

Hamiltonian:

$$\mathcal{H} = \beta H = \sum_{\alpha} K_{\alpha} S_{\alpha} \quad (1)$$

Truncate to nearest neighbor (NN) and next nearest neighbor (NNN) spin interactions:

$$\mathcal{H} \approx K_1 S_1 + K_2 S_2 = K_1 \sum_{\langle i,j \rangle} \sigma_i \sigma_j + K_2 \sum_{\langle\langle i,j \rangle\rangle} \sigma_i \sigma_j \quad (2)$$

Known critical temperature:

$$T_c = -\frac{2J}{\ln(1 + \sqrt{2})} \approx -2.27J \quad (3)$$

Expected critical coupling:

$$K_c = \frac{J}{T_c} = -\frac{\ln(1 + \sqrt{2})}{2} \approx -0.44 \quad (4)$$

Probability of spin flip:

$$P(\sigma_i \rightarrow -\sigma_i) = \exp(-\Delta\mathcal{E}) \quad (5)$$

$$\Delta\mathcal{E} = -2\sigma_i \left( K_1 \sum_{\langle j \rangle} \sigma_j + K_2 \sum_{\langle\langle j \rangle\rangle} \sigma_j \right) \quad (6)$$

Magnetization:

$$\mathcal{M} = \sum_i \sigma_i \quad (7)$$

Variances of energy and magnetization:

$$\sigma_{\mathcal{E}}^2 = \langle \mathcal{E}^2 \rangle - \langle \mathcal{E} \rangle^2 = \sum_{\alpha\beta} K_{\alpha} K_{\beta} (\langle S_{\alpha} S_{\beta} \rangle - \langle S_{\alpha} \rangle \langle S_{\beta} \rangle) \quad (8)$$

$$\sigma_{\mathcal{M},\alpha\beta}^2 = \langle \mathcal{M}^2 \rangle - \langle \mathcal{M} \rangle^2 = (\langle S_{\alpha} S_{\beta} \rangle - \langle S_{\alpha} \rangle \langle S_{\beta} \rangle) \quad (9)$$

Linearized transformation matrix:

$$T_{\alpha,\beta}^{(n)} = \frac{\partial K_{\alpha}^{(n+1)}}{\partial K_{\beta}^{(n)}} \quad (10)$$

Chain rule and invert:

$$\frac{\partial \langle S_{\gamma}^{(n+1)} \rangle}{\partial K_{\beta}^{(n)}} = \frac{\partial \langle S_{\gamma}^{(n+1)} \rangle}{\partial K_{\alpha}^{(n+1)}} \frac{\partial K_{\alpha}^{(n+1)}}{\partial K_{\beta}^{(n)}} \implies T_{\alpha,\beta}^{(n)} = \left[ \frac{\partial \langle S_{\gamma}^{(n+1)} \rangle}{\partial K_{\alpha}^{(n+1)}} \right]^{-1} \frac{\partial \langle S_{\gamma}^{(n+1)} \rangle}{\partial K_{\beta}^{(n)}} \quad (11)$$

where

$$\frac{\partial \langle S_{\gamma}^{(n+1)} \rangle}{\partial K_{\alpha}^{(n+1)}} = \langle S_{\gamma}^{(n+1)} S_{\alpha}^{(n+1)} \rangle - \langle S_{\gamma}^{(n+1)} \rangle \langle S_{\alpha}^{(n+1)} \rangle \quad (12)$$

$$\frac{\partial \langle S_{\gamma}^{(n+1)} \rangle}{\partial K_{\beta}^{(n)}} = \langle S_{\gamma}^{(n+1)} S_{\beta}^{(n)} \rangle - \langle S_{\gamma}^{(n+1)} \rangle \langle S_{\beta}^{(n)} \rangle \quad (13)$$