Unscented Kalman Filtering for Simultaneous Estimation of Attitude and Gyroscope Bias

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***Abstract*—We present an unscented Kalman filtering (UKF) algorithm for simultaneously estimating attitude and gyroscope bias from an inertial measurement unit (IMU). The algorithm is formulated as a discrete-time stochastic nonlinear filter, with state space given by the direct product matrix Lie group SO(3) $\times \mathbb{R}^3$, and observations in SO(3) reconstructed from IMU measurements of gravity and the earth's magnetic field. Computationally efficient implementations of our filter are made possible by formulating the state space dynamics and measurement equations in a way that leads to closed-form equations for covariance propagation and update. The resulting attitude estimates are invariant with respect to choice of fixed and moving reference frames. The performance advantages of our filter vis-à-vis existing state-of-the-art IMU attitude estimation algorithms are validated via numerical and hardware experiments involving both synthetic and real data.**

***Index Terms*—Attitude estimation, unscented Kalman filter, gyroscope bias, inertial measurement unit.**

I. Introduction

Estimating an object's orientation, or attitude, from an inertial measurement unit (IMU) attached to the object, arises in applications ranging from vehicle and robot navigation [1]–[3] to human pose tracking [4]. A typical IMU consists of a gyroscope, accelerometer, and magnetometer: the gyroscope measures angular rates (which are integrated to calculate the attitude), the accelerometer measures accelerations due to gravity and other external forces, and the magnetometer measures the earth's magnetic field. Because the gyroscopic measurements contain a time-varying bias error, they are augmented by accelerometer and magnetometer measurements. The challenges and benefits of simultaneously estimating the attitude and gyroscope bias from disparate sensor measurements are detailed in [5] and the cited references.

Notable among **deterministic filtering** methods for simultaneously estimating attitude and gyroscope bias are Mahony et al's series of nonlinear complementary filters (NCFs) [6]–[8]; these filters ensure almost global stability of the observer error, and their performance has been validated in numerous experimental scenarios. **Stochastic filtering** methods further take into account statistical characterizations of measurement and process noise, and include well-known and widely used methods such as the extended Kalman filter (EKF). More recently the unscented Kalman filter (UKF), despite its greater computational complexity, has been shown to outperform the EKF in a wide range of applications [9]–[11].

Because the underlying configuration space of rotations, represented by the group SO(3) of 3×3 real orthogonal matrices with unit determinant, is not a vector space but a curved space, the attitude estimation problem is fundamentally a nonlinear one. The straightforward but naive approach of expressing a rotation in terms of some suitable local coordinates (e.g., roll-pitch-yaw angles, Euler angles) is problematic at several levels: the local coordinates contain singularities that require special treatment (for example, when the pitch angle is 90 degrees), and the resulting estimates depend both on the choice of local coordinates as well as fixed and moving reference frames. If standard vector space filters are naively adapted to local coordinate representations of the attitude, not only are the equations for the state space dynamics and measurements both nonlinear and dependent on the choice of reference frames, but filtering performance is highly uneven throughout different regions of the configuration space.

Recent research has attempted to address the issue of coordinate and reference frame dependency through the use of differential geometric methods. Although computationally more involved than standard vector space filtering algorithms, when correctly formulated, these methods are invariant with respect to the choice of fixed and moving reference frames, and also independent of the choice of local coordinates used to parametrize the rotations. For estimation problems in which the underlying configuration space has the structure of a matrix Lie group (like SO(3)) coordinate-invariant versions of both the EKF [12]–[15], the UKF [16], [17], and also particle filtering methods [18] have been presented in the recent literature. Without exception, these general methods almost always include illustrative examples involving estimation on the rotation group, e.g., [14].

In this paper we address the problem of simultaneous estimation of attitude and gyroscope bias from a stochastic differential geometric perspective. When the assumed noise models are valid, the advantages of stochastic filtering methods over their deterministic counterparts are well-documented. For real-time applications, stochastic filtering methods require efficient calculation and propagation of covariances, which often prove to be difficult for systems with complex nonlinear state dynamics and measurements. Our contribution takes advantage of the coordinate- and frame-invariant properties

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of geometric filtering, and at the same time leads to a robust and computationally efficient stochastic UKF algorithm that can be implemented in real-time. These improvements in efficiency and robustness are achieved by formulating the state dynamics and measurements in a way that leads to closed-form equations for covariance propagation and update, and also by drawing upon efficient geometric algorithms for key steps of the geometric UKF algorithm.

The paper is organized as follows. After a brief review of geometric preliminaries in Section II, our UKF algorithm for simultaneously estimating attitude and gyroscope bias is described in Section III. Section IV details the calculation of the measurement noise covariance. Section V compares the performance of our geometric UKF algorithm against other existing state-of-the-art estimators for attitude and gyroscope bias [6], [19], [20], with detailed experiments involving both synthetic and real data validating the performance advantages of our geometric UKF algorithm.

II. Geometric Preliminaries

We first recall some basic facts and useful formulas about the rotation group SO(3) [21], [22]. Elements of SO(3) are represented by the 3×3 real matrices $\mathbf{R}$ satisfying $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = 1$, where $\mathbf{I}$ here denotes the 3×3 identity matrix. SO(3) is an example of a matrix Lie group; its associated Lie algebra, denoted so(3), is given by the set of 3×3 real skew-symmetric matrices of the form

$$[\boldsymbol{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix},$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T \in \mathbb{R}^3$. A fundamental connection between so(3) and SO(3) is the matrix exponential map $\exp$: so(3) $\to$ SO(3):

$$\begin{aligned} \exp([\boldsymbol{\omega}]) &= \sum_{m=0}^{\infty} \frac{[\boldsymbol{\omega}]^m}{m!} \\ &= \mathbf{I} + \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}[\boldsymbol{\omega}] + \frac{1-\cos\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|^2}[\boldsymbol{\omega}]^2, \end{aligned}$$

where $\|\cdot\|$ represents the standard Euclidean vector norm. The inverse of the exponential, or logarithm, of SO(3) is defined as follows: for any $\mathbf{R} \in$ SO(3) such that $\mathrm{tr}(\mathbf{R}) \neq -1$,

$$\log \mathbf{R} = \frac{\theta}{2\sin\theta}(\mathbf{R} - \mathbf{R}^T),$$

where θ satisfies $1 + 2\cos\theta = \mathrm{tr}(\mathbf{R}), |\theta| < \pi$ (here $\mathrm{tr}(\cdot)$ denotes the trace of a matrix). If $\mathrm{tr}(\mathbf{R}) = -1$, then the equation $\log \mathbf{R} = [\boldsymbol{\omega}]$ has two antipodal solutions $\pm\boldsymbol{\omega}$ that can be determined from the relation $\mathbf{R} = \mathbf{I} + (2/\pi^2)[\boldsymbol{\omega}]^2$. A straightforward calculation also establishes that $\|\log \mathbf{R}\|/\sqrt{2} = \theta$, where $\|\cdot\|$ denotes the Frobenius matrix norm.

The natural way to measure distances between two rotations $\mathbf{R}_1$ and $\mathbf{R}_2$ is via the formula

$$d(\mathbf{R}_1, \mathbf{R}_2) = \|\log(\mathbf{R}_1^T\mathbf{R}_2)\|.$$

The above distance metric is invariant with respect to left and right translations, or bi-invariant, in the sense that $d(\mathbf{R}_1, \mathbf{R}_2) = d(\mathbf{P}\mathbf{R}_1\mathbf{Q}, \mathbf{P}\mathbf{R}_2\mathbf{Q})$ for any $\mathbf{P}, \mathbf{Q} \in$ SO(3). With this notion of distance, the curve $\mathbf{R}(t)$ on SO(3) of shortest length (or minimal geodesic) that connects $\mathbf{R}_1 = \mathbf{R}(0)$ and $\mathbf{R}_2 = \mathbf{R}(1)$ is given by $\mathbf{R}(t) = \mathbf{R}_1\exp(\boldsymbol{\Omega}t)$, where $\boldsymbol{\Omega} = \log(\mathbf{R}_1^T\mathbf{R}_2) \in$ so(3).

Recalling that $\mathbb{R}^3$ is also trivially a Lie group under vector addition, the direct product SO(3) $\times\ \mathbb{R}^3$ can be given the structure of a Lie group via the product rule $(\mathbf{R}_1, \mathbf{b}_1)\cdot(\mathbf{R}_2, \mathbf{b}_2) = (\mathbf{R}_1\mathbf{R}_2, \mathbf{b}_1 + \mathbf{b}_2)$ and the inversion rule $(\mathbf{R}, \mathbf{b})^{-1} = (\mathbf{R}^T, -\mathbf{b})$.

Now define a random variable $\mathbf{X}$ on SO(3) as

$$\mathbf{X} := \exp([\boldsymbol{\eta}])\,\mathbf{X}_0, \tag{1}$$

where $\mathbf{X}_0 \in$ SO(3) is given and $\boldsymbol{\eta} \in \mathbb{R}^3$ is a zero-mean Gaussian with covariance $\mathbf{P}_\eta$, i.e., $\boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \mathbf{P}_\eta)$. We refer to $\boldsymbol{\eta}$ as ***right-translated exponential noise*** with **right-invariant covariance** $\mathbf{P}_\eta$. Alternatively, defining the random variable $\mathbf{X}$ on SO(3) as $\mathbf{X} = \mathbf{X}_0\exp([\boldsymbol{\zeta}])$, where $[\boldsymbol{\zeta}] \in$ so(3) and $\boldsymbol{\zeta} \sim \mathcal{N}(\mathbf{0}, \mathbf{P}_\zeta)$, we refer to $\boldsymbol{\zeta}$ as ***left-translated exponential noise*** with **left-invariant covariance** $\mathbf{P}_\zeta$. A straightforward calculation verifies that

$$\boldsymbol{\eta} = \mathbf{X}_0\boldsymbol{\zeta} \tag{2}$$

$$\mathbf{P}_\eta = \mathbf{X}_0\mathbf{P}_\zeta\mathbf{X}_0^T \tag{3}$$

Statistical and computational aspects of SO(3) exponential noise defined in this way are further discussed in [23], [24].

