2 Smooth Representations

In this section we will introduce some basic notions and results on representations of certain topological groups, including $\mathrm{GL}_n(\mathbb{Q}_p)$. The reason for this is because (1) to study representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, we will need to look at representations of and restrictions to certain subgroups, and (2) not much is gained by restricting to $\mathrm{GL}_2(\mathbb{Q}_p)$ at this stage.

Much of this chapter is based on Fiona Murnaghan's notes "Representations of reductive p-adic groups." The notes by Prasad and Raghuram entitled "Representation theory of $\mathrm{GL}(n)$ over non-archimedean local fields" contain similar material,

2.1 Some topological groups

Definition 2.1.1. *A* **topological group** *G is a group endowed with a topology such that the map $G \times G \to G$ given by $(x, y) \mapsto xy^{-1}$ is continuous.*

Here, of course, $G \times G$ is given the product topology. An alternative way to define topological group is to require that the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous.

Proposition 2.1.2. *Let $n \in \mathbb{N}$. Let $F = \mathbb{Q}_p$ or $\mathbb{R}$, or more generally, a ring which is a metric space. The the additive group of $n \times n$ matrices $M_n(F)$ over F is a topological group.*

Here, viewing $M_n(F) \simeq F^{n^2}$, we give $M_n(F)$ the product topology. Note that this is also a metric space.

Proof. Let U be an open set in $M_n(F)$, and let $V = \{(A, B) \in M_n(F) \times M_n(F) : A - B \in U\}$. We want to show V is open. Let $(A, B) \in V$. It suffices to show there there are open neighborhoods V_1 of A and V_2 of B such that $V_1 \times V_2 \subseteq V$.

Since U is open, there exist ϵ such that the ball $B_\epsilon(A - B) \subseteq U$. Taking $V_1 = B_{\epsilon/2}(A)$ and $V_2 = B_{\epsilon/2}(B)$ gives the claim. □

Proposition 2.1.3. *Let $F = \mathbb{Q}_p$ or R. Then $\mathrm{GL}_n(F) = \{A \in M_n(F) : \det(A) \neq 0\}$ is a topological group.*

Here we give $\mathrm{GL}_n(F)$ the subspace topology of $M_n(F)$.

Proof. First note that the coordinate maps $A = (a_{ij}) \mapsto a_{i_0,j_0}$ are continuous. Since multiplication and addition are continuous maps from $F \times F \to F$, a composition of these maps shows any polynomial in the coefficients of A is a continuous map.

Write $B = (b_{ij})$ and $AB^{-1} = C = (c_{ij})$. Each c_{ij} is a (well-defined) rational function of the a_{ij}'s and b_{ij}'s, and therefore $(A, B) \mapsto c_{ij}$ for any ij is a continuous map $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F) \to F$. Consequently $(A, B) \mapsto AB^{-1}$ is a continuous map $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F) \to M_n(F)$, whose image lies in the open set $\mathrm{GL}_n(F)$. Therefore the corresponding map $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)$ is continuous. □

Proposition 2.1.4. *Let G be a topological group and H a subgroup. Then H is also a topological group.*

Proof. Let U is a open set in G. Let V be the (open) preimage in $G \times G$ under the map $f(x, y) = xy^{-1}$. Then $V \cap H \times H$ is the preimage of $U \cap H$ under the restriction of f to $f_H : H \times H \to H$. Consequently f_H is continuous. □

Example 2.1.5. *Let $F = \mathbb{Q}_p$ or $\mathbb{R}$. Since* $\det : M_n(F) \to F$ *is given by a polynomial in the coordinates of $M_n(F)$, it is a continuous map. Consequently* $\mathrm{SL}_n(F) = \{A \in \mathrm{GL}_n(F) : \det(A) = 1\}$ *is an closed subgroup of* $\mathrm{GL}_n(F)$.

