Hamiltonian:

$$\mathcal{H} = \beta H = \sum_{\alpha} K_{\alpha} S_{\alpha} \tag{1}$$

Truncate to nearest neighbor (NN) and next nearest neighbor (NNN) spin interactions:

$$\mathcal{H} \approx K_1 S_1 + K_2 S_2 = K_1 \sum_{\langle i,j \rangle} \sigma_i \sigma_j + K_2 \sum_{\langle\langle i,j \rangle\rangle} \sigma_i \sigma_j \tag{2}$$

Known critical temperature:

$$T_c = -\frac{2J}{\ln(1+\sqrt{2})} \approx -2.27J \tag{3}$$

Expected critical coupling:

$$K_c = \frac{J}{T_c} = -\frac{\ln(1+\sqrt{2})}{2} \approx -0.44 \tag{4}$$

Probability of spin flip:

$$P(\sigma_i \to -\sigma_i) = \exp(-\Delta\mathcal{E}) \tag{5}$$

$$\Delta\mathcal{E} = -2\sigma_i \left(K_1 \sum_{\langle j \rangle} \sigma_j + K_2 \sum_{\langle\langle j \rangle\rangle} \sigma_j \right) \tag{6}$$

Magnetization:

$$\mathcal{M} = \sum_i \sigma_i \tag{7}$$

Variances of energy and magnetization:

$$\sigma_{\mathcal{E}}^2 = \langle \mathcal{E}^2 \rangle - \langle \mathcal{E} \rangle^2 = \sum_{\alpha\beta} K_\alpha K_\beta \big(\langle S_\alpha S_\beta \rangle - \langle S_\alpha \rangle \langle S_\beta \rangle \big) \tag{8}$$

$$\sigma_{\mathcal{M},\alpha\beta}^2 = \langle \mathcal{M}^2 \rangle - \langle \mathcal{M} \rangle^2 = \big(\langle S_\alpha S_\beta \rangle - \langle S_\alpha \rangle \langle S_\beta \rangle \big) \tag{9}$$

Linearized transformation matrix:

$$T_{\alpha,\beta}^{(n)} = \frac{\partial K_\alpha^{(n+1)}}{\partial K_\beta^{(n)}} \tag{10}$$

Chain rule and invert:

$$\frac{\partial \langle S_\gamma^{(n+1)} \rangle}{\partial K_\beta^{(n)}} = \frac{\partial \langle S_\gamma^{(n+1)} \rangle}{\partial K_\alpha^{(n+1)}} \frac{\partial K_\alpha^{(n+1)}}{\partial K_\beta^{(n)}} \implies T_{\alpha,\beta}^{(n)} = \left[\frac{\partial \langle S_\gamma^{(n+1)} \rangle}{\partial K_\alpha^{(n+1)}} \right]^{-1} \frac{\partial \langle S_\gamma^{(n+1)} \rangle}{\partial K_\beta^{(n)}} \tag{11}$$

where

$$\frac{\partial \langle S_\gamma^{(n+1)} \rangle}{\partial K_\alpha^{(n+1)}} = \left\langle S_\gamma^{(n+1)} S_\alpha^{(n+1)} \right\rangle - \left\langle S_\gamma^{(n+1)} \right\rangle \left\langle S_\alpha^{(n+1)} \right\rangle \tag{12}$$

$$\frac{\partial \langle S_\gamma^{(n+1)} \rangle}{\partial K_\beta^{(n)}} = \left\langle S_\gamma^{(n+1)} S_\beta^{(n)} \right\rangle - \left\langle S_\gamma^{(n+1)} \right\rangle \left\langle S_\beta^{(n)} \right\rangle \tag{13}$$

n	N = 16	N = 32	N = 64	N = 128
0	-0.4420811188	-0.4417429398	-0.4415445759	-0.4413292658
1	-0.4404791358	-0.4404577112	-0.4404935597	-0.4404266727
2	-0.4401272033	-0.4402638848	-0.4403468642	-0.4403131062
3	-	-0.4402975527	-0.4403307834	-0.4403016019
4	-	-	-0.4403116430	-0.4402985574
5	-	-	-	-0.4402807798

n	N = 16 @ $K_1^c(N)$	N = 16 @ K_1^c	N = 32 @ $K_1^c(N)$	N = 32 @ K_1^c
0	1.0366474702	1.0371936855	1.0359594271	1.0364461518
1	1.0055084088	1.0058615220	1.0030411491	1.0024276306
2	0.9850458988	0.9832937022	1.0042751683	1.0040962187
3	-	-	0.9833658883	0.9817702370

n	N = 64 @ $K_1^c(N)$	N = 64 @ K_1^c	N = 128 @ $K_1^c(N)$	N = 128 @ K_1^c
0	1.0340840722	1.0356603405	1.0343762634	1.0349921920
1	1.0013550363	1.0025377191	1.0018397044	1.0006737634
2	1.0021687106	1.0029573471	1.0019303651	1.0022232598
3	1.0029619937	1.0039006140	1.0025651221	1.0023411428
4	0.9820802731	0.9819506190	1.0035789853	1.0040042374
5	-	-	0.9814148379	0.9819137809