

Cat. theory Assignment 2 - Xiang Li

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2.1 Idempotent completion

1) $\text{Kar}(\text{primeVec})$ is equivalent to FdVec ; ie

$$\begin{array}{ccc} (V, p) & \xrightarrow{\alpha} & pV \\ f \downarrow & \xrightarrow{\quad} & \downarrow p \\ (W, q) & \xrightarrow{\quad} & qW \end{array} \quad \begin{array}{l} \text{is full, faithful and surjective} \\ \text{if } pV \xrightarrow{f} qW \end{array}$$

proof: well define: if $pV = pV'$, then $fV = fpV = fpV' = fV'$ \square

α is a functor $\alpha(1_{V,p}) = \alpha(p: V \rightarrow V) = \begin{pmatrix} pV \rightarrow pV \\ pV \rightarrow pV \end{pmatrix} = 1_{pV}$

$$\begin{array}{ccccc} (V, p) & \xrightarrow{f} & (W, q) & \xrightarrow{g} & (X, r) \\ \alpha f \downarrow & & \alpha g \downarrow & & \downarrow \alpha r \\ \alpha g \alpha f(p) & = & \alpha g(\alpha f(p)) & \xrightarrow{\alpha g} & \alpha g \alpha f(p) \end{array} \quad \begin{array}{l} \alpha g \alpha f(p) = \alpha g(f(p)) = \alpha(gf(p)) = \alpha(gf)(p) \\ \alpha g \alpha f(p) = \alpha(gf)(p) \end{array} \quad \square$$

α is surjective: given $W \in \text{FdVec}$ one can form $W \times K^n \in \text{primeVec}$ (for a suitable n) and $\alpha(W \times K^n, \pi_W) = W$

α is full given $h: pV \rightarrow qW$ define: $\tilde{h}: V \rightarrow W$
 $\tilde{h}: v \mapsto h(pv)$

- \tilde{h} is a map $(V, p) \rightarrow (W, q)$ since $\tilde{h}pv = h(ppv) = h(pv)$ and $q\tilde{h}v = qh(pv) = h(pv)$

- $h = \alpha(\tilde{h})$ since $\alpha(\tilde{h})(p) = \alpha(h(p)) = h(p)$

α is faithful:

$$(V, p) \xrightarrow{f} (W, q) \quad \text{if } \alpha f = \alpha f' \text{ then } fV = \alpha f pV = \alpha f' pV = f'V$$

2.1.2 $KarKar \cong KarC$

Notation: For maps f in C , sometimes f, \tilde{f} and \hat{f} denote the corresponding maps in $KarC$ and $KarKarC$.

$$\begin{array}{ccc} KarC & \xrightarrow{\alpha} & KarKarC \\ (X, \rho) & \xrightarrow{\quad} & (X, \rho, \tilde{\rho}) \\ \downarrow \tilde{f} & \xrightarrow{\quad} & \alpha(\tilde{f}) = \hat{f} \downarrow \\ (Y, \rho) & \xrightarrow{\quad} & (Y, \rho, \tilde{\rho}) \end{array}$$

The inverse natural transformation is $\beta: (X, \rho, \tilde{\rho}) \mapsto (X, \rho)$.
 $\alpha\beta(X, \rho, \tilde{\rho}) = (X, \rho, \tilde{\rho}) \cong (X, \rho, \tilde{\rho})$ See ess. surj below
 $\beta\alpha(X, \rho) = X, \rho$
 $\Rightarrow \beta$ is α 's inverse

Since \hat{f} and \tilde{f} are basically f α is like the identity on maps. Therefore α is full and faithful.

full: given $(X, \rho, \tilde{\rho})$. This implies $\tilde{f}\tilde{\rho} = \tilde{\rho} = \tilde{\rho}\tilde{f}$ and $f\rho = \rho = \rho f$.
 $\downarrow \tilde{f}$ And therefore $\tilde{f}: (X, \rho) \xrightarrow{f} (Y, \rho)$. $\alpha(\tilde{f}) = \hat{f}$
 $((Y, \rho), \tilde{\rho})$

faithful:

