(B)$ as a map in $\mathbf{Set}$. For $f \in \mathrm{Hom}_{\mathscr{A}}(B \to A)$, define

$$\tilde{x}_B(f) = \big(X(f)\big)(x).$$

Notice that $X(f)$ is a map $X(A) \to X(B)$ in $\mathbf{Set}$, so $X(f)(x) \in X(B)$ as required. We need to prove that the components satisfy the naturality condition. That is, for any map $B' \xrightarrow{g} B$ in $\mathscr{A}$, the square

$$\begin{tikzcd} H_A(B) = \mathrm{Hom}_{\mathscr{A}}(B, A) \arrow[r, "H_A g"] \arrow[d, "\tilde{x}_B"'] & H_A(B') = \mathrm{Hom}_{\mathscr{A}}(B' A) \arrow[d, "\tilde{x}_{B'}"] \\ X(B) \arrow[r, "Xg"] & X(B') \end{tikzcd}$$

[1] Without knowing this first, the isomorphism $\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]}(H_A \to X) \cong X(A)$ doesn't make sense. We need to know that $\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]}(H_A \to X)$ lives in **Set** before we can talk about isomorphisms between it and $X(A)$. Moreover, it's not possible to prove that $\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]}(H_A \to X)$ is a set without proving the Yoneda lemma, despite what Leinster implies. There is actually more going on here and what I've done isn't strictly correct. (I assume that $\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]}(H_A \to X)$ is a class and that a class which is in bijection with a set *is* a set.) See `https://math.stackexchange.com/questions/2245952/yoneda-lemma-why-is-mathscr-a-textop-textbfseth-a-x-a-set/2246284#2246284` if you're interested (especially the comments on HeinrichD's answer).

commutes. This follows by unwrapping definitions: for all $f \in \mathrm{Hom}_{\mathscr{A}}(B, A)$, we have

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[dd, mapsto] & f \circ g \arrow[d, mapsto] \\ & \left(X(f \circ g)\right)(x) \\ (Xf)(x) \arrow[r, mapsto] & (Xg \circ Xf)(x) \end{tikzcd}$$

and $X(f \circ g) = (Xg \circ Xf)$ by functorality. (Remember X is contravariant.) Thus, the square commutes.

We have now defined $(\hat{\ })$ and $(\tilde{\ })$. We will show that they are mutually inverse: for $x \in X(A)$,

$$\hat{\tilde{x}} = \tilde{x}_A(\mathrm{id}_A) = \left(X(\mathrm{id}_A)\right)(x) = \mathrm{id}_{X(A)}(x) = x.$$

Next, we want to show $\tilde{\hat{\alpha}} = \alpha$ for natural transformation $\alpha : H_A \to X$. This amounts to showing all the components are equal: $\tilde{\hat{\alpha}}_B = \alpha_B$ for B in $\mathscr{A}$. These are functions of sets $H_A(B) = \mathrm{Hom}_{\mathscr{A}}(B \to A) \to X(B)$, so to check equality, we need to verify

$$\tilde{\hat{\alpha}}_B(f) = \alpha_B(f)$$

for all $f \in \mathrm{Hom}_{\mathscr{A}}(B \to A)$. By definition, $\tilde{\hat{\alpha}}_B(f) = (Xf)(\hat{\alpha}) = (Xf)\left(\alpha_A(\mathrm{id}_A)\right)$. Also, we get that

$$(Xf)\left(\alpha_A(\mathrm{id}_A)\right) = \alpha_B\left(\left(H_A(f)\right)(\mathrm{id}_A)\right) = \alpha_B(f) \tag{1}$$

by naturality of α:

$$\begin{tikzcd} H_A(A) = \mathrm{Hom}_{\mathscr{A}}(A \to A) \arrow[r, "H_A(f) = - \circ f"] \arrow[d, "\alpha_A"] & H_A(B) = \mathrm{Hom}_{\mathscr{A}}(B \to A) \arrow[d, "\alpha_B"] \\ X(A) \arrow[r, "Xf"] & X(B). \end{tikzcd}$$

