PROBLEMS FOR THE CATEGORY THEORY READING COURSE, 2017

SCOTT MORRISON

1. Assignment 1, due end of Week 4

1.1. Functors.

(1) Leinster 1.2.21 (functors preserve isomorphisms)
(2) Leinster 1.2.27, 1.2.28b (full, faithful)

1.2. Natural transformations.

(1) In fdVec, show that the functors id and $**$ are naturally isomorphic.
(2) Show the the vertical composition of two natural transformations is in fact a natural transformation.
(3) Prove carefully that the horizontal composition of two natural transformations is again a natural transformation.
(4) Show that a functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories if and only if it is *fully faithful* and *essentially surjective*. Clearly state where you are using the axiom of choice, or add hypotheses so it is unnecessary.

1.3. Universal properties.

(1) Prove that two initial objects in a category are isomorphic.
(2) For each of the following categories, decide whether there is an initial, final, and/or zero object, and if so, describe them: FinSet, fdVec, Top, Top_* (pointed topological spaces), field extensions of a fixed field F, Graphs (your answer may depend on which class of graphs you consider), Semigroups, Groups.
(3) Describe the product of two objects as the terminal object in some category.
(4) Describe the tensor product of two vectors spaces as the initial object in some category.
(5) Describe both the product and coproduct in the following categories: FinSet, Top, Top_*, AbGroup, Group, Graphs.

1.4. Adjunctions.

(1) Consider the forgetful functor from abelian groups to groups. What is its left adjoint?
(2) In the category of finite dimensional vector spaces, show that $- \otimes V$ is biadjoint to $- \otimes V^*$.
(3) Prove that the 'hom-set isomorphism' and 'unit/counit' definitions of an adjunction are equivalent.

Date: Version of May 19, 2017.

2. Assignment 2, due at the end of Week 7:

2.1. Idempotent completion. The 'idempotent completion' $\mathrm{Kar}(C)$ (also called the 'Karoubi envelope') is defined as follows:

$$\mathrm{Obj}\,\mathrm{Kar}(C) = \left\{(X \in \mathrm{Obj}(C), p : X \to X) \mid |p^2 = p\right\}$$
$$\mathrm{Kar}(C)((X,p) \to (X',p')) = \{f \in C(X \to X') \mid |fp = f = p'f\}.$$

(1) Let primeVec denote the full subcategory of Vec consisting of vector spaces with prime dimensions. Show carefully that $\mathrm{Kar}(\mathrm{primeVec}) \cong \mathrm{Vec}$.
(2) Construct an equivalence $\iota_C : \mathrm{Kar}(\mathrm{Kar}(C)) \cong \mathrm{Kar}(C)$.
(3) Show that there is a fully faithful functor $C \to \mathrm{Kar}(C)$ given by $X \mapsto (X, 1_X)$.
(4) Note that given a functor $F : C \to \mathcal{D}$, there is functor $\mathrm{Kar}(F) : \mathrm{Kar}(C) \to \mathrm{Kar}(\mathcal{D})$ given by

$$\mathrm{Kar}(F)(X,p) = (F(X), F(p))$$
$$\mathrm{Kar}(F)(f : (X,p) \to (X',p')) = F(f).$$

(You might say that Kar is a 2-functor from CAT to itself — what does this mean at the level of natural transformations?) Given a functor $F : \mathrm{Kar}(C) \to \mathrm{Kar}(D)$, show that it is determined up to natural isomorphism by its restriction $F|_C : C \to \mathrm{Kar}(D)$, by showing $F \cong (\iota_{\mathcal{D}} \circ \mathrm{Kar}(F|_C))$.
(5) (Not on the problem set: there is a forgetful 2-functor CAT $\to$ SemiCat, the 2-category of 'semicategories' (categories without identities) and their functors and natural transformations. The idempotent completion gives a 2-functor SemiCAT $\to$ CAT. Are they an adjoint pair?)

2.2. Limits.

(1) Recall that given a functor $F : \mathcal{J} \to C$, the *limit* of F, written $\lim_{\mathcal{J}} F$ is a terminal object in the category of cones over F. Explicitly, a cone consists of
  (a) an object $X \in \mathrm{Obj}\,C$,
  (b) for each $j \in \mathrm{Obj}\,C$, a map $f_j : X \to F(j)$,
  (c) such that for any $g : j \to j'$, $F(g) \circ f_j = f_{j'}$.

Another way of saying this is that the category of cones is the comma category $(J \downarrow F)$, where here we interpret $\mathcal{J}$ as a functor $C \to \mathrm{Fun}(\mathcal{J} \to C)$ by $c \mapsto (j \mapsto c)$ and we interpret F as a functor $1 \to \mathrm{Fun}(\mathcal{J} \to C)$ by $1 \mapsto (j \mapsto F(j))$. Explain carefully why these are talking about the same thing!

