# PROBLEMS FOR THE CATEGORY THEORY READING COURSE, 2017

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## 1. Assignment 1, due end of Week 4

### 1.1. Functors.

(1) Leinster 1.2.21 (functors preserve isomorphisms)
(2) Leinster 1.2.27, 1.2.28b (full, faithful)

### 1.2. Natural transformations.

(1) In fdVec, show that the functors id and $**$ are naturally isomorphic.
(2) Show the the vertical composition of two natural transformations is in fact a natural transformation.
(3) Prove carefully that the horizontal composition of two natural transformations is again a natural transformation.
(4) Show that a functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories if and only if it is *fully faithful* and *essentially surjective*. Clearly state where you are using the axiom of choice, or add hypotheses so it is unnecessary.

### 1.3. Universal properties.

(1) Prove that two initial objects in a category are isomorphic.
(2) For each of the following categories, decide whether there is an initial, final, and/or zero object, and if so, describe them: FinSet, fdVec, Top, $\mathsf{Top}_*$ (pointed topological spaces), field extensions of a fixed field $F$, Graphs (your answer may depend on which class of graphs you consider), Semigroups, Groups.
(3) Describe the product of two objects as the terminal object in some category.
(4) Describe the tensor product of two vectors spaces as the initial object in some category.
(5) Describe both the product and coproduct in the following categories: FinSet, Top, $\mathsf{Top}_*$, AbGroup, Group, Graphs.

### 1.4. Adjunctions.

(1) Consider the forgetful functor from abelian groups to groups. What is its left adjoint?
(2) In the category of finite dimensional vector spaces, show that $- \otimes V$ is biadjoint to $- \otimes V^*$.
(3) Prove that the 'hom-set isomorphism' and 'unit/counit' definitions of an adjunction are equivalent.

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## 2. Assignment 2, due at the end of Week 7:

**2.1. Idempotent completion.** The 'idempotent completion' $\mathrm{Kar}(C)$ (also called the 'Karoubi envelope') is defined as follows:

$$\mathrm{Obj}\,\mathrm{Kar}(C) = \left\{(X \in \mathrm{Obj}(C), p : X \to X) \mid |p^2 = p\right\}$$
$$\mathrm{Kar}(C)((X, p) \to (X', p')) = \{f \in C(X \to X') \mid |fp = f = p'f\}\,.$$

(1) Let primeVec denote the full subcategory of Vec consisting of vector spaces with prime dimensions. Show carefully that $\mathrm{Kar}(\mathrm{primeVec}) \cong \mathrm{Vec}$.
(2) Construct an equivalence $\iota_C : \mathrm{Kar}(\mathrm{Kar}(C)) \cong \mathrm{Kar}(C)$.
(3) Show that there is a fully faithful functor $C \to \mathrm{Kar}(C)$ given by $X \mapsto (X, 1_X)$.
(4) Note that given a functor $F : C \to \mathcal{D}$, there is functor $\mathrm{Kar}(F) : \mathrm{Kar}(C) \to \mathrm{Kar}(\mathcal{D})$ given by

$$\mathrm{Kar}(F)(X, p) = (F(X), F(p))$$
$$\mathrm{Kar}(F)(f : (X, p) \to (X', p')) = F(f).$$

(You might say that Kar is a 2-functor from CAT to itself — what does this mean at the level of natural transformations?) Given a functor $F : \mathrm{Kar}(C) \to \mathrm{Kar}(D)$, show that it is determined up to natural isomorphism by its restriction $F|_C : C \to \mathrm{Kar}(D)$, by showing $F \cong (\iota_{\mathcal{D}} \circ \mathrm{Kar}(F|_C))$.
(5) (Not on the problem set: there is a forgetful 2-functor CAT $\to$ SemiCat, the 2-category of 'semicategories' (categories without identities) and their functors and natural transformations. The idempotent completion gives a 2-functor SemiCAT $\to$ CAT. Are they an adjoint pair?)

**2.2. Limits.**

(1) Recall that given a functor $F : \mathcal{J} \to C$, the *limit* of $F$, written $\lim_{\mathcal{J}} F$ is a terminal object in the category of cones over $F$. Explicitly, a cone consists of
  (a) an object $X \in \mathrm{Obj}\,C$,
  (b) for each $j \in \mathrm{Obj}\,C$, a map $f_j : X \to F(j)$,
  (c) such that for any $g : j \to j'$, $F(g) \circ f_j = f_{j'}$.