This establishes the bijection $(\hat{\ })$ and allows us to conclude that $\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]}(H_A \to X)$ is a set. It remains to show that it is natural in A and X. Let us spell out what this actually means. We have two functors

$$\begin{aligned} F : [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] \times \mathscr{A}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ (X, A) &\longmapsto \mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]}(H_A \to X) \\ (X \overset{\beta}{\Rightarrow} X', A \xrightarrow{f} A') &\longmapsto \left(\left(H_{A'} \overset{\alpha}{\Rightarrow} X\right) \mapsto \left(H_A \xRightarrow{\beta \circ \alpha \circ H_f} X'\right)\right) \\ G : [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] \times \mathscr{A}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ (X, A) &\longmapsto X(A) \\ (X \overset{\beta}{\Rightarrow} X', A \xrightarrow{f} A') &\longmapsto \left(X(A') \xrightarrow{\beta_A \circ Xf} X'(A)\right). \end{aligned}$$

We want to show that $(\hat{\ })$ is a natural transformation in two variables X, A between F and G. Exercise 1.3.29 in Leinster states that this is equivalent to checking naturality in each variable separately. That is, it suffices to show that

1. $(\hat{\ })$ is a natural transformation between F^X and G^X for every X in $[\mathscr{A}^{\text{op}}, \mathbf{Set}]$, and
2. $(\hat{\ })$ is a natural transformation between F_A and G_A for every A in $\mathscr{A}$[2].

To check 1., we need to check the following square commutes for each X in $[\mathscr{A}^{\text{op}}, \mathbf{Set}]$ and $B \xrightarrow{f} A$ in $\mathscr{A}$:

$$\begin{tikzcd} \mathrm{Hom}_{[\mathscr{A}^{\text{op}},\mathbf{Set}]}(H_A \to X) \arrow[r, "F^X(f)=-\circ H_f"] \arrow[d, "(\hat{\ })"'] & \mathrm{Hom}_{[\mathscr{A}^{\text{op}},\mathbf{Set}]}(H_B \to X) \arrow[d, "(\hat{\ })"] \\ X(A) \arrow[r, "G^X(f)=Xf"] & X(B). \end{tikzcd}$$

For a natural transformation $\alpha : H_A \Rightarrow X$, we have

$$\begin{tikzcd} \alpha \arrow[r, mapsto] \arrow[dd, mapsto] & \alpha \circ H_f \arrow[d, mapsto] \\ & (\alpha \circ H_f)_B(\mathrm{id}_B) \\ \alpha_A(\mathrm{id}_A) \arrow[r, mapsto] & (Xf)(\alpha_A(\mathrm{id}_A)). \end{tikzcd}$$

Using equation 1, we compute

$$(\alpha \circ H_f)_B(\mathrm{id}_B) = \alpha_B\big((H_f)_B(\mathrm{id}_B)\big) = \alpha_B(f \circ \mathrm{id}_B) = \alpha_B(f) = (Xf)(\alpha_A(\mathrm{id}_A)).$$

Thus, we have shown the naturality of $(\hat{\ })$ between F^X and G^X. To check naturality between F_A and G_A, we need that the square

$$\begin{tikzcd} \mathrm{Hom}_{[\mathscr{A}^{\text{op}},\mathbf{Set}]}(H_A \to X) \arrow[r, "F_A(\beta)=\beta\circ-"] \arrow[d, "(\hat{\ })"'] & \mathrm{Hom}_{[\mathscr{A}^{\text{op}},\mathbf{Set}]}(H_B \to X) \arrow[d, "(\hat{\ })"] \\ X(A) \arrow[r, "G_A(\beta)=\beta_A"] & X(B). \end{tikzcd}$$

commutes for all A in $\mathscr{A}$ and all natural transformations

$$\begin{tikzcd} \mathscr{A}^{\text{op}} \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "X'"'{name=D}] & \mathbf{Set}. \arrow[from=U, to=D, Rightarrow, "\beta"] \end{tikzcd}$$