2.3. The Yoneda embedding.

(1) Prove the Yoneda lemma.
(2) Explain why $\mathrm{Fun}(C^{\mathrm{op}} \to \mathcal{D}) = \mathrm{Fun}(C \to \mathcal{D}^{\mathrm{op}})$. Prove that $\mathrm{Set}^{\mathrm{op}} \not\cong \mathrm{Set}$, but that $\mathrm{fdVec}^{\mathrm{op}} \cong \mathrm{fdVec}$.

3. Assignment 3, due May 12:

(1) Prove that every monoidal category is monoidally equivalent to a strict monoidal category.

(Hint: given a monoidal category C, define a new monoidal category $\mathsf{List}C$, whose objects are lists of objects in C. In $\mathsf{List}C$, tensor product of objects is concatenation of lists, and the tensor unit is the empty list. There should be a functor $\mathsf{List}C$ to C defined by sending a list $[x_1, x_2, \ldots, x_n]$ to $(((1 \otimes x_1) \otimes x_2) \otimes \cdots \otimes x_n$. Your job is to describe what happens at the level of morphisms, and check that everything works.)

(2) Let C be a monoidal category. We say a 'monoid object' in C (or, as we gain confidence, just a monoid in C) is a tuple $(A \in \operatorname{Obj} C, \iota : 1 \to A, m : A \otimes A \to A)$ satisfying some conditions. Look up, or work out, what these conditions should be. (Hint: look at chapter 7 of Etingof, Gelaki, Nikshych, and Ostrik's book Tensor Categories.) You should be able to show that a monoid object in Vec is what is usually called an associative unital algebra.

(a) A 'module object' for a monoid object $A \in C$ is a tuple $(M, \triangleright : A \otimes M \to M)$ satisfying an appropriate condition (what is it?). A morphism f between module objects M and M' is a morphism between the underlying objects, such that $f \circ \triangleright_M = \triangleright_{M'} \circ (1_A \otimes f)$. Draw the string diagram corresponding to this axiom. Define composition of module morphisms, by imitating the definition for modules over a ring. Show that modules for a fixed monoid object form a category.

(3) Show that $\mathrm{Rep}G$, for G a finite group, forms a monoidal category.

(4) (For this part, you may assume we are looking at representations over the complex numbers.)

- If $\mathrm{Rep}G \cong \mathrm{Rep}H$, as categories, are G and H isomorphic?

  (Hint: no, give a counterexample — any pair of non-isomorphic groups with the same number of irreducible representations (equivalently, the same number of conjugacy classes) will do.)
- What about if $\mathrm{Rep}G \cong \mathrm{Rep}H$ as monoidal categories, and moreover this equivalence is compatible with the forgetful functors to Vec?

  (Hint: think about the monoidal automorphisms of the forgetful functor. You should prove that (not necessarily monoidal) automorphisms of the forgetful functor $\mathrm{Rep}G \to \mathrm{Vec}$ is a group isomorphic to $\mathbb{C}[G]^\times$, and then identify the subgroup of monoidal automorphisms is G is itself. To do the first part, show that any such automorphism is determined by its component on the regular representation $\mathbb{C}[G]$; for this you may like to use that the regular representation is faithful, or (more or less equivalently) that there is a surjective G-linear map from the regular representation to any irreducible representation of G.)

4. Assignment 4, due May 26:

4.1. **Abelian categories.**

(1) Show that in a rigid abelian category, the functor $- \otimes X$ is exact. (Proposition 2.1.8 of Bakalov-Kirillov, attached, gives a terse proof of this. I'd like you to carefully explain their steps. You may find the discussion at https://math.stackexchange.com/q/98877/ helpful.)

4.2. **Braided monoidal categories.**

(1) Explain how any object X in a spherical braided category gives an oriented framed link invariant.

Hints:

- If we define a function on link diagrams, what do we have to check to show that it defines a 'framed' link invariant? (Clue: it's Reidemeister moves II and III, and a modified version of the usual Reidemeister move I.)
- In a rigid category, we have morphisms
$$\mathrm{ev} : X^* \otimes X \to 1 \qquad\qquad \mathrm{coev} : 1 \to X \otimes X^*.$$
In a pivotal category, we further have natural isomorphisms $\tau_X : X \to X^{**}$, and using this we can construct morphisms $X \otimes X^* \to 1$ and $1 \to X^* \otimes X$. Now we can interpret any oriented link diagram as a morphism $1 \to 1$ in our favourite pivotal braided category, by labelling the strings with an object X, and interpreting cups and caps by evaluation and coevaluation maps (and their cousins defined with the pivotal structure), and crossings by the braiding maps.
- The axioms for a rigid category and for a pivotal category guarantee that this morphism is invariant under changing the link diagram by a planar isotopy.
- What about a braided pivotal category guarantees that this morphism is invariant under changing the link diagram by Reidemeister II and III?
- Finally, explain how the extra assumption that the category is spherical ensures invariance under the modified Reidemeister I move.