Now consider the element $(\mathbf{X}, \mathbf{b}) = (\exp([\boldsymbol{\eta}])\mathbf{X}_0, \mathbf{b}_0 + \mathbf{n}) \in$ SO(3) $\times\ \mathbb{R}^3$, where $[\boldsymbol{\eta}] \in$ so(3), $\mathbf{X}_0 \in$ SO(3), and $\mathbf{b}_0, \mathbf{n} \in \mathbb{R}^3$, with $\mathbf{X}_0, \mathbf{b}_0$ constant and $\boldsymbol{\eta}, \mathbf{n}$ zero-mean Gaussian random vectors. Define the six-dimensional zero-mean Gaussian $\boldsymbol{\epsilon} = (\boldsymbol{\eta}, \mathbf{n}) \sim \mathcal{N}(\mathbf{0}, \mathbf{P}_\epsilon)$, where $\mathbf{P}_\epsilon \in \mathbb{R}^{6\times 6}$ is the covariance of $\boldsymbol{\epsilon}$. The six-dimensional covariance $\mathbf{P}_\epsilon$ will play a prominent role in our later UKF algorithm; in particular, the off-diagonal elements of $\mathbf{P}_\epsilon$ will typically be non-zero since $\mathbf{X}$ and $\mathbf{b}$ may be correlated.

III. UKF Algorithm for Estimating Attitude and Gyroscope Bias

Before describing our geometric UKF algorithm, we fix notation, describe the sensor models and their underlying assumptions, and review Wahba's problem [25] and its solutions.

Let $\{\mathcal{I}\}$ be the inertial reference frame fixed to ground, and let $\{\mathcal{B}\}$ denote the body frame fixed to the moving IMU. Let $\boldsymbol{\omega}^m \in \mathbb{R}^3$ be the angular velocity measured by the IMU gyroscope with respect to frame $\{\mathcal{B}\}$. Denote by $\mathbf{a}, \mathbf{m} \in \mathbb{R}^3$ the IMU accelerometer and magnetometer measurements, respectively; like $\boldsymbol{\omega}^m$, both $\mathbf{a}$ and $\mathbf{m}$ are assumed measured with respect to the IMU frame $\{\mathcal{B}\}$. Further define the unit vectors $\mathbf{v}_1 := \mathbf{a}/\|\mathbf{a}\|$, $\mathbf{v}_2 := \mathbf{m}/\|\mathbf{m}\|$.

In what follows we assume that the IMU is suitably calibrated, and that the gravitational acceleration is dominant in the accelerometer measurement $\mathbf{a}$. Let $\mathbf{r}_1 \in \mathbb{R}^3$ be the unit vector in the opposite direction of gravity, and $\mathbf{r}_2 \in \mathbb{R}^3$ be the unit vector in the direction of the earth's magnetic field. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are not collinear, then $\mathbf{r}_i$ and $\mathbf{v}_i$ should satisfy $\mathbf{r}_i = \mathbf{R}\mathbf{v}_i$, $i = 1, 2$, for some rotation $\mathbf{R} \in$ SO(3) representing the orientation of the IMU frame $\{\mathcal{B}\}$ relative to the fixed frame $\{\mathcal{I}\}$. Since in practice IMU measurements are noisy, $\mathbf{R}$ is typically estimated as the solution to the following optimization problem (referred to in the literature as Wahba's Problem [5], [25], [26]:

$$\mathbf{R}^* = \underset{\mathbf{R}\in \mathrm{SO}(3)}{\arg\min} \sum_{i=1}^{2} w_i \|\mathbf{r}_i - \mathbf{R}\mathbf{v}_i\|^2, \tag{4}$$

where the w_i are positive weights. A popular choice for w_i is $w_i = 1/\sigma_i^2$, where σ_i^2 denotes the variance of $\mathbf{v}_i$ in the direction normal to $\mathbf{R}^T\mathbf{r}_i$ [27]. (Equivalently, the normalized weights $w_i = \sigma_{\text{tot}}^2/\sigma_i^2$, where $1/\sigma_{\text{tot}}^2 = \sum_{i=1}^{2}(1/\sigma_i^2)$ are also widely used [28].)

Wahba's Problem as defined by Equation (4) admits the following closed-form solution [26]:

$$\mathbf{R}^* = \mathbf{V}\mathbf{D}\mathbf{U}^T, \tag{5}$$

where $\mathbf{U}$ and $\mathbf{V}$ are obtained from the singular value decomposition (SVD) of $\mathbf{F} := \sum_{i=1}^{2} w_i \mathbf{v}_i \mathbf{r}_i^T = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$. The matrix $\mathbf{D}$ in Equation (5) is of the form $\mathbf{D} = \mathrm{diag}(1, 1, \det(\mathbf{V}\mathbf{U}^T))$. (See [5] for a review of alternative solutions to (5) and a discussion of their robustness and computational efficiency.)

A. State Space Dynamics and Measurements

Estimates of $\mathbf{R}$ obtained via the static optimization procedure described above do not take into account the state space dynamics of the object or process and measurement noise characteristics, and typically are inferior to estimates obtained via nonlinear stochastic filtering techniques. We now formulate the overall problem in a discrete-time stochastic filtering setting. First, the angular rates $\boldsymbol{\omega}_k^m \in \mathbb{R}^3$ measured by the gyroscope at time step k are assumed to have the form

$$\boldsymbol{\omega}_k^m = \boldsymbol{\omega}_k + \mathbf{b}_k + \boldsymbol{\eta}_k, \tag{6}$$

where $\boldsymbol{\omega}_k$ denotes the ground truth angular rate vector, $\mathbf{b}_k \in \mathbb{R}^3$ is a time-varying bias term, and $\boldsymbol{\eta}_k$ is zero-mean Gaussian noise. The state dynamics are then assumed to be of the form

$$\mathbf{R}_{k+1} = \mathbf{R}_k \exp([\boldsymbol{\omega}_k^m - \mathbf{b}_k - \boldsymbol{\eta}_k]h) \tag{7}$$

$$\mathbf{b}_{k+1} = \mathbf{b}_k + \mathbf{n}_k, \tag{8}$$

where h is the integration time step, and $\boldsymbol{\eta}_k, \mathbf{n}_k$ are independent zero-mean Gaussians with the following distributions: $\boldsymbol{\eta}_k \sim \mathcal{N}(\mathbf{0}, c\mathbf{I}),\ \mathbf{n}_k \sim \mathcal{N}(\mathbf{0}, d\mathbf{I})$, with $c, d > 0$.

We now derive a first-order linear approximation of the state dynamics (7) that leads to a closed-form expression for the covariance of $\mathbf{R}_{k+1}$ consistent with (1). From the Baker-Campbell-Hausdorff formula [29], given $[x], [y] \in \mathrm{so}(3)$, $\exp([\mathbf{x}])\exp([\mathbf{y}])$ can be written exactly in the form $\exp([\mathbf{x}])\exp([\mathbf{y}]) = \exp([\mathbf{z}])$, $[z] \in \mathrm{so}(3)$, where

$$[\mathbf{z}] = \log(\exp([\mathbf{x}])\exp([\mathbf{y}])) \tag{9}$$

$$= [\mathbf{x}] + [\mathbf{y}] + \frac{1}{2}[[\mathbf{x}], [\mathbf{y}]] + \frac{1}{12}[[\mathbf{x}], [[\mathbf{x}], [\mathbf{y}]]] + \frac{1}{12}[[\mathbf{y}], [[\mathbf{y}], [\mathbf{x}]]] + \cdots, \tag{10}$$

with the Lie bracket operator $[\cdot, \cdot] : \mathrm{so}(3) \times \mathrm{so}(3) \to \mathrm{so}(3)$ defined by the matrix commutator, i.e., $[[\mathbf{a}], [\mathbf{b}]] = [\mathbf{a}][\mathbf{b}] - [\mathbf{b}][\mathbf{a}]$.

Let $\mathbf{x}' = \mathbf{z} - \mathbf{y} \in \mathbb{R}^3$ and rewrite Equation (9) in the form

$$\exp([\mathbf{x}' + \mathbf{y}]) = \exp([\mathbf{x}])\exp([\mathbf{y}]). \tag{11}$$

Gathering only terms linear in $\mathbf{x}$ in Equation (10), the following approximation between $\mathbf{x}$ and $\mathbf{x}'$ holds for $\|\mathbf{x}\|$ sufficiently small [23]:

$$\mathbf{x} \approx \mathbf{J}_l(\mathbf{y})\mathbf{x}', \tag{12}$$

where $\mathbf{J}_l(\mathbf{y}) \in \mathbb{R}^{3\times 3}$ is given by

$$\mathbf{J}_l(\mathbf{y}) = \mathbf{I} + \left(\frac{1 - \cos\|\mathbf{y}\|}{\|\mathbf{y}\|^2}\right)[\mathbf{y}] + \left(\frac{\|\mathbf{y}\| - \sin\|\mathbf{y}\|}{\|\mathbf{y}\|^3}\right)[\mathbf{y}]^2. \tag{13}$$

The derivation of (12) is provided in Appendix A.

