Maximum Flow
Ch 26 CLRS

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Flow networks - overview

- Flow networks model the movement of a material from a source **s**, to a sink **t**
- Each edge in the network has a total **capacity** at which material can move
- As well as a current **flow** rate at which the material is moving
- Material is not allowed to accumulate at any of the vertices
    - **Flow conservation** - the rate at which the material enters the vertex should be the rate at which it leaves the vertex
- The goal of maximum flow problems is to determine the largest amount of material we can move through the network while abiding network capacity

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Flow networks - requirements

- For a flow network $G = (V, E)$
  - Directed graph
    - With capacity function c
  - Each edge $c(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$
  - No reverse edges
    - If E has an edge (u, v), there is no edge (v, u)
  - If $(u, v) \notin E$ then $c(u, v) = 0$
  - No self loops

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Flow networks - requirements

- For a flow network $G = (V, E)$
  - For each vertex $v \in V$ the follow network should have a path s $\rightsquigarrow$ v $\rightsquigarrow$ t
  - Each vertex, other than *s* has one entering edge
    - Therefore $|E| \geq |V| - 1$
- A flow in G is a real-valued function $f : V \times V \to \mathbb{R}$
  - **Capacity constraint:** For all $u, v \in V$, we require $0 \leq f(u, v) \leq c(u, v)$
  - **Flow conservation:** For all $u \in V - \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) .$$

When $(u, v) \notin E$, there can be no flow from u to v, and $f(u, v) = 0$.

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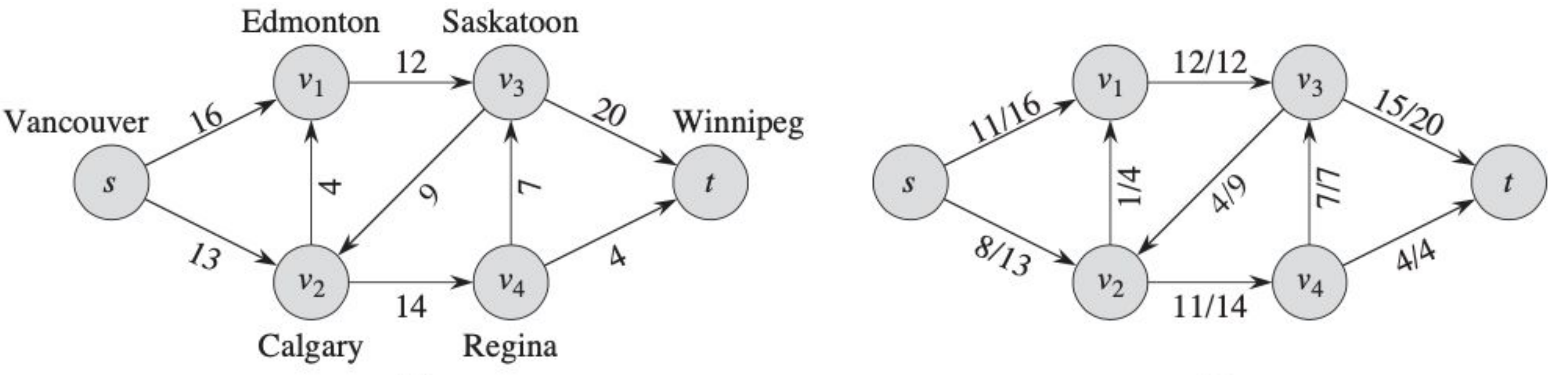
Flow networks



(a) (b)

- The nonnegative quantity of function *f(u, v)*, is a value | *f* | of a flow *f*

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) ,$$

  - In above e.g. flow *f* in *G* with value | *f* | = 19
  - Flow out of source should be 0, except when using residuals

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Residual networks

- For a flow network G and a flow f, the residual network $\boldsymbol{G_f}$ is a graph where edges are the remaining flow before exceeding capacity
- An edge has a residual capacity if
  - $c_f(u, v) = c(u, v) - f(u, v)$
- Only edges that can admit more flow are in G_f
  - Edges where the flow equals capacity are not in G_f and have $c_f(u, v) = 0$
- Residual G_f network may have edges that are not in G
  - To increase the total flow, may have to decrease flow on an edge
    - $c_f(v, u) = f(u, v)$

