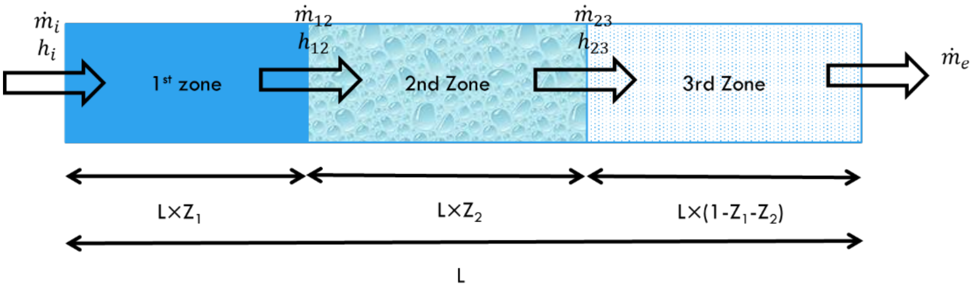

Figure 1: A schematic diagram of a heat exchanger having three zones

1 Appendix

1.1 MB formulation with Local States

In this section, a zone number is defined by refrigerant flow direction in a heat exchanger. For example, for the TPSH evaporators, $\bar{\rho}_1, \bar{\rho}_2$ mean densities for the first zone (TP), and second zone (SH), whereas they are those for SH and TP zones for the SHTP condensors. Mass and energy balances applied to each zone results in the following differential equations.

Single-zone formulation

$$\frac{\partial \bar{\rho}_1}{\partial P}\dot{P} + \frac{\partial \bar{\rho}_1}{\partial h_{e,1}}\dot{h}_e = (\dot{m}_i - \dot{m}_e)/(AL) - \frac{\partial \bar{\rho}_1}{\partial h_{i,1}}\dot{h}_i \tag{1}$$

$$\begin{aligned}&(\bar{h}_1\frac{\partial \bar{\rho}_1}{\partial P} - 1)\dot{P} + (\bar{h}_1\frac{\partial \bar{\rho}_1}{\partial h_{e,1}} + \frac{\bar{\rho}_1}{2})\dot{h}_e = \\ &(\dot{m}_i h_i - \dot{m}_e h_e + \dot{Q}_1)/(AL) - (\bar{h}_1\frac{\partial \bar{\rho}_1}{\partial h_{i,1}} + \frac{\bar{\rho}_1}{2})\dot{h}_i\end{aligned} \tag{2}$$

Two-zone formulation

$$\begin{aligned}&Z_1(\frac{\partial \bar{\rho}_1}{\partial P} + \frac{\partial \bar{\rho}_1}{\partial h_{e,1}}\frac{dh_{12}}{dP})\dot{P} + (\bar{\rho}_1 - \rho_{12})\dot{Z}_1 + Z_1\frac{\partial \bar{\rho}_1}{\partial h_{i,1}}\dot{h}_{i,1} \\ &= (\dot{m}_i - \dot{m}_{12})/(AL)\end{aligned} \tag{3}$$

$$\begin{aligned}&Z_1\left(\bar{h}_1\frac{\partial \bar{\rho}_1}{\partial P} - 1 + (\bar{h}_1\frac{\partial \bar{\rho}_1}{\partial h_{e,1}} + \frac{1}{2}\bar{\rho}_1)\frac{dh_{12}}{dP}\right)\dot{P} + (\overline{\rho h}_1 - \rho_{12}h_{12})\dot{Z}_1 \\ &+Z_1(\bar{h}_1\frac{\partial \bar{\rho}_1}{\partial h_{i,1}} + \frac{1}{2}\bar{\rho}_1)\dot{h}_i = (\dot{m}_i h_i - \dot{m}_{12}h_{12} + \dot{Q}_1)/(AL)\end{aligned} \tag{4}$$

$$\begin{aligned}&(1-Z_1)\left(\frac{\partial \bar{\rho}_2}{\partial P} + \frac{\partial \bar{\rho}_2}{\partial h_{i,2}}\frac{dh_{12}}{dP}\right)\dot{P} - (\bar{\rho}_2 - \rho_{12})\dot{Z}_1 + (1-Z_1)\frac{\partial \bar{\rho}_2}{\partial h_{e,2}}\dot{h}_e \\ &= (\dot{m}_{12} - \dot{m}_e)/(AL)\end{aligned} \tag{5}$$

