On Central Critical Values of the Degree Four L-functions for $\mathrm{GSp}(4)$: The Fundamental Lemma. III

Masaaki Furusawa

Kimball Martin

Joseph A. Shalika

Author address:

Department of Mathematics, Graduate School of Science, Osaka City University, Sugimoto 3–3–138, Sumiyoshi, Osaka 558–8585, Japan
E-mail address: furusawa@sci.osaka-cu.ac.jp

Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019-0315, USA
E-mail address: kmartin@math.ou.edu

Department of Mathematics, The Johns Hopkins University, 3400 North Charles Street, Baltimore, Maryland 21218–2689, USA

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Abstract

Some time ago, the first and third authors proposed two relative trace formulas to prove generalizations of Böcherer's conjecture on the central critical values of the degree four L-functions for GSp(4), and proved the relevant fundamental lemmas. Recently, the first and second authors proposed an alternative relative trace formula to approach the same problem which seems to possess several advantages over the first two, and proved the relevant fundamental lemma. In this paper the authors extend the latter fundamental lemma to the full Hecke algebra. The fundamental lemma is an equality of two local relative orbital integrals. In order to show that they are equal, the authors compute them explicitly for certain bases of the Hecke algebra and deduce the matching.

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Preface

One of the central themes of modern number theory is to investigate special values of automorphic L-functions and their relation to periods of automorphic forms. Central values are of considerable significance because of their relevance to the Birch & Swinnerton-Dyer conjecture and its generalizations. Here we cannot help mentioning the celebrated results of Waldspurger [**26, 27**] in the GL(2) case, which have seen many applications. Jacquet studied Waldspurger's results by his theory of relative trace formula in [**11, 12, 13**]. The relevant relative trace formulas have been explicated and extended by several authors [**1, 16, 21**].

Böcherer [**2**] proclaimed a conjecture concerning the central critical values of the spinor L-functions for Siegel eigen cusp forms of degree two. In the representation theoretical viewpoint, the relevant group is GSp(4), the group of 4 by 4 symplectic similitude matrices. In [**8**], the first and third authors proposed two relative trace formulas which should ultimately lead to a proof of Böcherer's conjecture and its generalization, and proved the fundamental lemmas for the unit element of the Hecke algebra. The reader is referred to [**8**, Introduction] for the statement of Böcherer's conjecture and to [**8**, Conjectures 1.10 and 1.11] for its generalizations, respectively. Böcherer's conjecture is closely related to a special case of the Gross-Prasad conjecture [**9**]. We refer to Prasad and Takloo-Bighash [**24**] for a proof of the local Gross-Prasad conjecture for Bessel models of GSp(4). We also refer to the important paper of Ichino and Ikeda [**10**] for a refined formulation of the global Gross-Prasad conjecture in the co-dimension one case.

To tackle the same problem, the first and second authors proposed another relative trace formula, which was inspired by a suggestion to the first author by Erez Lapid, and proved the fundamental lemma for the unit element of the Hecke algebra in [**6**]. The new relative trace formula seems to possess several advantages over the previous two. We refer to [**6**, Introduction] for a discussion of this conjectural relative trace formula and its advantages over the previous ones in [**8**].

In this paper we prove the extension of the fundamental lemma for the conjectural relative trace formula in [**8**] to the full Hecke algebra, which is an essential step towards our ultimate objective to prove the central-value formula. This paper is organized as follows. In Chapter 1, we introduce the necessary notation and state the main results. In Chapter 2, we recall some basic facts on Macdonald polynomials closely following [**18**]. Then we interpret the explicit formulas in [**4**] and [**3**] for the Whittaker model and the Bessel model, respectively, in terms of the Macdonald polynomials. We may reduce the relevant orbital integral for an element of the Hecke algebra to a finite linear combination of certain degenerate orbital integrals for the unit element using Fourier inversion. Here the linear coefficients appearing are explicitly given for elements of a certain basis of the Hecke algebra. Thus by computing the degenerate orbital integrals for the unit element, we may evaluate the two orbital integrals in the fundamental lemma explicitly for the full Hecke algebra. In Chapter 3 and Chapter 4, we compute the anisotropic Bessel orbital integral and the split Bessel orbital integral, respectively. In Chapter 5, we compute the Rankin-Selberg orbital integral and then we prove the matching by direct comparison.

The third author, Joseph A. Shalika passed away suddenly on September 18, 2010. The first and second authors would like to dedicate this paper to his memory, as a modest addition to the great legacy of his fundamental contributions to the modern theory of automorphic forms.

Masaaki Furusawa
Kimball Martin
Joseph A. Shalika

CHAPTER 1

Introduction

Notation

Let F be a non-archimedean local field whose residual characteristic is *not equal to two*. Let $\mathcal{O}$ denote the ring of integers in F and ϖ be a prime element of F. Let q denote the cardinality of the residue field $\mathcal{O}/\varpi\mathcal{O}$ and $|\cdot|$ denote the normalized absolute value on F, so that $|\varpi| = q^{-1}$. For $a \in F^\times$, $\mathrm{ord}\,(a)$ denotes the order of a. Hence we have $|a| = q^{-\mathrm{ord}(a)}$. Let ψ be an additive character of F of order zero, i.e. ψ is trivial on $\mathcal{O}$ but not on $\varpi^{-1}\mathcal{O}$.

Let E denote either the unique unramified quadratic extension of F, in the inert case, or $F \oplus F$, in the split case. When E is inert, we denote by $\mathcal{O}_E$ the ring of integers in E. Let $\kappa = \kappa_{E/F}$, i.e. κ is the unique unramified quadratic character of $F^\times$ in the inert case and κ is the trivial character of $F^\times$ in the split case. Let Ω be an unramified character of $E^\times$ and let $\omega = \Omega\,|_{F^\times}$. Then we may write $\Omega = \delta \circ \mathrm{N}_{E/F}$ where δ is an unramified character of $F^\times$ and we have $\omega = \delta^2$.

For a ring A and a positive integer n, $\mathrm{M}_n\,(A)$ denotes the set of n-by-n matrices with entries in A. For $X \in \mathrm{M}_n\,(A)$, we denote by ${}^tX$ its transpose. Let $\mathrm{Sym}^n\,(A)$ denote the set of n by n symmetric matrices with entries in A.

In general, for an algebraic group $\mathbb{G}$ defined over F, we also write $\mathbb{G}$ for its group of F-rational points.

Let G be $\mathrm{GSp}_4\,(F)$, the *group of four-by-four symplectic similitude matrices over* F, i.e.

$$G = \left\{ g \in \mathrm{GL}_4\,(F) \mid {}^tg\, J\, g = \lambda\,(g)\, J,\, \lambda\,(g) \in \mathbb{G}_m\,(F) \right\}, \quad J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

Let Z denote the center of G.

Let K be the maximal conpact subgroup $\mathrm{GSp}_4\,(\mathcal{O})$ of G. The *Hecke algebra* $\mathcal{H}$ of G is the space of compactly supported $\mathbb{C}$-valued bi-K-invariant functions on G, with the convolution product defined for $f_1, f_2 \in \mathcal{H}$ by

$$(f_1 * f_2)\,(x) = \int_G f_1\left(xg^{-1}\right) f_2\,(g)\, dg$$

where dg is the Haar measure on G normalized so that $\int_K dg = 1$. Let Ξ be the characteristic function of K. Then Ξ is the unit element of $\mathcal{H}$ with respect to the convolution product.

Let $W : \mathrm{GL}_2\,(F) \to \mathbb{C}$ denote the $\mathrm{GL}_2\,(\mathcal{O})$-fixed vector in the Whittaker model of the unramified principal series representation $\pi\,(1, \kappa)$ of $\mathrm{GL}_2\,(F)$ with respect to the upper unipotent subgroup and the additive character ψ, which is normalized so that $W\,(1) = 1$. Here we recall that for $a, b \in F^\times$, $x \in F$ and $k \in \mathrm{GL}_2\,(\mathcal{O})$, we have

$$W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}ab & 0\\ 0 & b\end{pmatrix}k\right)=\psi\left(-x\right)\kappa\left(b\right)W\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\tag{1.1}$$

and

$$W\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}=\begin{cases}|a|^{\frac{1}{2}}, & \text{when } E \text{ is inert and } \operatorname{ord}\left(a\right)\in 2\,\mathbb{Z}_{\geq 0},\\ |a|^{\frac{1}{2}}\left(1+\operatorname{ord}\left(a\right)\right), & \text{when } E \text{ splits and } \operatorname{ord}\left(a\right)\in \mathbb{Z}_{\geq 0},\\ 0, & \text{otherwise.}\end{cases}\tag{1.2}$$

1.1. Orbital Integrals

1.1.1. Rankin-Selberg type orbital integral. Let B be the standard Borel subgroup of G and $B=AN$ be its Levi decomposition. Thus A is the group of diagonal matrices in G and N consists of elements of G of the form

$$u\left(x_1,x_2,x_3,x_4\right)=\begin{pmatrix}1 & x_1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -x_1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & x_2 & x_3\\ 0 & 1 & x_3 & x_4\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\quad x_i\in F.$$

By abuse of notation, let ψ denote the non-degenerate character of N defined by

$$\psi\left[u\left(x_1,x_2,x_3,x_4\right)\right]=\psi\left(x_1+x_4\right).$$

Let H denote the subgroup of G consisting of elements of the form

$$\iota\left(h_1,h_2\right)=\begin{pmatrix}a_1 & 0 & b_1 & 0\\ 0 & a_2 & 0 & b_2\\ c_1 & 0 & d_1 & 0\\ 0 & c_2 & 0 & d_2\end{pmatrix},\quad \text{where } h_i=\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}\in \mathrm{GL}_2\left(F\right)$$

such that $\det h_1=\det h_2$.

DEFINITION 1.1. For $s\in F^{\times}$, $a\in F\setminus\{0,1\}$ and $f\in\mathcal{H}$, we define the Rankin-Selberg type orbital integral $I\left(s,a;f\right)$ by

$$I\left(s,a;f\right)=\int_{H_0\backslash H}\int_N\int_Z f\left(h^{-1}\,\bar{n}^{(s)}\,zn\right)W_{s,a}\left(h\right)\omega\left(z\right)\psi\left(n\right)\,dz\,dn\,dh\tag{1.3}$$

where

$$H_0^u=\left\{\iota\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\right)\,\middle|\,y\in F\right\},\quad \bar{n}^{(s)}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & s & 0 & 0\\ 0 & s & 1 & 0\\ 1 & 0 & 0 & s^{-1}\end{pmatrix},$$

$H_0=ZH_0^u$, and

$$\begin{aligned}W_{s,a}\left(\iota\left(h_1,h_2\right)\right)&=\delta^{-1}\left(s\left(1-a\right)\det h_2\right)\\ &\quad\cdot W\left(\begin{pmatrix}sa & 0\\ 0 & 1\end{pmatrix}h_1\right)W\left(\begin{pmatrix}s\left(1-a\right) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}h_2\right).\end{aligned}\tag{1.4}$$

1.1.2. Anisotropic Bessel orbital integral. Suppose that E is inert. Let us take $\eta \in \mathcal{O}_E^\times$ such that $E = F(\eta)$ and $d = \eta^2 \in F$. Then for $\alpha = a + b\eta \in E^\times$ where $a, b \in F$, we define $t_\alpha \in G$ by

$$t_\alpha = \begin{pmatrix} \left(\begin{smallmatrix} a & b \\ bd & a \end{smallmatrix}\right) & 0 \\ 0 & \left(\begin{smallmatrix} a & -bd \\ -b & a \end{smallmatrix}\right) \end{pmatrix}.$$

Let us denote by $T^{(a)}$ the anisotropic torus of G defined by $T^{(a)} = \{t_\alpha \mid \alpha \in E^\times\}$.

Let U be the unipotent radical of the upper Siegel parabolic subgroup of G, namely

$$U = \left\{ \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \,\middle|\, X \in \mathrm{Sym}^2(F) \right\}.$$

Let $\bar{U}$ be the opposite of U.

The upper anisotropic Bessel subgroup $R^{(a)}$ of G is defined by $R^{(a)} = T^{(a)}U$. Similarly the lower anisotropic Bessel subgroup $\bar{R}^{(a)}$ of G is defined by $\bar{R}^{(a)} = T^{(a)}\bar{U}$. We define a character $\tau^{(a)}$ of $R^{(a)}$ and a character $\xi^{(a)}$ of $\bar{R}^{(a)}$ by

$$\tau^{(a)}\left[\begin{pmatrix} \left(\begin{smallmatrix} a & b \\ bd & a \end{smallmatrix}\right) & 0 \\ 0 & \left(\begin{smallmatrix} a & -bd \\ -b & a \end{smallmatrix}\right) \end{pmatrix}\begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix}\right] = \Omega(a+b\eta)\cdot\psi\left[\mathrm{tr}\left(\begin{pmatrix} -d & 0 \\ 0 & 1 \end{pmatrix}X\right)\right] \tag{1.5}$$

and

$$\xi^{(a)}\left[\begin{pmatrix} \left(\begin{smallmatrix} a & b \\ bd & a \end{smallmatrix}\right) & 0 \\ 0 & \left(\begin{smallmatrix} a & -bd \\ -b & a \end{smallmatrix}\right) \end{pmatrix}\begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix}\right] = \Omega(a+b\eta)\cdot\psi\left[\mathrm{tr}\left(\begin{pmatrix} -d^{-1} & 0 \\ 0 & 1 \end{pmatrix}Y\right)\right], \tag{1.6}$$

respectively.

Definition 1.2. For $u \in E^\times$ such that $\mathrm{N}_{E/F}(u) \neq 1$, $\mu \in F^\times$ and $f \in \mathcal{H}$, we define the anisotropic Bessel orbital integral $\mathcal{B}^{(a)}(u,\mu;f)$ by

$$\mathcal{B}^{(a)}(u,\mu;f) = \int_{Z\backslash\bar{R}^{(a)}}\int_{R^{(a)}} f\left(\bar{r}\,A^{(a)}(u,\mu)\,r\right)\xi^{(a)}(\bar{r})\,\tau^{(a)}(r)\,dr\,d\bar{r} \tag{1.7}$$

where

$$A^{(a)}(u,\mu) = \begin{pmatrix} \left(\begin{smallmatrix} 1+a & -b \\ bd & 1-a \end{smallmatrix}\right) & 0 \\ 0 & \mu\ {}^t\left(\begin{smallmatrix} 1+a & -b \\ bd & 1-a \end{smallmatrix}\right)^{-1} \end{pmatrix} \tag{1.8}$$

for $u = a + b\eta$ with $a, b \in F$.

Remark 1.3. In [**6**], the anisotropic Bessel orbital integral $\mathcal{B}^{(a)}$ was simply called the Bessel orbital integral and was denoted by $\mathcal{B}$.

1.1.3. Split Bessel orbital integral. Let $T^{(s)}$ be the split torus of G defined by

$$T^{(s)} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \,\middle|\, a, b \in F^\times \right\}.$$

The upper split Bessel subgroup $R^{(s)}$ of G is defined by $R^{(s)} = T^{(s)}U$. Similarly the lower split Bessel subgroup $\bar{R}^{(s)}$ of G is defined by $\bar{R}^{(s)} = T^{(s)}\bar{U}$. We define a character $\tau^{(s)}$ of $R^{(s)}$ and $\xi^{(s)}$ of $\bar{R}^{(s)}$ by

$$\tau^{(s)}\left[\begin{pmatrix} a & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & a\end{pmatrix}\begin{pmatrix} 1_2 & X\\ 0 & 1_2\end{pmatrix}\right] = \delta\left(ab\right)\cdot\psi\left[\mathrm{tr}\left(\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} X\right)\right] \tag{1.9}$$

and

$$\xi^{(s)}\left[\begin{pmatrix} a & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & b & 0\\ 0 & 0 & 0 & a\end{pmatrix}\begin{pmatrix} 1_2 & 0\\ Y & 1_2\end{pmatrix}\right] = \delta\left(ab\right)\cdot\psi\left[\mathrm{tr}\left(\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} Y\right)\right], \tag{1.10}$$

respectively.

Definition 1.4. For $x \in F \setminus \{0,1\}$, $\mu \in F^\times$ and $f \in \mathcal{H}$, we define the split Bessel orbital integral $\mathcal{B}^{(s)}\left(x,\mu;f\right)$ by

$$\mathcal{B}^{(s)}\left(x,\mu;f\right) = \int_{Z\setminus\bar{R}^{(s)}}\int_{R^{(s)}} f\left(\bar{r}\,A^{(s)}\left(x,\mu\right)\,r\right)\,\xi^{(s)}\left(\bar{r}\right)\,\tau^{(s)}\left(r\right)\,dr\,d\bar{r} \tag{1.11}$$

where

$$A^{(s)}\left(x,\mu\right) = \begin{pmatrix} \left(\begin{smallmatrix} 1 & x\\ 1 & 1\end{smallmatrix}\right) & 0\\ 0 & \mu\ {}^t\left(\begin{smallmatrix} 1 & x\\ 1 & 1\end{smallmatrix}\right)^{-1}\end{pmatrix}. \tag{1.12}$$

Remark 1.5. In [**6**], the split Bessel orbital integral $\mathcal{B}^{(s)}$ was called the Novodvorsky orbital integral and was denoted by $\mathcal{N}$.

1.2. Matching

For $x \in F \setminus \{0,1\}$, $\mu \in F^\times$ and $f \in \mathcal{H}$, we define $\mathcal{I}\left(x,\mu;f\right)$ by

$$\mathcal{I}\left(x,\mu;f\right) = I\left(s,a;f\right) \quad \text{where} \quad s = -\frac{1-x}{4\mu},\ a = \frac{1}{1-x}.$$

The goal of this paper is to prove the following matching results.

Theorem 1.6 (Matching when E/F is inert). *For $x \in F \setminus \{0,1\}$, $\mu \in F^\times$ and $f \in \mathcal{H}$, the Rankin-Selberg type orbital integral $\mathcal{I}\left(x,\mu;f\right)$ vanishes unless* $\mathrm{ord}\left(x\right)$ *is even.*

When $x = \mathrm{N}_{E/F}\left(u\right)$ for $u \in E^\times$, we have

$$\mathcal{I}\left(x,\mu;f\right) = \delta^{-1}\left(\frac{x}{\mu^2}\right)\left|\frac{x}{\mu^2}\right|^{\frac{1}{2}}\mathcal{B}^{(a)}\left(u,\mu;f\right). \tag{1.13}$$

Theorem 1.7 (Matching when E/F is split). *For $x \in F \setminus \{0,1\}$, $\mu \in F^\times$ and $f \in \mathcal{H}$, we have*

$$\mathcal{I}\left(x,\mu;f\right) = \delta^{-1}\left(\frac{x}{\mu^2}\right)\left|\frac{x}{\mu^2}\right|^{\frac{1}{2}}\mathcal{B}^{(s)}\left(x,\mu;f\right). \tag{1.14}$$

These theorems were established in [**6**] in the special case where $f = \Xi$, the unit element of $\mathcal{H}$.

CHAPTER 2

Reduction Formulas

In this chapter, we prove the reduction formulas (2.41), (2.43) and (2.45), which express the orbital integrals for $f \in \mathcal{H}$ as finite linear combinations of degenerate orbital integrals for the unit element Ξ. We also show that when we take appropriate bases for $\mathcal{H}$, the coefficients in the linear combinations are explicitly given by the generalized Kostka numbers defined by (2.47). Finally we restate Theorems 1.6 and 1.7 as Theorem 2.17 and 2.18, respectively, which we shall prove in Chapter 5.

Here we remark that, in [**7**], the first and third author computed the Plancherel measures and proved the Fourier inversion formulas for the Whittaker and Bessel models by some direct computations based on the explicit formulas in [**4**] and [**3**] respectively, inspired by the work of Ye [**28, 29**]. The obstacle to proceed further was a lack of knowledge about the relationship between the Whittaker and Bessel models such as (2.47), which was provided by the theory of Macdonald polynomials. The relevance of the theory of Macdonald polynomials to the fundamental lemma came to our attention through the works of Mao and Rallis [**19, 20**] and Offen [**22, 23**].

2.1. Macdonald Polynomials

In this section we recall some general facts concerning the Macdonald polynomials, closely following the seminal paper of Macdonald [**18**].

2.1.1. Root system. Let V be a finite-dimensional vector space over $\mathbb{R}$ with a positive-definite symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let R be a root system in V, which is irreducible but not necessarily reduced, and let R^+ be its positive roots.

For $\alpha \in R$, let s_α denote the reflection associated with α, i.e.,

$$s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha \quad \text{for} \quad v \in V$$

where $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$, the co-root of α. Let W_R be the Weyl group of R, i.e., the group of orthogonal transformations of V generated by the s_α $(\alpha \in R)$.

Let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots determined by R^+. Let

$$Q = \sum_{i=1}^{n} \mathbb{Z}\, \alpha_i, \qquad Q^+ = \left\{ \sum_{i=1}^{n} m_i \alpha_i \in Q \,\middle|\, m_i \geq 0 \ \ (1 \leq i \leq n) \right\}$$

be the root lattice of R, its positive octant, respectively.

The weight lattice P, the set of dominant integral weights P^+ and the set of strictly dominant integral weights P^{++} are defined respectively by

$$P = \{ \lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R \},$$
$$P^+ = \left\{ \lambda \in P \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+ \right\},$$
$$P^{++} = \left\{ \lambda \in P \mid \langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in R^+ \right\}.$$

Let $\omega_1, \dots, \omega_n \in P$ be the fundamental weights given by the condition

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$$

where δ_{ij} denotes the Kronecker delta. The set P^+ is an integral cone with vertex 0, the set P^{++} is an integral cone with vertex $\rho = \sum_{i=1}^n \omega_i$ and the mapping $P^+ \ni \lambda \mapsto \lambda + \rho \in P^{++}$ is a bijection.

We define a partial order on P by

$$\lambda \geq \mu \quad \text{if and only if} \quad \lambda - \mu \in Q^+.$$

Let

$$R_1 = \{\alpha \in R \mid \alpha/2 \notin R\}, \qquad R_2 = \{\alpha \in R \mid 2\alpha \notin R\}.$$

When R is reduced, we have $R_1 = R_2 = R$. When R is not reduced, hence is of type BC_n, R_1 and R_2 are reduced root system of types B_n and C_n, respectively. Since P is the weight lattice of R_2, we have

$$\rho = \frac{1}{2} \sum_{\alpha \in R_2^+} \alpha, \quad \text{where } R_2^+ = R_2 \cap R^+.$$

2.1.2. Orbit sums and the Weyl characters. Let us introduce parameters t_α for $\alpha \in R_1$ (resp. $t_{2\alpha}^{1/2}$ for $2\alpha \in R \setminus R_1$) such that $t_\alpha = t_\beta$ (resp. $t_{2\alpha}^{1/2} = t_{2\beta}^{1/2}$) if $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$. Let $\mathbb{Z}[t]$ denote the ring of polynomials in the t_α and $t_{2\alpha}^{1/2}$ with integer coefficients. We also set $t_{2\alpha}^{1/2} = 1$ if $2\alpha \in V$ but $2\alpha \notin R$. Hence we have $t_\alpha = (t_\alpha^{1/2})^2 = 1$ if $\alpha \in V \setminus R$. Let $\mathbb{Q}(t)$ be the quotient field of $\mathbb{Z}[t]$, i.e., the field of rational functions of the t_α and $t_{2\alpha}^{1/2}$ with coefficients in $\mathbb{Q}$. Let us write $\mathbb{K}$ for $\mathbb{Q}(t)$. Let $\mathcal{A}$ be $\mathbb{K}[P]$, the group algebra of P over $\mathbb{K}$. We use the exponential notation, i.e., for each $\lambda \in P$, let e^λ denote the corresponding element of $\mathcal{A}$. The Weyl group W_R acts on $\mathcal{A}$ by $w\left(e^\lambda\right) = e^{w\lambda}$ for $w \in W_R$ and $\lambda \in P$. Let $\mathcal{A}^{W_R}$ denote the subalgebra of W_R-invariant elements of $\mathcal{A}$.

Since each W_R-orbit in P meets P^+ exactly once, the orbit sums

$$m_\lambda = \sum_{\mu \in W_R \cdot \lambda} e^\mu, \quad \text{where } W_R \cdot \lambda = \{w\lambda \mid w \in W_R\}, \tag{2.1}$$

for $\lambda \in P^+$ form a $\mathbb{K}$-basis of $\mathcal{A}^{W_R}$.

There is another $\mathbb{K}$-basis of $\mathcal{A}^{W_R}$ given by the Weyl characters. Let

$$\delta_R = \prod_{\alpha \in R_2^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right) = e^\rho \prod_{\alpha \in R_2^+} \left(1 - e^{-\alpha}\right).$$

Then we have $w\,\delta_R = \varepsilon(w) \cdot \delta_R$ for each $w \in W_R$, where $\varepsilon(w) = \pm 1$ denotes the signature of w. For each $\lambda \in P$, we define s_λ by

$$s_\lambda = \delta_R^{-1} \sum_{w \in W_R} \varepsilon(w)\, e^{w(\lambda+\rho)}. \tag{2.2}$$

Here we recall that if $s_\lambda \neq 0$, then there exists $w \in W_R$ and $\mu \in P^+$ such that

$$\mu + \rho = w(\lambda + \rho)$$

and we have $s_\lambda = \varepsilon(w) \cdot s_\mu$. On the other hand, when $\lambda \in P^+$, in terms of the orbit sums, we have

$$s_\lambda = m_\lambda + \sum_{\substack{\mu < \lambda \\ \mu \in P^+}} K_{\lambda\,\mu} m_\mu$$

where $K_{\lambda\,\mu}$ is a non-negative integer. Hence the Weyl characters s_λ ($\lambda \in P^+$) form another $\mathbb{K}$-basis of $\mathcal{A}^{W_R}$.

2.1.3. Scalar product. We define the scalar product $\langle\ ,\ \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$ by

$$\langle f, g\rangle = \frac{1}{|W_R|}\,[f\,\bar{g}\,\Delta]_1\,, \tag{2.3}$$

where

$$\bar{f} = \sum_{\lambda \in P} f_\lambda\, e^{-\lambda} \quad \text{for} \quad f = \sum_{\lambda \in P} f_\lambda\, e^{\lambda} \in \mathcal{A},$$

$$\Delta = \prod_{\alpha \in R} \frac{1 - t_{2\alpha}^{1/2}\, e^{\alpha}}{1 - t_{2\alpha}^{1/2}\, t_\alpha\, e^{\alpha}},$$

and

$$[h]_1 = h_0 \quad \text{for} \quad h = \sum_{\lambda \in P} h_\lambda\, e^{\lambda} \in \mathcal{A}.$$

Definition 2.1. For $\lambda \in P^+$, we define the Macdonald polynomial $P_\lambda \in \mathcal{A}^{W_R}$ by

$$P_\lambda = W_\lambda\,(t)^{-1} \sum_{w \in W_R} w \left(e^{\lambda} \prod_{\alpha \in R^+} \frac{1 - t_{2\alpha}^{1/2}\, t_\alpha\, e^{-\alpha}}{1 - t_{2\alpha}^{1/2}\, e^{-\alpha}} \right), \tag{2.4}$$

where

$$W_\lambda\,(t) = \sum_{w \in W_\lambda} t_w, \qquad W_\lambda = \{ w \in W_R \mid w\lambda = \lambda \}\,, \tag{2.5}$$

and

$$t_w = \prod_{\alpha \in R(w)} t_\alpha, \qquad R\,(w) = R^+ \cap \left(-wR^+\right). \tag{2.6}$$

Then Macdonald [**18**, §10] has shown the following.

Theorem 2.2. *The P_λ ($\lambda \in P^+$) is a unique $\mathbb{K}$-basis of $\mathcal{A}^{W_R}$ satisfying the following two conditions.*

(1) *P_λ is of the form*

$$P_\lambda = m_\lambda + \sum_{\substack{\mu < \lambda \\ \mu \in P^+}} a_{\lambda\,\mu}\, m_\mu, \quad a_{\lambda\,\mu} \in \mathbb{K}.$$

(2) *For $\lambda \neq \mu$, we have $\langle P_\lambda, P_\mu \rangle = 0$.*

It is also shown in [**18**, §10] that

$$|P_\lambda|^2 = \langle P_\lambda, P_\lambda \rangle = W_\lambda\,(t)^{-1} \tag{2.7}$$

and

$$P_\lambda = W_\lambda\,(t)^{-1} \sum_X \varphi_X\,(t)\ s_{\lambda - \sigma(X)}. \tag{2.8}$$

The summation in (2.8) is over all subsets X of R^+ satisfying the condition that

$$\alpha \in X \quad \text{implies} \quad 2\alpha \notin X$$

and for such X we write

$$\sigma(X) = \sum_{\alpha \in X} \alpha$$

and

$$\varphi_X(t) = \prod_{\alpha \in X} \varphi_\alpha(t), \quad \text{where } \varphi_\alpha(t) = \begin{cases} -t_{\alpha/2}\, t_\alpha & \text{if } 2\alpha \notin R, \\ (1 - t_\alpha)\, t_{2\alpha}^{1/2} & \text{if } 2\alpha \in R. \end{cases}$$

Here we recall that $t_{\alpha/2} = 1$ when $\alpha/2 \notin R$.

REMARK 2.3. The scalar product above is interpreted as follows. Let $R^\vee$ be the dual root system, i.e. $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$, and let $Q^\vee$ be the root lattice of $R^\vee$. Then we may regard each e^λ ($\lambda \in P$) as a character of the torus $T = V/Q^\vee$ by the rule:

$$e^\lambda(\dot{x}) = \exp\left(2\pi\sqrt{-1}\,\langle \lambda, x\rangle\right), \quad \text{where } \dot{x} \in T \text{ denotes the image of } x \in V.$$

When $\mathbb{K} \subset \mathbb{C}$, we may regard each element of $\mathcal{A}$ as a continuous function on T by linearity. Then for $f, g \in \mathcal{A} \cap \mathbb{R}[P]$ we have

$$\langle f, g\rangle = \frac{1}{|W_R|} \int_T f\, \bar{g}\, \Delta\, dt, \tag{2.9}$$

where the integration is with respect to the Haar measure dt on T normalized so that $\int_T dt = 1$ and $\bar{g}(\dot{x}) = \overline{g(\dot{x})}$, the complex conjugate of $g(\dot{x})$.

2.2. Explicit Formulas and the Macdonald Polynomials

Let us return to our situation where $G = \mathrm{GSp}_4(F)$.

2.2.1. Explicit formulas for the Bessel and Whittaker models. We recall that B denotes the standard Borel subgroup of G and $B = AN$ is its Levi decomposition. Let W_G be the Weyl group of G with respect to A, i.e. $W_G = N_G(A)/A$.

2.2.1.1. *Bessel and Whittaker functions.* Let $\hat{A}$ be the group of unramified characters of A. Let $\hat{A}_0$ be the subgroup of $\hat{A}$ consisting of $\chi \in \hat{A}$ such that $\chi\,|_Z = 1$. For $\chi \in \hat{A}$, let $I(\chi) = \mathrm{Ind}_B^G \chi$, i.e., $I(\chi)$ is the space of locally constant functions $\Phi : G \to \mathbb{C}$ which satisfy

$$\Phi(ang) = \delta_B(a)^{1/2}\, \chi(a)\, \Phi(g) \quad \text{for } a \in A,\ n \in N \text{ and } g \in G.$$

Here δ_B denotes the modulus function of B. The action π_χ of G on $I(\chi)$ is given by the right regular representation, i.e., $(\pi_\chi(g)\Phi)(x) = \Phi(xg)$ for $g, x \in G$. Let φ_χ be the K-fixed element in $I(\chi)$ with $\varphi_\chi(1) = 1$.

Suppose that $\chi \in \hat{A}$ is regular, i.e. $w\,\chi \neq \chi$ for $w \in W_G \setminus \{1\}$, and $\chi\,|_Z = \omega$. Then there exists a unique linear functional $H_\chi^{(a)}$(resp. $H_\chi^{(s)}$) $: I(\chi) \to \mathbb{C}$ such that

$$H_\chi^{(a)}(\pi_\chi(r)\Phi) = \tau^{(a)}(r) \cdot H_\chi^{(a)}(\Phi) \qquad \text{for } r \in R^{(a)},\ \Phi \in I(\chi)$$

$$(\text{resp. } H_\chi^{(s)}(\pi_\chi(r)\Phi) = \tau^{(s)}(r) \cdot H_\chi^{(s)}(\Phi) \qquad \text{for } r \in R^{(s)},\ \Phi \in I(\chi)),$$

and $H_\chi^{(a)}(\varphi_\chi) = H_\chi^{(s)}(\varphi_\chi) = 1$. We recall that the characters $\tau^{(a)}$ and $\tau^{(s)}$ are defined by (1.5) and (1.9) respectively. Then we define the anisotropic (resp. split) Bessel function $B_\chi^{(a)}$(resp. $B_\chi^{(s)}$) on G by

$$B_\chi^{(a)}(g) = H_\chi^{(a)}(\pi_\chi(g)\varphi_\chi) \qquad (\text{resp. } B_\chi^{(s)}(g) = H_\chi^{(s)}(\pi_\chi(g)\varphi_\chi)).$$

Similarly when $\chi \in \hat{A}$ is regular, let $\Omega_\chi : I(\chi) \to \mathbb{C}$ be the unique linear functional such that

$$\Omega_\chi\left(\pi_\chi(n)\,\Phi\right) = \psi(n)\,\Omega_\chi(\Phi) \qquad \text{for } n \in N,\ \Phi \in I(\chi),$$

and $\Omega_\chi(\varphi_\chi) = 1$. We define the *Whittaker function* W_χ on G by

$$W_\chi(g) = \Omega_\chi\left(\pi_\chi(g)\,\varphi_\chi\right).$$

2.2.1.2. *Explicit formulas.* Let us recall the explicit formulas for $B_\chi^{(a)}$ and $B_\chi^{(s)}$ from [**3**], and the explicit formula for W_χ from [**4**].

Let $\mathbf{\Lambda}$ be $X_*(A)$, the group of co-characters of A, regarded as an algebraic group over F. Then $\mathbf{\Lambda} \simeq \mathbb{Z}^3$ by identifying $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ with $\lambda \in \mathbf{\Lambda}$ defined by

$$\lambda(x) = x^{\lambda_3}\begin{pmatrix} x^{\lambda_1+\lambda_2} & 0 & 0 & 0 \\ 0 & x^{\lambda_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x^{\lambda_2} \end{pmatrix} \qquad \text{for } x \in F^\times.$$

Let us denote $\lambda(\varpi)$ by ϖ^λ for $\lambda \in \mathbf{\Lambda}$.

A double coset $R^{(a)}gK$ is called $\tau^{(a)}$-relevant when

$$\tau^{(a)}\left(gkg^{-1}\right) = 1 \qquad \text{for } k \in K \cap g^{-1}R^{(a)}\,g.$$

Since $B_\chi^{(a)}(rgk) = \tau^{(a)}(r)\;B_\chi^{(a)}(g)$ for $r \in R^{(a)}$ and $k \in K$, the function $B_\chi^{(a)}$ is supported on the $\tau^{(a)}$-relevant double cosets. Let

$$\mathbf{\Lambda}^- = \left\{\lambda = (\lambda_1, \lambda_2, 0) \in \mathbb{Z}^3 \mid \lambda_1 \geq \lambda_2 \geq 0\right\}. \tag{2.10}$$

Then as the representatives of the $\tau^{(a)}$-relevant double cosets, we may take

$$b_\lambda^{(a)} = \varpi^\lambda, \qquad \text{where } \lambda \in \mathbf{\Lambda}^-.$$

The $\tau^{(s)}$-relevant double cosets are similarly defined and we may take

$$b_\lambda^{(s)} = n_1\varpi^\lambda, \qquad \text{where } n_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \text{ and } \lambda \in \mathbf{\Lambda}^-$$

as the representatives of the $\tau^{(s)}$-relevant double cosets.

Similarly a double coset $ZNgK$ is called ψ-relevant when $\psi\left(gkg^{-1}\right) = 1$ for $k \in K \cap g^{-1}Ng$. Then the Whittaker function W_χ is supported on the ψ-relevant double cosets. As the representatives of the ψ-relevant double cosets, we may take

$$\varpi^\lambda, \qquad \text{where } \lambda \in \mathbf{\Lambda}^-.$$

Let us introduce some more notation in order to describe the explicit formulas. We define $\gamma_j \in \mathbf{\Lambda}$ $(1 \leq j \leq 4)$ by

$$\gamma_1 = (1, -1, 0), \quad \gamma_2 = (0, 2, -1), \quad \gamma_3 = \gamma_1 + \gamma_2, \quad \gamma_4 = 2\gamma_1 + \gamma_2$$

and $\delta_2, \delta_4 \in \mathbf{\Lambda}$ by

$$\delta_2 = (1, 0, 0), \quad \delta_4 = (0, 1, 0).$$

We recall that the Weyl group W_G is isomorphic to the group of permutations of $\{x_1, x_2, x_3, x_4\}$ preserving $x_1x_3 = x_2x_4$. Let us define $\chi_\delta : G \to \mathbb{C}^\times$ by

$$\chi_\delta(g) = \delta(\lambda(g)), \qquad \text{where } \lambda(g) \text{ is the similitude of } g. \tag{2.11}$$

For $\chi \in \hat{A}$ with $\chi\mid_Z = \omega = \delta^2$, we define $\chi_0 \in \hat{A}$ by $\chi_0 = (\chi_\delta\mid_Z)^{-1} \cdot \chi$. Then we have $\chi_0 \in \hat{A}_0$ and $I(\chi) = \chi_\delta \otimes I(\chi_0)$.

Then the explicit formulas for the Bessel functions in [**3**] and the one for the Whittaker function in [**4**] in our case are given as follows.

THEOREM 2.4 (Explicit formulas). *For $\lambda \in \mathbf{\Lambda}^-$, we have*

$$B_\chi^{(a)}\left(b_\lambda^{(a)}\right) = \frac{\chi_\delta\left(\varpi^\lambda\right)\cdot\delta_B\left(\varpi^\lambda\right)^{1/2}}{Q^{(a)}\left(q^{-1}\right)} \sum_{w\in W_G} w\left(\chi_0\left(\varpi^\lambda\right)^{-1}\cdot C^{(a)}\left(\chi_0\right)\right),$$

$$B_\chi^{(s)}\left(b_\lambda^{(s)}\right) = \frac{\chi_\delta\left(\varpi^\lambda\right)\cdot\delta_B\left(\varpi^\lambda\right)^{1/2}}{Q^{(s)}\left(q^{-1}\right)} \sum_{w\in W_G} w\left(\chi_0\left(\varpi^\lambda\right)^{-1}\cdot C^{(s)}\left(\chi_0\right)\right), \quad \textit{and}$$

$$W_\chi\left(\varpi^\lambda\right) = \chi_\delta\left(\varpi^\lambda\right)\cdot\delta_B\left(\varpi^\lambda\right)^{1/2} \sum_{w\in W_G} w\left(\chi_0\left(\varpi^\lambda\right)^{-1}\cdot C\left(\chi_0\right)\right).$$

Here

$$Q^{(a)}(t) = 1+t, \qquad Q^{(s)}(t) = 1-t,$$

$$C^{(a)}(\chi_0) = \prod_{1\le i\le 4} \frac{1}{1-\chi_0\left(\varpi^{\gamma_i}\right)} \cdot \prod_{j=2,4}\left(1-\chi_0\left(\varpi^{\gamma_j}\right)q^{-1}\right),$$

$$C^{(s)}(\chi_0) = \prod_{1\le i\le 4} \frac{1}{1-\chi_0\left(\varpi^{\gamma_i}\right)} \cdot \prod_{j=2,4}\left(1-\chi_0\left(\varpi^{\delta_j}\right)q^{-1/2}\right)^2, \quad \textit{and}$$

$$C(\chi_0) = \prod_{1\le i\le 4} \frac{1}{1-\chi_0\left(\varpi^{\gamma_i}\right)}.$$

For convenience, we note that $\delta_B(\varpi^\lambda) = q^{-3\lambda_1-\lambda_2}$, $\chi_\delta(\varpi^\lambda) = \delta(\varpi^{\lambda_1+\lambda_2})$ and $\chi_0(\varpi^\lambda) = \delta(\varpi^{\lambda_1+\lambda_2})^{-2}\chi(\varpi^\lambda)$.

2.2.2. Relation to the Macdonald polynomials. We interpret the explicit formulas in Theorem 2.4 in terms of the Macdonald polynomials.

As the real vector space V, we take

$$V = (\mathbf{\Lambda}\otimes_{\mathbb{Z}}\mathbb{R})/\mathbb{R}(0,0,1).$$

Let us denote the image of $x \in \mathbf{\Lambda}\otimes_{\mathbb{Z}}\mathbb{R}$ in V by $\bar{x}$. Then we have

$$\bar{\gamma}_1 = \bar{\delta}_2 - \bar{\delta}_4, \quad \bar{\gamma}_2 = 2\bar{\delta}_4, \quad \bar{\gamma}_3 = \bar{\delta}_2 + \bar{\delta}_4, \quad \bar{\gamma}_4 = 2\bar{\delta}_2.$$

We shall identify V with $\mathbb{R}^2$ by identifying $\bar{\delta}_2$ with $\epsilon_1 = (1,0)$ and $\bar{\delta}_4$ with $\epsilon_2 = (0,1)$, respectively. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the inner product on $V = \mathbb{R}^2$ defined by $\langle x, y\rangle = x\,{}^t y$. Then

$$R = R^+ \cup \left(-R^+\right), \quad \text{where } R^+ = \{\epsilon_1\pm\epsilon_2,\, \epsilon_1,\, \epsilon_2,\, 2\epsilon_1,\, 2\epsilon_2\}, \tag{2.12}$$

is a root system of type BC_2 in V and

$$R_2 = \{\alpha\in R \mid 2\alpha\notin R\} = R_2^+ \cup \left(-R_2^+\right), \text{ where } R_2^+ = \{2\epsilon_1,\, 2\epsilon_2,\, \epsilon_1\pm\epsilon_2\}, \tag{2.13}$$

is a root system of type C_2 in V. As for the weight lattice P, the set of dominant weights P^+ and the root lattice Q, we have

$$P = Q = \mathbb{Z}^2, \quad P^+ = \left\{\lambda = (\lambda_1,\lambda_2)\in\mathbb{Z}^2 \mid \lambda_1\ge\lambda_2\ge 0\right\}.$$

We shall identify P^+ with $\mathbf{\Lambda}^-$ defined by (2.10).

Let P_λ denote the Macdonald polynomial defined by (2.4), which corresponds to the root system R given by (2.12). Let us set the parameters $t_\alpha^{(w)}$, $t_\alpha^{(a)}$ and $t_\alpha^{(s)}$ for $\alpha \in R$, corresponding to the Whittaker case, the anisotropic Bessel case and the split Bessel case, respectively, as follows:

(2.14)
$$\begin{cases} t_\alpha^{(w)} = t_\alpha^{(a)} = t_\alpha^{(s)} = 0 & \text{if } \alpha \in \{\pm(\epsilon_1 \pm \epsilon_2)\}, \\ t_\alpha^{(w)} = 0,\ t_\alpha^{(a)} = 1,\ t_\alpha^{(s)} = -1 & \text{if } \alpha \in \{\pm\epsilon_1, \pm\epsilon_2\}, \\ (t_\alpha^{(w)})^{1/2} = 0,\ (t_\alpha^{(a)})^{1/2} = q^{-1/2},\ (t_\alpha^{(s)})^{1/2} = -q^{-1/2} & \text{if } \alpha \in \{\pm 2\epsilon_1, \pm 2\epsilon_2\}. \end{cases}$$

Here we note that by the Weyl denominator formula, we have

$$P_\lambda \mid_{t=t^{(w)}} = s_\lambda,$$

where s_λ is the Weyl character given by (2.2). Let us define $P_\lambda^{(a)}$ and $P_\lambda^{(s)}$ by

$$P_\lambda^{(a)} = P_\lambda \mid_{t=t^{(a)}} \quad \text{and} \quad P_\lambda^{(s)} = P_\lambda \mid_{t=t^{(s)}},$$

respectively.

For $\rho \in \hat{A}_0$, let $P_\lambda^{(a)}(\rho)$ denote the value of $P_\lambda^{(a)}$ evaluated at

$$e^{\epsilon_1} = \rho\left(\varpi^{\delta_2}\right) \quad \text{and} \quad e^{\epsilon_2} = \rho\left(\varpi^{\delta_4}\right).$$

We shall use similar notation for $P_\lambda^{(s)}$ and s_λ. Then the explicit formulas in Theorem 2.4 are rewritten as follows.

PROPOSITION 2.5. *For $\lambda \in P^+$, we have*

$$B_\chi^{(a)}\left(b_\lambda^{(a)}\right) = \frac{\chi_\delta\left(\varpi^\lambda\right) \cdot \delta_B\left(\varpi^\lambda\right)^{1/2} \cdot W_\lambda\left(t^{(a)}\right)}{Q^{(a)}\left(q^{-1}\right)} \cdot P_\lambda^{(a)}\left(\chi_0\right), \tag{2.15}$$

$$B_\chi^{(s)}\left(b_\lambda^{(s)}\right) = \frac{\chi_\delta\left(\varpi^\lambda\right) \cdot \delta_B\left(\varpi^\lambda\right)^{1/2} \cdot W_\lambda\left(t^{(s)}\right)}{Q^{(s)}\left(q^{-1}\right)} \cdot P_\lambda^{(s)}\left(\chi_0\right), \quad \textit{and} \tag{2.16}$$

$$W_\chi\left(\varpi^\lambda\right) = \chi_\delta\left(\varpi^\lambda\right) \cdot \delta_B\left(\varpi^\lambda\right)^{1/2} \cdot s_\lambda\left(\chi_0\right), \tag{2.17}$$

where

$$\begin{cases} W_\lambda\left(t^{(a)}\right) = W_\lambda\left(t^{(s)}\right) = 1 & \textit{when } \lambda_2 \geq 1, \\ W_\lambda\left(t^{(a)}\right) = Q^{(a)}\left(q^{-1}\right) \textit{ and } W_\lambda\left(t^{(s)}\right) = Q^{(s)}\left(q^{-1}\right) & \textit{when } \lambda_2 = 0. \end{cases} \tag{2.18}$$

PROOF. Except for (2.18), this is clear from Theorem 2.4. Let us write down the elements w_i $(1 \leq i \leq 8)$ of W_G in a way so that the action of each w_i on P is given respectively by

$$\begin{aligned} &w_1 : (\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2), && w_2 : (\lambda_1, \lambda_2) \mapsto (\lambda_2, \lambda_1), \\ &w_3 : (\lambda_1, \lambda_2) \mapsto (-\lambda_1, \lambda_2), && w_4 : (\lambda_1, \lambda_2) \mapsto (-\lambda_2, \lambda_1), \\ &w_5 : (\lambda_1, \lambda_2) \mapsto (\lambda_1, -\lambda_2), && w_6 : (\lambda_1, \lambda_2) \mapsto (\lambda_2, -\lambda_1), \\ &w_7 : (\lambda_1, \lambda_2) \mapsto (-\lambda_1, -\lambda_2), && w_8 : (\lambda_1, \lambda_2) \mapsto (-\lambda_2, -\lambda_1). \end{aligned}$$

Since $t_\alpha^{(a)} = t_\alpha^{(s)} = 0$ for $\alpha \in \{\pm(\epsilon_1 \pm \epsilon_2)\}$, in (2.6) we have

$$t_w^{(a)} = \prod_{\alpha \in R(w)} t_\alpha^{(a)} = 0 \quad \text{and} \quad t_w^{(s)} = \prod_{\alpha \in R(w)} t_\alpha^{(s)} = 0,$$

unless $w\{\epsilon_1 \pm \epsilon_2\} = \{\epsilon_1 \pm \epsilon_2\}$, i.e., $w = w_1, w_5$. On the other hand we have

$$t_{w_5}^{(a)} = t_{\epsilon_2}^{(a)}\, t_{2\epsilon_2}^{(a)} = q^{-1}, \qquad t_{w_5}^{(s)} = t_{\epsilon_2}^{(s)}\, t_{2\epsilon_2}^{(s)} = -q^{-1}$$

and $t_{w_1}^{(a)} = t_{w_1}^{(s)} = 1$. Since

$$W_\lambda = \{w \in W_G \mid w\lambda = \lambda\} = \begin{cases} \{w_1\} & \text{if } \lambda_1 > \lambda_2 > 0, \\ \{w_1, w_2\} & \text{if } \lambda_1 = \lambda_2 > 0, \\ \{w_1, w_5\} & \text{if } \lambda_1 > \lambda_2 = 0, \\ W_G & \text{if } \lambda_1 = \lambda_2 = 0, \end{cases}$$

for $\lambda = (\lambda_1, \lambda_2) \in P^+$, we see (2.5) yields (2.18). $\square$

Remark 2.6. Since the parameters are real in (2.14), we have $P_\lambda^{(a)} \in \mathbb{R}[P]$. Hence for $\rho \in \hat{A}_0$, the complex conjugate of $P_\lambda^{(a)}(\rho)$ is equal to $P_\lambda^{(a)}(\bar{\rho})$. On the other hand, since $\bar{\rho} = w_7\, \rho$ and $P_\lambda^{(a)}$ is W_G-invariant, we have $P_\lambda^{(a)}(\rho) = P_\lambda^{(a)}(\bar{\rho})$. It is similar for $P_\lambda^{(s)}$ and s_λ. Thus for $\rho \in \hat{A}_0$, we have

$$P_\lambda^{(a)}(\rho) = P_\lambda^{(a)}(\bar{\rho}) \in \mathbb{R}, \quad P_\lambda^{(s)}(\rho) = P_\lambda^{(s)}(\bar{\rho}) \in \mathbb{R}, \quad s_\lambda(\rho) = s_\lambda(\bar{\rho}) \in \mathbb{R}. \tag{2.19}$$

Let us write down formula (2.8) explicitly for $P_\lambda^{(a)}$ and $P_\lambda^{(s)}$.

Lemma 2.7. *For $\lambda = (\lambda_1, \lambda_2) \in P^+$, we have*

$$P_\lambda^{(a)} = s_\lambda - q^{-1}\, s_{(\lambda_1-2,\lambda_2)} - q^{-1}\, s_{(\lambda_1,\lambda_2-2)} + q^{-2}\, s_{(\lambda_1-2,\lambda_2-2)} \quad \text{if } \lambda_1 > \lambda_2, \tag{2.20}$$

$$P_\lambda^{(a)} = s_\lambda + q^{-1}\, s_{(\lambda_1-1,\lambda_1-1)} - q^{-1}\, s_{(\lambda_1,\lambda_1-2)} + q^{-2}\, s_{(\lambda_1-2,\lambda_1-2)} \quad \text{if } \lambda_1 = \lambda_2 \tag{2.21}$$

in the anisotropic case, and

$$\begin{aligned} P_\lambda^{(s)} = s_\lambda &+ q^{-1}\left(s_{(\lambda_1-2,\lambda_2)} + s_{(\lambda_1,\lambda_2-2)}\right) + q^{-2}\, s_{(\lambda_1-2,\lambda_2-2)} \\ &- 2q^{-1/2}\left(s_{(\lambda_1-1,\lambda_2)} + s_{(\lambda_1,\lambda_2-1)}\right) + 4q^{-1}\, s_{(\lambda_1-1,\lambda_2-1)} \\ &- 2q^{-3/2}\left(s_{(\lambda_1-2,\lambda_2-1)} + s_{(\lambda_1-1,\lambda_2-2)}\right) \quad \text{if } \lambda_1 > \lambda_2, \end{aligned} \tag{2.22}$$

$$\begin{aligned} P_\lambda^{(s)} = s_\lambda &+ 3q^{-1}\, s_{(\lambda_1-1,\lambda_1-1)} + q^{-2}\, s_{(\lambda_1-2,\lambda_1-2)} \\ &- 2q^{-1/2}\, s_{(\lambda_1,\lambda_1-1)} + q^{-1}\, s_{(\lambda_1,\lambda_1-2)} - 2q^{-3/2}\, s_{(\lambda_1-1,\lambda_1-2)} \quad \text{if } \lambda_1 = \lambda_2 \end{aligned} \tag{2.23}$$

in the split case. Here in (2.20), (2.21), (2.22) *and* (2.23), *we set*

$$s_\mu = 0 \quad \text{if} \quad \mu \notin P^+. \tag{2.24}$$

Proof. Since $t_\alpha^{(a)} = t_\alpha^{(s)} = 0$ for $\alpha = \epsilon_1 \pm \epsilon_2$ and $t_\alpha^{(a)} = 1$ for $\alpha = \epsilon_1, \epsilon_2$, we have $\varphi_X(t) = 0$ unless

$$X = \begin{cases} X_i \ (1 \le i \le 4) & \text{in the anisotropic case,} \\ X_j \ (1 \le j \le 9) & \text{in the split case,} \end{cases}$$

where

$$\begin{array}{lllll} X_1 = \emptyset, & X_2 = \{2\epsilon_1\}, & X_3 = \{2\epsilon_2\}, & X_4 = \{2\epsilon_1, 2\epsilon_2\}, & \\ X_5 = \{\epsilon_1\}, & X_6 = \{\epsilon_2\}, & X_7 = \{\epsilon_1, \epsilon_2\}, & X_8 = \{2\epsilon_1, \epsilon_2\}, & X_9 = \{\epsilon_1, 2\epsilon_2\}. \end{array}$$

Then we have

$$\varphi_{X_1}\left(t^{(a)}\right) = \varphi_{X_1}\left(t^{(s)}\right) = 1, \qquad \varphi_{X_2}\left(t^{(a)}\right) = \varphi_{X_3}\left(t^{(a)}\right) = -q^{-1},$$
$$\varphi_{X_2}\left(t^{(s)}\right) = \varphi_{X_3}\left(t^{(s)}\right) = q^{-1}, \qquad \varphi_{X_4}\left(t^{(a)}\right) = \varphi_{X_4}\left(t^{(s)}\right) = q^{-2},$$
$$\varphi_{X_5}\left(t^{(s)}\right) = \varphi_{X_6}\left(t^{(s)}\right) = -2q^{-1/2}, \qquad \varphi_{X_7}\left(t^{(s)}\right) = 4q^{-1},$$
$$\varphi_{X_8}\left(t^{(s)}\right) = \varphi_{X_9}\left(t^{(s)}\right) = -2q^{-3/2}.$$

For $\lambda = (\lambda_1, \lambda_2) \in P^+$, we have

$$s_{\lambda-\sigma(X_2)} = \begin{cases} s_{(\lambda_1-2,\lambda_2)} & \text{if } \lambda_1 - 2 \geq \lambda_2, \\ -s_{(\lambda_1-1,\lambda_1-1)} & \text{if } \lambda_1 = \lambda_2 \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

since

$$(\lambda_1 - 2, \lambda_2) + \rho = \begin{cases} (\lambda_1, \lambda_1) \notin W_G\ (\rho + P^+) & \text{if } \lambda_1 - 1 = \lambda_2, \\ w_2\left(\rho + (\lambda_1 - 1, \lambda_1 - 1)\right) & \text{if } \lambda_1 = \lambda_2 \geq 1, \\ (0,1) \notin W_G\ (\rho + P^+) & \text{if } \lambda_1 = \lambda_2 = 0, \end{cases}$$

where $\rho = \frac{1}{2}\sum_{\alpha \in R_2^+} \alpha = (2,1)$. Similarly for $\lambda = (\lambda_1, \lambda_2) \in P^+$, we have

$$s_{\lambda-\sigma(X_3)} = \begin{cases} s_{(\lambda_1,\lambda_2-2)} & \text{if } \lambda_2 \geq 2, \\ -s_{(\lambda_1,0)} & \text{if } \lambda_2 = 0, \\ 0 & \text{otherwise,} \end{cases} \quad s_{\lambda-\sigma(X_4)} = \begin{cases} s_{(\lambda_1-2,\lambda_2-2)} & \text{if } \lambda_2 \geq 2, \\ -s_{(\lambda_1-2,0)} & \text{if } \lambda_1 \geq 2,\ \lambda_2 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$s_{\lambda-\sigma(X_5)} = \begin{cases} s_{(\lambda_1-1,\lambda_2)} & \text{if } \lambda_1 - 1 \geq \lambda_2, \\ 0 & \text{otherwise,} \end{cases} \qquad s_{\lambda-\sigma(X_6)} = \begin{cases} s_{(\lambda_1,\lambda_2-1)} & \text{if } \lambda_2 \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$s_{\lambda-\sigma(X_7)} = \begin{cases} s_{(\lambda_1-1,\lambda_2-1)} & \text{if } \lambda_2 \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$s_{\lambda-\sigma(X_8)} = \begin{cases} s_{(\lambda_1-2,\lambda_2-1)} & \text{if } \lambda_1 - 1 \geq \lambda_2 \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$s_{\lambda-\sigma(X_9)} = \begin{cases} s_{(\lambda_1-1,\lambda_2-2)} & \text{if } \lambda_2 \geq 2, \\ -s_{(\lambda_1-1,0)} & \text{if } \lambda_1 \geq 1,\ \lambda_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the lemma follows. □

Corollary 2.8. *Let $\lambda = (\lambda_1, \lambda_2) \in P^+$.*

(1) *We put $m = \left[\frac{\lambda_1-\lambda_2}{2}\right]$ and $n = \left[\frac{\lambda_2}{2}\right]$. Here $[x]$ denotes the greatest integer less than or equal to x.*
Then we have

$$s_\lambda = \sum_{i=0}^{m}\sum_{j=0}^{n} q^{-i-j}\, P^{(a)}_{(\lambda_1-2i,\lambda_2-2j)} \quad \textit{if } \lambda_1 - \lambda_2 \textit{ is odd,} \tag{2.25}$$

and

$$s_\lambda = \sum_{i=0}^{m-1} \sum_{j=0}^{n} q^{-i-j} P^{(a)}_{(\lambda_1-2i,\lambda_2-2j)} + q^{-m} \sum_{k=0}^{\lambda_2} (-1)^k q^{-k} \sum_{l=0}^{\left[\frac{\lambda_2-k}{2}\right]} q^{-l} P^{(a)}_{(\lambda_2-k,\lambda_2-k-2l)} \quad \text{if } \lambda_1-\lambda_2 \text{ is even.} \tag{2.26}$$

(2) *We have*

$$s_\lambda = \sum_{i=0}^{\lambda_1-\lambda_2-1} \sum_{j=0}^{\lambda_2} (i+1)(j+1) q^{-\frac{i+j}{2}} P^{(s)}_{(\lambda_1-i,\lambda_2-j)} + (\lambda_1-\lambda_2+1) q^{-\frac{\lambda_1-\lambda_2}{2}} \sum_{k=0}^{\lambda_2} \sum_{l=0}^{\lambda_2-k} (l+1) q^{-k-\frac{l}{2}} P^{(s)}_{(\lambda_2-k,\lambda_2-k-l)}. \tag{2.27}$$

PROOF. Let us consider the anisotropic case first. For $\lambda = (\lambda_1,\lambda_2) \in P^+$, we define t_λ by

$$t_\lambda = s_{(\lambda_1,\lambda_2)} - q^{-1} s_{(\lambda_1-2,\lambda_2)}.$$

We set $t_\lambda = 0$ if $\lambda \notin P^+$. Then we have

$$s_\lambda = \sum_{i=0}^{m} q^{-i} t_{(\lambda_1-2i,\lambda_2)}. \tag{2.28}$$

On the other hand, by (2.20) and (2.21) we have

$$t_\lambda - q^{-1} t_{(\lambda_1,\lambda_2-2)} = \begin{cases} P^{(a)}_\lambda, & \text{if } \lambda_1 > \lambda_2, \\ P^{(a)}_\lambda - q^{-1} s_{(\lambda_1-1,\lambda_1-1)}, & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Hence

$$t_\lambda = \sum_{j=0}^{n} q^{-j} P^{(a)}_{(\lambda_1,\lambda_2-2j)} - \begin{cases} 0, & \text{if } \lambda_1 > \lambda_2, \\ q^{-1} s_{(\lambda_1-1,\lambda_1-1)}, & \text{if } \lambda_1 = \lambda_2. \end{cases} \tag{2.29}$$

Thus (2.25) holds when $\lambda_1 - \lambda_2$ is odd. Suppose that $\lambda_1 - \lambda_2$ is even. Then we have $\lambda_1 - \lambda_2 = 2m$ and the equality (2.28) reads

$$s_\lambda = \sum_{i=0}^{m-1} q^{-i} t_{(\lambda_1-2i,\lambda_2)} + q^{-m} s_{(\lambda_2,\lambda_2)}.$$

By (2.29) we have

$$s_{(\lambda_2,\lambda_2)} + q^{-1} s_{(\lambda_2-1,\lambda_2-1)} = \sum_{j=0}^{\left[\frac{\lambda_2}{2}\right]} q^{-j} P^{(a)}_{(\lambda_2,\lambda_2-2j)}.$$

Hence

$$s_{(\lambda_2,\lambda_2)} = \sum_{k=0}^{\lambda_2} (-1)^k q^{-k} \sum_{l=0}^{\left[\frac{\lambda_2-k}{2}\right]} q^{-l} P^{(a)}_{(\lambda_2-k,\lambda_2-k-2l)}.$$

Thus (2.26) holds.