Example 2.1.6. *Let $F = \mathbb{Q}_p$ or $\mathbb{R}$ and $G = \mathrm{GL}_n(F)$. Let P denote the set of upper triangular matrices in G (the standard Borel subgroup), A the group of diagonal matrices in G and N the set of upper triangular matrices in G with 1's on the diagonal (so $P = AN$). Then P, A, N are closed subgroups of G, since they are defined by equations on the matrix coefficients.*

More generally, one can consider subgroups of $\mathrm{GL}_n(F)$ defined by polynomial equations in the coefficients. Such groups are the prototypical examples of what are called *algebraic groups*. The most famous of these are the **classical groups**. These groups come in 4 types: linear, orthogonal, symplectic and unitary.

The linear groups are $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$, along with their projective versions: $\mathrm{PGL}_n(F) = \mathrm{GL}_n(F)/Z(\mathrm{GL}_n(F))$ and $\mathrm{PSL}_n(F) = \mathrm{SL}_n(F)/Z(\mathrm{SL}_n(F))$. Here $Z(G)$ denotes the center of G. Note $\mathrm{PSL}_n(F) \simeq \mathrm{PGL}_n(F)$ if F is algebraically closed, but not in general.

We will not say precisely what orthogonal, symplectic and unitary mean in general, but essentially they are subgroups of linear groups which preserve symmetric bilinear, skew-symmetric bilinear, and Hermitian forms, respectively. In any dimension, there are various (non-isomorphic) examples of each kind of group according to the classification of the appropriate types of forms on F^n. Here are a couple examples.

Example 2.1.7. *The* **special orthogonal group**

$$\mathrm{SO}_n(F) = \left\{ g \in \mathrm{SL}_n(F) : {}^t g g = I_n \right\}.$$

The **symplectic group**

$$\mathrm{Sp}_{2n}(F) = \left\{ g \in \mathrm{SL}_{2n}(F) : {}^t g J g = J \right\},$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. *Some authors denote* Sp_{2n} *by* Sp_n.

In Oklahoma, there is another algebraic group you should know, though it does not fall under the heading of "classical groups."

Example 2.1.8. *The* **symplectic similitude group**

$$\mathrm{GSp}_4(F) = \left\{ g \in \mathrm{GL}_4(F) : {}^t g \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \textit{ for some } \lambda(g) \in F \right\}.$$

This of course contains both $\mathrm{Sp}_4(F)$ *and the group $F^\times$ embedded diagonally in* $\mathrm{GL}_4(F)$, *but in general more than this.*

The most natural generalization of classical (elliptic) modular forms to higher dimensions is the notion of *Siegel modular forms*, which correspond to automorphic forms or representations on the symplectic groups Sp_{2n} and GSp_{2n}.

In low rank (when n is small) there are some coincidences among these algebraic groups, called *accidental isomorphisms*. We just mention a couple. The group $\mathrm{Sp}_2(F) = \mathrm{SL}_2(F)$ for any field F. Over $\mathbb{C}$, we have $\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{PGL}_2(\mathbb{C}) \simeq \mathrm{SO}_3(\mathbb{C})$ and $\mathrm{PGSp}_4(\mathbb{C}) := \mathrm{GSp}_4(\mathbb{C})/Z(\mathrm{GSp}_4(\mathbb{C})) \simeq \mathrm{SO}_5(\mathbb{C})$. Over other fields, these latter isomorphisms aren't exactly true, but these groups are still closely related, a fact which is often exploited in number theory and automorphic representations.

Example 2.1.9. *Let $F = \mathbb{Q}_p$ or $\mathbb{R}$, and E/F a quadratic field extension. One can define a topology on E by realizing $E \simeq F^2$ as a vector space over F. Then $M_2(E)$ and $\mathrm{GL}_2(E)$ are topological groups. Let σ be the nontrivial Galois automorphism of E/F and $\epsilon \in F^\times$. We define a* **quaternion algebra** *over F by*

$$D(F) = \left\{ \begin{pmatrix} a & b\epsilon \\ b^\sigma & a^\sigma \end{pmatrix} \in M_2(E) : a, b, \in E \right\}.$$

This is a 4-dimensional algebra over F, whose center is isomorphic to $F^\times$, and it will be a division algebra if and only if ϵ is not a square in F. If $F = \mathbb{R}$, then either $D(\mathbb{R}) \simeq M_2(\mathbb{R})$ if $\epsilon > 0$ or $D(\mathbb{R}) \simeq \mathbb{H}$, Hamilton's quaternions, if $\epsilon < 0$.