If $\|\boldsymbol{\eta}_k\| \ll 1$, then Equation (7) can be approximated by

$$\mathbf{R}_{k+1} \approx \mathbf{R}_k \exp([\boldsymbol{\eta}_k']h)\exp([\boldsymbol{\omega}_k^m - \mathbf{b}_k]h) \tag{14}$$

$$= \exp([\mathbf{l}_k])\mathbf{R}_k \exp([\boldsymbol{\omega}_k^m - \mathbf{b}_k]h), \tag{15}$$

where $\boldsymbol{\eta}_k' = -\mathbf{J}_l(\boldsymbol{\psi})\boldsymbol{\eta}_k$, $\mathbf{l}_k = \mathbf{R}_k\boldsymbol{\eta}_k' h$, $\boldsymbol{\psi} = (\boldsymbol{\omega}_k^m - \mathbf{b}_k)h$. In deriving (14), the first-order approximation given by Equation (12) is used. The relation $\mathbf{R}\exp([\boldsymbol{\omega}])\mathbf{R}^T = \exp([\mathbf{R}\boldsymbol{\omega}])$ for $\mathbf{R} \in \mathrm{SO}(3), [\boldsymbol{\omega}] \in \mathrm{so}(3)$ is used in the derivation of (15).

Note that $\mathbf{l}_k = -\mathbf{R}_k\mathbf{J}_l(\boldsymbol{\psi})\boldsymbol{\eta}_k h$ is itself a random variable, since it is a function of random variables $\boldsymbol{\eta}_k, \mathbf{R}_k$, and $\boldsymbol{\psi}$. If we assume that $\|\boldsymbol{\psi}\|$ is small—this is a reasonable assumption provided h is sufficiently small—then $\mathbf{J}_l(\boldsymbol{\psi}) \approx \mathbf{I} + \frac{1}{2}[\boldsymbol{\psi}] \approx \exp(\frac{1}{2}[\boldsymbol{\psi}])$ holds from the first-order approximation. Note that $\mathbf{l}_k$ can be approximated as an isotropic Gaussian multiplied by rotation matrices, i.e., $\mathbf{l}_k \sim \mathcal{N}(\mathbf{0}, (ch^2)\mathbf{I})$.

The measurement equations are assumed to be of the form

$$\mathbf{Y}_{k+1} = \exp([\mathbf{w}_{k+1}])\mathbf{R}_{k+1}, \tag{16}$$

where $\mathbf{Y}_{k+1} \in \mathrm{SO}(3)$ is calculated as a solution to Wahba's Problem (4) using IMU gravitational acceleration and magnetic field measurements. The measurement noise $\mathbf{w}_{k+1} \in \mathbb{R}^3$ is assumed to be zero-mean Gaussian, implying that the measurement vector statistics are rotationally symmetric about their true measurement vectors.

B. UKF Algorithm

We now present the geometric UKF algorithm for simultaneous attitude and gyroscope bias estimation. Let $\mathbf{R}_k$ and $\mathbf{b}_k$ respectively denote the attitude and the gyroscope bias at time step k, and $\mathbf{X}_k := (\mathbf{R}_k, \mathbf{b}_k) \in \mathrm{SO}(3) \times \mathbb{R}^3$.

1) Initialization: Let $\hat{\mathbf{X}}_{0|0} = (\hat{\mathbf{R}}_{0|0}, \hat{\mathbf{b}}_{0|0})$ be the initial state estimate. The right-invariant covariance of $\hat{\mathbf{X}}_{0|0}$, denoted $\hat{\mathbf{P}}_{0|0}$, is given. From Equation (5), $\hat{\mathbf{R}}_{0|0}$ is estimated by solving Wahba's Problem (4) from a pair of initial measurement vectors $(\mathbf{v}_1, \mathbf{v}_2)$.

2) Time Update:

- From the *a priori* state estimate $\hat{\mathbf{X}}_{k|k} = (\hat{\mathbf{R}}_{k|k}, \hat{\mathbf{b}}_{k|k})$ and its covariance $\mathbf{P}_{k|k}$, extract a set of sigma points $\mathcal{X}_k^{(i)} := (\mathcal{X}_{\mathbf{R},k}^{(i)}, \mathcal{X}_{\mathbf{b},k}^{(i)}) \in \mathrm{SO}(3) \times \mathbb{R}^3$, $i = 0, \ldots, 12$, as follows:

$$\mathcal{X}_k^{(0)} = (\hat{\mathbf{R}}_{k|k}, \hat{\mathbf{b}}_{k|k})$$

$$\mathcal{X}_k^{(i)} = (\exp([\gamma\mathbf{s}_i^{(a)}])\hat{\mathbf{R}}_{k|k}, \hat{\mathbf{b}}_{k|k} + \gamma\mathbf{s}_i^{(b)}), i = 1, \ldots, 6$$

$$\mathcal{X}_k^{(i+6)} = (\exp([-\gamma\mathbf{s}_i^{(a)}])\hat{\mathbf{R}}_{k|k}, \hat{\mathbf{b}}_{k|k} - \gamma\mathbf{s}_i^{(b)}), i = 1, \ldots, 6,$$

where following [10], the parameter γ is chosen as $\gamma = \sqrt{N_x + \lambda}$, with N_x set to the state dimension (six) and $\lambda = N_x(\alpha^2 - 1)$, $0 < \alpha < 1$; $\mathbf{s}_i \in \mathbb{R}^6$ is the i^{th} column vector of the lower-triangular matrix $\mathbf{S} \in \mathbb{R}^{6\times 6}$ in the Cholesky decomposition $\mathbf{P}_{k|k} = \mathbf{S}\mathbf{S}^T$, and $\mathbf{s}_i^{(a)}, \mathbf{s}_i^{(b)} \in \mathbb{R}^3$ are respectively the upper and lower halves of $\mathbf{s}_i$.

- Setting $\mathbf{l}_k = \mathbf{0}$ in Equation (15) and $\mathbf{n}_k = \mathbf{0}$ in Equation (8), define a set of sigma points $\{(\Upsilon^{(i)}_{\mathbf{R},k+1}, \Upsilon^{(i)}_{\mathbf{b},k+1}) \in \mathrm{SO}(3) \times \mathbb{R}^3 | i = 0, \ldots, 12\}$ as

$$\Upsilon^{(i)}_{\mathbf{R},k+1} = \mathcal{X}^{(i)}_{\mathbf{R},k} \exp([\boldsymbol{\omega}^m_k - \mathcal{X}^{(i)}_{\mathbf{b},k}]h) \tag{17}$$

$$\Upsilon^{(i)}_{\mathbf{b},k+1} = \mathcal{X}^{(i)}_{\mathbf{b},k}. \tag{18}$$

- Given the set of rotations $\{\Upsilon^{(0)}_{\mathbf{R},k+1}, \ldots, \Upsilon^{(12)}_{\mathbf{R},k+1}\}$ in SO(3), evaluate the weighted mean rotation $\bar{\Upsilon}_{\mathbf{R},k+1} \in \mathrm{SO}(3)$ using Algorithm 1. Taking advantage the rapid convergence of Algorithm 1 [18], [30], set the number of iterations in line 2 of the algorithm to $n = 3$ or 4. The weights $w_m^{(i)} \in \mathbb{R}$ in line 3 satisfy $\sum_{i=0}^{12} w_m^{(i)} = 1$.
- The gyroscope bias estimate $\bar{\Upsilon}_{\mathbf{b},k+1} \in \mathbb{R}^3$ is given by the weighted mean of $\{\Upsilon^{(0)}_{\mathbf{b},k+1}, \ldots, \Upsilon^{(12)}_{\mathbf{b},k+1}\}$ in $\mathbb{R}^3$, i.e., $\bar{\Upsilon}_{\mathbf{b},k+1} = \sum_{i=0}^{12} w_m^{(i)} \Upsilon_{\mathbf{b}}$. $\hat{\mathbf{X}}_{k+1|k} := (\hat{\mathbf{R}}_{k+1|k}, \hat{\mathbf{b}}_{k+1|k})$ is then

$$(\hat{\mathbf{R}}_{k+1|k}, \hat{\mathbf{b}}_{k+1|k}) = (\bar{\Upsilon}_{\mathbf{R},k+1}, \bar{\Upsilon}_{\mathbf{b},k+1}). \tag{19}$$

Algorithm 1: Weighted Intrinsic Mean on SO(3)

Input: Set of rotations $\{\mathbf{Z}_0, \ldots, \mathbf{Z}_{12}\}$ in SO(3)

1 $\mathbf{T} \leftarrow \mathbf{Z}_0$

2 **for** $j \leftarrow 0$ ***to*** n **do**

3 $\quad \boldsymbol{\Lambda} \leftarrow \sum_{i=0}^{12} w_m^{(i)} \log(\mathbf{Z}_i \mathbf{T}^{-1})$

4 $\quad \mathbf{T} \leftarrow \exp(\boldsymbol{\Lambda})\, \mathbf{T}$

5 **return T**

- Define the vectors $[\mathbf{q}_i^{(a)}] := \log(\Upsilon^{(i)}_{\mathbf{R},k+1} \bar{\Upsilon}^{-1}_{\mathbf{R},k+1}) \in \mathrm{so}(3)$ and $\mathbf{q}_i^{(b)} := \Upsilon^{(i)}_{\mathbf{b},k+1} - \bar{\Upsilon}_{\mathbf{b},k+1}$. Concatenate the two vectors $\mathbf{q}_i^{(a)}, \mathbf{q}_i^{(b)}$ into a single vector $\mathbf{q}_i = (\mathbf{q}_i^{(a)}, \mathbf{q}_i^{(b)}) \in \mathbb{R}^6$. The predicted covariance is given by

$$\mathbf{P}_{k+1|k} = \sum_{i=0}^{12} w_c^{(i)} \mathbf{q}_i \mathbf{q}_i^T + \mathbf{N}_k, \tag{20}$$

where $w_c^{(i)} \in \mathbb{R}$ are the weights and $\mathbf{N}_k = \begin{bmatrix} (ch^2)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & d\mathbf{I} \end{bmatrix}$ is the process noise covariance.