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Augmenting paths

- For a flow network *G* = (*V*, *E*) and a flow *f*
  - We can **increase** the **flow** on *(u, v)* by ***f'(u, v)*** but **decrease** it by ***f'(v, u)***
  - This is equivalent to decreasing flow, also called **cancellation**, by pushing flow in the reverse edge of the residual network

we define $f \uparrow f'$, the ***augmentation*** of flow f by f',

$$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E , \\ 0 & \text{otherwise} . \end{cases}$$

- The **residual capacity** of augmenting path *p* is the maximum amount by which we can increase the flow on **each edge**

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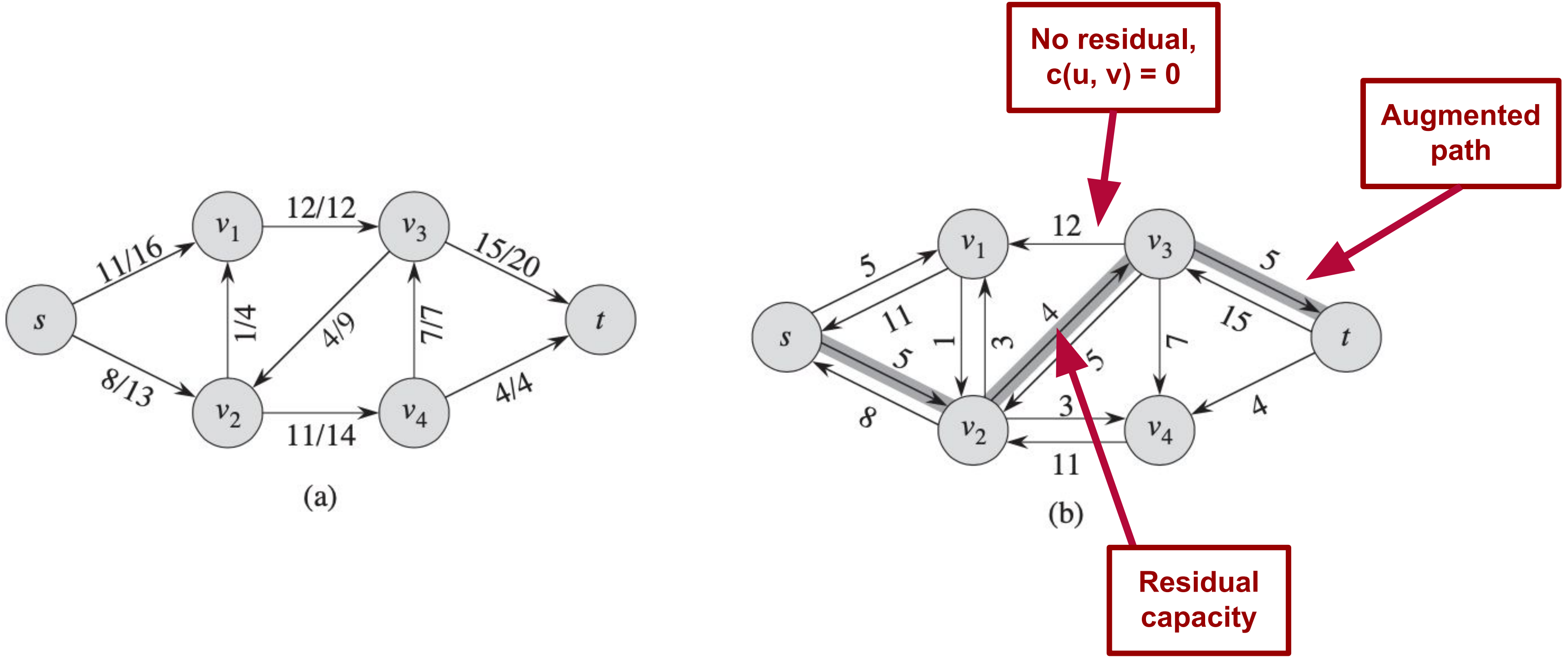
Residual networks with augmented paths