$$(1-Z_1)\left(\bar{h}_2\frac{\partial\bar{\rho}_2}{\partial P}-1+(\bar{h}_2\frac{\partial\bar{\rho}_2}{\partial h_{i,2}}+\frac{1}{2}\bar{\rho}_2)h_{P,12}\right)\dot{P}+ \tag{6}$$
$$(1-Z_1)(\bar{h}_2\frac{\partial\bar{\rho}_2}{\partial h_{e,2}}+\frac{1}{2}\bar{\rho}_2)\dot{h}_e-(\overline{\rho h}_2-\rho_{12}h_{12})\dot{Z}_1$$
$$=(\dot{m}_{12}h_{12}-\dot{m}_e h_e+\dot{Q}_2)/(AL)$$

<u>Three-zone formulation</u>

$$Z_1(\frac{\partial\bar{\rho}_1}{\partial P}+\frac{\partial\bar{\rho}_1}{\partial h_{e,1}}\frac{dh_{12}}{dP})\dot{P}+(\bar{\rho}_1-\rho_{12})\dot{Z}_1=(\dot{m}_i-\dot{m}_{12})/(AL)-Z_1\frac{\partial\bar{\rho}_1}{\partial h_{i,1}}\dot{h}_i \tag{7}$$

$$Z_1\left(\bar{h}_1\frac{\partial\bar{\rho}_1}{\partial P}-1+(\bar{h}_1\frac{\partial\bar{\rho}_1}{\partial h_{e,1}}+\frac{1}{2}\bar{\rho}_1)\frac{dh_{12}}{dP}\right)\dot{P}+(\overline{\rho h}_1-\rho_{12}h_{12})\dot{Z}_1 \tag{8}$$
$$=(\dot{m}_i h_i-\dot{m}_{12}h_{12}+\dot{Q}_1)/(AL)-Z_1(\bar{h}_1\frac{\partial\bar{\rho}_1}{\partial h_{i,1}}+\frac{1}{2}\bar{\rho}_1)\dot{h}_i$$

$$Z_2(\frac{\partial\bar{\rho}_2}{\partial P}+\frac{\partial\bar{\rho}_2}{\partial h_{e,2}}h_{P,23}+\frac{\partial\bar{\rho}_2}{\partial h_{i,2}}\frac{dh_{12}}{dP})\dot{P}+(\bar{\rho}_2-\rho_{23})\dot{Z}_2-(\rho_{23}-\rho_{12})\dot{Z}_1 \tag{9}$$
$$=(\dot{m}_{12}-\dot{m}_{23})/(AL)$$

$$Z_2\left((\bar{h}_2\frac{\partial\bar{\rho}_2}{\partial P}-1)+(\bar{h}_2\frac{\partial\bar{\rho}_2}{\partial h_{e,2}}+\frac{1}{2}\bar{\rho}_2)h_{P,23}+(\bar{h}_2\frac{\partial\bar{\rho}_2}{\partial h_{i,2}}+\frac{1}{2}\bar{\rho}_2)\frac{dh_{12}}{dP}\right)\dot{P} \tag{10}$$
$$+(\overline{\rho h}_2-\rho_{23}h_{23})\dot{Z}_2-(\rho_{23}h_{23}-\rho_{12}h_{12})\dot{Z}_1=(\dot{m}_{12}h_{12}-\dot{m}_{23}h_{23}+\dot{Q}_2)/(AL)$$

$$Z_3(\frac{\partial\bar{\rho}_3}{\partial P}+\frac{\partial\bar{\rho}_3}{\partial h_{i,3}}\frac{dh_{23}}{dP})\dot{P}+Z_3\frac{\partial\bar{\rho}_3}{\partial h_{e,3}}\dot{h}_e-(\bar{\rho}_3-\rho_{23})\dot{Z}_2-(\bar{\rho}_3-\rho_{23})\dot{Z}_1 \tag{11}$$
$$=(\dot{m}_{23}-\dot{m}_e)/(AL)$$