- Let $\mathbf{u}_i \in \mathbb{R}^6$ denote the i^{th} column vector of the lower-triangular matrix $\mathbf{U} \in \mathbb{R}^{6\times 6}$ in the Cholesky decomposition $\mathbf{P}_{k+1|k} = \mathbf{U}\mathbf{U}^T$. The upper and lower halves of $\mathbf{u}_i$ are respectively denoted $\mathbf{u}_i^{(a)} \in \mathbb{R}^3$ and $\mathbf{u}_i^{(b)} \in \mathbb{R}^3$. Redraw the sigma points $\mathcal{X}^{(i)}_{k+1} := (\mathcal{X}^{(i)}_{\mathbf{R},k+1}, \mathcal{X}^{(i)}_{\mathbf{b},k+1})$, $(i = 0, \ldots, 12)$ from $\hat{\mathbf{X}}_{k+1|k}$ and $\mathbf{P}_{k+1|k}$ as follows:

$$\mathcal{X}^{(0)}_{k+1} = (\hat{\mathbf{R}}_{k+1|k}, \hat{\mathbf{b}}_{k+1|k})$$
$$\mathcal{X}^{(i)}_{k+1} = (\exp([\gamma \mathbf{u}_i^{(a)}])\hat{\mathbf{R}}_{k|k}, \hat{\mathbf{b}}_{k|k} + \gamma \mathbf{u}_i^{(b)}), i = 1, \ldots, 6$$
$$\mathcal{X}^{(i+6)}_{k+1} = (\exp([-\gamma \mathbf{u}_i^{(a)}])\hat{\mathbf{R}}_{k|k}, \hat{\mathbf{b}}_{k|k} - \gamma \mathbf{u}_i^{(b)}), i = 1, \ldots, 6.$$

3) Measurement Update:

- If the IMU moves with high acceleration or is subject to magnetic disturbances, the accelerometer and magnetometer measurements may be corrupted and not satisfy our earlier assumptions. Appendix C summarizes some existing methods for addressing these disturbances.
- Setting $\mathbf{w}_{k+1} = \mathbf{0}$ in Equation (16), define the set of measurement sigma points $\mathcal{S}_{\mathcal{Y}} = \{\mathcal{Y}^{(i)}_{k+1} \in \mathrm{SO}(3) \mid i = 0, \ldots, 12\}$ as follows:

$$\mathcal{Y}^{(i)}_{k+1} = \mathcal{X}^{(i)}_{\mathbf{R},k+1} \quad (i = 0, \ldots, 12). \tag{21}$$

- The mean $\hat{\mathbf{Y}}_{k+1}$ of $\{\mathcal{Y}^{(0)}_{k+1}, \ldots, \mathcal{Y}^{(12)}_{k+1}\}$ is given by

$$\hat{\mathbf{Y}}_{k+1} = \hat{\mathbf{R}}_{k+1|k}, \tag{22}$$

where $\hat{\mathbf{R}}_{k+1|k}$ is given by Equation (19). The covariance of $\{\mathcal{Y}^{(0)}_{k+1}, \ldots, \mathcal{Y}^{(12)}_{k+1}\}$ is determined as

$$\mathbf{P}_{\mathbf{yy}} = \sum_{i=0}^{12} w_c^{(i)} \mathbf{z}_i \mathbf{z}_i^T, \tag{23}$$

where $[\mathbf{z}_i] := \log(\mathcal{Y}^{(i)}_{k+1} \hat{\mathbf{Y}}^{-1}_{k+1}) \in \mathrm{so}(3)$. The innovation covariance [9] is given by

$$\mathbf{P}_{\mathbf{vv}} = \mathbf{P}_{\mathbf{yy}} + \mathbf{W}_{k+1}, \tag{24}$$

where $\mathbf{W}_{k+1}$ is the right-invariant covariance of the solution to Wahba's Problem. In the next section we derive a closed-form expression for $\mathbf{W}_{k+1}$ from Equation (32).

- Define $[\mathbf{p}_i^{(a)}] := \log(\mathcal{X}^{(i)}_{\mathbf{R},k+1} \hat{\mathbf{R}}^{-1}_{k+1|k}) \in \mathrm{so}(3)$ and $\mathbf{p}_i^{(b)} := \mathcal{X}^{(i)}_{\mathbf{b},k+1} - \hat{\mathbf{b}}_{k+1|k} \in \mathbb{R}^3$, and $\mathbf{p}_i = (\mathbf{p}_i^{(a)}, \mathbf{p}_i^{(b)}) \in \mathbb{R}^6$. The associated covariance $\mathbf{P}_{\mathbf{xy}}$ is then calculated as

$$\mathbf{P}_{\mathbf{xy}} = \sum_{i=0}^{12} w_c^{(i)} \mathbf{p}_i \mathbf{z}_i^T. \tag{25}$$

- The Kalman gain is computed as $\mathbf{K} = \mathbf{P}_{\mathbf{xy}} \mathbf{P}^{-1}_{\mathbf{vv}}$. Define the innovation vector $\boldsymbol{\delta} \in \mathbb{R}^3$ as

$$[\boldsymbol{\delta}] := \log(\mathbf{Y}_{k+1} \hat{\mathbf{Y}}^{-1}_{k+1}) \in \mathrm{so}(3), \tag{26}$$

where $\mathbf{Y}_{k+1}$ and $\hat{\mathbf{Y}}_{k+1}$ are respectively given by Equations (16) and (22). Define $\boldsymbol{\phi}^{(a)} \in \mathbb{R}^3$ and $\boldsymbol{\phi}^{(b)} \in \mathbb{R}^3$ to be the upper and lower halves of $\boldsymbol{\phi} := \mathbf{K}\boldsymbol{\delta} \in \mathbb{R}^6$. The state and covariance are now updated according to

$$\hat{\mathbf{X}}_{k+1|k+1} = (\exp([\boldsymbol{\phi}^{(a)}])\hat{\mathbf{R}}_{k+1|k}, \hat{\mathbf{b}}_{k+1|k} + \boldsymbol{\phi}^{(b)}), \tag{27}$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{M}(\boldsymbol{\phi}^{(a)})(\mathbf{P}_{k+1|k} - \mathbf{K}\mathbf{P}_{\mathbf{yy}}\mathbf{K}^T)\mathbf{M}(\boldsymbol{\phi}^{(a)})^T, \tag{28}$$

where $\mathbf{M}(\boldsymbol{\phi}^{(a)}) \in \mathbb{R}^{6\times 6}$ is given by

$$\mathbf{M}(\boldsymbol{\phi}^{(a)}) = \begin{bmatrix} \mathbf{J}_l(\boldsymbol{\phi}^{(a)}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{29}$$

The justification for $\mathbf{M}(\boldsymbol{\phi}^{(a)})$ in Equation (28) is given in Appendix B.

IV. Measurement Noise Covariance

This section presents an algorithm for obtaining, from a set of noisy unit vector measurements of the gravity and magnetic field vectors, a full-rank measurement noise covariance matrix.

A. Covariances of the Solution to Wahba's Problem

In [27], Shuster provides the following first-order approximation to the left-invariant covariance of $\mathbf{R}$ in the solutio to Wahba's Problem (4):

$$(\sum_{i=1}^{2} \frac{1}{\sigma_i^2}(\mathbf{I} - \bar{\mathbf{A}}\mathbf{r}_i\mathbf{r}_i^T\bar{\mathbf{A}}^T))^{-1}, \tag{30}$$

where $\bar{\mathbf{A}} \in \mathrm{SO}(3)$ denotes the true value of $\mathbf{R}^T$, which is usually unknown. $\bar{\mathbf{A}}$ can be approximated by

$$\bar{\mathbf{A}} \approx \underset{\mathbf{A}\in\mathrm{SO}(3)}{\arg\min} \sum_{i=1}^{2} \frac{1}{\sigma_i^2}\|\mathbf{v}_i - \mathbf{A}\mathbf{r}_i\|^2. \tag{31}$$

In [27] it is asserted, without rigorous proof, that the left-invariant covariance of $\mathbf{R}$ is given by the inverse of the Fisher information matrix. Appendix E provides a more detailed and rigorous proof via the Cramer-Rao lower bound (CRLB).

Similarly, from Equation (30) the right-invariant covariance of $\mathbf{R}$ can be obtained as

$$(\sum_{i=1}^{2} \frac{1}{\sigma_i^2}(\mathbf{I} - \mathbf{r}_i\mathbf{r}_i^T))^{-1} \tag{32}$$

The above follows from a straightforward calculation combining Equations (3) and (30).

Note that the left-invariant covariance of $\mathbf{R}$ in Equation (30) is equivalent to the covariance of the solution to Wahba's Problem represented with respect to the IMU body frame. In contrast, the right-invariant covariance of $\mathbf{R}$ in Equation (32) is the covariance of the solution to Wahba's Problem represented with respect to the fixed ground frame. If values for $\sigma_i^2, \mathbf{r}_i$ are given, the right-invariant covariance of $\mathbf{R}$ in Equation (32) can be determined to be a constant matrix, independent of $\bar{\mathbf{A}}$. However, the left-invariant covariance of $\mathbf{R}$ in Equation (30) requires $\bar{\mathbf{A}}$, σ_i^2 and $\mathbf{r}_i$.

When the IMU is moving, $\bar{\mathbf{A}}$ is also changing, and the left-invariant covariance of $\mathbf{R}$ needs to be updated at every time step. The left-invariant covariance can be evaluated as the inverse of a matrix that varies with $\bar{\mathbf{A}}$, while the right-invariant covariance remains invariant. When the IMU motion involves both translation and rotation, measurements of the two direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are subject to greater errors, leading to less accurate estimates of $\bar{\mathbf{A}}$. For the reasons outlined above, our measurement noise covariance formula of Equation (32) is preferable to Shuster's formula (30) in the geometric UKF algorithm.

