Here are some examples. **Ab** is reflective in **Grp**. For, if $G/[G,G]$ is the usual factor-commutator group of a group G, then $\hom(G/[G,G],A)\cong\hom(G,A)$ for A abelian, and **Ab** is full in **Grp**. Or consider the category of all metric spaces X, with arrows uniformly continuous functions. The (full) subcategory of complete metric spaces is reflective; the reflector sends each metric space to its completion. Again, consider the category of all completely regular Hausdorff spaces (with arrows all continuous functions). The (full) subcategory of all compact Hausdorff spaces is reflective; the reflector sends each completely regular space to its Stone-Čech compactification.

A coreflective subcategory of **Ab** is the full subcategory of all torsion abelian groups (a group is torsion if all elements have finite order); the coreflector sends each abelian group A to the subgroup TA of all elements of finite order in A.

Exercises

1. Show that the table of dual statements (§ II.1) extends as follows:

Statement	*Dual statement*
$S, T\colon C\to B$ are functors	$S, T\colon C\to B$ are functors
T is full	T is full
T is faithful	T is faithful
$\eta\colon S\dot\to T$ is a natural transformation.	$\eta\colon T\dot\to S$ is a natural transformation.
$\langle F, G, \varphi\rangle\colon X\rightharpoonup A$ is an adjunction	$\langle G, F, \varphi^{-1}\rangle\colon A\rightharpoonup X$ is an adjunction
η is the unit of $\langle F, G, \varphi\rangle$.	η is the counit of $\langle G, F, \varphi^{-1}\rangle$.

2. Show that the torsion-free abelian groups form a full reflective subcategory of **Ab**.
3. If $\langle G, F, \varphi\rangle\colon X\rightharpoonup A$ is an adjunction with G full and every unit η_x a monic, then every η_x is also epi.
4. Show the following subcategories to be reflective:
   (a) The full subcategory of all partial orders in the category **Preord** of all preorders, with arrows all monotone functions.
   (b) The full subcategory of T_0-spaces in **Top**.
5. Given an adjunction $\langle F, G, \varphi\rangle\colon X\rightharpoonup A$, prove that G is faithful if and only if φ^{-1} carries epis to epis.
6. Given an adjunction $\langle F, G, \eta, \varepsilon\rangle$ with either F or G full, prove that $G\varepsilon\colon GFG\to G$ is invertible with inverse $\eta G\colon G\to GFG$.
7. If A is a full and reflective subcategory of B, prove that every functor $S\colon J\to A$ with a limit in B has a limit in A.

4. Equivalence of Categories

A functor $S\colon A\to C$ is an *isomorphism* of categories when there is a functor $T\colon C\to A$ (backwards) such that $ST=1\colon C\to C$ and $TS=1\colon A\to A$. In this case, the identity natural transformations

$\eta\colon 1\dot\to ST$ and $\varepsilon\colon TS\dot\to 1$ make $\langle T, S; \eta, \varepsilon\rangle\colon C\rightharpoonup A$ an adjunction. In other words, a two-sided inverse T of a functor S is a left-adjoint of S – and for that matter, T is also a right-adjoint of S.

There is a more general (and more useful) notion:

A functor $S\colon A\to C$ is an *equivalence of categories* (and the categories A and C are *equivalent*) when there is a functor $T\colon C\to A$ (backwards) and natural isomorphisms $ST\cong 1\colon C\to C$ and $TS\cong 1\colon A\to A$. In this case $T\colon C\to A$ is also an equivalence of categories. We shall soon see that T is then both a left adjoint and a right adjoint of S.

Here is an example. In any category C a *skeleton* of C is any full subcategory A such that each object of C is isomorphic (in C) to exactly one object of A. Then A is equivalent to C and the inclusion $K\colon A\to C$ is an equivalence of categories. For, select to each $c\in C$ an isomorphism $\theta_c\colon c\cong Tc$ with Tc an object of A. Then we can make T a functor $T\colon C\to A$ in exactly one way so that θ will become a natural isomorphism $\theta\colon 1\cong KT$. Moreover $TK\cong 1$, so K is indeed an equivalence: *A category is equivalent to (any one of) its skeletons.* For example, the category of all finite sets has as a skeleton the full subcategory with objects all finite ordinal numbers $0, 1, 2, \ldots, n, \ldots$. (Here 0 is the empty set and each $n=\{0, 1, \ldots, n-1\}$.)