Again, we compute this directly. For $\alpha : H_A \Rightarrow X$, we have

$$\begin{tikzcd} \alpha \arrow[r, mapsto] \arrow[dd, mapsto] & \beta \circ \alpha \arrow[d, mapsto] \\ & (\beta \circ \alpha)_A(\mathrm{id}_A) \\ \alpha_A(\mathrm{id}_A) \arrow[r, mapsto] & \beta_A(\alpha_A(\mathrm{id}_A)). \end{tikzcd}$$

[2] Recall that if $F : \mathscr{A} \times \mathscr{B} \to \mathscr{C}$ is a functor then for each A in $\mathscr{A}$, we have that $F^A : \mathscr{B} \to \mathscr{C}$ is a functor defined on objects by $F^A(B) = F(A, B)$ and on maps g in $\mathscr{B}$ by $F^A(g) = F(\mathrm{id}_A, g)$. Similarly, for each B in $\mathscr{B}$, the functor $F_B : \mathscr{A} \to \mathscr{C}$ is defined on objects by $F_B(A) = F(A, B)$ and on maps f in $\mathscr{A}$ by $F_B(f) = F(f, \mathrm{id}_B)$.

We get $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ by definition of composition in the functor category $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$. Thus, we have shown the naturality of $(\hat{\ })$ between F_A and G_A. $\square$

Part (2)

How does this part relate to the Yoneda lemma?

In this section we will explain why $[\mathscr{C}^{\mathrm{op}}, \mathscr{D}]$ is the same as $[\mathscr{C} \to \mathscr{D}^{\mathrm{op}}]$. (By $\mathrm{Fun}(\mathscr{C} \to \mathscr{D})$ I assume you mean the category of functors, not the *set* of functors.)

A functor is defined by what it does to objects and what it does to morphisms. A functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ sends objects in $\mathscr{C}$ to objects in $\mathscr{D}$. Secondly, F sends maps $C \xleftarrow{f} C'$ in $\mathscr{C}^{\mathrm{op}}$ to maps $F(C') \xrightarrow{F(f)} F(C)$ in $\mathscr{D}$. But $C \xleftarrow{f} C'$ in $\mathscr{C}^{\mathrm{op}}$ is simply a map $C \xrightarrow{f} C'$ in $\mathscr{C}$. So we can consider F as sending maps $C \xrightarrow{f} C'$ in $\mathscr{C}^{\mathrm{op}}$ to maps $F(C') \xrightarrow{F(f)} F(C)$ in $\mathscr{D}$.

In comparison, a functor $G : \mathscr{C} \to \mathscr{D}^{\mathrm{op}}$ sends objects in $\mathscr{C}$ to objects in $\mathscr{D}$. It sends maps $C \xrightarrow{f} C'$ in $\mathscr{C}$ to maps $G(C) \xrightarrow{G(f)} G(C')$ in $\mathscr{D}^{\mathrm{op}}$. Notice that $G(C) \xrightarrow{G(f)} G(C')$ in $\mathscr{D}^{\mathrm{op}}$ is simply a map $G(C) \xleftarrow{G(f)} G(C')$ in $\mathscr{D}$. Thus, we can consider G as sending maps $C \xrightarrow{f} C'$ in $\mathscr{C}$ to maps $G(C') \xrightarrow{G(f)} G(C)$ in $\mathscr{D}$. But this is exactly the same behaviour as a functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ described above. So we can consider functors $\mathscr{C} \to \mathscr{D}^{\mathrm{op}}$ as functors $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

Also, we can consider natural transformations

$$\begin{tikzcd} \mathscr{C}^{\mathrm{op}} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{D} \end{tikzcd}$$

as natural transformations

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow\beta"] & \mathscr{D}^{\mathrm{op}} \end{tikzcd}$$

in the following way. A natural transformation $\alpha : F \to G$ in $[\mathscr{C}^{\mathrm{op}}, \mathscr{D}]$ is defined to be the components $(F(C) \xrightarrow{\alpha_C} G(C))_{C \in \mathscr{C}^{\mathrm{op}}}$. The components are maps in $\mathscr{D}$. But the objects in $\mathscr{C}^{\mathrm{op}}$ are exactly the objects in $\mathscr{C}$. So we can equally describe α by the components $(F(C) \xrightarrow{\alpha_C} G(C))_{C \in \mathscr{C}}$. The naturality condition states that for every morphism $C \xrightarrow{f} D$ in $\mathscr{C}$, the square of morphisms in $\mathscr{D}$

$$\begin{tikzcd} F(C) \arrow[d, "\alpha_C"'] & F(D) \arrow[l, "F(f)"'] \arrow[d, "\alpha_D"] \\ G(C) & G(D) \arrow[l, "G(f)"] \end{tikzcd}$$

commutes.