The multiplicative group of $D(F)$ is

$$D^\times(F) = \left\{ \begin{pmatrix} a & b\epsilon \\ b^\sigma & a^\sigma \end{pmatrix} \in \mathrm{GL}_2(E) : a, b, \in E \right\}.$$

This construction of quaternion algebras works more generally (for instance, over $\mathbb{Q}$). Quaternion algebras have many applications to number theory and automorphic forms. One amazing theorem is the local Jacquet–Langlands correspondence, which provides a correspondence between (finite-dimensional) representations of $D^\times(\mathbb{Q}_p)$ and the (infinite-dimensional) "discrete series" representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

For fun, here are a couple simple exercises about subgroup and quotients of topological groups.

Exercise 2.1.10. *Let H be a subgroup of a topological group G. Show the closure of H is also a subgroup.*

Exercise 2.1.11. *Let H be an open subgroup of a topological groups G. Show H is also closed.*

Given a subgroup H of G, one often considers the quotient space G/H (with the quotient topology).

Exercise 2.1.12. *Let H be a subgroup of G. Then G/H the projection map $G \to G/H$ is open. If H is closed, then the singleton sets of G/H are closed.*

Exercise 2.1.13. *Let H be a normal subgroup of G. Then G/H is a topological group.*

2.2 l-groups

The topology of $\mathbb{Q}_p$, and therefore that of $\mathrm{GL}_2(\mathbb{Q}_p)$, is very special. In this section we define l-groups, which will be topological groups with similar topological properties.

Definition 2.2.1. *Let X be a topological space. We say $Y \subseteq X$ is* **connected** *if it is not a disjoint union of proper open subsets of Y (in the subspace topology). We say X is* **totally disconnected** *if no nonempty subsets are connected except singleton sets.*

Lemma 2.2.2. *$\mathbb{Q}_p$ is totally disconnected.*

Proof. Suppose $X \subseteq \mathbb{Q}_p$ is connected and contains more than one element. Let $x \in X$. For some r, the ball $B_r(x)$ of radius r about x does not contain X. Since $B_r(x)$ is both open and closed, we can partition X into 2 nonempty disjoint open subsets $U = B_r(x) \cap X$ and $V = B_r(x)^c \cap X$. $\square$

Exercise 2.2.3. *Suppose X and Y are totally disconnected. Show $X \times Y$ is also.*

Exercise 2.2.4. *Let Y be a subspace of X. If X is totally disconnected, so is Y.*

Corollary 2.2.5. *The groups $M_n(\mathbb{Q}_p)$ and $\mathrm{GL}_n(\mathbb{Q}_p)$, as well as their subgroups, are totally disconnected.*

To get a better sense of the topology of these groups, let's determine bases for the topologies of $M_n(\mathbb{Q}_p)$ and $\mathrm{GL}_n(\mathbb{Q}_p)$.

First, note that if G is a topological group, then for any $g \in G$, the map given by left multiplication by g is a homeomorphism. Consequently, to determine a basis of open sets for G, it suffices to determine a basis of open neighborhoods around the identity.

Now let's determine a basis of open neighborhoods around 0 for $M_n(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{n^2}$. Recall a basis of open (and also closed) neighborhoods around 0 in $\mathbb{Q}_p$ are the sets $p^j\mathbb{Z}_p = \overline{B}_{p^{-j}}(0)$ for $j \in \mathbb{Z}$. Consequently, a basis of open neighborhoods of 0 in $M_n(\mathbb{Q}_p)$ will be sets of the form $p^{j_1}\mathbb{Z}_p \times p^{j_2}\mathbb{Z}_p \times \cdots \times p^{j_{n^2}}Z_p$ for $j_1, \ldots, j_{n^2} \in \mathbb{Z}$. It is clear that each set of this form is both contained in and contains a set of the form $p^j M_n(\mathbb{Z}_p) = p^j\mathbb{Z}_p \times \cdots p^j\mathbb{Z}_p$. Hence a simpler basis of open neighborhoods of any $A \in M_n(\mathbb{Q}_p)$ is $\left\{A + p^j M_n(\mathbb{Z}_p) : j \in \mathbb{Z}\right\}$. Note each of these sets are also closed.