B. Determination of Parameters in the Covariance of $\mathbf{R}$

In this section we present an offline algorithm for determining the parameters in Equation (32), i.e., σ_i^2 and $\mathbf{r}_i$, $i = 1, 2$, from accelerometer and magnetometer measurements.

1) Constant Vectors $(\mathbf{r}_1, \mathbf{r}_2)$*:* Assign each axis of the inertial reference frame $\{\mathcal{I}\}$ as follows: The direction opposite to gravity is set to be the y-axis of $\{\mathcal{I}\}$, while the x-axis of $\{\mathcal{I}\}$ is orthogonal to both gravity and the magnetic field. With these assignments, $\mathbf{r}_1 = (0, 1, 0)^T$ and

$$\mathbf{r}_2 = (0, \cos(\phi), \sin(\phi))^T, \tag{33}$$

where ϕ is unknown and to be determined.

We assume that the IMU is stationary, and multiple measurement pairs are collected. Then $\hat{\mathbf{v}}_i := E(\mathbf{v}_i)$, $i = 1, 2$, can be calculated from Proposition 1 in Appendix D. Since $\mathbf{r}_1^T\mathbf{r}_2 \approx \hat{\mathbf{v}}_1^T\hat{\mathbf{v}}_2$, ϕ can be approximated as

$$\phi \approx \cos^{-1}(\hat{\mathbf{v}}_1^T\hat{\mathbf{v}}_2). \tag{34}$$

2) Variances (σ_1^2, σ_2^2)*:* Let the unit vector $\check{\mathbf{v}}_i$ denote the true value of the measured unit vector $\mathbf{v}_i$, $i = 1, 2$. The covariance of $\mathbf{v}_i$ is given by [31]

$$\mathbf{M}_t = \sigma_i^2(\mathbf{I} - \check{\mathbf{v}}_i\check{\mathbf{v}}_i^T). \tag{35}$$

Let the SVD of $\mathbf{M}_t$ be $\mathbf{M}_t = \mathbf{U}_t\mathbf{\Sigma}_t\mathbf{V}_t^T$, where in principle $\mathbf{\Sigma}_t = \mathrm{diag}(\sigma_i^2, \sigma_i^2, 0)$ and $\check{\mathbf{v}}_i$ is the corresponding direction for the singular value 0. Since in practice ground truth values of $\check{\mathbf{v}}_i$ are unavailable, an alternative method of determining σ_i^2 is needed. Assuming that the IMU is stationary and N measurements are available, the covariance of $\mathbf{v}_i$ can be estimated by

$$\mathbf{M}_a = \frac{1}{N}\sum_{j=1}^{N}(\mathbf{v}_i^{(j)} - \hat{\mathbf{v}}_i)(\mathbf{v}_i^{(j)} - \hat{\mathbf{v}}_i)^T, \tag{36}$$

where $\mathbf{v}_i^{(j)}$ denotes the j^{th} measurement vector obtained from the i^{th} sensor (sensor 1 is the accelerometer, while sensor 2 is the magnetometer). Let the SVD of $\mathbf{M}_a$ be $\mathbf{M}_a = \mathbf{U}_a\mathbf{\Sigma}_a\mathbf{V}_a^T$, where $\mathbf{\Sigma}_a = \mathrm{diag}(s_1, s_2, s_3)$ and $s_1 \geq s_2 \geq s_3$, $s_3 \approx 0$. $\mathbf{\Sigma}_a$ will typically be close to its theoretical value $\mathbf{\Sigma}_t$, in which case we can set

$$\sigma_i^2 = \frac{tr(\mathbf{M}_a)}{2}. \tag{37}$$

V. Experimental Results

In this section we compare the performance of our geometric UKF algorithm ("UKF on SO(3)") against other state-of-the-art methods ("UKF on Quaternion" [19], "EKF on Quaternion" [20], and the passive nonlinear complementary filter "NCF on SO(3)" [6]). Using both synthetic and real data in our experiments, both the convergence rate and accuracy of the attitude and gyroscope bias estimates are compared.

Ground-truth values of the attitude and gyroscope bias at time step k are denoted $\check{\mathbf{R}}_k$ and $\check{\mathbf{b}}_k$, respectively. In both simulations and real experiments, the filter update time step is set to $h_0 = 1/60$ seconds. Define

$$s_k := (180°/\pi)\|\log \check{\mathbf{R}}_k^{-1}\hat{\mathbf{R}}_{k|k}\| \tag{38}$$

$$d_k := \|\hat{\mathbf{b}}_{k|k} - \check{\mathbf{b}}\|, \tag{39}$$

where s_k and d_k represent the estimation errors of the attitude and gyroscope bias at time step k, respectively.

The weighting factors $w_m^{(i)}$ and $w_c^{(i)}$ in Section III-B are set to

$$w_m^{(0)} = \frac{\lambda}{\lambda + N_x}, \quad w_c^{(0)} = \frac{\lambda}{\lambda + N_x} + (1 - \alpha^2 + \beta) \tag{40}$$

$$w_m^{(i)} = w_c^{(i)} = \frac{1}{2(\lambda + N_x)}, \quad (i = 1, \dots, 2N_x). \tag{41}$$

α in Equation (40) is set to 0.9, and β is set to two for a Gaussian prior [10].

A. Synthetic Data

In our numerical simulation experiments, the vectors in Equation (4) are set to $\mathbf{r}_1 = (0, 1, 0)^T$ and $\mathbf{r}_2 = (0, \cos(\phi_s), \sin(\phi_s))^T$, where $\phi_s = 2.4$ radians. The ground truth value $\breve{\mathbf{R}}_1 \in \mathrm{SO(3)}$ is set randomly to be the initial attitude.

For realistic simulation, we first collect a set of real angular rate measurements $\breve{\boldsymbol{\omega}}_k$ from an actual gyroscope (L3G4200D) at the sampling rate $1/h_0 = 60$ Hz. From $\breve{\mathbf{R}}_1$, true attitude matrices can be iteratively generated by

$$\breve{\mathbf{R}}_{k+1} = \breve{\mathbf{R}}_k \exp([\breve{\boldsymbol{\omega}}_k] h_0).$$

The ground-truth value of the initial gyroscope bias is set to be $\breve{\mathbf{b}}_0 = (-0.06, 0.3, 0.3)^T$ radian/seconds. We then generate a set of synthetic data as follows:

$$\boldsymbol{\omega}_k^m = \breve{\boldsymbol{\omega}}_k + \breve{\mathbf{b}}_k + \boldsymbol{\eta}_{\omega,k} \tag{42}$$

$$\breve{\mathbf{b}}_k = \breve{\mathbf{b}}_{k-1} + \boldsymbol{\eta}_{\mathbf{b},k-1} \tag{43}$$

$$\mathbf{v}_{1,k} = (\breve{\mathbf{R}}_k^T \mathbf{r}_1 + \boldsymbol{\eta}_{\mathbf{v}_{1,k}}) / \|\breve{\mathbf{R}}_k^T \mathbf{r}_1 + \boldsymbol{\eta}_{\mathbf{v}_{1,k}}\| \tag{44}$$

$$\mathbf{v}_{2,k} = (\breve{\mathbf{R}}_k^T \mathbf{r}_2 + \boldsymbol{\eta}_{\mathbf{v}_{2,k}}) / \|\breve{\mathbf{R}}_k^T \mathbf{r}_2 + \boldsymbol{\eta}_{\mathbf{v}_{2,k}}\|, \tag{45}$$

where the Gaussian noise vectors have the following distributions: $\boldsymbol{\eta}_{\omega,k} \sim \mathcal{N}(\mathbf{0}, \sigma_0^2 \mathbf{I}), \boldsymbol{\eta}_{\mathbf{b},k} \sim \mathcal{N}(\mathbf{0}, \sigma_1^2 \mathbf{I}), \boldsymbol{\eta}_{\mathbf{v}_{1,k}} \sim \mathcal{N}(\mathbf{0}, \sigma_2^2 \mathbf{I})$, and $\boldsymbol{\eta}_{\mathbf{v}_{2,k}} \sim \mathcal{N}(\mathbf{0}, \sigma_3^2 \mathbf{I})$, $k = 1, \dots, N$. Here $\sigma_0 = (1.1 \times 10^{-3}/h_0)$ radian/seconds, $\sigma_1 = (1.0 \times 10^{-5})$ radian/seconds, $\sigma_2 = 1.00 \times 10^{-2}$, and $\sigma_3 = 1.58 \times 10^{-2}$.

To simulate the large initial estimation errors of gyroscope bias and attitude, we set $\hat{\mathbf{b}}_{1|1} = \mathbf{0}$ and $\hat{\mathbf{R}}_{1|1} = \breve{\mathbf{R}}_1 \exp([\mathbf{a}_1])$, where $\mathbf{a}_1 = (3.13/\sqrt{3})(1, 1, 1)^T$. The noise covariances $\mathbf{N}_k$ in Equation (20) and $\mathbf{W}_{k+1}$ in (24) of the proposed attitude estimator ("UKF on SO(3)") are set as follows: $\mathbf{N}_k = \begin{bmatrix} (\sigma_0 h_0)^2 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma_1^2 \mathbf{I} \end{bmatrix}$ and $\mathbf{W}_{k+1} = (\frac{1}{\sigma_2^2}(\mathbf{I} - \mathbf{r}_1 \mathbf{r}_1^T) + \frac{1}{\sigma_3^2}(\mathbf{I} - \mathbf{r}_2 \mathbf{r}_2^T))^{-1}$.