Now, a natural transformation $\beta : G \to F$ in $[\mathscr{C}, \mathscr{D}^{\mathrm{op}}]$ is defined to be the components

$$(G(C) \xrightarrow{\beta_C} F(C))_{C \in \mathscr{C}}$$

of maps in $\mathscr{D}^{op}$. Equally, we can describe the components to be maps $(F(C) \xrightarrow{\beta_C} G(C))_{C \in \mathscr{C}}$ in $\mathscr{D}$. The naturality condition states that for every morphism $C \xrightarrow{f} D$ in $\mathscr{C}$, the square of morphisms in $\mathscr{D}^{\text{op}}$

$$\begin{tikzcd} G(C) \arrow[r, "G(f)"] \arrow[d, "\beta_C"'] & G(D) \arrow[d, "\beta_D"] \\ F(C) \arrow[r, "F(f)"] & F(D) \end{tikzcd}$$

commutes. This is equivalent to the statement that the square of morphisms in $\mathscr{D}$

$$\begin{tikzcd} G(C) & G(D) \arrow[l, "G(f)"'] \\ F(C) \arrow[u, "\beta_C"] & F(D) \arrow[l, "F(f)"] \arrow[u, "\beta_D"'] \end{tikzcd}$$

commutes. But this description of β is exactly the same as the description of a natural transformation $\alpha : F \to G$ in $[\mathscr{C}^{\text{op}}, \mathscr{D}]$. So we can interpret natural transformations $G \to F$ in $[\mathscr{C}, \mathscr{D}^{\text{op}}]$ as natural transformations $F \to G$ in $[\mathscr{C}^{\text{op}}, \mathscr{D}]$, and visa versa.

I talked to Kie Seng about the second half of this part.

Claim 4. *We have the following non-equivalence*

$$\mathbf{Set}^{\text{op}} \not\simeq \mathbf{Set}.$$

To prove this we need a lemma.

Lemma 4. *An equivalence between categories sends initial objects to initial objects and terminal objects to terminal objects.*

I looked at `https://math.stackexchange.com/a/840065` when proving this.

Proof. Suppose $F : \mathscr{C} \to \mathscr{D}$ is an equivalence and I an initial object of $\mathscr{C}$. We will show that $F(I)$ is initial in $\mathscr{D}$.

Let D be an object in $\mathscr{D}$. Since F is essentially surjective, there is some object C in $\mathscr{C}$ and some isomorphism $F(C) \xrightarrow{\phi} D$. (Note that it is not correct to say that $F(I) \xrightarrow{F(f)} F(C) \xrightarrow{\phi} D$ is the only map $F(I) \to D$, since we do not know that the isomorphism ϕ is the only map $F(C) \to D$.)

We claim that we have a bijection

$$\begin{aligned} \text{Hom}_{\mathscr{D}}(F(I) \to F(C)) &\longrightarrow \text{Hom}_{\mathscr{D}}(F(I) \to D) \\ f &\longmapsto \phi \circ f \end{aligned}$$

We see that this map is injective since if $\phi \circ f = \phi \circ g$ then $f = \phi^{-1} \circ \phi \circ f = \phi^{-1} \circ \phi \circ g = g$. Also, given $F(I) \xrightarrow{g} D$, we have that $\phi^{-1} \circ g$ is a map $F(I) \to F(C)$ with $\phi \circ (\phi^{-1} \circ g) = g$. So the map is also surjective.

Moreover, since F is full and faithful, we have a bijection

$$\text{Hom}_{\mathscr{C}}(I \to C) \longrightarrow \text{Hom}_{\mathscr{D}}(F(I) \to F(C)).$$

(It suffices to have a surjection, since we know that $\mathrm{Hom}_{\mathscr{C}}(I \to C)$ is a singleton.)