Another way of saying this is that the category of cones is the comma category $(J \downarrow F)$, where here we interpret $\mathcal{J}$ as a functor $C \to \mathrm{Fun}(\mathcal{J} \to C)$ by $c \mapsto (j \mapsto c)$ and we interpret $F$ as a functor $1 \to \mathrm{Fun}(\mathcal{J} \to C)$ by $1 \mapsto (j \mapsto F(j))$. Explain carefully why these are talking about the same thing!

**2.3. The Yoneda embedding.**

(1) Prove the Yoneda lemma.
(2) Explain why $\mathrm{Fun}(C^{\mathrm{op}} \to \mathcal{D}) = \mathrm{Fun}(C \to \mathcal{D}^{\mathrm{op}})$. Prove that $\mathrm{Set}^{\mathrm{op}} \ncong \mathrm{Set}$, but that $\mathrm{fdVec}^{\mathrm{op}} \cong \mathrm{fdVec}$.

## 3. Assignment 3, due May 12:

(1) Prove that every monoidal category is monoidally equivalent to a strict monoidal category.

(Hint: given a monoidal category $\mathcal{C}$, define a new monoidal category $\mathsf{List}\mathcal{C}$, whose objects are lists of objects in $\mathcal{C}$. In $\mathsf{List}\mathcal{C}$, tensor product of objects is concatenation of lists, and the tensor unit is the empty list. There should be a functor $\mathsf{List}\mathcal{C}$ to $\mathcal{C}$ defined by sending a list to the tensor product of the elements of that list. Your job is to describe what happens at the level of morphisms, and check that everything works.)

(2) Find an example of a monoidal functor which is not naturally isomorphic to any strict monoidal functor.

(Hint: consider categories $\mathrm{Vec}^{\omega}G$, where $G$ is a finite group, and $\omega \in H^3(G, k^\times)$ is a 3-cocycle. In particular, consider $G = \mathbb{Z}/2\mathbb{Z}$. The category $\mathrm{Vec}\mathbb{Z}/2\mathbb{Z}$ can be made into a monoidal category in two distinct ways: either with the 'obvious' associator, or with the associator $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ which, when $Z$ is 1-dimensional, is either the identity if $Z$ is in the even grading, or minus the identity if $Z$ is in the odd grading. If you're going to use this example you should explain carefully why this associator works. You should be able to show that there are *no* strict monoidal functors between these two versions of $\mathrm{Vec}\mathbb{Z}/2\mathbb{Z}$, but nevertheless there are non-strict monoidal functors, with a non-trivial tensorator; you should write one down explicitly.)

(3) Let $\mathcal{C}$ be a monoidal category. We say a 'monoid object' in $\mathcal{C}$ (or, as we gain confidence, just a monoid in $\mathcal{C}$) is a tuple $(A \in \operatorname{Obj}\mathcal{C}, \iota : 1 \to A, m : A \otimes A \to A)$ satisfying some conditions. Look up, or work out, what these conditions should be. (Hint: look at chapter 7 of Etingof, Gelaki, Nikshych, and Ostrik's book Tensor Categories.) You should be able to show that a monoid object in Vec is what is usually called an associative unital algebra.

   (a) A 'module object' for a monoid object $A \in \mathcal{C}$ is a tuple $(M, \triangleright : A \otimes M \to M)$ satisfying an appropriate condition (what is it?). A morphism $f$ between module objects $M$ and $M'$ is a morphism between the underlying objects, such that $f \circ \triangleright_M = \triangleright_{M'} \circ (1_A \otimes f)$. Draw the string diagram corresponding to this axiom. Define composition of module morphisms, by imitating the definition for modules over a ring. Show that modules for a fixed monoid object form a category.

(4) Show that $\mathrm{Rep}G$, for $G$ a finite group, forms a monoidal category.

(5) If $\mathrm{Rep}G \cong \mathrm{Rep}H$, as categories, are $G$ and $H$ isomorphic? (Hint: no, give a counterexample.) What about if $\mathrm{Rep}G \cong \mathrm{Rep}H$ as monoidal categories, and moreover this equivalence is compatible with the forgetful functors to Vec? (Hint: think about the monoidal automorphisms of the forgetful functor.) What if we drop the condition about compatibility with the forgetful functors? (Hint: google 'isocategorical group'; you can answer this with just an appropriate reference to the literature, or do it yourself.)

## 4. Assignment 4, due May 26:

### 4.1. **Abelian categories.**

(1) Show if $F : \mathcal{C} \to \mathcal{D}$ is a fully faithful functor, then $f \in \mathcal{C}(X \to Y)$ is a monomorphism if and only if $F(f)$ is. (Similarly for epimorphisms.)
(2) Use this, and the Yoneda embedding, to show that in a rigid abelian category, the functor $- \otimes X$ is exact. (See Proposition 2.1.8 of Bakalov-Kirillov, attached, if you need some help; they don't explain how they are using Yoneda, however! You may cite the literature for any lemmas you like.)