Let us consider the split case. For $\lambda = (\lambda_1,\lambda_2) \in P^+$, let

$$u_\lambda = s_\lambda - q^{-\frac{1}{2}} s_{(\lambda_1-1,\lambda_2)}, \quad v_\lambda = u_\lambda - q^{-\frac{1}{2}} u_{(\lambda_1-1,\lambda_2)}, \quad w_\lambda = v_\lambda - q^{-\frac{1}{2}} v_{(\lambda_1,\lambda_2-1)}$$

and we set $u_\lambda = v_\lambda = w_\lambda = 0$ if $\lambda \notin P^+$. Thus we have

$$\begin{aligned} s_\lambda &= \sum_{i=0}^{\lambda_1-\lambda_2} q^{-\frac{i}{2}} u_{(\lambda_1-i,\lambda_2)} = \sum_{i=0}^{\lambda_1-\lambda_2} \left(\sum_{j=0}^{\lambda_1-\lambda_2-i} q^{-\frac{i+j}{2}} v_{(\lambda_1-i-j,\lambda_2)} \right) \\ &= (\lambda_1-\lambda_2+1)\, q^{-\frac{\lambda_1-\lambda_2}{2}} s_{(\lambda_2,\lambda_2)} + \sum_{i=0}^{\lambda_1-\lambda_2-1} (i+1)\, q^{-\frac{i}{2}} v_{(\lambda_1-i,\lambda_2)} \end{aligned}$$

since $v_{(\lambda_2,\lambda_2)} = s_{(\lambda_2,\lambda_2)}$. We may rewrite (2.22) and (2.23) as

$$w_\lambda - q^{-\frac{1}{2}} w_{(\lambda_1,\lambda_2-1)} = \begin{cases} P_\lambda^{(s)} & \text{if } \lambda_1 > \lambda_2, \\ P_\lambda^{(s)} + q^{-1} s_{(\lambda_1-1,\lambda_1-1)} & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Thus

$$w_\lambda = \sum_{i=0}^{\lambda_2} q^{-\frac{i}{2}} P^{(s)}_{(\lambda_1,\lambda_2-i)} + \begin{cases} 0 & \text{if } \lambda_1 > \lambda_2, \\ q^{-1} s_{(\lambda_1-1,\lambda_1-1)} & \text{if } \lambda_1 = \lambda_2 \end{cases}$$

and hence

$$v_\lambda = \sum_{j=0}^{\lambda_2} (j+1)\, q^{-\frac{j}{2}} P^{(s)}_{(\lambda_1,\lambda_2-j)} + \begin{cases} 0 & \text{if } \lambda_1 > \lambda_2, \\ q^{-1} s_{(\lambda_1-1,\lambda_1-1)} & \text{if } \lambda_1 = \lambda_2. \end{cases} \tag{2.30}$$

When $\lambda_1 = \lambda_2$, the equality (2.30) reads

$$s_{(\lambda_2,\lambda_2)} - q^{-1} s_{(\lambda_2-1,\lambda_2-1)} = \sum_{j=0}^{\lambda_2} (j+1)\, q^{-\frac{j}{2}} P^{(s)}_{(\lambda_2,\lambda_2-j)}.$$

Hence

$$s_{(\lambda_2,\lambda_2)} = \sum_{k=0}^{\lambda_2} \sum_{l=0}^{\lambda_2-k} (l+1)\, q^{-k-\frac{l}{2}} P^{(s)}_{(\lambda_2-k,\lambda_2-k-l)}.$$

Thus (2.27) holds. □

2.2.3. Inversion formulas.

2.2.3.1. *Bessel case.* Let $\mathcal{B}^{(a)}$ (resp. $\mathcal{B}^{(s)}$) denote the space of $\mathbb{C}$-valued functions h on G satisfying the following two conditions:

(2.31) $h(rgk) = \tau^{(a)}(r) \cdot h(g)$ for $r \in R^{(a)}$, $g \in G$ and $k \in K$

$\Big($resp. $h(rgk) = \tau^{(s)}(r) \cdot h(g)$ for $r \in R^{(s)}$, $g \in G$ and $k \in K\Big)$; and

(2.32) $|h|$ is compactly supported on $R^{(a)}\backslash G$ (resp. $R^{(s)}\backslash G$).

For $\rho \in \hat{A}_0$, we write $\tilde{\rho}$ for $(\chi_\delta\,|_A) \cdot \rho \in \hat{A}$. We recall that χ_δ is the character of G defined by (2.11). Then for $h \in \mathcal{B}^{(a)}$ (resp. $h \in \mathcal{B}^{(s)}$), we define its Fourier transform $\hat{h} : \hat{A}_0 \to \mathbb{C}$ by

$$\hat{h}(\rho) = \int_{R^{(a)}\backslash G} h(g)\, \overline{B^{(a)}_{\tilde{\rho}}(g)}\, dg \quad \left(\text{resp. } \int_{R^{(s)}\backslash G} h(g)\, \overline{B^{(s)}_{\tilde{\rho}}(g)}\, dg\right)$$

where the measure is normalized so that $R^{(a)}\backslash R^{(a)}K$ (resp. $R^{(s)}\backslash R^{(s)}K$) has unit volume.

Then the following inversion formulas hold.

PROPOSITION 2.9. *For $h \in \mathcal{B}^{(a)}$, we have the inversion formula*

$$h(g) = \int_{\hat{A}_0} \hat{h}(\rho)\, B^{(a)}_{\tilde{\rho}}(g)\, d\nu^{(a)}. \tag{2.33}$$

Here the Plancherel measure $d\nu^{(a)}$ is given by

$$d\nu^{(a)} = \frac{Q^{(a)}\left(q^{-1}\right)}{|W_G|} \cdot \frac{d\rho}{C^{(a)}(\rho)\, C^{(a)}(\bar{\rho})}$$

where $d\rho$ is the Haar measure on $\hat{A}_0$ such that $\int_{\hat{A}_0} d\rho = 1$.

Similarly for $h \in \mathcal{B}^{(s)}$, we have

$$h(g) = \int_{\hat{A}_0} \hat{h}(\rho)\, B^{(s)}_{\tilde{\rho}}(g)\, d\nu^{(s)}, \tag{2.34}$$

where

$$d\nu^{(s)} = \frac{Q^{(s)}\left(q^{-1}\right)}{|W_G|} \cdot \frac{d\rho}{C^{(s)}(\rho)\, C^{(s)}(\bar{\rho})}.$$

PROOF. Since the proofs are identical, here we prove (2.33) only. As a basis for $\mathcal{B}^{(a)}$, we take h_λ ($\lambda \in P^+$) defined by

$$h_\lambda(g) = \begin{cases} \tau^{(a)}(r) & \text{if } g = r b^{(a)}_\lambda k \text{ where } r \in R^{(a)} \text{ and } k \in K, \\ 0 & \text{otherwise.} \end{cases} \tag{2.35}$$

Then we have

$$\hat{h}_\lambda(\rho) = \left(\int_{R^{(a)} \backslash R^{(a)} b^{(a)}_\lambda K} dg \right) \cdot \overline{B^{(a)}_{\tilde{\rho}}\left(b^{(a)}_\lambda\right)}.$$

Hence it is enough for us to show that, for $\lambda, \lambda' \in P^+$, we have

$$\int_{\hat{A}_0} B^{(a)}_{\tilde{\rho}}\left(b^{(a)}_{\lambda'}\right) \overline{B^{(a)}_{\tilde{\rho}}\left(b^{(a)}_\lambda\right)}\, d\nu^{(a)} = \begin{cases} \left(\int_{R^{(a)} \backslash R^{(a)} b^{(a)}_\lambda K} dg \right)^{-1} & \text{if } \lambda = \lambda', \\ 0 & \text{if } \lambda \neq \lambda'. \end{cases} \tag{2.36}$$

By Theorem 2.2 and (2.7), the left hand side of (2.36) is equal to

$$\frac{\chi_\delta\left(\varpi^{\lambda' - \lambda}\right) \cdot \delta_B\left(\varpi^{\lambda' + \lambda}\right)^{1/2} \cdot W_{\lambda'}\left(t^{(a)}\right) \cdot W_\lambda\left(t^{(a)}\right)}{Q^{(a)}\left(q^{-1}\right)} \langle P^{(a)}_{\lambda'}, P^{(a)}_\lambda \rangle$$
$$= \begin{cases} \dfrac{\delta_B\left(\varpi^\lambda\right) \cdot W_\lambda\left(t^{(a)}\right)}{Q^{(a)}\left(q^{-1}\right)} & \text{if } \lambda = \lambda', \\ 0 & \text{if } \lambda \neq \lambda'. \end{cases}$$

On the other hand it is easily seen (e.g. [**5**, (3.5.3) Lemma]) that

$$\int_{R^{(a)} \backslash R^{(a)} b^{(a)}_\lambda K} dg = \delta_B\left(\varpi^\lambda\right)^{-1} \cdot Q^{(a)}\left(q^{-1}\right)^{e(\lambda)} \tag{2.37}$$

for $\lambda = (\lambda_1, \lambda_2) \in P^+$, where

$$e(\lambda) = \begin{cases} 1 & \text{if } \lambda_2 > 0, \\ 0 & \text{if } \lambda_2 = 0. \end{cases}$$

Hence (2.36) follows from (2.18). $\square$

2.2.3.2. *Whittaker case.* Let $\mathcal{W}_\omega$ denote the space of functions $h : G \to \mathbb{C}$ satisfying:

$$h(zngk) = \omega(z)\,\psi(n)\,h(g) \quad \text{for } z \in Z,\ n \in N,\ g \in G \text{ and } k \in K; \text{ and} \tag{2.38}$$

$$|h| \text{ is compactly supported on } ZN\backslash G. \tag{2.39}$$

Then by an argument identical to the one in the Bessel case, we have the following inversion formula.

Proposition 2.10. *For $h \in \mathcal{W}_\omega$, we have*

$$h(g) = \int_{\hat{A}_0} \hat{h}(\rho)\; W_{\tilde{\rho}}(g)\; d\nu. \tag{2.40}$$

Here the Fourier transform $\hat{h} : \hat{A}_0 \to \mathbb{C}$ is defined by

$$\hat{h}(\rho) = \int_{ZN\backslash G} h(g)\; \overline{W_{\tilde{\rho}}(g)}\, dg$$

where the measure is normalized so that $\int_{ZN\backslash ZNK} dg = 1$ and the Plancherel measure $d\nu$ is given by

$$d\nu = \frac{1}{|W_G|} \cdot \frac{d\rho}{C(\rho)\; C(\bar{\rho})}.$$

2.3. Reduction Formulas for the Orbital Integrals

We keep the notation of Section 2.2, so $G = \mathrm{GSp}(4)$.

2.3.1. Satake isomorphism. Let $\mathcal{H}^{(A)}$ denote the Hecke algebra of A, i.e. $\mathcal{H}^{(A)}$ is the space of compactly supported $\mathbb{C}$-valued bi-$A(\mathcal{O})$-invariant functions on A, with the convolution product defined for $h_1, h_2 \in \mathcal{H}^{(A)}$ by

$$(h_1 * h_2)(x) = \int_A h_1\left(xa^{-1}\right) h_2(a)\, da$$

where the Haar measure da on A is normalized so that $\int_{A(\mathcal{O})} da = 1$. Then $\mathcal{H}^{(A)}$ is isomorphic to the group algebra $\mathbb{C}[\mathbf{\Lambda}]$, where $\mathbf{\Lambda}$ is the group of co-characters of A, by identifying $\lambda \in \mathbf{\Lambda}$ with the characteristic function e^λ of the coset $\varpi^\lambda A(\mathcal{O})$.

For $f \in \mathcal{H}$, its Satake transform $Sf : A \to \mathbb{C}$ is defined by

$$Sf(a) = \delta_B(a)^{1/2} \int_N f(an)\, dn$$

where the Haar measure dn on N is normalized so that $\int_{N(\mathcal{O})} dn = 1$. For $\rho \in \hat{A}$, we define $\omega_\rho : \mathcal{H} \to \mathbb{C}$ by

$$\omega_\rho(f) = \int_A Sf(a)\; \rho(a)\, da.$$

Here we note that

$$\omega_\rho(f) = \sum_{\lambda \in \mathbf{\Lambda}} a_\lambda\, \rho\left(\varpi^\lambda\right) \quad \text{when} \quad Sf = \sum_{\lambda \in \mathbf{\Lambda}} a_\lambda\, e^\lambda,$$

i.e. $Sf \mapsto \omega_\rho(f)$ is evaluation at ρ.

The following theorem is fundamental.

THEOREM 2.11 (Satake isomorphism [**25**]). *The Satake transform is an algebra isomorphism from $\mathcal{H}$ onto the subalgebra $\mathbb{C}[\mathbf{\Lambda}]^{W_G}$ of $\mathbb{C}[\mathbf{\Lambda}]$ consisting of the elements invariant by the Weyl group W_G.*

Let $G_0 = \mathrm{PGSp}_4(F) = G/Z$ and let $\mathcal{H}_0$ be the Hecke algebra of G_0. Let $\mathbf{\Lambda}_0$ denote the group of co-characters of $A_0 = A/Z$. Then the natural homomorphism $\mathbf{\Lambda} \to \mathbf{\Lambda}_0$ induces a surjective algebra homomorphism $\phi : \mathcal{H} \to \mathcal{H}_0$. The following lemma is immediate from the definition of ϕ.

LEMMA 2.12. *For $f \in \mathcal{H}$ and $\rho \in \hat{A}_0$, we have*

$$\omega_\rho(f) = \omega_\rho(\phi(f)).$$

2.3.2. Reduction formulas.

PROPOSITION 2.13 (Reduction formula for the anisotropic Bessel orbital integral). *For $f \in \mathcal{H}$, $\mu \in F^\times$, and $u \in E^\times$ such that $\mathrm{N}_{E/F}(u) \neq 1$, we have*

$$\begin{aligned}(2.41)\quad &\mathcal{B}^{(a)}(u,\mu;f)\\ &= \sum_{\lambda \in P^+} \delta_B\left(\varpi^\lambda\right)^{-1}\left(1+q^{-1}\right)^{e(\lambda)} \mathcal{B}^{(a)}(\lambda;u,\mu)\int_{\hat{A}_0} \omega_\rho\left(\phi\left(f\cdot\chi_\delta\right)\right) B_{\tilde{\rho}}^{(a)}\left(b_\lambda^{(a)}\right) d\nu^{(a)}\end{aligned}$$

where

$$(2.42)\quad \mathcal{B}^{(a)}(\lambda;u,\mu) = \int_{Z\backslash\bar{R}^{(a)}}\int_{R^{(a)}} \Xi\left(\bar{r}\,A^{(a)}(u,\mu)\,r\,b_\lambda^{(a)}\right)\xi^{(a)}(\bar{r})\,\tau^{(a)}(r)\,dr\,d\bar{r}.$$

Before proving the proposition, we prove the following lemma.

LEMMA 2.14. *For $f \in \mathcal{H}$, Let $\Psi_f^{(a)} : G \to \mathbb{C}$ be the Bessel transform of f defined by*

$$\Psi_f^{(a)}(g) = \int_{R^{(a)}} f(gr)\,\tau^{(a)}(r)\,dr.$$

Then we have

$$\Psi_f^{(a)}(g) = \int_{\hat{A}_0} \omega_\rho\left(\phi\left(f\cdot\chi_\delta\right)\right)\, B_{\tilde{\rho}}^{(a)}\left(g^{-1}\right)\, d\nu^{(a)}.$$

PROOF. Let us define $h_f : G \to \mathbb{C}$ by $h_f(g) = \Psi_f^{(a)}\left(g^{-1}\right)$. Since $h_f \in \mathcal{B}^{(a)}$,

$$\Psi_f^{(a)}(g) = h_f\left(g^{-1}\right) = \int_{\hat{A}_0} \hat{h}_f(\rho)\, B_{\tilde{\rho}}^{(a)}\left(g^{-1}\right) d\nu^{(a)}$$

by Proposition 2.9. Here

$$\begin{aligned}\hat{h}_f(\rho) &= \int_{R^{(a)}\backslash G} \Psi_f^{(a)}\left(g^{-1}\right)\,\overline{B_{\tilde{\rho}}^{(a)}(g)}\,dg = \int_G f\left(g^{-1}\right)\,\overline{B_{\tilde{\rho}}^{(a)}(g)}\,dg\\ &= \int_K\int_G f\left(g^{-1}k\right)\,\overline{B_{\tilde{\rho}}^{(a)}(g)}\,dg\,dk = \int_G f(g)\left(\int_K \overline{B_{\tilde{\rho}}^{(a)}\left(kg^{-1}\right)}\,dk\right)dg.\end{aligned}$$

We note that

$$\int_K \overline{B_{\tilde{\rho}}^{(a)}\left(kg^{-1}\right)}\,dk = \overline{\Gamma_{\tilde{\rho}}\left(g^{-1}\right)} = \Gamma_{\tilde{\rho}}(g),$$

where Γ_χ denotes the zonal spherical function on G associated to $\chi \in \hat{A}$. Hence

$$\hat{h}_f(\rho) = \int_G f(g)\, \Gamma_{\tilde{\rho}}(g)\, dg = \int_G f(g)\, \chi_\delta(g)\, \Gamma_\rho(g)\, dg = \omega_\rho(f \cdot \chi_\delta).$$

Since $\rho \in \hat{A}_0$, we have $\omega_\rho(f \cdot \chi_\delta) = \omega_\rho(\phi(f \cdot \chi_\delta))$, and the lemma holds. $\square$

Proof of Proposition 2.13. We may write (1.7) as

$$\mathcal{B}^{(a)}(u, \mu; f) = \int_{Z \backslash \bar{R}^{(a)}} \Psi_f^{(a)}\left(\bar{r}\, A^{(a)}(u, \mu)\right) \xi^{(a)}(\bar{r})\, d\bar{r}.$$

Let us write $h_f = \sum_{\lambda \in P^+} a_\lambda\, h_\lambda$ where h_λ is defined by (2.35). Then we have

$$a_\lambda = h_f\left(b_\lambda^{(a)}\right) = \int_{\hat{A}_0} \omega_\rho(\phi(f \cdot \chi_\delta))\; B_{\tilde{\rho}}^{(a)}\left(b_\lambda^{(a)}\right) d\nu^{(a)}$$

by Lemma 2.14 and

$$\mathcal{B}^{(a)}(u, \mu; f) = \sum_{\lambda \in P^+} a_\lambda \int_{Z \backslash \bar{R}^{(a)}} h_\lambda\left(A^{(a)}(u, \mu)^{-1}\, \bar{r}^{-1}\right) \xi^{(a)}(\bar{r})\, d\bar{r}.$$

For $r \in R^{(a)}$, we have

$$A^{(a)}(u, \mu)^{-1}\, \bar{r}^{-1} \in r\, b_\lambda^{(a)}\, K \Longleftrightarrow \bar{r}\, A^{(a)}(u, \mu)\; r\, b_\lambda^{(a)} \in K.$$

When this holds, for $r' \in R^{(a)}$, we have

$$\bar{r}\, A^{(a)}(u, \mu)\; r'\, b_\lambda^{(a)} \in K \Longleftrightarrow r^{-1} r' \in R^{(a)} \cap b_\lambda^{(a)}\, K\, (b_\lambda^{(a)})^{-1}.$$

Here we note that

$$\left(\int_{R^{(a)} \cap b_\lambda^{(a)}\, K\, (b_\lambda^{(a)})^{-1}} dr\right)^{-1} = \delta_B\left(\varpi^\lambda\right)^{-1}\, \left(1 + q^{-1}\right)^{e(\lambda)}$$

by (2.37). Thus we have

$$\begin{aligned} &h_\lambda\left(A^{(a)}(u, \mu)^{-1}\, \bar{r}^{-1}\right) \\ &\qquad = \delta_B\left(\varpi^\lambda\right)^{-1}\, \left(1 + q^{-1}\right)^{e(\lambda)} \int_{R^{(a)}} \Xi\left(\bar{r}\, A^{(a)}(u, \mu)\; r\, b_\lambda^{(a)}\right) \tau^{(a)}(r)\, dr, \end{aligned}$$

and the proposition holds. $\square$

Similarly we have the following two propositions.

Proposition 2.15 (Reduction formula for the split Bessel orbital integral). *For $f \in \mathcal{H}$, $x \in F \setminus \{0, 1\}$, and $\mu \in F^\times$, we have*

$$\begin{aligned} &\mathcal{B}^{(s)}(x, \mu; f) \qquad (2.43) \\ &= \sum_{\lambda \in P^+} \delta_B\left(\varpi^\lambda\right)^{-1} \left(1 - q^{-1}\right)^{e(\lambda)} \mathcal{B}^{(s)}(\lambda; x, \mu) \int_{\hat{A}_0} \omega_\rho(\phi(f \cdot \chi_\delta))\, B_{\tilde{\rho}}^{(s)}\left(b_\lambda^{(s)}\right) d\nu^{(s)} \end{aligned}$$

where

$$\mathcal{B}^{(s)}(\lambda; x, \mu) = \int_{Z \backslash \bar{R}^{(s)}} \int_{R^{(s)}} \Xi\left(\bar{r}\, A^{(s)}(x, \mu)\; r\, b_\lambda^{(s)}\right) \xi^{(s)}(\bar{r})\, \tau^{(s)}(r)\, dr\, d\bar{r}. \tag{2.44}$$

Proposition 2.16 (Reduction formula for the Rankin-Selberg type orbital integral). *For $f \in \mathcal{H}$, $s \in F^\times$ and $a \in F \setminus \{0,1\}$, we have*

$$I(s,a;f) = \sum_{\lambda \in P^+} \delta_B\left(\varpi^\lambda\right)^{-1} I(\lambda; s, a) \int_{\hat{A}_0} \omega_\rho\left(\phi\left(f \cdot \chi_\delta\right)\right) W_{\tilde{\rho}}\left(\varpi^\lambda\right) d\nu \tag{2.45}$$

where

$$I(\lambda; s, a) = \int_{H_0 \backslash H} \int_N \int_Z \Xi\left(h^{-1} \bar{n}^{(s)}\, z\, n\, \varpi^\lambda\right) W_{s,a}(h)\, \omega(z)\, \psi(n)\, dz\, dn\, dh. \tag{2.46}$$

2.3.3. Restating the theorems. For $x \in F \setminus \{0,1\}$, $\mu \in F^\times$ and $\lambda \in P^+$, we define $\mathcal{I}(\lambda; x, \mu)$ by

$$\mathcal{I}(\lambda; x, \mu) = I(\lambda; s, a), \quad \text{where } s = -\frac{1-x}{4\mu},\ a = \frac{1}{1-x}.$$

For $\lambda, \lambda' \in P^+$, let $k^{(a)}_{\lambda\,\lambda'}$ (resp. $k^{(s)}_{\lambda\,\lambda'}$) denote the generalized Kostka number defined by

$$s_\lambda = \sum_{\lambda' \in P^+} k^{(a)}_{\lambda\,\lambda'}\, P^{(a)}_{\lambda'} = \sum_{\lambda' \in P^+} k^{(s)}_{\lambda\,\lambda'}\, P^{(s)}_{\lambda'}. \tag{2.47}$$

We have computed $k^{(a)}_{\lambda\,\lambda'}$ and $k^{(s)}_{\lambda\,\lambda'}$ in Corollary 2.8 explicitly. Since $\{s_\lambda\}_{\lambda \in P^+}$ is a basis for $\mathcal{H}_0$, it is enough to prove (1.13) and (1.14) for $f \in \mathcal{H}$ such that $\phi(f \cdot \chi_\delta) = s_\lambda$. Thus we may restate Theorem 1.6 and Theorem 1.7 as follows, respectively.

Theorem 2.17 (Matching when E/F is inert). *For $x \in F \setminus \{0,1\}$, $\mu \in F^\times$ and $\lambda \in P^+$, the orbital integral $\mathcal{I}(\lambda; x, \mu)$ vanishes unless* ord(x) *is even.*

When $x = \mathrm{N}_{E/F}(u)$ for $u \in E^\times$, we have

(2.48)

$$\delta^{-1}\left(\frac{x}{\mu^2}\right) \left|\frac{x}{\mu^2}\right|^{\frac{1}{2}} \sum_{\lambda' \in P^+} k^{(a)}_{\lambda\,\lambda'} \left(1+q^{-1}\right)^{e(\lambda')} \delta_B\left(\varpi^{\lambda'}\right)^{-\frac{1}{2}} \chi_\delta\left(\varpi^{\lambda'}\right) \mathcal{B}^{(a)}(\lambda'; u, \mu)$$
$$= \delta_B\left(\varpi^\lambda\right)^{-\frac{1}{2}} \chi_\delta\left(\varpi^\lambda\right) \mathcal{I}(\lambda; x, \mu).$$

Theorem 2.18 (Matching when E/F is split). *For $x \in F \setminus \{0,1\}$, $\mu \in F^\times$ and $\lambda \in P^+$, we have*

(2.49)

$$\delta^{-1}\left(\frac{x}{\mu^2}\right) \left|\frac{x}{\mu^2}\right|^{\frac{1}{2}} \sum_{\lambda' \in P^+} k^{(s)}_{\lambda\,\lambda'} \left(1-q^{-1}\right)^{e(\lambda')} \delta_B\left(\varpi^{\lambda'}\right)^{-\frac{1}{2}} \chi_\delta\left(\varpi^{\lambda'}\right) \mathcal{B}^{(s)}(\lambda'; x, \mu)$$
$$= \delta_B\left(\varpi^\lambda\right)^{-\frac{1}{2}} \chi_\delta\left(\varpi^\lambda\right) \mathcal{I}(\lambda; x, \mu).$$

CHAPTER 3

Evaluation of the Anisotropic Bessel Orbital Integral

In the first section we shall recall some facts concerning Gauss sums, Kloosterman sums, Salié sums and matrix argument Kloosterman sums. Then we prove the functional equation (3.20) for the anisotropic Bessel orbital integral. In the second section, we shall explicitly evaluate the degenerate anisotropic Bessel orbital integral defined by (2.42).

Throughout this chapter, E denotes the unique unramified quadratic extension of F and σ denotes the unique non-trivial element of the Galois group of E over F.

3.1. Preliminaries

3.1.1. Gauss sum, Kloosterman sum and Salié sum. We refer to [**8**, Chapter 2] for the proofs.

3.1.1.1. *Gauss sum.* Let $\psi^{(1)}$ be the additive character of $\mathbb{F}_q = \mathcal{O}/\varpi\mathcal{O}$ defined by $\psi^{(1)}(\bar{x}) = \psi\left(\varpi^{-1}x\right)$ where $\mathcal{O} \ni x \mapsto \bar{x} \in \mathcal{O}/\varpi\mathcal{O}$ denotes the natural homomorphism.

Definition 3.1. We define a character $\operatorname{sgn} : \mathbb{F}_q^\times \to \mathbb{C}^\times$ by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{when } x \in \left(\mathbb{F}_q^\times\right)^2, \\ -1 & \text{when } x \notin \left(\mathbb{F}_q^\times\right)^2. \end{cases}$$

We also denote by sgn a character of $\mathcal{O}_F^\times$ defined by $\operatorname{sgn}(x) = \operatorname{sgn}(\bar{x})$ for $x \in \mathcal{O}_F^\times$.

Then the *Gauss sum* $\mathfrak{G}(\operatorname{sgn})$ attached to the character sgn is defined by

$$\mathfrak{G}(\operatorname{sgn}) = \sum_{y \in \mathbb{F}_q^\times} \operatorname{sgn}(y) \cdot \psi^{(1)}(y)$$

and for $a \in F^\times$ with $\operatorname{ord}(a) = -n < 0$, we define $C(a) \in \mathbb{C}^\times$ by

$$C(a) = |a|^{\frac{1}{2}} \cdot \begin{cases} 1 & \text{when } n \text{ is even,} \\ q^{-\frac{1}{2}} \operatorname{sgn}(\varpi^n a)\, \mathfrak{G}(\operatorname{sgn}) & \text{when } n \text{ is odd.} \end{cases}$$

For a positive integer n, let us simply write C_n for $C(\varpi^{-n})$.

Here we note that since $\mathfrak{G}(\operatorname{sgn})^2 = q \cdot \operatorname{sgn}(-1)$, we have

$$C_m\, C_n = q^{-\frac{m+n}{2}} \operatorname{sgn}(-1)^m \quad \text{for } m, n > 0 \text{ with } m \equiv n \pmod 2. \tag{3.1}$$

Proposition 3.2. *For $a, b \in F$, let $\mathcal{G}(a, b)$ be the Gaussian integral defined by*

$$\mathcal{G}(a, b) = \int_{\mathcal{O}} \psi\left(ax^2 + 2bx\right) dx$$

where dx denotes the Haar measure on F normalized so that $\displaystyle\int_{\mathcal{O}} dx = 1$.

Then the following assertions hold.

(1) *When $|a| \le 1$, we have* $\mathcal{G}(a,b) = \begin{cases} 1 & \text{when } |b| \le 1, \\ 0 & \text{when } |b| > 1. \end{cases}$
(2) *When $1 < |a| < |b|$, we have $\mathcal{G}(a,b) = 0$.*
(3) *When $|a| > 1$ and $|a| \ge |b|$, we have $\mathcal{G}(a,b) = C(a) \cdot \psi\left(-a^{-1}b^2\right)$.*

3.1.1.2. *Kloosterman sum and Salié sum.*

Definition 3.3. For $r, s \in F^\times$, we define the *Kloosterman sum* $\mathcal{K}l\,(r,s)$ by

$$\mathcal{K}l\,(r,s) = \int_{\mathcal{O}_F^\times} \psi\left(r\varepsilon + s\varepsilon^{-1}\right) d\varepsilon \tag{3.2}$$

and the *Salié sum* $\mathcal{S}(r,s)$ by

$$\mathcal{S}(r,s) = \int_{\mathcal{O}^\times} \operatorname{sgn}(\varepsilon) \cdot \psi\left(r\varepsilon + s\varepsilon^{-1}\right) d\varepsilon. \tag{3.3}$$

Here $d\varepsilon$ denotes the restriction of the normalized Haar measure on F. Note $d\varepsilon$ restricts to the multiplicative Haar measure on $\mathcal{O}^\times$ such that $\displaystyle\int_{\mathcal{O}^\times} d\varepsilon = 1 - q^{-1}$.

It is clear from the definition that we have

$$\mathcal{K}l\,(r,s) = \mathcal{K}l\,(s,r), \quad \mathcal{S}(r,s) = \mathcal{S}(s,r)$$

and

$$\mathcal{K}l\,(r\varepsilon', s) = \mathcal{K}l\,(r, s\varepsilon'), \quad \mathcal{S}(r\varepsilon', s) = \operatorname{sgn}(\varepsilon') \cdot \mathcal{S}(r, s\varepsilon') \quad \text{for } \varepsilon' \in \mathcal{O}^\times.$$

Proposition 3.4. *The Kloosterman sum $\mathcal{K}l\,(r,s)$ and the Salié sum $\mathcal{S}(r,s)$ are evaluated explicitly as follows, excluding the case of $\mathcal{K}l\,(r,s)$ when $|r| = |s| = q$.*

(1) *Suppose that $|r| > |s|$.*
   (a) *When $|r| \le 1$, we have $\mathcal{K}l\,(r,s) = 1 - q^{-1}$ and $\mathcal{S}(r,s) = 0$.*
   (b) *When $|r| = q$, we have $\mathcal{K}l\,(r,s) = -q^{-1}$ and $\mathcal{S}(r,s) = C(r)$.*
   (c) *When $|r| > q$, we have $\mathcal{K}l\,(r,s) = \mathcal{S}(r,s) = 0$.*
(2) *Suppose that $|r| = |s|$.*
   (a) *When $|r| = |s| \le 1$, we have $\mathcal{K}l\,(r,s) = 1 - q^{-1}$ and $\mathcal{S}(r,s) = 0$.*
   (b) *When $|r| = |s| = q^n$ with $n \ge 2$, $\mathcal{K}l\,(r,s)$ vanishes unless $rs \in (F^\times)^2$. When $rs \in (F^\times)^2$, we have*

$$\mathcal{K}l\,(r,s) = C\left(\sqrt{rs}\right) \psi\left(2\sqrt{rs}\right) + C\left(-\sqrt{rs}\right) \psi\left(-2\sqrt{rs}\right).$$

   (c) *When $|r| = |s| = q^n$ with $n \ge 1$, $\mathcal{S}(r,s)$ vanishes unless $rs \in (F^\times)^2$. Suppose that $rs \in (F^\times)^2$.*
      (i) *When n is even, we have*

$$\mathcal{S}(r,s) = C(r) \left\{ \operatorname{sgn}\left(\frac{\sqrt{rs}}{r}\right) \psi\left(2\sqrt{rs}\right) + \operatorname{sgn}\left(-\frac{\sqrt{rs}}{r}\right) \psi\left(-2\sqrt{rs}\right) \right\}.$$

      (ii) *When n is odd, we have*

$$\mathcal{S}(r,s) = C(r) \left\{ \psi\left(2\sqrt{rs}\right) + \psi\left(-2\sqrt{rs}\right) \right\}.$$

Corollary 3.5. *Let $r, s \in F^\times$.*

(1) *When* $|rs| \leq q$, *we have*

$$\mathcal{K}l\,(r,s) = \begin{cases} 1-q^{-1} & \textit{when } \max\{|r|,|s|\} \leq 1, \\ -q^{-1} & \textit{when } \max\{|r|,|s|\} = q, \\ 0 & \textit{otherwise.} \end{cases}$$

(2) *When* $|rs| \geq q^2$, *the Kloosterman sum* $\mathcal{K}l\,(r,s)$ *vanishes unless* $|r| = |s|$.

For our later use, we introduce the following definition.

DEFINITION 3.6. For $x \in F \setminus \{0,1\}$ and $\mu \in F^\times$, we put

$$m = \mathrm{ord}\,(x)\,, \quad \varepsilon_x = \varpi^{-m}x, \quad n = -\mathrm{ord}\,(\mu)\,, \quad \varepsilon_\mu = \varpi^n\mu.$$

Then we define $\mathcal{K}l_i = \mathcal{K}l_i\,(x,\mu)$ for $i = 1,2$ by

$$\mathcal{K}l_1 = \mathcal{K}l_1\,(x,\mu) = \begin{cases} \mathcal{K}l\left(\dfrac{2\varpi^{\frac{m-n}{2}}}{1-x}, \dfrac{-2\varpi^{\frac{m-n}{2}}\varepsilon_\mu\varepsilon_x}{1-x}\right) & \text{if } m \equiv n \pmod 2, \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

and

$$\mathcal{K}l_2 = \mathcal{K}l_2\,(x,\mu) = \begin{cases} \mathcal{K}l\left(\dfrac{2\varpi^{\frac{-n}{2}}}{1-x}, \dfrac{-2\varpi^{\frac{-n}{2}}\varepsilon_\mu}{1-x}\right) & \text{if } n \equiv 0 \pmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Here we note that

$$\mathcal{K}l_1\,(x,\mu) = \mathcal{K}l_2\,(x,\mu)\,, \quad \text{if } m = 0 \text{ and } n \leq 0, \tag{3.6}$$

since $m = 0$ and $n \leq 0$ is even implies

$$\frac{2\varpi^{\frac{-n}{2}}x}{1-x} \in \frac{2\varpi^{\frac{-n}{2}}}{1-x} + \mathcal{O},$$

which, in turn, implies

$$\mathcal{K}l_1\,(x,\mu) = \mathcal{K}l\left(\frac{2\varpi^{\frac{-n}{2}}x}{1-x}, \frac{-2\varpi^{\frac{-n}{2}}\varepsilon_\mu}{1-x}\right).$$

DEFINITION 3.7. For an integer i, we define $\mathcal{K}l_1\,(x,\mu,i)$ and $\mathcal{K}l_2\,(x,\mu;i)$ by

$$\mathcal{K}l_1\,(x,\mu;i) = \mathcal{K}l\left(\frac{2\varpi^i x}{1-x}, \frac{-2\varpi^{-i}\mu}{1-x}\right) \tag{3.7}$$

and

$$\mathcal{K}l_2\,(x,\mu;i) = \mathcal{K}l\left(\frac{2\varpi^i}{1-x}, \frac{-2\varpi^{-i}\mu}{1-x}\right). \tag{3.8}$$

It is clear that we have

$$\mathcal{K}l_1\,(x,\mu;i) = \mathcal{K}l_2\,(x,\mu x;i+m) \tag{3.9}$$

where $m = \mathrm{ord}\,(x)$.

The following proposition, which is an immediate consequence of Proposition 3.4, will be repeatedly used.

PROPOSITION 3.8. *Let $x \in \mathcal{O} \setminus \{0,1\}$ and $\mu \in F^\times$. We put*

$$m = \mathrm{ord}\,(x)\,, \quad m' = \mathrm{ord}\,(1-x)\,, \quad n = -\mathrm{ord}\,(\mu)\,.$$

Then for an integer i, $\mathcal{K}l_1\,(x,\mu;i)$ and $\mathcal{K}l_2\,(x,\mu;i)$ are evaluated as follows.

(1) *Suppose that $m' = 0$.*

(a) *We have*

$$\mathcal{K}l_1\,(x,\mu;i) = \begin{cases} \mathcal{K}l_1 & \textit{if } n \ge m+2,\ m-n \textit{ is even and } i = \frac{-m-n}{2}, \\ -q^{-1} & \textit{if } n \le m+1 \textit{ and } i = -m-1,\ -n+1, \\ 1-q^{-1} & \textit{if } n \le m \textit{ and } -m \le i \le -n, \\ 0 & \textit{otherwise.} \end{cases}$$

(b) *We have*

$$\mathcal{K}l_2\,(x,\mu;i) = \begin{cases} \mathcal{K}l_2 & \textit{if } n \ge 2,\ n \textit{ is even and } i = \frac{-n}{2}, \\ -q^{-1} & \textit{if } n \le 1 \textit{ and } i = -1,\ -n+1, \\ 1-q^{-1} & \textit{if } n \le 0 \textit{ and } 0 \le i \le -n, \\ 0 & \textit{otherwise.} \end{cases}$$

(2) *Suppose that $m' \ge 1$.*

(a) *We have*

$$\mathcal{K}l_1\,(x,\mu;i) = \begin{cases} \mathcal{K}l_1 & \textit{if } 2m'+n \ge 2,\ n \textit{ is even and } i = \frac{-n}{2}, \\ -q^{-1} & \textit{if } 2m'+n \le 1 \textit{ and } i = m'-1,\ -m'-n+1, \\ 1-q^{-1} & \textit{if } 2m'+n \le 0 \textit{ and } m' \le i \le -m'-n, \\ 0 & \textit{otherwise.} \end{cases}$$

(b) *We have*

$$\mathcal{K}l_2\,(x,\mu;i) = \begin{cases} \mathcal{K}l_2 & \textit{if } 2m'+n \ge 2,\ n \textit{ is even and } i = \frac{-n}{2}, \\ -q^{-1} & \textit{if } 2m'+n \le 1 \textit{ and } i = m'-1,\ -m'-n+1, \\ 1-q^{-1} & \textit{if } 2m'+n \le 0 \textit{ and } m' \le i \le -m'-n, \\ 0 & \textit{otherwise.} \end{cases}$$

3.1.2. Matrix argument Kloosterman sum. We define a matrix argument Kloosterman sum as follows.

DEFINITION 3.9. For $A \in \mathrm{GL}_2\,(F)$ and $S, T \in \mathrm{Sym}^2\,(F)$, we define the matrix argument Kloosterman sum $\mathcal{K}l\,(A;S,T)$ by

$$\mathcal{K}l\,(A;S,T) = \int_{\mathrm{Sym}^2(F)} \int_{\mathrm{Sym}^2(F)} \Xi \left[\begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \right] \psi\left\{ \mathrm{tr}\,(SX+TY) \right\} dX\, dY \tag{3.10}$$

where dX is the Haar measure on $\mathrm{Sym}^2\,(F)$ normalized so that $\int_{\mathrm{Sym}^2(\mathcal{O})} dX = 1$. Here, as before, Ξ denotes the characteristic function of $K = \mathrm{GSp}_4\,(\mathcal{O})$.

LEMMA 3.10. *The matrix argument Kloosterman $\mathcal{K}l\,(A;S,T)$ vanishes unless $A \in \mathrm{GL}_2\,(F) \cap \mathrm{M}_2\,(\mathcal{O})$ and $S, T \in \mathrm{Sym}^2\,(\mathcal{O})$.*

Proof. If $A \notin \mathrm{M}_2(\mathcal{O})$, the integrand of (3.10) is zero and $\mathcal{K}l(A; S, T)$ vanishes. When $Y_0 \in \mathrm{Sym}^2(\mathcal{O})$, the function Ξ is left invariant by $\begin{pmatrix} 1_2 & 0 \\ Y_0 & 1_2 \end{pmatrix}$. Hence we have

$$\mathcal{K}l(A; S, T) = \mathcal{K}l(A; S, T) \cdot \int_{\mathrm{Sym}^2(\mathcal{O})} \psi\{\mathrm{tr}(TY_0)\}\, dY_0.$$

Thus $\mathcal{K}l(A; S, T)$ vanishes unless $T \in \mathrm{Sym}^2(\mathcal{O})$. Similarly $\mathcal{K}l(A; S, T)$ vanishes unless $S \in \mathrm{Sym}^2(\mathcal{O})$. □

For $A \in \mathrm{GL}_2(F) \cap \mathrm{M}_2(\mathcal{O})$, we define the domain $\mathcal{S}_A \subset \mathrm{Sym}^2(F)$ by

$$\mathcal{S}_A = \left\{ Y \in \mathrm{Sym}^2(F) \mid \begin{pmatrix} A & 0 \\ YA & {}^tA^{-1} \end{pmatrix} \in KU \right\}. \tag{3.11}$$

Proposition 3.11. *Let $A \in \mathrm{GL}_2(F) \cap \mathrm{M}_2(\mathcal{O})$ and $S, T \in \mathrm{Sym}^2(\mathcal{O})$.*

(1) *We have*

$$\mathcal{K}l(A; S, T) = \int_{\mathcal{S}_A} \psi\{\mathrm{tr}(SX_Y + TY)\}\, dY. \tag{3.12}$$

Here $X_Y \in \mathrm{Sym}^2(F)$ is chosen so that $\begin{pmatrix} A & 0 \\ YA & {}^tA^{-1} \end{pmatrix}\begin{pmatrix} 1_2 & X_Y \\ 0 & 1_2 \end{pmatrix} \in K$ for each $Y \in \mathcal{S}_A$.

(2) *Let $C = k_1^{-1} A k_2^{-1}$, where $k_1, k_2 \in \mathrm{GL}_2(\mathcal{O})$. Then we have*

$$\mathcal{K}l(A; S, T) = \mathcal{K}l\left(C; {}^tk_2^{-1} S k_2^{-1}, k_1^{-1} T\, {}^tk_1^{-1}\right). \tag{3.13}$$

Proof. It is clear that for $Y \in \mathcal{S}_A$, we have

$$\left\{ X \in \mathrm{Sym}^2(F) \mid \begin{pmatrix} A & 0 \\ YA & {}^tA^{-1} \end{pmatrix}\begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \in K \right\} = X_Y + \mathrm{Sym}^2(\mathcal{O}).$$

The equality (3.12) follows from (3.10) since $S \in \mathrm{Sym}^2(\mathcal{O})$.

The equality (3.13) follows by changes of variables $X \mapsto k_2^{-1} S\, {}^tk_2^{-1}$ and $Y \mapsto {}^tk_1^{-1} Y k_1^{-1}$ in (3.10). □

Here let us recall the following lemma [**8**, Lemma 4.9].

Lemma 3.12. *For $g = (g_{ij}) \in G$, we have $g \in KU$ if and only if the following three conditions are satisfied:*

$$\lambda(g) \in \mathcal{O}^\times, \text{ where } \lambda(g) \text{ denotes the similitude of } g; \tag{3.14}$$

$$\max_{1 \le i \le 4}\{|g_{i1}|\} \le 1, \quad \max_{1 \le i \le 4}\{|g_{i2}|\} \le 1; \text{ and} \tag{3.15}$$

$$\max_{1 \le k < l \le 4}\{|A_{kl}|\} = 1, \text{ where } A_{kl} = \det\begin{pmatrix} g_{k1} & g_{k2} \\ g_{l1} & g_{l2} \end{pmatrix}. \tag{3.16}$$

By the theory of elementary divisors and (3.13), we may assume that A is diagonal. When A is diagonal, the domain $\mathcal{S}_A$ is given explicitly as follows by Lemma 3.12.

Lemma 3.13. *Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ such that $|a| \le |b| \le 1$.*

(1) *When $|a| = |b| = 1$, we have $\mathcal{S}_A = \mathrm{Sym}^2(\mathcal{O})$. For $Y \in \mathcal{S}_A$, we may take $X_Y = 0$.*

(2) *When* $|a| < |b| = 1$, *we have*

$$\mathcal{S}_A = \left\{ \begin{pmatrix} a^{-1}r & s \\ s & t \end{pmatrix} \mid r \in \mathcal{O}^\times,\, s \in \mathcal{O},\, t \in \mathcal{O} \right\}.$$

For $Y = \begin{pmatrix} a^{-1}r & s \\ s & t \end{pmatrix} \in \mathcal{S}_A$, *we may take* $X_Y = \begin{pmatrix} -a^{-1}r^{-1} & 0 \\ 0 & 0 \end{pmatrix}$.

(3) *When* $|b| < 1$, *we have*

$$\mathcal{S}_A = \left\{ Y \in \mathrm{Sym}^2(F) \mid YA \in \mathrm{GL}_2(\mathcal{O}) \right\}.$$

For $Y \in \mathcal{S}_A$, *we may take* $X_Y = -A^{-1}Y^{-1}A^{-1}$. *The domain* $\mathcal{S}_A$ *is given explicitly as follows.*

(a) *When* $|a| < |b| < 1$, *we have*

$$\mathcal{S}_A = \left\{ \begin{pmatrix} a^{-1}r & b^{-1}s \\ b^{-1}s & b^{-1}t \end{pmatrix} \mid r \in \mathcal{O}^\times,\, s \in \mathcal{O},\, t \in \mathcal{O}^\times \right\}.$$

(b) *When* $|a| = |b| < 1$, *we have*

$$\mathcal{S}_A = \left\{ \begin{pmatrix} a^{-1}r & b^{-1}s \\ b^{-1}s & b^{-1}t \end{pmatrix} \mid r \in \mathcal{O},\, s \in \mathcal{O},\, t \in \mathcal{O},\, rt - ab^{-1}s^2 \in \mathcal{O}^\times \right\}.$$

Corollary 3.14. *Let* $a, b \in F^\times$ *such that* $|a| \le |b| \le 1$. *Let* $S = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$ *and* $T = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}$ *where* $a_i, b_i \in \mathcal{O}$ $(1 \le i \le 3)$.

(1) *When* $|a| = |b| = 1$, *we have*

$$\mathcal{K}l\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; S, T \right) = 1. \tag{3.17}$$

(2) *When* $|a| < |b| = 1$, *we have*

$$\mathcal{K}l\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; S, T \right) = |a|^{-1} \cdot \mathcal{K}l\left(-a_1 a^{-1}, b_1 a^{-1}\right). \tag{3.18}$$

3.1.3. Functional equation. Here we prove the functional equation satisfied by the anisotropic Bessel orbital integral $\mathcal{B}^{(a)}(u, \mu; f)$ defined by (1.7). First we remark the following.

Proposition 3.15. *Suppose that* $u, v \in E^\times$ *satisfy* $\mathrm{N}_{E/F}(u) = \mathrm{N}_{E/F}(v) \neq 1$. *Then for* $\mu \in F^\times$ *and* $f \in \mathcal{H}$, *we have*

$$\mathcal{B}^{(a)}(u, \mu; f) = \mathcal{B}^{(a)}(v, \mu; f).$$

Proof. When $v = a + b\eta$ with $a, b \in F$, let

$$g_v = 1_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t'_v, \quad \text{where } t'_v = \begin{pmatrix} a & b \\ bd & a \end{pmatrix}.$$

Then we have

$$A^{(a)}(v, \mu) = \begin{pmatrix} g_v & 0 \\ 0 & \mu\, {}^t g_v^{-1} \end{pmatrix}.$$

Since $\mathrm{N}_{E/F}(u) = \mathrm{N}_{E/F}(v)$, there exists $\varepsilon \in \mathcal{O}_E^\times$ such that $v = u\,\varepsilon\varepsilon^{-\sigma}$. Hence

$$g_v = 1_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t'_{\varepsilon^{-\sigma}}\, t'_u t'_\varepsilon = t'_{\varepsilon^{-1}} \left(1_2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t'_u \right) t'_\varepsilon = t'_{\varepsilon^{-1}}\, g_u\, t'_\varepsilon,$$

and thus

$$A^{(a)}(v,\mu) = t'_{\varepsilon^{-1}}\, A^{(a)}(u,\mu)\, t'_{\varepsilon}. \tag{3.19}$$

The rest is clear. □

The orbital integral $\mathcal{B}^{(a)}(u,\mu)$ satisfies the following functional equation.

PROPOSITION 3.16. *For $u \in E^{\times}$ such that $\mathrm{N}_{E/F}(u) \neq 1$, $\mu \in F^{\times}$ and $f \in \mathcal{H}$, we have*

$$\mathcal{B}^{(a)}\left(u^{-1}, \mu x^{-1}; f\right) = \delta(x) \cdot \mathcal{B}^{(a)}(u,\mu; f) \tag{3.20}$$

where $x = \mathrm{N}_{E/F}(u)$.

PROOF. First we note that

$$A^{(a)}(u,\mu) = A^{(a)}\left(u^{-\sigma}, \mu x^{-1}\right)\, a_0\, t_u \quad \text{where } a_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since $\tau^{(a)}\left(a_0^{-1} r a_0\right) = \tau^{(a)}(r)$ for $r \in R^{(a)}$ and $a_0 \in K$, we have the proposition. □

Thus it is enough for us to evaluate $\mathcal{B}^{(a)}(\lambda; u, \mu)$ when $|u| \leq 1$.

3.2. Evaluation

Let us evaluate the degenerate orbital integral $\mathcal{B}^{(a)}(\lambda; u, \mu)$ explicitly.

3.2.1. Rewriting the integral. First we shall rewrite $\mathcal{B}^{(a)}(\lambda; u, \mu)$ in the following form for our subsequent evaluation.

