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CATEGORY THEORY ASSIGNMENT TWO

1 Idempotent Completion

The 'idempotent completion' $\mathrm{Kar}(\mathcal{C})$ (also called the 'Karoubi envelope') is defined as follows:

$$\begin{aligned} \mathrm{Obj}\,\mathrm{Kar}(\mathcal{C}) &= \left\{(X \in \mathrm{Obj}(\mathcal{C}), p : X \to X) \mid |p^2 = p\right\} \\ \mathrm{Kar}(\mathcal{C})((X,p) \to (X',p')) &= \{f \in \mathcal{C}(X \to X') \mid |fp = f = p'f\}\,. \end{aligned}$$

Question 1.1. Let $\mathsf{primeVec}$ denote the full subcategory of Vec consisting of vector spaces with prime dimensions. Show carefully that $\mathrm{Kar}(\mathsf{primeVec}) \cong \mathsf{Vec}$.

First observe that if V is a vector space and $f : V \to V$ is a linear transformation such that $f^2 = f$, then $V \cong fV \oplus (1-f)V$. This is because if $v \in V$ is in the image of f, then there exists $w \in V$ such that $f(w) = v$, so $f(f(w)) = v$ so $f(v) = v$. Then for $v \in V$ is not in the image of f, it must be in the kernel of f, so $(1-f)(v) = v$. Now suppose there exists $v \in V, v \neq 0$ such that $v \in fV \cap (1-f)V$, since f and $(1-f)$ are all idempotent, we know $f(v) = v$ and $(1-f)v = v$. But we know that $(1-f)f = 0$ so $fV \cap (1-f)V = \{0\}$.

Now consider $f : (V,p) \to (W,p')$ a morphism $\mathrm{Kar}(\mathsf{Vec})$, we can see f as a morphism from $pV \oplus (1-p)V \to p'W \oplus (1-p')W$. Since$fp = f = p'f$, the kernel of f is $(1-p)V$ and the range is $p'W$.

Define a functor $F : \mathrm{Kar}(\mathsf{primeVec}) \to \mathsf{Vec}$ to send (V,p) to pV, and $f : (X,p) \to (X',p')$ to $F(f) : pX \to p'X'$ where we restrict f to the domain pX and cut down the range to $p'X'$ (note that this is possible due to our previous observations). It is easy to see that F is indeed functorial and sends identities and to identities. Given $W \in \mathsf{Vec}$ with dimension n then there exist prime $q > n$, let V be a vector space with dimension q and $f : V \to V$ the projection into a n-dimensional subspace, then $F(V,q) \simeq W$, so F is essentially surjective. Let $(V,p),(V',p') \in \mathrm{Kar}(\mathsf{primeVec})$. For any $f : pV \to p'V' \in \mathsf{Vec}(pV,p'V')$, we can extend this to $f' : V \to V'$ by setting $f'((1-p)V) = 0$, then $f'p = p'f' = f'$. So F is full. Suppose $f, f' \in \mathrm{Kar}(\mathsf{primeVec})((V,p),(V',p'))$ and $F(f) = F(f')$, then $f = f'$ since f, f' sends $(1-p)V$ to 0 and $fp = f'p'$. So F is also faithful and we have an equivalence of categories.

Question 1.2. Construct an equivalence $\iota_{\mathcal{C}} : \mathrm{Kar}(\mathrm{Kar}(\mathcal{C})) \cong \mathrm{Kar}(\mathcal{C})$.

Define a functor $F : \operatorname{Kar}\operatorname{Kar}(\mathcal{C}) \to \operatorname{Kar}(\mathcal{C})$, that sends $((X,p),q)$ to (X,q) and $f : ((X,p),q) \to ((X',p'),q')$ to $f : (X,q) \to (X',q')$ (Functoriality is easy to see).