\tilde{f} undelices $\alpha(\tilde{f})$

ess. surj:

$$(X, \rho, \tilde{\rho}) \xrightleftharpoons[\rho]{\rho} (X, \rho, \tilde{\rho})$$

is an isomorphism since $id_{(X, \rho)} = \rho = \rho^2$

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & KarC \\ x & \xrightarrow{\quad} & (x, 1_x) \\ \downarrow g & \xrightarrow{\quad} & g \downarrow \\ y & \xrightarrow{\quad} & (y, 1_y) \end{array}$$

$\alpha(g)$ well defined because of the identity property of $1_x, 1_y$.
 $\alpha(g)$ is basically g . Therefore α is full and faithful.

2.1.3

2.1.4

$$I: \text{Kar } C \rightarrow \text{Kar } D \cong L_D \circ \text{Kar } F_C$$

1. Understand the functors.

This only a ^{collection of ideas} draft, not a solution.
I only came up with a one-sided inverse on $F(x, p)$.

$$F: \text{Kar } C \rightarrow \text{Kar } D$$

$$(x, p) \mapsto F(x, p)$$

$$h \mapsto Fh$$

$$F_C: C \xrightarrow{F_C} \text{Kar } D$$

$$x \mapsto (x, 1_x) \mapsto F(x, 1_x)$$

$$\downarrow g \quad \downarrow g \quad \downarrow Fg$$

$$x' \mapsto (x', 1_{x'}) \mapsto F(x', 1_{x'})$$

$$L_D \circ \text{Kar } F_C: \text{Kar } C \xrightarrow{\text{Kar } F_C} \text{Kar } D \xrightarrow{L_D} \text{Kar } D$$

$$(x, p) \mapsto (F_C(x), F_C(p)) \mapsto F(x, 1_x)$$

$$\downarrow g \quad \downarrow Fg \quad \downarrow Fg$$

$$(x', p') \mapsto (F(x', 1_{x'}), F(p')) \mapsto F(x', 1_{x'})$$

2. Construct Natural Transformations $(F(x, p) \xrightarrow{Fp} F(x, 1_x))_{(x, p) \in \text{Kar } C}$

well defined:

Fp is a map: p is a map $(x, p) \xrightarrow{p} (x, 1_x)$ because $p^2 = p = 1_x p$
Since F is a functor Fp is a map $F(x, p) \xrightarrow{Fp} F(x, 1_x)$ in $\text{Kar } D$

Fp is a natural transformation

given $(x, p) \xrightarrow{h} (y, q)$ we have to check

$$F(x, p) \xrightarrow{Fh} F(y, q)$$

$$\downarrow Fp \quad \downarrow Fq$$

$$F(x, 1_x) \xrightarrow{Fh \circ p} F(y, 1_y)$$

This is true since $Fq \circ Fh = F(qh) = F(hp) = Fh \circ Fp$

Fp is a self inverse isomorphism?

$Fp \circ Fp$ is the identity on $F(x, p)$ since $\text{id}_{(x, p)} = p$ and F preserves isomorphisms.
How about $Fp \circ Fp \stackrel{?}{=} \text{id}_{F(x, 1_x)} = F(1_x)$?

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2.2 Cones

Define a functor: Cones

$$\longrightarrow (\text{val} \downarrow \tilde{F}) = \text{comma category}$$

object: $X \in C, (x \xrightarrow{f_j} F(j))_{j \in J} \longmapsto (X, \cdot, x = \text{val}_X(j) \xrightarrow{f_j} F(j))_{j \in J}$

