Automorphic Representations
Fall 2011 Notes

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Preface

This notes are the sequel to my Modular Forms notes from Spring 2011.

Recall the basic setup. The upper half plane $\mathfrak{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ with the hyperbolic metric has as its orientation-preserving isometry group $\mathrm{PSL}_2(\mathbb{R})$, acting by linear fractional transformations. Let Γ be a congruence subgroup of $\mathrm{PSL}_2(\mathbb{R})$, such as $\mathrm{PSL}_2(\mathbb{Z})$. Then a modular form f of weight k for Γ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ which satisfies the transformation law

$$f\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right] = (cz+d)^k f(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ z \in \mathfrak{H}.$$

(It must also satisfy a holomorphy condition at the cusps.)

Since $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathfrak{H}$ and the stabilizer of the point $i \in \mathfrak{H}$ is

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\},$$

we may identify $\mathfrak{H}$ with the quotient $\mathrm{PSL}_2(\mathbb{R})/\mathrm{SO}(2)$ via the map $\mathrm{PSL}_2(\mathbb{R}) \to \mathfrak{H}$ given by

$$g \mapsto g \cdot i.$$

Therefore, we can think of f on $\mathrm{PSL}_2(\mathbb{R})$ which is right-invariant by $\mathrm{SO}(2)$ and has a left-transformation property by Γ.

Consequently, the function $\phi_f : \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{C}$ given by

$$\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (cz+d)^{-k} f\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is left-invariant under Γ, i.e., ϕ_f is a function of $\Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$. We call ϕ_f a *(classical) automorphic form.* Now ϕ_f is not right-invariant by $\mathrm{SO}(2)$, but one can check that it has a right-transformation property by $\mathrm{SO}(2)$. So the passage from modular forms to classical modular forms trades right-$\mathrm{SO}(2)$-invariance with left-Γ-invariance, as well as the (left) Γ transformation law for a (right) $\mathrm{SO}(2)$ transformation law.

For simplicity, say $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Then one can reformulate things adelically because

$$Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K \simeq \Gamma\backslash\mathrm{PSL}_2(\mathbb{R}).$$

Here Z denotes the center of GL(2), the *adèles* $\mathbb{A}$ are a certain subring of $\mathbb{R} \times \prod_p \mathbb{Q}_p$, and K is a certain nice subgroup of $\mathrm{GL}_2(\mathbb{A})$. (One could work with $\mathrm{PSL}_2(\mathbb{A})$ or $\mathrm{PGL}_2(\mathbb{A})$ instead, but it will be most convenient to work with $\mathrm{GL}_2(\mathbb{A})$.)

This means we can lift ϕ_f to a function of $\mathrm{GL}_2(\mathbb{A})$, where it is now called an *adelic automorphic form*. In fact, $\phi_f \in L^2(Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$. Now for any group G and subgroup Γ, G acts on $L^2(\Gamma\backslash G)$ by right translation, i.e.,

$$R(g)\phi(x) = \phi(xg), \quad x, g \in G.$$

Here R is called the *right regular representation* of G on $L^2(\Gamma\backslash G)$.

Consequently, ϕ_f generates a representation π of $\mathrm{GL}_2(\mathbb{A})$ on a subspace V of $L^2(Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}))$. Namely

$$V = \langle R(g)\phi_f \rangle$$

$$\pi(g)\phi = R(g)\phi \in V, \quad g \in G, \, \phi \in V.$$

In other words, π is the restriction of the right regular representation R to the space V spanned by the translates of ϕ_f.

While all of this may seem excessively complicated, there are two big advantages of this approach. First, it allows one to unify the notions of (elliptic) modular forms, Hilbert modular forms, Siegel modular forms, and various other generalizations, by viewing them as automorphic form or representations on $G(\mathbb{A})$ for an appropriate group G.

Secondly, the representation π has a decomposition $\pi = \otimes_p \pi_p \otimes \pi_\infty$ where each π_p is an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and π_∞ is a representation of $\mathrm{GL}_2(\mathbb{R})$. Consequently, this allows us to study modular forms via the representation theory of GL_2.

This subject has two parts: the local theory (e.g., the study of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$) and the global theory (the study of automorphic forms and representations on $\mathrm{GL}_2(\mathbb{A})$). While it is the global theory that has the direct applications to number theory, it is the local theory that carries the arithmetic information. Namely, the Fourier coefficients a_p for a modular form f are encoded in the local components π_p of the corresponding global automorphic representation π.

See my note "A brief over of modular and automorphic forms" for a somewhat broader and more detailed introduction than the above.

One of the main goals of the course is to understand, at least roughly, how the passage from modular forms to automorphic representations looks—in particular, given f, know what the local components π_p are. Consequently, most of the course will be devoted to understanding the representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Even so, we will not be able to prove everything, but will have to take some results for granted.

In the last part of the course, we will sketch the global theory and discuss applications of automorphic representations.

Let me emphasize that this is a very rich field, where many branches of mathematics come together. On the one hand, this is what makes it so interesting, but on the other, it means it requires a large amount of background material to understand. Fortunately, there are many problems you can work on with knowledge of only a small portion of this field. Therefore, approach this subject with much patience and good humour. Understand a little bit at a time. I hope that at the end, you will have some rough idea of the subject, and be able to start reading some of the literature on your own.