PROPOSITION 3.17. *Let $u \in E^{\times}$ such that $\mathrm{N}_{E/F}(u) \neq 1$. Let $\mu \in F^{\times}$ and let $n = -\mathrm{ord}(\mu)$. Put $\varepsilon_\mu = \varpi^n \mu \in \mathcal{O}^{\times}$. Let $\lambda = (\lambda_1, \lambda_2) \in P^{+}$.*

(1) *The orbital integral $\mathcal{B}^{(a)}(\lambda; u, \mu)$ vanishes unless*

$$n \equiv \lambda_1 + \lambda_2 \pmod 2. \tag{3.21}$$

(2) *When the condition (3.21) holds, we have*

$$\mathcal{B}^{(a)}(\lambda; u, \mu) = q^{-3\lambda_1} \omega(\varpi)^{n_1 - \lambda_1} \int_{\mathcal{O}_E^{\times}} \mathcal{B}^{*}\left(\lambda; u\varepsilon\varepsilon^{-\sigma}, \mu\right)\, d^{\times}\varepsilon \tag{3.22}$$

where $n_1 = (n + \lambda_1 - \lambda_2)/2$, $d^{\times}\varepsilon$ is the Haar measure on $\mathcal{O}_E^{\times}$ normalized such that $\mathcal{O}_E^{\times}$ has volume 1, and

$$\mathcal{B}^{*}(\lambda; v, \mu) = \mathcal{K}l\left(g_v A_{\lambda,\mu}; \begin{pmatrix} -\varpi^{\lambda_1+\lambda_2} d & 0 \\ 0 & \varpi^{\lambda_1-\lambda_2} \end{pmatrix}, \varepsilon_\mu \begin{pmatrix} -d^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right), \tag{3.23}$$

where we put

$$A_{\lambda,\mu} = \begin{pmatrix} \varpi^{n_1+\lambda_2} & 0 \\ 0 & \varpi^{n_1} \end{pmatrix}.$$

The orbital integral $\mathcal{B}^{*}(\lambda; v, \mu)$ vanishes unless

$$\varpi^{n_1}(1 - v) \in \mathcal{O}_E \quad \text{and} \quad n_1 + \lambda_2 \geq 0. \tag{3.24}$$

(3) *When* $\lambda_2 = 0$, *we have*

$$\mathcal{B}^*(\lambda; u, \mu) = \mathcal{B}^*(\lambda; v, \mu) \quad \textit{if} \quad \mathrm{N}_{E/F}(u) = \mathrm{N}_{E/F}(v) \tag{3.25}$$

and

$$\mathcal{B}^{(a)}(\lambda; u, \mu) = q^{-3\lambda_1} \omega(\varpi)^{n_1 - \lambda_1} \, \mathcal{B}^*(\lambda; u, \mu). \tag{3.26}$$

Proof. Since $T^{(a)} = Z\, T_K^{(a)}$ where $T_K^{(a)} = T^{(a)} \cap K$, we may write (2.42) as

$$\mathcal{B}^{(a)}(\lambda; u, \mu) = \int_{\bar{U}} \int_Z \int_{T_K^{(a)}} \int_U \Xi\left(z\bar{n}\, A^{(a)}(u, \mu)\; tn\, \varpi^\lambda\right) \xi^{(a)}(\bar{n})\, \tau^{(a)}(zn)\, dz\, dt\, dn\, d\bar{n}.$$

For $t \in T_K^{(a)}$, we have

$$\Xi\left(z\bar{n}\, A^{(a)}(u, \mu)\; tn\, \varpi^\lambda\right) = \Xi\left[z\left(t^{-1}\bar{n}t\right)\left(t^{-1}A^{(a)}(u, \mu)\, t\right)\varpi^\lambda\left(\varpi^{-\lambda} n \varpi^\lambda\right)\right].$$

Thus by changes of variables $\bar{n} \mapsto t\bar{n}t^{-1}$ and $n \mapsto \varpi^\lambda n \varpi^{-\lambda}$, we have

$$\mathcal{B}^{(a)}(\lambda; u, \mu) = q^{-3\lambda_1} \int_{\bar{U}} \int_Z \int_{T_K^{(a)}} \int_U \Xi\left[z\bar{n}\, \left(t^{-1}A^{(a)}(u, \mu)\, t\right)\varpi^\lambda\, n\right] \xi^{(a)}(\bar{n})\, \omega(z)\, \tau^{(a)}\left(\varpi^\lambda n \varpi^{-\lambda}\right)\, dz\, dt\, dn\, d\bar{n}. \tag{3.27}$$

By considering the similitude, the integrand of (3.27) is zero unless $z^2\mu\varpi^{\lambda_1+\lambda_2}$ belongs to $\mathcal{O}^\times$. Thus $\mathcal{B}^{(a)}(\lambda; u, \mu)$ vanishes unless $n \equiv \lambda_1 + \lambda_2 \pmod 2$.

When $n \equiv \lambda_1 + \lambda_2 \pmod 2$, we may write (3.27) as

$$\mathcal{B}^{(a)}(\lambda; u, \mu) = q^{-3\lambda_1} \omega(\varpi)^{n_1 - \lambda_1} \int_{\bar{U}} \int_{T_K^{(a)}} \int_U \Xi\left[\varpi^{n_1 - \lambda_1}\bar{n}\left(t^{-1}A^{(a)}(u, \mu)\, t\right)\varpi^\lambda\, n\right] \xi^{(a)}(\bar{n})\; \tau^{(a)}\left(\varpi^\lambda n \varpi^{-\lambda}\right)\, dt\, dn\, d\bar{n}. \tag{3.28}$$

After the change of variable $\bar{n} \mapsto \begin{pmatrix} 1_2 & 0 \\ 0 & \varepsilon_\mu \cdot 1_2 \end{pmatrix} \bar{n} \begin{pmatrix} 1_2 & 0 \\ 0 & \varepsilon_\mu \cdot 1_2 \end{pmatrix}^{-1}$, we note (3.22) follows from (3.19).

Let $v = a + b\eta$, where $a, b \in F$. Then by Lemma 3.10, the integral $\mathcal{B}^*(\lambda; v, \mu)$ vanishes unless

$$g_v A_{\lambda,\mu} = \begin{pmatrix} \varpi^{n_1+\lambda_2}(1+a) & \varpi^{n_1} b \\ -\varpi^{n_1+\lambda_2} db & \varpi^{n_1}(1-a) \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}),$$

i.e.,

$$\varpi^{n_1+\lambda_2}(1+v) \in \mathcal{O}_E \quad \text{and} \quad \varpi^{n_1}(1-v) \in \mathcal{O}_E.$$

Since

$$2\varpi^{n_1+\lambda_2} = \varpi^{n_1+\lambda_2}(1+v) + \varpi^{\lambda_2} \cdot \varpi^{n_1}(1-v),$$

the integral $\mathcal{B}^*(\lambda; v, \mu)$ vanishes unless the condition (3.24) holds.

When $\lambda_2 = 0$, by changes of variables $\bar{n} \mapsto t^{-1}\bar{n}t$ and $n \mapsto t^{-1}nt$ in (3.28), we have

$$\mathcal{B}^{(a)}(\lambda; u, \mu) = q^{-3\lambda_1} \omega(\varpi)^{-k} \int_{\bar{U}} \int_U \Xi\left[\varpi^{-k}\bar{n}\, A^{(a)}(u, \mu)\, \varpi^\lambda\, n\right] \xi^{(a)}(\bar{n})\; \tau^{(a)}\left(\varpi^\lambda n \varpi^{-\lambda}\right)\, dn\, d\bar{n},$$

since $t \in T_K^{(a)}$ commutes with ϖ^λ. Hence (3.26) and (3.25) hold. ☐

We now evaluate the orbital integral $\mathcal{B}^*(\lambda; v, \mu)$.

PROPOSITION 3.18. *Let $v \in E^{\times} \cap \mathcal{O}_E$ such that $\mathrm{N}_{E/F}(v) \neq 1$. We put $m' = \mathrm{ord}\left(1 - \mathrm{N}_{E/F}(v)\right)$. Let $\mu \in F^{\times}$ and let $n = -\mathrm{ord}(\mu)$. We put $\varepsilon_\mu = \varpi^n \mu$. Let $\lambda = (\lambda_1, \lambda_2) \in P^+ \setminus \{(0,0)\}$. Suppose that $n \equiv \lambda_1 + \lambda_2 \pmod 2$, $n_1 + \lambda_2 \geq 0$ and $\varpi^{n_1}(1-v) \in \mathcal{O}_E$, where $n_1 = (n + \lambda_1 - \lambda_2)/2$. Take an elementary divisor decomposition $g_v A_{\lambda,\mu} = k_1 C k_2$ such that $k_1, k_2 \in \mathrm{GL}_2(\mathcal{O})$ and $C = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $\mathrm{ord}(\alpha) \geq \mathrm{ord}(\beta) \geq 0$. We put*

$$S = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} = {}^t k_2^{-1} \begin{pmatrix} -\varpi^{\lambda_1+\lambda_2} d & 0 \\ 0 & \varpi^{\lambda_1-\lambda_2} \end{pmatrix} k_2^{-1}, \tag{3.29}$$

$$T = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix} = \varepsilon_\mu \, k_1^{-1} \begin{pmatrix} -d^{-1} & 0 \\ 0 & 1 \end{pmatrix} {}^t k_1^{-1}. \tag{3.30}$$

Then the orbital integral $\mathcal{B}^*(\lambda; v, \mu)$ is evaluated explicitly as follows.

(1) *When $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta) = 0$, we have*

$$\mathcal{B}^*(\lambda; v, \mu) = 1. \tag{3.31}$$

(2) *When $\mathrm{ord}(\alpha) > \mathrm{ord}(\beta) = 0$, we have*

$$\mathcal{B}^*(\lambda; v, \mu) = \begin{cases} |\alpha|^{-1} \cdot \mathcal{K}l\left(-a_1\alpha^{-1}, b_1\alpha^{-1}\right) & \textit{if } \mathrm{ord}(a_1) = 0, \\ -1 & \textit{if } \mathrm{ord}(a_1) > 0,\ \mathrm{ord}(\alpha) = 1, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.32}$$

(3) *When $\mathrm{ord}(\alpha) > \mathrm{ord}(\beta) > 0$, the orbital integral $\mathcal{B}^*(\lambda; v, \mu)$ vanishes unless $\mathrm{ord}(\beta) = 1$ and $\mathrm{ord}(a_1) = 0$.*
When $\mathrm{ord}(\beta) = 1$ and $\mathrm{ord}(a_1) = 0$, we have

$$\mathcal{B}^*(\lambda; v, \mu) = \begin{cases} -q\,|\alpha|^{-1} \cdot \mathcal{K}l\left(-a_1\alpha^{-1}, b_1\alpha^{-1}\right), & \textit{if } \mathrm{ord}(a_3) > 0, \\ -q\,|\alpha|^{-1} \cdot \mathcal{K}l\left(\alpha^{-1}, -\alpha^{-1}\left(a_1 b_1 + 2a_2 b_2 \alpha\beta^{-1}\right)\right) & \textit{if } \mathrm{ord}(a_3) = 0. \end{cases} \tag{3.33}$$

(4) *When $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta) > 0$ and $\mathrm{ord}(a_1) > 0$, we have*

$$\mathcal{B}^*(\lambda; v, \mu) = \begin{cases} -q^2 \cdot \mathcal{K}l\left(\alpha^{-1}, -a_3 b_3 \alpha\beta^{-2}\right) & \textit{if } \mathrm{ord}(\alpha) = 1, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.34}$$

REMARK 3.19. As we shall see in the proof of Corollary 3.20, we may assume that $\mathrm{ord}(a_1) > 0$ whenever $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta) > 0$ without loss of generality.

PROOF. By (3.13) and (3.23), we have

$$\mathcal{B}^*(\lambda; v, \mu) = \mathcal{K}l(C; S, T).$$

Here we note that in (3.30) we have

$$\mathrm{ord}(b_1) = \mathrm{ord}(b_3) = 0$$

since $b_2^2 - b_1 b_3 = \varepsilon_\mu^2 (\det k_1)^{-2} d^{-1}$ is not a square in $\mathcal{O}^{\times}$.

In the first case, (3.31) is nothing but (3.17).

In the second case, we have

$$\mathcal{B}^*(\lambda; v, \mu) = |\alpha|^{-1} \cdot \mathcal{K}l\left(-a_1\alpha^{-1}, b_1\alpha^{-1}\right)$$

by (3.18). Then (3.32) follows from Proposition 3.4.

Let us consider the third case. When $\mathrm{ord}(\alpha) > \mathrm{ord}(\beta) > 0$, we have

$$\mathcal{B}^*(\lambda; v, \mu) = |\alpha\beta^2|^{-1} \int_{\mathcal{O}^\times}\int_{\mathcal{O}}\int_{\mathcal{O}^\times} \psi\left(\frac{-a_1\alpha^{-1}t + 2a_2\beta^{-1}s - a_3\beta^{-1}r}{rt - \alpha\beta^{-1}s^2}\right) \psi\left(b_1\alpha^{-1}r + 2b_2\beta^{-1}s + b_3\beta^{-1}t\right) dr\,ds\,dt \tag{3.35}$$

by Lemma 3.13. By a change of variable $r \mapsto \left(r + \alpha\beta^{-1}s^2\right)t^{-1}$ in (3.35), we have

$$\begin{aligned}\mathcal{B}^*(\lambda; v, \mu) = |\alpha\beta^2|^{-1} &\int_{\mathcal{O}^\times}\int_{\mathcal{O}}\int_{\mathcal{O}^\times} \psi\left(b_1\beta^{-1}s^2t^{-1} + 2b_2\beta^{-1}s + b_3\beta^{-1}t - a_3\beta^{-1}t^{-1}\right)\\ &\psi\left\{b_1\alpha^{-1}rt^{-1} - \alpha^{-1}\left(a_1 - 2a_2\alpha\beta^{-1}st^{-1} + a_3\alpha^2\beta^{-2}s^2t^{-2}\right)r^{-1}t\right\} dr\,ds\,dt.\end{aligned}$$

Then by changes of variables $r \mapsto rt$ and $s \mapsto st$, we have

$$\mathcal{B}^*(\lambda; v, \mu) = |\alpha\beta^2|^{-1}\int_{\mathcal{O}} \mathcal{K}l(A_1, A_2)\cdot\mathcal{K}l(B_1, B_2)\,ds \tag{3.36}$$

where

$$A_1 = \beta^{-1}\left(b_1s^2 + 2b_2s + b_3\right), \qquad B_1 = b_1\alpha^{-1},$$
$$A_2 = -a_3\beta^{-1}, \qquad B_2 = -\alpha^{-1}\left(a_1 - 2a_2\alpha\beta^{-1}s + a_3\alpha^2\beta^{-2}s^2\right).$$

Suppose that $\mathrm{ord}(a_1) > 0$. Then we have $\mathrm{ord}(B_1) = -\mathrm{ord}(\alpha) \le -2$ and $\mathrm{ord}(B_2) > -\mathrm{ord}(\alpha)$. Hence $\mathcal{K}l(B_1, B_2) = 0$ by Proposition 3.4.

Suppose that $\mathrm{ord}(a_1) = 0$ and $\mathrm{ord}(a_3) > 0$. Then we also have $\mathrm{ord}(a_2) > 0$ since

$$a_2^2 - a_1a_3 = \varpi^{2\lambda_1}d\left(\det k_2\right)^{-2}, \quad \text{where } \lambda_1 > 0. \tag{3.37}$$

For $s \in \mathcal{O}$, we have

$$b_1s^2 + 2b_2s + b_3 = \varepsilon_\mu\,(s, 1)\,k_1^{-1}\begin{pmatrix}-d^{-1} & 0\\ 0 & 1\end{pmatrix}{}^t k_1^{-1}\begin{pmatrix}s\\ 1\end{pmatrix} \in \mathcal{O}^\times.$$

Hence $\mathcal{K}l(A_1, A_2)$ vanishes unless $\mathrm{ord}(\beta) = 1$. When $\mathrm{ord}(\beta) = 1$, we have

$$\mathcal{B}^*(\lambda; v, \mu) = -q\,|\alpha|^{-1}\cdot\mathcal{K}l\left(-a_1\alpha^{-1}, b_1\alpha^{-1}\right)$$

since $B_2 + \alpha^{-1}a_1 \in \mathcal{O}$.

Suppose that $\mathrm{ord}(a_1) = \mathrm{ord}(a_3) = 0$. Then we also have $\mathrm{ord}(a_2) = 0$ by (3.37). We rewrite (3.36) as

$$\begin{aligned}\mathcal{B}^*(\lambda; v, \mu) = |\alpha\beta^2|^{-1} &\int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times} \psi\left\{b_3\beta^{-1}t - a_1\alpha^{-1}r^{-1} + \alpha^{-1}\left(b_1rt - a_3\alpha\beta^{-1}\right)t^{-1}\right\}\\ &\left(\int_{\mathcal{O}}\psi\left\{\beta^{-1}r^{-1}\left(b_1rt - a_3\alpha\beta^{-1}\right)s^2 + 2\beta^{-1}r^{-1}\left(b_2rt + a_2\right)s\right\}ds\right)dr\,dt.\end{aligned} \tag{3.38}$$

By applying Proposition 3.2 to the inner integral of (3.38), we have

$$\begin{aligned}\mathcal{B}^*(\lambda; v, \mu) = |\alpha\beta^2|^{-1}C\left(\beta^{-1}b_1\right)&\int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times}\psi\left\{\frac{-\beta^{-1}r^{-1}\left(b_2rt + a_2\right)^2}{b_1rt - a_3\alpha\beta^{-1}}\right\}\\ &\mathrm{sgn}(t)^{\mathrm{ord}(\beta)}\,\psi\left\{b_3\beta^{-1}t - a_1\alpha^{-1}r^{-1} + \alpha^{-1}\left(b_1rt - a_3\alpha\beta^{-1}\right)t^{-1}\right\}dr\,dt.\end{aligned}$$

The change of variable $t \mapsto r^{-1}t$ gives

$$\mathcal{B}^*(\lambda; v, \mu) = |\alpha\beta^2|^{-1} C\left(\beta^{-1}b_1\right) \int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times} \psi\left\{\frac{-\beta^{-1}r^{-1}\left(b_2t + a_2\right)^2}{b_1t - a_3\alpha\beta^{-1}}\right\}$$
$$\operatorname{sgn}(rt)^{\operatorname{ord}(\beta)}\,\psi\left\{\left(b_3\beta^{-1}t - a_1\alpha^{-1}\right)r^{-1} + \alpha^{-1}r\left(b_1t - a_3\alpha\beta^{-1}\right)t^{-1}\right\}\, dr\, dt.$$

Another change of variable $r \mapsto r\left(b_1t - a_3\alpha\beta^{-1}\right)^{-1}$ yields

$$\mathcal{B}^*(\lambda; v, \mu) = |\alpha\beta^2|^{-1} C\left(\beta^{-1}\right) \int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times}$$
$$\psi\left\{\alpha^{-1}rt^{-1} - \alpha^{-1}\left(a_1b_1 + 2a_2b_2\alpha\beta^{-1} + \alpha^2\beta^{-2}a_3b_3\right)r^{-1}t\right\}$$
$$\operatorname{sgn}(r)^{\operatorname{ord}(\beta)}\,\psi\left\{\beta^{-1}\left(b_1b_3 - b_2^2\right)r^{-1}t^2 + \beta^{-1}\left(a_1a_3 - a_2^2\right)r^{-1}\right\}\, dr\, dt.$$

By one more change of variable $t \mapsto rt$, we have

$$\mathcal{B}^*(\lambda; v, \mu) = |\alpha\beta^2|^{-1} C\left(\beta^{-1}\right) \int_{\mathcal{O}^\times} \tag{3.39}$$
$$\psi\left\{\alpha^{-1}t^{-1} - \alpha^{-1}\left(a_1b_1 + 2a_2b_2\alpha\beta^{-1} + \alpha^2\beta^{-2}a_3b_3\right)t\right\}$$
$$\left(\int_{\mathcal{O}^\times} \operatorname{sgn}(r)^{\operatorname{ord}(\beta)}\,\psi\left\{\beta^{-1}\left(b_1b_3 - b_2^2\right)t^2r + \beta^{-1}\left(a_1a_3 - a_2^2\right)r^{-1}\right\}\, dr\right) dt.$$

By Proposition 3.4, the inner integral of (3.39) vanishes unless $\operatorname{ord}(\beta) = 1$. When $\operatorname{ord}(\beta) = 1$, we have

$$\mathcal{B}^*(\lambda; v, \mu) = -q\,|\alpha|^{-1} \cdot \mathcal{K}l\left(\alpha^{-1}, -\alpha^{-1}\left(a_1b_1 + 2a_2b_2\alpha\beta^{-1}\right)\right) \tag{3.40}$$

since $\operatorname{ord}\left(\alpha\beta^{-2}\right) = \operatorname{ord}(\alpha) - 2 \geq 0$. Thus we have shown (3.33).

Let us consider the fourth case, i.e., when $\operatorname{ord}(\alpha) = \operatorname{ord}(\beta) > 0$ and $\operatorname{ord}(a_1) > 0$. By Lemma 3.13, we have $\mathcal{B}^*(\lambda; v, \mu) = \mathcal{B}_1^* + \mathcal{B}_2^*$ where

$$\mathcal{B}_1^* = |\alpha|^{-3} \int_{\varpi\mathcal{O}}\int_{\mathcal{O}^\times}\int_{\mathcal{O}} \psi\left(\frac{-a_1\alpha^{-1}t + 2a_2\beta^{-1}s - a_3\beta^{-1}r}{rt - \alpha\beta^{-1}s^2}\right) \tag{3.41}$$
$$\psi\left(b_1\alpha^{-1}r + 2b_2\beta^{-1}s + b_3\beta^{-1}t\right)\, dr\, ds\, dt$$

and

$$\mathcal{B}_2^* = |\alpha|^{-3} \int_{\{r\in\mathcal{O}^\times,\, s\in\mathcal{O},\, t\in\mathcal{O} \mid rt - \alpha\beta^{-1}s^2 \in \mathcal{O}^\times\}} \tag{3.42}$$
$$\psi\left(\frac{-a_1\alpha^{-1}t + 2a_2\beta^{-1}s - a_3\beta^{-1}r}{rt - \alpha\beta^{-1}s^2} + b_1\alpha^{-1}r + 2b_2\beta^{-1}s + b_3\beta^{-1}t\right)\, dr\, ds\, dt.$$

Let us compute $\mathcal{B}_1^*$. In the integrand of (3.41), we note that

$$\frac{-a_1\alpha^{-1}\left(t + \alpha\varpi^{-1}x\right) + 2a_2\beta^{-1}s - a_3\beta^{-1}r}{r\left(t + \alpha\varpi^{-1}x\right) - \alpha\beta^{-1}s^2} - \frac{-a_1\alpha^{-1}t + 2a_2\beta^{-1}s - a_3\beta^{-1}r}{rt - \alpha\beta^{-1}s^2}$$
$$= \frac{\alpha\beta^{-1}\varpi^{-1}x\left(a_1s^2 - 2a_2sr + a_3r^2\right)}{\left\{r\left(t + \alpha\varpi^{-1}x\right) - \alpha\beta^{-1}s^2\right\}\left(rt - \alpha\beta^{-1}s^2\right)} \in \mathcal{O}$$

for $x \in \mathcal{O}$, since $\operatorname{ord}(a_1) > 0$. Thus by substituting t by $t + \alpha\varpi^{-1}x$ in (3.41) and integrating over $x \in \mathcal{O}$, we have

$$\mathcal{B}_1^* = \mathcal{B}_1^* \cdot \int_{\mathcal{O}} \psi\left(b_3\beta^{-1}\alpha\varpi^{-1}x\right)\, dx = 0.$$

As for $\mathcal{B}_2^*$, by a change of variable $t \mapsto r^{-1}t$ in (3.42), we have

$$\mathcal{B}_2^* = |\alpha|^{-3} \int_{\{r\in\mathcal{O}^\times,\, s\in\mathcal{O},\, t\in\mathcal{O} \mid t-\alpha\beta^{-1}s^2\in\mathcal{O}^\times\}} \psi\left(\frac{-a_1\alpha^{-1}r^{-1}t + 2a_2\beta^{-1}s - a_3\beta^{-1}r}{t-\alpha\beta^{-1}s^2} + b_1\alpha^{-1}r + 2b_2\beta^{-1}s + b_3\beta^{-1}r^{-1}t\right) dr\,ds\,dt.$$

By another change of variable $t \mapsto t + \alpha\beta^{-1}s^2$, we have

$$\mathcal{B}_2^* = |\alpha|^{-3} \int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times} \psi\left(-a_1\alpha^{-1}r^{-1} - a_3\beta^{-1}rt^{-1} + b_1\alpha^{-1}r + b_3\beta^{-1}r^{-1}t\right) \left(\int_{\mathcal{O}} \psi\left\{\beta^{-1}r^{-1}t^{-1}\left(b_3\alpha\beta^{-1}t - a_1\right)s^2 + 2\beta^{-1}t^{-1}\left(a_2+b_2t\right)s\right\}\,ds\right) dr\,dt.$$

Applying Proposition 3.2 to the inner integral yields

$$\mathcal{B}_2^* = |\alpha|^{-3}\, C\left(b_3\alpha\beta^{-2}\right) \int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times} \operatorname{sgn}(r)^{\operatorname{ord}(\alpha)}\, \psi\left\{\frac{-\beta^{-1}rt^{-1}\left(a_2+b_2t\right)^2}{b_3\alpha\beta^{-1}t - a_1}\right\} \psi\left\{\alpha^{-1}\left(b_3\alpha\beta^{-1}t - a_1\right)r^{-1} + \alpha^{-1}\left(b_1t - a_3\alpha\beta^{-1}\right)rt^{-1}\right\} dr\,dt.$$

Then the change of variable $r \mapsto r\left(b_3\alpha\beta^{-1}t - a_1\right)$ gives

$$\mathcal{B}_2^* = |\alpha|^{-3}\, C\left(\beta^{-1}\right) \int_{\mathcal{O}^\times} \operatorname{sgn}(r)^{\operatorname{ord}(\alpha)}\, \psi\left\{\alpha^{-1}r^{-1} - \alpha^{-1}r\left(a_1b_1 + 2a_2b_2\alpha\beta^{-1} + a_3b_3\alpha^2\beta^{-2}\right)\right\} \left(\int_{\mathcal{O}^\times} \operatorname{sgn}(t)^{\operatorname{ord}(\alpha)}\, \psi\left\{\beta^{-1}rt\left(b_1b_3 - b_2^2\right) + \beta^{-1}rt^{-1}\left(a_1a_3 - a_2^2\right)\right\}\,dt\right) dr.$$

We use one last change of variable $t \mapsto rt$ to get

$$\mathcal{B}_2^* = |\alpha|^{-3}\, C\left(\beta^{-1}\right) \int_{\mathcal{O}^\times} \psi\left\{\alpha^{-1}r^{-1} - \alpha^{-1}r\left(a_1b_1 + 2a_2b_2\alpha\beta^{-1} + a_3b_3\alpha^2\beta^{-2}\right)\right\} \left(\int_{\mathcal{O}^\times} \operatorname{sgn}(t)^{\operatorname{ord}(\beta)}\, \psi\left\{\beta^{-1}r^2t\left(b_1b_3 - b_2^2\right) + \beta^{-1}t^{-1}\left(a_1a_3 - a_2^2\right)\right\}\,dt\right) dr. \tag{3.43}$$

By Propostion 3.4, the inner integral of (3.43) vanishes unless $\operatorname{ord}(\beta) = 1$. When $\operatorname{ord}(\beta) = 1$, we have

$$\mathcal{B}_2^* = -q^2 \cdot \mathcal{K}l\left(\alpha^{-1}, -a_3b_3\alpha\beta^{-2}\right)$$

since we also have $\operatorname{ord}(a_2) > 0$ by $\operatorname{ord}\left(a_1a_3 - a_2^2\right) = 2\lambda_1 > 0$. Thus we have (3.34). □

Corollary 3.20. *Let $v \in E^\times \cap \mathcal{O}_E$ such that $\mathrm{N}_{E/F}(v) \neq 1$. We write*

$$v = a + b\,\eta \quad \text{where} \quad a, b \in \mathcal{O}.$$

Let $m' = \operatorname{ord}\left(1 - \mathrm{N}_{E/F}(v)\right)$. Let $\mu \in F^\times$ and let $n = -\operatorname{ord}(\mu)$. We put $\varepsilon_\mu = \varpi^n\mu$.

Let $\lambda = (\lambda_1, \lambda_2) \in P^+ \setminus \{(0,0)\}$. Suppose that

$$n \equiv \lambda_1 + \lambda_2 \pmod 2, \quad n_1 + \lambda_2 \geq 0 \quad \text{and} \quad \varpi^{n_1}(1 - v) \in \mathcal{O}_E$$

where $n_1 = (n + \lambda_1 - \lambda_2)/2$.

(1) *When* $m'=0$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=\begin{cases}1 & \text{when } n\le -1 \text{ and } \lambda=(-n,0),\\ -1 & \text{when } n\le 0 \text{ and } \lambda=(1-n,1),\\ q & \text{when } n\le 1 \text{ and } \lambda=(2-n,0),\\ 0 & \text{otherwise.}\end{cases} \tag{3.44}$$

(2) *When* $m'>0$, $v=a\in F^\times$, $\operatorname{ord}(1-a)=0$ *and* $\lambda_2=0$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=\begin{cases}-1 & \text{when } m'=1,\ n\le -1 \text{ and } \lambda=(-n,0),\\ 0 & \text{otherwise.}\end{cases} \tag{3.45}$$

(3) *When* $m'>0$, $\operatorname{ord}(1-a)=0$, *and* $\lambda_2>0$, *the orbital integral* $\mathcal{B}^*(\lambda;v,\mu)$ *vanishes.*

(4) *When* $m'>0$, $\operatorname{ord}(1-a)>0$, $\operatorname{ord}(b)<\min\{\lambda_2,m'\}$, *and* $\lambda_2>0$, *the orbital integral* $\mathcal{B}^*(\lambda;v,\mu)$ *vanishes.*

(5) *Suppose that* $m'>0$, $\operatorname{ord}(1-a)>0$, *and* $m'>\operatorname{ord}(b)\ge\lambda_2>0$.

(a) *When* n *is even,* $-2m'+2\le n\le -2$, *and* $\lambda=\left(\frac{-n}{2},\frac{-n}{2}\right)$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=q^{m'+\frac{n}{2}}\cdot\mathcal{K}l_2\left(\mathrm{N}_{E/F}(v),\mu\right). \tag{3.46}$$

(b) *When* $n\le -2m'+1$ *and* $\lambda=(1-m'-n,m'-1)$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=-1.$$

(c) *When* n *is even,* $-2m'+4\le n\le 0$, *and* $\lambda=\left(\frac{2-n}{2},\frac{2-n}{2}\right)$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=-q^{m'+\frac{n}{2}+1}\cdot\mathcal{K}l_2\left(\mathrm{N}_{E/F}(v),\mu\right). \tag{3.47}$$

(d) *Otherwise the orbital integral* $\mathcal{B}^*(\lambda;v,\mu)$ *vanishes.*

(6) *Suppose that* $m'>0$, $\operatorname{ord}(1-a)>0$, *and* $\min\{\lambda_2,\operatorname{ord}(b)\}\ge m'$.

(a) *When* $n\le -2m'$ *and* $\lambda=(-m'-n,m')$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=1.$$

(b) *When* $n\le -2m'$ *and* $\lambda=(1-m'-n,m'+1)$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=-1.$$

(c) *When* $n\le -2m'+2$ *and* $\lambda=(2-m'-n,m')$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=-q^2\cdot\mathcal{K}l_2\left(\mathrm{N}_{E/F}(v),\mu;m'-1\right). \tag{3.48}$$

(d) *Otherwise the orbital integral* $\mathcal{B}^*(\lambda;v,\mu)$ *vanishes.*

(7) *Suppose that* $m'>0$, $\operatorname{ord}(1-a)>0$, *and* $\operatorname{ord}(b)\ge m'>\lambda_2>0$.

(a) *When* n *is even,* $-2m'+2\le n\le -2$, *and* $\lambda=\left(\frac{-n}{2},\frac{-n}{2}\right)$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=q^{m'+\frac{n}{2}}\cdot\mathcal{K}l_2\left(\mathrm{N}_{E/F}(v),\mu\right). \tag{3.49}$$

(b) *When* $n\le -2m'+1$ *and* $\lambda=(1-m'-n,m'-1)$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=-1.$$

(c) *When* n *is even,* $-2m'+4\le n\le 0$, *and* $\lambda=\left(\frac{2-n}{2},\frac{2-n}{2}\right)$, *we have*

$$\mathcal{B}^*(\lambda;v,\mu)=-q^{m'+\frac{n}{2}+1}\cdot\mathcal{K}l_2\left(\mathrm{N}_{E/F}(v),\mu\right). \tag{3.50}$$

(d) *Otherwise the orbital integral* $\mathcal{B}^*(\lambda;v,\mu)$ *vanishes.*

PROOF. By Proposition 3.18, the orbital integral $\mathcal{B}^*(\lambda; v, \mu)$ vanishes except for the following five cases:

$$\text{when } \operatorname{ord}(\alpha) = \operatorname{ord}(\beta) = 0; \tag{3.51}$$

$$\text{when } \operatorname{ord}(\alpha) > \operatorname{ord}(\beta) = 0 \text{ and } \operatorname{ord}(a_1) = 0; \tag{3.52}$$

$$\text{when } \operatorname{ord}(\alpha) = 1 > \operatorname{ord}(\beta) = 0, \text{ and } \operatorname{ord}(a_1) > 0; \tag{3.53}$$

$$\text{when } \operatorname{ord}(\alpha) > \operatorname{ord}(\beta) = 1 \text{ and } \operatorname{ord}(a_1) = 0; \quad \text{and} \tag{3.54}$$

$$\text{when } \operatorname{ord}(\alpha) = \operatorname{ord}(\beta) = 1 \text{ and } \operatorname{ord}(a_1) > 0. \tag{3.55}$$

Suppose that $m' = 0$. Then we may take $k_1 = g_v$, $k_2 = 1_2$,

$$C = \begin{pmatrix} \varpi^{n_1+\lambda_2} & 0 \\ 0 & \varpi^{n_1} \end{pmatrix}, \quad S = \varpi^{\lambda_1-\lambda_2} \begin{pmatrix} -\varpi^{2\lambda_2} d & 0 \\ 0 & 1 \end{pmatrix}, \text{ and}$$
$$T = \frac{1}{\left(1 - \mathrm{N}_{E/F}(v)\right)^2} \begin{pmatrix} -\varepsilon_\mu d^{-1} \mathrm{N}_{E/F}(1-v) & -\varepsilon_\mu \operatorname{tr}_{E/F}\left(\eta^{-1} v\right) \\ -\varepsilon_\mu \operatorname{tr}_{E/F}\left(\eta^{-1} v\right) & \varepsilon_\mu \mathrm{N}_{E/F}(1+v) \end{pmatrix}.$$

Then $\operatorname{ord}(\alpha) = n_1 + \lambda_2 \geq \operatorname{ord}(\beta) = n_1$ and $\operatorname{ord}(a_1) = \lambda_1 + \lambda_2 > 0$. Thus we have (3.44).

Suppose that $m' > 0$, $v = a \in F^\times$, $\operatorname{ord}(1-a) = 0$, *and* $\lambda_2 = 0$. Then we may take $k_1 = k_2 = 1_2$,

$$C = \varpi^{n_1} \begin{pmatrix} 1+a & 0 \\ 0 & 1-a \end{pmatrix}, \quad S = \varpi^{\lambda_1} \begin{pmatrix} -d & 0 \\ 0 & 1 \end{pmatrix}, \text{ and} \quad T = \varepsilon_\mu \begin{pmatrix} -d^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\operatorname{ord}(\alpha) = n_1 + m' > \operatorname{ord}(\beta) = n_1$ and $\operatorname{ord}(a_1) = \lambda_1 > 0$. Thus we have (3.45).

Suppose that $m' > 0$, $\operatorname{ord}(1-a) = 0$, *and* $\lambda_2 > 0$. Then we may take

$$k_1 = \begin{pmatrix} 1 & \frac{b}{1-a} \\ 0 & 1 \end{pmatrix}, \quad k_2 = \begin{pmatrix} 1 & 0 \\ \frac{-\varpi^{\lambda_2} db}{1-a} & 1 \end{pmatrix},$$
$$C = \frac{\varpi^{n_1}}{1-a} \begin{pmatrix} \varpi^{\lambda_1}\left(1 - \mathrm{N}_{E/F}(v)\right) & 0 \\ 0 & (1-a)^2 \end{pmatrix},$$
$$S = \frac{\varpi^{\lambda_1-\lambda_2}}{(1-a)^2} \begin{pmatrix} -\varpi^{2\lambda_2} d\, \mathrm{N}_{E/F}(1-v) & \varpi^{\lambda_2} db\,(1-a) \\ \varpi^{\lambda_2} db\,(1-a) & (1-a)^2 \end{pmatrix}, \text{ and}$$
$$T = \frac{\varepsilon_\mu}{(1-a)^2} \begin{pmatrix} -d^{-1} \mathrm{N}_{E/F}(1-v) & -b(1-a) \\ -b(1-a) & (1-a)^2 \end{pmatrix}.$$

Then $\operatorname{ord}(\alpha) = n_1 + m' + \lambda_1 \geq n_1 + 2 > \operatorname{ord}(\alpha) = n_1$ and $\operatorname{ord}(a_1) = \lambda_1 + \lambda_2 > 0$. Hence $\mathcal{B}^*(\lambda; v, \mu)$ vanishes.

Before moving on to the remaining four cases, we observe the following lemma.

LEMMA 3.21. *Suppose that* $m' > 0$ *and* $\operatorname{ord}(1-a) > 0$.

(1) *When* $\operatorname{ord}(b) < m'$, *we have* $0 < \operatorname{ord}(b) < \operatorname{ord}(1-a)$.
(2) *When* $\operatorname{ord}(b) \geq m'$, *we have* $\operatorname{ord}(1-a) = \operatorname{ord}(1-v) = m'$.

PROOF OF LEMMA 3.21. Since

$$1 - \mathrm{N}_{E/F}(v) = (1+a)(1-a) + db^2, \tag{3.56}$$

we have $\operatorname{ord}(b) > 0$. We also note that $1 + a = 2 - (1-a) \in \mathcal{O}^\times$.

Suppose that $\operatorname{ord}(1-a) \leq \operatorname{ord}(b) < m'$. Then by (3.56), we have $m' = \operatorname{ord}(1-a)$. This is a contradiction and hence we have $\operatorname{ord}(b) < \operatorname{ord}(1-a)$ when $\operatorname{ord}(b) < m'$.

Suppose that $\operatorname{ord}(b) \geq m'$. Then (3.56) implies that $m' = \operatorname{ord}(1-a)$. Since $1-v = (1-a) - b\eta$, it is clear that $\operatorname{ord}(1-v) \geq \operatorname{ord}(1-a)$. On the other hand, we have $\operatorname{ord}(1-v) \leq \operatorname{ord}(1-a)$ since $2(1-a) = \operatorname{tr}_{E/F}(1-v)$. $\square$

Suppose that $m' > 0$, $\operatorname{ord}(1-a) > 0$, $\operatorname{ord}(b) < \min\{\lambda_2, m'\}$, *and* $\lambda_2 > 0$. Then we may take

$$\begin{aligned}
k_1 &= \begin{pmatrix} 0 & 1 \\ 1 & \frac{1-a}{b} \end{pmatrix}, \quad k_2 = \begin{pmatrix} 1 & 0 \\ \frac{\varpi^{\lambda_2}(1+a)}{b} & 1 \end{pmatrix}, \\
C &= \frac{\varpi^{n_1}}{b} \begin{pmatrix} -\varpi^{\lambda_2}\left(1 - \mathrm{N}_{E/F}(v)\right) & 0 \\ 0 & b^2 \end{pmatrix}, \\
S &= \frac{\varpi^{\lambda_1-\lambda_2}}{b^2} \begin{pmatrix} \varpi^{2\lambda_2}\,\mathrm{N}_{E/F}(1+v) & -\varpi^{\lambda_2}(1+a)\,b \\ -\varpi^{\lambda_2}(1+a)\,b & b^2 \end{pmatrix}, \text{ and} \\
T &= \frac{\varepsilon_\mu}{db^2} \begin{pmatrix} -\mathrm{N}_{E/F}(1-v) & (1-a)\,b \\ (1-a)\,b & -b^2 \end{pmatrix}.
\end{aligned}$$

Then $\operatorname{ord}(\alpha) = n_1 + \lambda_2 + m' - \operatorname{ord}(b) \geq n_1 + \operatorname{ord}(b) + 2 > \operatorname{ord}(\alpha) = n_1 + \operatorname{ord}(b)$ and $\operatorname{ord}(a_1) = \lambda_1 + \lambda_2 - 2\operatorname{ord}(b) \geq 2$. Hence $\mathcal{B}^*(\lambda; v, \mu)$ vanishes.

Suppose that $m' > 0$, $\operatorname{ord}(1-a) > 0$, *and* $m' > \operatorname{ord}(b) \geq \lambda_2 > 0$. Then we may take

$$\begin{aligned}
k_1 &= \begin{pmatrix} 0 & 1 \\ 1 & \frac{-bd}{1+a} \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{\varpi^{-\lambda_2} b}{1+a} \end{pmatrix}, \\
C &= \frac{\varpi^{n_1+\lambda_2}}{1+a} \begin{pmatrix} \frac{1-\mathrm{N}_{E/F}(v)}{\varpi^{\lambda_2}} & 0 \\ 0 & (1+a)^2 \end{pmatrix}, \\
S &= \frac{\varpi^{\lambda_1-\lambda_2}}{(1+a)^2} \begin{pmatrix} \mathrm{N}_{E/F}(1+v) & \varpi^{\lambda_2}(1+a)\,bd \\ \varpi^{\lambda_2}(1+a)\,bd & -\varpi^{2\lambda_2}(1+a)^2 d \end{pmatrix}, \text{ and} \\
T &= \frac{\varepsilon_\mu}{(1+a)^2 d} \begin{pmatrix} d\,\mathrm{N}_{E/F}(1+v) & -(1+a)\,bd \\ -(1+a)\,bd & -(1+a)^2 \end{pmatrix}.
\end{aligned}$$

Then $\operatorname{ord}(\alpha) = n_1 + m' > \operatorname{ord}(\beta) = n_1 + \lambda_2$ and $\operatorname{ord}(a_1) = \lambda_1 - \lambda_2$. In case (a), by Proposition 3.18, we have

$$\mathcal{B}^*(\lambda; v, \mu) = q^{m'+\frac{n}{2}} \cdot \mathcal{K}l\left(\frac{-\varpi^{-\frac{n}{2}}\,\mathrm{N}_{E/F}(1+v)}{(1+a)\left(1-\mathrm{N}_{E/F}(v)\right)}, \frac{\varpi^{-\frac{n}{2}}\varepsilon_\mu\,\mathrm{N}_{E/F}(1+v)}{(1+a)\left(1-\mathrm{N}_{E/F}(v)\right)} \right).$$

Since

$$\frac{\varpi^{-\frac{n}{2}}\,\mathrm{N}_{E/F}(1+v)}{(1+a)\left(1-\mathrm{N}_{E/F}(v)\right)} - \frac{2\varpi^{-\frac{n}{2}}}{1-\mathrm{N}_{E/F}(v)} = \frac{-\varpi^{-\frac{n}{2}}}{1+a} \in \mathcal{O},$$

we have (3.46). The rest is clear.

Suppose that $m' > 0$, $\operatorname{ord}(1-a) > 0$, *and* $\min\{\lambda_2, \operatorname{ord}(b)\} \geq m'$. Then we may take

$$k_1 = \begin{pmatrix} 1 & \frac{b}{1-\mathrm{N}_{E/F}(v)} \\ \frac{-db}{1+a} & \frac{1-a}{1-\mathrm{N}_{E/F}(v)} \end{pmatrix}, \quad k_2 = 1_2,$$
$$C = \varpi^{n_1} \begin{pmatrix} \varpi^{\lambda_2}(1+a) & 0 \\ 0 & 1 - \mathrm{N}_{E/F}(v) \end{pmatrix},$$
$$S = \varpi^{\lambda_1 - \lambda_2} \begin{pmatrix} -\varpi^{2\lambda_2} d & 0 \\ 0 & 1 \end{pmatrix}, \text{ and}$$
$$T = \varepsilon_\mu \begin{pmatrix} \frac{-(1+a)^2 \mathrm{N}_{E/F}(1-v)}{d\left(1-\mathrm{N}_{E/F}(v)\right)^2} & \frac{-2(1+a)b}{1-\mathrm{N}_{E/F}(v)} \\ \frac{-2(1+a)b}{1-\mathrm{N}_{E/F}(v)} & \mathrm{N}_{E/F}(1+v) \end{pmatrix}.$$

Then $\operatorname{ord}(\alpha) = n_1 + \lambda_2 \geq \operatorname{ord}(\beta) = n_1 + m'$ and $\operatorname{ord}(a_1) = \lambda_1 + \lambda_2 > 0$. In the case (c), by Proposition 3.18, we have

$$\mathcal{B}^*(\lambda; v, \mu) = -q^2 \cdot \mathcal{K}l\left(\varpi^{-1}, \frac{\varpi^{1-n}\varepsilon_\mu \mathrm{N}_{E/F}(1+v)}{\left(1 - \mathrm{N}_{E/F}(v)\right)^2}\right).$$

Since $\varpi^{-m'}\left(1 - \mathrm{N}_{E/F}(v)\right) \in \mathcal{O}^\times$, we have

$$\mathcal{B}^*(\lambda; v, \mu) = -q^2 \cdot \mathcal{K}l\left(\frac{\varpi^{m'-1}}{1 - \mathrm{N}_{E/F}(v)}, \frac{\varpi^{1-m'-n}\varepsilon_\mu \mathrm{N}_{E/F}(1+v)}{1 - \mathrm{N}_{E/F}(v)}\right).$$

Since $\mathrm{N}_{E/F}(1+v) = (1+a)^2 - db^2 \equiv 2^2 \pmod{\varpi\mathcal{O}}$ and $1 - m' - n \geq m' - 1$, we have (3.48). The rest is clear.

Suppose that $m' > 0$, $\operatorname{ord}(1-a) > 0$, *and* $\operatorname{ord}(b) \geq m' > \lambda_2 > 0$. Then we may take

$$k_1 = \begin{pmatrix} \frac{b}{1-\mathrm{N}_{E/F}(v)} & 1 \\ \frac{1-a}{1-\mathrm{N}_{E/F}(v)} & \frac{-db}{1+a} \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$C = \varpi^{n_1} \begin{pmatrix} 1 - \mathrm{N}_{E/F}(v) & 0 \\ 0 & \varpi^{\lambda_2}(1+a) \end{pmatrix},$$
$$S = \varpi^{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & 0 \\ 0 & -\varpi^{2\lambda_2} d \end{pmatrix}, \text{ and}$$
$$T = \varepsilon_\mu \begin{pmatrix} \mathrm{N}_{E/F}(1+v) & \frac{-2(1+a)b}{1-\mathrm{N}_{E/F}(v)} \\ \frac{-2(1+a)b}{1-\mathrm{N}_{E/F}(v)} & \frac{-(1+a)^2 \mathrm{N}_{E/F}(1-v)}{d\left(1-\mathrm{N}_{E/F}(v)\right)^2} \end{pmatrix}.$$

Then $\operatorname{ord}(\alpha) = n_1 + m' > \operatorname{ord}(\beta) = n_1 + \lambda_2$ and $\operatorname{ord}(a_1) = \lambda_1 - \lambda_2$. By Proposition 3.18, we have

$$\mathcal{B}^*(\lambda; v, \mu) = q^{m' + \frac{n}{2}} \cdot \mathcal{K}l\left(\frac{-\varpi^{-\frac{n}{2}}}{1 - \mathrm{N}_{E/F}(v)}, \frac{\varpi^{-\frac{n}{2}}\varepsilon_\mu \mathrm{N}_{E/F}(1+v)}{1 - \mathrm{N}_{E/F}(v)}\right),$$

in the case (a), and

$$\mathcal{B}^*(\lambda; v, \mu) = -q^{m' + \frac{n}{2} + 1} \cdot \mathcal{K}l\left(\frac{-\varpi^{-\frac{n}{2}}}{1 - \mathrm{N}_{E/F}(v)}, \frac{\varpi^{-\frac{n}{2}}\varepsilon_\mu \mathrm{N}_{E/F}(1+v)}{1 - \mathrm{N}_{E/F}(v)}\right),$$

in the case (c), respectively. Here we have $1+v=(1+a)+b\eta\equiv 2 \pmod{\varpi^{m'}\mathcal{O}_E}$ by Lemma 3.21. Thus we have (3.49) and (3.50). The rest is clear. $\square$

Finally let us evaluate $\mathcal{B}^{(a)}(\lambda;u,\mu)$.

PROPOSITION 3.22. *Let $u\in E^\times\cap\mathcal{O}_E$ such that $\mathrm{N}_{E/F}(u)\neq 1$. We put $x=\mathrm{N}_{E/F}(u)$. Let $m'=\mathrm{ord}(1-x)$. Let $\mu\in F^\times$ and let $n=-\mathrm{ord}(\mu)$.*

Then for $\lambda=(\lambda_1,\lambda_2)\in P^+\setminus\{(0,0)\}$, the integral $\mathcal{B}^{(a)}(\lambda;u,\mu)$ is evaluated as follows.

(1) *When $m'=0$, we have*

$$\mathcal{B}^{(a)}(\lambda;u,\mu)=\begin{cases} q^{3n}\omega(\varpi)^n & \text{if } n\leq -1 \text{ and } \lambda=(-n,0),\\ -q^{3n-3}\omega(\varpi)^{n-1} & \text{if } n\leq 0 \text{ and } \lambda=(1-n,1),\\ q^{3n-5}\omega(\varpi)^{n-1} & \text{if } n\leq 1 \text{ and } \lambda=(2-n,0),\\ 0 & \text{otherwise.}\end{cases} \tag{3.57}$$

(2) *Suppose that $m'>0$.*

(a) *When $n\geq -2m'+2$, we have*

$$\begin{aligned}\mathcal{B}^{(a)}&(\lambda;u,\mu)\\ &=\begin{cases} \dfrac{-q^{m'+\frac{5n}{2}-3}\omega(\varpi)^{n-1}}{1+q^{-1}}\cdot\mathcal{K}l_2 & \text{if } n\leq 0,\ n \text{ is even},\ \lambda=\left(\frac{2-n}{2},\frac{2-n}{2}\right),\\ \dfrac{q^{m'+\frac{5n}{2}}\omega(\varpi)^{n}}{1+q^{-1}}\cdot\mathcal{K}l_2 & \text{if } n\leq -2,\ n \text{ is even},\ \lambda=\left(\frac{-n}{2},\frac{-n}{2}\right),\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{3.58}$$

(b) *When $n=-2m'+1$, we have*

$$\mathcal{B}^{(a)}(\lambda;u,\mu)=\begin{cases} -q^{-3}\omega(\varpi)^{-1} & \text{if } m'=1,\ \lambda=(1,0),\\ \dfrac{-q^{-4m'+1}\,\omega(\varpi)^{1-2m'}}{1+q^{-1}} & \text{if } m'\geq 2,\ \lambda=(m',m'-1),\\ \dfrac{q^{-4m'-2}\,\omega(\varpi)^{-2m'}}{1+q^{-1}} & \text{if } \lambda=(m'+1,m'),\\ 0 & \text{otherwise.}\end{cases} \tag{3.59}$$

(c) *When $n\leq -2m'$, we have*

$$\mathcal{B}^{(a)}(\lambda;u,\mu)=\begin{cases} \dfrac{q^{2m'+3n}\,\omega(\varpi)^n}{1+q^{-1}} & \text{if } \lambda=(-m'-n,m'),\\ -q^{3n}\,\omega(\varpi)^n & \text{if } m'=1,\ \lambda=(-n,0),\\ \dfrac{-q^{2m'+3n-2}\,\omega(\varpi)^n}{1+q^{-1}} & \text{if } m'\geq 2,\ \lambda=(-m'-n+1,m'-1),\\ \dfrac{-q^{2m'+3n-3}\,\omega(\varpi)^{n-1}}{1+q^{-1}} & \text{if } \lambda=(-m'-n+1,m'+1),\\ \dfrac{q^{2m'+3n-5}\,\omega(\varpi)^{n-1}}{1+q^{-1}} & \text{if } \lambda=(2-m'-n,m'),\\ 0 & \text{otherwise.}\end{cases} \tag{3.60}$$

Proof. When $m' > 0$, we may assume that $u = a_0 \in \mathcal{O}^\times$ with $\operatorname{ord}(1-a_0) = 0$ and $\operatorname{ord}(1+a_0) = m'$, since $\mathcal{B}^{(a)}(\lambda; u, \mu) = \mathcal{B}^{(a)}(\lambda; u', \mu)$ when $\mathrm{N}_{E/F}(u) = \mathrm{N}_{E/F}(u')$.

Here we observe the following lemma.

Lemma 3.23. *Let $a_0 \in \mathcal{O}^\times$ such that $\operatorname{ord}(1-a_0) = 0$ and $\operatorname{ord}(1+a_0) > 0$. Let $\varepsilon = a_\varepsilon + b_\varepsilon \eta \in \mathcal{O}_E^\times$ with $a_\varepsilon, b_\varepsilon \in \mathcal{O}$. Let $v = a_0\, \varepsilon \varepsilon^{-\sigma}$. Let us write $v = a_v + b_v \eta$ where $a_v, b_v \in \mathcal{O}$. Then we have $\operatorname{ord}(1-a_v) > 0$ if and only if $\operatorname{ord}(a_\varepsilon) > 0$. Moreover, when $\operatorname{ord}(a_\varepsilon) > 0$, we have $\operatorname{ord}(b_v) = \operatorname{ord}(a_\varepsilon)$.*

Proof. Since $v = a_0\, \varepsilon^2\, \mathrm{N}_{E/F}(\varepsilon)^{-1}$, we have

$$a_v = a_0 \left(a_\varepsilon^2 + b_\varepsilon^2 d\right) \mathrm{N}_{E/F}(\varepsilon)^{-1} \quad \text{and} \quad b_v = 2 a_0 a_\varepsilon b_\varepsilon\, \mathrm{N}_{E/F}(\varepsilon)^{-1}.$$

Hence $1 - a_v = \left\{(1-a_0)\, a_\varepsilon^2 - b_\varepsilon^2 d\,(1+a_0)\right\} \mathrm{N}_{E/F}(\varepsilon)^{-1}$ where $\operatorname{ord}(1-a_0) = 0$ and $\operatorname{ord}(1+a_0) > 0$. Thus we have $\operatorname{ord}(1-a_v) > 0$ if and only if $\operatorname{ord}(a_\varepsilon) > 0$. When $\operatorname{ord}(a_\varepsilon) > 0$, we have $b_\varepsilon \in \mathcal{O}^\times$ since $\varepsilon \in \mathcal{O}_E^\times$. Hence we have $\operatorname{ord}(b_v) = \operatorname{ord}(a_\varepsilon)$. □

We return to the proof of the proposition. For a positive integer r, we note that

$$\int_{\left\{\varepsilon \in \mathcal{O}_E^\times \mid \varepsilon = a_\varepsilon + b_\varepsilon \eta,\, a_\varepsilon \in \varpi^r \mathcal{O},\, b_\varepsilon \in \mathcal{O}^\times\right\}} d^\times \varepsilon = \frac{1}{1-q^{-2}} \int_{\varpi^r \mathcal{O}} \int_{\mathcal{O}^\times} da_\varepsilon\, db_\varepsilon = \frac{q^{-r}}{1+q^{-1}}.$$

Then the rest of the assertions follow from Corollary 3.20 and Lemma 3.23. □

We recall that $\mathcal{B}^{(a)}(\lambda; u, \mu)$ for $\lambda = (0,0)$ is given as follows (cf. [**6**, Proposition 6]).

Proposition 3.24. *Let $u \in E^\times \cap \mathcal{O}_E$ such that $\mathrm{N}_{E/F}(u) \neq 1$. We put $x = \mathrm{N}_{E/F}(u)$. Let $m = \operatorname{ord}(x)$ and let $m' = \operatorname{ord}(1-x)$. Let $\mu \in F^\times$ and let $n = -\operatorname{ord}(\mu)$.*

Then the integral $\mathcal{B}^{(a)}(0; u, \mu)$ is evaluated as follows.