This means

$$C \xrightarrow{\text{val}} \text{Fun}(J, C) \xleftarrow{\tilde{F}} 1$$

$$x \mapsto \left(\begin{array}{c} \text{val}_X: J \rightarrow C \\ j \mapsto x \\ j, h \mapsto \text{id}_x \\ j, h' \mapsto x \end{array} \right) \quad \begin{array}{c} F \longleftarrow 1 \\ j, \text{id} \longleftarrow \text{id}_j \\ F \longleftarrow 1 \end{array}$$

satisfying for $g: x \rightarrow x'$

$$\begin{array}{ccc} \text{val}_X \xrightarrow{\text{val}_g = (- \circ g)_{j \in J}} & \text{val}_{X'} & \text{ie } \forall j \in J: f_j = f'_j \circ g \\ \downarrow (f_j)_j & \downarrow (f'_j)_j & \\ F & \xrightarrow{\text{id}} & F \end{array}$$

map: $\begin{array}{c} X, (x \xrightarrow{f_j} F(j))_j \\ \downarrow g \\ X', (x' \xrightarrow{f'_j} F(j))_j \end{array} \longmapsto (g, \text{id}) \begin{array}{c} (X, \cdot, (f_j)_j) \\ g \downarrow \quad \downarrow \text{id} \\ (X', \cdot, (f'_j)_j) \end{array}$

satisfying $f'_j = f_j \circ g$ (2)

This functor is full: Since 1 only has the identity map, all maps in $\text{val} \downarrow \tilde{F}$ are of the form (g, id) .

g is a preimage as

faithful:

(2) and (1) are the same
let (g, id) and (g', id) be arbitrary maps.

$$\text{Then } (g, \text{id}) = (g', \text{id}) \iff g = g'$$

surjective: $(X', \cdot, (x' = \text{val}_{X'}(j) \xrightarrow{f'_j} F(j))_j)$ has the preimage $X, (x \xrightarrow{f_j} F(j))_j$ because

for $h: j \rightarrow j'$ $x \xrightarrow{f_j} F(j) \xrightarrow{Fh} F(j')$ is satisfied ~~as~~ due to the naturality $f_{j'} = f_j \circ h$

$$\text{of } (f_j)_j: \quad \begin{array}{ccc} x = \text{val}(x(j)) & \xrightarrow{f_j} & F_j \\ \text{id}_x \downarrow & \curvearrowright & \downarrow f_h \\ x = \text{val}(x(j')) & \xrightarrow{f_{j'}} & F_{j'} \end{array}$$

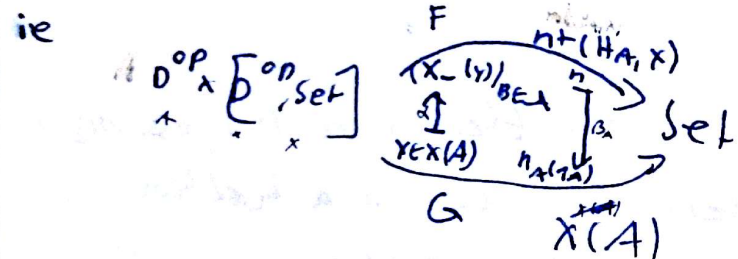
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\Rightarrow The category of Cones is equivalent to the comma category

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Yoneda Lemma 2.3.1

Let \mathcal{D} be a locally small category. Then $[\mathcal{D}^{op}, \text{Set}](H_A, x) \cong x(A)$ naturally in $A \in \mathcal{D}, x \in [\mathcal{D}^{op}, \text{Set}]$



α and β are natural transformations and mutually inverse.

F denotes the functor

$$\begin{array}{ccccc}
 \mathcal{D}^{op} \times [\mathcal{D}^{op}, \text{Set}] & \xrightarrow{H_A \times 1} & [\mathcal{D}^{op}, \text{Set}]^{op} \times [\mathcal{D}^{op}, \text{Set}] & \xrightarrow{\text{Hom}_{[\mathcal{D}^{op}, \text{Set}]}} & \text{Set} \\
 (A, x) & \longmapsto & (H_A, x) & \longmapsto & \text{nt}(H_A, x) \\
 \uparrow \text{no } f & & \downarrow (- \circ f) = (f \circ -, y) & & \downarrow g \circ \text{no}(f \circ -) \\
 (A', x') & \longmapsto & (H_{A'}, x') & \longmapsto & \text{nt}(H_{A'}, x')
 \end{array}$$