Given $(X,p) \in \operatorname{Kar}(\mathcal{C})$, then $F((X,p),p) = (X,p)$, so F is essentially surjective. Given $((X,p),q),((X',p'),q') \in \operatorname{Kar}\operatorname{Kar}(\mathcal{C})$ and $f \in \operatorname{Kar}(\mathcal{C})((X,q),(X',q'))$. $((X,p),q),((X',p'),q') \in \operatorname{Kar}\operatorname{Kar}(\mathcal{C})$ implies:

$$qp = q \tag{1}$$

$$p'q' = q' \tag{2}$$

And $f \in \operatorname{Kar}(\mathcal{C})((X,q),(X',q'))$ implies:

$$fq = q'f = f \tag{3}$$

By [3], $fp = fqp$, and by [1] $fqp = fq$, so $fp = f$. Similarly, $p'f = f$. Then we can also think of f as a morphism from $((X,p),q)$ to $((X',p'),q')$, and $F(f) = f$, so F is full. It is obvious that F is faithful. So we have an equivalence of categories.

Question 1.3. Show that there is a fully faithful functor $\mathcal{C} \to \operatorname{Kar}(\mathcal{C})$ given by $X \mapsto (X, 1_X)$.

Let F be the functor that send X to $(X,1_X)$ and $f : X \to Y$ to $f : (X,1_X) \to (Y,1_Y)$ this makes sense since $f \cdot 1_X = 1_Y \cdot f = f$

Given $f : X \to Y$, $g : Y \to Z$, then its easy to see that $F(g \cdot f) = F(g) \cdot F(f)$. It is also easy to see that $F(1_X) = id_{(X,1_X)}$. Given $f : (X,1_X) \to (Y,1_Y)$, then the underlying map $f : X \to Y$ is a preimage of f, so F is full. The fact that F is faithful follows directly from the definition.

Question 1.4.

Note that given a functor $F : \mathcal{C} \to \mathcal{D}$, there is functor $\operatorname{Kar}(F) : \operatorname{Kar}(\mathcal{C}) \to \operatorname{Kar}(\mathcal{D})$ given by

$$\operatorname{Kar}(F)(X,p) = (F(X), F(p))$$
$$\operatorname{Kar}(F)(f : (X,p) \to (X',p')) = F(f).$$

Given a functor $F : \operatorname{Kar}(C) \to \operatorname{Kar}(D)$, show that it is determined up to natural isomorphism by its restriction $F|_{\mathcal{C}} : C \to \operatorname{Kar}(D)$, by showing $F \cong (\iota_{\mathcal{D}} \circ \operatorname{Kar}(F|_{\mathcal{C}}))$.

For (X,p) in $\operatorname{Kar}(C)$, for simiplicity, we denote X, the first projection, by $\pi_1(X,p)$, similarly p, the second projection, by $\pi_2(X,p)$. For $(X,p) \in \operatorname{Kar} C$, we can write $\iota_{\mathcal{D}} \circ \operatorname{Kar}(F|_{\mathcal{C}})(X,p)$ as $(\pi_1(F(X,\mathrm{id})), \pi_2(F(p : (X,\mathrm{id}) \to (X,\mathrm{id}))))$. Now define:

$$\alpha_{(X,p)} : \iota_{\mathcal{D}} \circ \operatorname{Kar}(F|_{\mathcal{C}})(X,p) \to F(X,p)$$

to be the map $F(p : (X, \mathrm{id}) \to (X, p))$. Note that this definition indeed makes sense as it defines a map from the first projection of $F(X, \mathrm{id})$ to the first projection of $F(X, p)$. Furthermore, since $(X, p) \in \operatorname{Kar} C$ so $p \cdot p = p$. Thus pre and post composing by p gives back p.

Now define $\beta_{(X,p)} : F(X, p) \to \iota_{\mathcal{D}} \circ \operatorname{Kar}(F|_{\mathcal{C}})(X, p)$ to be $F(p : (X, p) \to (X, \mathrm{id}))$. Then by functoriality $\beta_{(X,p)} \circ \alpha_{(X,p)} = F(p : (X, \mathrm{id}) \to (X, \mathrm{id}))$, but $p : (X, \mathrm{id}) \to (X, \mathrm{id})$ is the identity on $(\pi_1(F(X, \mathrm{id})), \pi_2(F(p : (X, \mathrm{id}) \to (X, \mathrm{id}))))$, so $F(p : (X, \mathrm{id}) \to (X, \mathrm{id})) = \mathrm{id}_{\iota_{\mathcal{D}} \circ \operatorname{Kar}(F|_{\mathcal{C}})(X,p)}$. Similarly, we observe that $\alpha_{(X,p)} \circ \beta_{(X,p)} = F(p : (X, p) \to (X, p) = F(\mathrm{id}_{(X,p)}) = \mathrm{id}_{F(X,p)}$.