A category is called *skeletal* when any two isomorphic objects are identical; i.e., when the category is its own skeleton.

An *adjoint equivalence* of categories is an adjunction $\langle T, S; \eta, \varepsilon\rangle\colon C\rightharpoonup A$ in which both the unit $\eta\colon I\dot\to ST$ and the counit $\varepsilon\colon TS\dot\to I$ are natural isomorphisms: $I\cong ST$, $TS\cong I$. Then η^{-1} and ε^{-1} are also natural isomorphisms, and the triangular identities $\varepsilon T\cdot T\eta=1$, $S\varepsilon\cdot\eta S=1$ can be written as $T\eta^{-1}\cdot\varepsilon^{-1}T=1$, $\eta^{-1}S\cdot S\varepsilon^{-1}=1$, respectively. These identities then state that $\langle S, T, \varepsilon^{-1}, \eta^{-1}\rangle\colon A\rightharpoonup C$ is an adjunction with $\varepsilon^{-1}\colon I\dot\to TS$ as unit and $\eta^{-1}\colon ST\dot\to I$ as counit. Thus in an adjoint equivalence $\langle T, S, -, -\rangle$ the functor $T\colon C\to A$ is the left adjoint of $S\colon A\to C$ with unit η and at the same time T is the right adjoint of S, with unit ε^{-1}.

We can now state the main facts about equivalence.

Theorem 1. *The following properties of a functor $S\colon A\to C$ are logically equivalent:*

i) *S is an equivalence of categories,*

ii) *S is part of an adjoint equivalence $\langle T, S; \eta, \varepsilon\rangle\colon C\rightharpoonup A$,*

iii) *S is full and faithful, and each object $c\in C$ is isomorphic to Sa for some object $a\in A$.*

Proof. Trivially, (ii) implies (i). To prove that (i) implies (iii), note that $ST\cong 1$ shows that each $c\in C$ has the form $c\cong S(Tc)$ for an $a=Tc\in A$. The natural isomorphism $\theta\colon TS\cong I$ gives for each $f\colon a\to a'$ the commutative square

$$\begin{tikzcd} TSa \arrow[r, "\theta_a"] \arrow[d, "TSf"'] & a \arrow[d, "f"] \\ TSa' \arrow[r, "\theta_{a'}"'] & a' . \end{tikzcd}$$

Hence $f=\theta_{a'}\circ TSf\circ\theta_a^{-1}$; it follows that S is faithful. Symmetrically, $ST\cong I$ proves T faithful. To show S full, consider any $h: Sa\to Sa'$ and set $f=\theta_{a'}\circ Th\circ\theta_a^{-1}$. Then the square above commutes also with Sf replaced by h, so $TSf=Th$. Since T is faithful, $Sf=h$, which means that S is full.

To prove that (iii) implies (ii) we must construct from S a (left) adjoint T. For each $c\in C$ we can choose some object $a_0=T_0c\in A$ and an isomorphism η_c:

$$\begin{tikzcd} \eta_c : c \arrow[r, phantom, "\cong"] \arrow[dr, "f"'] & S(T_0 c) \arrow[d, "Sg"] \\ & Sa \end{tikzcd} \qquad g: T_0 c\to a .$$

For every arrow $f: c\to Sa$, the composite $f\circ\eta_c^{-1}$ has the form Sg for some g because S is full; this g is unique because S is faithful. In other words, $f=Sg\circ\eta_c$ for a unique g, so η_c is universal from c to S. Therefore T_0 can be made a functor $T: C\to A$ in exactly one way so that $\eta: I\dot{\to} ST$ is natural, and then T is the left adjoint of S with unit the isomorphism η. As with any adjunction, $S\varepsilon_a\cdot\eta_{Sa}=1$ (put $c=Sa$, $f=1$ in the diagram above). Thus $S\varepsilon_a=(\eta_{Sa})^{-1}$ is invertible. Since S is full and faithful, the counit ε_a is also invertible. Therefore $\langle T,S;\eta,\varepsilon\rangle: C\rightharpoonup A$ is an adjoint equivalence, and the proof is complete.