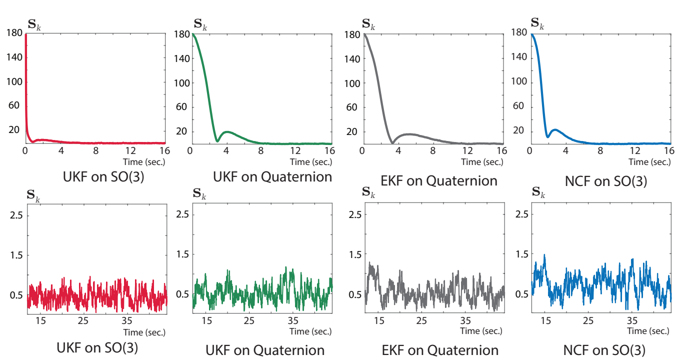

Fig. 1. Simulation experiments: Attitude estimation errors (in degrees) over the time intervals $t \in [0, 16]$ seconds (top) and $t \in [12, 44]$ seconds (bottom).

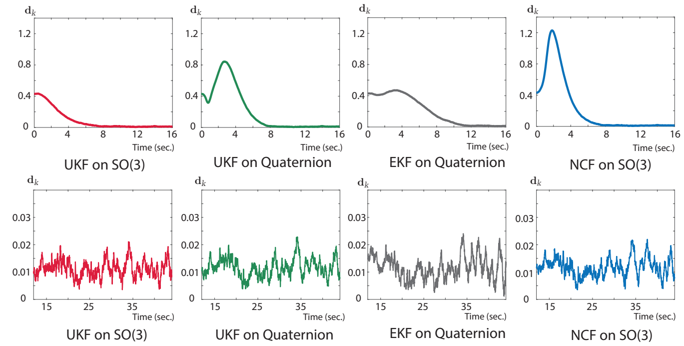

Fig. 2. Simulation experiments: Gyroscope bias estimate errors (in radian/seconds) over the time intervals $t \in [0, 16]$ seconds (top) and $t \in [12, 44]$ seconds (bottom).

From the simulation results shown in Figures 1 and 2, it can be seen that the proposed algorithm ("UKF on SO(3)") converges most rapidly over the time interval $t \in [0, 14]$ seconds. To more reliably assess the accuracy of each estimator, we generate 500 sets of synthetic data using Equations (42)-(45). Figure 3 shows the histograms of estimation errors of the attitudes and the slowly time-varying gyroscope biases. Tables I and II summarize the experimental results corresponding to Figures 3(a)-(b). From Figure 3(b) and Table II, it can be seen that the gyroscope bias estimates show similar performance for all estimators. In terms of attitude estimates, "UKF on SO(3)" is the most accurate among the estimators (see Figure 3(a) and Table I).

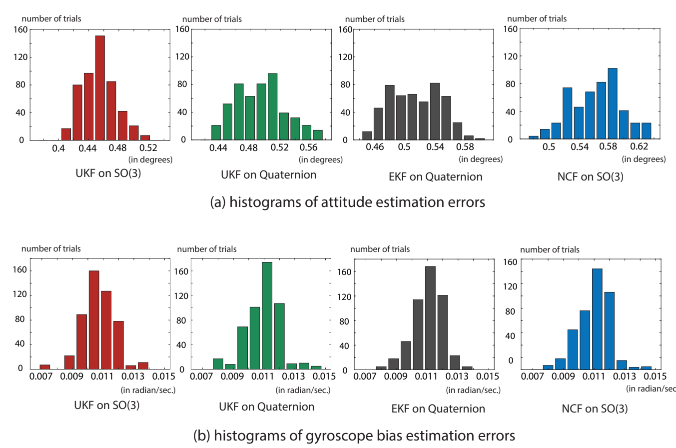

Fig. 3. Simulation experiments: Histograms of estimation errors over the time interval $t \in [12, 44]$ seconds (averaged over 500 trials).

TABLE I
AVERAGE AND STANDARD DEVIATION OF ATTITUDE ESTIMATION ERRORS (IN DEGREES) OVER THE TIME INTERVAL $t \in [12, 44]$ SECONDS (AVERAGED OVER 500 TRIALS).

	UKF on SO(3)	UKF on Quaternion	EKF on Quaternion	NCF on SO(3)
Average	**0.45**	0.49	0.51	0.57
Standard deviation	**0.02**	0.03	0.03	0.03

B. Real Experiments

The IMU for real experiments consists of an L3G4200D gyroscope, LIS3LV02DQ accelerometer, HMC5883L magne-

TABLE II
AVERAGE AND STANDARD DEVIATION OF GYROSCOPE BIAS ESTIMATION ERRORS (IN RADIAN/SECONDS) DURING TIME INTERVAL $t \in [12, 44]$ SECONDS OVER 500 TRIALS.

	UKF on SO(3)	UKF on Quaternion	EKF on Quaternion	NCF on SO(3)
Average	**0.011**	0.011	0.011	0.011
Standard deviation	**0.001**	0.001	0.001	0.001

tometer, and Cortex-M3$^{\text{TM}}$ microcontroller. In real experiments, ground-truth values of the slowly time-varying gyroscope bias are unknown. We therefore assume that the gyroscope bias is initially unknown, but near-constant over short time durations. If the IMU is stationary, then the gyroscope bias, denoted $\breve{\mathbf{b}}$, can be temporarily captured by averaging a set of gyroscope data over a certain time interval [32].

Keeping the IMU stationary, the variance σ_i^2 of the unit vector $\mathbf{v}_{i,k}$, $i = 1, 2$, can be calculated from Equation (37); in our experiments we obtain the values $\sigma_1^2 = 8.95 \times 10^{-5}$ and $\sigma_2^2 = 1.911 \times 10^{-4}$. Denoting by ϕ_r the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, i.e., $\phi_r = \cos^{-1}(\mathbf{r}_1^T \mathbf{r}_2)$, we obtain $\phi_r = 2.486$ radians using Proposition 1 of Appendix D and Equation (34). The noise covariances $\mathbf{N}_k$ in Equation (20) and $\mathbf{W}_{k+1}$ in (24) of the proposed attitude estimator ("UKF on SO(3)") are set as follows: $\mathbf{N}_k = \begin{bmatrix} (2.0\times10^{-9})\mathbf{I} & \mathbf{0} \\ \mathbf{0} & (3.0\times10^{-11})\mathbf{I} \end{bmatrix}$ and $\mathbf{W}_{k+1} = (\sum_{i=1}^{2} \frac{1}{\sigma_i^2}(\mathbf{I} - \mathbf{r}_i \mathbf{r}_i^T))^{-1}$.

To obtain the ground-truth value of the attitude $\breve{\mathbf{R}}_k$ at time step k, we use the optical motion capture system OptiTrack$^{\text{TM}}$ consisting of multiple networked infrared cameras. The IMU and four reflective markers are first rigidly attached to a plastic plate. A set of real data $\{(\boldsymbol{\omega}_k^m, \mathbf{v}_{1,k}, \mathbf{v}_{2,k}) \mid k = 1, \ldots, N_r\}$ obtained from the moving IMU, and the ground-truth attitude $\breve{\mathbf{R}}_k$ obtained from the OptiTrack$^{\text{TM}}$ infrared camera system, are synchronously saved into files at a sampling rate $1/h_0 = 60$ Hz. Here the number of measurements N_r is set to 3000. For fair comparison among filters, we perform experiments with real data under the condition of negligible disturbances.

To evaluate the convergence rate and accuracy of each filter when the initial estimation errors of the gyroscope bias and attitude are large, we set the initial estimates as follows: $\hat{\mathbf{b}}_{1|1} = \breve{\mathbf{b}} + (1/h_0)(-0.001, 0.005, 0.005)^T = \breve{\mathbf{b}} + (-0.06, 0.3, 0.3)^T$ (radian/seconds) and $\hat{\mathbf{R}}_{1|1} \leftarrow \breve{\mathbf{R}}_1 \exp([\mathbf{a}_1])$, where $\mathbf{a}_1 = (3.13/\sqrt{3})(1, 1, 1)^T$. Recall that $\breve{\mathbf{b}}$ can be obtained under the stationary IMU assumption.

Like our earlier simulation results, Figures 4 and 5 show that the proposed method ("UKF on SO(3)") converges the most rapidly, whereas other methods show slow convergence rates and relatively large overshoots. To further experimentally verify these results, we collect nine additional sets of real data. As shown in Table III, "UKF on SO(3)" demonstrates superior performance compared to existing methods in terms of the accuracy of attitude estimates.

We also measure, at every time step, the computation times for each filter—all implemented in C++ and executed on a desktop computer with Intel$^{\text{TM}}$ i5-4670 (3.4GHz) CPU. The computation times for each estimator are averaged over N_r steps. From Table IV it can be seen that "NCF on SO(3)" is the fastest among the estimators. Computation times for "UKF on SO(3)" are similar to those for "Quaternion UKF".

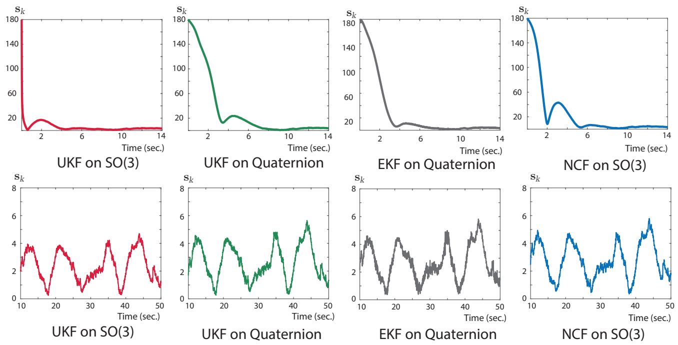

Fig. 4. Real experiments: Attitude estimate errors (in degrees) over the time interval $t \in [0, 14]$ seconds (top) and $t \in [10, 50]$ seconds. (bottom).