$$Z_3\left((\bar{h}_3\frac{\partial\bar{\rho}_3}{\partial P}-1)+(\bar{h}_3\frac{\partial\bar{\rho}_3}{\partial h_{i,3}}+\frac{1}{2}\bar{\rho}_3)\frac{dh_{23}}{dP}\right)\dot{P}+Z_3(\bar{h}_3\frac{\partial\bar{\rho}_3}{\partial h_{e,3}}+\frac{1}{2}\bar{\rho}_3)\dot{h}_e \tag{12}$$
$$-(\overline{\rho h}_3-\rho_{23}h_{23})\dot{Z}_2-(\overline{\rho h}_3-\rho_{23}h_{23})\dot{Z}_1=(\dot{m}_{23}h_{23}-\dot{m}_e h_e+\dot{Q}_3)/(AL)$$

After algebraic manipulation, one can reduce intermediate variables of $\dot{m}_{12}$ or $\dot{m}_{23}$ as follows.

<u>One-zone formulation</u>

$$E_{1z}(x_{1z})\dot{x}_{1z}=f_{1z}(x_{1z},u_{1z}) \tag{13}$$

where

$$x_{1z}=\left[P,h_e,T_{t1}\right]^T\ u_{1z}=\left[\dot{m}_i,\dot{m}_e,h_i,\dot{h}_i,T_{env,1}\right]^T, \tag{14}$$

$$E_{1z}(x_{1z})=\begin{bmatrix}\bar{\rho}_{P1} & \bar{\rho}_{he1} & 0\\ (\bar{h}_1\bar{\rho}_{P1}-1) & (\bar{h}_1\bar{\rho}_{he1}+\frac{\bar{\rho}_1}{2}) & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$
f_{1z}(x_{1z}, u_{1z}) = \begin{bmatrix} (\dot{m}_i - \dot{m}_e)/(AL) - \bar{\rho}_{hi1}\dot{h}_{i1} \\ (\dot{m}_i h_i - \dot{m}_e h_e + \alpha_{r1} A_{s,i}(T_{t1} - T_1))/(AL) - (\bar{h}_1 \bar{\rho}_{hi1} + \frac{\bar{\rho}_1}{2})\dot{h}_{i1} \\ (\alpha_{r1} A_{s,i}(T_1 - T_{t1}) + \alpha_{a1} A_{s,o}(T_{env,1} - T_{t1}))/C_t \end{bmatrix}
$$

<u>Two-zone formulation</u>

$$
E_{2z}(x)\dot{x}_{2z} = f_{2z}(x_{2z}, u_{2z}) \tag{15}
$$

where

$$
x_{2z} = \left[P, h_e, Z_1, T_{t1}, T_{t2}\right]^T \; u_{2z} = \left[\dot{m}_i, \dot{m}_e, h_i, \dot{h}_i, T_{env,1}, T_{env,2}\right]^T, \tag{16}
$$

$$
E_{2z}(x) =
$$

$$
\begin{bmatrix} Z_1(\bar{\rho}_{P,1} + \bar{\rho}_{he,1} h_{P,12}) + Z_2\left(\bar{\rho}_{P,2} + \bar{\rho}_{hi,2} h_{P,12}\right) & Z_2 \bar{\rho}_{he,2} & \bar{\rho}_1 - \bar{\rho}_2 & & \\ Z_1\left(\left((\bar{h}_1 - h_{12})\bar{\rho}_{P,1} - 1\right) + \left((\bar{h}_1 - h_{12})\bar{\rho}_{he,1} + \frac{1}{2}\bar{\rho}_1\right) h_{P,12}\right) & & \bar{\rho}_1(\bar{h}_1 - h_{12}) & & \\ Z_2\left(\left((\bar{h}_2 - h_{12})\bar{\rho}_{P,2} - 1\right) + \left((\bar{h}_2 - h_{12})\bar{\rho}_{hi,2} + \frac{1}{2}\bar{\rho}_2\right) h_{P,12}\right) & Z_2((\bar{h}_2 - h_{12})\bar{\rho}_{he,2} + \frac{1}{2}\bar{\rho}_2) & -\bar{\rho}_2(\bar{h}_2 - h_{12}) & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}
$$

