YEAR MONTH DAY

1.

(a) Observation equation

: $y_t = y_t^* + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$

Transition equation

: $$\begin{bmatrix} y_{t+1}^* \\ \phi_1 y_t^* + \theta \zeta_{t+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & 1 \\ \phi_1 & 0 \end{bmatrix} \begin{bmatrix} y_t^* \\ \phi_1 y_{t-1}^* + \theta \cdot \zeta_t \end{bmatrix} + \begin{bmatrix} 1 \\ \theta \end{bmatrix} \zeta_{t+1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mu$$

where $\zeta_t \sim N(0, \sigma_\zeta^2)$

$y_1^* \sim N(a_1, P_1)$.

(b) We can write the model as $y_t = I y_t^* + \varepsilon_t$,

where I is a identity matrix.

Assume that a_1, P_1 are known, from $y_1^* \sim N(a_1, P_1)$

and, let $Y_t = (y_1, \cdots, y_{t-1})'$.

We need to obtain $a_{t|t} = E(y_t^* | Y_t)$, $P_{t|t} = Var(y_t^* | Y_t)$

$a_{t+1} = E(y_{t+1}^* | Y_t)$, $P_{t+1} = Var(y_{t+1}^* | Y_t)$.

Assume that $y_t^* | Y_t \sim N(a_{t|t}, P_{t|t})$

$y_{t+1}^* | Y_t \sim N(a_{t+1}, P_{t+1})$.

Let $v_t = y_t - E(y_t | Y_{t-1}) = y_t - I a_t = y_t - a_t$.

When Y_{t-1} and v_t are fixed, Y_t is also fixed.

Thus, $a_{t|t} = E(y_t^* | Y_t) = E(y_t^* | Y_{t-1}, v_t)$

$a_{t+1} = E(y_{t+1}^* | Y_t) = E(y_{t+1}^* | Y_{t-1}, v_t)$

Lemma

: Suppose that $\begin{pmatrix} x \\ y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}' & \Sigma_{yy} \end{pmatrix}\right)$

Then, $x \mid y \sim N(\mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{xy}')$

Taking x and y as y_t^* and v_t, we can obtain

$$a_{t|t} = E(y_t^* \mid Y_t) = E(y_t^* \mid Y_{t-1}, v_t)$$

$$= E(y_t^* \mid Y_{t-1}) + Cov(y_t^*, v_t \mid Y_{t-1}) Var(v_t \mid Y_{t-1})^{-1} v_t.$$

and, $Cov(y_t^*, v_t \mid Y_{t-1}) = E(y_t^*(I y_t^* + \varepsilon_t - I a_t)' \mid Y_{t-1})$

$$= E(y_t^*(y_t^* - a_t)' I \mid Y_{t-1}) = P_t I' = P_t.$$

$$Var(v_t \mid Y_{t-1}) = Var(I y_t^* + \varepsilon_t - I a_t \mid Y_{t-1})$$

$$= I P_t I' + \sigma_\varepsilon^2 = P_t + \sigma_\varepsilon^2 = F_t, \text{ say.}$$

Thus, $a_{t|t} = a_t + P_t F_t^{-1} v_t$.

Using Lemma, we can obtain

$$P_{t|t} = Var(y_t^* \mid Y_t) = Var(y_t^* \mid Y_{t-1}, v_t).$$

$$= Var(y_t^* \mid Y_{t-1}) - Cov(y_t^*, v_t \mid Y_{t-1}) Var(v_t \mid Y_{t-1})^{-1} Cov(y_t^*, v_t \mid Y_{t-1})'.$$

$$= P_t - P_t F_t^{-1} P_t'$$

Now, we can develop recursion for a_{t+1} and P_{t+1}.

$$a_{t+1} = E(\phi_1 y_t^* + \theta z_t \mid Y_t) = \phi_1 a_{t|t}.$$

$$P_{t+1} = Var(\phi_1 y_t^* + \theta z_t \mid Y_t) = \phi_1 Var(y_t^* \mid Y_t)\phi_1' + \theta\sigma_z^2\theta'$$

$$= \phi_1 P_{t|t}\phi_1' + \theta\sigma_z^2\theta'$$

And,

$$a_{t+1} = \phi_1(a_t + P_t F_t^{-1} v_t) = \phi_1 a_t + K_t v_t$$

where $K_t = \phi_1 P_t I' F_t^{-1} = \phi_1 P_t F_t^{-1}$

$$P_{t+1} = \Phi_1 (P_t - P_t F_t^{-1} P_t') \Phi_1' + \theta \sigma_\eta^2 \theta'$$
$$= \Phi_1 P_t (\Phi_1 - K_t)' + \theta \sigma_\eta^2 \theta'$$