G denotes the functor

$$\begin{array}{ccc}
 \mathcal{D}^{op} \times [\mathcal{D}^{op}, \text{Set}] & \longrightarrow & \text{Set} \\
 (A, x) & \longmapsto & x(A) \\
 \uparrow \text{no } f & & \downarrow g_{A'}(xf -) \\
 (A', x') & \longmapsto & x'(A')
 \end{array}$$

2.3.1 (continued)

β is natural in X
 Let $g: X \rightarrow X'$ be a map in $[D^{op}, \text{Set}]$ (i.e. a natural transformation)

$$\begin{array}{ccc}
 n_1 & \xrightarrow{\quad} & g_B(n_B(-)) \\
 \downarrow \beta_A & \searrow g_{A,0} & \downarrow \beta_A \\
 n_1(H_A, X) & \xrightarrow{\quad} & n_1(H_A, X') \\
 \downarrow \beta_A & & \downarrow \beta_A \\
 X(A) & \xrightarrow{g_A} & X'(A) \\
 \downarrow & & \downarrow \\
 n_A(1_A) & \xrightarrow{\quad} & g_A(n_A(1_A))
 \end{array}$$

β is natural in X and A

It follows from the following lemma:



$$\begin{array}{ccccc}
 F(A, X) & \xrightarrow{b} & F(A', X) & \xrightarrow{d} & F(A', X') \\
 \downarrow a & & \downarrow c & & \downarrow g \\
 G(A, X) & \xrightarrow{e} & G(A', X) & \xrightarrow{f} & G(A', X')
 \end{array}$$

commutativity follows from the naturality in A and X :
 $gdb = feb = fca$

2.3.2

$$\begin{array}{ccc}
 \text{fdvec}^{op} & \xrightarrow{\quad} & \text{fdvec} \\
 V & \xrightarrow{\alpha} & V^* \\
 f & \mapsto & (- \circ f) \equiv f^* \\
 V^* & \xleftarrow{\beta} & V \\
 f^* & \xleftarrow{\quad} & f
 \end{array}$$

$$\alpha(\beta(V)) = V^{**} = \beta(\alpha(V)) \\
 \text{id}_V(V) = V$$

Since V and V^{**} are naturally isomorphic α and β are part of the equivalence

Since $\text{fdvec} \xrightarrow{\quad} \text{fdvec}^{op}$ is a functor, it is also functor $\text{fdvec}^{op} \rightarrow \text{fdvec}$ because of $\text{Functor}(C^{op} \rightarrow D) = \text{Fun}(C \rightarrow D^{op})$
 proof: $\varepsilon: C^{op} \xrightarrow{\alpha} D$. Then in D : $\alpha(fg) = \alpha(gf) = \alpha g \alpha f$

$$\text{given two maps } f, g \text{ in } C: \alpha(fg) = \alpha g \alpha f \text{ in } D \\
 = \alpha f \alpha g \text{ in } D^{op}$$

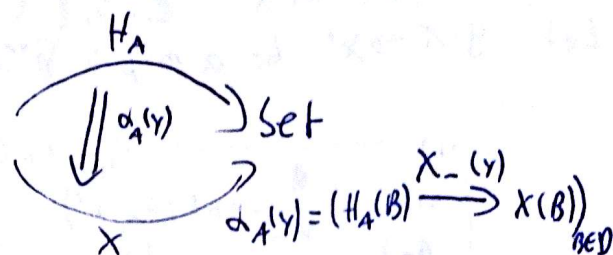
The identity property is trivial.