### 4.2. **Braided monoidal categories.**

(1) Explain how any object $X$ in a pivotal braided category gives an oriented link invariant.
(2) Describe how the Temperley-Lieb category has the structure of a pivotal braided category.
(3) Calculate the invariant of the trefoil corresponding to the object 1 in Temperley-Lieb. (What is this invariant usually called?)
(4) Show that in $\mathrm{Kar}(TL)$, we have $(2, 1_2) \cong (2, f^{(2)}) \oplus (0, 1_0)$. (Here $f^{(2)}$ denotes the second Jones-Wenzl idempotent.)
(5) Calculate the invariant of the unknot corresponding to the object $(2, f^{(2)})$ in (the idempotent completion of) Temperley-Lieb.
(6) We say a monoid $(A, m)$ in a braided monoidal category is commutative if $m \circ \beta = m$. Define a monoidal structure on the category of modules for a commutative monoid.

LEMMA 2.1.6. *Suppose that $V$ has a dual $V^*$. Then there exist canonical isomorphisms*

$$\mathrm{Hom}(U \otimes V, W) = \mathrm{Hom}(U, W \otimes V^*), \tag{2.1.13}$$

$$\mathrm{Hom}(U, V \otimes W) = \mathrm{Hom}(V^* \otimes U, W). \tag{2.1.14}$$

PROOF. To $\psi \in \mathrm{Hom}(U \otimes V, W)$ we associate the composition

$$U \xrightarrow{\mathrm{id} \otimes i_V} U \otimes V \otimes V^* \xrightarrow{\psi \otimes \mathrm{id}} W \otimes V^*$$

which is an element of $\mathrm{Hom}(U, W \otimes V^*)$.

Similarly, to $\varphi \in \mathrm{Hom}(U, W \otimes V^*)$ we assign

$$U \otimes V \xrightarrow{\varphi \otimes \mathrm{id}} W \otimes V^* \otimes V \xrightarrow{\mathrm{id} \otimes e_V} W.$$

One can easily check that these two maps are inverse to each other, establishing (2.1.13). The proof of (2.1.14) is similar. $\square$

In particular, if both $U$ and $V$ have duals, then by Lemma 2.1.6

$$\mathrm{Hom}(U, V) = \mathrm{Hom}(V^*, U^*) = \mathrm{Hom}(\mathbf{1}, V \otimes U^*). \tag{2.1.15}$$

(In the language of abstract nonsense, this means that the category $\mathcal{C}$ has internal Hom's when it has duals.) For $f \in \mathrm{Hom}(U, V)$ its image in $\mathrm{Hom}(V^*, U^*)$ via the isomorphism (2.1.15) will be denoted by $f^*$.

If the right dual $*$ exists for all objects in $\mathcal{C}$, then by (2.1.15) it is a contravariant functor, i.e., a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ where $\mathcal{C}^{\mathrm{op}}$ is the opposite (or dual) category to $\mathcal{C}$. (Recall that $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ but with all arrows reversed.)

EXERCISE 2.1.7. Show that, in a rigid category, $*$ is an equivalence of categories $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$.

Rigidity is a very restrictive requirement. As an illustration, let us prove the following proposition.

PROPOSITION 2.1.8. *In an abelian rigid monoidal category the tensor product functor $\otimes$ is exact, i.e., for any short exact sequence $0 \to U \to V \to W \to 0$ and an object $X$, the sequences*

$$0 \to U \otimes X \to V \otimes X \to W \otimes X \to 0$$

*and*

$$0 \to X \otimes U \to X \otimes V \to X \otimes W \to 0$$

*are exact.*

PROOF. The sequence

$$0 \to U \otimes X \to V \otimes X \to W \otimes X$$

is exact iff

$$0 \to \mathrm{Hom}(Y, U \otimes X) \to \mathrm{Hom}(Y, V \otimes X) \to \mathrm{Hom}(Y, W \otimes X)$$

is exact for any object $Y$. But by (2.1.12, 2.1.13), $\mathrm{Hom}(Y, U \otimes X) = \mathrm{Hom}(Y \otimes {}^*X, U)$. Since the functor $\mathrm{Hom}(Y, -)$ is left exact, it follows that $- \otimes X$ is left exact. Using Exercise 2.1.7 (or repeating the same argument with duals), we see that it is also right exact. $\square$

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