Next, let's determine a basis of open neighborhoods around 1 for $\mathrm{GL}_n(\mathbb{Q}_p)$. By the above remarks, one basis is

$$\left\{(1 + p^j M_n(\mathbb{Z}_p)) \cap \mathrm{GL}_n(\mathbb{Q}_p) : j \in \mathbb{Z}\right\}.$$

In fact, it suffices to restrict j to be sufficiently large, say $j \geq 1$. For $j \geq 1$, put

$$K_j = (1 + p^j M_n(\mathbb{Z}_p)) \cap \mathrm{GL}_n(\mathbb{Q}_p).$$

Then $\{K_j\}$ is a basis of open (and closed) neighborhoods of 1 in $\mathrm{GL}_n(\mathbb{Q}_p)$

It is a general fact that if a space has a basis of open neighborhoods which are also closed, then the space is totally disconnected.

In fact, these sets K_j have more structure.

Exercise 2.2.6. *Show K_j is an open subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. Further show, that*

$$K_j \subseteq K_0 := \mathrm{GL}_2(\mathbb{Z}_p) = \{g \in \mathrm{GL}_n(\mathbb{Z}_p) \cap M_n(\mathbb{Z}_p) : |\det g|_p = 1.\}.$$

Note that K_0 is an open compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. To see that it is open (and closed), note that $M_n(\mathbb{Z}_p)$ is open (and closed) in $M_n(\mathbb{Q}_p)$, so its restriction to $\mathrm{GL}_n(\mathbb{Q}_p)$ is also. Next observe that the preimage of $(1/p, p)$ under the map $\det : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}^\times$ is the open (and closed) set $\{g \in GL_n(\mathbb{Z}_p) : |\det g|_p = 1\}$. Thus K_0 is open (and closed).

To see that K_0 is compact, observe it is the restriction of the compact subgroup $M_n(\mathbb{Z}_p)$ of $M_n(\mathbb{Q}_p)$ to the closed subset $\{g \in M_n(\mathbb{Q}_p) : |\det g|_p = 1\}$. One can similarly see each K_j is compact, and we have a family of inclusions

$$\cdots \subset K_2 \subset K_1 \subset K_0.$$

It turns out that K_0 is a maximal compact open subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$.

Maximal compact open subgroups play an important role in our theory. In the case of $\mathrm{GL}_n(\mathbb{Q}_p)$, there is only one maximal compact open subgroup, up to conjugacy, but this is not true for other algebraic groups. This is one feature that makes the theory simpler for $\mathrm{GL}_n(\mathbb{Q}_p)$.

Definition 2.2.7. *An* l**-group** *is a locally compact totally disconnected Hausdorff topological group.*

Example 2.2.8. *Since* $\mathbb{Q}_p$ *is locally compact Hausdorff,* $M_n(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{n^2}$ *is also. Since an open subset of a locally compact space Hausdorff is locally compact Hausdorff, so is* $\mathrm{GL}_n(\mathbb{Q}_p)$*. Consequently,* $M_n(\mathbb{Q}_p)$ *and* $\mathrm{GL}_n(\mathbb{Q}_p)$ *are* l*-groups. Note that* $M_n(\mathbb{R})$ *and* $\mathrm{GL}_n(\mathbb{R})$ *are not, since* $\mathbb{R}$ *is not totally disconnected.*

Exercise 2.2.9. *Let* $G = \mathrm{GL}_2(\mathbb{Q}_p)$*, and consider the subgroups* $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$*,* $A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ *and* $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$*. Show* P*,* A *and* N *are* l*-groups.*

In fact, all of the matrix groups we considered last section, taken over $F = \mathbb{Q}_p$, are l-groups.