Therefore, we have a bijection between $\mathrm{Hom}_{\mathscr{C}}(I \to C)$ and $\mathrm{Hom}_{\mathscr{D}}(F(I) \to D)$ for any object D and so $F(I)$ must be initial.

The proof of the second half of the lemma is analogous. □

Proof of claim 4. Suppose $F : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ is an equivalence. We know that $\emptyset$ is initial in $\mathbf{Set}$ and the set $\mathbf{1}$ of one object is initial in $\mathbf{Set}^{\mathrm{op}}$. By the above lemma, $F(\emptyset) \cong \mathbf{1}$.

Also, $\mathbf{1}$ is terminal in $\mathbf{Set}$ and $\emptyset$ is terminal in $\mathbf{Set}^{\mathrm{op}}$. So $F(\mathbf{1}) \cong \emptyset$. Now, suppose $S \neq \emptyset$ is a set. There exists a map $\mathbf{1} \xrightarrow{f} S$ in $\mathbf{Set}$. Then $F(S) \xrightarrow{F(f)} F(\mathbf{1})$ is also a map in $\mathbf{Set}$. Since $F(\mathbf{1}) \cong \emptyset$, this implies we have a map $F(S) \to \emptyset$ in $\mathbf{Set}$. Therefore, $F(S) = \emptyset$.

Combining the results of the preceding two paragraphs, we get that, for any set R,

$$F(R) \cong \begin{cases} \mathbf{1} & \text{if } S = \emptyset, \\ \emptyset & \text{if } S \neq \emptyset. \end{cases}$$

This implies F is not essentially surjective. □

Claim 5. *We have the following equivalence*

$$\mathbf{fdVec}^{\mathrm{op}} \cong \mathbf{fdVec}.$$

Proof. This follows from the fact that $[\mathscr{C}^{\mathrm{op}}, \mathscr{D}] = [\mathscr{C}, \mathscr{D}^{\mathrm{op}}]$ for any categories $\mathscr{C}$ and $\mathscr{D}$. Consider the dual functor $(\ )^* : \mathbf{fdVec} \to \mathbf{fdVec}^{\mathrm{op}}$. We can also consider it as a functor $(\ )^* : \mathbf{fdVec}^{\mathrm{op}} \to \mathbf{fdVec}$. We will show that there are natural isomorphisms $(\ )^* \circ (\ )^* \to \mathrm{id}_{\mathbf{fdVec}}$ and $\mathrm{id}_{\mathbf{fdVec}^{\mathrm{op}}} \to (\ )^* \circ (\ )^*$.

In example 1.3.4, Leinster constructs a natural isomorphism

$$\begin{tikzcd} \mathbf{fdVec} \arrow[r, bend left=50, "(\ )^* \circ (\ )^*"] \arrow[r, bend right=50, "\mathrm{id}_{\mathbf{fdVec}}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{fdVec} \end{tikzcd}$$

This gives us one of the required natural isomorphisms. We can rewrite this as

$$\begin{tikzcd} (\mathbf{fdVec}^{\mathrm{op}})^{\mathrm{op}} \arrow[r, bend left=50, "(\ )^* \circ (\ )^*"] \arrow[r, bend right=50, "\mathrm{id}_{\mathbf{fdVec}}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{fdVec} \end{tikzcd}$$

Applying the result $[(\mathbf{fdVec}^{\mathrm{op}})^{\mathrm{op}}, \mathbf{fdVec}] = [\mathbf{fdVec}^{\mathrm{op}}, \mathbf{fdVec}^{\mathrm{op}}]$, we get that

$$\begin{tikzcd} \mathbf{fdVec}^{\mathrm{op}} \arrow[r, bend left=50, "(\ )^* \circ (\ )^*"] \arrow[r, bend right=50, "\mathrm{id}_{\mathbf{fdVec}^{\mathrm{op}}}"'] \arrow[r, phantom, "\Uparrow\alpha"] & \mathbf{fdVec}^{\mathrm{op}} \end{tikzcd}$$

is also a natural isomorphism. This is the other required natural isomorphism. □