1. *The integral $\mathcal{B}^{(a)}(0; u, \mu)$ vanishes unless $n \geq 0$ and n is even.*
2. *When $n = 0$ and $m' = 0$, we have $\mathcal{B}^{(a)}(0; u, \mu) = 1$.*
3. *When $n = 0$ and $m' > 0$, we have $\mathcal{B}^{(a)}(0; u, \mu) = q^{m'}\, \mathcal{K}l_2$.*
4. *When $m \geq n > 0$ and n is even, we have*
$$\mathcal{B}^{(a)}(0; u, \mu) = \delta(\varpi)^n\, q^n \left\{(-1)^{\frac{n}{2}} \cdot \mathcal{K}l_2 + 1 + q^{-1}\right\}.$$
5. *When $n > m$ and n is even, we have*
$$\mathcal{B}^{(a)}(0; u, \mu) = \delta(\varpi)^n\, q^{m'+n} \left\{(-1)^{\frac{m-n}{2}} \cdot \mathcal{K}l_1 + (-1)^{\frac{n}{2}} \cdot \mathcal{K}l_2\right\}.$$

For a fixed pair (u, μ), we regard $\mathcal{B}^{(a)}(\lambda; u, \mu)$ as a function on P^+ and we simply write it as $\mathcal{B}^{(a)}(\lambda)$. Then by Proposition 3.22 and Proposition 3.24, the function $\mathcal{B}^{(a)}$ is expressed as follows.

Proposition 3.25. *Let $u \in E^\times \cap \mathcal{O}_E$ such that $\mathrm{N}_{E/F}(u) \neq 1$. We put $x = \mathrm{N}_{E/F}(u)$. Let $m = \operatorname{ord}(x)$ and let $m' = \operatorname{ord}(1-x)$. Let $\mu \in F^\times$ and let $n = -\operatorname{ord}(\mu)$. For $\lambda = (\lambda_1, \lambda_2) \in P^+$, we define $C_a(\lambda)$ by*

$$C_a(\lambda) = \left(1+q^{-1}\right)^{-e(\lambda)} q^{m'+n-2\lambda_1-\lambda_2}\, \delta(\varpi)^{n-\lambda_1-\lambda_2}. \tag{3.61}$$

For $(a,b) \in \mathbb{Z}^2$, let $P_{(a,b)}$ denote the characteristic function of the set $\{(a,b)\} \cap P^+$.

Then the function $\mathcal{B}^{(a)}$ on P^+ is expressed as follows.

(1) *Suppose that* $m' = 0$.

(a) *When* $n \geq 3$ *and* n *is odd, we have* $\mathcal{B}^{(a)} = 0$.

(b) *When* $n \geq 2$ *and* n *is even, we have*

$$C_a^{-1} \cdot \mathcal{B}^{(a)} = \begin{cases} P_{(0,0)} \left\{ (-1)^{\frac{m-n}{2}} \cdot \mathcal{K}l_1 + (-1)^{\frac{n}{2}} \cdot \mathcal{K}l_2 \right\} & \text{if } n > m, \\ P_{(0,0)} \left\{ 1 + q^{-1} + (-1)^{\frac{n}{2}} \cdot \mathcal{K}l_2 \right\} & \text{if } n \leq m. \end{cases}$$

(c) *When* $n \leq 1$, *we have*

$$C_a^{-1} \cdot \mathcal{B}^{(a)} = P_{(-n,0)} + q^{-1} P_{(2-n,0)} - \left(1 + q^{-1}\right) P_{(1-n,1)}.$$

(2) *Suppose that* $m' > 0$.

(a) *When* $n \geq -2m' + 3$ *and* n *is odd, we have* $\mathcal{B}^{(a)} = 0$.

(b) *When* $n \geq -2m' + 2$ *and* n *is even, we have*

$$C_a^{-1} \cdot \mathcal{B}^{(a)} = \begin{cases} P_{(0,0)} \, (-1)^{\frac{n}{2}} \left(\mathcal{K}l_1 + \mathcal{K}l_2\right) & \text{if } n \geq 2, \\ \left(P_{\left(\frac{-n}{2}, \frac{-n}{2}\right)} - P_{\left(\frac{2-n}{2}, \frac{2-n}{2}\right)} \right) \cdot \mathcal{K}l_2 & \text{if } n \leq 0. \end{cases}$$

(c) *When* $n \leq -2m' + 1$, *we have*

$$C_a^{-1} \cdot \mathcal{B}^{(a)} = \left(P_{(-m'-n,m')} - P_{(1-m'-n,m'+1)} \right) + q^{-1} \left(P_{(2-m'-n,m')} - P_{(1-m'-n,m'-1)} \right).$$

CHAPTER 4

Evaluation of the Split Bessel Orbital Integral

In the first section, we prove the functional equation (4.1) for the split Bessel orbital integral. Then we rewrite the degenerate split Bessel orbital integral defined by (2.44) in a form suitable for our subsequent evaluation and perform some preliminary computations. In the second section, we evaluate the degenerate split Bessel orbital integrals explicitly.

4.1. Preliminaries

4.1.1. Functional equation. We note the following functional equation satisfied by the split Bessel orbital integral $\mathcal{B}^{(s)}(x,\mu;f)$ defined in (1.11).

PROPOSITION 4.1. *For $x \in F \setminus \{0,1\}$, $\mu \in F^\times$ and $f \in \mathcal{H}$, we have the functional equation*

$$\mathcal{B}^{(s)}\left(x^{-1},\mu x^{-1};f\right) = \delta(x)\cdot\mathcal{B}^{(s)}(x,\mu;f). \tag{4.1}$$

PROOF. For $A^{(s)}(x,\mu)$ defined by (1.12), we have

$$w_0\,A^{(s)}(x,\mu) = A^{(s)}\left(x^{-1},\mu x^{-1}\right)\begin{pmatrix}1&0&0&0\\0&x&0&0\\0&0&x&0\\0&0&0&1\end{pmatrix}, \quad \text{where } w_0 = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

Since w_0 stabilizes $\xi^{(s)}$, we have

$$\begin{aligned}\mathcal{B}^{(s)}(x,\mu;f) &= \int_{Z\backslash\bar{R}^{(s)}}\int_{R^{(s)}} f\left[\bar{r}\,A^{(s)}\left(x^{-1},\mu x^{-1}\right)\begin{pmatrix}1&0&0&0\\0&x&0&0\\0&0&x&0\\0&0&0&1\end{pmatrix}r\right]\xi^{(s)}(\bar{r})\,\tau^{(s)}(r)\,dr\,d\bar{r}\\ &= \delta(x)^{-1}\cdot\mathcal{B}^{(s)}\left(x^{-1},\mu x^{-1};f\right).\end{aligned}$$

□

Thus it is enough for us to evaluate $\mathcal{B}^{(s)}(\lambda;x,\mu)$ when $|x|\le 1$.

4.1.2. Rewriting the integral. We rewrite the degenerate orbital integral $\mathcal{B}^{(s)}(\lambda;x,\mu)$ defined by (2.44).

PROPOSITION 4.2. *Let $x \in \mathcal{O}\setminus\{0,1\}$, $\mu \in F^\times$ and $\lambda = (\lambda_1,\lambda_2) \in P^+$. Let $m = \mathrm{ord}(x)$ and $m' = \mathrm{ord}(1-x)$. Let $n = -\mathrm{ord}(\mu)$ and we put $\varepsilon_\mu = \varpi^n\mu \in \mathcal{O}^\times$.*

For a given pair (x,μ), the function $\mathcal{B}^{(s)}(\lambda) = \mathcal{B}^{(s)}(\lambda;x,\mu)$ on P^+ is supported on the set

$$P^+(x,\mu) = \left\{(\lambda_1,\lambda_2)\in P^+ \mid \lambda_1+\lambda_2 \ge -n,\ \lambda_1 \ge -m'-n\right\}. \tag{4.2}$$

For $\lambda = (\lambda_1,\lambda_2) \in P^+(x,\mu)$, we have

$$\mathcal{B}^{(s)}(\lambda) = q^{-3\lambda_1}\delta(\varpi)^{n-\lambda_1-\lambda_2}\sum_{\substack{0\le i\le m+n+\lambda_1+\lambda_2\\0\le j\le n+\lambda_1+\lambda_2}}\mathcal{N}_\lambda^{i,j}(x,\mu), \tag{4.3}$$

where

$$\mathcal{N}_\lambda^{i,j}(x,\mu) = \int_{\mathcal{O}^\times} \mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(\varepsilon), \varepsilon_\mu \varepsilon\right) d^\times \varepsilon,$$

$$A_\lambda^{i,j}(\varepsilon) = \begin{pmatrix} \varpi^j \varepsilon & \varpi^{j-\lambda_2}\varepsilon + \varpi^{n+\lambda_1-i} x \\ \varpi^i \varepsilon & \varpi^{i-\lambda_2}\varepsilon + \varpi^{n+\lambda_1-j} \end{pmatrix},$$

and

$$\mathcal{N}_\lambda^*(A,\gamma) = \mathcal{K}l\left(A; \begin{pmatrix} 0 & \varpi^{\lambda_1} \\ \varpi^{\lambda_1} & 2\varpi^{\lambda_1-\lambda_2} \end{pmatrix}, \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}\right) \tag{4.4}$$

for $A \in \mathrm{GL}_2(F)$ *and* $\gamma \in \mathcal{O}^\times$.

Before proving the proposition, we note some basic properties of the integral $\mathcal{N}_\lambda^*(A,\gamma)$.

LEMMA 4.3. *Suppose that* $A \in \mathrm{M}_2(\mathcal{O}) \cap \mathrm{GL}_2(F)$ *and* $\gamma \in \mathcal{O}^\times$.

(1) *For* $\varepsilon_1, \varepsilon_2 \in \mathcal{O}^\times$, *we have*

$$\mathcal{N}_\lambda^*\left(\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} A, \gamma\varepsilon_1\varepsilon_2\right) = \mathcal{N}_\lambda^*(A,\gamma). \tag{4.5}$$

(2) *We have*

$$\mathcal{N}_\lambda^*\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A, \gamma\right) = \mathcal{N}_\lambda^*(A,\gamma). \tag{4.6}$$

(3) *For* $\varepsilon_1, \varepsilon_2 \in \mathcal{O}^\times$ *with* $\varepsilon_1 - \varepsilon_2 \in \varpi^{\lambda_2}\mathcal{O}$, *we have*

$$\mathcal{N}_\lambda^*\left(A\begin{pmatrix} \varepsilon_1 & \varpi^{-\lambda_2}(\varepsilon_1-\varepsilon_2) \\ 0 & \varepsilon_2 \end{pmatrix}, \gamma\varepsilon_1\varepsilon_2\right) = \mathcal{N}_\lambda^*(A,\gamma). \tag{4.7}$$

(4) *We have*

$$\mathcal{N}_\lambda^*\left(A\begin{pmatrix} -1 & 0 \\ \varpi^{\lambda_2} & 1 \end{pmatrix}, \gamma\right) = \mathcal{N}_\lambda^*(A,\gamma). \tag{4.8}$$

PROOF. We recall that

$$\mathcal{N}_\lambda^*(A,\gamma) = \int_{\mathrm{Sym}^2(F)}\int_{\mathrm{Sym}^2(F)} \Xi\left[\begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix}\right] \psi\left[\mathrm{tr}\left\{\varpi^{\lambda_1}\begin{pmatrix} 0 & 1 \\ 1 & 2\varpi^{-\lambda_2} \end{pmatrix} X + \gamma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y\right\}\right] dX\, dY. \tag{4.9}$$

By the change of variable

$$\begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \mapsto \begin{pmatrix} 1_2 & 0 \\ 0 & \gamma 1_2 \end{pmatrix}^{-1}\begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix}\begin{pmatrix} 1_2 & 0 \\ 0 & \gamma 1_2 \end{pmatrix}$$

in (4.9), we have

$$\mathcal{N}_\lambda^*(A,\gamma) = \int_{\mathrm{Sym}^2(F)}\int_{\mathrm{Sym}^2(F)} \Xi\left[\begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix}\right] \psi\left[\mathrm{tr}\left\{\gamma\varpi^{\lambda_1}\begin{pmatrix} 0 & 1 \\ 1 & 2\varpi^{-\lambda_2} \end{pmatrix} X + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y\right\}\right] dX\, dY. \tag{4.10}$$

Then (4.5) follows from the change of variable

$$\begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \mapsto \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & \varepsilon_1^{-1} & 0 \\ 0 & 0 & 0 & \varepsilon_2^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & \varepsilon_1^{-1} & 0 \\ 0 & 0 & 0 & \varepsilon_2^{-1} \end{pmatrix}$$

in (4.9) since

$$\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} = \varepsilon_1 \varepsilon_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Similarly (4.6), (4.7) and (4.8) hold since we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \varepsilon_1 & 0 \\ \varpi^{-\lambda_2} (\varepsilon_1 - \varepsilon_2) & \varepsilon_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2\varpi^{-\lambda_2} \end{pmatrix} \begin{pmatrix} \varepsilon_1 & \varpi^{-\lambda_2} (\varepsilon_1 - \varepsilon_2) \\ 0 & \varepsilon_2 \end{pmatrix} = \varepsilon_1 \varepsilon_2 \begin{pmatrix} 0 & 1 \\ 1 & 2\varpi^{-\lambda_2} \end{pmatrix},$$

and

$$\begin{pmatrix} -1 & \varpi^{\lambda_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2\varpi^{-\lambda_2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ \varpi^{\lambda_2} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2\varpi^{-\lambda_2} \end{pmatrix},$$

respectively. □

PROOF OF PROPOSITION 4.2. We may write (2.44) as

$$\mathcal{B}^{(s)}(\lambda) = \int_Z \int_{F^\times} \int_{F^\times} \int_{\bar{U}} \int_U \Xi \left[z\bar{n} \left(\begin{smallmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{smallmatrix} \right) A^{(s)}(x, \mu) \left(\begin{smallmatrix} b & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \end{smallmatrix} \right) n b_\lambda^{(s)} \right] \delta\left(abz^2\right) \xi^{(s)}(\bar{n})\, \tau^{(s)}(n)\, dz\, d^\times a\, d^\times b\, d\bar{n}\, dn.$$

Making the change of variable $n \mapsto b_\lambda^{(s)}\, n\, (b_\lambda^{(s)})^{-1}$, we have

$$\mathcal{B}^{(s)}(\lambda) = q^{-3\lambda_1} \int_Z \int_{F^\times} \int_{F^\times} \delta\left(abz^2\right) \cdot \mathcal{N}_\lambda^* \left[z\varpi^{\lambda_1} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b\varpi^{\lambda_2} & b \\ 0 & 1 \end{pmatrix}, z^2 ab\mu\varpi^{\lambda_1+\lambda_2} \right] dz\, d^\times a\, d^\times b \tag{4.11}$$

where

$$\mathcal{N}_\lambda^*(A, \gamma) = \int_{\mathrm{Sym}^2(F)} \int_{\mathrm{Sym}^2(F)} \Xi \left[\begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \gamma \cdot {}^t A^{-1} \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \right] \psi\left[\mathrm{tr}\left\{ \varpi^{\lambda_1} \begin{pmatrix} 0 & 1 \\ 1 & 2\varpi^{-\lambda_2} \end{pmatrix} X + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y \right\} \right] dX\, dY. \tag{4.12}$$

The integrand of (4.12) vanishes unless $A \in \mathrm{M}_2(\mathcal{O})$ and $\gamma \in \mathcal{O}^\times$. Then by another change of variable,

$$\begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \mapsto \begin{pmatrix} 1_2 & 0 \\ 0 & \gamma \cdot 1_2 \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ Y & 1_2 \end{pmatrix} \begin{pmatrix} 1_2 & 0 \\ 0 & \gamma \cdot 1_2 \end{pmatrix}^{-1},$$

we have (4.4).

Put $\mathrm{ord}(az) = n-i$ and $\mathrm{ord}(z) = n-j$. Then $z^2 ab\mu\varpi^{\lambda_1+\lambda_2} \in \mathcal{O}^\times$ implies that $b = \varpi^{i+j-n-\lambda_1-\lambda_2}\varepsilon$ for some $\varepsilon \in \mathcal{O}^\times$. By (4.5), we may assume that $z = \varpi^{n-j}$ and $a = \varpi^{-i+j}$ in (4.11). Thus we have

$$\mathcal{B}^{(s)}(\lambda) = q^{-3\lambda_1} \delta(\varpi)^{n-\lambda_1-\lambda_2} \sum_{i,j \in \mathbb{Z}} \mathcal{N}_\lambda^{i,j}(x, \mu).$$

Suppose that $A_\lambda^{i,j}(\varepsilon) \in \mathrm{M}_2(\mathcal{O})$. Then from the first row, we have $i \geq 0$ and $j \geq 0$. Since $\lambda_2 \geq 0$, the entries of ϖ^{λ_2} time the second row minus the first row, i.e., $\varpi^{n+\lambda_1+\lambda_2-i}x$ and $\varpi^{n+\lambda_1+\lambda_2-j}$, belong to $\mathcal{O}$. Thus we have $i \leq m+n+\lambda_1+\lambda_2$ and $j \leq n+\lambda_1+\lambda_2$. Hence (4.3) holds. We also note that $\det A_\lambda^{i,j}(\varepsilon) = \varpi^{n+\lambda_1}(1-x)$. Thus $\mathcal{B}^{(s)}(\lambda)$ vanishes unless (4.2) holds. $\square$

REMARK 4.4. Here we remark that for $\varepsilon' \in \mathcal{O}^\times$ such that $\varepsilon' - 1 \in \varpi^{\lambda_2}\mathcal{O}$, we have

$$A_\lambda^{i,j}(\varepsilon\varepsilon') = A_\lambda^{i,j}(\varepsilon)\begin{pmatrix} \varepsilon' & \varpi^{-\lambda_2}(\varepsilon'-1) \\ 0 & 1 \end{pmatrix}.$$

Hence by (4.7) we have

$$\mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(\varepsilon\varepsilon'), \varepsilon_\mu\varepsilon\varepsilon'\right) = \mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(\varepsilon), \varepsilon_\mu\varepsilon\right). \tag{4.13}$$

In particular when $\lambda_2 = 0$, we have

$$\mathcal{B}^{(s)}(\lambda) = q^{-3\lambda_1}\delta(\varpi)^{n-\lambda_1}\sum_{\substack{0\leq i\leq m+n+\lambda_1 \\ 0\leq j\leq n+\lambda_1}} \mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(1), \varepsilon_\mu\right). \tag{4.14}$$

We also note the following symmetry in the summation in (4.3).

LEMMA 4.5. *For $i' = m+n+\lambda_1+\lambda_2-i$ and $j' = n+\lambda_1+\lambda_2-j$, we have*

$$\mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(\varepsilon), \varepsilon_\mu\varepsilon\right) = \mathcal{N}_\lambda^*\left(A_\lambda^{i',j'}\left(\varepsilon_x\varepsilon^{-1}\right), \varepsilon_\mu\varepsilon_x\varepsilon^{-1}\right) \tag{4.15}$$

and

$$\mathcal{N}_\lambda^{i,j}(x,\mu) = \mathcal{N}_\lambda^{i',j'}(x,\mu), \tag{4.16}$$

under the assumptions present in (4.3).

PROOF. Let us write $x = \varpi^m\varepsilon_x$. Then we have

$$\begin{pmatrix} \varepsilon_x\varepsilon^{-1} & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_\lambda^{i,j}(\varepsilon)\begin{pmatrix} -1 & 0 \\ \varpi^{\lambda_2} & 1 \end{pmatrix} = A_\lambda^{i',j'}\left(\varepsilon_x\varepsilon^{-1}\right). \tag{4.17}$$

Hence by Lemma 4.3, we have (4.15) and then (4.16) follows. $\square$

Let us evaluate $\mathcal{N}_\lambda^*(A,\varepsilon)$ explicitly.

PROPOSITION 4.6. *Suppose that $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}) \cap \mathrm{GL}_2(F)$ and $\varepsilon \in \mathcal{O}^\times$. We put $\|A\| = \max\{|\alpha|,|\beta|,|\gamma|,|\delta|\}$ and $\Delta = \det A$.*

Then for $\lambda = (\lambda_1,\lambda_2) \in P^+ \setminus \{(0,0)\}$, the integral $\mathcal{N}_\lambda^*(A,\varepsilon)$ is evaluated as follows.

(1) *Suppose that $\|A\| = 1$.*
  (a) *When $|\Delta| = 1$, we have*

$$\mathcal{N}_\lambda^*(A,\varepsilon) = 1. \tag{4.18}$$

(b) *When* $|\Delta| < 1$*, we have*

(4.19)
$$\mathcal{N}^*_\lambda(A,\varepsilon) = \begin{cases} |\Delta|^{-1}\cdot \mathcal{K}l\left(\dfrac{-2\gamma}{\Delta}, \dfrac{2\varpi^{\lambda_1-\lambda_2}\varepsilon\left(\varpi^{\lambda_2}\beta-\alpha\right)}{\Delta}\right) & \textit{when } |\alpha| = 1, \\ |\Delta|^{-1}\cdot \mathcal{K}l\left(\dfrac{-2\delta}{\Delta}, \dfrac{2\varpi^{\lambda_1-\lambda_2}\varepsilon\alpha\beta^{-1}\left(\varpi^{\lambda_2}\beta-\alpha\right)}{\Delta}\right) & \textit{when } |\beta| = 1, \\ |\Delta|^{-1}\cdot \mathcal{K}l\left(\dfrac{-2\alpha}{\Delta}, \dfrac{2\varpi^{\lambda_1-\lambda_2}\varepsilon\left(\varpi^{\lambda_2}\delta-\gamma\right)}{\Delta}\right) & \textit{when } |\gamma| = 1, \\ |\Delta|^{-1}\cdot \mathcal{K}l\left(\dfrac{-2\beta}{\Delta}, \dfrac{2\varpi^{\lambda_1-\lambda_2}\varepsilon\gamma\delta^{-1}\left(\varpi^{\lambda_2}\delta-\gamma\right)}{\Delta}\right) & \textit{when } |\delta| = 1. \end{cases}$$

(2) *Suppose that* $\|A\| < 1$.

(a) *When* $|\Delta| = \|A\|^2$*, the integral* $\mathcal{N}^*_\lambda(A,\varepsilon)$ *vanishes unless* $\|A\| = q^{-1}$*. When* $\|A\| = q^{-1}$*, we have*

$$\mathcal{N}^*_\lambda(A,\varepsilon) = \begin{cases} q^2\cdot\mathcal{K}l\left(\dfrac{2\alpha}{\Delta}, \dfrac{2\varepsilon\gamma}{\Delta}\right) & \textit{when } \lambda_1=\lambda_2 \textit{ and } |\alpha| = |\gamma|, \\ -q & \textit{otherwise.} \end{cases} \tag{4.20}$$

(b) *When* $|\Delta| < \|A\|^2$*, the integral* $\mathcal{N}^*_\lambda(A,\varepsilon)$ *vanishes unless*

$$\|A\| = |\alpha| = |\gamma| = q^{-1} \qquad \textit{and} \qquad \lambda_1 = \lambda_2. \tag{4.21}$$

When (4.21) *holds, we have*

$$\mathcal{N}^*_\lambda(A,\varepsilon) = |\Delta|^{-1}\cdot\mathcal{K}l\left(\frac{-2\gamma}{\Delta}, \frac{2\varepsilon\left(\varpi^{\lambda_2}\beta-\alpha\right)}{\Delta}\right). \tag{4.22}$$

Proof. By (4.6), we may assume that $\|A\| = |\alpha|$ or $\|A\| = |\delta|$. When $\|A\| = |\alpha|$, we have $A = k_1Ck_2$ where

$$k_1 = \begin{pmatrix} 0 & 1 \\ 1 & \alpha^{-1}\gamma \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 & 1 \\ 1 & \alpha^{-1}\beta \end{pmatrix}, \quad C = \begin{pmatrix} \Delta\alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \tag{4.23}$$

and $\mathcal{N}^*_\lambda(A,\varepsilon) = \mathcal{K}l\,(C;S,T)$ with

$$S = \varpi^{\lambda_1-\lambda_2}\begin{pmatrix} 2\left(1-\varpi^{\lambda_2}\alpha^{-1}\beta\right) & \varpi^{\lambda_2} \\ \varpi^{\lambda_2} & 0 \end{pmatrix}, \quad T = \varepsilon\begin{pmatrix} -2\alpha^{-1}\gamma & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.24}$$

by (3.13). Similarly when $\|A\| = |\delta|$, we have $A = k_1Ck_2$, where

$$k_1 = \begin{pmatrix} 0 & 1 \\ 1 & \alpha^{-1}\gamma \end{pmatrix}, \quad k_2 = \begin{pmatrix} 0 & 1 \\ 1 & \alpha^{-1}\beta \end{pmatrix}, \quad C = \begin{pmatrix} \Delta\delta^{-1} & 0 \\ 0 & \delta \end{pmatrix} \tag{4.25}$$

and $\mathcal{N}^*_\lambda(A,\varepsilon) = \mathcal{K}l\,(C;S,T)$ with

(4.26)
$$S = \varpi^{\lambda_1-\lambda_2}\begin{pmatrix} 2\gamma\delta^{-1}\left(\gamma\delta^{-1}-\varpi^{\lambda_2}\right) & \varpi^{\lambda_2}-2\gamma\delta^{-1} \\ \varpi^{\lambda_2}-2\gamma\delta^{-1} & 2 \end{pmatrix}, \quad T = \varepsilon\begin{pmatrix} -2\beta\delta^{-1} & 1 \\ 1 & 0 \end{pmatrix}.$$

In both cases we write $C = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $S = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$ and $T = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}$. Here we note that $b_2 \in \mathcal{O}^\times$ and $b_3 = 0$.

Suppose that $\|A\| = 1$. Then (4.18) and (4.19) follow from Corollary 3.14 immediately.

Suppose that $\|A\| < 1$ and $|\Delta| = \|A\|^2$. Then we have $|a| = |b| = \|A\|$. We also have $\|A\| = |\alpha|$ or $\|A\| = |\gamma|$. By noting that the right hand side of (4.20) is symmetric with respect to α and γ, we may assume that $\|A\| = |\alpha|$ by (4.6). Then by Lemma 3.13 we have $\mathcal{N}_\lambda^* (A, \varepsilon) = \mathcal{N}_1^* + \mathcal{N}_2^*$, where

$$\mathcal{N}_1^* = |a|^{-3} \int_{\mathcal{O}^\times} \int_{\varpi\mathcal{O}} \int_{\mathcal{O}^\times} \psi\left(\frac{-a_1 a^{-1} t + 2a_2 b^{-1} s}{rt - ab^{-1}s^2} + b_1 a^{-1} r + 2b_2 b^{-1} s\right) dr\, ds\, dt \tag{4.27}$$

and

$$\mathcal{N}_2^* = |a|^{-3} \int\int\int_{\{r\in\mathcal{O},\, s\in\mathcal{O}^\times,\, t\in\mathcal{O} | rt - ab^{-1}s^2 \in \mathcal{O}^\times\}} \psi\left(\frac{-a_1 a^{-1} t + 2a_2 b^{-1} s}{rt - ab^{-1}s^2} + b_1 a^{-1} r + 2b_2 b^{-1} s\right) dr\, ds\, dt. \tag{4.28}$$

Let us evaluate $\mathcal{N}_1^*$. By the change of variable $r \mapsto \left(r + ab^{-1}s^2\right) t^{-1}$ in (4.27), we have

$$\mathcal{N}_1^* = |a|^{-3} \int_{\mathcal{O}^\times} \int_{\mathcal{O}^\times} \psi\left(-a_1 a^{-1} r^{-1} t + b_1 a^{-1} r t^{-1}\right) \left(\int_{\varpi\mathcal{O}} \psi\left\{b_1 b^{-1} t^{-1} s^2 + 2b^{-1}\left(a_2 r^{-1} + b_2\right) s\right\} ds\right) dr\, dt. \tag{4.29}$$

The inner integral of (4.29) is given by $q^{-1} \cdot \mathcal{G}(M, N)$ where $M = b_1 b^{-1} t^{-1} \varpi^2$ and $N = b^{-1}\left(a_2 r^{-1} + b_2\right)\varpi$. Since $\mathrm{ord}\,(a_2) > 0 = \mathrm{ord}\,(b_2)$, we have

$$\mathrm{ord}\,(N) = 1 - \mathrm{ord}\,(b) < 2 - \mathrm{ord}\,(b) + \mathrm{ord}\,(b_1) = \mathrm{ord}\,(M).$$

By Proposition 3.2, the inner integral vanishes unless $\mathrm{ord}\,(b) = 1$. When $\mathrm{ord}\,(b) = 1$, we have

$$\mathcal{N}_1^* = q^2 \int_{\mathcal{O}^\times} \int_{\mathcal{O}^\times} \psi\left(-a_1 a^{-1} r^{-1} t + b_1 a^{-1} r t^{-1}\right) dr\, dt.$$

Then the change of variable $r \mapsto rt$ gives

$$\mathcal{N}_1^* = q^2\left(1 - q^{-1}\right) \cdot \mathcal{K}l\left(\frac{2\varpi^{\lambda_1 - \lambda_2}\alpha}{\Delta}, \frac{2\varepsilon\gamma}{\Delta}\right). \tag{4.30}$$

As for $\mathcal{N}_2^*$, by changes of variables $r \mapsto ab^{-1}rs$ and $t \mapsto st$ in (4.28), we have

$$\mathcal{N}_2^* = |a|^{-3} \int_{\mathcal{O}^\times} \int\int_{\{r\in\mathcal{O},\, t\in\mathcal{O} | rt - 1 \in \mathcal{O}^\times\}} \psi\left(\frac{-a_1 a^{-2} b s^{-1} t + 2a_2 a^{-1} s^{-1}}{rt - 1} + b_1 b^{-1} r s + 2b_2 b^{-1} s\right) ds\, dr\, dt.$$

Let us write $\mathcal{N}_2^* = \mathcal{N}_{2,1}^* + \mathcal{N}_{2,2}^*$ where

$$\mathcal{N}_{2,1}^* = |a|^{-3} \int_{r\in\mathcal{O}} \int_{t\in\varpi\mathcal{O}} \left(\int_{\mathcal{O}^\times} \psi\left(M' s + N' s^{-1}\right)\, ds\right) dr\, dt \tag{4.31}$$

with $M' = b^{-1}(b_1 r + 2b_2)$, $N' = a^{-1}\left(2a_2 - a_1 a^{-1} bt\right)(rt-1)^{-1}$, and

$$\mathcal{N}^*_{2,2} = |a|^{-3} \int_{\mathcal{O}^\times} \int_{t\in\mathcal{O}^\times} \int_{\{r\in\mathcal{O}|rt-1\in\mathcal{O}^\times\}} \psi\left(\frac{-a_1 a^{-2} bs^{-1} t + 2a_2 a^{-1} s^{-1}}{rt-1} + b_1 b^{-1} rs + 2b_2 b^{-1} s\right) ds\, dt\, dr. \tag{4.32}$$

Let us evaluate $\mathcal{N}^*_{2,1}$. When $\mathrm{ord}\,(b_1) > 0$, the integral $\mathcal{N}^*_{2,1}$ vanishes unless $\mathrm{ord}\,(b) = 1$ by Proposition 3.4 since $\mathrm{ord}\,(M') = -\mathrm{ord}\,(b) < \mathrm{ord}\,(N')$. When $\mathrm{ord}\,(b) = 1$, we have $\mathcal{N}^*_{2,1} = -q$. Suppose that $\mathrm{ord}\,(b_1) = 0$. Then we write

$$\mathcal{N}^*_{2,1} = \sum_{i=1}^{\infty} \mathcal{N}^*_{2,1,i}, \quad \mathcal{N}^*_{2,1,i} = |a|^{-3} \int_{r\in\mathcal{O}} \int_{t\in\varpi^i\mathcal{O}^\times} \left(\int_{\mathcal{O}^\times} \psi\left(M's + N's^{-1}\right) ds\right) dr\, dt.$$

By introducing new variables $u = \varpi^{-i} t$ and $v = rt - 1$, we have

$$\mathcal{N}^*_{2,1,i} = |a|^{-3} \int_{s\in\mathcal{O}^\times} \int_{u\in\mathcal{O}^\times} \int_{v\in -1+\varpi^i\mathcal{O}} \psi\left(-a_1 a^{-2} bs^{-1} \varpi^i uv^{-1} + 2a_2 a^{-1} s^{-1} v^{-1}\right) \psi\left\{b_1 b^{-1} \varpi^{-i} u^{-1}(v+1)s + 2b_2 b^{-1} s\right\} ds\, du\, dv.$$

The change of variable $u \mapsto uv$ yields

$$\mathcal{N}^*_{2,1,i} = |a|^{-3} \int_{s\in\mathcal{O}^\times} \int_{u\in\mathcal{O}^\times} \psi\left(-a_1 a^{-2} bs^{-1} \varpi^i u + b_1 b^{-1} \varpi^{-i} u^{-1} s + 2b_2 b^{-1} s\right) \left(\int_{v\in -1+\varpi^i\mathcal{O}} \psi\left\{b^{-1}\varpi^{-i}\left(b_1 u^{-1} s + 2a_2 a^{-1} bs^{-1}\varpi^i\right) v^{-1}\right\} dv\right) ds\, du.$$

Here the inner integral vanishes. Thus we have shown that

$$\mathcal{N}^*_{2,1} = \begin{cases} -q & \text{when } \mathrm{ord}\,(b) = 1 \text{ and } |\alpha| > |\gamma|, \\ 0 & \text{otherwise.} \end{cases} \tag{4.33}$$

As for $\mathcal{N}^*_{2,2}$, by replacing r by $w = rt - 1$ in (4.32), we have

$$\mathcal{N}^*_{2,2} = |a|^{-3} \int_{\mathcal{O}^\times} \int_{\mathcal{O}^\times} \int_{\mathcal{O}^\times} \psi\left\{-a_1 a^{-2} bs^{-1} tw^{-1} + 2a_2 a^{-1} s^{-1} w^{-1} + b_1 b^{-1} st^{-1}(w+1) + 2b_2 b^{-1} s\right\} ds\, dt\, dw.$$

By the change of variable $t \mapsto st$, we have

$$\mathcal{N}^*_{2,2} = |a|^{-3} \int_{\mathcal{O}^\times} \int_{\mathcal{O}^\times} \psi\left\{-a_1 a^{-2} btw^{-1} + b_1 b^{-1} t^{-1}(w+1)\right\} \left(\int_{\mathcal{O}^\times} \psi\left(2a_2 a^{-1} s^{-1} w^{-1} + 2b_2 b^{-1} s\right) ds\right) dt\, dw.$$

Since $\mathrm{ord}\,(a_2) = \lambda_1 > 0 = \mathrm{ord}\,(b_2)$, the inner integral vanishes unless $\mathrm{ord}\,(b) = 1$ by Proposition 3.4. When $\mathrm{ord}\,(b) = 1$, we have

$$\mathcal{N}^*_{2,2} = -q^2 \int_{\mathcal{O}^\times} \int_{\mathcal{O}^\times} \psi\left(-a_1 a^{-2} btw^{-1} + b_1 b^{-1} t^{-1} w + b_1 b^{-1} t^{-1}\right) dt\, dw.$$

Then the change of variable $w \mapsto tw$ gives

$$\mathcal{N}^*_{2,2} = -q^2 \left(\int_{\mathcal{O}^\times} \psi\left(b_1 b^{-1} t^{-1}\right) dt\right) \cdot \mathcal{K}l\left(\frac{2\varpi^{\lambda_1-\lambda_2}\alpha}{\Delta}, \frac{2\varepsilon\gamma}{\Delta}\right).$$

Hence we have shown that

(4.34)
$$\mathcal{N}_{2,2}^{*}=\begin{cases}-q^{2}\left(1-q^{-1}\right)\cdot\mathcal{K}l\left(\dfrac{2\varpi^{\lambda_1-\lambda_2}\alpha}{\Delta},\dfrac{2\varepsilon\gamma}{\Delta}\right) & \text{if ord}\,(b)=1 \text{ and } |\alpha|>|\gamma|,\\ q\cdot\mathcal{K}l\left(\dfrac{2\varpi^{\lambda_1-\lambda_2}\alpha}{\Delta},\dfrac{2\varepsilon\gamma}{\Delta}\right) & \text{if ord}\,(b)=1 \text{ and } |\alpha|=|\gamma|,\\ 0 & \text{otherwise.}\end{cases}$$

Thus (4.20) follows from (4.30), (4.33) and (4.34).

Suppose that $\|A\|<1$ and $|\Delta|<\|A\|^2$. Then we have $|a|<|b|=\|A\|$. In a way similar how we obtained (3.38), we have

(4.35)
$$\mathcal{N}_{\lambda}^{*}(A,\varepsilon)=|ab^{2}|^{-1}\int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times}\psi\left\{-a_1a^{-1}r^{-1}+a^{-1}\left(b_1rt-a_3ab^{-1}\right)t^{-1}\right\}$$
$$\left(\int_{\mathcal{O}}\psi\left\{b^{-1}r^{-1}\left(b_1rt-a_3ab^{-1}\right)s^{2}+2b^{-1}r^{-1}\left(b_2rt+a_2\right)s\right\}\,ds\right)dr\,dt.$$

When $\mathrm{ord}\,(a_2)>0$, the inner integral of (4.35) vanishes unless $\mathrm{ord}\,(b_1)=0$ by Proposition 3.2. When $\mathrm{ord}\,(b_1)=0$, in a way similar to how we obtained (3.40), we have

$$\mathcal{N}_{\lambda}^{*}(A,\varepsilon)=\begin{cases}q\,|a|^{-1}\cdot\mathcal{K}l\left(a^{-1},-a^{-1}a_1b_1\right) & \text{when ord}\,(b)=1\text{ and ord}\,(a_1)=0,\\ 0 & \text{otherwise,}\end{cases}$$

by Proposition 3.4 since $\mathrm{ord}\,(a)\geq 2$. Hence when $\|A\|=|\alpha|$, the integral $\mathcal{N}_{\lambda}^{*}(A,\varepsilon)$ vanishes unless $|\alpha|=|\gamma|=q^{-1}$ and $\lambda_1=\lambda_2$. When so, (4.8) holds. When $\|A\|=|\delta|$ and $\mathrm{ord}\,(a_2)>0$, we also have $\mathrm{ord}\,(a_1)>0$ and $\mathcal{N}_{\lambda}^{*}(A,\varepsilon)$ vanishes. On the other hand, when $\|A\|=|\delta|$ and $\mathrm{ord}\,(a_2)=0$, we may use (4.6) to assume $\|A\|=|\alpha|$. Hence the integral $\mathcal{N}_{\lambda}^{*}(A,\varepsilon)$ vanishes unless $|\gamma|=|\alpha|=q^{-1}$ and $\lambda_1=\lambda_2$. Thus we finish the proof of the proposition. □

4.2. Evaluation

Let us evaluate the degenerate split Bessel orbital integral $\mathcal{B}^{(s)}(\lambda;x,\mu)$ explicitly. Until the end of this chapter, we fix the notation as follows. Let $x\in\mathcal{O}\setminus\{0,1\}$, $\mu\in F^\times$ and $\lambda=(\lambda_1,\lambda_2)\in P^+\setminus\{(0,0)\}$. Let $m=\mathrm{ord}\,(x)$ and $m'=\mathrm{ord}\,(1-x)$. Put $\varepsilon_x=\varpi^{-m}x$. Let $n=-\mathrm{ord}\,(\mu)$ and put $\varepsilon_\mu=\varpi^{n}\mu$. As in the previous section, for a fixed pair (x,μ), we regard $\mathcal{B}^{(s)}(\lambda;x,\mu)$ as a function on P^+ and we write $\mathcal{B}^{(s)}(\lambda)$ for $\mathcal{B}^{(s)}(\lambda;x,\mu)$.

Before going further, let us introduce some notation. For $\lambda=(\lambda_1,\lambda_2)\in P^+$, let

(4.36)
$$C_s(\lambda)=\left(1-q^{-1}\right)^{-e(\lambda)}q^{m'+n-2\lambda_1-\lambda_2}\delta\left(\varpi\right)^{n-\lambda_1-\lambda_2}.$$

For $(c,d)\in P^+$, we define a subset $\mathcal{L}(c,d)$ of P^+ by

$$\mathcal{L}(c,d)=\left\{(\lambda_1,\lambda_2)\in P^+\mid\lambda_1\geq c,\ \lambda_2=d\right\}$$

and let $L_{(c,d)}$ denote the characteristic function of the set $\mathcal{L}(c,d)$.

4.2.1. The first case. First let us evaluate $\mathcal{B}^{(s)}$ on P_0^+ where

$$P_0^+ = \left\{(\lambda_1, \lambda_2) \in P^+ \mid \lambda_2 = 0\right\}. \tag{4.37}$$

We recall that $\mathcal{B}^{(s)}(0)$ is given as follows by [**6**, Proof of Theorem 2].

LEMMA 4.7. (1) *The integral $\mathcal{B}^{(s)}(0)$ vanishes unless $n \geq 0$.*

(2) *Suppose that $m' = 0$.*

(a) *When $n \geq m+2$, we have*

$$C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{\lambda=0} = (n+1)\,\mathcal{K}l_1 + (m+n+1)\,\mathcal{K}l_2.$$

(b) *When $m+1 \geq n \geq 2$, we have*

$$C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{\lambda=0} = (n+1)\left\{(m-n+1) - (m-n+3)q^{-1}\right\} + (m+n+1)\mathcal{K}l_2.$$

(c) *When $n = 1$, we have $C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{\lambda=0} = 2\left\{m - (m+2)q^{-1}\right\}$.*

(d) *When $n = 0$, we have $C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{\lambda=0} = m+1$.*

(3) *Suppose that $m' > 0$.*

(a) *When $n \geq 1$, we have $C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{\lambda=0} = (n+1)\,(\mathcal{K}l_1 + \mathcal{K}l_2)$.*

(b) *When $n = 0$, we have $C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{\lambda=0} = \mathcal{K}l_1$.*

Let us evaluate $\mathcal{B}^{(s)} \mid_{P_0^+}$.

PROPOSITION 4.8. *The function $\mathcal{B}^{(s)} \mid_{P_0^+}$ is expressed as follows.*

(1) *Suppose that $m' = 0$.*

(a) *When $n \geq m+2$, we have*

$$C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{P_0^+} = 2\,(\mathcal{K}l_1 + \mathcal{K}l_2) \cdot L_{(1,0)} + \{(n+1)\mathcal{K}l_1 + (m+n+1)\mathcal{K}l_2\} \cdot P_{(0,0)}.$$

(b) *When $m+1 \geq n \geq 2$, we have*

$$\begin{aligned} C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{P_0^+} &= 2\left\{(m-n+1) - (m-n+3)\,q^{-1} + \mathcal{K}l_2\right\} \cdot L_{(1,0)} \\ &\quad + \left[(n+1)\left\{(m-n+1) - (m-n+3)q^{-1}\right\} + (m+n+1)\mathcal{K}l_2\right] \cdot P_{(0,0)}. \end{aligned}$$

(c) *When $n = 1$, we have*

$$\begin{aligned} C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{P_0^+} &= \left\{2m - (3m+7)\,q^{-1}\right\} \cdot P_{(1,0)} + 2\left\{m - (m+3)\,q^{-1}\right\} \cdot L_{(2,0)} \\ &\quad + 2\left\{m - (m+2)q^{-1}\right\} \cdot P_{(0,0)}. \end{aligned}$$

(d) *When $n \leq 0$, we have*

$$\begin{aligned} C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{P_0^+} &= (m+1) \cdot P_{(-n,0)} + 2\left\{(m+1) - (m+2)\,q^{-1}\right\} \cdot P_{(1-n,0)} \\ &\quad + \left\{2\,(m+1) - (3m+7)\,q^{-1}\right\} \cdot P_{(2-n,0)} \\ &\quad + 2\left\{(m+1) - (m+3)\,q^{-1}\right\} \cdot L_{(3-n,0)}. \end{aligned}$$

(2) *Suppose that $m' \geq 1$.*

(a) *When $n \geq 1$, we have*

$$C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{P_0^+} = 2\,(\mathcal{K}l_1 + \mathcal{K}l_2) \cdot L_{(1,0)} + (n+1)(\mathcal{K}l_1 + \mathcal{K}l_2) \cdot P_{(0,0)}.$$

(b) *When $n = 0$, we have*

$$C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{P_0^+} = 2\,\mathcal{K}l_1 \cdot L_{(1,0)} + \mathcal{K}l_1 \cdot P_{(0,0)}.$$

(c) *When $n \leq -1$, we have*

$$C_s^{-1} \cdot \mathcal{B}^{(s)} \mid_{P_0^+} = \begin{cases} -q^{-1} \cdot P_{(-n,0)} - 2q^{-1} \cdot L_{(1-n,0)} & \textit{if } m' = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since we already have Lemma 4.7, we may assume that $\lambda_1 \geq 1$. By (4.14), we have

$$\mathcal{B}^{(s)}(\lambda) = C_s(\lambda) \cdot |\Delta| \sum_{\substack{0 \leq i \leq m+n+\lambda_1 \\ 0 \leq j \leq n+\lambda_1}} \mathcal{N}_\lambda^*\left(A^{i,j}, \varepsilon_\mu\right), \tag{4.38}$$

where

$$A^{i,j} = \begin{pmatrix} \varpi^j & \varpi^j + \varpi^{n+\lambda_1-i}x \\ \varpi^i & \varpi^i + \varpi^{n+\lambda_1-j} \end{pmatrix}, \quad \text{with } \Delta = \det A^{i,j} = \varpi^{n+\lambda_1}(1-x).$$

Suppose that $n + \lambda_1 = 0$. Then we have $\|A^{i,0}\| = 1$. When $m' = 0$, we have $|\Delta| = 1$ and

$$\mathcal{B}^{(s)}(\lambda) = C_s(\lambda) \sum_{0 \leq i \leq m} \mathcal{N}_\lambda^*\left(A^{i,0}, \varepsilon_\mu\right) = (m+1) \cdot C_s(\lambda)$$

by (4.18). When $m' \geq 1$, we have $m = 0$ and

$$\mathcal{B}^{(s)}(\lambda) = C_s(\lambda) \cdot \mathcal{K}l_1(x, \mu; 0) = \begin{cases} -q^{-1} \cdot C_s(\lambda) & \text{if } m' = 1, \\ 0 & \text{otherwise,} \end{cases}$$

by (4.19) and Proposition 3.8.

Suppose that $n + \lambda_1 \geq 1$. Then since $|\Delta| < 1$ and $\lambda_1 > \lambda_2 = 0$, the integral $\mathcal{N}_\lambda^*\left(A^{i,j}, \varepsilon_\mu\right)$ vanishes unless

$$\|A^{i,j}\| = q^{-1} \quad \text{and} \quad |\Delta| = q^{-2} \tag{4.39}$$

or

$$\|A^{i,j}\| = 1. \tag{4.40}$$

When (4.39) holds, we have $0 < j < n + \lambda_1$. Hence $n + \lambda_1 \geq 2$. Thus (4.39) holds if and only if

$$n + \lambda_1 = 2, \quad m' = 0, \quad 1 \leq i \leq m+1, \quad j = 1. \tag{4.41}$$

Then we have

$$C_s(\lambda) \cdot |\Delta| \sum_{i=1}^{m+1} \mathcal{N}_\lambda^*\left(A^{i,1}, \varepsilon_\mu\right) = -(m+1)\, q^{-1} \cdot C_s(\lambda)$$

by (4.20).

Equation (4.40) holds if and only if $i = 0,\ m + n + \lambda_1$ or $j = 0,\ n + \lambda_1$. For i such that $0 \leq i \leq m + n + \lambda_1$, we have

$$\mathcal{N}_\lambda^*\left(A^{i,0}, \varepsilon_\mu\right) = |\Delta|^{-1} \cdot \mathcal{K}l_1(x, \mu; \lambda_1 - i).$$

by (4.19). Then by Proposition 3.8, we have

$$\sum_{i=0}^{m+n+\lambda_1} \mathcal{K}l_1\left(x,\mu;\lambda_1-i\right) = \sum_{i'=-m-n}^{\lambda_1} \mathcal{K}l_1\left(x,\mu;i'\right)$$
$$= \begin{cases} \mathcal{K}l_1 & \text{if } n \geq m+2 \text{ and } m-n \text{ is even,} \\ (m-n+1)-(m-n+3)\,q^{-1} & \text{if } m+1 \geq n \geq 1, \\ (m+1)-(m+2)\,q^{-1} & \text{if } n \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

when $m'=0$, and

$$\sum_{i=0}^{n+\lambda_1} \mathcal{K}l_1\left(x,\mu;\lambda_1-i\right) = \begin{cases} \mathcal{K}l_1 & \text{if } n \geq 0 \text{ and } n \text{ is even,} \\ -q^{-1} & \text{if } n \leq -1 \text{ and } m'=1, \\ 0 & \text{otherwise,} \end{cases}$$

when $m'>0$. Similarly for j such that $1 \leq j \leq n+\lambda_1-1$, we have

$$\mathcal{N}_\lambda^*\left(A^{0,j},\varepsilon_\mu\right) = |\Delta|^{-1}\cdot \mathcal{K}l_2\left(x,\mu;\lambda_1-j\right)$$

and

$$\sum_{j=1}^{n+\lambda_1-1} \mathcal{K}l_2\left(x,\mu;\lambda_1-j\right) = \begin{cases} \mathcal{K}l_2 & \text{if } n \geq 2 \text{ and } n \text{ is even,} \\ -q^{-1} & \text{if } n \leq 1 \text{ and } m'=0, \\ 0 & \text{otherwise,} \end{cases}$$

by Proposition 3.8. The rest is clear by summing up all the contributions, taking Lemma 4.5 into account. □

4.2.2. The second case. We shall consider the case when $\lambda_2 \geq 1$. This case is much more elaborate than the previous one. Let

$$P_+^+ = \left\{(\lambda_1,\lambda_2) \in P^+ \mid \lambda_2 \geq 1\right\}. \tag{4.42}$$

For $\lambda = (\lambda_1,\lambda_2) \in P_+^+$, we put

$$a = n+\lambda_1+\lambda_2. \tag{4.43}$$

We recall that for (i,j) such that $0 \leq i \leq a+m$ and $0 \leq j \leq a$, and, $\varepsilon \in \mathcal{O}^\times$, we have

$$A_\lambda^{i,j}\left(\varepsilon\right) = \begin{pmatrix} \varpi^j\varepsilon & \left(\varpi^j\varepsilon+\varpi^{a+m-i}\varepsilon_x\right)\varpi^{-\lambda_2} \\ \varpi^i\varepsilon & \left(\varpi^i\varepsilon+\varpi^{a-j}\right)\varpi^{-\lambda_2} \end{pmatrix},$$

where we note that

$$\det A_\lambda^{i,j}\left(\varepsilon\right) = \varpi^{a-\lambda_2}\varepsilon\left(1-x\right), \quad \operatorname{ord}\left(\det A_\lambda^{i,j}\left(\varepsilon\right)\right) = a-\lambda_2+m'. \tag{4.44}$$

Let us define subsets $\mathcal{A}_k$ $(1 \leq k \leq 4)$ of $\mathrm{M}_2\left(\mathcal{O}\right)$ by

$$\mathcal{A}_1 = \left\{A \in \mathrm{M}_2\left(\mathcal{O}\right) \mid \|A\| = |\det A| = 1\right\} = \mathrm{GL}_2\left(\mathcal{O}\right),$$
$$\mathcal{A}_2 = \left\{A \in \mathrm{M}_2\left(\mathcal{O}\right) \mid \|A\| = 1,\ |\det A| < 1\right\},$$
$$\mathcal{A}_3 = \left\{A \in \mathrm{M}_2\left(\mathcal{O}\right) \mid \|A\| = q^{-1},\ |\det A| = q^{-2}\right\},$$
$$\mathcal{A}_4 = \left\{A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{M}_2\left(\mathcal{O}\right) \mid \|A\| = |\alpha| = |\gamma| = q^{-1},\ |\det A| < q^{-2}\right\}.$$

By Proposition 4.6, the support of $\mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(\varepsilon), \varepsilon_\mu \varepsilon\right)$, as a function of ε, is contained in $\left\{\varepsilon \in \mathcal{O}^\times \mid A_\lambda^{i,j}(\varepsilon) \in \bigcup_{k=1}^4 \mathcal{A}_k\right\}$. Hence we have

$$\mathcal{B}^{(s)}(\lambda; x, \mu) = \sum_{k=1}^4 \mathcal{N}_\lambda^{(k)}(x, \mu),$$

where

$$\mathcal{N}_\lambda^{(k)}(x, \mu) = \left(1 - q^{-1}\right) |\Delta| \, q^{\lambda_2} \, C_s(\lambda) \sum_{\substack{0 \le i \le a+m \\ 0 \le j \le a}} \mathcal{N}_\lambda^{i,j,(k)}(x, \mu),$$

$$\mathcal{N}_\lambda^{i,j,(k)}(x, \mu) = \int_{\left\{\varepsilon \in \mathcal{O}^\times \mid A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_k\right\}} \mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(\varepsilon), \varepsilon_\mu \varepsilon\right) d^\times \varepsilon$$

and

$$\Delta = \varpi^{a-\lambda_2}(1-x).$$

Here we note that

$$\mathcal{N}_\lambda^{i,j,(k)}(x, \mu) = \mathcal{N}_\lambda^{i',j',(k)}(x, \mu), \tag{4.45}$$

where $i' = a + m - i$ and $j' = a - j$, by (4.15) and (4.17).

Evaluation of $\mathcal{N}_\lambda^{(1)}(x, \mu)$. First we note the following lemma.