2.3 Smooth representations

The representation $\pi : \mathrm{GL}_2(\mathbb{R}) \to \mathrm{GL}_2(\mathbb{C})$ given by inclusion is a perfectly reasonable one, in that it should be continuous and smooth (as a map of real manifolds). That is, varying $g \in \mathrm{GL}_2(\mathbb{R})$ continuously or smoothly does the same thing to $\pi(g)$. A less trivial example is the symmetric square representation $\mathrm{Sym}^2 : \mathrm{GL}_2(F) \to \mathrm{GL}_3(F)$ given in Example 1.2.14.

On the other hand, one can take the metric completion $\mathbb{C}_p$ of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and this is abstractly isomorphic to $\mathbb{C}$ as a field. Hence (with the axiom of choice), one has an injective ring homomorphism $\iota : \mathbb{Q}_p \to \mathbb{C}$. This extends to an homomorphism $\pi : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathrm{GL}_n(\mathbb{C})$, which is algebraically a representation, but not one with any utility, for it completely disregards the metric/topological structures of both $\mathbb{Q}_p$ and $\mathbb{C}$.

Therefore, when dealing with a topological group G, it makes sense to consider **continuous representations.** Namely, suppose V is a finite-dimensional vector space over C. As above, one can give $\mathrm{GL}(V)$ which is isomorphic to $\mathrm{GL}_n(\mathbb{C})$ for some n, a topology and consider continuous homomorphisms $\pi : G \to \mathrm{GL}(V)$.

If G is not compact, then there will be infinite-dimensional irreducible representations of G, so one needs a way to put a topology on $\mathrm{GL}(V)$ for infinite-dimensional V (which one often takes to be Hilbert or at least Banach, and replaces $\mathrm{GL}(V)$ with the group of invertible bounded linear operators). These ideas are at the core of functional analysis.

However, for l-groups, it turns out that the stronger notion of *smooth representations* is easier to define. While it is easy to see what it means for a function of $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{GL}_2(\mathbb{C})$ to be smooth—namely, being smooth (real or complex) in each coordinate—it is less obvious what smoothness should mean for a function of $\mathbb{Q}_p$, let alone $\mathrm{GL}_2(\mathbb{Q}_p)$. Though not obvious, the correct definition is quite simple.

For the rest of this chapter, unless otherwise stated, **we assume G is an l-group.**

Definition 2.3.1. *Let* $G = \mathbb{Q}_p$*, or more generally any* l*-group. Then a function* $f : G \to \mathbb{C}^n$ *is* **smooth** *if it is locally constant, i.e., given any* $x \in G$*, there exists an open neighborhood* U *of* x *such that* $f(y) = f(x)$ *for all* $y \in U$*.*

From the point of view of functional or harmonic analysis, this really is the right analogue of smooth functions on $\mathbb{R}$ or $\mathbb{C}$. While we will not discuss this now, here is another characterization of smoothness.

Proposition 2.3.2. *Let G be an l-group, and $f : G \to \mathbb{C}^n$. Then f is smooth if and only if for any subset $U \subseteq \mathbb{C}^n$, the preimage $f^{-1}(U)$ is open in G.*

Proof. First suppose f is smooth. Consider the case when $U = U_z = \{z\}$ is a singleton set, let $V = f^{-1}(U)$ and take $x \in V$. Then there is an open neighborhood V_x of x such that $f(y) = a$ for all $y \in V_x$. By definition, $V_x \subseteq V$, so V is open.

For an arbitrary (nonempty) U, $f^{-1}(U)$ is a union of preimages $f^{-1}(U_a)$ of singleton sets, which we now know are open.

Now suppose $f^{-1}(U)$ is open for any $U \subseteq \mathbb{C}^n$. In particular, $f^{-1}(U)$ is open when $U = \{z\}$ is a singleton set. This means if $x \in G$ such that $f(z) = x$, then $f^{-1}(U)$ is an open neighborhood of x on which f is constant. Hence f is smooth. □

In particular, this means f is continuous, and we see that being locally constant is much stronger than just being continuous. Of course, on connected sets, such as $\mathbb{R}$ or $\mathbb{C}$, locally constant just means constant. However, on l-groups, there are a plethora of useful locally constant functions.

