LEMMA 2.1.6. *Suppose that V has a dual V^*. Then there exist canonical isomorphisms*

$$\mathrm{Hom}(U \otimes V, W) = \mathrm{Hom}(U, W \otimes V^*), \tag{2.1.13}$$

$$\mathrm{Hom}(U, V \otimes W) = \mathrm{Hom}(V^* \otimes U, W). \tag{2.1.14}$$

PROOF. To $\psi \in \mathrm{Hom}(U \otimes V, W)$ we associate the composition

$$U \xrightarrow{\mathrm{id} \otimes i_V} U \otimes V \otimes V^* \xrightarrow{\psi \otimes \mathrm{id}} W \otimes V^*$$

which is an element of $\mathrm{Hom}(U, W \otimes V^*)$.

Similarly, to $\varphi \in \mathrm{Hom}(U, W \otimes V^*)$ we assign

$$U \otimes V \xrightarrow{\varphi \otimes \mathrm{id}} W \otimes V^* \otimes V \xrightarrow{\mathrm{id} \otimes e_V} W.$$

One can easily check that these two maps are inverse to each other, establishing (2.1.13). The proof of (2.1.14) is similar. $\square$

In particular, if both U and V have duals, then by Lemma 2.1.6

$$\mathrm{Hom}(U, V) = \mathrm{Hom}(V^*, U^*) = \mathrm{Hom}(\mathbf{1}, V \otimes U^*). \tag{2.1.15}$$

(In the language of abstract nonsense, this means that the category $\mathcal{C}$ has internal Hom's when it has duals.) For $f \in \mathrm{Hom}(U, V)$ its image in $\mathrm{Hom}(V^*, U^*)$ via the isomorphism (2.1.15) will be denoted by f^*.

If the right dual $*$ exists for all objects in $\mathcal{C}$, then by (2.1.15) it is a contravariant functor, i.e., a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ where $\mathcal{C}^{\mathrm{op}}$ is the opposite (or dual) category to $\mathcal{C}$. (Recall that $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ but with all arrows reversed.)

EXERCISE 2.1.7. Show that, in a rigid category, $*$ is an equivalence of categories $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$.

Rigidity is a very restrictive requirement. As an illustration, let us prove the following proposition.

PROPOSITION 2.1.8. *In an abelian rigid monoidal category the tensor product functor $\otimes$ is exact, i.e., for any short exact sequence $0 \to U \to V \to W \to 0$ and an object X, the sequences*

$$0 \to U \otimes X \to V \otimes X \to W \otimes X \to 0$$

and

$$0 \to X \otimes U \to X \otimes V \to X \otimes W \to 0$$

are exact.

PROOF. The sequence

$$0 \to U \otimes X \to V \otimes X \to W \otimes X$$

is exact iff

$$0 \to \mathrm{Hom}(Y, U \otimes X) \to \mathrm{Hom}(Y, V \otimes X) \to \mathrm{Hom}(Y, W \otimes X)$$

is exact for any object Y. But by (2.1.12, 2.1.13), $\mathrm{Hom}(Y, U \otimes X) = \mathrm{Hom}(Y \otimes {}^*X, U)$. Since the functor $\mathrm{Hom}(Y, -)$ is left exact, it follows that $- \otimes X$ is left exact. Using Exercise 2.1.7 (or repeating the same argument with duals), we see that it is also right exact. $\square$