LEMMA 4.9. *We have $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_1$ if and only if one of the following conditions holds:*

$$a + m' = \lambda_2, \quad i = a + m, \quad j = 0, \quad \varepsilon \in -\varepsilon_x + \varpi^{\lambda_2}\mathcal{O}; \quad \textit{or} \tag{4.46}$$

$$a + m' = \lambda_2, \quad i = 0, \quad j = a, \quad \varepsilon \in -1 + \varpi^{\lambda_2}\mathcal{O}. \tag{4.47}$$

PROOF. By considering the determinant, we have $a - \lambda_2 + m' = 0$. It is clear that we have $\min\{i, j\} = 0$ by looking at the first row of $A_\lambda^{i,j}(\varepsilon)$. When $j = 0$, $\left(\varepsilon + \varpi^{a+m-i}\varepsilon_x\right)\varpi^{-\lambda_2} \in \mathcal{O}$ implies that $i = a+m$ and $\varepsilon \in -\varepsilon_x + \varpi^{\lambda_2}\mathcal{O}$. Conversely when (4.46) holds, we have

$$\left(\varpi^{a+m}\varepsilon + \varpi^a\right)\varpi^{-\lambda_2} = \varpi^{a-\lambda_2}(1-x) + \varpi^{a+m-\lambda_2}(\varepsilon + \varepsilon_x) \in \mathcal{O},$$

and hence $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_1$. The other case is similar. $\square$

LEMMA 4.10. *For $(\lambda_1, \lambda_2) \in P_+^+$, the integral $\mathcal{N}_\lambda^{(1)}(x, \mu)$ is evaluated as follows.*

(1) *When $m' = 0$ and $n \le -1$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(1)}(x, \mu) = 2 \sum_{i=1}^{-n} P_{(-n,i)}(\lambda).$$

(2) *When $m' \ge 1$ and $2m' + n \le 0$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(1)}(x, \mu) = 2 \sum_{i=m'+1}^{-m'-n} P_{(-m'-n,i)}(\lambda) + P_{(-m'-n,m')}(\lambda).$$

(3) *The integral $\mathcal{N}_\lambda^{(1)}(x, \mu)$ vanishes otherwise.*

PROOF. We note that $a + m' = \lambda_2$ is equivalent to $\lambda_1 = -m' - n$. We also have $a = \lambda_2 - m' \ge 0$. Since $\lambda_1 \ge \lambda_2$, the integral $\mathcal{N}_\lambda^{(1)}(x, \mu)$ vanishes unless $-m' - n \ge m'$, i.e. $2m' + n \le 0$. The rest is clear from Lemma 4.9. $\square$

Evaluation of $\mathcal{N}_\lambda^{(2)}(x,\mu)$. We shall evaluate $\mathcal{N}_\lambda^{(2)}(x,\mu)$ explicitly. This is the main term among the four terms $\mathcal{N}_\lambda^{(k)}(x,\mu)$ $(1 \le k \le 4)$. Let us define subsets $\mathcal{A}_{2,l}$ $(1 \le l \le 4)$ of $\mathcal{A}_2$ by

$$\mathcal{A}_{2,1} = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \setminus \mathrm{GL}_2(\mathcal{O}), \qquad \mathcal{A}_{2,2} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O}^\times & \mathcal{O} \end{pmatrix} \setminus \mathrm{GL}_2(\mathcal{O}),$$
$$\mathcal{A}_{2,3} = \begin{pmatrix} \varpi\mathcal{O} & \mathcal{O}^\times \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix}, \qquad \mathcal{A}_{2,4} = \begin{pmatrix} \varpi\mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O}^\times \end{pmatrix}.$$

For l such that $1 \le l \le 4$, let

$$\mathcal{N}_\lambda^{i,j,(2,l)}(x,\mu) = \int_{\left\{\varepsilon \in \mathcal{O}^\times \mid A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,l}\right\}} \mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(\varepsilon), \varepsilon_\mu \varepsilon\right) d^\times \varepsilon.$$

Then by (4.15) and (4.17), we have

$$\mathcal{N}_\lambda^{i,j,(2,l)}(x,\mu) = \mathcal{N}_\lambda^{i',j',(2,l+1)}(x,\mu), \quad i' = a+m-i, \quad j' = a-j, \tag{4.48}$$

for $l = 1,3$. We also put

$$\mathcal{N}_\lambda^{i,j,(2,5)}(x,\mu) = \int_{\left\{\varepsilon \in \mathcal{O}^\times \mid A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,1} \cap \mathcal{A}_{2,2}\right\}} \mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(\varepsilon), \varepsilon_\mu \varepsilon\right) d^\times \varepsilon$$

and

$$\mathcal{N}_\lambda^{i,j,(2,6)}(x,\mu) = \int_{\left\{\varepsilon \in \mathcal{O}^\times \mid A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,3} \cap \mathcal{A}_{2,4}\right\}} \mathcal{N}_\lambda^*\left(A_\lambda^{i,j}(\varepsilon), \varepsilon_\mu \varepsilon\right) d^\times \varepsilon.$$

Then we have

$$\mathcal{N}_\lambda^{(2)}(x,\mu) = 2\cdot\mathcal{N}_\lambda^{(2,1)}(x,\mu) + 2\cdot\mathcal{N}_\lambda^{(2,3)}(x,\mu) - \mathcal{N}_\lambda^{(2,5)}(x,\mu) - \mathcal{N}_\lambda^{(2,6)}(x,\mu), \tag{4.49}$$

where

$$\mathcal{N}_\lambda^{(2,l)}(x,\mu) = \left(1-q^{-1}\right) q^{\lambda_2} |\Delta| \cdot C_s(\lambda) \sum_{\substack{0 \le i \le a+m \\ 0 \le j \le a}} \mathcal{N}_\lambda^{i,j,(2,l)}(x,\mu).$$

Evaluation of $\mathcal{N}_\lambda^{(2,1)}(x,\mu)$ and $\mathcal{N}_\lambda^{(2,5)}(x,\mu)$. First we note the following lemma.

Lemma 4.11. (1) *We have $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,1}$ if and only if:*

$$a \ge 0, \quad a+m' > \lambda_2, \quad i = a+m, \quad j = 0, \quad \varepsilon \in -\varepsilon_x + \varpi^{\lambda_2}\mathcal{O}. \tag{4.50}$$

(2) *We have $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,2}$ if and only if:*

$$a \ge 0, \quad a+m' > \lambda_2, \quad i = 0, \quad j = a, \quad \varepsilon \in -1 + \varpi^{\lambda_2}\mathcal{O}. \tag{4.51}$$

(3) *We have $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,1} \cap \mathcal{A}_{2,2}$ if and only if*

$$a = 0, \quad m' > \lambda_2, \quad i = j = 0, \quad \varepsilon \in -\varepsilon_x + \varpi^{\lambda_2}\mathcal{O}. \tag{4.52}$$

Proof. The proof is similar to the one given for Lemma 4.9. □

Lemma 4.12. *The integral $\mathcal{N}_\lambda^{(2,1)}(x,\mu)$ is evaluated as follows for $\lambda \in P_+^+$.*

(1) *When $m' = 0$ and $n = 0$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,1)}(x,\mu) = -q^{-1} \cdot L_{(1,1)}(\lambda).$$

(2) *When $m'=0$ and $n \leq -1$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,1)}(x,\mu) = \left(1-q^{-1}\right) \sum_{i=1}^{-n} L_{(1-n,i)}(\lambda) - q^{-1} \cdot L_{(1-n,1-n)}(\lambda).$$

(3) *When $m' \geq 1$, $2m'+n \geq 2$, $n \leq -2$, and n is even, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,1)}(x,\mu) = \mathcal{K}l_1 \cdot L_{(\frac{-n}{2},\frac{-n}{2})}(\lambda).$$

(4) *When $m' \geq 2$ and $2m'+n \leq 1$, we have*

$$\begin{aligned} C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,1)}(x,\mu) &= -q^{-1} \cdot L_{(1-m'-n,m'-1)}(\lambda) \\ &\quad + \left(1-q^{-1}\right) \cdot \sum_{i=m'}^{-m'-n} L_{(1-m'-n,i)}(\lambda) - q^{-1} \cdot L_{(1-m'-n,1-m'-n)}(\lambda). \end{aligned}$$

(5) *When $m'=1$ and $n \leq -1$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,1)}(x,\mu) = \left(1-q^{-1}\right) \sum_{i=1}^{-1-n} L_{(-n,i)}(\lambda) - q^{-1} \cdot L_{(-n,-n)}(\lambda).$$

(6) *The integral $\mathcal{N}_\lambda^{(2,1)}(x,\mu)$ vanishes otherwise.*

LEMMA 4.13. *The integral $\mathcal{N}_\lambda^{(2,5)}(x,\mu)$ is evaluated as follows for $\lambda \in P_+^+$.*

(1) *When $m' \geq 1$, $2m'+n \geq 2$, $n \leq -2$ and n is even, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,5)}(x,\mu) = \mathcal{K}l_1 \cdot P_{(\frac{-n}{2},\frac{-n}{2})}(\lambda).$$

(2) *When $m' \geq 2$ and $2m'+n \leq 1$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,5)}(x,\mu) = -q^{-1} \cdot P_{(1-m'-n,m'-1)}(\lambda).$$

(3) *The integral $\mathcal{N}_\lambda^{(2,5)}(x,\mu)$ vanishes otherwise.*

PROOF OF LEMMA 4.12 AND LEMMA 4.13. By Proposition 4.6 and (4.50), we have

$$\mathcal{N}_\lambda^{(2,1)}(x,\mu) = C_s(\lambda) \cdot \mathcal{K}l_1(x,\mu;\lambda_2)$$

if $n+\lambda_1+\lambda_2 \geq 0$ and $m'+n+\lambda_1 > 0$, and it vanishes otherwise. Similarly we have

$$\mathcal{N}_\lambda^{(2,5)}(x,\mu) = C_s(\lambda) \cdot \mathcal{K}l_1(x,\mu;\lambda_2)$$

if $n+\lambda_1+\lambda_2 = 0$ and $m' > \lambda_2$, and it vanishes otherwise. Let us define subsets $\mathcal{D}_+^{(2,1)}$ (resp. $\mathcal{D}_+^{(2,5)}$) of P_+^+ by

$$\mathcal{D}_+^{(2,1)} = \left\{(\lambda_1,\lambda_2) \in P_+^+ \mid \lambda_1+\lambda_2 \geq -n,\ \lambda_1 \geq -m'-n+1\right\}$$

$$(\text{resp. } \mathcal{D}_+^{(2,5)} = \left\{(\lambda_1,\lambda_2) \in P_+^+ \mid \lambda_1+\lambda_2 = -n,\ \lambda_1 \geq -m'-n+1\right\}).$$

Let $F_+^{(2,1)}$ (resp. $F_+^{(2,5)}$) denote the characteristic function of the set $\mathcal{D}_+^{(2,1)}$ (resp. $\mathcal{D}_+^{(2,5)}$). Then we have

$$\mathcal{N}_\lambda^{(2,1)}(x,\mu) = \mathcal{K}l_1(x,\mu;\lambda_2) \cdot C_s(\lambda)\ F_+^{(2,1)}(\lambda) \tag{4.53}$$

and

$$\mathcal{N}_\lambda^{(2,5)}(x,\mu) = \mathcal{K}l_1(x,\mu;\lambda_2) \cdot C_s(\lambda)\ F_+^{(2,5)}(\lambda). \tag{4.54}$$

Since $\lambda_2 \geq 1$, by Proposition 3.8, we have

$$\mathcal{K}l_1\left(x, \mu; \lambda_2\right) = \begin{cases} -q^{-1} & \text{if } m' = 0,\ n \leq 0 \text{ and } \lambda_2 = 1 - n, \\ 1 - q^{-1} & \text{if } m' = 0,\ n \leq -1 \text{ and } \lambda_2 \leq -n, \\ \mathcal{K}l_1 & \text{if } m' \geq 1,\ 2m' + n \geq 2,\ n \text{ is even, and, } \lambda_2 = \frac{-n}{2}, \\ -q^{-1} & \text{if } m' \geq 1,\ 2m' + n \leq 1 \text{ and } \lambda_2 = m' - 1,\ 1 - m' - n, \\ 1 - q^{-1} & \text{if } m' \geq 1,\ 2m' + n \leq 0 \text{ and } m' \leq \lambda_2 \leq -m' - n, \\ 0 & \text{otherwise.} \end{cases}$$

□

Evaluation of $\mathcal{N}_\lambda^{(2,3)}(x, \mu)$ and $\mathcal{N}_\lambda^{(2,6)}(x, \mu)$. First we note the following lemma.

LEMMA 4.14. (1) *We have $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,3}$ if and only if one of the following conditions holds:*

$$i = a + m - j, \quad 1 \leq j \leq \min\left\{\lambda_2 - 1, \frac{a+m}{2}, a - \lambda_2 + m'\right\}, \quad \varepsilon \in -\varepsilon_x + \varpi^{\lambda_2 - j}\mathcal{O}^\times; \tag{4.55}$$

$$i = a - j, \quad \frac{a}{2} < j < \min\left\{\lambda_2, a\right\}, \quad m' = \lambda_2 - j, \quad \varepsilon \in -1 + \varpi^{j + \lambda_2 - a}\mathcal{O}; \tag{4.56}$$

$$a \geq 2\lambda_2, \quad \lambda_2 \leq i < a + m - \lambda_2, \quad j = \lambda_2; \tag{4.57}$$

$$a \geq 2\lambda_2, \quad i = a + m - \lambda_2, \quad j = \lambda_2, \quad \varepsilon \in \mathcal{O}^\times \setminus \left(-\varepsilon_x + \varpi\mathcal{O}\right); \tag{4.58}$$

$$m' = 0, \quad 1 \leq i = a - \lambda_2 < \lambda_2, \quad j = \lambda_2, \quad \varepsilon \in -1 + \varpi^{2\lambda_2 - a}\mathcal{O}; \quad \textit{or} \tag{4.59}$$

$$i = a + m - \lambda_2, \quad \lambda_2 < j \leq a - \lambda_2. \tag{4.60}$$

(2) *We have $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,3} \cap \mathcal{A}_{2,4}$ if and only if one of the following conditions holds:*

$$m' = 0, \quad i = m + \lambda_2, \quad 1 \leq j = a - \lambda_2 < \lambda_2, \quad \varepsilon \in -\varepsilon_x + \varpi^{2\lambda_2 - a}\mathcal{O}^\times; \tag{4.61a}$$

$$m' > 0, \quad i = a - j, \quad \max\left\{0, a - \lambda_2\right\} < j < \min\left\{\lambda_2, a\right\}, \quad \varepsilon \in \left(-\varepsilon_x + \varpi^{\lambda_2 - j}\mathcal{O}^\times\right) \cap \left(-1 + \varpi^{j + \lambda_2 - a}\mathcal{O}^\times\right); \tag{4.61b}$$

$$a = 2\lambda_2, \quad \lambda_2 < i < \lambda_2 + m, \quad j = \lambda_2; \tag{4.62a}$$

$$a > 2\lambda_2, \quad i = j = \lambda_2; \tag{4.62b}$$

$$a = 2\lambda_2, \quad i = j = \lambda_2, \quad m > 0, \quad \varepsilon \in \mathcal{O}^\times \setminus \left(-1 + \varpi\mathcal{O}\right); \tag{4.62c}$$

$$a = 2\lambda_2, \quad i = \lambda_2 + m, \quad m > 0, \quad j = \lambda_2, \quad \varepsilon \in \mathcal{O}^\times \setminus \left(-\varepsilon_x + \varpi\mathcal{O}\right); \tag{4.63a}$$

$$a = 2\lambda_2, \quad i = j = \lambda_2, \quad m = 0, \quad \varepsilon \in \mathcal{O}^\times \setminus \left\{\left(-\varepsilon_x + \varpi\mathcal{O}\right) \cup \left(-1 + \varpi\mathcal{O}\right)\right\}; \tag{4.63b}$$

$$m' = 0, \quad 1 \leq i = a - \lambda_2 < \lambda_2, \quad j = \lambda_2, \quad \varepsilon \in -1 + \varpi^{2\lambda_2 - a}\mathcal{O}^\times; \quad \textit{or} \tag{4.64}$$

$$i = a + m - \lambda_2, \quad \lambda_2 < j = a - \lambda_2. \tag{4.65}$$

(3) *For integers i and j satisfying $0 < i < a+m$ and $0 < j < a$, and, $\varepsilon \in \mathcal{O}^\times$, we have*

$$A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,4} \Longleftrightarrow A_\lambda^{i',j'}\left(\varepsilon^{-1}\varepsilon_x\right) \in \mathcal{A}_{2,3}, \tag{4.66}$$

where $i' = a + m - i$ and $j' = a - j$.

Proof. Let $a_\lambda^{i,j}(\varepsilon)$ (resp. $b_\lambda^{i,j}(\varepsilon)$) denote the $(1,2)$-entry (resp. $(2,2)$-entry) of $\mathcal{A}_\lambda^{i,j}(\varepsilon)$, i.e.

$$a_\lambda^{i,j}(\varepsilon) = \varpi^{j-\lambda_2}\varepsilon + \varpi^{a+m-i-\lambda_2}\varepsilon_x, \quad b_\lambda^{i,j}(\varepsilon) = \varpi^{i-\lambda_2}\varepsilon + \varpi^{a-j-\lambda_2}.$$

Here we note that

$$b_\lambda^{i,j}\left(\varepsilon^{-1}\varepsilon_x\right) = \varepsilon^{-1}\, a_\lambda^{i',j'}(\varepsilon), \tag{4.67}$$

and hence (4.66) holds.

It is clear that we have $\mathcal{A}_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,3}$ (resp. $\mathcal{A}_{2,3} \cap \mathcal{A}_{2,4}$) if and only if

$$0 < i < a+m, \quad 0 < j < a, \quad a_\lambda^{i,j}(\varepsilon) \in \mathcal{O}^\times, \quad b_\lambda^{i,j}(\varepsilon) \in \mathcal{O} \quad (\text{resp. } b_\lambda^{i,j}(\varepsilon) \in \mathcal{O}^\times). \tag{4.68}$$

Suppose that $j < \lambda_2$ in (4.68). Then we have $a_\lambda^{i,j}(\varepsilon) \in \mathcal{O}^\times$ if and only if $i = a + m - j$ and $\varepsilon + \varepsilon_x \in \varpi^{\lambda_2 - j}\mathcal{O}^\times$. Then

$$b_\lambda^{a+m-j,j}(\varepsilon) = \varpi^{a-\lambda_2-j}\left\{\varpi^{m+(\lambda_2-j)} \cdot a_\lambda^{a+m-j,j}(\varepsilon) + (1-x)\right\}.$$

Thus we have $b_\lambda^{a+m-j,j}(\varepsilon) \in \mathcal{O}$ if and only if

$$a - 2j + m \geq 0, \quad a - \lambda_2 - j + m' \geq 0; \tag{4.69}$$

or

$$a - 2j < 0, \quad \lambda_2 - j = m', \quad \varepsilon + 1 \in \varpi^{j+\lambda_2-a}\mathcal{O}. \tag{4.70}$$

Here we note that (4.70) implies $a_\lambda^{a-j,j}(\varepsilon) \in \mathcal{O}^\times$, since

$$a_\lambda^{a-j,j}(\varepsilon) = \varpi^{-m'}(\varepsilon + 1) - \varpi^{-m'}(1-x),$$

where $-m' + (j + \lambda_2 - a) = 2j - a > 0$. Thus when $j < \lambda_2$, $\mathcal{A}_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,3}$ if and only if (4.55) or (4.56) holds. The converse is clear.

Suppose that $j < \lambda_2$ and $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,3} \cap \mathcal{A}_{2,4}$. Then we have

$$b_\lambda^{a+m-j,j}(\varepsilon) = \varpi^{a-\lambda_2-j}(\varpi^m\varepsilon + 1) \in \mathcal{O}^\times$$

if and only if

$$j = a - \lambda_2, \quad \varpi^m\varepsilon + 1 \in \mathcal{O}^\times; \tag{4.71}$$

or

$$m = 0, \quad a - j - \lambda_2 < 0, \quad \varepsilon + 1 \in \varpi^{j+\lambda_2-a}\mathcal{O}^\times. \tag{4.72}$$

Since $\varpi^m\varepsilon + 1 = \varpi^m(\varepsilon + \varepsilon_x) + (1-x)$, we have (4.61a) or (4.61b). The converse is clear.

Suppose that $j = \lambda_2$ in (4.68). Then we have $a_\lambda^{i,\lambda_2}(\varepsilon) \in \mathcal{O}^\times$ if and only if

$$a + m - i - \lambda_2 > 0; \tag{4.73}$$

or

$$i = a + m - \lambda_2, \quad \varepsilon \in \mathcal{O}^\times \setminus (-\varepsilon_x + \varpi\mathcal{O})\,. \tag{4.74}$$

On the other hand, we have $b_\lambda^{i,\lambda_2}(\varepsilon) \in \mathcal{O}$ if and only if

$$i - \lambda_2 \geq 0, \quad a - 2\lambda_2 \geq 0; \tag{4.75}$$

or

$$i = a - \lambda_2 < \lambda_2, \quad \varepsilon + 1 \in \varpi^{2\lambda_2 - a}\mathcal{O}. \tag{4.76}$$

Suppose that (4.73) and (4.76) hold. Then $m > 0$ and hence $m' = 0$. Suppose that (4.74) and (4.76) hold. Then $m = 0$ and

$$1 - x = (\varepsilon + 1) - (\varepsilon + \varepsilon_x) \in \mathcal{O}^\times,$$

i.e., $m' = 0$. Conversely when (4.59) holds, we have (4.73) (resp. (4.74)) if $m > 0$ (resp. if $m = 0$). Thus when $j = \lambda_2$, one of (4.57), (4.58) and (4.59) holds and the converse also holds.

Suppose that $j = \lambda_2$ and $A_\lambda^{i,\lambda_2}(\varepsilon) \in \mathcal{A}_{2,3} \cap \mathcal{A}_{2,4}$. We have $b_\lambda^{i,\lambda_2}(\varepsilon) \in \mathcal{O}^\times$ if and only if

$$i = \lambda_2, \quad a - 2\lambda_2 > 0; \tag{4.77}$$

$$i - \lambda_2 > 0, \quad a = 2\lambda_2; \tag{4.78}$$

$$i = \lambda_2, \quad a = 2\lambda_2, \quad \varepsilon \in \mathcal{O}^\times \setminus (-1 + \varpi\mathcal{O})\,; \tag{4.79}$$

or

$$i = a - \lambda_2 < \lambda_2, \quad \varepsilon + 1 \in \varpi^{2\lambda_2 - a}\mathcal{O}^\times. \tag{4.80}$$

Hence when (4.57) (resp. (4.58)) holds, we have $b_\lambda^{i,\lambda_2}(\varepsilon) \in \mathcal{O}^\times$ if and only if one of (4.62) (resp. (4.63)) holds. When (4.59) holds, we have $b_\lambda^{i,\lambda_2}(\varepsilon) \in \mathcal{O}^\times$ if and only if (4.64) holds by (4.80).

Suppose that $j > \lambda_2$ in (4.81). We have $a_\lambda^{i,j}(\varepsilon) \in \mathcal{O}^\times$ if and only if

$$i = a + m - \lambda_2. \tag{4.81}$$

Then we have $b_\lambda^{a+m-\lambda_2,j}(\varepsilon) \in \mathcal{O}$ (resp. $\mathcal{O}^\times$) if and only if

$$a - \lambda_2 - j \geq 0 \quad (\text{resp. } a - \lambda_2 - j = 0) \tag{4.82}$$

since $(a + m - \lambda_2) - \lambda_2 > (a + m - \lambda_2) - j \geq a - \lambda_2 - j$. Hence when $j > \lambda_2$, (4.60) (resp. (4.65)) holds if $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_{2,3}$ (resp. $\mathcal{A}_{2,3} \cap \mathcal{A}_{2,4}$). The converse is clear. $\square$

LEMMA 4.15. *The integral* $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$ *is evaluated as follows for* $\lambda \in P_+^+$.

(1) *Suppose that* $m' = 0$.

(a) *When* $n \geq m + 2$*, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,3)}(x,\mu) = -(\mathcal{K}l_1 + \mathcal{K}l_2) \cdot L_{(1,1)}.$$

(b) *When* $n = m + 1 \geq 2$*, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,3)}(x,\mu) = \left(2q^{-1} - \mathcal{K}l_2\right) \cdot L_{(1,1)}.$$

(c) *When* $m \geq n \geq 2$*, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,3)}(x,\mu) = \left\{-(m - n + 1) + (m - n + 3)\,q^{-1} - 2\mathcal{K}l_2\right\} \cdot L_{(1,1)}.$$

(d) *When $n = 1$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,3)}(x,\mu) = q^{-1} \cdot L_{(2,2)} + \left\{-m + (m+2)\, q^{-1}\right\} \cdot L_{(1,1)}.$$

(e) *When $n = 0$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,3)}(x,\mu) = q^{-1} \cdot L_{(3,3)} - \left(1 - q^{-1}\right) \cdot L_{(2,2)} + \left\{-m + (m+2)\, q^{-1}\right\} \cdot L_{(1,1)}.$$

(f) *When $n \leq -1$, we have*

$$\begin{aligned} C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,3)}(x,\mu) &= q^{-1} \cdot L_{(3-n,3-n)} - \left(1 - q^{-1}\right) \cdot L_{(2-n,2-n)} \\ &\quad - \left(1 - q^{-1}\right) \cdot \sum_{i=2}^{1-n} L_{(1-n,i)} + \left\{-m + (m+2)\, q^{-1}\right\} \cdot L_{(1-n,1)}. \end{aligned}$$

(2) *Suppose that $m' \geq 1$.*

(a) *When $2m' + n \geq 2$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,3)}(x,\mu) = \begin{cases} -\left(\mathcal{K}l_1 + \mathcal{K}l_2\right) \cdot L_{(1,1)} & \textit{if } n > 0 \textit{ and } n \textit{ is even,} \\ -\mathcal{K}l_1 \cdot L_{\left(\frac{4-n}{2}, \frac{4-n}{2}\right)}, & \textit{if } n \leq 0 \textit{ and } n \textit{ is even,} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases}$$

(b) *When $2m' + n = 1$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,3)}(x,\mu) = q^{-1} \cdot L_{(m'+1,m'+1)} + q^{-1} \cdot L_{(m'+2,m'+2)}.$$

(c) *When $2m' + n \leq 0$, we have*

$$\begin{aligned} C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,3)}(x,\mu) &= q^{-1} \cdot L_{(1-m'-n,m'+1)} \\ &\quad - \left(1 - q^{-1}\right) \cdot \sum_{i=m'+2}^{1-m'-n} L_{(1-m'-n,i)} - \left(1 - q^{-1}\right) \cdot L_{(2-m'-n,2-m'-n)} \\ &\quad + q^{-1} \cdot L_{(3-m'-n,3-m'-n)}. \end{aligned}$$

LEMMA 4.16. *The integral $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ is evaluated as follows for $\lambda \in P_+^+$.*

(1) *When $m' = 0$ and $n \leq 0$, we have*

$$\begin{aligned} C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,6)}(x,\mu) &= \left\{-(m+1) + (m+3)\, q^{-1}\right\} \cdot P_{(1-n,1)} \\ &\quad - 2\left(1 - q^{-1}\right) \cdot \sum_{i=2}^{1-n} P_{(1-n,i)}. \end{aligned}$$

(2) *When $m' \geq 1$ and $2m' + n \leq 0$, we have*

$$\begin{aligned} &C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(2,6)}(x,\mu) \\ &\quad = -\left(1 - 2q^{-1}\right) \cdot P_{(1-m'-n,m'+1)} - 2\left(1 - q^{-1}\right) \cdot \sum_{i=m'+2}^{1-m'-n} P_{(1-m'-n,i)}. \end{aligned}$$

(3) *The integral $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ vanishes otherwise.*

PROOF OF LEMMA 4.15 AND LEMMA 4.16. First we shall compute the contributions from each type of the domains in Lemma 4.14.

Type (4.55) *contribution to* $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$. Suppose that (4.55) holds. Then we have

$$\mathcal{N}_\lambda^{a+m-j,j,(2,3)}(x,\mu)=\frac{|\Delta|^{-1}}{1-q^{-1}}$$
$$\cdot\int_{-\varepsilon_x+\varpi^{\lambda_2-j}\mathcal{O}^\times}\int_{\mathcal{O}^\times}\psi\left(\frac{-2\varpi^{-j}\left(\varpi^m\varepsilon+1\right)\varepsilon_1}{1-x}+\frac{2\varpi^{\lambda_1+\lambda_2+j-a}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{(1-x)\left(\varepsilon+\varepsilon_x\right)}\right)d\varepsilon\,d\varepsilon_1.$$

By the change of variable $\varepsilon_2=\varpi^{j-\lambda_2}\left(\varepsilon+\varepsilon_x\right)\varepsilon_1$, we have

$$\mathcal{N}_\lambda^{a+m-j,j,(2,3)}(x,\mu)=\frac{|\Delta|^{-1}\cdot q^{j-\lambda_2}}{1-q^{-1}}\int_{\mathcal{O}^\times}\left(\int_{\mathcal{O}^\times}\psi\left(-2\varpi^{-j}\varepsilon_1\right)\,d\varepsilon_1\right)$$
$$\cdot\,\psi\left(\frac{-2\varpi^{m+\lambda_2-2j}\varepsilon_2+2\varpi^{\lambda_1+2j-a}\varepsilon_\mu\varepsilon_x\varepsilon_2^{-1}}{1-x}\right)d\varepsilon_2.$$

Since $j\geq 1$, the inner integral vanishes unless $j=1$. When $j=1$, we have

$$\mathcal{N}_\lambda^{a+m-1,1,(2,3)}(x,\mu)=\frac{-|\Delta|^{-1}\cdot q^{-\lambda_2}}{1-q^{-1}}\cdot\mathcal{K}l_1\left(x,\mu;\lambda_2-2\right).$$

Thus the condition for the type (4.55) contribution to $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$ occurs, and the contribution is given by:

$$\lambda_1\geq 1-m'-n,\quad \lambda_2\geq 2\quad\text{and}\quad \lambda_1+\lambda_2\geq 2-m-n; \tag{4.83a}$$

$$-\mathcal{K}l_1\left(x,\mu;\lambda_2-2\right)\cdot C_s(\lambda). \tag{4.83b}$$

Here we note that by Proposition 3.8, under the condition (4.83a), we have

(4.84) $\quad\mathcal{K}l_1\left(x,\mu;\lambda_2-2\right)$
$$=\begin{cases}-q^{-1} & \text{if } m'=0,\ n\leq 1,\ \lambda_2=3-n,\\ 1-q^{-1} & \text{if } m'=0,\ n\leq 0,\ 2\leq\lambda_2\leq 2-n,\\ \mathcal{K}l_1 & \text{if } m'\geq 1,\ 2m'+n\geq 2,\ n\leq 0,\ n \text{ is even},\ \lambda_2=\frac{4-n}{2},\\ -q^{-1} & \text{if } m'\geq 1,\ 2m'+n=1,\ \lambda_2=m'+1,\ m'+2,\\ -q^{-1} & \text{if } -n\geq 2m'\geq 2,\ \lambda_1\geq 1-m'-n,\ \lambda_2=m'+1,\\ -q^{-1} & \text{if } -n\geq 2m'\geq 2,\ \lambda_2=3-m'-n,\\ 1-q^{-1} & \text{if } -n\geq 2m'\geq 2,\ \lambda_1\geq 1-m'-n,\ 2-m'-n\geq\lambda_2\geq m'+2,\\ 0 & \text{otherwise.}\end{cases}$$

Type (4.61a) *contribution to* $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$. Similarly when (4.61a) holds, the integral $\mathcal{N}_\lambda^{m+\lambda_2,n+\lambda_1,(2,6)}(x,\mu)$ vanishes unless $n+\lambda_1=1$. When $n+\lambda_1=1$, we have

$$\mathcal{N}_\lambda^{m+\lambda_2,1,(2,6)}(x,\mu)=\frac{-|\Delta|^{-1}\cdot q^{-\lambda_2}}{1-q^{-1}}\cdot\mathcal{K}l_1\left(x,\mu;\lambda_2-2\right). \tag{4.85}$$

Hence the condition for the type (4.61a) contribution to $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ occurs, and this contribution is given by:

$$m'=0,\quad \lambda_1=1-n\geq\lambda_2\geq 2; \tag{4.86a}$$

$$-\mathcal{K}l_1\left(x,\mu;\lambda_2-2\right)\cdot C_s(\lambda). \tag{4.86b}$$

By (4.84), we have

$$\mathcal{K}l_1 \left(x, \mu; \lambda_2 - 2\right) = 1 - q^{-1}$$

under the condition (4.86a).

Type (4.61b) *contribution to* $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$. When (4.61b) holds, we have

$$\mathcal{N}_\lambda^{a-j,j,(2,6)}(x,\mu) = \frac{|\Delta|^{-1}}{1-q^{-1}} \int_{\left(-\varepsilon_x + \varpi^{\lambda_2 - j}\mathcal{O}^\times\right) \cap \left(-1 + \varpi^{j+\lambda_2-a}\mathcal{O}^\times\right)} \mathcal{K}l\left(\frac{-2\varpi^{-j}(\varepsilon+1)}{1-x}, \frac{2\varpi^{\lambda_1+\lambda_2+j-a}\varepsilon_\mu\varepsilon_x}{(1-x)(\varepsilon+\varepsilon_x)}\right) d\varepsilon.$$

In the integrand, we have

$$\operatorname{ord}\left(\frac{-2\varpi^{-j}(\varepsilon+1)}{1-x}\right) = \lambda_2 - a - m' < \lambda_1 + 2j - a - m' = \operatorname{ord}\left(\frac{2\varpi^{\lambda_1+\lambda_2+j-a}\varepsilon_\mu\varepsilon_x}{(1-x)(\varepsilon+\varepsilon_x)}\right).$$

Hence

$$\mathcal{K}l\left(\frac{-2\varpi^{-j}(\varepsilon+1)}{1-x}, \frac{2\varpi^{\lambda_1+\lambda_2+j-a}\varepsilon_\mu\varepsilon_x}{(1-x)(\varepsilon+\varepsilon_x)}\right) = \begin{cases} -q^{-1} & \text{if } \lambda_2 - a - m' = -1, \\ 1 - q^{-1} & \text{if } \lambda_2 - a - m' \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\lambda_2 - a - m' \geq 0$. Then we have

$$j + \lambda_2 - a \geq j + m' > m'.$$

Hence the non-emptiness of the set

$$\left(-\varepsilon_x + \varpi^{\lambda_2 - j}\mathcal{O}^\times\right) \cap \left(-1 + \varpi^{j+\lambda_2-a}\mathcal{O}^\times\right)$$

implies that $\lambda_2 - j = m'$. Then $j = \lambda_2 - m' \geq a$. This contradicts the condition (4.61b).

Suppose that $\lambda_2 - a - m' = -1$, i.e. $\lambda_1 = 1 - m' - n$. Then we have

$$\max\{0, a - \lambda_2\} = 0 \quad \text{and} \quad \max\{\lambda_2, a\} = \lambda_2.$$

Since $j + \lambda_2 - a = j + m' - 1 \geq m'$, we have

$$\int_{\left(-\varepsilon_x + \varpi^{\lambda_2 - j}\mathcal{O}^\times\right) \cap \left(-1 + \varpi^{j+m'-1}\mathcal{O}^\times\right)} d\varepsilon = \begin{cases} q^{1-\lambda_2}\left(1 - 2q^{-1}\right) & \text{if } \lambda_2 - j = j + m' - 1 = m', \\ q^{1-\lambda_2}\left(1 - q^{-1}\right) & \text{if } \lambda_2 - j > m' = j + m' - 1, \\ q^{1-\lambda_2}\left(1 - q^{-1}\right) & \text{if } \lambda_2 - j = m' < j + m' - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence we have

$$\mathcal{N}_\lambda^{a-j,j,(2,6)}(x,\mu) = \begin{cases} -\left(1 - 2q^{-1}\right) \cdot C_s(\lambda) & \text{if } \lambda_2 = m' + 1,\ j = 1, \\ -\left(1 - q^{-1}\right) \cdot C_s(\lambda) & \text{if } \lambda_2 \geq m' + 2,\ j = 1,\ \lambda_2 - m'. \end{cases}$$

Thus the condition for the type (4.61b) contribution to $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ occurs. This contribution is given by:

(4.87a) $$m' \geq 1, \quad \lambda_1 = 1 - m' - n \geq \lambda_2 \geq m' + 1;$$

(4.87b) $$\begin{cases} -\left(1 - 2q^{-1}\right) \cdot C_s(\lambda), & \text{if } \lambda_2 = m' + 1, \\ -2\left(1 - q^{-1}\right) \cdot C_s(\lambda), & \text{if } \lambda_2 \geq m' + 2. \end{cases}$$

Type (4.56) *contribution to* $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$. When (4.56) holds, we have

$$\mathcal{N}_\lambda^{a-j,j,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}}{1-q^{-1}}$$
$$\cdot \int_{-1+\varpi^{\lambda_2-a+j}\mathcal{O}} \int_{\mathcal{O}^\times} \psi\left(\frac{-2\varpi^{-j}(\varepsilon+1)\varepsilon_1}{1-x} + \frac{2\varpi^{\lambda_1+\lambda_2+j-a}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{(1-x)(\varepsilon+\varepsilon_x)}\right) d\varepsilon\, d\varepsilon_1.$$

In the integrand, we have

$$\operatorname{ord}\left(\frac{2\varpi^{\lambda_1+\lambda_2+j-a}\varepsilon_\mu\varepsilon_x}{(1-x)(\varepsilon+\varepsilon_x)}\right) = \lambda_1-\lambda_2+3j-a > 0.$$

By the change of variable $\varepsilon+1 = \varpi^{\lambda_2-a+j}\delta$, we have

$$\mathcal{N}_\lambda^{a-j,j,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}\cdot q^{a-j-\lambda_2}}{1-q^{-1}} \int_{\mathcal{O}^\times}\left(\int_{\mathcal{O}} \psi\left(\frac{-2\varpi^{\lambda_2-a}\varepsilon_1\delta}{1-x}\right) d\delta\right) d\varepsilon_1.$$

Here the inner integral vanishes since

$$\operatorname{ord}\left(\frac{-2\varpi^{\lambda_2-a}\varepsilon_1}{1-x}\right) = j-a < 0.$$

Thus there is no type (4.56) contribution to $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$.

Type (4.57) *contribution to* $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$. When (4.57) holds, we have

$$\mathcal{N}_\lambda^{i,\lambda_2,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}}{1-q^{-1}}$$
$$\cdot \int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times} \psi\left(\frac{-2\varpi^{i-a}\left(\varepsilon+\varpi^{a-i-\lambda_2}\right)\varepsilon_1}{1-x} + \frac{2\varpi^{\lambda_1+\lambda_2+m-i}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{(1-x)\left(\varepsilon+\varpi^{a+m-\lambda_2-i}\varepsilon_x\right)}\right) d\varepsilon\, d\varepsilon_1.$$

By the change of variable $\varepsilon_2 = \varepsilon + \varpi^{a+m-\lambda_2-i}\varepsilon_x$, we have

$$\mathcal{N}_\lambda^{i,\lambda_2,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}}{1-q^{-1}} \cdot \mathcal{K}l\left(\frac{-2\varpi^{i-a}}{1-x}, \frac{2\varpi^{\lambda_1+\lambda_2+m-i}\varepsilon_\mu\varepsilon_x}{1-x}\right)$$
$$\cdot \int_{\mathcal{O}^\times} \psi\left(-2\varpi^{-\lambda_2}\varepsilon_1\right) d\varepsilon_1.$$

Since $\lambda_2 \geq 1$, the integral vanishes unless $\lambda_2 = 1$. Thus the condition for the type (4.57) contribution to $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$ exists. This contribution is given by:

$$\lambda_1 \geq \max\{1-n, 2-m-n\}, \quad \lambda_2 = 1; \tag{4.88a}$$

$$-C_s(\lambda) \sum_{i=1}^{\lambda_1+m+n-1} \mathcal{K}l_1(x,\mu;\lambda_1+1-i) = -C_s(\lambda) \sum_{j=2-m-n}^{\lambda_1} \mathcal{K}l_1(x,\mu;j) \tag{4.88b}$$

By Proposition 3.8, under the condition (4.88a), we have

$$\sum_{j=2-m-n}^{\lambda_1} \mathcal{K}l_1\left(x,\mu;j\right) = \tag{4.89}$$

$$\begin{cases} \mathcal{K}l_1 & \text{if } m'=0,\ n\geq m+2,\ n\geq 3,\ m-n \text{ is even}, \\ -2q^{-1} & \text{if } m'=0,\ n=m+1\geq 3, \\ -q^{-1} & \text{if } m'=0,\ n=m+1=2, \\ (m-n+1)-(m-n+3)\,q^{-1} & \text{if } m'=0,\ 3\leq n\leq m, \\ (m-1)-mq^{-1} & \text{if } m'=0,\ m\geq 1,\ n\leq \min\{2,m\}, \\ \mathcal{K}l_1 & \text{if } m'\geq 1,\ n\geq 4,\ n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Type (4.62a) *contribution to* $\mathcal{N}_\lambda^{(2,6)}\left(x,\mu\right)$. By the computation above, the condition for the type (4.62a) contribution to $\mathcal{N}_\lambda^{(2,6)}\left(x,\mu\right)$ exists, and its contribution is given by:

$$m\geq 2,\quad \lambda_1=1-n\geq\lambda_2=1; \tag{4.90a}$$

$$-C_s\left(\lambda\right)\sum_{i=2}^{m}\mathcal{K}l_1\left(x,\mu;2-n-i\right)=-C_s\left(\lambda\right)\sum_{j=2-m-n}^{-n}\mathcal{K}l_1\left(x,\mu;j\right) \tag{4.90b}$$

By Proposition 3.8, under the condition (4.90a), we have

$$\sum_{j=2-m-n}^{-n}\mathcal{K}l_1\left(x,\mu;j\right)=\left(m-1\right)\left(1-q^{-1}\right), \tag{4.91}$$

since $n\leq 0$ and $-m<2-m-n\leq -n$.

Type (4.62b) *contribution to* $\mathcal{N}_\lambda^{(2,6)}\left(x,\mu\right)$. By the computation above, the condition for the type (4.62b) contribution to $\mathcal{N}_\lambda^{(2,6)}\left(x,\mu\right)$ is given by:

$$\lambda_1\geq 2-n,\quad \lambda_2=1 \tag{4.92a}$$

$$-C_s\left(\lambda\right)\,\mathcal{K}l_1\left(x,\mu;\lambda_1\right). \tag{4.92b}$$

By Proposition 3.8, under the condition (4.92a), we have

$$\mathcal{K}l_1\left(x,\mu;\lambda_1\right)=0. \tag{4.93}$$

Type (4.62c) *contribution to* $\mathcal{N}_\lambda^{(2,6)}\left(x,\mu\right)$. When (4.62c) holds, we have

$$\mathcal{N}_\lambda^{\lambda_2,\lambda_2,(2,6)}\left(x,\mu\right)=\frac{|\Delta|^{-1}}{1-q^{-1}}$$
$$\cdot\int_{\mathcal{O}^\times\setminus(-1+\varpi\mathcal{O})}\int_{\mathcal{O}^\times}\psi\left(\frac{-2\varpi^{-\lambda_2}\left(\varepsilon+1\right)\varepsilon_1}{1-x}+\frac{2\varpi^{\lambda_1+m}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{\left(1-x\right)\left(\varepsilon+\varpi^m\varepsilon_x\right)}\right)d\varepsilon\,d\varepsilon_1.$$

In the integrand, we have

$$\operatorname{ord}\left(\frac{2\varpi^{\lambda_1+m}\varepsilon_\mu\varepsilon_x}{\left(1-x\right)\left(\varepsilon+\varpi^m\varepsilon_x\right)}\right)=\lambda_1+m>0.$$

Hence

$$\mathcal{N}_{\lambda}^{\lambda_2,\lambda_2,(2,6)}(x,\mu) = \frac{|\Delta|^{-1}}{1-q^{-1}} \int_{\mathcal{O}^\times \setminus (-1+\varpi\mathcal{O})} \left(\int_{\mathcal{O}^\times} \psi\left(\frac{-2\varpi^{-\lambda_2}(\varepsilon+1)\varepsilon_1}{1-x} \right) d\varepsilon_1 \right) d\varepsilon$$

where the inner integral vanishes unless $\lambda_2 = 1$. Thus the condition for the type (4.62c) contribution to $\mathcal{N}_{\lambda}^{(2,6)}(x,\mu)$ exists, and its contribution are given by:

$$m > 0, \quad \lambda_1 = 1-n \geq \lambda_2 = 1; \tag{4.94a}$$

$$-\left(1-2q^{-1}\right) \cdot C_s(\lambda). \tag{4.94b}$$

Type (4.58) *contribution to* $\mathcal{N}_{\lambda}^{(2,3)}(x,\mu)$. When (4.58) holds, by a similar computation, we have

$$\mathcal{N}_{\lambda}^{a+m-\lambda_2,\lambda_2,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}}{1-q^{-1}} \int_{\mathcal{O}^\times} \int_{\mathcal{O}^\times \setminus (\varepsilon_x+\varpi\mathcal{O})} \psi\left(\frac{-2\varpi^{m-\lambda_2}\varepsilon_1\varepsilon_2 + 2\varpi^{\lambda_2-n}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}\varepsilon_2^{-1}}{1-x} \right) \psi\left(-2\varpi^{-\lambda_2}\varepsilon_1\right) d\varepsilon_1\, d\varepsilon_2.$$

By the change of variable $\varepsilon_1 \mapsto \varepsilon_1\varepsilon_2^{-1}$, we have

$$\mathcal{N}_{\lambda}^{a+m-\lambda_2,\lambda_2,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}}{1-q^{-1}} \int_{\mathcal{O}^\times} \psi\left(\frac{-2\varpi^{m-\lambda_2}\varepsilon_1 + 2\varpi^{\lambda_2-n}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{1-x} \right) \left(\int_{\mathcal{O}^\times \setminus (\varepsilon_x+\varpi\mathcal{O})} \psi\left(-2\varpi^{-\lambda_2}\varepsilon_1\varepsilon_2^{-1}\right) d\varepsilon_2 \right) d\varepsilon_1.$$

Here

$$\int_{\mathcal{O}^\times \setminus (\varepsilon_x+\varpi\mathcal{O})} \psi\left(-2\varpi^{-\lambda_2}\varepsilon_1\varepsilon_2^{-1}\right) d\varepsilon_2 = \begin{cases} -q^{-1}\left\{1+\psi\left(-2\varpi^{-1}\varepsilon_1\varepsilon_x^{-1}\right)\right\} & \text{if } \lambda_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

When $\lambda_2 = 1$, we have

$$\mathcal{N}_{\lambda}^{a+m-1,1,(2,3)}(x,\mu) = \frac{-|\Delta|^{-1} \cdot q^{-1}}{1-q^{-1}} \cdot \left\{\mathcal{K}l_1(x,\mu;-1) + \mathcal{K}l_2(x,\mu;-1)\right\}.$$

Thus the condition for the type (4.58) contribution to $\mathcal{N}_{\lambda}^{(2,3)}(x,\mu)$ exists, and this contribution is given by:

$$\lambda_1 \geq 1-n, \quad \lambda_2 = 1; \tag{4.95a}$$

$$-\left\{\mathcal{K}l_1(x,\mu;-1) + \mathcal{K}l_2(x,\mu;-1)\right\} \cdot C_s(\lambda). \tag{4.95b}$$

By Proposition 3.8, we have

$$\mathcal{K}l_1(x,\mu;-1)=\begin{cases}\mathcal{K}l_1 & \text{if } m'=0,\ m=0,\ n=2,\\ -q^{-1} & \text{if } m'=0,\ m\geq 1,\ n=2,\\ -q^{-1} & \text{if } m'=0,\ m=0,\ n\leq 1,\\ 1-q^{-1} & \text{if } m'=0,\ n\leq 1\leq m,\\ \mathcal{K}l_1 & \text{if } m'\geq 1,\ n=2,\\ 0 & \text{otherwise,}\end{cases} \tag{4.96a}$$

$$\mathcal{K}l_2(x,\mu;-1)=\begin{cases}\mathcal{K}l_2 & \text{if } m'=0,\ n=2,\\ -q^{-1} & \text{if } m'=0,\ n\leq 1,\\ \mathcal{K}l_2 & \text{if } m'\geq 1,\ n=2,\\ 0 & \text{otherwise.}\end{cases} \tag{4.96b}$$

Type (4.63a) *contribution to* $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$. Similarly the type (4.63a) contribution to $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ exists only when

$$m>0,\quad \lambda_1=1-n\geq\lambda_2=1;$$

and it is given by

$$-\left\{\mathcal{K}l_1(x,\mu;-1)+\mathcal{K}l_2(x,\mu;-1)\right\}\cdot C_s(\lambda).$$

Since $n=1-\lambda_1\leq 0$ and $m>0$, we have

$$\mathcal{K}l_1(x,\mu;-1)=1-q^{-1},\quad \mathcal{K}l_2(x,\mu;-1)=-q^{-1},$$

by Proposition 3.8. Hence the condition for the type (4.63a) contribution to $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ exists, and this contribution is given by:

$$m>0,\quad \lambda_1=1-n\geq\lambda_2=1; \tag{4.97a}$$

$$-\left(1-2q^{-1}\right)\cdot C_s(\lambda). \tag{4.97b}$$

Type (4.63b) *contribution to* $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$. When (4.63b) holds, we have

$$\mathcal{N}_\lambda^{\lambda_2,\lambda_2,(2,6)}(x,\mu)=\frac{|\Delta|^{-1}}{1-q^{-1}}\int_{\mathcal{O}^\times\setminus\{(-\varepsilon_x+\varpi\mathcal{O})\cup(-1+\varpi\mathcal{O})\}}\int_{\mathcal{O}^\times}\psi\left(\frac{-2\varpi^{-\lambda_2}(\varepsilon+1)\varepsilon_1}{1-x}+\frac{2\varpi^{\lambda_1}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{(1-x)(\varepsilon+\varepsilon_x)}\right)d\varepsilon\, d\varepsilon_1.$$

In the integrand, we have

$$\operatorname{ord}\left(\frac{-2\varpi^{-\lambda_2}(\varepsilon+1)}{1-x}\right)=-m'-\lambda_2<\operatorname{ord}\left(\frac{2\varpi^{\lambda_1}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{(1-x)(\varepsilon+\varepsilon_x)}\right)=-m'+\lambda_1.$$

Hence $\mathcal{N}_\lambda^{\lambda_2,\lambda_2,(2,6)}(x,\mu)$ vanishes unless $m'=0$ and $\lambda_2=1$. Thus the condition for the type (4.63b) contribution to $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ exists, and its contribution is given by:

$$m=m'=0,\quad \lambda_1=1-n\geq\lambda_2=1; \tag{4.98a}$$

$$-\left(1-3q^{-1}\right)\cdot C_s(\lambda). \tag{4.98b}$$

Type (4.59) *contribution to* $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$. When (4.59) holds, we have

$$\mathcal{N}_\lambda^{a-\lambda_2,\lambda_2,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}}{1-q^{-1}}$$
$$\cdot \int_{-1+\varpi^{2\lambda_2-a}\mathcal{O}} \int_{\mathcal{O}^\times} \psi\left(\frac{-2\varpi^{-\lambda_2}(\varepsilon+1)\varepsilon_1}{1-x} + \frac{2\varpi^{\lambda_1+2\lambda_2-a+m}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{(1-x)(\varepsilon+\varpi^m\varepsilon_x)}\right) d\varepsilon\, d\varepsilon_2.$$

In the integrand, we have

$$\operatorname{ord}\left(\frac{2\varpi^{\lambda_1+2\lambda_2-a+m}\varepsilon_\mu\varepsilon_x}{(1-x)(\varepsilon+\varpi^m\varepsilon_x)}\right) = \lambda_1+2\lambda_2-a+m>0.$$

By a change of variable $\varepsilon+1=\varpi^{2\lambda_2-a}\delta$, we have

$$\mathcal{N}_\lambda^{a-\lambda_2,\lambda_2,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}\cdot q^{a-2\lambda_2}}{1-q^{-1}} \int_{\mathcal{O}^\times}\left(\int_{\mathcal{O}} \psi\left(\frac{-2\varpi^{\lambda_2-a}\varepsilon_1\delta}{1-x}\right) d\delta\right) d\varepsilon_1.$$

Here the inner integral vanishes since $\lambda_2-a<0$ and thus

$$\mathcal{N}_\lambda^{a-\lambda_2,\lambda_2,(2,3)}(x,\mu) = 0. \tag{4.99}$$

Hence there is no type (4.59) contribution to $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$.

Type (4.64) *contribution to* $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$. When (4.64) holds, as above, we have

$$\mathcal{N}_\lambda^{a-\lambda_2,\lambda_2,(2,6)}(x,\mu) = \frac{|\Delta|^{-1}\cdot q^{a-2\lambda_2}}{1-q^{-1}} \int_{\mathcal{O}^\times}\left(\int_{\mathcal{O}^\times} \psi\left(\frac{-2\varpi^{\lambda_2-a}\varepsilon_1\varepsilon_2}{1-x}\right) d\varepsilon_2\right) d\varepsilon_1$$

and the inner integral vanishes unless $\lambda_2-a=-1$. Hence the condition for the type (4.64) contribution to $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ exists, and this contribution is given by:

$$m'=0, \quad \lambda_1=1-n\ge\lambda_2\ge 2; \tag{4.100a}$$

$$-\left(1-q^{-1}\right) C_s(\lambda). \tag{4.100b}$$

Type (4.60) *contribution to* $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$. When (4.60) holds, we have

$$\mathcal{N}_\lambda^{a+m-\lambda_2,j,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}}{1-q^{-1}}$$
$$\cdot \int_{\mathcal{O}^\times}\int_{\mathcal{O}^\times} \psi\left(\frac{-2\left(\varpi^{m-\lambda_2}\varepsilon+\varpi^{-j}\right)\varepsilon_1}{1-x} + \frac{2\varpi^{j-n}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{(1-x)(\varepsilon_x+\varpi^{j-\lambda_2}\varepsilon)}\right) d\varepsilon\, d\varepsilon_1.$$

Making the change of variable $\varepsilon_2=\varepsilon_x+\varpi^{j-\lambda_2}\varepsilon$, we have

$$\mathcal{N}_\lambda^{a+m-\lambda_2,j,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}\cdot q^{-\lambda_2+j}}{1-q^{-1}}$$
$$\cdot \int_{\mathcal{O}^\times}\int_{\varepsilon_x+\varpi^{j-\lambda_2}\mathcal{O}^\times} \psi\left(-2\varpi^{-j}\varepsilon_1 + \frac{-2\varpi^{m-j}\varepsilon_1\varepsilon_2+2\varpi^{j-n}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}\varepsilon_2^{-1}}{1-x}\right) d\varepsilon_1\, d\varepsilon_2.$$

Another change of variable $\varepsilon_1\mapsto\varepsilon_1\varepsilon_2^{-1}$ gives

$$\mathcal{N}_\lambda^{a+m-\lambda_2,j,(2,3)}(x,\mu) = \frac{|\Delta|^{-1}\cdot q^{-\lambda_2+j}}{1-q^{-1}} \int_{\mathcal{O}^\times} \psi\left(\frac{-2\varpi^{m-j}\varepsilon_1+2\varpi^{j-n}\varepsilon_\mu\varepsilon_x\varepsilon_1^{-1}}{1-x}\right)$$
$$\left(\int_{\varepsilon_x+\varpi^{j-\lambda_2}\mathcal{O}^\times} \psi\left(-2\varpi^{-j}\varepsilon_1\varepsilon_2^{-1}\right) d\varepsilon_2\right) d\varepsilon_1.$$

Here the inner integral is given by

$$\int_{\varepsilon_x+\varpi^{j-\lambda_2}\mathcal{O}^\times} \psi\left(-2\varpi^{-j}\varepsilon_1\varepsilon_2^{-1}\right) d\varepsilon_2 = \begin{cases} -q^{-j}\cdot\psi\left(-2\varpi^{-j}\varepsilon_1\varepsilon_x^{-1}\right) & \text{if } \lambda_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the condition for the type (4.60) contribution to $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$ occurs, and this contribution is given by:

$$\lambda_1 \geq 2-n, \quad \lambda_2 = 1; \tag{4.101a}$$

$$-C_s(\lambda)\sum_{j=2}^{n+\lambda_1}\mathcal{K}l_2(x,\mu;-j) = -C_s(\lambda)\sum_{i=-n-\lambda_1}^{-2}\mathcal{K}l_2(x,\mu;i). \tag{4.101b}$$

By Proposition 3.8, we have

$$\sum_{i=-n-\lambda_1}^{-2}\mathcal{K}l_2(x,\mu;i) = \begin{cases} \mathcal{K}l_2 & \text{if } m'=0,\ n\geq 4,\ n \text{ is even}, \\ \mathcal{K}l_2 & \text{if } m'\geq 1,\ n\geq 4,\ n \text{ is even}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.102}$$

Type (4.65) *contribution to* $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$. By the computation above, the condition for the type (4.65) contribution to $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ is given by:

$$\lambda_1 \geq 2-n, \quad \lambda_2 = 1; \tag{4.103a}$$

$$-C_s(\lambda)\cdot\mathcal{K}l_2(x,\mu;-n-\lambda_1). \tag{4.103b}$$

By Proposition 3.8, under the condition (4.103a), we have

$$\mathcal{K}l_2(x,\mu;-n-\lambda_1) = 0. \tag{4.104}$$

Evaluation of $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$. The contributions to the term $\mathcal{N}_\lambda^{(2,3)}(x,\mu)$ are given by (4.83), (4.88), (4.95) and (4.101). By adding up the contributions, we have Lemma 4.15.

Evaluation of $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$. The contributions to the term $\mathcal{N}_\lambda^{(2,6)}(x,\mu)$ are given by (4.86), (4.87), (4.90), (4.94), (4.97), (4.98) and (4.100). By adding up the contributions, we have Lemma 4.16.