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Cuts

- As a part of the max-flow min-cut theorem, a flow is only at maximum when its residual network no longer has augmenting paths
  - s must be a part of set S, $s \in S$
  - t must be a part of set T, $t \in T$
- For flow *f*, the **net flow *f(S, T)*** across cut *(S, T)*
  - We consider the flow going from *S* to *T* minus the flow from *T* to *S*

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u) .$$

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Cuts

- The **capacity** of the cut *(S, T)*
    - We consider only flow going from *S* to *T*, not from *T* to *S*

$$c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v) \ .$$

- A **minimum cut** of a network is a cut where the capacity is minimum over all the cut options in the network

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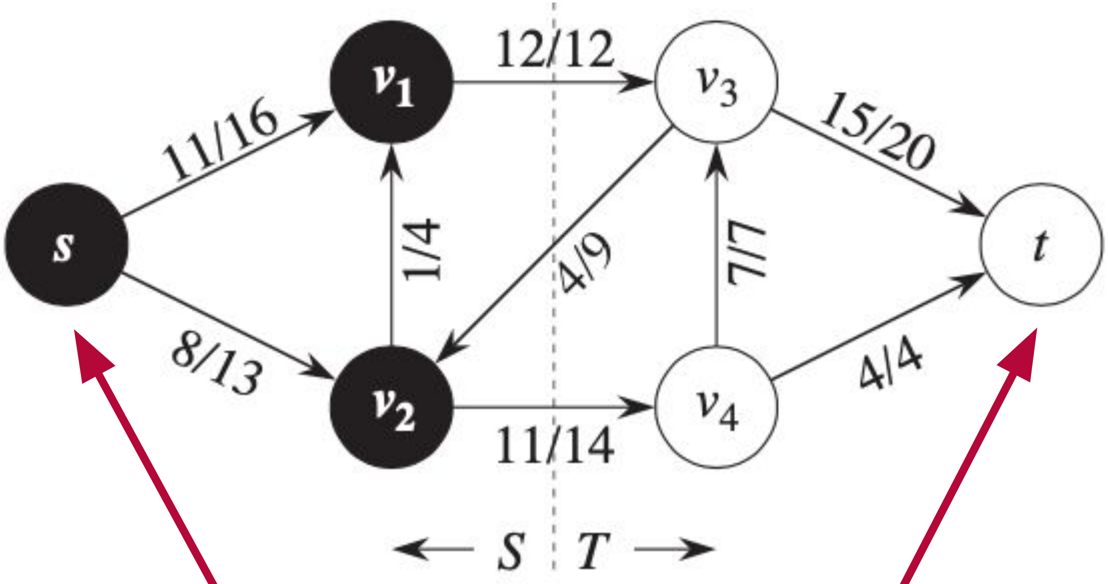
Minimum cut

Net flow *f(S, T)* = 19
Capacity c(S, T) = 26



s and t in separate partitions

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Min-cut max-flow theorem

If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

Proof (1) ⇒ (2)

Proof (2) ⇒ (3)

Proof (3) ⇒ (1)

Cuts

- As a part of the max-flow min-cut theorem, a flow is only at maximum when its residual network no longer has augmenting paths
  - s must be a part of set S, s ∈ S
  - t must be a part of set T, t ∈ T
- For flow *f*, the **net flow *f(S, T)*** across cut *(S, T)*
  - We consider the flow going from *S* to *T* minus the flow from *T* to *S*

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u) .$$

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Ford-Fulkerson algorithm

- We find some augmenting path p
  - Use p to modify the flow f
  - By replacing f by $f \uparrow f_p$, we get a new flow of $|f| + |f_p|$
- Running time is ***O(E)***

FORD-FULKERSON(G, s, t)

```
for each edge (u, v) ∈ G.E
    (u, v).f = 0
while there exists a path p from s to t in the residual network G_f
    c_f(p) = min {c_f(u, v) : (u, v) is in p}
    for each edge (u, v) in p
        if (u, v) ∈ E
            (u, v).f = (u, v).f + c_f(p)
        else (v, u).f = (v, u).f − c_f(p)
```