\exists : replace C and D by C^{op} and D^{op} and use $C^{opop} = C$

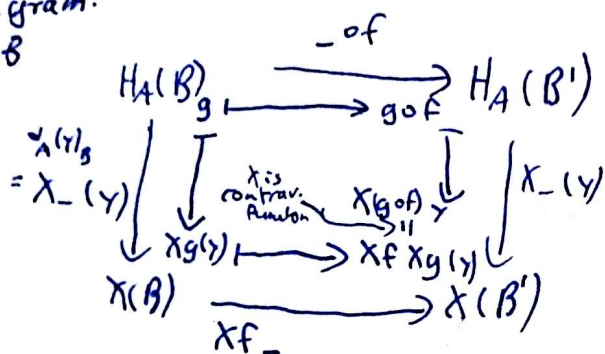
α is well-defined

ie for $A \in D, X \in \mathcal{K}(A)$ $\alpha_A(Y)$ is a nat. tr.

0^{op}



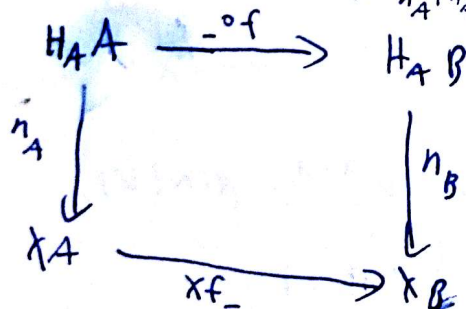
check nat. diagram:
given $f: B' \rightarrow B$



$\alpha \circ \beta = id$

given $A \in D^{op}, X \in [D^{op}, Set]$ we need to check that for any $n \in nat(H_A, X)$ ie $n = (H_A(B) \xrightarrow{\eta_B} X(B))_{B \in D}$
 $\alpha \circ n$ equals $\alpha \circ \beta(n) = (H_A(B) \xrightarrow{X_{\beta(n)}} X(B))_{B \in D}$

proof. For $f \in H_A(B)$ $\eta_B(f) = Xf \eta_A(1_A)$ follows from the naturality of n and $1_A \in H_A(A)$



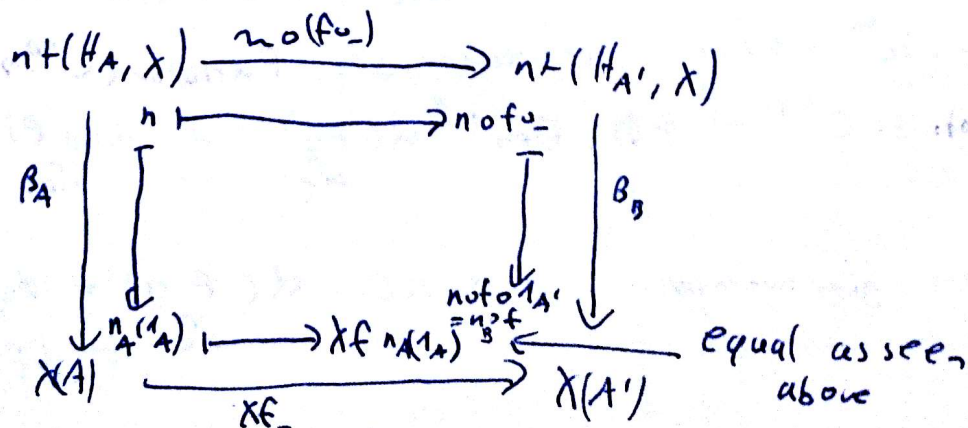
$\beta \circ \alpha = id$

given $X, A, Y \in \mathcal{K}(A)$:

$$\beta \circ \alpha(Y) = X_{1_A}(Y) = 1_{X(A)}(Y) = Y$$

β is natural in A

Let $f: A' \rightarrow A$ in D . Then



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2.3.2 continued

$\text{Set}^{\text{op}} \neq \text{Set}$

If two categories are equivalent, then there is a full, ess. surj. and faithful functor F between them. Therefore there is a bijection between maps $A \rightarrow A'$ and maps $FA \rightarrow FA'$. It follows that A is terminal iff FA is terminal.

In Set there is a map from its terminal object $*$ to a non-terminal object X . (Any function with codomain $\neq \emptyset$). If there is an equivalence this must also be true in Set^{op} .

This would mean that there is a function from the initial object $*$ in $\text{Set}(\emptyset)$ to a non-initial object (non-empty set). However, there is no such function.