□

Now the evaluation of $\mathcal{N}_\lambda^{(2)}(x,\mu)$ is just given by the linear combination of $\mathcal{N}_\lambda^{(2,l)}(x,\mu)$, for $l = 1,3,5,6$, expressed in (4.49).

Evaluation of $\mathcal{N}_\lambda^{(3)}(x,\mu)$.

LEMMA 4.17. *We have $A_\lambda^{i,j}(\varepsilon)\in\mathcal{A}_3$ if and only if one of the following conditions holds:*

$$a+m' = \lambda_2+2, \quad i = a+m-1 \geq 1, \quad j=1, \quad \varepsilon\in -\varepsilon_x+\varpi^{\lambda_2}\mathcal{O}; \tag{4.105}$$

$$a+m' = \lambda_2+2, \quad i=1, \quad j=a-1\geq 1, \quad \varepsilon\in -1+\varpi^{\lambda_2}\mathcal{O}. \tag{4.106}$$

PROOF. By considering the determinant, we have $a-\lambda_2+m'=2$. It is clear that we have $\min\{i,j\}=1$ by looking at the first row of $A_\lambda^{i,j}(\varepsilon)$. Suppose that $j=1$. Then

$$\left(\varpi\varepsilon+\varpi^{a+m-i}\varepsilon_x\right)\varpi^{-\lambda_2}\in\varpi\mathcal{O}$$

implies that $a+m-i=1$ and $\varepsilon \in -\varepsilon_x + \varpi^{\lambda_2}\mathcal{O}$. Conversely suppose that (4.105) holds. Then we have

$$\left(\varpi^{a+m-1}\varepsilon + \varpi^{a-1}\right)\varpi^{-\lambda_2} = \varpi^{a-\lambda_2-1}(1-x) + \varpi^{a+m-\lambda_2-1}(\varepsilon+\varepsilon_x) \in \varpi\mathcal{O}$$

and hence $A_\lambda^{i,j}(\varepsilon) \in \mathcal{A}_3$. The other case is similar. $\square$

LEMMA 4.18. *For $(\lambda_1, \lambda_2) \in P_+^+$, the integral $\mathcal{N}_\lambda^{(3)}(x,\mu)$ is evaluated as follows.*

(1) *When $m' = 0$ and $n \leq 1$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(3)}(x,\mu) = -2q^{-1} \cdot \sum_{i=1}^{2-n} P_{(2-n,i)}(\lambda).$$

(2) *When $m' \geq 1$ and $2m' + n \leq 1$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(3)}(x,\mu) = -2q^{-1} \sum_{i=m'+1}^{2-m'-n} P_{(2-m'-n,i)}(\lambda) - q^{-1} \cdot P_{(2-m'-n,m')}(\lambda).$$

(3) *When $m' \geq 1$ and $2m' + n = 2$, we have*

$$C_s(\lambda)^{-1} \cdot \mathcal{N}_\lambda^{(3)}(x,\mu) = \mathcal{K}l_1 \cdot P_{\left(\frac{2-n}{2}, \frac{2-n}{2}\right)}(\lambda).$$

(4) *The integral $\mathcal{N}_\lambda^{(3)}(x,\mu)$ vanishes otherwise.*

PROOF. We note that $a + m' = \lambda_2 + 2$ is equivalent to $\lambda_1 = -m' - n + 2$. We also have $\lambda_2 = m' + a - 2$ where $a \geq 2$. Since $\lambda_1 \geq \lambda_2$, the integral $\mathcal{N}_\lambda^{(3)}(x,\mu)$ vanishes unless $-m' - n + 2 \geq m'$, i.e., $n + 2m' \leq 2$. When $m' = 0$, we have $\lambda_1 = -n + 2 \geq 1$ and hence $n \leq 1$.

When $m' = 0$ and $n \leq 1$, we have $a = \lambda_2 + 2 \geq 3$. Hence $i \neq j$ in (4.105) nor (4.106). Thus

$$\begin{aligned}\mathcal{N}_\lambda^{(3)}(x,\mu) &= \left(1-q^{-1}\right)|\Delta|\, q^{\lambda_2}\, C_s(\lambda) \\ &\quad \cdot \left\{\mathcal{N}_\lambda^{a+m-1,1,(3)}(x,\mu) + \mathcal{N}_\lambda^{1,a-1,(3)}(x,\mu)\right\} = -2q^{-1} \cdot C_s(\lambda).\end{aligned}$$

Suppose that $m' \geq 1$ and $2m'+n \leq 2$. Since $a-1 = \lambda_2 - m' + 1$, the conditions (4.105) and (4.106) coincide if and only if $\lambda_2 = m'$. Hence when $\lambda_1 = -m'-n+2 \geq \lambda_2 \geq m'+1$, we have

$$\mathcal{N}_\lambda^{(3)}(x,\mu) = -2q^{-1} \cdot C_s(\lambda).$$

When $\lambda_1 = -m' - n + 2 > \lambda_2 = m'$, we have

$$\mathcal{N}_\lambda^{(3)}(x,\mu) = -q^{-1} \cdot C_s(\lambda).$$

When $\lambda_1 = -m' - n + 2 = \lambda_2 = m'$, we have

$$\begin{aligned}\mathcal{N}_\lambda^{(3)}(x,\mu) &= \left(1-q^{-1}\right)|\Delta|\, q^{m'}\, C_s(\lambda)\, \mathcal{N}_\lambda^{1,1,(3)}(x,\mu) \\ &= q^{m'}\, C_s(\lambda) \int_{-1+\varpi^{m'}\mathcal{O}} \mathcal{K}l\left(\frac{2\varpi^{m'-1}}{1-x}, \frac{2\varpi^{m'-1}\varepsilon_\mu\varepsilon}{1-x}\right) d\varepsilon.\end{aligned}$$

Since

$$\frac{2\varpi^{m'-1}\varepsilon_\mu\varepsilon}{1-x} \in \frac{-2\varpi^{m'-1}\varepsilon_\mu}{1-x} + \mathcal{O}$$

for $\varepsilon \in -1+\varpi^{m'}\mathcal{O}$, we have

$$\mathcal{N}_\lambda^{(3)}(x,\mu) = C_s(\lambda)\cdot \mathcal{K}l\left(\frac{-2\varpi^{m'-1}}{1-x}, \frac{2\varpi^{m'-1}\varepsilon_\mu}{1-x}\right).$$

Here we note that $m'-1=-\frac{n}{2}$ and $\mathcal{K}l_1=\mathcal{K}l_2$ by (3.6). $\square$

Evaluation of $\mathcal{N}_\lambda^{(4)}(x,\mu)$.

LEMMA 4.19. *Suppose that $\lambda_1=\lambda_2$. Then we have $A_\lambda^{i,j}(\varepsilon)\in\mathcal{A}_4$ if and only if*

$$n\le 0 \text{ even},\quad \lambda_1=\lambda_2=\frac{2-n}{2}<m',\quad i=j=1,\quad \varepsilon\in -1+\varpi^{\frac{2-n}{2}}\mathcal{O}. \tag{4.107}$$

PROOF. It is clear that $i=j=1$. Then $\left(\varpi\varepsilon+\varpi^{a-1}\right)\varpi^{-\lambda_2}\in\varpi\mathcal{O}$ implies that $a=2$ and $\varepsilon\in-1+\varpi^{\lambda_2}\mathcal{O}$. Hence n is even and $\lambda_1=\lambda_2=\frac{2-n}{2}$. Since $\lambda_2>0$, we have $n\le 0$. By considering the determinant, we have $a-\lambda_2+m'=\frac{n+2}{2}+m'>2$, i.e., $m'>\frac{2-n}{2}$. Conversely when (4.107) holds, we have $m=0$, since $m'>0$ and

$$(\varpi\varepsilon+\varpi\varepsilon_x)\,\varpi^{\frac{n-2}{2}} = (\varpi\varepsilon+\varpi)\,\varpi^{\frac{n-2}{2}} - (\varpi-\varpi\varepsilon_x)\,\varpi^{\frac{n-2}{2}}\in\varpi\mathcal{O}.$$

Hence we have $A_\lambda^{1,1}(\varepsilon)\in\mathcal{A}_4$. $\square$

LEMMA 4.20. *For $(\lambda_1,\lambda_2)\in P_+^+$, the integral $\mathcal{N}_\lambda^{(4)}(x,\mu)$ is evaluated as follows.*

(1) *When $m'\ge 1$, $2m'+n\ge 4$, and $n\le 0$ is even, we have*

$$C_s(\lambda)^{-1}\cdot\mathcal{N}_\lambda^{(4)}(x,\mu)=\mathcal{K}l_1\cdot P_{\left(\frac{2-n}{2},\frac{2-n}{2}\right)}.$$

(2) *The integral $\mathcal{N}_\lambda^{(4)}(x,\mu)$ vanishes otherwise.*

PROOF. In the first case, when $\lambda_1=\lambda_2=\frac{2-n}{2}$, we have

$$\begin{aligned}\mathcal{N}_\lambda^{(4)}(x,\mu)&=\left(1-q^{-1}\right)|\Delta|\,q^{\frac{2-n}{2}}\,C_s(\lambda)\,\mathcal{N}_\lambda^{1,1,(4)}(x,\mu)\\ &=q^{\frac{2-n}{2}}\,C_s(\lambda)\int_{-1+\varpi^{\frac{2-n}{2}}\mathcal{O}}\mathcal{K}l\left(\frac{-2\varpi^{\frac{-n}{2}}}{1-x},\frac{2\varpi^{\frac{-n}{2}}\varepsilon_\mu\varepsilon_x}{1-x}\right)d\varepsilon=\mathcal{K}l_1\cdot C_s(\lambda).\end{aligned}$$

$\square$

4.2.3. Evaluation of $\mathcal{B}^{(s)}$. Thus by Lemmas 4.10, 4.12, 4.13, 4.15, 4.16, 4.18 and 4.20, we may evaluate $\mathcal{B}^{(s)}\mid_{P_+^+}$. Together with Proposition 4.8, we have the following proposition.

PROPOSITION 4.21. *When $m'=0$, the function $\mathcal{B}^{(s)}$ on P^+ is evaluated as follows.*

(1) *When $n\ge m+2$, we have*

$$\begin{aligned}C_s^{-1}\cdot\mathcal{B}^{(s)}=2\left(\mathcal{K}l_1+\mathcal{K}l_2\right)\cdot L_{(0,0)}+\{(n-1)\mathcal{K}l_1+(m+n-1)\mathcal{K}l_2\}\cdot P_{(0,0)}&\\ -2\left(\mathcal{K}l_1+\mathcal{K}l_2\right)\cdot L_{(1,1)}.&\end{aligned}$$

(2) *When $n=m+1\ge 2$, we have*

$$\begin{aligned}C_s^{-1}\cdot\mathcal{B}^{(s)}=-2\left(2q^{-1}-\mathcal{K}l_2\right)\cdot L_{(0,0)}+2(n-1)\left(\mathcal{K}l_2-q^{-1}\right)\cdot P_{(0,0)}&\\ +2\left(2q^{-1}-\mathcal{K}l_2\right)\cdot L_{(1,1)}.&\end{aligned}$$

(3) *When $m \geq n \geq 2$, we have*

$$C_s^{-1} \cdot \mathcal{B}^{(s)} = -2\left\{-(m-n+1)+(m-n+3)\,q^{-1}-\mathcal{K}l_2\right\} \cdot L_{(0,0)} + \left[(n-1)\left\{(m-n+1)-(m-n+3)q^{-1}\right\}+(m+n-1)\mathcal{K}l_2\right] \cdot P_{(0,0)} + 2\left\{-(m-n+1)+(m-n+3)\,q^{-1}-\mathcal{K}l_2\right\} \cdot L_{(1,1)}.$$

(4) *When $n = 1$, we have*

$$C_s^{-1} \cdot \mathcal{B}^{(s)} = 2q^{-1} \cdot P_{(0,0)} - (m+1)q^{-1} \cdot P_{(1,0)} + 2\left\{m-(m+3)q^{-1}\right\} \cdot L_{(0,0)} - 2q^{-1} \cdot P_{(1,1)} + 2\left\{-m+(m+2)\,q^{-1}\right\} \cdot L_{(1,1)} + 2q^{-1} \cdot L_{(2,2)}.$$

(5) *When $n \leq 0$, we have*

$$\begin{aligned} C_s^{-1} \cdot \mathcal{B}^{(s)} &= 2q^{-1} \cdot L_{(3-n,3-n)} - 2\left(1-q^{-1}\right) \cdot L_{(2-n,2-n)} - 2 \cdot L_{(1-n,1-n)} \\ &\quad + 2 \cdot V_{(-n,0)} + 2\left(1-q^{-1}\right) \cdot V_{(1-n,0)} - 2q^{-1} \cdot V_{(2-n,0)} \\ &\quad + \left\{(1-m)+(m+1)\,q^{-1}\right\} \cdot P_{(1-n,1)} + 2\left\{(1-m)+(m+1)\,q^{-1}\right\} \cdot L_{(2-n,1)} \\ &\quad + (m-1) \cdot P_{(-n,0)} + 2\left\{m-(m+1)q^{-1}\right\} \cdot P_{(1-n,0)} \\ &\quad + \left\{2(m+1)-(3m+5)q^{-1}\right\} \cdot P_{(2-n,0)} + 2\left\{(m+1)-(m+3)q^{-1}\right\} \cdot L_{(3-n,0)}. \end{aligned}$$

Here for $(a,b) \in \mathbb{Z}^2$, $V_{(a,b)}$ denotes the characteristic function of the set

$$\left\{(\lambda_1,\lambda_2) \in P^+ \mid \lambda_1 = a \geq \lambda_2 \geq b\right\}. \tag{4.108}$$

Proposition 4.22. *When $m' \geq 1$, the function $\mathcal{B}^{(s)}$ on P^+ is evaluated as follows.*

(1) *Suppose that $2m'+n \geq 2$. Then the function $\mathcal{B}^{(s)}$ vanishes when n is odd. When n is even, we have*

$$C_s^{-1} \cdot \mathcal{B}^{(s)} = \begin{cases} (\mathcal{K}l_1+\mathcal{K}l_2)\left\{-2 \cdot L_{(1,1)} + 2 \cdot L_{(0,0)} + (n-1) \cdot P_{(0,0)}\right\} & \text{if } n \geq 2, \\ \mathcal{K}l_1 \cdot \left\{-2 \cdot L_{(\frac{4-n}{2},\frac{4-n}{2})} + P_{(\frac{2-n}{2},\frac{2-n}{2})} - P_{(\frac{-n}{2},\frac{-n}{2})} + 2 \cdot L_{(\frac{-n}{2},\frac{-n}{2})}\right\} & \text{if } n \leq 0. \end{cases}$$

(2) *When $2m'+n = 1$, we have*

$$\begin{aligned} C_s^{-1} \cdot \mathcal{B}^{(s)} &= 2q^{-1} \cdot L_{(m'+2,m'+2)} + 2q^{-1} \cdot L_{(m'+1,m'+1)} - 2q^{-1} \cdot P_{(m'+1,m'+1)} \\ &\quad - q^{-1} \cdot P_{(m'+1,m')} - 2q^{-1} \cdot L_{(m',m')} \\ &\quad + 2q^{-1} \cdot P_{(m'-1,m'-1)} + q^{-1} \cdot P_{(m',m'-1)} - 2q^{-1} \cdot L_{(m'-1,m'-1)}. \end{aligned}$$

(3) *When $2m'+n \leq 0$, we have*

$$\begin{aligned} C_s^{-1} \cdot \mathcal{B}^{(s)} &= 2q^{-1} \cdot L_{(3-m'-n,3-m'-n)} - 2\left(1-q^{-1}\right) \cdot L_{(2-m'-n,2-m'-n)} \\ &\quad - 2 \cdot L_{(1-m'-n,1-m'-n)} \\ &\quad + 2 \cdot V_{(-m'-n,m')} + 2\left(1-q^{-1}\right) \cdot V_{(1-m'-n,m')} - 2q^{-1} \cdot V_{(2-m'-n,m')} \\ &\quad + P_{(1-m'-n,m'+1)} + 2 \cdot L_{(2-m'-n,m'+1)} \\ &\quad - P_{(-m'-n,m')} + \left(2-q^{-1}\right) \cdot P_{(2-m'-n,m')} + 2\left(1-q^{-1}\right) \cdot L_{(3-m'-n,m')} \\ &\quad - q^{-1} \cdot P_{(1-m'-n,m'-1)} - 2q^{-1} \cdot L_{(2-m'-n,m'-1)}. \end{aligned}$$

CHAPTER 5

Evaluation of the Rankin-Selberg Orbital Integral

In the first section we evaluate the degenerate Rankin-Selberg orbital integral $I(\lambda; s, a)$ defined by (2.46). In the second section we verify the matching theorems.

5.1. Preliminaries

5.1.1. Rewriting the integral. By the Iwasawa decomposition

$$H = N_H \, T_H \, K_H \text{ where } K_H = H(\mathcal{O}),$$

$$N_H = \left\{ \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y & 0 & 1 \end{pmatrix} \mid x, y \in F \right\}, \quad T_H = \left\{ z' \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & bc & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid z', b, c \in F^\times \right\},$$

a Haar measure on H is given by $|c|^2 \, dn_H \, dt_H \, dk_H$. Then we may rewrite (2.46) as

$$I(\lambda; s, a) = \int_Z \int_N \int_F \int_{(F^\times)^2} \Xi \left[\iota \left(\begin{pmatrix} b & cx \\ 0 & c \end{pmatrix}, \begin{pmatrix} bc & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} \bar{n}_s \, z \, n \, \varpi^\lambda \right]$$
$$\omega(z) \; \psi(n) \; \psi(-sax) \; \delta^{-1}(s(1-a)\, bc) \; \kappa(b)$$
$$W\left(sabc^{-1}\right) \, W\left(s(1-a)\, b^{-1} c^{-1}\right) \, |c|^2 \, d^\times b \, d^\times c \, dx \, dn \, dz.$$

For $\lambda = (\lambda_1, \lambda_2) \in P^+$, let

$$\|\lambda\| = \lambda_1 + \lambda_2.$$

Then the condition that the similitude

$$\lambda \left[\iota \left(\begin{pmatrix} b & cx \\ 0 & c \end{pmatrix}, \begin{pmatrix} bc & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} \bar{n}_s \, z \, n \, \varpi^\lambda \right] = z^2 b^{-1} c^{-1} q^{-\|\lambda\|} \in \mathcal{O}^\times$$

implies that $\operatorname{ord}(c) = \operatorname{ord}\left(z^2 b^{-1}\right) + \|\lambda\|$. Thus we have

$$I(\lambda; s, a) = q^{-2\|\lambda\|} \, \delta^{-1}\left(\varpi^{\|\lambda\|} s(1-a)\right) \int_{F^\times} \int_N \int_F \int_{F^\times} \psi(n) \; \psi(-sax) \; \kappa(b)$$
$$\Xi \left[\iota \left(\begin{pmatrix} zb^{-1} & -zb^{-1}x \\ 0 & \varpi^{-\|\lambda\|} z^{-1} b \end{pmatrix}, \begin{pmatrix} \varpi^{-\|\lambda\|} z^{-1} & 0 \\ 0 & z \end{pmatrix} \right) \bar{n}_s \, n \, \varpi^\lambda \right]$$
$$W\left(\varpi^{-\|\lambda\|} sab^2 z^{-2}\right) \, W\left(\varpi^{-\|\lambda\|} s(1-a)\, z^{-2}\right) \, |z^2 b^{-1}|^2 \, d^\times b \, dx \, dn \, d^\times z.$$

Replacing b by zb^{-1}, we have

$$I(\lambda; s, a) = q^{-2\|\lambda\|} \, \delta^{-1}\left(\varpi^{\|\lambda\|} s(1-a)\right) \int_{F^\times} \int_N \int_F \int_{F^\times} \psi(n) \; \psi(-sax) \; \kappa\left(zb^{-1}\right)$$
$$\Xi \left[\iota \left(\begin{pmatrix} b & -bx \\ 0 & \varpi^{-\|\lambda\|} b^{-1} \end{pmatrix}, \begin{pmatrix} \varpi^{-\|\lambda\|} z^{-1} & 0 \\ 0 & z \end{pmatrix} \right) \bar{n}_s \, n \, \varpi^\lambda \right]$$
$$W\left(\varpi^{-\|\lambda\|} sab^{-2}\right) \, W\left(\varpi^{-\|\lambda\|} s(1-a)\, z^{-2}\right) \, |zb|^2 \, d^\times b \, dx \, dn \, d^\times z.$$

For $b, z \in F^\times$ and $x, y \in F$, let

$$A_\lambda (b, z, x, y) = \iota \left(\begin{pmatrix} b & -bx \\ 0 & \varpi^{-\|\lambda\|} b^{-1} \end{pmatrix}, \begin{pmatrix} \varpi^{-\|\lambda\|} z^{-1} & 0 \\ 0 & z \end{pmatrix} \right) \bar{n}_s \begin{pmatrix} 1 & y & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -y & 1 \end{pmatrix} \varpi^\lambda \tag{5.1}$$
$$= \begin{pmatrix} \varpi^{\|\lambda\|} b & \varpi^{\lambda_1} b\,(y - sx) & -bx & 0 \\ 0 & \varpi^{-\lambda_2} s z^{-1} & 0 & 0 \\ 0 & \varpi^{-\lambda_2} b^{-1} s & \varpi^{-\|\lambda\|} b^{-1} & 0 \\ \varpi^{\|\lambda\|} z & \varpi^{\lambda_1} yz & -s^{-1} yz & \varpi^{\lambda_2} s^{-1} z \end{pmatrix}.$$

Then we have

$$\begin{aligned} I\,(\lambda; s, a) = q^{-2\|\lambda\|}\, \delta^{-1} \left(\varpi^{\|\lambda\|} s\,(1-a) \right) \int_{F^\times} \int_U \int_F \int_F \int_{F^\times} \psi\,(u)\; \psi\,(y - sax) \\ \kappa\left(zb^{-1}\right)\, \Xi\left[A_\lambda\,(b, z, x, y)\,\left(\varpi^{-\lambda}\, u\, \varpi^\lambda\right)\right] \\ W\left(\varpi^{-\|\lambda\|} sab^{-2}\right) W\left(\varpi^{-\|\lambda\|} s\,(1-a)\, z^{-2}\right) |zb|^2\, d^\times b\, dx\; dy\, du\, d^\times z \end{aligned}$$

where U denotes the unipotent radical of the upper Siegel parabolic subgroup of G. Replacing u by $\varpi^\lambda\, u\, \varpi^{-\lambda}$, we have

$$\begin{aligned} I\,(\lambda; s, a) = q^{-2\|\lambda\| - 3\lambda_1}\, \delta^{-1} \left(\varpi^{\|\lambda\|} s\,(1-a) \right) \int_{F^\times} \int_U \int_F \int_F \int_{F^\times} \\ \psi\left(\varpi^\lambda\, u\, \varpi^{-\lambda}\right)\, \psi\,(y - sax)\; \kappa\left(zb^{-1}\right)\, \Xi\left[A_\lambda\,(b, z, x, y)\; u\right] \\ W\left(\varpi^{-\|\lambda\|} sab^{-2}\right) W\left(\varpi^{-\|\lambda\|} s\,(1-a)\, z^{-2}\right) |zb|^2\, d^\times b\, dx\; dy\, du\, d^\times z. \end{aligned} \tag{5.2}$$

Let us examine the support of the integral (5.2).

LEMMA 5.1. *For the matrix $A_\lambda\,(b, z, x, y)$, we have*

$$A_\lambda\,(b, z, x, y)\; u \in K \text{ for some } u \in U \tag{5.3}$$

if and only if

$$\max\{|b|, |z|\} = q^{\|\lambda\|}, \tag{5.4}$$
$$\max\left\{|\varpi^{\lambda_1} b\,(y - sx)|, |\varpi^{-\lambda_2} s z^{-1}|, |\varpi^{-\lambda_2} b^{-1} s|, |\varpi^{\lambda_1} yz|\right\} \le 1, \tag{5.5}$$
$$\max\left\{|\varpi^{\lambda_1} sbz^{-1}|, |\varpi^{\|\lambda\| + \lambda_1} sbxz|, |\varpi^{\lambda_1} b^{-1} sz|\right\} = 1. \tag{5.6}$$

PROOF. By Lemma 3.12, the condition (5.3) is equivalent to (5.4), (5.5) and

$$\max\left\{|\varpi^{\lambda_1} sbz^{-1}|, |\varpi^{\lambda_1} s|, |\varpi^{\|\lambda\| + \lambda_1} sbxz|, |\varpi^{\lambda_1} b^{-1} sz|\right\} = 1. \tag{5.7}$$

By noting that $\left(\varpi^{\lambda_1} sbz^{-1}\right) \cdot \left(\varpi^{\lambda_1} b^{-1} sz\right) = \left(\varpi^{\lambda_1} s\right)^2$, the condition (5.6) is equivalent to the condition (5.7). □

From (5.7), we have the following vanishing condition on $I\,(\lambda; s, a)$.

LEMMA 5.2. *The orbital integral $I\,(\lambda; s, a)$ vanishes unless $|s| \le q^{\lambda_1}$.*

5.1.2. Evaluation of $I(\lambda; s, a)$ when $|s| = q^{\lambda_1}$. The integral $I(\lambda; s, a)$ is evaluated easily as follows when $|s| = q^{\lambda_1}$.

Proposition 5.3. *When $|s| = q^{\lambda_1}$, we have*

$$I(\lambda; s, a) = q^{-2\lambda_1}\,\delta^{-1}\left(\varpi^{\lambda_2}(1-a)\right) W\left(\varpi^{\lambda_2}a\right) W\left(\varpi^{\lambda_2}(1-a)\right). \tag{5.8}$$

Proof. When $|s| = q^{\lambda_1}$, the condition (5.7) implies that $|b| = |z|$. It is readily seen that the condition (5.4) is equivalent to $|b| = |z| = q^{\|\lambda\|}$, $|x| \le q^{-\|\lambda\|}$ and $y \le q^{-\lambda_2}$. Then we have $A_\lambda(b, z, x, y) \in K$. Hence we have $A_\lambda(b, z, x, y)\, u \in K$ for $u \in U$ if and only if $u \in U \cap K$. Thus we have

$$I(\lambda; s, a) = q^{\|\lambda\|-2\lambda_1}\,\delta^{-1}\left(\varpi^{\lambda_2}(1-a)\right) W\left(\varpi^{\lambda_2}a\right) W\left(\varpi^{\lambda_2}(1-a)\right) \int_{\varpi^{\|\lambda\|}\mathcal{O}} \psi(-sax)\, dx$$

where

$$\int_{\varpi^{\|\lambda\|}\mathcal{O}} \psi(-sax)\, dx = \begin{cases} q^{-\|\lambda\|}, & \text{if } |\varpi^{\lambda_2}a| \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $W\left(\varpi^{\lambda_2}a\right)$ vanishes unless $|\varpi^{\lambda_2}a| \le 1$, the equality (5.8) holds. □

5.1.3. Evaluation of $I(\lambda; s, a)$ when $|s| < q^{\lambda_1}$. First we put

$$\operatorname{ord}(s) = h, \quad \operatorname{ord}(1-a) = k, \quad \operatorname{ord}(a) = k'. \tag{5.9}$$

By the condition (5.4), the orbital integral $I(\lambda; s, a)$ is a sum of three integrals $I^{(j)}(\lambda; s, a)$ $(j = 0, 1, 2)$, supported on

$$|b| = |z| = q^{\|\lambda\|}; \quad |b| = q^{\|\lambda\|} > |z|; \quad |b| < |z| = q^{\|\lambda\|},$$

respectively.

5.1.3.1. *Evaluation of $I^{(0)}(\lambda; s, a)$.* In the domain where $|b| = |z| = q^{\|\lambda\|}$, we may assume that $b = z = \varpi^{-\|\lambda\|}$ in the integrand of (5.2).

For $A_\lambda\left(\varpi^{-\|\lambda\|}, \varpi^{-\|\lambda\|}, x, y\right)$, the condition (5.3) holds if and only if $|\varpi^{-\lambda_2}sx| = 1$, i.e. $\operatorname{ord}(x) = \lambda_2 - h$, and $|y| \le q^{-\lambda_2}$. Then we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A_\lambda\left(\varpi^{-\|\lambda\|}, \varpi^{-\|\lambda\|}, x, y\right) = \begin{pmatrix} 1 & \varpi^{-\lambda_2}(y - sx) & -\varpi^{-\|\lambda\|}x & 0 \\ 1 & \varpi^{-\lambda_2}y & -\varpi^{-\|\lambda\|}s^{-1}y & \varpi^{-\lambda_1}s^{-1} \\ 0 & \varpi^{\lambda_1}s & 1 & 0 \\ 0 & -\varpi^{\lambda_1}s & 0 & 0 \end{pmatrix}$$

where $\begin{pmatrix} 1 & \varpi^{-\lambda_2}(y - sx) \\ 1 & \varpi^{-\lambda_2}y \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O})$. Hence for $S \in \mathrm{Sym}^2(F)$, we have

$$A_\lambda\left(\varpi^{-\|\lambda\|}, \varpi^{-\|\lambda\|}, x, y\right) \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if

$$\begin{pmatrix} 1 & \varpi^{-\lambda_2}(y - sx) \\ 1 & \varpi^{-\lambda_2}y \end{pmatrix} S + \begin{pmatrix} -\varpi^{-\|\lambda\|}x & 0 \\ -\varpi^{-\|\lambda\|}s^{-1}y & \varpi^{-\lambda_1}s^{-1} \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}) \tag{5.10}$$

and

$$\begin{pmatrix} 0 & \varpi^{\lambda_1} s \\ 0 & -\varpi^{\lambda_1} s \end{pmatrix} S + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2\left(\mathcal{O}\right). \tag{5.11}$$

The condition (5.10) is equivalent to

$$S \in \begin{pmatrix} \varpi^{-\|\lambda\|}\left(2s^{-1}y - s^{-2}x^{-1}y^2\right) & \varpi^{-\lambda_1}\left(s^{-2}x^{-1}y - s^{-1}\right) \\ \varpi^{-\lambda_1}\left(s^{-2}x^{-1}y - s^{-1}\right) & -\varpi^{-\lambda_1+\lambda_2} s^{-2}x^{-1} \end{pmatrix} + \mathrm{Sym}^2\left(\mathcal{O}\right). \tag{5.12}$$

Since the elements in the second column lie in $\varpi^{-\lambda_1}s^{-1}\mathcal{O}$, the condition (5.12) implies (5.11). Hence we have

$$I^{(0)}\left(\lambda; s, a\right) = q^{-\lambda_1+\lambda_2}\, \delta^{-1}\left(\varpi^{\|\lambda\|} s\left(1-a\right)\right) W\left(\varpi^{\|\lambda\|} sa\right) W\left(\varpi^{\|\lambda\|} s\left(1-a\right)\right)$$
$$\int_{\varpi^{\lambda_2 - h}\mathcal{O}^\times} \psi\left(-s^{-2}x^{-1} - sax\right) dx.$$

By a change of variable $x \mapsto -\varpi^{\lambda_2}s^{-1}x$, we have

$$I^{(0)}\left(\lambda; s, a\right) = q^{h-\lambda_1}\, \delta^{-1}\left(\varpi^{\|\lambda\|} s\left(1-a\right)\right) W\left(\varpi^{\|\lambda\|} sa\right) W\left(\varpi^{\|\lambda\|} s\left(1-a\right)\right) \cdot \mathcal{K}l\left(\varpi^{\lambda_2}a, \varpi^{-\lambda_2}s^{-1}\right). \tag{5.13}$$

Here we note that we have

$$\delta\left(1-a\right) \cdot I^{(0)}\left(\lambda; s, a\right) = \delta\left(a\right) \cdot I^{(0)}\left(\lambda; -s, 1-a\right) \tag{5.14}$$

since

$$\mathcal{K}l\left(\varpi^{\lambda_2}(1-a), -\varpi^{-\lambda_2}s^{-1}\right) = \mathcal{K}l\left(-\varpi^{\lambda_2}a, -\varpi^{-\lambda_2}s^{-1}\right) = \mathcal{K}l\left(\varpi^{\lambda_2}a, \varpi^{-\lambda_2}s^{-1}\right).$$

5.1.3.2. *Evaluation of* $I^{(1)}\left(\lambda; s, a\right)$. In the domain where $|b| = q^{\|\lambda\|} > |z|$, we may assume that $b = \varpi^{-\|\lambda\|}$ in the integrand of (5.2). For $A_\lambda\left(\varpi^{-\|\lambda\|}, z, x, y\right)$, the condition (5.3) is equivalent to

$$\max\left\{|\varpi^{-\lambda_2}sz^{-1}|, |\varpi^{\lambda_1}sxz|\right\} = 1, \qquad |\varpi^{-\lambda_2}\left(y - sx\right)| \le 1 \tag{5.15}$$

since $\varpi^{\lambda_1}yz = \varpi^{\|\lambda\|}z \cdot \varpi^{-\lambda_2}\left(y-sx\right) + \varpi^{\lambda_1}sxz$. By a change of variable $y \mapsto y + sx$, the second condition of (5.15) is equivalent to $y \in \varpi^{\lambda_2}\mathcal{O}$. Splitting the first condition into two separate cases

$$|\varpi^{-\lambda_2}sz^{-1}| = 1 \ge |\varpi^{\lambda_1}sxz| \quad \text{or} \quad |\varpi^{-\lambda_2}sz^{-1}| < 1 = |\varpi^{\lambda_1}sxz|,$$

we may write $I^{(1)}\left(\lambda; s, a\right) = I^{(1,1)}\left(\lambda; s, a\right) + I^{(1,2)}\left(\lambda; s, a\right)$.

For $I^{(1,1)}\left(\lambda; s, a\right)$, we may assume that $z = \varpi^{-\lambda_2}s$ and we have

$$\begin{aligned} I^{(1,1)}\left(\lambda; s, a\right) = {} & q^{-3\lambda_1+2\lambda_2-2h}\, \delta^{-1}\left(\varpi^{\|\lambda\|} s\left(1-a\right)\right) \kappa\left(\varpi\right)^{\lambda_1+h} \\ & W\left(\varpi^{\|\lambda\|} sa\right) W\left(\varpi^{\lambda_2-\lambda_1}s^{-1}\left(1-a\right)\right) \int_{x \in \varpi^{-\lambda_1+\lambda_2}s^{-2}\mathcal{O}} \int_{y \in \varpi^{\lambda_2}\mathcal{O}} \int_U \\ & \psi\left(\varpi^{\lambda}\, u\, \varpi^{-\lambda}\right) \psi\left(s\left(1-a\right)x\right) \Xi\left[A_\lambda\left(\varpi^{-\|\lambda\|}, \varpi^{-\lambda_2}s, x, y+sx\right) u\right] dx\, dy\, du. \end{aligned} \tag{5.16}$$

In (5.16), we have

$$A_\lambda\left(\varpi^{-\|\lambda\|}, \varpi^{-\lambda_2}s, x, y+sx\right) = \begin{pmatrix} 1 & \varpi^{-\lambda_2}y & -\varpi^{-\|\lambda\|}x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \varpi^{\lambda_1}s & 1 & 0 \\ \varpi^{\lambda_1}s & \varpi^{\lambda_1-\lambda_2}s(y+sx) & -\varpi^{-\lambda_2}(y+sx) & 1 \end{pmatrix}$$

where $\begin{pmatrix} 1 & \varpi^{-\lambda_2} y \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O})$. Hence for $S \in \mathrm{Sym}^2(F)$, we have

$$A_\lambda\left(\varpi^{-\|\lambda\|}, \varpi^{-\lambda_2} s, x, y+sx\right)\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if

$$S \in \begin{pmatrix} \varpi^{-\|\lambda\|} x & 0 \\ 0 & 0 \end{pmatrix} + \mathrm{Sym}^2(\mathcal{O}) \tag{5.17}$$

and

$$\begin{pmatrix} 0 & \varpi^{\lambda_1} s \\ \varpi^{\lambda_1} s & \varpi^{\lambda_1-\lambda_2} s(y+sx) \end{pmatrix} S + \begin{pmatrix} 1 & 0 \\ -\varpi^{-\lambda_2}(y+sx) & 1 \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}). \tag{5.18}$$

Here (5.17) implies (5.18). Thus by a change of variable $x \mapsto \varpi^{-\lambda_1+\lambda_2} s^{-2} x$,

$$\begin{aligned} I^{(1,1)}(\lambda; s, a) &= q^{-2\lambda_1}\, \delta^{-1}\left(\varpi^{\|\lambda\|} s\,(1-a)\right)\, \kappa(\varpi)^{\lambda_1+h} \\ &\quad W\left(\varpi^{\|\lambda\|} sa\right)\, W\left(\varpi^{-\lambda_1+\lambda_2} s^{-1}(1-a)\right) \int_{\mathcal{O}} \psi\left(\varpi^{-\lambda_1+\lambda_2} s^{-1}(1-a)\, x\right)\, dx. \end{aligned}$$

By the non-vanishing condition of the latter Whittaker value, we have

$$\begin{aligned} I^{(1,1)}(\lambda; s, a) &= q^{-2\lambda_1}\, \delta^{-1}\left(\varpi^{\|\lambda\|} s\,(1-a)\right)\, \kappa(\varpi)^{\lambda_1+h} \\ &\qquad \cdot W\left(\varpi^{\|\lambda\|} sa\right)\, W\left(\varpi^{-\lambda_1+\lambda_2} s^{-1}(1-a)\right). \end{aligned} \tag{5.19}$$

As for $I^{(1,2)}(\lambda; s, a)$, we have $|\varpi^{-\lambda_2} s| < |z| < q^{\|\lambda\|}$. Splitting into separate cases according to $\mathrm{ord}(z)$, we have

$$I^{(1,2)}(\lambda; s, a) = \sum_{j=1}^{h+\lambda_1-1} I^{(1,2,j)}(\lambda; s, a)$$

where

$$\begin{aligned} I^{(1,2,j)}(\lambda; s, a) &= q^{-\lambda_1+2\lambda_2-2j}\, \delta^{-1}\left(\varpi^{\|\lambda\|} s\,(1-a)\right)\, \kappa(\varpi)^{j} \\ &\quad W\left(\varpi^{\|\lambda\|} sa\right)\, W\left(\varpi^{\|\lambda\|-2j} s\,(1-a)\right) \int_{x \in \varpi^{\lambda_2-j} s^{-1} \mathcal{O}^\times} \int_{y \in \varpi^{\lambda_2} \mathcal{O}} \int_U \\ &\quad \psi\left(\varpi^{\lambda}\, u\, \varpi^{-\lambda}\right)\, \psi\left(s\,(1-a)\, x\right)\, \Xi\left[A_\lambda\left(\varpi^{-\|\lambda\|}, \varpi^{j-\|\lambda\|}, x, y+sx\right)\, u\right]\, dx\, dy\, du. \end{aligned} \tag{5.20}$$

In (5.20), we have

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A_\lambda\left(\varpi^{-\|\lambda\|}, \varpi^{j-\|\lambda\|}, x, y+sx\right) \\ &\qquad = \begin{pmatrix} 1 & \varpi^{-\lambda_2} y & -\varpi^{-\|\lambda\|} x & 0 \\ \varpi^{j} & \varpi^{j-\lambda_2}(y+sx) & -\varpi^{j-\|\lambda\|} s^{-1}(y+sx) & \varpi^{j-\lambda_1} s^{-1} \\ 0 & \varpi^{\lambda_1} s & 1 & 0 \\ 0 & -\varpi^{\lambda_1-j} s & 0 & 0 \end{pmatrix} \end{aligned}$$

where $\begin{pmatrix} 1 & \varpi^{-\lambda_2} y \\ \varpi^j & \varpi^{j-\lambda_2}(y+sx) \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O})$. Hence for $S \in \mathrm{Sym}^2(F)$, we have

$$A_\lambda\left(\varpi^{-\|\lambda\|}, \varpi^{j-\|\lambda\|}, x, y+sx\right) \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if

$$\begin{pmatrix} 1 & \varpi^{-\lambda_2} y \\ \varpi^j & \varpi^{j-\lambda_2}(y+sx) \end{pmatrix} S + \begin{pmatrix} -\varpi^{-\|\lambda\|} x & 0 \\ -\varpi^{j-\|\lambda\|} s^{-1}(y+sx) & \varpi^{j-\lambda_1} s^{-1} \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}) \tag{5.21}$$

and

$$\begin{pmatrix} 0 & \varpi^{\lambda_1} s \\ 0 & -\varpi^{\lambda_1 - j} s \end{pmatrix} S + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}). \tag{5.22}$$

The condition (5.21) is equivalent to

$$S \in \begin{pmatrix} \varpi^{-\|\lambda\|}\left(x - s^{-2} x^{-1} y^2\right) & \varpi^{-\lambda_1} s^{-2} x^{-1} y \\ \varpi^{-\lambda_1} s^{-2} x^{-1} y & -\varpi^{-\lambda_1+\lambda_2} s^{-2} x^{-1} \end{pmatrix} + \mathrm{Sym}^2(\mathcal{O})$$

and this implies (5.22). Thus by a change of variable $x \mapsto \varpi^{\lambda_2 - j} s^{-1} x$, we have

$$\begin{aligned} I^{(1,2,j)}(\lambda; s, a) = {} & q^{h-\lambda_1-j}\, \delta^{-1}\left(\varpi^{\|\lambda\|} s(1-a)\right) \kappa(\varpi)^j \\ & \cdot W\left(\varpi^{\|\lambda\|} sa\right) W\left(\varpi^{\|\lambda\|-2j} s(1-a)\right) \mathcal{K}l\left(\varpi^{\lambda_2-j}(1-a), -\varpi^{j-\lambda_2} s^{-1}\right). \end{aligned} \tag{5.23}$$

Hence from (5.19) and (5.23), we have

$$\begin{aligned} I^{(1)}(\lambda; s, a) = {} & q^{-2\lambda_1}\, \delta^{-1}\left(\varpi^{\|\lambda\|} s(1-a)\right) W\left(\varpi^{\|\lambda\|} sa\right) \\ & \cdot \kappa(\varpi)^{h+\lambda_1}\, W\left(\varpi^{\|\lambda\| - 2(h+\lambda_1)} s(1-a)\right) \\ & + q^{h-\lambda_1}\, \delta^{-1}\left(\varpi^{\|\lambda\|} s(1-a)\right) W\left(\varpi^{\|\lambda\|} sa\right) \\ & \cdot \sum_{j=1}^{h+\lambda_1-1} q^{-j}\, \kappa(\varpi)^j\, W\left(\varpi^{\|\lambda\|-2j} s(1-a)\right) \mathcal{K}l\left(\varpi^{\lambda_2-j}(1-a), -\varpi^{j-\lambda_2} s^{-1}\right). \end{aligned} \tag{5.24}$$

5.1.3.3. *Evaluation of* $I^{(2)}(\lambda; s, a)$. When $|b| < |z| = q^{\|\lambda\|}$, we may assume that $z = \varpi^{-\|\lambda\|}$. Then for $A_\lambda\left(b, \varpi^{-\|\lambda\|}, x, y\right)$, the condition (5.3) is equivalent to

$$\max\left\{|\varpi^{\lambda_1} sbx|, |\varpi^{-\lambda_2} sb^{-1}|\right\} = 1, \qquad |\varpi^{-\lambda_2} y| \leq 1. \tag{5.25}$$

By separating the first condition into two cases

$$|\varpi^{-\lambda_2} sb^{-1}| = 1 \geq |\varpi^{\lambda_1} sbx| \qquad \text{or} \qquad |\varpi^{-\lambda_2} sb^{-1}| < 1 = |\varpi^{\lambda_1} sbx|,$$

we may write $I^{(2)}(\lambda; s, a) = I^{(2,1)}(\lambda; s, a) + I^{(2,2)}(\lambda; s, a)$.

For $I^{(2,1)}(\lambda; s, a)$, we may also assume that $b = \varpi^{-\lambda_2} s$. We note that when $x \in \varpi^{-\lambda_1+\lambda_2} s^{-2} \mathcal{O}$ and $y \in \varpi^{\lambda_2} \mathcal{O}$, we have

$$A_\lambda\left(\varpi^{-\lambda_2} s, \varpi^{-\|\lambda\|}, x, y\right) = \begin{pmatrix} \varpi^{\lambda_1} s & \varpi^{\lambda_1-\lambda_2} s(y - sx) & -\varpi^{-\lambda_2} sx & 0 \\ 0 & \varpi^{\lambda_1} s & 0 & 0 \\ 0 & 1 & \varpi^{-\lambda_1} s^{-1} & 0 \\ 1 & \varpi^{-\lambda_2} y & -\varpi^{-\|\lambda\|} s^{-1} y & \varpi^{-\lambda_1} s^{-1} \end{pmatrix}$$

where $\begin{pmatrix} 0 & 1 \\ 1 & \varpi^{-\lambda_2} y \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O})$. Hence for $S \in \mathrm{Sym}^2(F)$, we have

$$A_\lambda\left(\varpi^{-\lambda_2}s, \varpi^{-\|\lambda\|}, x, y\right)\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if

$$\begin{pmatrix} \varpi^{\lambda_1}s & \varpi^{\lambda_1-\lambda_2}s(y-sx) \\ 0 & \varpi^{\lambda_1}s \end{pmatrix} S + \begin{pmatrix} -\varpi^{-\lambda_2}sx & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}) \tag{5.26}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & \varpi^{-\lambda_2}y \end{pmatrix} S + \begin{pmatrix} \varpi^{-\lambda_1}s^{-1} & 0 \\ -\varpi^{-\|\lambda\|}s^{-1}y & \varpi^{-\lambda_1}s^{-1} \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}). \tag{5.27}$$

The condition (5.27) is equivalent to

$$S \in \begin{pmatrix} 2\varpi^{-\|\lambda\|}s^{-1}y & -\varpi^{-\lambda_1}s^{-1} \\ -\varpi^{-\lambda_1}s^{-1} & 0 \end{pmatrix} + \mathrm{Sym}^2(\mathcal{O})$$

and this implies (5.26). Hence

$$\begin{aligned} I^{(2,1)}(\lambda; s, a) = {}& q^{-3\lambda_1+\lambda_2-2h}\,\delta^{-1}\left(\varpi^{\|\lambda\|}s(1-a)\right)\kappa(\varpi)^{\lambda_1+h} \\ & W\left(\varpi^{-\lambda_1+\lambda_2}s^{-1}a\right)W\left(\varpi^{\|\lambda\|}s(1-a)\right)\int_{\varpi^{-\lambda_1+\lambda_2}s^{-2}\mathcal{O}}\psi(-sax)\,dx. \end{aligned} \tag{5.28}$$

Here we note that the non-vanishing of the former Whittaker value implies the value of the integral in (5.28) to be $q^{\lambda_1-\lambda_2+2h}$. Thus

$$\begin{aligned} I^{(2,1)}(\lambda; s, a) = {}& q^{-2\lambda_1}\,\delta^{-1}\left(\varpi^{\|\lambda\|}s(1-a)\right)\kappa(\varpi)^{\lambda_1+h} \\ & \cdot W\left(\varpi^{-\lambda_1+\lambda_2}s^{-1}a\right)W\left(\varpi^{\|\lambda\|}s(1-a)\right). \end{aligned} \tag{5.29}$$

Separating into cases according to $\mathrm{ord}(b)$, we have

$$I^{(2,2)}(\lambda; s, a) = \sum_{j=1}^{h+\lambda_1-1} I^{(2,2,j)}(\lambda; s, a)$$

where

$$\begin{aligned} I^{(2,2,j)}(\lambda; s, a) = {}& q^{-\lambda_1+2\lambda_2-2j}\,\delta^{-1}\left(\varpi^{\|\lambda\|}s(1-a)\right)\kappa(\varpi)^{j} \\ & W\left(\varpi^{\|\lambda\|-2j}sa\right)W\left(\varpi^{\|\lambda\|}s(1-a)\right)\int_{x\in\varpi^{\lambda_2-j}s^{-1}\mathcal{O}^\times}\int_{y\in\varpi^{\lambda_2}\mathcal{O}}\int_U \\ & \psi\left(\varpi^{\lambda}\,u\,\varpi^{-\lambda}\right)\psi(-sax)\,\Xi\left[A_\lambda\left(\varpi^{j-\|\lambda\|}, \varpi^{-\|\lambda\|}, x, y\right)u\right]dx\,dy\,du. \end{aligned} \tag{5.30}$$

In (5.30), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A_\lambda\left(\varpi^{j-\|\lambda\|}, \varpi^{-\|\lambda\|}, x, y\right)$$

$$= \begin{pmatrix} \varpi^j & \varpi^{j-\lambda_2}(y-sx) & -\varpi^{j-\|\lambda\|}x & 0 \\ 1 & \varpi^{-\lambda_2}y & -\varpi^{-\|\lambda\|}s^{-1}y & \varpi^{-\lambda_1}s^{-1} \\ 0 & \varpi^{\lambda_1-j}s & \varpi^{-j} & 0 \\ 0 & -\varpi^{\lambda_1}s & 0 & 0 \end{pmatrix}$$

where $\begin{pmatrix} \varpi^j & \varpi^{j-\lambda_2}(y-sx) \\ 1 & \varpi^{-\lambda_2}y \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O})$. Hence for $S \in \mathrm{Sym}^2(F)$, we have

$$A_\lambda\left(\varpi^{j-\|\lambda\|}, \varpi^{-\|\lambda\|}, x, y\right)\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \in K$$

if and only if

$$\begin{pmatrix} \varpi^j & \varpi^{j-\lambda_2}(y-sx) \\ 1 & \varpi^{-\lambda_2}y \end{pmatrix} S + \begin{pmatrix} -\varpi^{j-\|\lambda\|}x & 0 \\ -\varpi^{-\|\lambda\|}s^{-1}y & \varpi^{-\lambda_1}s^{-1} \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}) \tag{5.31}$$

and

$$\begin{pmatrix} 0 & \varpi^{\lambda_1-j}s \\ 0 & -\varpi^{\lambda_1}s \end{pmatrix} S + \begin{pmatrix} \varpi^{-j} & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}). \tag{5.32}$$

The condition (5.31) is equivalent to

$$S \in \begin{pmatrix} \varpi^{-\|\lambda\|}s^{-2}x^{-1}\left(2sxy-y^2\right) & \varpi^{-\lambda_1}s^{-2}x^{-1}(y-sx) \\ \varpi^{-\lambda_1}s^{-2}x^{-1}(y-sx) & -\varpi^{-\lambda_1+\lambda_2}s^{-2}x^{-1} \end{pmatrix} + \mathrm{Sym}^2(\mathcal{O})$$

and this implies (5.32). Thus by a change of variable $x \mapsto \varpi^{\lambda_2-j}s^{-1}x$, we have

$$\begin{aligned} I^{(2,2,j)}(\lambda; s, a) = q^{h-\lambda_1-j}\,\delta^{-1}\left(\varpi^{\|\lambda\|}s(1-a)\right)\,\kappa(\varpi)^j \\ \cdot W\left(\varpi^{\|\lambda\|-2j}sa\right) W\left(\varpi^{\|\lambda\|}s(1-a)\right) \mathcal{K}l\left(\varpi^{\lambda_2-j}a, \varpi^{j-\lambda_2}s^{-1}\right). \end{aligned} \tag{5.33}$$

Hence we have

$$\begin{aligned} I^{(2)}(\lambda; s, a) = {} & q^{-2\lambda_1}\,\delta^{-1}\left(\varpi^{\|\lambda\|}s(1-a)\right) W\left(\varpi^{\|\lambda\|}s(1-a)\right) \\ & \cdot \kappa(\varpi)^{h+\lambda_1}\; W\left(\varpi^{\|\lambda\|-2(h+\lambda_1)}sa\right) \\ & + q^{h-\lambda_1}\,\delta^{-1}\left(\varpi^{\|\lambda\|}s(1-a)\right) W\left(\varpi^{\|\lambda\|}s(1-a)\right) \\ & \cdot \sum_{j=1}^{h+\lambda_1-1} q^{-j}\,\kappa(\varpi)^j\; W\left(\varpi^{\|\lambda\|-2j}sa\right) \mathcal{K}l\left(\varpi^{\lambda_2-j}a, \varpi^{j-\lambda_2}s^{-1}\right). \end{aligned} \tag{5.34}$$

By comparing (5.19) with (5.29), (5.23) with (5.33), and, (5.24) with (5.34), we have

$$\delta(1-a)\cdot I^{(1,1)}(\lambda; s, a) = \delta(a)\cdot I^{(2,1)}(\lambda; -s, 1-a), \tag{5.35}$$

$$\begin{aligned} & \delta(1-a)\cdot I^{(1,2,j)}(\lambda; s, a) \\ & \qquad = \delta(a)\cdot I^{(2,2,j)}(\lambda; -s, 1-a) \quad (1 \le j \le h+\lambda_1-1) \end{aligned} \tag{5.36}$$

and

$$\delta(1-a)\cdot I^{(1)}(\lambda;s,a)=\delta(a)\cdot I^{(2)}(\lambda;-s,1-a) \tag{5.37}$$

respectively.

We also obtain the following functional equation for $I_\lambda(s,a)$ itself.

PROPOSITION 5.4. *For $s\in F^\times$ and $a\in F\backslash\{0,1\}$, the function $\delta^{-1}(a)\cdot I(\lambda;s,a)$ is invariant under the transformation $(s,a)\mapsto(-s,1-a)$, i.e.*

$$\delta^{-1}(1-a)\cdot I(\lambda;-s,1-a)=\delta^{-1}(a)\cdot I(\lambda;s,a). \tag{5.38}$$

PROOF. Both sides of (5.38) vanish when $|s|>q^{\lambda_1}$. When $|s|=q^{\lambda_1}$, (5.38) is clear from (5.8). When $|s|<q^{\lambda_1}$, (5.38) follows from (5.14) and (5.37). □

5.1.4. Preparation for the matching. We recall that we use the bijection

$$(F\setminus\{0,1\})\times F^\times\ni(x,\mu)\xrightarrow{\approx}\left(-\frac{1-x}{4\mu},\frac{1}{1-x}\right)\in F^\times\times(F\setminus\{0,1\})$$

whose inverse is given by

$$(s,a)\mapsto\left(-\frac{1-a}{a},-\frac{1}{4sa}\right)$$

for the matching. We also recall that we put

$$\mathcal{I}(\lambda;x,\mu)=I(\lambda;s,a)\quad\text{where}\quad s=-\frac{1-x}{4\mu}\quad\text{and}\quad a=\frac{1}{1-x},$$

for $x\in F\setminus\{0,1\}$, $\mu\in F^\times$ and $\lambda\in P^+$. The dictionary between the two sets of parameters

$$m=\operatorname{ord}(x),\quad m'=\operatorname{ord}(1-x),\quad n=-\operatorname{ord}(\mu)$$

and

$$h=\operatorname{ord}(s),\quad k=\operatorname{ord}(1-a),\quad k'=\operatorname{ord}(a),$$

is given by

$$m=k-k',\quad m'=-k',\quad n=h+k'$$

and

$$h=m'+n,\quad k=m-m',\quad k'=-m'.$$

Then we may rewrite the results obtained so far for $I(\lambda;s,a)$ as follows in terms of $\mathcal{I}(\lambda;x,\mu)$.

PROPOSITION 5.5. *Let $x\in F\setminus\{0,1\}$, $\mu\in F^\times$ and $\lambda=(\lambda_1,\lambda_2)\in P^+$. We put $m=\operatorname{ord}(x)$, $m'=\operatorname{ord}(1-x)$ and $n=-\operatorname{ord}(\mu)$. Let us write $x=\varpi^m\varepsilon_x$ and $\mu=\varpi^{-n}\varepsilon_\mu$. Let $l(\lambda_1)=\lambda_1+m'+n$.*

Then the integral $\mathcal{I}(\lambda;x,\mu)$ is evaluated as follows.