$$
f_{2z}(x, u) = \begin{bmatrix} (\dot{m}_{i,1} - \dot{m}_{e,2})/AL - Z_1 \bar{\rho}_{hi,1}\dot{h}_{i,1} \\ (\dot{m}_{i,1} h_{i,1} - \dot{m}_{i,1} h_{12} + \alpha_{r1} A_{s,i} Z_1(T_{t1} - T_1))/(AL) - Z_1\left((\bar{h}_1 - h_{12})\rho_{hi,1} + \frac{1}{2}\bar{\rho}_1\right)\dot{h}_{i,1} \\ (\dot{m}_{e,2} h_{12} - \dot{m}_{e,2} h_{e,2} + \alpha_{r2} A_{s,i} Z_2(T_{t2} - T_2))/(AL) \\ \left(\alpha_{r1} A_{s,i}(T_1 - T_{t1}) + \alpha_{a1} A_{s,o}(T_{env,1} - T_{t1})\right)/C_t \\ \left(\alpha_{r2} A_{s,i}(T_2 - T_{t2}) + \alpha_{a2} A_{s,o}(T_{env,2} - T_{t2})\right)/C_t \end{bmatrix}
$$

Three-zone formulation

$$\mathbf{E}_{3z}(x)\dot{x} = \mathbf{f}_{3z}(x,u) \tag{17}$$

where

$$x = \left[P, h_e, Z_1, Z_2, T_{t1}, T_{t2}, T_{t3}\right]^T \tag{18}$$

$$u = \left[\dot{m}_i, \dot{m}_e, h_i, \dot{h}_i, T_{env,1}, T_{env,2}, T_{env,3}\right]^T . \tag{19}$$

$$\mathbf{E}_{3z}(x) =$$

$$\begin{bmatrix}
Z_1(\bar{\rho}_{P,1} + \bar{\rho}_{he,1}h_{P,12}) + Z_2(\bar{\rho}_{P,2} + \bar{\rho}_{he,2}h_{P,23} + \bar{\rho}_{hi,2}h_{P,12}) + Z_3(\bar{\rho}_{P,3} + \bar{\rho}_{hi,3}h_{P,23}) & Z_3\bar{\rho}_{he,3} & (\bar{\rho}_1 - \bar{\rho}_3) & (\bar{\rho}_2 - \bar{\rho}_3) & & & \\
Z_1\left(\left((\bar{h}_1 - h_{12})\bar{\rho}_{P,1} - 1\right) + \left((\bar{h}_1 - h_{12})\bar{\rho}_{he,1} + \frac{1}{2}\bar{\rho}_1\right) h_{P,12}\right) & & \bar{\rho}_1(\bar{h}_1 - h_{12}) & 0 & & & \\
\left\{ \begin{array}{l} Z_2\left((\bar{h}_2\bar{\rho}_{P,2} - 1) + (\bar{h}_2\bar{\rho}_{he,2} + \frac{1}{2}\bar{\rho}_2)h_{P,23} + (\bar{h}_2\bar{\rho}_{hi,2} + \frac{1}{2}\bar{\rho}_2)h_{P,12}\right) \\ +Z_1(\bar{\rho}_{P,1} + \bar{\rho}_{he,1}h_{P,12})h_{12} + Z_3(\bar{\rho}_{P,3} + \bar{\rho}_{hi,3}h_{P,23})h_{23} \end{array} \right\} & Z_3\bar{\rho}_{he,3}h_{23} & \bar{\rho}_1 h_{12} - \bar{\rho}_3 h_{23} & \overline{\rho h}_2 - \bar{\rho}_3 h_{23} & & & \\
Z_3\left((\bar{h}_3 - h_{23})\bar{\rho}_{P,3} - 1\right) + \left((\bar{h}_3 - h_{23})\bar{\rho}_{hi,3} + \frac{1}{2}\bar{\rho}_3\right) h_{P,23}\right) & Z_3((\bar{h}_3 - h_{23})\bar{\rho}_{he,3} + \frac{1}{2}\bar{\rho}_3 & -\bar{\rho}_3(\bar{h}_3 - h_{23}) & -\bar{\rho}_3(\bar{h}_3 - h_{23}) & & & \\
 & & & & 1 & & \\
 & & & & & 1 & \\
 & & & & & & 1
\end{bmatrix}$$