Now it remains to show that α is a natural transformation, let $g \in \operatorname{Kar} C((X, p), (Y, q))$, then we can view g as a morphism in $\mathcal{C}$ from X to Y satisfying $gp = qg = g$. Consider the following diagram:

$$\begin{tikzcd}
(\pi_1(F(X, \mathrm{id})), \pi_2(F(p : (X, \mathrm{id}) \to (X, \mathrm{id})))) \arrow[r, "F(g:(X,\mathrm{id})\to(Y,\mathrm{id}))"] \arrow[d, "F(p:(X,\mathrm{id})\to(X,p))"] & (\pi_1(F(Y, \mathrm{id})), \pi_2(F(q : (Y, \mathrm{id}) \to (Y, \mathrm{id})))) \arrow[d, "F(q:(Y,\mathrm{id})\to(Y,q))"] \\
F(X, p) \arrow[r, "F(g:(X,\mathrm{id})\to(Y,\mathrm{id}))"] & F(Y, q)
\end{tikzcd}$$

By functoriality, going clockwise gives us $F(q \circ g : (X, \mathrm{id}) \to (Y, q))$ and going anticlockwise gives us $F(g \circ p : (X, \mathrm{id}) \to (Y, q))$, but $q \circ g = g \circ p$ by assumption, so the diagram commutes. Therefore α is the natural isomorphism we want.

2 Limits

Question 2.1. Recall that given a functor $F : \mathcal{J} \to \mathcal{C}$, the *limit* of F, written $\lim_{\mathcal{J}} F$ is a terminal object in the category of cones over F. Explicitly, a cone consists of

1. an object $X \in \operatorname{Obj} \mathcal{C}$,
2. for each $j \in \operatorname{Obj} \mathcal{C}$, a map $f_j : X \to F(j)$,
3. such that for any $g : j \to j'$, $F(g) \circ f_j = f_{j'}$.

Another way of saying this is that the category of cones is the comma category $(J \downarrow F)$, where here we interpret $\mathcal{J}$ as a functor $\mathcal{C} \to \operatorname{Fun}(\mathcal{J} \to \mathcal{C})$ by $c \mapsto (j \mapsto c)$ and we interpret F as a functor $1 \to \operatorname{Fun}(\mathcal{J} \to \mathcal{C})$ by $1 \mapsto (j \mapsto F(j))$. Explain carefully why these are talking about the same thing!

Let us first unpack the definition of the comma category $(J \downarrow F)$. The objects of this category are triples $(X, 1, \alpha : \mathcal{J}(X) \to 1(1))$ where $X \in \mathcal{C}$, and α is a natural transformation from $\mathcal{J}(X)$ to F. Since there are only 1 object in 1, we can simplify and think of the objects in this category as just tuples $(X, \alpha : \mathcal{J}(X) \to 1(1))$. α is a natural transformation, so for all $f \in \mathcal{J}(j, j')$, we have:

$$\begin{tikzcd}
\mathcal{J}(X)(j) \arrow[r, "\mathcal{J}(X)(f)"] \arrow[d, "\alpha j"] & \mathcal{J}(X)(j') \arrow[d, "\alpha_{j'}"] \\
F(j) \arrow[r, "F(f)"] & F(j')
\end{tikzcd}$$

commutes.

