catlvm : CATegorical Latent Variable Model

Upward-Downward Algorithm

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1 Hidden Markov tree

2 Hierarchical latent class model

$$\pi_k = \mathsf{Pr}\left(C_1 = k\right), \quad \tau_{k|l}^{(u)} = \mathsf{Pr}\left(C_u = k | C_{\rho(u)} = l\right), \text{ and } \rho_{mr|k}^{(u)} = \mathsf{Pr}\left(Y_{m(u)} = r | C_u = k\right)$$

Upward recursion

$$\begin{aligned}
\xi_{i(u)}(k) &= \mathsf{Pr}\left(\mathbf{Y}_{i(u)} = \mathbf{y}_{i(u)} | C_u = k\right) = \prod_{m=1}^{M} \rho_{my_m|k}^{(u)} \\
\lambda_{i(u)}(k) &= \xi_{i(u)}(k), \quad \text{where} \quad \mathbf{c}(u) = \emptyset \\
\lambda_{i(u,\rho(u))}(l) &= \sum_k \tau_{k,l}^{(u)} \lambda_{i(u)}(k) \\
\lambda_{i(v)}(k) &= \left\{ \prod_{u \in \mathbf{c}(v)} \lambda_{i(u,v)}(k) \right\} \xi_{i(v)}(k)
\end{aligned}$$

Downward recursion

$$\alpha_{i(u)}(k) = \sum_l \frac{\tau_{k|l}^{(u)} \alpha_{i(\rho(u))}(l) \lambda_{i(\rho(u))}(l)}{\lambda_{i(u,\rho(u))}(l)}$$

Posterior probabilities

$$\begin{aligned}
\theta_{i(u)}(k) = \mathsf{Pr}\left(C_u = k | \bar{\mathbf{Y}}_i = \bar{\mathbf{y}}_i\right) &= \frac{\mathsf{Pr}\left(C_u = k, \bar{\mathbf{Y}}_i = \bar{\mathbf{y}}_i\right)}{\mathsf{Pr}\left(\bar{\mathbf{Y}}_i = \bar{\mathbf{y}}_i\right)} \\
&= \frac{\mathsf{Pr}\left(C_u = k, \bar{\mathbf{Y}}_{i(1\backslash u)} = \bar{\mathbf{y}}_{i(1\backslash u)}\right) \times \mathsf{Pr}\left(\bar{\mathbf{Y}}_{i(u)} = \bar{\mathbf{y}}_{i(u)} | C_u = k\right)}{\mathsf{Pr}\left(\bar{\mathbf{Y}}_i = \bar{\mathbf{y}}_i\right)} \\
&= \frac{\alpha_{i(u)}(k) \lambda_{i(u)}(k)}{\sum_j \alpha_{i(v)}(j) \lambda_{i(v)}(j)}
\end{aligned}$$

$$\theta_{i(\rho(u),u)}(k,l) = \mathsf{Pr}\left(C_u = k, C_{\rho(u)} = l | \bar{\mathbf{Y}}_i = \bar{\mathbf{y}}_i\right) = \frac{\lambda_{i(u)}(k) \tau_{k|l}^{(u)} \alpha_{i(\rho(u))}(j) \lambda_{i(\rho(u))}(j)}{\lambda_{i(u,\rho(u))}(j) \sum_j \alpha_{i(v)}(j) \lambda_{i(v)}(j)}$$

3 Hessian

$$\frac{\partial \log L}{\partial \beta_{qj} \partial \beta_{pk}} = \sum_i x_{ip} x_{iq} \left\{ \zeta_{kj} (\theta_{ik} - \pi_k) - (\theta_{ik}\theta_{ij} - \pi_k \pi_j) \right\}$$

$$\frac{\partial \log L}{\partial \beta_{qj} \partial \beta_{pk|l}} = \sum_i x_{ip} x_{iq} \left(\theta_{i(k,l)} - \tau_{k|l} \theta_{i(l)} \right) \left(\zeta_{kl} - \theta_{i(j)} \right)$$

$$\frac{\partial \log L}{\partial \beta_{qj|m} \partial \beta_{pk|l}} = \sum_i x_{ip} x_{iq} \left\{ \zeta_{ml} \left[\left(\theta_{i(k,l)} - \tau_{k|l} \theta_{i(l)} \right) \left(\zeta_{jk} - \tau_{j|l} \right) - \tau_{k|l} \left(\theta_{i(j,l)} - \tau_{j|l} \theta_{i(l)} \right) \right] - \left(\theta_{i(j,m)} - \tau_{j|m} \theta_{i(m)} \right) \left(\theta_{i(k,l)} - \right.\right.$$