$$\mathbf{f}_{3z}(x,u) = \begin{bmatrix}
(\dot{m}_i - \dot{m}_e)/(AL) - Z_1\bar{\rho}_{hi,1}\dot{h} \\
(\dot{m}_i h_i - \dot{m}_i h_{12} + \dot{Q}_1)/(AL) - Z_1((\bar{h}_1 - h_{12})\bar{\rho}_{hi,1} + \frac{1}{2}\bar{\rho}_1)\dot{h}_i \\
(\dot{m}_i h_{12} - \dot{m}_e h_{23} + \dot{Q}_2)/(AL) - Z_1\bar{\rho}_{hi,1}\dot{h}_i h_{12} \\
(\dot{m}_e(h_{23} - h_e) + \dot{Q}_3)/(AL) \\
\left(\alpha_{r1}A_{s,i}(T_1 - T_{t1}) + \alpha_{a1}A_{s,o}(T_{env,1} - T_{t1})\right)/C_t \\
\left(\alpha_{r2}A_{s,i}(T_2 - T_{t2}) + \alpha_{a2}A_{s,o}(T_{env,2} - T_{t2})\right)/C_t \\
\left(\alpha_{r3}A_{s,i}(T_3 - T_{t3}) + \alpha_{a3}A_{s,o}(T_{env,3} - T_{t3})\right)/C_t
\end{bmatrix}$$

1.2 MB formulation with Global State

There is a mismatch between states and inputs between the expressions of 13, 15 and 17. This section derives MB formulations that have consistent states and inputs.

Let us define x and u as follows.

$$x = \left[P, h_e, Z_1, Z_2, T_{t1}, T_{t2}, T_{t3}\right]^T \tag{20}$$

$$u = \left[\dot{m}_i, \dot{m}_e, h_i, \dot{h}_i, T_{env,1}, T_{env,2}, T_{env,3}\right]^T. \tag{21}$$

Our final one-zone moving boundary model is defined in the following form by aggregating pseudo state dynamics for inactive states for the 1-zone case, i.e. $\dot{Z}_1 = 0, \dot{Z}_2 = -\dot{Z}_1, \dot{T}_{t2} = 0, \dot{T}_{t3} = 0$, to Eqn. (13).

$$\mathbf{E}_{1z}(x)\dot{x} = \mathbf{f}_{1z}(x, u) \tag{22}$$

Likewise, our final two-zone moving boundary model is defined in the following form by aggregating pseudo state dynamics for inactive states for the 2-zone case, i.e. $\dot{Z}_2 = -\dot{Z}_1, \dot{T}_{t3} = 0$, to Eqn. (15).

$$\mathbf{E}_{2z}(x)\dot{x} = \mathbf{f}_{2z}(x, u) \tag{23}$$

The final three-zone model is identical to Eqn. (17), that is

$$\mathbf{E}_{3z}(x)\dot{x} = \mathbf{f}_{3z}(x, u) \tag{24}$$

where $\mathbf{E}_{3z} = E_{3z}, \mathbf{f}_{3z} = f_{3z}$.

1.3 Mode Enumeration

For the one-zone model, there are three possible modes, i.e. SC, TP and SP. Meanwhile, there are two modes, i.e. SCTP and TPSH in case of evaporator or SHTP and TPSC in case of condensor, for the two-zone model, and one mode, i.e. SCTPSH in case of evaporator or SHTPSC in case of condensor, for the three-zone model.

We denote the six dynamic models for either evaporator and condensor uniformly as

$$\mathbf{E}_j\dot{x} = \mathbf{f}_j(x, u) \tag{25}$$

or

$$\dot{x} = F_j(x, u), \quad j \in \{1, \cdots, 6\} \tag{26}$$

where $F_j(x, u) = \mathbf{E}_j^{-1}(x)\mathbf{f}_j(x, u)$ where j represents the mode number defined in Fig. **??** for evaporator. For a condensor, the mode number 1 means SHTPSC, 2 is TPSC and 3 represents SHTP.