But $\mathcal{J}(X)(j) = \mathcal{J}(X)(j') = X$ by definition and $\mathcal{J}(X)(f) = \mathrm{id}_X$, so we can simplify the diagram to:

$$\begin{tikzcd}
X \arrow[d, "\alpha j"] \arrow[dr, "\alpha_{j'}"] & \\
F(j) \arrow[r, "F(f)"] & F(j')
\end{tikzcd} \tag{4}$$

Now consider morphisms in the comma category, since the only morphism in 1 is the identity, we can think of morphisms from (X, α) to (Y, β) as the collection $\{g \in \mathcal{C}(X, Y)\}$ satisfying the following commutative diagram:

$$\begin{tikzcd}
\mathcal{J}(X) \arrow[r, "\mathcal{J}(X)(g)"] \arrow[d, "\alpha"] & \mathcal{J}(Y) \arrow[d, "\beta"] \\
1(1) \arrow[r, "\mathrm{id}"] & 1(1)
\end{tikzcd}$$

Specializing to $j \in \mathcal{J}$, this is saying:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha j"] & X' \arrow[dl, "\beta_j"] \\
F(j) &
\end{tikzcd} \tag{5}$$

commutes.

Observe that every (X, α) in the comma category correspond uniquely to a cone by setting α_j as the maps f_j, and morphisms of the comma category correspond uniquely to a morphism of cones since they satisfy [5].

Given a cone and let $\beta_i = f_i$, it follows from the definition f_i in a cone that β satisfies [4], so β is indeed a natural transformation. So (X, β) is an object of the comma category. It is also easy to see that morphisms of cones correspond to morphisms of the comma category since they both satisfies [5]. Furthermore, note that if we reinterpreted (X, β) back to a cone, we will get exactly what we started with. It is obvious that this also applies to morphisms.

So objects and morphisms in the comma category is in bijection with cones and morphisms of cones.

3 The Yoneda embedding

Question 3.1. Prove the Yoneda lemma: Let $\mathcal{A}$ be a locally small category. Then

$$[\mathcal{A}^{\text{op}}, \mathsf{Set}](H_A, X) \cong X(A)$$

Define a function $(\hat{\ })$: $[\mathcal{A}^{\text{op}}, \mathsf{Set}](H_A, X) \to X(A)$ to send $\alpha : H_A \to X$ to $\alpha_A(\text{Id}_A)$. Define another function $(\tilde{\ }) : X(A) \to [\mathcal{A}^{\text{op}}, \mathsf{Set}](H_A, X)$, to send $x \in X(A)$, to the natural transformation $\tilde{x} : H_A \to X$ defined by $\tilde{x}_B : \mathcal{A}(B, A) \to X(B)$ where $\tilde{x}_B(f) = (X(f))(x)$ for $f \in \mathcal{A}(B, A)$.

Now we need to show that $\tilde{x}$ is indeed a natural transformation. Let $g : B' \to B$ in $\mathcal{A}$, then is suffices to show that

$$\begin{tikzcd}
\mathcal{A}(B, A) \arrow[r, "- \circ g"] \arrow[d, "\tilde{x}_B"] & \mathcal{A}(B', A) \arrow[d, "\tilde{x}_{B'}"] \\
X(B) \arrow[r, "X(g)"] & X(B')
\end{tikzcd}$$

commutes.

Let $f \in \mathcal{A}(B, A)$, then going clockwise in the diagram gives $X(f \circ g)(x)$. which by functoriality equals $X(g) \circ X(f)(x)$, but this is exactly what we obtain by going anticlockwise, so the diagram commutes.

Let $x \in X(A)$, then $\hat{\tilde{x}} = \tilde{x}_A(\text{Id}_A) = x$.

Let $\alpha : H_A \to X$, we want to show that $\tilde{\hat{\alpha}} = \alpha$, it suffices to show that these two natural transformations agree on each components, but since $H_A(B)$ is a set, this is the same as saying, for all $B \in \mathcal{A}$ and $f \in \mathcal{A}(B, A)$:

$$\tilde{\hat{\alpha}}_B(f) = \alpha_B(f)$$

The left hand side $\tilde{\hat{\alpha}}_B(f) = (X(f))(\tilde{\alpha}) = (X(f))(\alpha_A(\mathrm{id}_A))$

Since α is a natural transformation:

$$\begin{array}{ccc} \mathcal{A}(A,A) & \xrightarrow{-\circ f} & \mathcal{A}(B,A) \\ \Big\downarrow{\scriptstyle \alpha_A} & & \Big\downarrow{\scriptstyle \alpha_B} \\ X(A) & \xrightarrow{X(f)} & X(B) \end{array}$$

commutes.