(1) *The integral $\mathcal{I}(\lambda;x,\mu)$ vanishes unless $l(\lambda_1)\geq 0$.*
(2) *Suppose that $l(\lambda_1)\geq 0$. Then we have*

$$\mathcal{I}(\lambda;x,\mu)=\mathcal{I}^{(1)}(\lambda;x,\mu) \tag{5.39}$$

when $l(\lambda_1)=0$, and, we have

$$\mathcal{I}(\lambda;x,\mu)=\mathcal{I}^{(1)}(\lambda;x,\mu)+\mathcal{I}^{(2)}(\lambda;x,\mu)+\sum_{s\in\mathbb{Z}}\mathcal{I}^{(1,s)}(\lambda;x,\mu)+\sum_{t\in\mathbb{Z}}\mathcal{I}^{(2,t)}(\lambda;x,\mu) \tag{5.40}$$

when $l(\lambda_1) > 0$. Here

$$\mathcal{I}^{(1)}(\lambda; x, \mu) = q^{-2\lambda_1} \kappa(\varpi)^{l(\lambda_1)} \delta^{-1}\left(\varpi^{\|\lambda\|+m+n}\right) \cdot W\left(\varpi^{\|\lambda\|+n}\right) W\left(\varpi^{\|\lambda\|+m+n-2l(\lambda_1)}\right), \tag{5.41}$$

$$\mathcal{I}^{(2)}(\lambda; x, \mu) = q^{-2\lambda_1} \kappa(\varpi)^{l(\lambda_1)} \delta^{-1}\left(\varpi^{\|\lambda\|+m+n}\right) \cdot W\left(\varpi^{\|\lambda\|+m+n}\right) W\left(\varpi^{\|\lambda\|+n-2l(\lambda_1)}\right), \tag{5.42}$$

$$\mathcal{I}^{(1,s)}(\lambda; x, \mu) = q^{-\lambda_1+m'+n-s} \kappa(\varpi)^{s} \delta^{-1}\left(\varpi^{\|\lambda\|+m+n}\right) W\left(\varpi^{\|\lambda\|+n}\right) \cdot W\left(\varpi^{\|\lambda\|+m+n-2s}\right) \mathcal{K}l_1(x, \mu; \lambda_2 - s) \quad \text{if } 1 \le s \le l(\lambda_1) - 1 \tag{5.43}$$

and $\mathcal{I}^{(1,s)}(\lambda; x, \mu) = 0$ otherwise, and,

$$\mathcal{I}^{(2,t)}(\lambda; x, \mu) = q^{-\lambda_1+m'+n-t} \kappa(\varpi)^{t} \delta^{-1}\left(\varpi^{\|\lambda\|+m+n}\right) W\left(\varpi^{\|\lambda\|+m+n}\right) \cdot W\left(\varpi^{\|\lambda\|+n-2t}\right) \mathcal{K}l_2(x, \mu; \lambda_2 - t) \quad \text{if } 0 \le t \le l(\lambda_1) - 1 \tag{5.44}$$

and $\mathcal{I}^{(2,t)}(\lambda; x, \mu) = 0$ otherwise.

(3) *The functional equation*

$$\mathcal{I}\left(\lambda; x^{-1}, \mu x^{-1}\right) = \delta(x) \cdot \mathcal{I}(\lambda; x, \mu) \tag{5.45}$$

holds.

Proof. This is clear from the computations in the previous subsection. Note that the indexing of the summands in (5.40) is slightly different from the one used in the previous subsection. □

Corollary 5.6. *The orbital integral $\mathcal{I}(\lambda; x, \mu)$ vanishes unless*

$$l(\lambda_1) = \lambda_1 + m' + n \ge 0 \quad \text{and} \quad \|\lambda\| + n = \lambda_1 + \lambda_2 + n \ge 0. \tag{5.46}$$

Moreover, in the inert case, the integral $\mathcal{I}(\lambda; x, \mu)$ vanishes unless

$$m \text{ is even and } \|\lambda\| + n \text{ is even.} \tag{5.47}$$

Proof. We proved the first condition in Proposition 5.5. The rest of the conditions follow from the appearance of the product of the Whittaker values of the form

$$W\left(\varpi^{\|\lambda\|+m+n-2i}\right) W\left(\varpi^{\|\lambda\|+n-2j}\right) \quad \text{where } i \ge 0 \text{ and } j \ge 0$$

in the formulas expressing the summands of $\mathcal{I}(\lambda; x, \mu)$ and (1.1). □

5.2. Matching in the Inert Case

We shall prove Theorem 2.17 in this section. By Corollary 5.6, the integral $\mathcal{I}(\lambda; x, \mu)$ vanishes unless m is even. Also the functional equations (3.20) and (5.45) are compatible with (2.48). Thus our task here is to show (2.48) when $m \in 2\,\mathbb{Z}_{\geq 0}$.

For a fixed pair (x, μ), we regard $\mathcal{I}(\lambda; x, \mu)$ as a function on P^+ and denote it as $\mathcal{I}(\lambda)$. We denote the summands of $\mathcal{I}(\lambda)$ in a similar way.

First let us express $\mathcal{I}(\lambda)$ more explicitly. For $\lambda = (\lambda_1, \lambda_2) \in P^+$, let us define $C(\lambda)$ by

$$C(\lambda) = q^{-\frac{m}{2}+m'-2\lambda_1-\lambda_2}\,\delta(\varpi)^{-m-n-\lambda_1-\lambda_2}. \tag{5.48}$$

Let $\mathcal{Z}_+$ denote the characteristic function of $\mathbb{Z}_{\geq 0}$, the set of non-negative integers. Then by the Whittaker value formula (1.1), we have

(5.49)
$$\begin{aligned} C^{-1}\cdot\mathcal{I}^{(1)} &= (-1)^{l(\lambda_1)}\,\mathcal{Z}_+\left(\frac{\|\lambda\|+n}{2}\right)\mathcal{Z}_+\left(\|\lambda\|+m+n-2l(\lambda_1)\right)\mathcal{Z}_+\left(l(\lambda_1)\right) \\ &= (-1)^{\lambda_1+m'+n}\,\mathcal{Z}_+\left(\frac{\|\lambda\|+n}{2}\right)\mathcal{Z}_+(-\lambda_1+\lambda_2+m-2m'-n)\mathcal{Z}_+(\lambda_1+m'+n), \end{aligned}$$

$$\begin{aligned} C^{-1}\cdot\mathcal{I}^{(2)} &= (-1)^{l(\lambda_1)}\,\mathcal{Z}_+\left(\frac{\|\lambda\|+n-2l(\lambda_1)}{2}\right)\mathcal{Z}_+\left(l(\lambda_1)-1\right) \\ &= (-1)^{\lambda_1+m'+n}\,\mathcal{Z}_+\left(\frac{-\lambda_1+\lambda_2-2m'-n}{2}\right)\mathcal{Z}_+\left(\lambda_1+m'+n-1\right), \end{aligned} \tag{5.50}$$

$$\begin{aligned} C^{-1}\cdot\mathcal{I}^{(1,s)} = (-1)^{s}\,\mathcal{Z}_+&\left(\frac{\|\lambda\|+n}{2}\right)\mathcal{Z}_+\left(\|\lambda\|+m+n-2s\right) \\ &\cdot\mathcal{Z}_+(s-1)\,\mathcal{Z}_+\left(l(\lambda_1)-s-1\right)\cdot\mathcal{K}l_1\left(x,\mu;\lambda_2-s\right) \end{aligned} \tag{5.51}$$

and

$$\begin{aligned} C^{-1}\cdot\mathcal{I}^{(2,t)} = (-1)^{t}\,\mathcal{Z}_+&\left(\frac{\|\lambda\|+n-2t}{2}\right) \\ &\cdot\mathcal{Z}_+(t)\,\mathcal{Z}_+\left(l(\lambda_1)-t-1\right)\cdot\mathcal{K}l_2\left(x,\mu;\lambda_2-t\right). \end{aligned} \tag{5.52}$$

Thus we have

$$\begin{aligned} C^{-1}\cdot\mathcal{I} = C^{-1}&\cdot\mathcal{I}^{(1)} + C^{-1}\cdot\mathcal{I}^{(2)} \\ &+\sum_{i\in\mathbb{Z}}\mathcal{J}^{(1,i)}\cdot\mathcal{K}l_1(x,\mu;i)+\sum_{j\in\mathbb{Z}}\mathcal{J}^{(2,j)}\cdot\mathcal{K}l_2(x,\mu;j) \end{aligned} \tag{5.53}$$

where

$$\begin{aligned} \mathcal{J}^{(1,i)} = (-1)^{\lambda_2-i}\,\mathcal{Z}_+&\left(\frac{\|\lambda\|+n}{2}\right)\mathcal{Z}_+\left(\lambda_1-\lambda_2+m+n+2i\right) \\ &\cdot\mathcal{Z}_+(\lambda_2-i-1)\,\mathcal{Z}_+\left(\lambda_1-\lambda_2+m'+n+i-1\right) \end{aligned} \tag{5.54}$$

and

$$\begin{aligned} \mathcal{J}^{(2,j)} = (-1)^{\lambda_2-j}\,\mathcal{Z}_+&\left(\frac{\lambda_1-\lambda_2+n+2j}{2}\right) \\ &\cdot\mathcal{Z}_+(\lambda_2-j)\,\mathcal{Z}_+\left(\lambda_1-\lambda_2+m'+n+j-1\right). \end{aligned} \tag{5.55}$$

DEFINITION 5.7. (1) For $(c,d) \in \mathbb{Z}^2$, let $F^{(a)}_{(c,d)}$ be the characteristic function of the set

$$\mathcal{D}^{(a)}_1(c,d) = \left\{(\lambda_1,\lambda_2) \in P^+ \mid \lambda_1 \geq c,\ \lambda_1 - \lambda_2 \leq c-d,\ \|\lambda\| \equiv c-d \pmod 2\right\}.$$

(2) For $(c,d) \in \mathbb{Z}^2$, let $G^{(a)}_{(c,d)}$ be the characteristic function of the set

$$\mathcal{D}^{(a)}_2(c,d) = \left\{(\lambda_1,\lambda_2) \in P^+ \mid \lambda_2 \geq d,\ \lambda_1 - \lambda_2 \geq c-d,\ \|\lambda\| \equiv c-d \pmod 2\right\}.$$

(3) For $r \in \mathbb{Z}$, we define $\epsilon(r) \in \{0,1\}$ by

$$\epsilon(r) = \begin{cases} 0, & \text{if } r \text{ is even,} \\ 1, & \text{if } r \text{ is odd.} \end{cases}$$

Let us express $C^{-1} \cdot \mathcal{I}^{(1)}$, $C^{-1} \cdot \mathcal{I}^{(2)}$, $\mathcal{J}^{(1,i)}$ and $\mathcal{J}^{(2,j)}$ more concretely. As for the latter two, it is enough to consider the cases below because of the vanishing condition on $\mathcal{K}l_1(x,\mu;i)$ and $\mathcal{K}l_2(x,\mu;j)$ in Proposition 3.8.

LEMMA 5.8. (1) *We have*

$$C^{-1} \cdot \mathcal{I}^{(1)} = (-1)^{\lambda_1+m'+n}\ F^{(a)}_{(-m'-n,-m+m')}. \tag{5.56}$$

In particular $C^{-1} \cdot \mathcal{I}^{(1)}$ *vanishes if* $n > m-2m'$.

(2) *We have*

$$C^{-1} \cdot \mathcal{I}^{(2)} = (-1)^{\lambda_1+m'+n}\ F^{(a)}_{(-m'-n+1,m'+1)}. \tag{5.57}$$

In particular $C^{-1} \cdot \mathcal{I}^{(2)}$ *vanishes if* $n > -2m'$.

(3) (a) *When* $n \geq m-2m'+2$ *and* n *is even, we have*

$$\mathcal{J}^{\left(1,\frac{-m-n}{2}\right)} = (-1)^{\lambda_2+\frac{m+n}{2}}\ G^{(a)}_{\left(\frac{2-m-n}{2},\frac{2-m-n}{2}\right)}. \tag{5.58}$$

(b) *When* $n \leq m-2m'+1$ *and* $-m+m'-1 \leq i \leq -m'-n+1$, *we have*

$$\mathcal{J}^{(1,i)} = (-1)^{\lambda_2-i}\ G^{(a)}_{(3-m'-n-\epsilon(i-m'),i+1)}. \tag{5.59}$$

(4) (a) *When* $n \geq -2m'+2$ *and* n *is even, we have*

$$\mathcal{J}^{\left(2,\frac{-n}{2}\right)} = (-1)^{\lambda_2+\frac{n}{2}}\ G^{(a)}_{\left(\frac{-n}{2},\frac{-n}{2}\right)}. \tag{5.60}$$

(b) *When* $n \leq -2m'+1$ *and* $m'-1 \leq j \leq -m'-n+1$, *we have*

$$\mathcal{J}^{(2,j)} = (-1)^{\lambda_2-j}\ G^{(a)}_{(2-m'-n-\epsilon(j-m'),j)}. \tag{5.61}$$

PROOF. We note that

$$\|\lambda\| + n = \begin{cases} (-\lambda_1+\lambda_2-2m'-n) + 2\left(\lambda_1+m'+n\right), & \text{if } m' > 0, \\ (\lambda_1+n)+\lambda_2, & \text{if } m' = 0. \end{cases}$$

Hence (5.56) holds.

The equality (5.57) is clear from (5.50).

Suppose that $n \geq m-2m'+2$ and n is even. Then we have

$$\mathcal{J}^{\left(1,\frac{-m-n}{2}\right)} = (-1)^{\lambda_2+\frac{m+n}{2}}\ \mathcal{Z}_+\left(\frac{\|\lambda\|+n}{2}\right)\ \mathcal{Z}_+\left(\lambda_2 - \frac{2-m-n}{2}\right).$$

Here we note that $\lambda_2 - \frac{2-m-n}{2} \geq 0$ implies that $\|\lambda\| + n \geq 0$. If $m' = 0$, we have $n \geq m+2 > 0$ and hence $\|\lambda\| + n > 0$. If $m' > 0$, we have $m = 0$ and hence $\lambda_2 + \frac{n-2}{2} \geq 0$. Thus $\|\lambda\| + n \geq 2\lambda_2 + n \geq 2$. Hence (5.58) holds.

Suppose that $n \le m - 2m' + 1$ and $-m + m' - 1 \le i \le -m' - n + 1$. Then we have

$$\mathcal{J}^{(1,i)} = (-1)^{\lambda_2 - i} \ \mathcal{Z}_+ \left(\frac{\|\lambda\| + n}{2} \right) \ \mathcal{Z}_+ \left(\lambda_1 - \lambda_2 + m' + n + i - 1 \right) \ \mathcal{Z}_+ \left(\lambda_2 - i - 1 \right)$$

since $m + n + 2i \ge m' + n + i - 1$. When $m' = 0$, we note that

$$\|\lambda\| + n = \begin{cases} (\lambda_1 - \lambda_2 + n + i - 1) + 2\,(\lambda_2 - i - 1) + (i + 3)\,, & \text{if } i \ge 0, \\ (\lambda_1 - \lambda_2 + n + i - 1) + 2\lambda_2 + (1 - i), & \text{if } i < 0. \end{cases}$$

When $m' > 0$, we note that $m = 0$ and

$$\|\lambda\| + n = (\lambda_1 - \lambda_2 + m' + n + i - 1) + 2\,(\lambda_2 - i - 1) + (i - m' + 3).$$

Thus (5.59) holds.

Suppose that $n \ge -2m' + 2$ and n is even. Then we have

$$\mathcal{J}^{\left(2, \frac{-n}{2}\right)} = (-1)^{\lambda_2 + \frac{n}{2}} \ \mathcal{Z}_+ \left(\frac{\lambda_1 - \lambda_2}{2} \right) \ \mathcal{Z}_+ \left(\lambda_2 + \frac{n}{2} \right)$$

by (5.55). Hence (5.60) holds.

Suppose that $n \le -2m' + 1$ and $m' - 1 \le j \le -m' - n + 1$. Since we have $n + 2j \ge m' + n + j - 1$, the equality (5.61) follows from (5.55). $\square$

Let us prove the matching in the inert case.

For a function f on P^+, let us define a function $T^{(a)}(f)$ on P^+ by

$$T^{(a)}(f) = T_1^{(a)}(f) + T_2^{(a)}(f)$$

where

$$T_1^{(a)}(f)(\lambda_1, \lambda_2) = \sum_{\substack{\lambda_1' \equiv \lambda_1 \ (\mathrm{mod}\ 2) \\ \lambda_1 \ge \lambda_1' > \lambda_2}} \ \sum_{\substack{\lambda_2' \equiv \lambda_2 \ (\mathrm{mod}\ 2) \\ \lambda_2 \ge \lambda_2' \ge 0}} f\left(\lambda_1', \lambda_2'\right), \tag{5.62}$$

$$T_2^{(a)}(f)(\lambda_1, \lambda_2) = 0 \quad \text{if } \|\lambda\| \text{ is odd} \tag{5.63}$$

and

$$T_2^{(a)}(f)(\lambda_1, \lambda_2) = \sum_{\lambda_1' = 0}^{\lambda_2} \ \sum_{\substack{\lambda_2' \equiv \lambda_1' \ (\mathrm{mod}\ 2) \\ \lambda_1' \ge \lambda_2' \ge 0}} (-1)^{\lambda_2 - \lambda_1'} \ f\left(\lambda_1', \lambda_2'\right) \quad \text{if } \|\lambda\| \text{ is even}. \tag{5.64}$$

Since we have

$$\delta^{-1} \left(\frac{x}{\mu^2} \right) \left| \frac{x}{\mu^2} \right|^{\frac{1}{2}} q^{-\frac{i+j}{2}} \ \left(1 + q^{-1}\right)^{e(\lambda')} \ \delta_B \left(\varpi^{\lambda'} \right)^{-\frac{1}{2}} \ \chi_\delta \left(\varpi^{\lambda'} \right) \ C_a\left(\lambda'\right)$$
$$= \delta_B \left(\varpi^{\lambda} \right)^{-\frac{1}{2}} \ \chi_\delta \left(\varpi^{\lambda} \right) \ C\left(\lambda\right)$$

for $\lambda' = (\lambda_1 - i, \lambda_2 - j)$, our task of proving (2.48) is reduced to show the equality

$$T^{(a)} \left(C_a^{-1} \cdot \mathcal{B}^{(a)} \right) = C^{-1} \cdot \mathcal{I} \tag{5.65}$$

as functions on P^+.

Let $\mathcal{B}' = T^{(a)} \left(C_a^{-1} \cdot \mathcal{B}^{(a)} \right)$ and let $\mathcal{I}' = C^{-1} \cdot \mathcal{I}$. We recall that we proved

$$\mathcal{B}'(0,0) = \mathcal{I}'(0,0) \tag{5.66}$$

as Theorem 1 in [**6**].

Let us prove (5.65). We note the following two lemmas first.

LEMMA 5.9. *For $(c,d) \in \mathbb{Z}^2$, let $H^{(a)}_{(c,d)}$ denote the characteristic function of the set*

$$\mathcal{D}_3^{(a)}(c,d) = \left\{(\lambda_1,\lambda_2) \in P^+ \mid \lambda_1 \geq c > \lambda_2 \geq d,\ \lambda_1 \equiv c \,(\mathrm{mod}\, 2),\ \lambda_2 \equiv d \,(\mathrm{mod}\, 2)\right\}.$$

Then for $(c,d) \in P^+$, we have

$$T^{(a)}\left(P_{(c,d)}\right) = H^{(a)}_{(c,d)} + \begin{cases} (-1)^{\lambda_2 - c}\ G^{(a)}_{(c,c)}, & \text{if } c-d \text{ is even,} \\ 0, & \text{if } c-d \text{ is odd.} \end{cases}$$

PROOF. It is clear that we have $T_1^{(a)}\left(P_{(c,d)}\right) = H^{(a)}_{(c,d)}$ and

$$T_2^{(a)}\left(P_{(c,d)}\right) = \begin{cases} (-1)^{\lambda_2 - c}\ G^{(a)}_{(c,c)}, & \text{if } c-d \text{ is even,} \\ 0, & \text{if } c-d \text{ is odd.} \end{cases}$$

□

LEMMA 5.10. (1) *For $(c,d) \in P^+$, we have*

$$H^{(a)}_{(c,d)} + H^{(a)}_{(c+1,d+1)} = F^{(a)}_{(c,d)} + G^{(a)}_{(c+2,d)} - \begin{cases} G^{(a)}_{(c,c)}, & \text{if } c-d \text{ is even,} \\ G^{(a)}_{(c+2,c+1)}, & \text{if } c-d \text{ is odd.} \end{cases} \tag{5.67}$$

(2) *For $(c,d) \in P^+$ with $d > 0$, we have*

$$H^{(a)}_{(c,d)} + H^{(a)}_{(c+1,d-1)} = F^{(a)}_{(c,d)} + G^{(a)}_{(c+1,d-1)} - \begin{cases} G^{(a)}_{(c,c)}, & \text{if } c-d \text{ is even,} \\ G^{(a)}_{(c+2,c+1)}, & \text{if } c-d \text{ is odd.} \end{cases} \tag{5.68}$$

PROOF. Clear. □

5.2.1. Proof of (5.65) when $m' = 0$.

5.2.1.1. *When $n \geq m+2$.* By Proposition 3.25 and Lemma 5.9, we have

$$\mathcal{B}' = \begin{cases} (-1)^{\lambda_2}\ G^{(a)}_{(0,0)} \cdot \left\{(-1)^{\frac{m-n}{2}} \cdot \mathcal{K}l_1 + (-1)^{\frac{n}{2}} \cdot \mathcal{K}l_2\right\}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, we have

$$\mathcal{I}' = \begin{cases} (-1)^{\lambda_2}\ G^{(a)}_{(0,0)} \left\{(-1)^{\frac{m-n}{2}} \cdot \mathcal{K}l_1 + (-1)^{\frac{n}{2}} \cdot \mathcal{K}l_2\right\}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

by Lemma 5.8. Hence (5.65) holds.

5.2.1.2. *When $m+1 \geq n \geq 2$.* By Proposition 3.25 and Lemma 5.9, we have

$$\mathcal{B}' = \begin{cases} (-1)^{\lambda_2}\ G^{(a)}_{(0,0)} \cdot \left\{1 + q^{-1} + (-1)^{\frac{n}{2}} \cdot \mathcal{K}l_2\right\}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, by Lemma 5.8 we have $\mathcal{I}'=0$ when $n=m+1$ and

$$\mathcal{I}' = (-1)^{\lambda_1+n}\ F^{(a)}_{(0,-m+n)} + \left(1-q^{-1}\right)(-1)^{\lambda_2} \sum_{-m\le i\le -n} (-1)^i\ G^{(a)}_{(3-n-\epsilon(i),i+1)} + (-1)^{\lambda_2}\ G^{(a)}_{(m-n+2,0)}\cdot q^{-1} + (-1)^{\lambda_2+n}\ G^{(a)}_{(1-\epsilon(n+1),0)}\cdot q^{-1} + \begin{cases} (-1)^{\lambda_2+\frac{n}{2}}\ G^{(a)}_{(0,0)}\cdot \mathcal{K}l_2, & \text{if } n \text{ is even,}\\ 0, & \text{if } n \text{ is odd,}\end{cases}$$

when $m\ge n\ge 2$. Here when $-m<i<-2$ and i is odd, we have

$$-G^{(a)}_{(2-n,i+1)} + G^{(a)}_{(3-n,i+2)} = -G^{(a)}_{(1-n-i,0)} + G^{(a)}_{(1-n-i,0)} = 0 \tag{5.69}$$

and hence

$$\sum_{-m\le i\le -n} (-1)^i\ G^{(a)}_{(3-n-\epsilon(i),i+1)} = \begin{cases} G^{(a)}_{(2+m-n,0)}, & \text{if } n \text{ is even,}\\ G^{(a)}_{(2+m-n,0)} - G^{(a)}_{(1,0)}, & \text{if } n \text{ is odd.}\end{cases}$$

Thus when n is odd, we have

$$\mathcal{I}' = -(-1)^{\lambda_1}\ F^{(a)}_{(0,-m+n)} + (-1)^{\lambda_2}\ G^{(a)}_{(m-n+2,0)} - (-1)^{\lambda_2}\ G^{(a)}_{(1,0)} = 0$$

since

$$-(-1)^{\lambda_1-\lambda_2}\ F^{(a)}_{(0,-m+n)} + G^{(a)}_{(m-n+2,0)} = F^{(a)}_{(0,-m+n)} + G^{(a)}_{(m-n+2,0)} = G^{(a)}_{(1,0)}.$$

When n is even, we have

$$\mathcal{I}' = (-1)^{\lambda_2}\left(F^{(a)}_{(0,-m+n)} + G^{(a)}_{(m-n+2,0)} + G^{(a)}_{(0,0)}\cdot q^{-1} + (-1)^{\frac{n}{2}}\ G^{(a)}_{(0,0)}\cdot\mathcal{K}l_2\right)$$

where $F^{(a)}_{(0,-m+n)} + G^{(a)}_{(m-n+2,0)} = G^{(a)}_{(0,0)}$. Thus (5.65) holds.

5.2.1.3. *When $n\le 1$.* By Proposition 3.25 and Lemma 5.9, we have

$$\mathcal{B}' = (-1)^{\lambda_2}\left\{\left(H^{(a)}_{(-n,0)} + H^{(a)}_{(1-n,1)}\right) + q^{-1}\left(H^{(a)}_{(1-n,1)} + H^{(a)}_{(2-n,0)}\right)\right\} + \begin{cases} (-1)^{\lambda_2}\left\{G^{(a)}_{(-n,-n)} + q^{-1}\, G^{(a)}_{(2-n,2-n)} + \left(1+q^{-1}\right)\ G^{(a)}_{(1-n,1-n)}\right\}, & \text{if } n \text{ is even,}\\ 0, & \text{if } n \text{ is odd.}\end{cases} \tag{5.70}$$

On the other hand, we have

$$\begin{aligned}\mathcal{I}' = {}&(-1)^{\lambda_1+n}\left(F^{(a)}_{(-n,-m)} + F^{(a)}_{(-n+1,1)}\right) + (-1)^{\lambda_2}\ G^{(a)}_{(2+m-n,0)}\cdot q^{-1}\\ &+ (-1)^{\lambda_2+n}\ G^{(a)}_{(3-n-\epsilon(n+1),2-n)}\cdot q^{-1} + \left(1-q^{-1}\right)\sum_{-m\le i\le -n}(-1)^{\lambda_2+i}\ G^{(a)}_{(3-n-\epsilon(i),i+1)}\\ &+ (-1)^{\lambda_2}\ G^{(a)}_{(2-n,0)}\cdot q^{-1} + (-1)^{\lambda_2+n}\ G^{(a)}_{(2-n-\epsilon(n+1),-n+1)}\cdot q^{-1}\\ &+ \left(1-q^{-1}\right)\sum_{0\le j\le -n}(-1)^{\lambda_2+j}\ G^{(a)}_{(2-n-\epsilon(j),j)}.\end{aligned} \tag{5.71}$$

When $n=1$, the equality (5.70) becomes

$$\mathcal{B}' = q^{-1}\, H^{(a)}_{(1,0)}.$$

As for $\mathcal{I}'$, when $m=0$, we have

$$\mathcal{I}' = (-1)^{\lambda_2}\left(G^{(a)}_{(1,0)} - G^{(a)}_{(2,1)}\right)\cdot q^{-1} = q^{-1}\, H^{(a)}_{(1,0)}.$$

When $m > 0$, we have

$$\mathcal{I}' = (-1)^{\lambda_2}\ F^{(a)}_{(0,1-m)} + (-1)^{\lambda_2} \left(G^{(a)}_{(m+1,0)} - G^{(a)}_{(2,1)}\right) \cdot q^{-1} \\ + \left(1 - q^{-1}\right)(-1)^{\lambda_2} \sum_{-m \le i \le -1} (-1)^i\ G^{(a)}_{(2-\epsilon(i),i+1)}.$$

Here by (5.69), we have

$$\sum_{-m \le i \le -1} (-1)^i\ G^{(a)}_{(2-\epsilon(i),i+1)} = G^{(a)}_{(m+1,0)} - G^{(a)}_{(1,0)}.$$

Thus

$$\begin{aligned} \mathcal{I}' &= (-1)^{\lambda_2} \left\{F^{(a)}_{(0,1-m)} + G^{(a)}_{(m+1,0)} - q^{-1}\, G^{(a)}_{(2,1)} - \left(1 - q^{-1}\right) G^{(a)}_{(1,0)}\right\} \\ &= (-1)^{\lambda_2} \left(G^{(a)}_{(1,0)} - G^{(a)}_{(2,1)}\right) \cdot q^{-1} = q^{-1}\, H^{(a)}_{(1,0)} \end{aligned}$$

since $F^{(a)}_{(0,1-m)} + G^{(a)}_{(m+1,0)} = G^{(a)}_{(1,0)}$. Hence (5.65) holds when $n = 1$.

Suppose that $n \le 0$. By (5.69), when $m > 0$, we have

$$\sum_{-m \le i \le -1} (-1)^i\ G^{(a)}_{(3-n-\epsilon(i),i+1)} = G^{(a)}_{(2+m-n,0)} - G^{(a)}_{(2-n,0)}. \tag{5.72}$$

We remark that (5.72) is valid also when $m = 0$. We also note that when i is odd, we have

$$G^{(a)}_{(3-n-\epsilon(i),i+1)} = G^{(a)}_{(2-n,i+1)} = G^{(a)}_{(2-n-\epsilon(i+1),i+1)}.$$

Hence we have

$$\begin{aligned} &\sum_{-m \le i \le -n} (-1)^i\ G^{(a)}_{(3-n-\epsilon(i),i+1)} + \sum_{0 \le j \le -n} (-1)^j\ G^{(a)}_{(2-n-\epsilon(j),j)} \\ =&G^{(a)}_{(2+m-n,0)} + (-1)^n\ G^{(a)}_{(3-n-\epsilon(n),1-n)} - \sum_{1 \le k \le \left[\frac{1-n}{2}\right]} \left(G^{(a)}_{(1-n,2k-1)} - G^{(a)}_{(3-n,2k-1)}\right). \end{aligned}$$

Here we have

$$\sum_{1 \le k \le \left[\frac{1-n}{2}\right]} \left(G^{(a)}_{(1-n,2k-1)} - G^{(a)}_{(3-n,2k-1)}\right) = \begin{cases} F^{(a)}_{(1-n,1)} - F^{(a)}_{(1-n,1-n)}, & \text{if } n \text{ is even}, \\ F^{(a)}_{(1-n,1)}, & \text{if } n \text{ is odd} \end{cases}$$

where $F^{(a)}_{(1-n,1-n)} = G^{(a)}_{(1-n,1-n)} - G^{(a)}_{(3-n,1-n)}$. Thus

$$\mathcal{I}' = (-1)^{\lambda_2} \left(F^{(a)}_{(-n,-m)} + G^{(a)}_{(2+m-n,0)} + q^{-1}\, F^{(a)}_{(1-n,1)} + q^{-1}\, G_{(2-n,0)}\right) \\ + (-1)^{\lambda_2+n} \cdot \begin{cases} (G^{(a)}_{(1-n,1-n)} + q^{-1}\, G^{(a)}_{(2-n,2-n)}), & \text{if } n \text{ is even}, \\ (G^{(a)}_{(2-n,1-n)} + q^{-1}\, G^{(a)}_{(3-n,2-n)}), & \text{if } n \text{ is odd}. \end{cases}$$

Since $F^{(a)}_{(-n,-m)} + G^{(a)}_{(2+m-n,0)} = F^{(a)}_{(-n,0)} + G^{(a)}_{(2-n,0)}$, we have

$$\begin{aligned}\mathcal{I}' = (-1)^{\lambda_2} &\left\{ F^{(a)}_{(-n,0)} + q^{-1} F^{(a)}_{(1-n,1)} + \left(1+q^{-1}\right) G_{(2-n,0)} \right\} \\ &+ (-1)^{\lambda_2+n} \cdot \begin{cases} \left(G^{(a)}_{(1-n,1-n)} + q^{-1} G^{(a)}_{(2-n,2-n)} \right), & \text{if } n \text{ is even,} \\ \left(G^{(a)}_{(2-n,1-n)} + q^{-1} G^{(a)}_{(3-n,2-n)} \right), & \text{if } n \text{ is odd.} \end{cases}\end{aligned}$$

Hence by (5.67) and (5.68), we have (5.65).

5.2.2. Proof of (5.65) when $m' > 0$.

5.2.2.1. *When $n + 2m' = 2$.* When n is odd, we have $\mathcal{B}' = \mathcal{I}' = 0$ by Proposition 3.25 and Lemma 5.8. Hence (5.65) holds.

When n is even, we have

$$\mathcal{B}' = \begin{cases} (-1)^{\lambda_2+\frac{n}{2}} \; G^{(a)}_{(0,0)} \cdot (\mathcal{K}l_1 + \mathcal{K}l_2)\,, & \text{if } n \geq 2, \\ (-1)^{\lambda_2+\frac{n}{2}} \left(G^{(a)}_{\left(\frac{-n}{2},\frac{-n}{2}\right)} + G^{(a)}_{\left(\frac{2-n}{2},\frac{2-n}{2}\right)} \right) \cdot \mathcal{K}l_2, & \text{if } n \leq 0 \end{cases}$$

by Proposition 3.25 and Lemma 5.9. Then noting (3.6) when $n \leq 0$, the equality (5.65) holds by Lemma 5.8.

5.2.2.2. *When $n + 2m' = 1$.* We have

$$\mathcal{B}' = \left(H^{(a)}_{(m'+1,m')} - H^{(a)}_{(m',m'-1)} \right) \cdot q^{-1}$$

by Proposition 3.25 and Lemma 5.9. On the other hand, we have

$$\mathcal{I}' = (-1)^{\lambda_2+m'} \left\{ \left(G^{(a)}_{(m'+1,m')} - G^{(a)}_{(m'+2,m'+1)} \right) + \left(G^{(a)}_{(m',m'-1)} - G^{(a)}_{(m'+1,m')} \right) \right\}$$

by Lemma 5.8. Since $H^{(a)}_{(d+1,d)} = G^{(a)}_{(d+1,d)} - G^{(a)}_{(d+2,d+1)}$ for $d \geq 0$, we have $\mathcal{B}' = \mathcal{I}'$.

5.2.2.3. *When $n + 2m' \leq 0$.* We put $r = -m' - n$ and $s = m'$. By Proposition 3.25 and Lemma 5.9, we have

$$\begin{aligned}\text{(5.73)}\quad \mathcal{B}' &= (-1)^{\lambda_2+s} \left\{ \left(H^{(a)}_{(r,s)} + H^{(a)}_{(r+1,s+1)} \right) + q^{-1} \left(H^{(a)}_{(r+1,s-1)} + H^{(a)}_{(r+2,s)} \right) \right\} \\ &+ \begin{cases} (-1)^{\lambda_2+r} \left\{ G^{(a)}_{(r,r)} + (1+q^{-1}) G^{(a)}_{(r+1,r+1)} + q^{-1} G^{(a)}_{(r+2,r+2)} \right\}, & \text{if } r-s \text{ is even,} \\ 0, & \text{if } r-s \text{ is odd.} \end{cases}\end{aligned}$$

On the other hand, we have

(5.74)

$$\begin{aligned}\mathcal{I}' = (-1)^{\lambda_1+r} &\left(F^{(a)}_{(r,s)} + F^{(a)}_{(r+1,s+1)} \right) + (-1)^{\lambda_2+s} q^{-1} \left(G^{(a)}_{(r+2,s)} + G^{(a)}_{(r+1,s-1)} \right) \\ &+ (-1)^{\lambda_2+r} q^{-1} \left(G^{(a)}_{(r+3-\epsilon(r+s+1),r+2)} + G^{(a)}_{(r+2-\epsilon(r+s+1),r+1)} \right) \\ &+ \left(1-q^{-1}\right) (-1)^{\lambda_2+s} \sum_{0\leq j\leq r-s} (-1)^j \left(G^{(a)}_{(r+3-\epsilon(j),s+j+1)} + G^{(a)}_{(r+2-\epsilon(j),s+j)} \right)\end{aligned}$$

by Lemma 5.8. When j is odd, we have

$$G^{(a)}_{(r+3-\epsilon(j),s+j+1)} = G^{(a)}_{(r+2,s+j+1)} = G^{(a)}_{(r+2-\epsilon(j+1),s+j+1)}.$$

Hence

$$\begin{aligned}\sum_{0\le j\le r-s} &(-1)^j \left(G^{(a)}_{(r+3-\epsilon(j),s+j+1)} + G^{(a)}_{(r+2-\epsilon(j),s+j)}\right)\\ &= (-1)^{r-s}\, G^{(a)}_{(r+3-\epsilon(r-s),r+1)} + G^{(a)}_{(r+2,s)}\\ &\qquad - \sum_{1\le k\le\left[\frac{r-s+1}{2}\right]} \left(G^{(a)}_{(r+1,s+2k-1)} - G^{(a)}_{(r+3,s+2k-1)}\right).\end{aligned}$$

Further we have

$$\begin{aligned}\sum_{1\le k\le\left[\frac{r-s+1}{2}\right]} &\left(G^{(a)}_{(r+1,s+2k-1)} - G^{(a)}_{(r+3,s+2k-1)}\right)\\ &= \begin{cases} F^{(a)}_{(r+1,s+1)} - F^{(a)}_{(r+1,r+1)}, & \text{if } r-s \text{ is even},\\ F^{(a)}_{(r+1,s+1)}, & \text{if } r-s \text{ is odd}.\end{cases}\end{aligned}$$

Noting that $G^{(a)}_{(r+3,r+1)} + F^{(a)}_{(r+1,r+1)} = G^{(a)}_{(r+1,r+1)}$ when $r-s$ is even, we have

$$\begin{aligned}\mathcal{I}' = (-1)^{\lambda_2+s} &\left(F^{(a)}_{(r,s)} + q^{-1}\, F^{(a)}_{(r+1,s+1)} + G^{(a)}_{(r+2,s)} + q^{-1}\, G^{(a)}_{(r+1,s-1)}\right)\\ &+ (-1)^{\lambda_2+r}\cdot \begin{cases}\left(G^{(a)}_{(r+1,r+1)} + q^{-1}\, G^{(a)}_{(r+2,r+2)}\right), & \text{if } r-s \text{ is even},\\ \left(G^{(a)}_{(r+2,r+1)} + q^{-1}\, G^{(a)}_{(r+3,r+2)}\right), & \text{if } r-s \text{ is odd}.\end{cases}\end{aligned}$$

By (5.67) and noting

$$F^{(a)}_{(r+1,s+1)} + G^{(a)}_{(r+1,s-1)} = F^{(a)}_{(r+1,s-1)} + G^{(a)}_{(r+3,s-1)},$$

we have $\mathcal{B}' = \mathcal{I}'$.

Thus we establish (5.65) in all cases.

5.3. Matching in the Split Case

We shall prove Theorem 2.18 in this section. As in the inert case, it is enough for us to prove (2.49) for $x \in \mathcal{O} \setminus \{1, 0\}$ since the functional equations (4.1) and (5.45) are compatible with (2.49).

In order to express $\mathcal{I}(\lambda)$ in the split case explicitly, let us introduce the following functions.

Definition 5.11. (1) For $(c,d) \in \mathbb{Z}^2$, let $F^{(s)}_{(c,d)}$ be the characteristic function of the set

$$\mathcal{D}^{(s)}_1(c,d) = \left\{(\lambda_1,\lambda_2) \in P^+ \mid \lambda_1 \ge c,\ \lambda_1 - \lambda_2 \le c - d\right\}.$$

(2) For $(c,d) \in \mathbb{Z}^2$, let $G^{(s)}_{(c,d)}$ be the characteristic function of the set

$$\mathcal{D}^{(s)}_2(c,d) = \left\{(\lambda_1,\lambda_2) \in P^+ \mid \lambda_2 \ge d,\ \lambda_1 - \lambda_2 \ge c - d\right\}.$$

(3) For $(c,d) \in \mathbb{Z}^2$, let $C_{(c,d)}$ be the function on P^+ defined by

$$C_{(c,d)}\,(\lambda_1,\lambda_2) = (\lambda_1+\lambda_2+c)\,(\lambda_1-\lambda_2+d)\,.$$

Then by the Whittaker value formula (1.1), the function $C^{-1}\cdot\mathcal{I}$, where C is defined by (5.48), is expressed as

$$C^{-1}\cdot\mathcal{I}=C^{-1}\cdot\mathcal{I}^{(1)}+C^{-1}\cdot\mathcal{I}^{(2)}+\sum_{i\in\mathbb{Z}}\mathcal{J}^{(1,i)}\cdot\mathcal{K}l_1(x,\mu;i)+\sum_{j\in\mathbb{Z}}\mathcal{J}^{(2,j)}\cdot\mathcal{K}l_2(x,\mu;j),$$

where $\mathcal{I}^{(1)}$, $\mathcal{I}^{(2)}$ and $\mathcal{J}^{(l,i)}$ $(l=1,2)$ with i such that $\mathcal{K}l_l(x,\mu;i)\neq 0$ are given as follows.

LEMMA 5.12. (1) *We have*

$$C^{-1}\cdot\mathcal{I}^{(1)}=-C_{(n+1,-m+2m'+n-1)}\,F^{(s)}_{(-m'-n,-m+m')}. \tag{5.75}$$

In particular $C^{-1}\cdot\mathcal{I}^{(1)}$ vanishes if $n>m-2m'$.

(2) *We have*

$$C^{-1}\cdot\mathcal{I}^{(2)}=-C_{(m+n+1,2m'+n-1)}\,F^{(s)}_{(-m'-n+1,m'+1)}. \tag{5.76}$$

In particular $C^{-1}\cdot\mathcal{I}^{(2)}$ vanishes if $n>-2m'$.

(3) (a) *When $n\geq m-2m'+2$ and n is even, we have*

$$\mathcal{J}^{\left(1,\frac{-m-n}{2}\right)}=C_{(n+1,1)}\,G^{(s)}_{\left(\frac{2-m-n}{2},\frac{2-m-n}{2}\right)}. \tag{5.77}$$

(b) *When $n\leq m-2m'+1$ and $-m+m'-1\leq i\leq -m'-n+1$, we have*

$$\mathcal{J}^{(1,i)}=C_{(n+1,m+n+2i+1)}\,G^{(s)}_{(2-m'-n,i+1)}. \tag{5.78}$$

(4) (a) *When $n\geq -2m'+2$ and n is even, we have*

$$\mathcal{J}^{\left(2,\frac{-n}{2}\right)}=C_{(m+n+1,1)}\,G^{(s)}_{\left(\frac{-n}{2},\frac{-n}{2}\right)}. \tag{5.79}$$

(b) *When $n\leq -2m'+1$ and $m'-1\leq j\leq -m'-n+1$, we have*

$$\mathcal{J}^{(2,j)}=C_{(m+n+1,n+2j+1)}\,G^{(s)}_{(1-m'-n,j)}. \tag{5.80}$$

PROOF. The proof is similar to the one for Lemma 5.8. □

For a function f on P^+, let us define a function $T^{(s)}(f)$ on P^+ by

$$T^{(s)}(f)=T_1^{(s)}(f)+T_2^{(s)}(f)$$

where

$$T_1^{(s)}(f)\,(\lambda_1,\lambda_2)=\sum_{\lambda_1\geq\lambda_1'>\lambda_2\geq\lambda_2'\geq 0}(\lambda_1-\lambda_1'+1)\,(\lambda_2-\lambda_2'+1)\;f\,(\lambda_1',\lambda_2') \tag{5.81}$$

and

$$T_2^{(s)}(f)\,(\lambda_1,\lambda_2)=(\lambda_1-\lambda_2+1)\sum_{\lambda_2\geq\lambda_1'\geq\lambda_2'\geq 0}(\lambda_1'-\lambda_2'+1)\;f\,(\lambda_1',\lambda_2')\,. \tag{5.82}$$

Since we have

$$\delta^{-1}\left(\frac{x}{\mu^2}\right)\left|\frac{x}{\mu^2}\right|^{\frac{1}{2}}q^{-\frac{i+j}{2}}\,\left(1-q^{-1}\right)^{e(\lambda')}\,\delta_B\left(\varpi^{\lambda'}\right)^{-\frac{1}{2}}\,\chi_\delta\left(\varpi^{\lambda'}\right)\,C_s\,(\lambda')$$
$$=\delta_B\left(\varpi^{\lambda}\right)^{-\frac{1}{2}}\,\chi_\delta\left(\varpi^{\lambda}\right)\,C\,(\lambda)$$

for $\lambda'=(\lambda_1-i,\lambda_2-j)$, our task of proving (2.49) is reduced to show the equality

$$T^{(s)}\left(C_s^{-1}\cdot\mathcal{B}^{(s)}\right)=C^{-1}\cdot\mathcal{I} \tag{5.83}$$

as functions on P^+.

Let $\mathcal{B}' = T^{(s)}\left(C_s^{-1}\cdot\mathcal{B}^{(s)}\right)$ and let $\mathcal{I}' = C^{-1}\cdot\mathcal{I}$. We recall that we proved

$$\mathcal{B}'(0,0) = \mathcal{I}'(0,0) \tag{5.84}$$

as Theorem 2 in [**6**].

Let us prove (5.83). As in the inert case, first we note the following lemma.

Lemma 5.13. *For $(c,d)\in\mathbb{Z}^2$, let $H^{(s)}_{(c,d)}$ be the characteristic function of the set*

$$\mathcal{D}_3^{(s)}(c,d) = \left\{(\lambda_1,\lambda_2)\in P^+ \mid \lambda_1 \geq c > \lambda_2 \geq d\right\}.$$

(1) *For $(c,d)\in P^+$, we have*

$$\begin{aligned} T^{(s)}\left(P_{(c,d)}\right) = (\lambda_1 - c+1)\,(\lambda_2-d+1)\;H^{(s)}_{(c,d)} \\ + (c-d+1)\,(\lambda_1-\lambda_2+1)\;G^{(s)}_{(c,c)}. \end{aligned} \tag{5.85}$$

(2) *For $(c,d)\in P^+$, we have*

$$\begin{aligned} T^{(s)}\left(L_{(c,d)}\right) = \frac{(\lambda_1-c+1)(\lambda_1-c+2)(\lambda_2-d+1)}{2}\cdot H^{(s)}_{(c,d)} \\ + \frac{\lambda_1-\lambda_2+1}{2}\left\{(\lambda_1-c+1)(\lambda_2-d+1)+(c-d+1)(\lambda_2-c+1)\right\}\cdot G^{(s)}_{(c,c)}. \end{aligned} \tag{5.86}$$

In particular, when $c=d$, we have

$$\begin{aligned} T^{(s)}\left(L_{(c,c)}\right) &= \frac{(\lambda_1-\lambda_2+1)(\lambda_1-c+2)(\lambda_2-c+1)}{2}\cdot G^{(s)}_{(c,c)} \\ &= \frac{(\lambda_1-\lambda_2+1)(\lambda_1-c+2)(\lambda_2-c+1)}{2}\cdot G^{(s)}_{(c-1,c-1)}. \end{aligned} \tag{5.87}$$

(3) *For $(c,d)\in P^+$, we have*

$$\begin{aligned} T^{(s)}\left(V_{(c,d)}\right) = \frac{(\lambda_1-c+1)(\lambda_2-d+1)(\lambda_2-d+2)}{2}\cdot H^{(s)}_{(c,d)} \\ + \frac{(\lambda_1-\lambda_2+1)(c-d+1)(c-d+2)}{2}\cdot G^{(s)}_{(c,c)}. \end{aligned} \tag{5.88}$$

Remark 5.14. Since

$$\begin{aligned} H^{(s)}_{(c,d)} &= H^{(s)}_{(c+1,d)} + V_{(c,d)} - L_{(c,c)} \\ &= H^{(s)}_{(c+2,d)} + V_{(c+1,d)} + V_{(c,d)} - L_{(c+1,c+1)} - L_{(c,c)}, \end{aligned}$$

we have

$$\begin{aligned} T^{(s)}\left(P_{(c,d)}\right) = (\lambda_1-c+1)\,(\lambda_2-d+1)\;H^{(s)}_{(c+1,d)} \\ + (c-d+1)\,(\lambda_1-\lambda_2+1)\;G^{(s)}_{(c+1,c+1)} + (\lambda_2-d+1)\,V_{(c,d)} \end{aligned} \tag{5.89}$$

and

$$\begin{aligned} T^{(s)}\left(P_{(c,d)}\right) = (\lambda_1-c+1)\,(\lambda_2-d+1)\;H^{(s)}_{(c+2,d)} \\ + (c-d+1)\,(\lambda_1-\lambda_2+1)\;G^{(s)}_{(c+2,c+2)} \\ - (\lambda_1-d+2)\,L_{(c+1,c+1)} + (\lambda_2-d+1)\left\{V_{(c,d)} + 2\,V_{(c+1,d)}\right\} \end{aligned} \tag{5.90}$$

by (5.85). Similarly we have

$$(5.91)\quad T^{(s)}\left(L_{(c+1,d)}\right) = \frac{(\lambda_1-c)(\lambda_1-c+1)(\lambda_2-d+1)}{2}\cdot H^{(s)}_{(c,d)} + \frac{\lambda_1-\lambda_2+1}{2}\left\{(\lambda_1-c)(\lambda_2-d+1)+(c-d+2)(\lambda_2-c)\right\}\cdot G^{(s)}_{(c,c)}$$

by (5.86),

$$(5.92)\quad T^{(s)}\left(V_{(c,d)}\right) = \frac{(\lambda_1-c+1)(\lambda_2-d+1)(\lambda_2-d+2)}{2}\cdot H^{(s)}_{(c+1,d)} + \frac{(\lambda_1-\lambda_2+1)(c-d+1)(c-d+2)}{2}\cdot G^{(s)}_{(c+1,c+1)} + \frac{(\lambda_2-d+1)(\lambda_2-d+2)}{2}\cdot V_{(c,d)}$$

and

$$(5.93)\quad T^{(s)}\left(V_{(c,d)}\right) = \frac{(\lambda_1-c+1)(\lambda_2-d+1)(\lambda_2-d+2)}{2}\cdot H^{(s)}_{(c+2,d)} + \frac{(\lambda_1-\lambda_2+1)(c-d+1)(c-d+2)}{2}\cdot G^{(s)}_{(c+2,c+2)} + \frac{(\lambda_2-d+1)(\lambda_2-d+2)}{2}\cdot\left(V_{(c,d)}+2\,V_{(c+1,d)}\right) - \frac{(c-d+2)(2\lambda_1-c-d+3)}{2}\cdot L_{(c+1,c+1)}$$

by (5.88).

Proof of Lemma 5.13. It is clear that we have

$$T_1^{(s)}\left(P_{(c,d)}\right) = (\lambda_1-c+1)\,(\lambda_2-d+1)\ H^{(s)}_{(c,d)}$$

and

$$T_2^{(s)}\left(P_{(c,d)}\right) = (c-d+1)\,(\lambda_1-\lambda_2+1)\ G^{(s)}_{(c,c)}.$$

Thus (5.85) holds.

When $\lambda_1 \geq c > \lambda_2 \geq d$, we have

$$T_1^{(s)}\left(L_{(c,d)}\right)(\lambda_1,\lambda_2) = \sum_{\substack{\lambda_1\geq\lambda_1'\geq c\\ \lambda_2'=d}} (\lambda_1-\lambda_1'+1)\,(\lambda_2-\lambda_2'+1) = \frac{(\lambda_1-c+1)(\lambda_1-c+2)(\lambda_2-d+1)}{2}$$

and $T_2^{(s)}\left(L_{(c,d)}\right)(\lambda_1,\lambda_2)=0$. When $\lambda_2 \geq c$, we have

$$T_1^{(s)}\left(L_{(c,d)}\right)(\lambda_1,\lambda_2) = \sum_{\substack{\lambda_1\geq\lambda_1'\geq \lambda_2+1\\ \lambda_2'=d}} (\lambda_1-\lambda_1'+1)\,(\lambda_2-\lambda_2'+1)$$

$$= \frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_2+1)(\lambda_2-d+1)}{2}$$

and

$$\begin{aligned}T_2^{(s)}\left(L_{(c,d)}\right)(\lambda_1,\lambda_2) &= (\lambda_1-\lambda_2+1)\sum_{\substack{\lambda_2\geq\lambda_1'\geq c\\ \lambda_2'=d}}(\lambda_1'-\lambda_2'+1)\\ &=\frac{(\lambda_1-\lambda_2+1)(\lambda_2-c+1)}{2}\left\{(\lambda_2-d+1)+(c-d+1)\right\}.\end{aligned}$$

Thus (5.86) holds. The second equality of (5.87) holds since

$$(\lambda_2-c+1)\ G^{(s)}_{(c,c)} = (\lambda_2-c+1)\ G^{(s)}_{(c-1,c-1)}.$$

When $\lambda_1\geq c>\lambda_2\geq d$, we have

$$\begin{aligned}T_1^{(s)}\left(V_{(c,d)}\right)(\lambda_1,\lambda_2) &= \sum_{\substack{\lambda_1'=c\\ \lambda_2\geq\lambda_2'\geq d}}(\lambda_1-c+1)(\lambda_2-\lambda_2'+1)\\ &=\frac{(\lambda_1-c+1)(\lambda_2-d+1)(\lambda_2-d+2)}{2}\end{aligned}$$

and $T_2^{(s)}\left(V_{(c,d)}\right)(\lambda_1,\lambda_2)=0$. When $\lambda_2\geq c$, we have $T_1^{(s)}\left(V_{(c,d)}\right)(\lambda_1,\lambda_2)=0$ and

$$\begin{aligned}T_2^{(s)}\left(V_{(c,d)}\right)(\lambda_1,\lambda_2) &=(\lambda_1-\lambda_2+1)\sum_{\substack{\lambda_1'=c\\ c\geq\lambda_2'\geq d}}(\lambda_1'-\lambda_2'+1)\\ &=\frac{(\lambda_1-\lambda_2+1)(c-d+1)(c-d+2)}{2}.\end{aligned}$$

Thus (5.88) holds. □

5.3.1. Proof of (5.83) when $2m'+n\geq 2$.

5.3.1.1. *When $m'=0$ and $n\geq m+2$.* By Proposition 4.21, we have

$$\begin{aligned}\mathcal{B}' = -2(\mathcal{K}l_1+\mathcal{K}l_2)\cdot T^{(s)}\left(L_{(1,1)}\right) &+ 2(\mathcal{K}l_1+\mathcal{K}l_2)\cdot T^{(s)}\left(L_{(0,0)}\right)\\ &+\left\{(n-1)\mathcal{K}l_1+(m+n-1)\mathcal{K}l_2\right\}\cdot T^{(s)}\left(P_{(0,0)}\right).\end{aligned}$$

By Lemma 5.13, we have

$$\begin{cases}T^{(s)}\left(L_{(1,1)}\right)=\frac{1}{2}(\lambda_1-\lambda_2+1)(\lambda_1+1)\lambda_2\cdot G^{(s)}_{(0,0)},\\ T^{(s)}\left(L_{(0,0)}\right)=\frac{1}{2}(\lambda_1-\lambda_2+1)(\lambda_1+2)(\lambda_2+1)\cdot G^{(s)}_{(0,0)},\\ T^{(s)}\left(P_{(0,0)}\right)=(\lambda_1-\lambda_2+1)\ G^{(s)}_{(0,0)}.\end{cases}\tag{5.94}$$

Hence

$$\mathcal{B}' = \left(C_{(n+1,1)}\,\mathcal{K}l_1 + C_{(m+n+1,1)}\,\mathcal{K}l_2\right)\cdot G^{(s)}_{(0,0)}.$$

Thus (5.83) holds by Lemma 5.12.