(2) Describe how the Temperley-Lieb category has the structure of a spherical braided category.

(3) Calculate the invariant of the trefoil corresponding to the object 1 in Temperley-Lieb. (What is this invariant usually called?)

(4) Show that in $\mathrm{Kar}(TL)$, we have $(2, 1_2) \cong (2, f^{(2)}) \oplus (0, 1_0)$. Here $f^{(2)}$ denotes the second Jones-Wenzl idempotent:
$$f^{(2)} = \Big)\;\Big(- \frac{1}{\delta} \asymp .$$

(5) Calculate the invariant of the unknot corresponding to the object $(2, f^{(2)})$ in (the idempotent completion of) Temperley-Lieb.

(6) We say a monoid (A, m) in a braided monoidal category is commutative if $m \circ \beta = m$, where β denotes the braiding. Define a monoidal structure on the category of modules for a commutative monoid.

LEMMA 2.1.6. *Suppose that V has a dual V^*. Then there exist canonical isomorphisms*

$$\mathrm{Hom}(U \otimes V, W) = \mathrm{Hom}(U, W \otimes V^*), \tag{2.1.13}$$

$$\mathrm{Hom}(U, V \otimes W) = \mathrm{Hom}(V^* \otimes U, W). \tag{2.1.14}$$

PROOF. To $\psi \in \mathrm{Hom}(U \otimes V, W)$ we associate the composition

$$U \xrightarrow{\mathrm{id} \otimes i_V} U \otimes V \otimes V^* \xrightarrow{\psi \otimes \mathrm{id}} W \otimes V^*$$

which is an element of $\mathrm{Hom}(U, W \otimes V^*)$.

Similarly, to $\varphi \in \mathrm{Hom}(U, W \otimes V^*)$ we assign

$$U \otimes V \xrightarrow{\varphi \otimes \mathrm{id}} W \otimes V^* \otimes V \xrightarrow{\mathrm{id} \otimes e_V} W.$$

One can easily check that these two maps are inverse to each other, establishing (2.1.13). The proof of (2.1.14) is similar. □

In particular, if both U and V have duals, then by Lemma 2.1.6

$$\mathrm{Hom}(U, V) = \mathrm{Hom}(V^*, U^*) = \mathrm{Hom}(\mathbf{1}, V \otimes U^*). \tag{2.1.15}$$

(In the language of abstract nonsense, this means that the category $\mathcal{C}$ has internal Hom's when it has duals.) For $f \in \mathrm{Hom}(U, V)$ its image in $\mathrm{Hom}(V^*, U^*)$ via the isomorphism (2.1.15) will be denoted by f^*.

If the right dual $*$ exists for all objects in $\mathcal{C}$, then by (2.1.15) it is a contravariant functor, i.e., a functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ where $\mathcal{C}^{\mathrm{op}}$ is the opposite (or dual) category to $\mathcal{C}$. (Recall that $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ but with all arrows reversed.)

EXERCISE 2.1.7. Show that, in a rigid category, $*$ is an equivalence of categories $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$.

Rigidity is a very restrictive requirement. As an illustration, let us prove the following proposition.

PROPOSITION 2.1.8. *In an abelian rigid monoidal category the tensor product functor $\otimes$ is exact, i.e., for any short exact sequence $0 \to U \to V \to W \to 0$ and an object X, the sequences*

$$0 \to U \otimes X \to V \otimes X \to W \otimes X \to 0$$

and

$$0 \to X \otimes U \to X \otimes V \to X \otimes W \to 0$$

are exact.

PROOF. The sequence

$$0 \to U \otimes X \to V \otimes X \to W \otimes X$$

is exact iff

$$0 \to \mathrm{Hom}(Y, U \otimes X) \to \mathrm{Hom}(Y, V \otimes X) \to \mathrm{Hom}(Y, W \otimes X)$$

is exact for any object Y. But by (2.1.12, 2.1.13), $\mathrm{Hom}(Y, U \otimes X) = \mathrm{Hom}(Y \otimes {}^*X, U)$. Since the functor $\mathrm{Hom}(Y, -)$ is left exact, it follows that $- \otimes X$ is left exact. Using Exercise 2.1.7 (or repeating the same argument with duals), we see that it is also right exact. □

(Scott Morrison) Mathematical Sciences Institute, Australian National University
E-mail address: scott.morrison@anu.edu.au