Lesson 6

METHOD OF SEPARATION OF VARIABLES

In this method, it is assumed that a solution can be expressed as a product or a sum of unknown functions each of which depends on only one of the independent variables

The success of the method depends on being able to write the resulting eqn so that one side depends only on one variable while the other side depends on the remaining variable so that each side must be a constant

By repetition of this, the unknown functions can be determined
The main idea is to convert the given pde into several ode's and then obtain solution by familiar techniques
eg
- separation of variable
- homogeneous equations
- exact eqns
- linear eqns etc

This method is the most powerful method of a wide class of pde's where boundary conditions are given ie Boundary value problem (BVP)

Given pde of the form

$$Pp + Qq = R \quad \text{--- (i)}$$

or

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + Cu = 0 \quad \text{--- (ii)}$$

Soln:

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q$$

$$Z(x,y) \text{ or } U(x,y)$$

Let the soln be $u(x,y)$

(i) Using a product
$u(x,y) = X(x) \cdot Y(y)$ --- ii

The method of soln is to substitute eqn (ii) into eqn I and proceed to solve separately for x and y

ii) Using sum
Let $u(x,y) = X(x) + Y(y)$

Given boundary conditions are to be used to obtain a particular solution

Using separable product
$u(x,y) = X(x) Y(y)$

$\frac{\partial u}{\partial x} = X'Y$ $\quad \frac{\partial u}{\partial y}$ XY'

Example

Solve the boundary value problem

$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ $\quad u(0,y) = 8e^{-3y}$

Using method of separation of variables

Soln:

$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ --- (1)

Let $u(x,y) = X(x) \cdot Y(y)$ ... (2)

$\frac{\partial u}{\partial x} = X'Y$, $\frac{\partial u}{\partial y} = XY'$ ... (3)

substitute in original eqn

$X'Y = 4XY'$

$\frac{X'}{4X} = \frac{Y'}{Y} = c$

2 odes

$\frac{X'}{4X} = c$ and $\frac{Y'}{Y} = C$

Recall $\int \frac{f'(x)}{f(x)} = \ln f(x)$

$\Rightarrow \frac{1}{4}\ln X = Cx + A$

Using logs

$\frac{1}{4}\int \frac{X'}{X} = \int c$

$\frac{1}{4}\log X = cx + \log A$

$\log X = 4Cx + \log A$

$\log X = 4Cx \log e + \log A$

$X = Ae^{4Cx}$

$\ln X = 4Cx + A$

$X = Ae^{Bx}$ --- (4)

$\ln Y = Cy + D$

$Y = De^{cy}$ --- (5)

Recall $u(x,y) = X(x) \cdot Y(y)$

substitute (4) and (5) into 2

$u(x,y) = Ae^{Bx} \cdot De^{cy}$

Let $AD = M$

$u(x,y) = Me^{(Bx+cy)}$

But $B = 4C$

$u(x,y) = Me^{c(4x+y)}$

$u(0,y) = 8e^{-3y}$

$u(0,y) = Me^{c(4x+y)}$

$= Me^{cy}$

$= 8e^{-3y}$

$\therefore M = 8 \quad c = -3$

Particular soln is

$u(x,y) = 8e^{-3(4x+y)}$

Summary

$\frac{\partial u}{\partial x} = \frac{4\partial u}{\partial y}$

$P = 4Q$

2) Solve the Initial Value Problem

$u_x + 2u_y = 0 \qquad u(0,y) = (e^{-2y}).4$

Soln

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

$u(x,y) = X(x).Y(y)$

$u_x = X'Y \qquad u_y = XY'$

substitute

$X'Y + 2XY' = 0$

$X'Y = -2XY'$

$$\frac{X'}{2X} = -\frac{Y'}{Y} = k$$

$$\Rightarrow \frac{X'}{2X} = k$$

$\frac{1}{2} \ln X = kx + A$

$X = Be^{2kx}$

$$\Rightarrow -\frac{Y'}{Y} = k \qquad = \frac{Y'}{Y} = -k$$

$\ln Y = -ky + c$

$Y = e^{-ky+c}$

$Y = e^{-ky}.e^{c}$

$Y = De^{-ky}$

$u(x,y) = X(x)\,Y(y)$

$Be^{2kx}.De^{-ky}$

$= Ee^{k(2x-y)}$

$u(0,y) = Ee^{k(0-y)}$

$= Ee^{k(-y)}$

$4e^{-2y} = Ee^{-ky}$

$E = 4\,; \quad k = (-2) \Rightarrow 2$

Particular soln

$u(x,y) = 4e^{-2(2x-y)}$

$= 4e^{4x-2y}$

Exercise

1. $\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$

2. $3u_x + 2u_y = 0$

$u(x,0) = 4e^{-x}$

3. $u_x + u = u_y$

$u(x,0) = 4e^{-3x}$

4. $4u_x + u_y = 3u$

$, u = 3e^{-y} - e^{5y}$ where $x = 0$