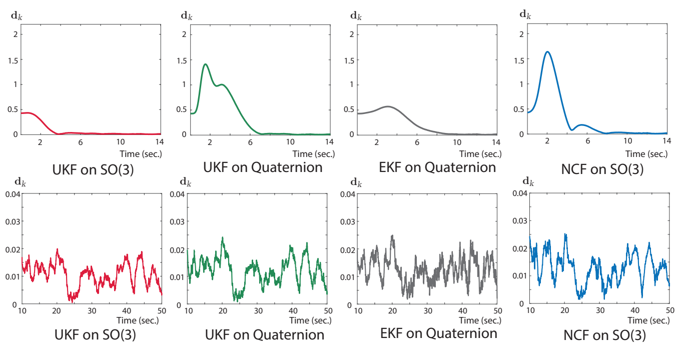

Fig. 5. Real experiments: Gyroscope bias estimate errors (in radian/seconds) over the time intervals $t \in [0, 14]$ seconds (top) and $t \in [10, 50]$ seconds (bottom).

TABLE III
RESULTS OF REAL EXPERIMENTS: AVERAGE ERRORS OVER THE TIME INTERVAL $t \in [10, 50]$ SECONDS (AVERAGED OVER TEN EXPERIMENTS).

Average of attitude errors (in degrees)				Average of gyroscope bias errors (in radian/seconds)			
UKF on SO(3)	UKF on Quaternion	EKF on Quaternion	NCF on SO(3)	**UKF on SO(3)**	UKF on Quaternion	EKF on Quaternion	NCF on SO(3)
2.60	2.69	2.71	2.76	**0.012**	0.012	0.012	0.012

TABLE IV
AVERAGE COMPUTATION TIMES FOR EACH FILTER (IN MICRO-SECONDS)

	UKF on SO(3)	UKF on Quaternion	EKF on Quaternion	NCF on SO(3)
Average time	**8.1**	7.9	6.8	0.2

VI. CONCLUSION

This paper has presented a geometric unscented Kalman filtering algorithm for simultaneously estimating attitude and gyroscope bias from an inertial measurement unit. Drawing upon the Lie group properties of the set of rotation matrices SO(3), we derive a dsicrete-time stochastic nonlinear filtering algorithm evolving on SO(3) $\times\mathbb{R}^3$. One of the key features of our algorithm is to express observations as elements of SO(3), by determining the rotation corresponding to the IMU's gravitational acceleration and magnetic field vector measurements as a solution to Wahba's Problem. By doing so, first-order linear approximations of the state dynamics and measurement equations lead to closed-form equations for covariance propagation and update. These in turn lead to computationally efficient implementations of our filter, with the resulting attitude estimates invariant with respect to the choice of fixed and moving reference frames. Extensive numerical simulation and hardware experiments have demonstrated the superior convergence behavior and estimation accuracy of our proposed algorithm compared to existing state-of-the-art IMU estimators for attitude and gyroscope bias.

Appendix A
First-Order Approximation of Exponential Map

Given $[\mathbf{x}], [\mathbf{y}] \in \mathrm{so}(3)$, let $[\mathbf{z}] \in \mathrm{so}(3)$ satisfy

$$\exp([\mathbf{z}]) = \exp([\mathbf{x}])\exp([\mathbf{y}]). \tag{46}$$

From the Baker-Campbell-Hausdorff formula [29], we have

$$\begin{aligned}[\mathbf{z}] &= \log(\exp([\mathbf{x}])\exp([\mathbf{y}])) \\ &= [\mathbf{x}] + [\mathbf{y}] + \frac{1}{2}[[\mathbf{x}],[\mathbf{y}]] + \frac{1}{12}[[\mathbf{x}],[[\mathbf{x}],[\mathbf{y}]]] \\ &\quad + \frac{1}{12}[[\mathbf{y}],[[\mathbf{y}],[\mathbf{x}]]] + \cdots .\end{aligned}$$

The Lie bracket operator $[\cdot,\cdot] : \mathrm{so}(3) \times \mathrm{so}(3) \to \mathrm{so}(3)$ is defined as $[[\mathbf{a}],[\mathbf{b}]] = [\mathbf{a}][\mathbf{b}] - [\mathbf{b}][\mathbf{a}]$ for $[\mathbf{a}],[\mathbf{b}] \in \mathrm{so}(3)$. $[\mathbf{c}] = [[\mathbf{a}],[\mathbf{b}]] \in \mathrm{so}(3)$ also admits the vector representation $\mathbf{c} = [\mathbf{a}]\mathbf{b} \in \mathbb{R}^3$.

If we assume that $\|\mathbf{x}\|$ is small, then by gathering only terms linear in $\mathbf{x}$, the following approximation holds [23]:

$$\mathbf{z} \approx \mathbf{y} + \sum_{n=0}^{\infty} \frac{B_n}{n!}[\mathbf{y}]^n \mathbf{x}, \tag{47}$$

where B_n are the Bernoulli numbers ($B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, ...). The Bernoulli numbers satisfy the following series expression: $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$ for any scalar $x \neq 0$.

Letting $[\mathbf{x}'] = [\mathbf{z}] - [\mathbf{y}] \in \mathrm{so}(3)$, we have

$$\exp([\mathbf{x}'] + [\mathbf{y}]) = \exp([\mathbf{x}])\exp([\mathbf{y}]), \tag{48}$$

with

$$\mathbf{x} \approx \mathbf{J}_l(\mathbf{y})\mathbf{x}', \tag{49}$$

where

$$\begin{aligned}\mathbf{J}_l(\mathbf{y}) &= (\sum_{n=0}^{\infty} \frac{B_n}{n!}[\mathbf{y}]^n)^{-1} &(50)\\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}[\mathbf{y}]^n &(51)\\ &= \int_0^1 \exp([\mathbf{y}]s)\ ds &(52)\end{aligned}$$

denotes the left Jacobian of SO(3) on $\mathbf{y}$ [23]. The closed-form formula of $\mathbf{J}_l(\mathbf{y})$ is given by

$$\mathbf{J}_l(\mathbf{y}) = \mathbf{I} + \left(\frac{1 - \cos\|\mathbf{y}\|}{\|\mathbf{y}\|^2}\right)[\mathbf{y}] + \left(\frac{\|\mathbf{y}\| - \sin\|\mathbf{y}\|}{\|\mathbf{y}\|^3}\right)[\mathbf{y}]^2. \tag{53}$$

Appendix B
UKF Covariance Update on $\mathrm{SO}(3) \times \mathbb{R}^3$

From Equation (1), a random variable $\mathbf{R} \in \mathrm{SO}(3)$ can be defined as

$$\mathbf{R} := \exp([\boldsymbol{\varphi}])\,\hat{\mathbf{R}}, \tag{54}$$

where $\boldsymbol{\varphi} \sim \mathcal{N}(\mathbf{0}, \mathbf{P}_{\boldsymbol{\varphi}})$ is the right-translated exponential noise and $\hat{\mathbf{R}} \in \mathrm{SO}(3)$ is the state estimate. We refer to $\mathbf{P}_{\boldsymbol{\varphi}}$ as the right-invariant covariance of $\mathbf{R}$.

The right-translated exponential noise after the time update as described in Section III-B2 is assumed to be zero-mean Gaussian, with covariance $\mathbf{P}_{k+1|k}$ calculated by Equation (20). Special caution is required when computing $\mathbf{P}_{k+1|k+1}$, which is the *a posteriori* right-invariant covariance of $(\mathbf{R}_{k+1}, \mathbf{b}_{k+1})$ after the measurement update. If one implements the measurement update as in standard vector space UKF, the state $(\mathbf{R}_{k+1}, \mathbf{b}_{k+1})$ is given by

$$\begin{aligned}\mathbf{R}_{k+1} &= \exp([\boldsymbol{\xi}^{(a)}])\,\hat{\mathbf{R}}_{k+1|k}, &(55)\\ \mathbf{b}_{k+1} &= \hat{\mathbf{b}}_{k+1|k} + \boldsymbol{\xi}^{(b)} &(56)\end{aligned}$$

where $\boldsymbol{\xi}^{(a)}, \boldsymbol{\xi}^{(b)} \in \mathbb{R}^3$ refer to the upper and lower halves of $\boldsymbol{\xi} \sim \mathcal{N}(\boldsymbol{\phi}, \mathbf{P}_{k+1|k} - \mathbf{K}\mathbf{P}_{\mathbf{yy}}\mathbf{K}^T)$. However, since $\boldsymbol{\phi} \neq \mathbf{0}$ in general, there exists a discrepancy between the random variable models (54) and (55). Equation (55) is there reformulated to conform to (54) (i.e., to satisfy the property of "zero-mean" right-translated exponential noise). Assume that $(\mathbf{R}_{k+1}, \mathbf{b}_{k+1})$ can be represented as

$$\begin{aligned}\mathbf{R}_{k+1} &= \exp([\boldsymbol{\epsilon}'^{(a)}])\,\hat{\mathbf{R}}_{k+1|k+1}, &(57)\\ \mathbf{b}_{k+1} &= \hat{\mathbf{b}}_{k+1|k+1} + \boldsymbol{\epsilon}'^{(b)} &(58)\end{aligned}$$

where $\boldsymbol{\epsilon}' \sim \mathcal{N}(\mathbf{0}, \mathbf{P}_{\boldsymbol{\epsilon}'})$ and $\mathbf{P}_{k+1|k+1} = \mathbf{P}_{\boldsymbol{\epsilon}'}$. We now find $\mathbf{P}_{\boldsymbol{\epsilon}'}$.