Example 2.3.3. *The p-adic absolute value $|\cdot|_p : \mathbb{Q}_p^\times \to \mathbb{R}$ is smooth. To see this, consider any $x \in \mathbb{Q}_p$. Write $|x|_p = p^{-n}$ for some $n \in \mathbb{Z}$. Then the preimage of p^{-n} under the absolute value is the open (and closed) set $p^n\mathbb{Z}_p^\times$.*

Note that $|\cdot|_p$ is not smooth on $\mathbb{Q}_p$ because the preimage of 0 is just a single point, i.e., there is no open neighborhood of 0 on which the absolute value is constant. Just to remark on the analogy with the reals, the usual absolute value on $\mathbb{R}$ is smooth (infinitely differentiable) when restricted to $\mathbb{R}^\times$ but of course not at 0.

Exercise 2.3.4. *Show the map $g \mapsto |\det g|_p : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{R}$ is smooth.*

Identifying $M_m(\mathbb{C})$ with $\mathbb{C}^{m^2}$, we have a notion of $f : G \to M_m(\mathbb{C})$ being smooth, where G is an l-group. It is easy to see that $f : G \to M_m(\mathbb{C})$ being smooth is equivalent to each coordinate function being smooth. Given a finite-dimensional vector space V, we can identify $\mathrm{GL}(V)$ with a (topological) subspace of some $M_m(\mathbb{C})$. Hence we may say $f : G \to \mathrm{GL}(V)$ is a smooth function if it is locally constant.

Thus, at least when V is a finite-dimensional vector space over $\mathbb{C}$, the notion of a smooth representation $\pi : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathrm{GL}(V)$ of an l-group should mean that f is a homomorphism which is smooth, or equivalently each coordinate function is smooth. There is another way to characterize this condition, which will be more useful for us.

Definition 2.3.5. *Let V be a complex vector space. We say a representation $\pi : G \to \mathrm{GL}(V)$ is* **smooth** *if*

$$Stab_G(v) := \{g \in G : \pi(g)v = v\}$$

is an open subgroup of G for all $v \in V$.

Recall that by Exercise 2.1.11, open subgroups of topological groups are also closed.

Definition 2.3.6. *Let $\pi : G \to \mathrm{GL}(V)$ be a representation of G on a Hilbert space V with inner product $\langle \cdot, \cdot \rangle$. A* **matrix coefficient** *of π is a function $f : G \to \mathbb{C}$ of the form*

$$f(g) = \langle \pi(g)v, v' \rangle$$

for some fixed $v, v' \in V$.

Example 2.3.7. *Let $V = \mathbb{C}^n$ with the standard inner product and $e_1, \cdots, e_n$ the standard basis. For a representation π of G on V, and $g \in G$, let $A = (a_{ij}) = \pi(g)$. Then the matrix coefficient*

$$f_{ij}(g) = \langle \pi(g)e_i, e_j \rangle$$

is simply given by

$$f_{ij}(g) = a_{ji}.$$

Hence this definition of matrix coefficients generalizes the notion for finite-dimensional representations.

Proposition 2.3.8. *Let π be a smooth representation of G on a Hilbert space V, and $f : G \to \mathbb{C}$ be a matrix coefficient. Then f is a smooth function of G.*

Proof. Write $f(g) = \langle \pi(g)v, v' \rangle$ for some $v, v' \in V$. Fix $g \in G$ and let $K = Stab_G(v)$. Then for any $k \in K$, we have

$$f(gk) = \langle \pi(gk)v, v' \rangle = \langle \pi(g)\pi(k)v, v' \rangle = \langle \pi(g)v, v' \rangle = f(g).$$

Since $1 \in K$, this means gK is an open neighborhood of g on which f is locally constant, i.e., smooth. □

This makes our definition of smooth representation seem more reasonable. In particular, it shows that if V is finite dimensional, the smooth representation π is a smooth (locally constant) function from G into $\mathrm{GL}(V)$. In fact these criteria are equivalent, as the following exercise shows.

Exercise 2.3.9. *Let π be a representation of G on $V = \mathbb{C}^n$. Suppose every matrix coefficient of π is a smooth function. Show π is a smooth representation.*

This is in fact true when V is an arbitrary Hilbert space.