A note on semiparametric efficient estimation

- Let $g(\theta : X, Y)$ be an estimating equation for θ_0 satisfying

$$E\{g(\theta_0; X, Y)\} = 0. \tag{1}$$

- We can estimate θ by

$$\hat{\theta} = \arg\min_{\theta} \hat{U}(\theta)' \{E(gg')\}^{-1} \hat{U}(\theta), \tag{2}$$

  where $\hat{U}(\theta) = n^{-1} \sum_{i=1}^{n} g(\theta; x_i, y_i)$.

- The asymptotic variance of $\hat{\theta}$, the solution to (2), can be expressed as

$$V(\hat{\theta}) \doteq n^{-1} \left\{ E\left(\frac{\partial}{\partial \theta'} g\right)' \{E(gg')\}^{-1} E\left(\frac{\partial}{\partial \theta'} g\right) \right\}^{-1}$$

- From (1), we can obtain

$$E\left\{\frac{\partial}{\partial \theta'} g(\theta; X, Y)\right\} = -E\left\{g(\theta; X, Y) S_{\text{eff}}(\theta; X, Y)\right\} \tag{3}$$

  where $S_{\text{eff}}(\theta; x, y)$ is the efficient score function for θ. This is a version of the second Bartlett identity.

- From (3), using the Cauchy-Schwartz inequality, we can obtain

$$V(\hat{\theta}) \geq n^{-1} \left[E\{S_{\text{eff}}(\theta; X, Y)^{\otimes 2}\}\right]^{-1}$$

  with the equality iff

$$g^*(\theta; x, y) \propto S_{\text{eff}}(\theta; x, y). \tag{4}$$

- Now, note that (4) means that $S_{\text{eff}}(\theta; x, y)$ can be found in the class of $g(\theta; x, y)$. In this case, we can combine (3) and (4) to get

$$E\left(\frac{\partial}{\partial \theta'} g\right) = -E(gg') \tag{5}$$

  which can be used as an equation for finding g^*, the optimal estimating function achieving the lower bound of the asymptotic variance.

- For example, consider the case of the regression problem of the form

$$g(\theta; x, y) = \{y - m(x; \theta)\} b(x; \theta)$$

  where $m(x; \theta) = E(Y \mid x)$ is a known function with unknown parameter θ and $b(x; \theta)$ is an unknown function. In this case, the optimal g can be completely determined by finding the optimal b.

- Let's use (5) to find the optimal $b^*(x)$. Note that (5) can be expressed as

$$E\{b(x)\dot{m}(x)'\} = E\{v(x)b(x)b(x)'\} \tag{6}$$

  where $\dot{m}(x) = \partial m(x; \theta)/\theta$ and $v(x) = E[\{y - m(x; \theta)\}^2 \mid x]$.

- From (6), we can obtain

$$b^*(x) = \dot{m}(x)/v(x)$$

  and

$$g^*(\theta; x, y) = \{y - m(x; \theta)\} b^*(x; \theta),$$

  which leads to the efficient score function.

- Therefore, under a certain class of semiparametric problems, we can directly use (5) as the equation for finding the optimal estimator.