Define the vector $\boldsymbol{\epsilon} \in \mathbb{R}^6$ by $\boldsymbol{\epsilon} := \boldsymbol{\xi} - \boldsymbol{\phi}$. $\boldsymbol{\epsilon}$ has the following distribution: $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{P}_{\boldsymbol{\epsilon}})$, where

$$\mathbf{P}_{\boldsymbol{\epsilon}} = \mathbf{P}_{k+1|k} - \mathbf{K}\mathbf{P}_{\mathbf{yy}}\mathbf{K}^T. \tag{59}$$

Since $\boldsymbol{\xi} = \boldsymbol{\epsilon} + \boldsymbol{\phi}$, Equation (55) can be rewritten as

$$\mathbf{R}_{k+1} = \exp([\boldsymbol{\epsilon}^{(a)} + \boldsymbol{\phi}^{(a)}])\,\hat{\mathbf{R}}_{k+1|k}. \tag{60}$$

Substituting Equation (27) into (57), we have

$$\mathbf{R}_{k+1} = \exp([\boldsymbol{\epsilon}'^{(a)}])\,\exp([\boldsymbol{\phi}^{(a)}])\,\hat{\mathbf{R}}_{k+1|k}. \tag{61}$$

Combining Equations (60) and (61) leads to

$$\exp([\boldsymbol{\epsilon}^{(a)} + \boldsymbol{\phi}^{(a)}]) = \exp([\boldsymbol{\epsilon}'^{(a)}])\,\exp([\boldsymbol{\phi}^{(a)}]), \tag{62}$$

and $\boldsymbol{\epsilon}^{(b)} = \boldsymbol{\xi}^{(b)} - \boldsymbol{\phi}^{(b)} = \boldsymbol{\epsilon}'^{(b)}$ holds by equating (56) and (58) using Equation (27). If $\|\boldsymbol{\epsilon}\| \ll 1$, from the first-order approximation derived from the Baker-Campbell-Hausdorff formula in Appendix A, it follows that

$$\boldsymbol{\epsilon}' \approx \mathbf{M}(\boldsymbol{\phi})\boldsymbol{\epsilon},$$

where

$$\mathbf{M}(\boldsymbol{\phi}) = \begin{bmatrix} \mathbf{J}_l(\boldsymbol{\phi}^{(a)}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{63}$$

and $\mathbf{J}_l(\boldsymbol{\phi}^{(a)})$ denotes the left Jacobian of SO(3) at $\boldsymbol{\phi}^{(a)}$, with corresponding closed-form equation given by (13). Finally we have

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{\boldsymbol{\epsilon}'} \approx \mathbf{M}(\boldsymbol{\phi})\mathbf{P}_{\boldsymbol{\epsilon}}\mathbf{M}(\boldsymbol{\phi})^T, \tag{64}$$

where $\mathbf{P}_{\boldsymbol{\epsilon}}$ is given by Equation (59). This justifies Equation (28) in Section III-B3. ([33] and [34] propose slightly different algorithms from (64): the former proposes a method for covariance correction of the quaternion state, while the latter takes a first-order approximation of both $\boldsymbol{\phi}^{(a)}$ and the noise vector $\boldsymbol{\epsilon}^{(a)}$ in the derivation. In contrast, Equation (64) is derived solely from the first-order approximation of $\boldsymbol{\epsilon}$.)

Remark 1. *If the left-invariant noise is adopted [12], the right Jacobian should be used in the covariance update equation.*

Appendix C
Motion and Magnetic Disturbances

If a triaxial accelerometer is subject to large accelerations, it outputs the vector sum of the negative gravitational acceleration vector and other accelerations due to external forces; the resulting acceleration vector measurement is expressed in the moving frame $\{\mathcal{B}\}$ attached to the IMU. In [35] these additional acceleration terms are referred to as motion disturbances. In magnetically disturbed environments, the measurement of a triaxial magnetometer deviates from the local magnetic field expressed in frame $\{\mathcal{B}\}$ coordinates.

To detect these disturbances, a number of reliability functions have been proposed [8], [35]. In [36] it is claimed that checking only the norms of the calibrated outputs of the accelerometers and magnetometers is in many cases sufficient for practical purposes. Let $\tilde{\mathbf{v}}_i \in \mathbb{R}^3$, $i = 1, 2$ be the unnormalized calibrated output vector of the three-axis accelerometer or magnetometer at a particular instant. If $|\|\tilde{\mathbf{v}}_i\| - 1| > \gamma_i$ for some positive threshold value γ_i, the disturbance is regarded as detected; otherwise no disturbance is presumed to exist.

When dealing with motion or magnetic disturbances in stochastic attitude filtering, two methods are commonly used:

- Adaptation of noise covariances [37]: If a disturbance is detected, then the noise covariance of the Kalman filter is adjusted.
- Measurement reconstruction with a vector selector [38]: If a disturbance is detected, then $\tilde{\mathbf{v}}_i$ is replaced by $\hat{\mathbf{R}}_{k+1|k}^T \mathbf{r}_i$. Here, $\hat{\mathbf{R}}_{k+1|k}$ is given by Equation (19).

In our estimator, the measurement reconstruction method with a vector selector is used.

Appendix D
Extrinsic Mean of Unit Vectors

Proposition 1. *Given a set of N unit vectors in $\mathbb{R}^d$, denoted $\mathcal{S}_v = \{\mathbf{v}_i \in \mathbb{R}^d \mid \|\mathbf{v}_i\| = 1, i = 1, \ldots, N\}$, the extrinsic mean of $\mathcal{S}_v$ is defined as $\mathbf{v}^* := \arg\min_{\mathbf{v}} \sum_{i=1}^N \|\mathbf{v}_i - \mathbf{v}\|^2$ subject to $\|\mathbf{v}\| = 1$. If $\mathbf{m} := \sum_{i=1}^N \mathbf{v}_i \neq \mathbf{0}$, then $\mathbf{v}^* = \mathbf{m}/\|\mathbf{m}\|$.*

Proof. Defining $L(\mathbf{v}, \lambda) = \sum_{i=1}^N \|\mathbf{v}_i - \mathbf{v}\|^2 + \lambda(\mathbf{v}^T\mathbf{v} - 1)$ where $\lambda > 0$, the first-order necessary conditions for optimality ($\frac{\partial L(\mathbf{v}^*,\lambda)}{\partial \mathbf{v}^*} = 0$ and $\frac{\partial L(\mathbf{v}^*,\lambda)}{\partial \lambda} = 0$) yield the result. □

Appendix E
Proof of Equation (30)

Given the inverse $\bar{\mathbf{A}} \in$ SO(3) of the true attitude, consider the following slightly modified version of the optimization problem of Equation (4):

$$\boldsymbol{\theta}^* = \underset{\boldsymbol{\theta} \in \mathbb{R}^3}{\arg\min} \sum_{i=1}^2 \frac{1}{\sigma_i^2} \|\mathbf{v}_i - \exp([\boldsymbol{\theta}])\, \bar{\mathbf{A}}\mathbf{r}_i\|^2, \tag{65}$$

where $\mathbf{v}_i = \bar{\mathbf{A}}\mathbf{r}_i + \Delta\mathbf{v}_i$, and $\Delta\mathbf{v}_i$ denotes the zero-mean measurement noise. The covariance of the random variable $\Delta\mathbf{v}_i$ is given by Equation (35), and $\exp([\boldsymbol{\theta}])\bar{\mathbf{A}}$ corresponds to the inverse of the optimization variable $\mathbf{R}$ in (4). Assuming that $\Delta\mathbf{v}_i$ is small, the solution $\boldsymbol{\theta}^*$ will be located near the origin. Under the first-order approximation $\exp([\boldsymbol{\theta}]) \approx \mathbf{I} + [\boldsymbol{\theta}]$, the objective function can be approximated as

$$\boldsymbol{\theta}^* = \underset{\boldsymbol{\theta} \in \mathbb{R}^3}{\arg\min} \sum_{i=1}^2 \frac{1}{\sigma_i^2} \|\Delta\mathbf{v}_i + [\bar{\mathbf{A}}\mathbf{r}_i]\boldsymbol{\theta}\|^2. \tag{66}$$

Equation (66) corresponds to a linear least-squares estimation problem, with the optimal estimate given as a linear function of $\Delta\mathbf{v}_i$:

$$\boldsymbol{\theta}^* = \sum_{i=1}^2 \mathbf{J}_i \Delta\mathbf{v}_i,$$

where

$$\mathbf{J}_i = \mathbf{M}^{-1}(\frac{1}{\sigma_i^2}[\mathbf{A}\mathbf{r}_i]), \tag{67}$$

and

$$\mathbf{M} := \sum_{i=1}^2 \frac{1}{\sigma_i^2}(\mathbf{I} - \bar{\mathbf{A}}\mathbf{r}_i\mathbf{r}_i^T\bar{\mathbf{A}}^T). \tag{68}$$

Here $\mathbf{M}$ denotes the Fisher information matrix [27]. Since Equation (66) has the form of a linear least-squares estimation problem, the covariance of $\boldsymbol{\theta}^*$ achieves the Cramer-Rao lower bound [39]. The covariance of $\boldsymbol{\theta}^*$ is therefore given by

$$E(\boldsymbol{\theta}^*\boldsymbol{\theta}^{*T}) = \sum_{i=1}^2 \mathbf{J}_i E(\Delta\mathbf{v}_i\Delta\mathbf{v}_i^T)\mathbf{J}_i^T \tag{69}$$

$$= \mathbf{M}^{-1}, \tag{70}$$

where $E(\boldsymbol{\theta}^*) = 0$ is used. Since $\mathbf{R} = \bar{\mathbf{A}}^{-1}\exp(-[\boldsymbol{\theta}])$ holds, the left-invariant covariance of $\mathbf{R}$ in Equation (4) is the same as the covariance of $\boldsymbol{\theta}$. This completes the proof.

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