In this proof, suppose that A is a full subcategory of C and that $S=K: A\to C$ is the insertion. For objects $a\in A\subset C$ we can then choose $a_0=a=Ka$ and η_{Ka} the identity. Then $K\varepsilon_a=1$, hence $\varepsilon_a=1$ for all a. This proves

Proposition 2. *If A is a full subcategory of C and every $c\in C$ is isomorphic (in C) to some object of A, then the insertion $K: A\to C$ is an equivalence and is part of an adjoint equivalence $\langle T,K;\eta,1\rangle: C\rightharpoonup A$ with counit the identity. Therefore A is reflective in C.*

This includes in particular the case already noted, when A is a skeleton of C.

A functor $F: X\to A$ is said to be a *left-adjoint-left-inverse* of $G: A\to X$ when there is an adjunction $\langle F,G;\eta,1\rangle: X\rightharpoonup A$ with counit the identity. This means (Exercise 4) that G is an isomorphism of A to a reflective subcategory of X. In the case of the Proposition 2 just above, we have shown that the insertion $A\to C$ has a left-adjoint-left-inverse.

Duality theorems in functional analysis are often instances of equivalences. For example, let **CAb** be the category of compact topological abelian groups, and let P assign to each such group G its character group PG, consisting of all continuous homomorphisms $G\to\mathbf{R}/\mathbf{Z}$. The Pontrjagin duality theorem asserts that $P: \mathbf{CAb}\to\mathbf{Ab}^{\mathrm{op}}$ is an equivalence of categories. Similarly, the Gelfand-Naimark theorem states that the functor C which assigns to each compact Hausdorff space X its abelian C^*-algebra of continuous complex-valued functions is an equivalence of categories (see Negrepontis [1971]).

Exercises

1. Prove: (a) Any two skeletons of a category C are isomorphic.
   (b) If A_0 is a skeleton of A and C_0 a skeleton of C, then A and C are equivalent if and only if A_0 and C_0 are isomorphic.
2. (a) Prove: the composite of two equivalences $D\to C$, $C\to A$ is an equivalence.
   (b) State and prove the corresponding fact for adjoint equivalences.
3. If $S: A\to C$ is full, faithful, and surjective on objects (each $c\in C$ is $c=Sa$ for some $a\in A$), prove that there is an adjoint equivalence $\langle T,S;1,\varepsilon\rangle: C\rightharpoonup A$ with unit the identity (and thence that T is a left-adjoint-right-inverse of S).
4. Given a functor $G: A\to X$, prove the three following conditions logically equivalent:
   (a) G has a left-adjoint-left-inverse,
   (b) G has a left adjoint, and is full, faithful, and injective on objects.
   (c) There is a full reflective subcategory Y of X and an isomorphism $H: A\cong Y$ such that $G=KH$, where $K: Y\to X$ is the insertion.
5. If J is a connected category and $\Delta: C\to C^J$ has a left adjoint (colimit), show that this left adjoint is a left-adjoint-left-inverse.

5. Adjoints for Preorders

Recall that a preorder P is a set $P=\{p,p',\dots\}$ equipped with a reflexive and transitive binary relation $p\leqq p'$, and that preorders may be regarded as categories so that order-preserving functions become functors. An order-reversing function $\bar{L}$ on P to Q is then a functor $L: P\to Q^{\mathrm{op}}$.

Theorem 1. *(Galois connections are adjoint pairs). Let P, Q be two preorders and $L: P\to Q^{\mathrm{op}}$, $R: Q^{\mathrm{op}}\to P$ two order-preserving functions. Then L (regarded as a functor) is a left adjoint to R if and only if, for all $p\in P$ and $q\in Q$,*

$$Lp\geqq q \text{ in } Q \text{ if and only if } p\leqq Rq \text{ in } P. \tag{1}$$

When this is the case, there is exactly one adjunction φ making L the left adjoint of R. For all p and q, $p\leqq RLp$ and $LRq\geqq q$; hence also

$$Lp\geqq LRLp\geqq Lp, \qquad Rq\leqq RLRq\leqq Rq. \tag{2}$$