State estimation x_t can be defined as
$x_t = y_t^* - a_t$ with $Var(x_t) = P_t$.
Then,
$$v_t = y_t - a_t = I y_t^* + \varepsilon_t - a_t = y_t^* + \varepsilon_t - a_t$$
$$= x_t + \varepsilon_t$$
Thus, $x_{t+1} = y_{t+1}^* - a_{t+1} = \Phi_1 y_t^* + \theta \eta_t - \Phi_1 a_t - K_t v_t$
$$= \Phi_1 y_t^* + \theta \eta_t - K_t x_t - K_t \varepsilon_t$$
$$= L_t x_t + \theta \eta_t - K_t \varepsilon_t$$
where $L_t = \Phi_1 - K_t$

In summary,
$$v_t = y_t - a_t$$
$$F_t = P_t + \sigma_\varepsilon^2$$
$$a_{t|t} = a_t + P_t F_t^{-1} v_t$$
$$P_{t|t} = P_t - P_t F_t^{-1} P_t'$$
$$a_{t+1} = \Phi_1 a_t + K_t v_t$$
$$P_{t+1} = \Phi_1 P_t (\Phi_1 - K_t)' + \theta \sigma_\varepsilon^2 \theta'$$
$$x_{t+1} = L_t x_t + \theta \eta_t - K_t \varepsilon_t.$$

(c)

We need to obtain $\hat{y}_t^* = E(y_t^* \mid Y_n)$

Let $v_{t:n} = (v_t', \cdots, v_n')'$, Y_n is fixed when Y_{t-1} and $v_{t:n}$ are fixed.

Using Lemma from (b), $\hat{y}_t^* = E(y_t^* \mid Y_n) = E(y_t^* \mid Y_{t-1}, v_{t:n})$

$= a_t = \sum_{j=t}^{n} Cov(y_t^*, v_j) F_j^{-1} v_j$

where $F_j = Var(v_j, Y_{t-1})$

Then, $Cov(y_t^*, v_j \mid Y_{t-1}) = E(y_t^* v_j' \mid Y_{t-1}) = E[y_t^* (x_j + \varepsilon_j)' \mid Y_{t-1}]$

$= E(y_t^* x_j' \mid Y_{t-1})$

And,

$E(y_t^* x_t' \mid Y_{t-1}) = E(y_t^* (y_t^* a_t)' \mid Y_{t-1}) = P_t$.

$E(y_t^* x_{t+1}' \mid Y_{t-1}) = E[y_t^* (L_t x_t + \theta \zeta_t - k_t \varepsilon_t)' \mid Y_{t-1}] = P_t L_t'$

$E(y_t^* x_{t+2}' \mid Y_{t-1}) = P_t L_t' L_{t+1}'$

$\vdots$

$E(y_t^* x_n' \mid Y_{t-1}) = P_t L_t' \cdots L_{n-1}' = Cov(y_t^*, v_j \mid Y_{t-1})$

Then, we can write $\hat{y}_t^* = a_t + P_t r_{t-1}$

where $r_{n-1} = F_n^{-1} v_n$,

$r_{t-1} = F_t^{-1} v_t + L_t' F_{t+1}^{-1} v_{t+1} + \cdots + L_t' L_{t+1}' \cdots L_{n-1}' F_n^{-1} v_n$

$\{r_t\}$ satisfies the backward recursion

$r_{t-1} = F_t^{-1} v_t + L_t' r_t$ with $r_n = 0$

Applying Lemma to the conditional joint distribution of $y_t^*, v_{t:n}$ given Y_{t-1},

$$V_t = Var(y_t^* \mid Y_{t-1}, v_{t:n}) = P_t - \sum_{j=t}^{n} cov(y_t^*, v_j \mid Y_{t-1}) F_j^{-1} cov(y_t^* v_j \mid Y_{t-1})'$$

$$= P_t - P_t N_{t-1} P_t$$

where $N_{t-1} = F_t^{-1} + L_t' F_{t+1}^{-1} L_t + \cdots + L_t' L_{t+1}' \cdots L_{n-1}' F_n^{-1} L_{n-1} \cdots L_t$.

$\{N_t\}$ satisfies the recursion

$$N_{t-1} = F_t^{-1} + L_t' N_t L_t \quad \text{with } N_n = 0$$

In Summary,

$$r_{t-1} = F_t^{-1} v_t + L_t' r_t$$

$$N_{t-1} = F_t^{-1} + L_t' N_t L_t$$

$$\hat{y}_t^* = a_t + P_t r_{t-1}$$

$$V_t = P_t - P_t N_{t-1} P_t$$

with $r_n = N_n = 0$