So

$$\alpha_B(f) = \alpha_B(\mathrm{id}_A \circ f) = (X(f))(\alpha_A(\mathrm{id}_A)) \tag{6}$$

So indeed $\tilde{\hat{\alpha}} = \alpha$.

It remains to show naturality of $(\tilde{\ })$ and $(\hat{\ })$, but we know that these are inverses of each other, so if we know that $(\hat{\ })$ is a natural transformation, then $(\tilde{\ })$ is also a natural transformation. Thus it suffices to show that $(\hat{\ })$ is natural, i.e. $(\hat{\ })$ is natural in H_A and X.

Naturality in H_A translate to the following diagram, given $f \in \mathcal{A}(B, A)$:

$$\begin{array}{ccc} [\mathcal{A}^{\mathrm{op}}, \mathsf{Set}](H_A, X) & \xrightarrow{-\circ H_f} & [\mathcal{A}^{\mathrm{op}}, \mathsf{Set}](H_B, X) \\ \Big\downarrow{\scriptstyle (\tilde{\ })} & & \Big\downarrow{\scriptstyle (\tilde{\ })} \\ X(A) & \xrightarrow{\qquad X(f) \qquad} & X(B) \end{array}$$

Let $\alpha : H_A \to X$, then going clockwise yields $(\alpha \circ H_f)_B(\mathrm{id}_B)$ and going anti-clockwise yields $X(f)(\alpha_A(\mathrm{id}_A))$. By definition of H_f we know $(\alpha \circ H_f)_B(\mathrm{id}_B) = \alpha_B \cdot (H_f)_B(\mathrm{id}_B) = \alpha)_B(f \circ \mathrm{id}_B) = \alpha_B(f)$. By [6], this equals $X(f)(\alpha_A(\mathrm{id}_A))$. So the diagram commutes.

Naturality in X translate to the following diagram, given θ natural transformation form $X \to X'$, $X, X' \in [\mathcal{A}^{\mathrm{op}}, \mathsf{Set}]$ and for all $A \in \mathcal{A}$:

$$\begin{array}{ccc} [\mathcal{A}^{\mathrm{op}}, \mathsf{Set}](H_A, X) & \xrightarrow{\theta\circ -} & [\mathcal{A}^{\mathrm{op}}, \mathsf{Set}](H_A, X') \\ \Big\downarrow{\scriptstyle (\tilde{\ })} & & \Big\downarrow{\scriptstyle (\tilde{\ })} \\ X(A) & \xrightarrow{\qquad \theta_A \qquad} & X'(A) \end{array}$$

Let $\alpha : H_A \to X$. Then going clockwise gives $(\theta \circ \alpha)_A(\mathrm{id}_A) = \theta_A(\alpha_A(\mathrm{id}_A))$, which is what we obtain going anticlockwise. So the diagram commutes.

Question 3.2. Explain why $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathcal{D}) = \mathrm{Fun}(\mathcal{C} \to \mathcal{D}^{\mathrm{op}})$. Prove that $\mathsf{Set}^{\mathrm{op}} \not\cong \mathsf{Set}$, but that $\mathsf{fdVec}^{\mathrm{op}} \cong \mathsf{fdVec}$.

Define $\alpha : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathcal{D}) \to \mathrm{Fun}(\mathcal{C} \to \mathcal{D}^{\mathrm{op}})$, to send a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ to αF such that $\alpha F(c) = F(c)$ for $c \in \mathcal{C}$ and $\alpha F(f : c \to c') = F(f) : F(c') \to F(c)$. It is easy to see that αF indeed a functor (satisfies functoriality and sends identity to identity) as F is a functor. Define $\beta : \mathrm{Fun}(\mathcal{C} \to \mathcal{D}^{\mathrm{op}}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathcal{D})$ to send a functor $G : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ to βG such that $\beta G(c) = G(c)$ for $c \in \mathcal{C}$ and $\beta G(f : c \to c') = G(f) : F(c') \to F(c)$. By similar reasoning as before, we know that the definition β makes sense.