5.3.1.2. *When $m'=0$ and $n=m+1\geq 2$.* By Proposition 4.21 and (5.94), we have

$$\mathcal{B}' = \left(\mathcal{K}l_2\,C_{(2n,1)} - 2q^{-1}\,C_{(n+1,1)}\right)\cdot G^{(s)}_{(0,0)}.$$

On the other hand, by Lemma 5.12, we have

$$\mathcal{I}' = \mathcal{K}l_2\,C_{(2n,1)}\cdot G^{(s)}_{(0,0)} - q^{-1}\left(C_{(n+1,0)}\,G^{(s)}_{(1,0)} + C_{(n+1,2)}\,G^{(s)}_{(0,0)}\right).$$

Here we have

$$C_{(n+1,0)}\,G^{(s)}_{(1,0)} + C_{(n+1,2)}\,G^{(s)}_{(0,0)} = \left(C_{(n+1,0)}+C_{(n+1,2)}\right)G^{(s)}_{(0,0)} = 2\,C_{(n+1,1)}\,G^{(s)}_{(0,0)}.$$

Thus (5.83) holds.

5.3.1.3. *When $m' = 0$ and $m \geq n \geq 2$.* By Proposition 4.21 and (5.94), we have

$$\mathcal{B}' = \left[\mathcal{K}l_2\, C_{(m+n+1,1)} - \left\{-(m-n+1) + (m-n+3)q^{-1}\right\} C_{(n+1,1)}\right] \cdot G^{(s)}_{(0,0)}.$$

On the other hand, by Lemma 5.12, we have

$$\begin{aligned}\mathcal{I}' = \mathcal{K}l_2\, C_{(m+n+1,1)} \cdot G^{(s)}_{(0,0)} - C_{(n+1,-m+n-1)}\, F^{(s)}_{(0,-m+n)} \\ + \left(1-q^{-1}\right) \sum_{0\leq i\leq m-n+2} C_{(n+1,m-n-2i+3)}\, G^{(s)}_{(i,0)} \\ - \left(C_{(n+1,-m+n-1)}\, G^{(s)}_{(m-n+2,0)} + C_{(n+1,m-n+3)}\, G^{(s)}_{(0,0)}\right).\end{aligned}$$

Here we have

$$\begin{aligned}&C_{(n+1,-m+n-1)}\, F^{(s)}_{(0,-m+n)} + C_{(n+1,-m+n-1)}\, G^{(s)}_{(m-n+2,0)} \\ =&C_{(n+1,-m+n-1)} \left\{G^{(s)}_{(0,0)} - \left(G^{(s)}_{(m-n+1,0)} - G^{(s)}_{(m-n+2,0)}\right)\right\} = C_{(n+1,-m+n-1)}\, G^{(s)}_{(0,0)}.\end{aligned}$$

Thus (5.83) holds if we show that

$$\sum_{0\leq i\leq m-n+2} C_{(n+1,m-n-2i+3)}\, G^{(s)}_{(i,0)} = (m-n+3)\, C_{(n+1,1)}\, G^{(s)}_{(0,0)}. \tag{5.95}$$

For $(\lambda_1, \lambda_2) \in P^+$, put $a = \lambda_1 - \lambda_2$. If $a \geq m-n+2$, we have

$$(\|\lambda\| + n + 1) \sum_{0\leq i\leq m-n+2} (a + m - n - 2i + 3) = (\|\lambda\| + n + 1)(m - n + 3)(a + 1).$$

If $0 \leq a < m - n + 2$, we have

$$(\|\lambda\| + n + 1) \sum_{0\leq i\leq a} (a + m - n - 2i + 3) = (\|\lambda\| + n + 1)(a + 1)(m - n + 3).$$

Thus (5.95) holds.

5.3.1.4. *When $m' > 0$ and $2m' + n \geq 2$.* When n is odd, we have $\mathcal{B}' = \mathcal{I}' = 0$ by Proposition 4.22 and Lemma 5.12. Suppose that $n \geq 2$ and n is even. By Proposition 4.22 and (5.94), we have

$$\mathcal{B}' = (\mathcal{K}l_1 + \mathcal{K}l_2)\; C_{(n+1,1)}\, G^{(s)}_{(0,0)}$$

and (5.83) holds. Suppose that $n \geq 0$. Then by Lemma 5.12 and (3.6), we have

$$\mathcal{I}' = \mathcal{K}l_1\, C_{(n+1,1)} \left(G^{(s)}_{\left(\frac{2-n}{2},\frac{2-n}{2}\right)} + G^{(s)}_{\left(\frac{-n}{2},\frac{-n}{2}\right)}\right).$$

On the other hand, by Proposition 4.22 and Lemma 5.13, we have

$$\begin{aligned}\mathcal{B}' = \mathcal{K}l_1\, (\lambda_1 - \lambda_2 + 1) \left\{\left(\lambda_1 + \frac{n}{2} + 2\right)\left(\lambda_2 + \frac{n}{2} + 1\right) - 1\right\} G^{(s)}_{\left(\frac{-n}{2},\frac{-n}{2}\right)} \\ - \mathcal{K}l_1\, (\lambda_1 - \lambda_2 + 1) \left\{\left(\lambda_1 + \frac{n}{2}\right)\left(\lambda_2 + \frac{n}{2} - 1\right) - 1\right\} G^{(s)}_{\left(\frac{2-n}{2},\frac{2-n}{2}\right)}.\end{aligned}$$

By substituting $G^{(s)}_{\left(\frac{-n}{2},\frac{-n}{2}\right)} = G^{(s)}_{\left(\frac{2-n}{2},\frac{2-n}{2}\right)} + L_{\left(\frac{-n}{2},\frac{-n}{2}\right)}$, we have

$$\mathcal{B}' = \mathcal{K}l_1 \left\{2\, C_{(n+1,1)}\, G^{(s)}_{\left(\frac{2-n}{2},\frac{2-n}{2}\right)} + \left(\lambda_1 + \frac{n+2}{2}\right)^2 L_{\left(\frac{-n}{2},\frac{-n}{2}\right)}\right\} = \mathcal{I}'.$$

Thus (5.83) holds.

5.3.2. Proof of (5.83) **when** $2m'+n\le 1$**.** For $k=1,2$, we put

$$\mathcal{I}'_k = C^{-1}\cdot \mathcal{I}^{(k)} + \sum_{i\in\mathbb{Z}} \mathcal{J}^{(k,i)}\cdot \mathcal{K}l_k(x,\mu;i).$$

Then we have $\mathcal{I}'=\mathcal{I}'_1+\mathcal{I}'_2$. By Lemma 5.12, when $m'=0$, we have

$$\begin{aligned}\mathcal{I}'_1 = &-C_{(n+1,-m+n-1)}\,F^{(s)}_{(-n,-m)}\\ &+\left(1-q^{-1}\right)\sum_{-m\le i\le -n} C_{(n+1,m+n+2i+1)}\,G^{(s)}_{(2-n,i+1)}\\ &-q^{-1}\left(C_{(n+1,-m+n-1)}\,G^{(s)}_{(2+m-n,0)}+C_{(n+1,m-n+3)}\,G^{(s)}_{(2-n,2-n)}\right)\end{aligned}\tag{5.96}$$

and

$$\begin{aligned}\mathcal{I}'_2 = &-C_{(m+n+1,n-1)}\,F^{(s)}_{(1-n,1)}\\ &+\left(1-q^{-1}\right)\sum_{0\le j\le -n} C_{(m+n+1,n+2j+1)}\,G^{(s)}_{(1-n,j)}\\ &-q^{-1}\left(C_{(m+n+1,n-1)}\,G^{(s)}_{(2-n,0)}+C_{(m+n+1,-n+3)}\,G^{(s)}_{(1-n,1-n)}\right).\end{aligned}\tag{5.97}$$

Similarly, when $m'>0$, we have

$$\begin{aligned}\mathcal{I}'_1 = &-C_{(n+1,2m'+n-1)}\,F^{(s)}_{(-m'-n,m')}\\ &+\left(1-q^{-1}\right)\sum_{m'\le i\le -m'-n} C_{(n+1,n+2i+1)}\,G^{(s)}_{(2-m'-n,i+1)}\\ &-q^{-1}\left(C_{(n+1,2m'+n-1)}\,G^{(s)}_{(2-m'-n,m')}+C_{(n+1,-2m'-n+3)}\,G^{(s)}_{(2-m'-n,2-m'-n)}\right)\end{aligned}\tag{5.98}$$

and

$$\begin{aligned}\mathcal{I}'_2 = &-C_{(n+1,2m'+n-1)}\,F^{(s)}_{(1-m'-n,m'+1)}\\ &+\left(1-q^{-1}\right)\sum_{m'\le j\le -m'-n} C_{(n+1,n+2j+1)}\,G^{(s)}_{(1-m'-n,j)}\\ &-q^{-1}\left(C_{(n+1,2m'+n-1)}\,G^{(s)}_{(1-m'-n,m'-1)}+C_{(n+1,-2m'-n+3)}\,G^{(s)}_{(1-m'-n,1-m'-n)}\right).\end{aligned}\tag{5.99}$$

Before proceeding further, we prove the following two lemmas.

Lemma 5.15. *Suppose that $m\ge 1$ and $n\le 1$. Let*

$$S_0(m,n)=\sum_{-m\le i\le -1} C_{(n+1,m+n+2i+1)}\,G^{(s)}_{(1-n-i,0)}.$$

Then we have

$$S_0(m,n)=(m+1)\,C_{(n+1,n-1)}\,G^{(s)}_{(2-n,0)}-C_{(n+1,-m+n-1)}\,G^{(s)}_{(m-n+1,0)}.\tag{5.100}$$

Remark 5.16. When $n\le 1$, if we substitute $m=0$ in the right hand side of (5.100), we have

$$C_{(n+1,n-1)}\left(G^{(s)}_{(2-n,0)}-G^{(s)}_{(1-n,0)}\right)=0.$$

PROOF. We rewrite

$$S_0(m,n) = \sum_{-n+2\le j\le m-n+1} C_{(n+1,m-n-2j+3)}\, G^{(s)}_{(j,0)}.$$

Suppose $\lambda_1-\lambda_2 \ge 2-n$, for otherwise $S_0(m,n)$ vanishes. Then

$$\begin{aligned} S_0(m,n) &= \sum_{2-n\le j\le k} (\lambda_1+\lambda_2+n+1)(\lambda_1-\lambda_2+m-n-2j+3) \\ &= (\lambda_1+\lambda_2+n+1)(k+n-1)(\lambda_1-\lambda_2-k+m+1), \end{aligned}$$

where $k=\min\{1+m-n, \lambda_1-\lambda_2\}$. Consequently

$$\begin{aligned} S_0(m,n) = (m+1)(\lambda_1+\lambda_2+n+1)(\lambda_1-\lambda_2+n-1)\left(G^{(s)}_{(2-n,0)} - G^{(s)}_{(1+m-n,0)}\right) \\ + m(\lambda_1+\lambda_2+n+1)(\lambda_1-\lambda_2+n)G^{(s)}_{(1+m-n,0)}. \end{aligned}$$

This is equivalent to (5.100). □

LEMMA 5.17. *Suppose that $2m'+n\le 0$. Let*

$$S_1(m,m',n) = \sum_{m'\le i\le -m'-n} C_{(n+1,m+n+2i+1)}\, G^{(s)}_{(2-m'-n,i+1)}$$

and

$$S_2(m,m',n) = \sum_{m'\le i\le -m'-n} C_{(m+n+1,n+2i+1)}\, G^{(s)}_{(1-m'-n,i)}.$$

Then we have

$$\begin{aligned} S_1(m,m',n) &= -(m+1)C_{(n+1,2m'+n-1)}G^{(s)}_{(2-m'-n,m'+1)} \\ &+ (\lambda_1+m'+n-1)(\lambda_2+m-m'+1)(\lambda_1+\lambda_2+n+1)H^{(s)}_{(2-m'-n,m'+1)} \\ &+ (m-2m'-n+2)C_{(n+1,0)}G^{(s)}_{(3-m'-n,2-m'-n)}. \end{aligned} \tag{5.101}$$

and

$$\begin{aligned} S_2(m,m',n) &= -C_{(m+n+1,2m'+n-1)}G^{(s)}_{(1-m'-n,m')} \\ &+ (\lambda_1+m'+n)(\lambda_2-m'+2)(\lambda_1+\lambda_2+m+n+1)H^{(s)}_{(1-m'-n,m')} \\ &+ (2-2m'-n)C_{(m+n+1,0)}G^{(s)}_{(2-m'-n,1-m'-n)} \end{aligned} \tag{5.102}$$

PROOF. First consider

$$S_1(m,m',n) = (\lambda_1+\lambda_2+n+1)\sum_{m'\le i\le -m'-n}(\lambda_1-\lambda_2+m+n+2i+1)G^{(s)}_{(2-m'-n,i+1)}.$$

This vanishes unless $\lambda_2\ge m'+1$, so assume this. When $\lambda_1-\lambda_2\ge 1-2m'-n$, this sum on the right is

$$\sum_{m'\le i\le k}(\lambda_1-\lambda_2+m+n+2i+1) = (1+k-m')(\lambda_1-\lambda_2+k+m+m'+n+1),$$

where $k=\min\{-m'-n,\lambda_2-1\}$. When $1\le\lambda_1-\lambda_2\le -2m'-n$, the sum on the right is

$$\sum_{1-m'-n+\lambda_2-\lambda_1\le i\le k}(\lambda_1-\lambda_2+m+n+2i+1) = (m-m'+k+2)(\lambda_1-\lambda_2+m'+n+k)$$

where $k = \min\{-m' - n, \lambda_2 - 1\}$. Hence

$$\frac{S_1(m, m', n)}{(\lambda_1 + \lambda_2 + n + 1)} =$$
$$(\lambda_1 + m + m' + n)(\lambda_2 - m')\left(G^{(s)}_{(2-m'-n,m'+1)} - G^{(s)}_{(3-3m'-2n,2-m'-n)}\right)$$
$$+ (1 - 2m' - n)(\lambda_1 - \lambda_2 + m + 1)G^{(s)}_{(3-3m'-2n,2-m'-n)}$$
$$+ (\lambda_1 + m' + n - 1)(\lambda_2 + m - m' + 1)$$
$$\times \left(H^{(s)}_{(2-m'-n,m'+1)} - G^{(s)}_{(2-m'-n,m'+1)} + G^{(s)}_{(3-3m'-2n,2-m'-n)}\right)$$
$$+ (m - 2m' - n + 2)(\lambda_1 - \lambda_2)\left(G^{(s)}_{(3-m'-n,2-m'-n)} - G^{(s)}_{(3-3m'-2n,2-m'-n)}\right)$$

Note that the coefficients of $G^{(s)}_{(3-3m'-2n,2-m'-n)}$ sum to 0, and we have (5.101).

Now consider

$$\frac{S_2(m, m', n)}{(\lambda_1 + \lambda_2 + m + n + 1)} = \sum_{m' \le i \le -m'-n} (\lambda_1 - \lambda_2 + n + 2i + 1)G^{(s)}_{(1-m'-n,i)}.$$

Assume $\lambda_2 \ge m'$. When $\lambda_1 - \lambda_2 \ge -2m' - n$, this equals

$$\sum_{m' \le i \le k} (\lambda_1 - \lambda_2 + n + 2i + 1) = (1 - m' + k)(\lambda_1 - \lambda_2 + m' + n + 1 + k)$$

where $k = \min\{-m' - n, \lambda_2\}$. Similarly, when $1 \ge \lambda_1 - \lambda_2 < -2m' - n$, the above sum is

$$\sum_{1-m'-n-\lambda_1+\lambda_2 \le i \le k} (\lambda_1 - \lambda_2 + n + 2i + 1) = (2 - m' + k)(\lambda_1 - \lambda_2 + m' + n + k)$$

where $k = \min\{-m' - n, \lambda_2\}$. Hence

$$\frac{S_2(m, m', n)}{(\lambda_1 + \lambda_2 + m + n + 1)} =$$
$$(\lambda_1 + m' + n + 1)(\lambda_2 - m' + 1)\left(G^{(s)}_{(1-m'-n,m')} - G^{(s)}_{(2-m'-2n,1-m'-n)}\right)$$
$$+ (1 - 2m' - n)(\lambda_1 - \lambda_2 + 1)G^{(s)}_{(2-m'-2n,1-m'-n)}$$
$$+ (\lambda_1 + m' + n)(\lambda_2 - m' + 2)\left(H^{(s)}_{(1-m'-n,m')} - G^{(s)}_{(1-m'-n,m')} + G^{(s)}_{(2-m'-2n,1-m'-n)}\right)$$
$$+ (2 - 2m' - n)(\lambda_1 - \lambda_2)\left(G^{(s)}_{(2-m'-n,1-m'-n)} - G^{(s)}_{(2-m'-2n,1-m'-n)}\right)$$

Note that the coefficients of $G^{(s)}_{(2-m'-2n,1-m'-n)}$ sum to 0, and we have (5.102). $\square$

5.3.2.1. *When $m' = 0$ and $n = 1$.* By Lemma 5.15, we have

$$\mathcal{I}'_1 = -C_{(2,-m)}\, F^{(s)}_{(0,1-m)} + \left(1 - q^{-1}\right)\left\{(m + 1)C_{(2,0)}\, G^{(s)}_{(1,0)} - C_{(2,-m)}\, G^{(s)}_{(m,0)}\right\}$$
$$- q^{-1}\left(C_{(2,-m)}\, G_{(m+1,0)} + C_{(2,m+2)}\, G^{(s)}_{(1,1)}\right)$$

and

$$\mathcal{I}'_2 = -q^{-1}\left(C_{(m+2,0)}\, G^{(s)}_{(1,0)} + C_{(m+2,2)}\, G^{(s)}_{(0,0)}\right).$$

Since

$$\begin{aligned}
&- C_{(2,-m)}\, F^{(s)}_{(0,1-m)} - \left(1-q^{-1}\right) C_{(2,-m)}\, G^{(s)}_{(m,0)} - q^{-1} C_{(2,-m)}\, G_{(m+1,0)} \\
&= - C_{(2,-m)} \left(F^{(s)}_{(0,1-m)} + G^{(s)}_{(m,0)}\right) + q^{-1}\, C_{(2,-m)} \left(G^{(s)}_{(m,0)} - G^{(s)}_{(m+1,0)}\right) \\
&= - C_{(2,-m)}\, G^{(s)}_{(0,0)},
\end{aligned}$$

$C_{(2,0)}\, G^{(s)}_{(1,0)} = C_{(2,0)}\, G^{(s)}_{(0,0)}$, and $C_{(m+2,0)}\, G^{(s)}_{(1,0)} = C_{(m+2,0)}\, G^{(s)}_{(0,0)}$, we have

$$\begin{aligned}
(5.103)\quad \mathcal{I}' &= -q^{-1}\, C_{(2,m+2)}\, G^{(s)}_{(1,1)} \\
&+ \left\{-C_{(2,-m)} + \left(1-q^{-1}\right)(m+1)\, C_{(2,0)} - q^{-1} C_{(m+2,0)} - q^{-1} C_{(m+2,2)}\right\} G^{(s)}_{(0,0)}.
\end{aligned}$$

As for $\mathcal{B}'$, first we note that

$$T^{(s)}\left(P_{(1,0)}\right) = \lambda_1(\lambda_2+1)\, H^{(s)}_{(1,0)} + 2(\lambda_1-\lambda_2+1)\, G^{(s)}_{(1,1)}$$

where $\lambda_1\, H^{(s)}_{(1,0)} = \lambda_1 \left(G^{(s)}_{(0,0)} - G^{(s)}_{(1,1)}\right)$. Hence

$$\begin{aligned}
T^{(s)}\left(P_{(1,0)}\right) &= (\lambda_1\lambda_2+\lambda_1) \left(G^{(s)}_{(0,0)} - G^{(s)}_{(1,1)}\right) + 2(\lambda_1-\lambda_2+1)\, G^{(s)}_{(1,1)} \\
&= (\lambda_1-\lambda_2)\, G^{(s)}_{(0,0)} + (\lambda_1-\lambda_2+2)\, G^{(s)}_{(1,1)},
\end{aligned}$$

since $\lambda_2\, G^{(s)}_{(1,1)} = \lambda_2\, G^{(s)}_{(0,0)}$. Therefore we have

$$\begin{aligned}
(5.104)\quad \mathcal{B}' &= 2q^{-1}(\lambda_1-\lambda_2+1)\, G^{(s)}_{(0,0)} - (m+1)q^{-1}(\lambda_1-\lambda_2)\, G^{(s)}_{(0,0)} \\
&\qquad - (m+1)q^{-1}(\lambda_1-\lambda_2+2)\, G^{(s)}_{(1,1)} \\
&+ \left\{m-(m+3)q^{-1}\right\}(\lambda_1-\lambda_2+1)(\lambda_1+2)(\lambda_2+1)\, G^{(s)}_{(0,0)} - 2q^{-1}(\lambda_1-\lambda_2+1)\, G^{(s)}_{(1,1)} \\
&- \left\{m-(m+2)q^{-1}\right\}(\lambda_1-\lambda_2+1)(\lambda_1+1)\lambda_2\, G^{(s)}_{(0,0} + q^{-1}(\lambda_1-\lambda_2+1)\lambda_1(\lambda_2-1)\, G^{(s)}_{(1,1)}
\end{aligned}$$

by Proposition 4.21 and Lemma 5.13. By substituting $G^{(s)}_{(0,0)} = G^{(s)}_{(1,1)} + L_{(0,0)}$ into (5.103) and (5.104), we obtain

$$\begin{aligned}
&\mathcal{B}' = \mathcal{I}' \\
&= (\lambda_1-\lambda_2+1)\left[\left\{m-(m+4)q^{-1}\right\}(\lambda_1+\lambda_2) + 2\left\{m-2(m+2)q^{-1}\right\}\right] G^{(s)}_{(1,1)} + \\
&\left[\left\{m-(m+3)q^{-1}\right\}\lambda_1^2 + \left\{3m-4(m+2)q^{-1}\right\}\lambda_1 + 2\left\{m-(m+2)q^{-1}\right\}\right] L_{(0,0)}.
\end{aligned}$$

5.3.2.2. *When $m'>0$ and $2m'+n=1$.* By a compuataion similar to the one in the previous case, we have

$$\begin{aligned}
&\mathcal{B}' = \mathcal{I}' = -4q^{-1}\, C_{(2-2m',1)}\, G^{(s)}_{(m'+1,m'+1)} \\
&- q^{-1}(\lambda_1-m'+2)(3\lambda_1-3m'+2)\, L_{(m',m')} - q^{-1}(\lambda_1-m'+1)^2\, L_{(m'-1,m'-1)}.
\end{aligned}$$

5.3.2.3. *When $m'=0$ and $n\le 0$.* Let us put $a=-n$.

Recall from (5.96) and (5.97),

$$\begin{aligned}
(5.105)\quad \mathcal{I}_1' &= -C_{(n+1,-m+n-1)}\, F^{(s)}_{(-n,-m)} + \left(1-q^{-1}\right)\left(S_0(m,n) + S_1(m,0,n)\right) \\
&\qquad - q^{-1}\left(C_{(n+1,-m+n-1)}\, G^{(s)}_{(2+m-n,0)} + C_{(n+1,m-n+3)}\, G^{(s)}_{(2-n,2-n)}\right)
\end{aligned}$$

and

$$\begin{aligned}(5.106)\quad \mathcal{I}_2' = &-C_{(m+n+1,n-1)}\, F^{(s)}_{(1-n,1)} + \left(1-q^{-1}\right) S_2(m,0,n)\\ &- q^{-1}\left(C_{(m+n+1,n-1)}\, G^{(s)}_{(2-n,0)} + C_{(m+n+1,-n+3)}\, G^{(s)}_{(1-n,1-n)}\right).\end{aligned}$$

By Lemmas 5.15 and 5.17, we have

$$\begin{aligned}\mathcal{I}_1' = &-C_{(1-a,-m-a-1)}\left(F^{(s)}_{(a,-m)} + G^{(s)}_{(a+m+1,0)}\right)\\ &- q^{-1}C_{(1-a,a+m+3)}G^{(s)}_{(a+2,a+2)} + (1-q^{-1})(m+1)C_{(1-a,-1-a)}G^{(s)}_{(a+2,0)}\\ &- (1-q^{-1})(m+1)C_{(1-a,-1-a)}G^{(s)}_{(a+2,1)}\\ &+ (1-q^{-1})(m+a+2)C_{(1-a,0)}G^{(s)}_{(a+3,a+2)}\\ &+ (1-q^{-1})(\lambda_1-a-1)(\lambda_2+m+1)(\lambda_1+\lambda_2-a+1)H^{(s)}_{(a+2,1)}.\end{aligned}$$

Note

$$C_{(1-a,0)}G^{(s)}_{(a+3,a+2)} = C_{(1-a,0)}G^{(s)}_{(a+2,a+2)},$$

$$C_{(1-a,-1-a)}G^{(s)}_{(a+2,1)} = C_{(1-a,-1-a)}\left(G^{(s)}_{(a+2,0)} - L_{(a+2,0)}\right),$$

and

$$F^{(s)}_{(a,-m)} + G^{(s)}_{(a+m+1,0)} = H^{(s)}_{(a+2,0)} + G^{(s)}_{(a+2,a+2)} + V_{(a+1,0)} + V_{(a,0)}.$$

Therefore the above simplifies to

$$\begin{aligned}(5.107)\quad \mathcal{I}_1' = &-C_{(1-a,-m-a-1)}\left(V_{(a+1,0)} + V_{(a,0)}\right)\\ &+ \left\{(m+a+1) - q^{-1}(m+a+3)\right\} C_{(1-a,1)}G^{(s)}_{(a+2,a+2)}\\ &+ \left\{(\lambda_1-a)(\lambda_2+m) - q^{-1}(\lambda_1-a-1)(\lambda_2+m+1)\right\}(\lambda_1+\lambda_2-a+1)H^{(s)}_{(a+2,0)}\end{aligned}$$

Similarly, by Lemma 5.17, we have

$$\begin{aligned}\mathcal{I}_2' = &-C_{(m-a+1,-1-a)}\left(F^{(s)}_{(a+1,1)} + G^{(s)}_{(a+1,0)}\right)\\ &+ q^{-1}C_{(m-a+1,-1-a)}\left(G^{(s)}_{(a+1,0)} - G^{(s)}_{(a+2,0)}\right)\\ &+ \left\{(2+a)C_{(m-a+1,0)} - q^{-1}(a+3)C_{(m-a+1,1)}\right\} G^{(s)}_{(a+1,a+1)}\\ &+ \left(1-q^{-1}\right)(\lambda_1-a)(\lambda_2+2)(\lambda_1+\lambda_2+m-a+1)H^{(s)}_{(a+1,0)}.\end{aligned}$$

Since

$$F^{(s)}_{(a+1,1)} + G^{(s)}_{(a+1,0)} = G^{(s)}_{(a+1,a+1)} + H^{(s)}_{(a+1,0)},$$

we obtain

$$\begin{aligned}\mathcal{I}_2' = &\left\{(a+1) - q^{-1}(a+3)\right\} C_{(m-a+1,1)}G^{(s)}_{(a+1,a+1)}\\ &+ \left\{(\lambda_1-a+1)(\lambda_2+1) - q^{-1}(\lambda_1-a)(\lambda_2+2)\right\}(\lambda_1+\lambda_2+m-a+1)H^{(s)}_{(a+1,0)}.\end{aligned}$$

We can rewrite this as

$$\mathcal{I}_2' = \left\{(a+1) - q^{-1}(a+3)\right\} C_{(m-a+1,1)} \left(G^{(s)}_{(a+2,a+2)} + L_{(a+1,a+1)}\right)$$
$$+ \left\{(\lambda_1 - a + 1)(\lambda_2 + 1) - q^{-1}(\lambda_1 - a)(\lambda_2 + 2)\right\}$$
$$\times (\lambda_1 + \lambda_2 + m - a + 1) \left(H^{(s)}_{(a+2,0)} + V_{(a+1,0)} - L_{(a+1,a+1)}\right),$$

which simplifies to

$$\mathcal{I}_2' = \left\{(a+1) - q^{-1}(a+3)\right\} C_{(m-a+1,1)} \left(G^{(s)}_{(a+2,a+2)}\right) \tag{5.108}$$
$$+ \left\{(\lambda_1 - a + 1)(\lambda_2 + 1) - q^{-1}(\lambda_1 - a)(\lambda_2 + 2)\right\} (\lambda_1 + \lambda_2 + m - a + 1) H^{(s)}_{(a+2,0)}$$
$$+ \left\{2(\lambda_2 + 1) - q^{-1}(\lambda_2 + 2)\right\} (\lambda_2 + m + 2) V_{(a+1,0)}$$
$$- (\lambda_1 + 2)(\lambda_1 + m + 2) L_{(a+1,a+1)}$$

Combining (5.107) and (5.108), we get

$$\mathcal{I}' = (\lambda_2 + 1)(\lambda_2 + m + 1) V_{(a,0)} \tag{5.109}$$
$$+ \left\{(\lambda_2 + 2)(\lambda_2 + m) + \left\{2(\lambda_2 + 1) - q^{-1}(\lambda_2 + 2)\right\} (\lambda_2 + m + 2)\right\} V_{(a+1,0)}$$
$$- (\lambda_1 + 2)(\lambda_1 + m + 2) L_{(a+1,a+1)}$$
$$m(\lambda_1 + \lambda_2 + 2) + 2(a+1)(\lambda_1 + \lambda_2 - a + 1) G^{(s)}_{(a+2,a+2)}$$
$$- q^{-1} \left\{m(\lambda_1 + \lambda_2 + 4) + 2(a+3)(\lambda_1 + \lambda_2 - a + 1)\right\} G^{(s)}_{(a+2,a+2)}$$
$$+ (\lambda_1 + \lambda_2 - a + 1) \left\{(\lambda_1 - a)(\lambda_2 + m) - q^{-1}(\lambda_1 - a - 1)(\lambda_2 + m + 1)\right\} H^{(s)}_{(a+2,0)}$$
$$+ (\lambda_1 + \lambda_2 + m - a + 1) \left\{(\lambda_1 - a + 1)(\lambda_2 + 1) - q^{-1}(\lambda_1 - a)(\lambda_2 + 2)\right\} H^{(s)}_{(a+2,0)}.$$

On the other hand, by Proposition 4.21, we have

$$C_s^{-1} \cdot \mathcal{B}^{(s)} =$$
$$2q^{-1} L_{(a+3,a+3)} - 2(1 - q^{-1}) L_{(a+2,a+2)} - 2L_{(a+1,a+1)} \tag{5.110}$$
$$+2V_{(a,1)} + 2(1 - q^{-1}) V_{(a+1,1)} - 2q^{-1} V_{(a+2,1)} \tag{5.111}$$
$$+ \left\{(1 - m) + (m+1)q^{-1}\right\} P_{(a+1,1)} \tag{5.112}$$
$$+(m+1) P_{(a,0)} + 2q^{-1} P_{(a+1,0)} - (m+1) q^{-1} P_{(a+2,0)} \tag{5.113}$$
$$+2 \left\{(1 - m) + (m+1)q^{-1}\right\} L_{(a+2,1)} \tag{5.114}$$
$$+2 \left\{(m+1) - (m+3)q^{-1}\right\} L_{(a+1,0)}. \tag{5.115}$$

We compute $\mathcal{B}'$ by Lemma 5.13, noting Remark 5.14.

First observe that (5.110) plus (5.111) can be written as

$$X_{(a,1)} - q^{-1} X_{(a+1,1)},$$

where

$$X_{(c,d)} = 2\left(V_{(c,d)} + V_{(c+1,d)} - L_{(c+1,c+1)} - L_{(c+2,c+2)}\right).$$

Since

$$2T^{(s)} \left(L_{(c,c)} + L_{(c+1,c+1)}\right) =$$
$$(\lambda_1 - \lambda_2 + 1) \left\{(\lambda_1 - c + 2)(\lambda_2 - c + 1) + (\lambda_1 - c + 1)(\lambda_2 - c)\right\} G^{(s)}_{(c,c)}$$

and

$$2T^{(s)}\left(V_{(c,d)}+V_{(c+1,d)}\right)=(\lambda_2-d+1)(\lambda_2-d+2)\left\{(2\lambda_1-2c+1)H^{(s)}_{(c+1,d)}+V_{(c,d)}\right\}$$
$$+2(\lambda_1-\lambda_2+1)(c-d+2)^2G^{(s)}_{(c+1,c+1)},$$

we see

$$\begin{aligned}T^{(s)}\left(X_{(c,d)}\right)&=(\lambda_2-d+1)(\lambda_2-d+2)\left\{(2\lambda_1-2c+1)H^{(s)}_{(c+1,d)}+V_{(c,d)}\right\}\\&+(\lambda_1-\lambda_2+1)\left\{2(c-d+2)^2-(\lambda_1-c+1)(\lambda_2-c)\right.\\&\left.-(\lambda_1-c)(\lambda_2-c-1)\right\}G^{(s)}_{(c+1,c+1)}.\end{aligned}\tag{5.116}$$

Hence the contribution to $T^{(s)}\left(C_s^{-1}\cdot\mathcal{B}^{(s)}\right)$ from (5.110) and (5.111) is

$$\begin{aligned}&\lambda_2(\lambda_2+1)\left\{(2\lambda_1-2a+1)H^{(s)}_{(a+1,1)}+V_{(a,1)}\right\}\\&+(\lambda_1-\lambda_2+1)\left\{2(a+1)^2-(\lambda_1-a+1)(\lambda_2-a)-(\lambda_1-a)(\lambda_2-a-1)\right\}G^{(s)}_{(a+1,a+1)}\\&-q^{-1}\lambda_2(\lambda_2+1)\left\{(2\lambda_1-2a-1)H^{(s)}_{(a+2,1)}+V_{(a+1,1)}\right\}\\&-q^{-1}(\lambda_1-\lambda_2+1)\left\{2(a+2)^2-(\lambda_1-a)(\lambda_2-a-1)\right.\\&\left.-(\lambda_1-a-1)(\lambda_2-a-2)\right\}G^{(s)}_{(a+2,a+2)}.\end{aligned}$$

We may rewrite this as

$$\begin{aligned}&\lambda_2(\lambda_2+1)\left\{\left(3-q^{-1}\right)V_{(a+1,0)}+V_{(a,0)}\right\}-(\lambda_1-a)(\lambda_2+2)L_{(a+1,a+1)}\\&+(1-q^{-1})\lambda_2(\lambda_2+1)(2\lambda_1-2a-1)H^{(s)}_{(a+2,0)}\\&+(\lambda_1-\lambda_2+1)\left\{2(a+1)^2-(\lambda_1-a+1)(\lambda_2-a)-(\lambda_1-a)(\lambda_2-a-1)\right\}G^{(s)}_{(a+2,a+2)}\\&-q^{-1}(\lambda_1-\lambda_2+1)\left\{2(a+2)^2-(\lambda_1-a)(\lambda_2-a-1)\right.\\&\left.-(\lambda_1-a-1)(\lambda_2-a-2)\right\}G^{(s)}_{(a+2,a+2)}.\end{aligned}\tag{5.117}$$

The contribution from (5.112) to $T^{(s)}\left(C_s^{-1}\cdot\mathcal{B}^{(s)}\right)$ is

$$\left\{(1-m)+(m+1)q^{-1}\right\}\left\{(\lambda_1-a)\lambda_2H^{(s)}_{(a+1,1)}+(a+1)(\lambda_1-\lambda_2+1)G^{(s)}_{(a+1,a+1)}\right\},$$

which we rewrite as

$$\begin{aligned}&\left\{(1-m)+(m+1)q^{-1}\right\}(\lambda_1-a)\lambda_2\left(H^{(s)}_{(a+2,0)}+V_{(a+1,0)}\right)\\&+\left\{(1-m)+(m+1)q^{-1}\right\}(a+1)(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)}.\end{aligned}\tag{5.118}$$

The contribution from (5.113) is

$$
\begin{aligned}
(m+1)&\Big\{(\lambda_1-a+1)(\lambda_2+1)H^{(s)}_{(a+1,0)}+(\lambda_2+1)V_{(a,0)}\\
&\qquad +(a+1)(\lambda_1-\lambda_2+1)G^{(s)}_{(a+1,a+1)}\Big\}\\
&+2q^{-1}\Big\{(\lambda_1-a)(\lambda_2+1)H^{(s)}_{(a+2,0)}+(\lambda_2+1)V_{(a+1,0)}\\
&\qquad +(a+2)(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)}\Big\}\\
-(m+1)q^{-1}&\Big\{(\lambda_1-a-1)(\lambda_2+1)H^{(s)}_{(a+2,0)}+(a+3)(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)}\Big\}
\end{aligned}
$$

We will rewrite this as

$$
\begin{aligned}
(5.119)\quad &(\lambda_2+1)\left\{\left((m+1)(\lambda_1-a+1)+2q^{-1}\right)V_{(a+1,0)}+(m+1)V_{(a,0)}\right\}\\
&\qquad -(m+1)(\lambda_2+2)L_{(a+1,a+1)}\\
&+(\lambda_2+1)\left\{(m+1)(\lambda_1-a+1)+q^{-1}\left((\lambda_1-a)(1-m)+(m+1)\right)\right\}H^{(s)}_{(a+2,0)}\\
&+(\lambda_1-\lambda_2+1)\left\{(a+1)(m+1)+q^{-1}(a+1-m(a+3))\right\}G^{(s)}_{(a+2,a+2)}.
\end{aligned}
$$

Finally, the contribution from (5.114) and (5.115) is

$$
\begin{aligned}
&\left\{(1-m)+(m+1)q^{-1}\right\}(\lambda_1-a-1)(\lambda_1-a)\lambda_2H^{(s)}_{(a+2,1)}\\
&\quad+\left\{(1-m)+(m+1)q^{-1}\right\}(\lambda_1-\lambda_2+1)\left\{(\lambda_1-a-1)\lambda_2\right.\\
&\qquad\left.+(a+2)(\lambda_2-a-1)\right\}G^{(s)}_{(a+2,a+2)}\\
&\quad+\left\{(m+1)-(m+3)q^{-1}\right\}(\lambda_1-a)(\lambda_1-a+1)(\lambda_2+1)H^{(s)}_{(a+1,0)}\\
&\quad+\left\{(m+1)-(m+3)q^{-1}\right\}(\lambda_1-\lambda_2+1)\left\{(\lambda_1-a)(\lambda_2+1)\right.\\
&\qquad\left.+(a+2)(\lambda_2-a)\right\}G^{(s)}_{(a+1,a+1)},
\end{aligned}
$$

which we may rewrite as

$$
\begin{aligned}
(5.120)\quad &2(\lambda_2+1)\left\{(m+1)-(m+3)q^{-1}\right\}V_{(a+1,0)}\\
&\quad+(\lambda_1-a)\left\{2\left(1-q^{-1}\right)(\lambda_1-a+1)(\lambda_2+1)\right.\\
&\qquad\left.+(\lambda_1+2\lambda_2-a+1)\left\{(m-1)-(m+1)q^{-1}\right\}\right\}H^{(s)}_{(a+2,0)}\\
&\quad+\big[\left\{(m+1)(\lambda_1+\lambda_2+2)+2(\lambda_1+1)\lambda_2-2(a+1)(a+2)\right\}\\
&\qquad-q^{-1}\left\{(m+3)(\lambda_1+\lambda_2+2)+2(\lambda_1+1)\lambda_2-2(a+1)(a+2)\right\}\big]\\
&\qquad\times(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)}.
\end{aligned}
$$

Then summing up (5.117), (5.118), (5.119), and (5.120) yields (5.109), i.e., $\mathcal{B}'=\mathcal{I}'$.

5.3.2.4. *When $m'>0$ and $2m'+n\le 0$.* Let us put $a=-m'-n$ and $b=m'$. Observe that $2m'+n\le 0$ and $m'>0$ means $0<b\le a$.

Recall

$$\begin{aligned}\mathcal{I}_1' &= -C_{(n+1,2m'+n-1)}\, F^{(s)}_{(-m'-n,m')} + \left(1-q^{-1}\right) S_1(0,m',n)\\ &\quad - q^{-1}\left(C_{(n+1,2m'+n-1)}\, G^{(s)}_{(2-m'-n,m')} + C_{(n+1,-2m'-n+3)}\, G^{(s)}_{(2-m'-n,2-m'-n)}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_2' &= -C_{(n+1,2m'+n-1)}\, F^{(s)}_{(1-m'-n,m'+1)} + \left(1-q^{-1}\right) S_2(0,m',n)\\ &- q^{-1}\left(C_{(n+1,2m'+n-1)}\, G^{(s)}_{(1-m'-n,m'-1)} + C_{(n+1,-2m'-n+3)}\, G^{(s)}_{(1-m'-n,1-m'-n)}\right).\end{aligned}$$

By Lemma 5.17, we have

$$\begin{aligned}\mathcal{I}_1' &= -C_{(1-a-b,b-a-1)}\, F^{(s)}_{(a,b)}\\ &\qquad - \left(1-q^{-1}\right) C_{(1-a-b,b-a-1)} G^{(s)}_{(a+2,b+1)}\\ &\quad + \left(1-q^{-1}\right)(\lambda_1-a-1)(\lambda_2-b+1)(\lambda_1+\lambda_2-a-b+1) H^{(s)}_{(a+2,b+1)}\\ &\qquad + \left(1-q^{-1}\right)(2+a-b) C_{(1-a-b,0)} G^{(s)}_{(a+2,a+2)}\\ &\qquad - q^{-1}\left(C_{(1-a-b,b-a-1)}\, G^{(s)}_{(a+2,b)} + C_{(1-a-b,a-b+3)}\, G^{(s)}_{(a+2,a+2)}\right)\end{aligned}$$

Note

$$F^{(s)}_{(a,b)} + G^{(s)}_{(a,b-1)} = G^{(s)}_{(a,a)} + H^{(s)}_{(a,b-1)}$$

so

$$\begin{aligned}F^{(s)}_{(a,b)} + G^{(s)}_{(a+2,b+1)} &= G^{(s)}_{(a,a)} + H^{(s)}_{(a,b-1)} - L_{(a,b-1)} - L_{(a+1,b)}\\ &= G^{(s)}_{(a+2,a+2)} + H^{(s)}_{(a+2,b+1)} + V_{(a,b)} + V_{(a+1,b+1)}.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{I}_1' &= -C_{(1-a-b,b-a-1)}\left(G^{(s)}_{(a+2,a+2)} + H^{(s)}_{(a+2,b+1)} + V_{(a,b)} + V_{(a+1,b+1)}\right)\\ &\qquad + q^{-1} C_{(1-a-b,b-a-1)}\left(G^{(s)}_{(a+2,b+1)} - G^{(s)}_{(a+2,b)}\right)\\ &\quad + \left(1-q^{-1}\right)(\lambda_1-a-1)(\lambda_2-b+1)(\lambda_1+\lambda_2-a-b+1) H^{(s)}_{(a+2,b+1)}\\ &\qquad + \left\{(2+a-b) C_{(1-a-b,0)} - q^{-1}(3+a-b) C_{(1-a-b,1)}\right\} G^{(s)}_{(a+2,a+2)}.\end{aligned}$$

We can rewrite this as

$$\begin{aligned}\mathcal{I}_1' &= -C_{(1-a-b,b-a-1)}\left(H^{(s)}_{(a+2,b+1)} + V_{(a,b)} + V_{(a+1,b+1)}\right)\\ &\qquad - q^{-1}(\lambda_1-a+1)(\lambda_1-a-1) L_{(a+1,b)}\\ &\quad + \left(1-q^{-1}\right)(\lambda_1-a-1)(\lambda_2-b+1)(\lambda_1+\lambda_2-a-b+1) H^{(s)}_{(a+2,b+1)}\\ &\qquad + \left\{(1+a-b) - q^{-1}(3+a-b)\right\} C_{(1-a-b,1)} G^{(s)}_{(a+2,a+2)}\end{aligned} \tag{5.121}$$

Similarly

$$\begin{aligned}\mathcal{I}_2' = -C_{(1-a-b,b-a-1)}\, H^{(s)}_{(a+1,b)} \\ + q^{-1} C_{(1-a-b,b-a-1)} \left(G^{(s)}_{(a+1,b)} - G^{(s)}_{(a+1,b-1)}\right) \\ + \left(1-q^{-1}\right)(\lambda_1 - a)(\lambda_2 - b + 2)(\lambda_1 + \lambda_2 + 1 - a - b) H^{(s)}_{(a+1,b)} \\ + \left\{(1+a-b)C_{(1-a-b,1)} - q^{-1}(3+a-b)C_{(1-a-b,1)}\right\} G^{(s)}_{(a+1,a+1)}\end{aligned}$$

We rewrite this as

(5.122)

$$\begin{aligned}\mathcal{I}_2' = \left\{(\lambda_1 - a + 1)(\lambda_2 - b + 1) - q^{-1}(\lambda_1 - a)(\lambda_2 - b + 2)\right\}(\lambda_1 + \lambda_2 - a - b + 1) \\ \times \left(H^{(s)}_{(a+2,b+1)} + L_{(a+1,b)} + V_{(a+1,b+1)} - L_{(a+1,a+1)}\right) \\ - q^{-1}(\lambda_1 - a)^2 L_{(a+1,b-1)} \\ + \left\{(1+a-b) - q^{-1}(3+a-b)\right\} C_{(1-a-b,1)} \left(G^{(s)}_{(a+2,a+2)} + L_{(a+1,a+1)}\right).\end{aligned}$$

Combining (5.121) and (5.122) gives

$$\begin{aligned}\mathcal{I}' = (\lambda_2 - b + 1)^2 V_{(a,b)} \\ + \left\{\left(1-q^{-1}\right)(\lambda_2 - b + 2) + 2(\lambda_2 - b)\right\}(\lambda_2 - b + 2) V_{(a+1,b+1)} \\ - q^{-1}(\lambda_1 - a)^2 L_{(a+1,b-1)} \\ + \left\{\left(1-q^{-1}\right)(\lambda_1 - a + 1) - 2q^{-1}(\lambda_1 - a - 1)\right\}(\lambda_1 - a + 1) L_{(a+1,b)} \\ - (\lambda_1 - b + 2)^2 L_{(a+1,a+1)} \\ + \{(\lambda_1 - a)(\lambda_2 - b) + (\lambda_1 - a + 1)(\lambda_2 - b + 1)\}(\lambda_1 + \lambda_2 - a - b + 1) H^{(s)}_{(a+2,b+1)} \\ -q^{-1}\{(\lambda_1 - a - 1)(\lambda_2 - b + 1) + (\lambda_1 - a)(\lambda_2 - b + 2)\}(\lambda_1 + \lambda_2 - a - b + 1) H^{(s)}_{(a+2,b+1)} \\ + 2\left\{(a-b+1) - q^{-1}(a-b+3)\right\} C_{(1-a-b,1)} G^{(s)}_{(a+2,a+2)}.\end{aligned} \tag{5.123}$$

From Proposition 4.22, we have

$$C_s^{-1} \cdot \mathcal{B}^{(s)} =$$

$$2q^{-1} \cdot L_{(a+3,a+3)} - 2\left(1-q^{-1}\right) \cdot L_{(a+2,a+2)} - 2 \cdot L_{(a+1,a+1)} \tag{5.124}$$

$$+ 2 \cdot V_{(a,b)} + 2\left(1-q^{-1}\right) \cdot V_{(a+1,b)} - 2q^{-1} \cdot V_{(a+2,b)} \tag{5.125}$$

$$+ 2 \cdot L_{(a+2,b+1)} + 2\left(1-q^{-1}\right) \cdot L_{(a+2,b)} - 2q^{-1} \cdot L_{(a+2,b-1)} \tag{5.126}$$

$$+ P_{(a+1,b+1)} - P_{(a,b)} + q^{-1}\left(P_{(a+2,b)} - P_{(a+1,b-1)}\right). \tag{5.127}$$

With $X_{(c,d)}$ as in the previous section, note that (5.124) plus (5.125) is

$$X_{(a,b)} - q^{-1} X_{(a+1,b)}.$$

Hence by (5.116) the contribution to $T^{(s)}\left(C_s^{-1}\cdot\mathcal{B}^{(s)}\right)$ from (5.124) and (5.125) is

$$\begin{aligned}
&(\lambda_2-b+1)(\lambda_2-b+2)\left\{(2\lambda_1-2a+1)H^{(s)}_{(a+1,b)}+V_{(a,b)}\right\}\\
&\quad+\left\{2(a-b+2)^2-(\lambda_1-a+1)(\lambda_2-a)-(\lambda_1-a)(\lambda_2-a-1)\right\}\\
&\qquad\times(\lambda_1-\lambda_2+1)G^{(s)}_{(a+1,a+1)}\\
&\quad-q^{-1}(\lambda_2-b+1)(\lambda_2-b+2)\left\{(2\lambda_1-2a-1)H^{(s)}_{(a+2,b)}+V_{(a+1,b)}\right\}\\
&\quad-q^{-1}\left\{2(a-b+3)^2-(\lambda_1-a)(\lambda_2-a-1)-(\lambda_1-a-1)(\lambda_2-a-2)\right\}\\
&\qquad\times(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)}
\end{aligned}$$

We may rewrite this as

$$\begin{aligned}
&(\lambda_2-b+1)(\lambda_2-b+2)\\
&\times\left\{(2\lambda_1-2a+1)\left(H^{(s)}_{(a+2,b+1)}+L_{(a+1,b)}+V_{(a+1,b+1)}-L_{(a+1,a+1)}\right)+V_{(a,b)}\right\}\\
&+(\lambda_1-\lambda_2+1)\left\{2(a-b+2)^2-(\lambda_1-a+1)(\lambda_2-a)-(\lambda_1-a)(\lambda_2-a-1)\right\}\\
&\qquad\times\left(L_{(a+1,a+1)}+G^{(s)}_{(a+2,a+2)}\right)\\
&-q^{-1}(\lambda_2-b+1)(\lambda_2-b+2)\left\{(2\lambda_1-2a-1)\left(H^{(s)}_{(a+2,b+1)}+L_{(a+1,b)}\right)+V_{(a+1,b+1)}\right\}\\
&-q^{-1}\left\{2(a-b+3)^2-(\lambda_1-a)(\lambda_2-a-1)-(\lambda_1-a-1)(\lambda_2-a-2)\right\}\\
&\qquad\times(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)},
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&2\left\{(2\lambda_1-2a+1)-(2\lambda_1-2a-1)q^{-1}\right\}L_{(a+1,b)}\\
&\quad-(\lambda_1-b+1)(\lambda_1-b+2)L_{(a+1,a+1)}\\
&\quad+(\lambda_2-b+1)(\lambda_2-b+2)\left\{\left(3-q^{-1}\right)V_{(a+1,b+1)}+V_{(a,b)}\right\}\\
&\quad+(\lambda_2-b+1)(\lambda_2-b+2)\left\{(2\lambda_1-2a+1)\left(1-q^{-1}\right)+2q^{-1}\right\}H^{(s)}_{(a+2,b+1)}\\
&\quad+\Big[\left\{2(a-b+2)^2-(\lambda_1-a+1)(\lambda_2-a)-(\lambda_1-a)(\lambda_2-a-1)\right\}\\
&\quad-q^{-1}\left\{2(a-b+3)^2-(\lambda_1-a)(\lambda_2-a-1)-(\lambda_1-a-1)(\lambda_2-a-2)\right\}\Big]\\
&\qquad\times(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)}.
\end{aligned}\tag{5.128}$$

Next note that

$$\begin{aligned}
T^{(s)}\left(2L_{(a+2,d)}+2L_{(a+2,d+1)}\right)&=(\lambda_1-a-1)(\lambda_1-a)L_{(a+2,d)}\\
&\quad+(\lambda_1-a-1)(\lambda_1-a)(2\lambda_2-2d+1)H^{(s)}_{(a+2,d+1)}\\
&\quad+\{(\lambda_1-a-1)(2\lambda_2-2d+1)+(\lambda_2-a-1)(2a-2d+5)\}\\
&\qquad\times(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)}.
\end{aligned}$$

Thus the contribution to $T^{(s)}\left(C_s^{-1}\cdot\mathcal{B}^{(s)}\right)$ from (5.126) is

$$\begin{aligned}&(\lambda_1-a-1)(\lambda_1-a)\left(L_{(a+2,b)}-q^{-1}L_{(a+2,b-1)}\right)\\&+(\lambda_1-a-1)(\lambda_1-a)\left\{(2\lambda_2-2b+1)H^{(s)}_{(a+2,b+1)}-q^{-1}(2\lambda_2-2b+3)H^{(s)}_{(a+2,b)}\right\}\\&\qquad+\left[\{(\lambda_1-a-1)(2\lambda_2-2b+1)+(\lambda_2-a-1)(2a-2b+5)\}\right.\\&\qquad\left.-q^{-1}\{(\lambda_1-a-1)(2\lambda_2-2b+3)+(\lambda_2-a-1)(2a-2b+7)\}\right]\\&\qquad\qquad\times(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)}.\end{aligned}$$

We may rewrite this as

(5.129)
$$\begin{aligned}&(\lambda_1-a-1)(\lambda_1-a)\left(1-3q^{-1}\right)L_{(a+1,b)}-q^{-1}(\lambda_1-a-1)(\lambda_1-a)L_{(a+1,b-1)}\\&\quad+(\lambda_1-a-1)(\lambda_1-a)\left\{(2\lambda_2-2b+1)\left(1-q^{-1}\right)-2q^{-1}\right\}H^{(s)}_{(a+2,b+1)}\\&\qquad+\left[\{(\lambda_1-a-1)(2\lambda_2-2b+1)+(\lambda_2-a-1)(2a-2b+5)\}\right.\\&\qquad\left.-q^{-1}\{(\lambda_1-a-1)(2\lambda_2-2b+3)+(\lambda_2-a-1)(2a-2b+7)\}\right]\\&\qquad\qquad\times(\lambda_1-\lambda_2+1)G^{(s)}_{(a+2,a+2)}.\end{aligned}$$

Now note that

$$H^{(s)}_{(c,d)}=H^{(s)}_{(c+1,d+1)}+L_{(c,d)}+V_{(c,d+1)}-L_{(c,c)}$$

implies

$$\begin{aligned}T^{(s)}\left(P_{(c+1,d+1)}-P_{(c,d)}\right)=&-(\lambda_1-c+1)L_{(c,d)}-(\lambda_2-d+1)V_{(c,d+1)}\\&-(\lambda_1+\lambda_2-c-d+1)H^{(s)}_{(c+1,d+1)}.\end{aligned}$$

Hence the contribution from (5.127) is

$$\begin{aligned}&-(\lambda_1-a+1)L_{(a,b)}-q^{-1}(\lambda_1-a)L_{(a+1,b-1)}\\&\qquad-(\lambda_2-b+1)V_{(a,b+1)}-q^{-1}(\lambda_2-b+2)V_{(a+1,b)}\\&\quad-(\lambda_1+\lambda_2-a-b+1)H^{(s)}_{(a+1,b+1)}-q^{-1}(\lambda_1+\lambda_2-a-b+1)H^{(s)}_{(a+2,b)}.\end{aligned}$$

We can rewrite this as

$$\begin{aligned}&P_{(a,b)}-2q^{-1}P_{(a+1,b)}-(\lambda_1-a+1)L_{(a,b)}-q^{-1}(\lambda_1-a)L_{(a+1,b-1)}\\&\qquad-(\lambda_2-b+1)V_{(a,b)}-q^{-1}(\lambda_2-b+2)V_{(a+1,b+1)}\\&\qquad\qquad-(\lambda_1+\lambda_2-a-b+1)\\&\quad\times\left\{H^{(s)}_{(a+2,b+1)}+V_{(a+1,b+1)}-L_{(a+1,a+1)}+q^{-1}\left(H^{(s)}_{(a+2,b+1)}+L_{(a+2,b)}\right)\right\},\end{aligned}$$

or

(5.130)
$$\begin{aligned}&-q^{-1}(\lambda_1-a)L_{(a+1,b-1)}-\left(1+q^{-1}\right)(\lambda_1-a+1)L_{(a+1,b)}+(\lambda_1-b+2)L_{(a+1,a+1)}\\&\qquad-(\lambda_2-b+1)V_{(a,b)}-\left(1+q^{-1}\right)(\lambda_2-b+2)V_{(a+1,b+1)}\\&\qquad\qquad-(\lambda_1+\lambda_2-a-b+1)\left(1+q^{-1}\right)H^{(s)}_{(a+2,b+1)}.\end{aligned}$$

Summing up (5.128), (5.129) and (5.130) yields (5.123).

Thus we establish (5.83) in all cases.

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