Consider $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. For all $c \in \mathcal{C}$, $\beta\alpha F(c) = F(c)$. For all $f \in \mathcal{A}(c, c')$, $\beta\alpha F(f : c \to c') = \alpha F(f) : \beta\alpha F(c') \to \beta\alpha F(c') = F(f) : \beta\alpha F(c) \to \beta\alpha F(c) = F(f) : F(c) \to (c')$. Similarly one can see that $\beta\alpha(G) = G$. So these two sets are indeed equal.

Lemma 3.1. *If $F : \mathcal{C} \to \mathcal{D}$ is full and essentially surjective, then it sends terminal objects to terminal objects.*

Let T be a object of $\mathcal{C}$, then for all $d \in \mathcal{D}$, there exists $c \in \mathcal{C}$ such that $\varphi : d \to F(c)$ is an isomphism. Since T is a terminal object, there exist unique morphism $f : c \to T$, so $F(f) \cdot \varphi$ gives a morphism from d to $F(T)$. Suppose there exist another morphism $g : d \to F(T)$, F is full so $g \cdot \varphi^{-1} : F(c) \to F(T)$ correponds to some $h : c \to T$ such that $F(h) = g \cdot \varphi^{-1}$. But T terminal, therefore $h = f$, so $F(f) = g \cdot \varphi^{-1}$. $F(T)$ is terminal.

Now suppose Set and $\mathsf{Set}^{\mathrm{op}}$ are equivalent, then there exist a fully faithful, essentially surjective functor $F : \mathsf{Set} \to \mathsf{Set}^{\mathrm{op}}$. By the lemma, F must send $\{*\}$ to a terminal object of $\mathsf{Set}^{\mathrm{op}}$, but terminal objects of $\mathsf{Set}^{\mathrm{op}}$ are just the initial objects of Set, so $F(\{*\}) = \emptyset$. Let $f \in \mathsf{Set}(\{*\}, A)$, where A is a non-empty set. Then $F(f) : \emptyset \to F(A)$ can be seen as a morphism in Set from $F(A)$ to $\emptyset$, but the only morphism mapping to the empty set is the empty map $e : \emptyset \to \emptyset$, so $F(A) = \emptyset$. Then for all $A \in \mathcal{A}$, $F(A)$ is either $\emptyset$ (when $A \neq \emptyset$) or $F(\emptyset)$ (when $A = \emptyset$). So F cannot be essentially surjective, a contradiction. So $\mathsf{Set}^{\mathrm{op}} \not\cong \mathsf{Set}$.

Consider the functor $F : \mathsf{fdVec} \to \mathsf{fdVec}^{op}$ that sends V to V^* the space of linear functionals from V to $\mathbb{K}$ and sends $f : V \to W$ to $F(f) : W^* \to V^*$ define as follows: let $v \in V$, $\varphi \in W^*$, then $f^*(\varphi)(v) = \varphi f(v)$. Also, define $G : \mathsf{fdVec}^{op} \to \mathsf{fdVec}$ to send V to V^* and $f : V \to W$ to $G(f) : W^* \to V^*$ in the same way as above. From the previous assignment we know that $G \cdot F(V) : V \to V^{**}$ is naturally isomorphic, so $G \cdot F$ is a natural isomorphism. We know $F \cdot G(V)$ also sends $V \in \mathsf{fdVec}^{op}$ to $V^{**} \in \mathsf{fdVec}^{op}$. So it suffices to show that V^{**} is naturally isomorphic to V in fdVec. Let $f \in \mathsf{fdVec}(V, W)$, we already know V is naturally isomorphic to V^{**}, so there exist a natural isomorphism α such that:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha_V"] & W \arrow[d, "\alpha_W"] \\ V^{**} \arrow[r, "f^{**}"] & W^{**} \end{tikzcd}$$

commutes. But α_V and α_W are isomorphisms, so we have:

$$\begin{tikzcd} V \arrow[r, "f"] & W \\ V^{**} \arrow[r, "f^{**}"] \arrow[u, "\alpha_V^{-1}"] & W^{**} \arrow[u, "\alpha_W^{-1}"'] \end{tikzcd}$$

But V is isomorphic to V^{**} implies that V^{**} is isomorphic to V, so $\mathsf{fdVec}^{\mathrm{op}} \cong \mathsf{fdVec}$.