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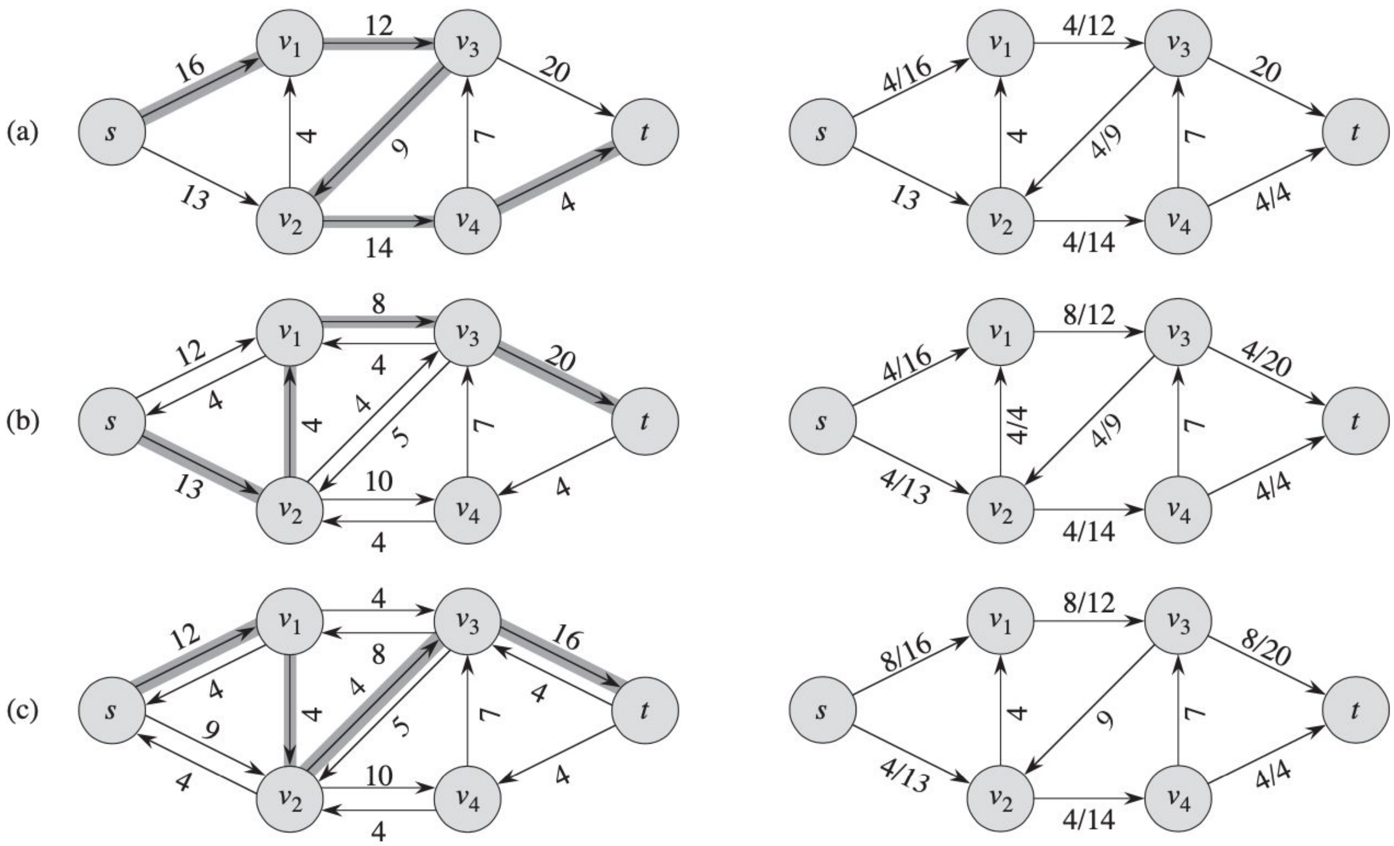
Ford-Fulkerson algorithm



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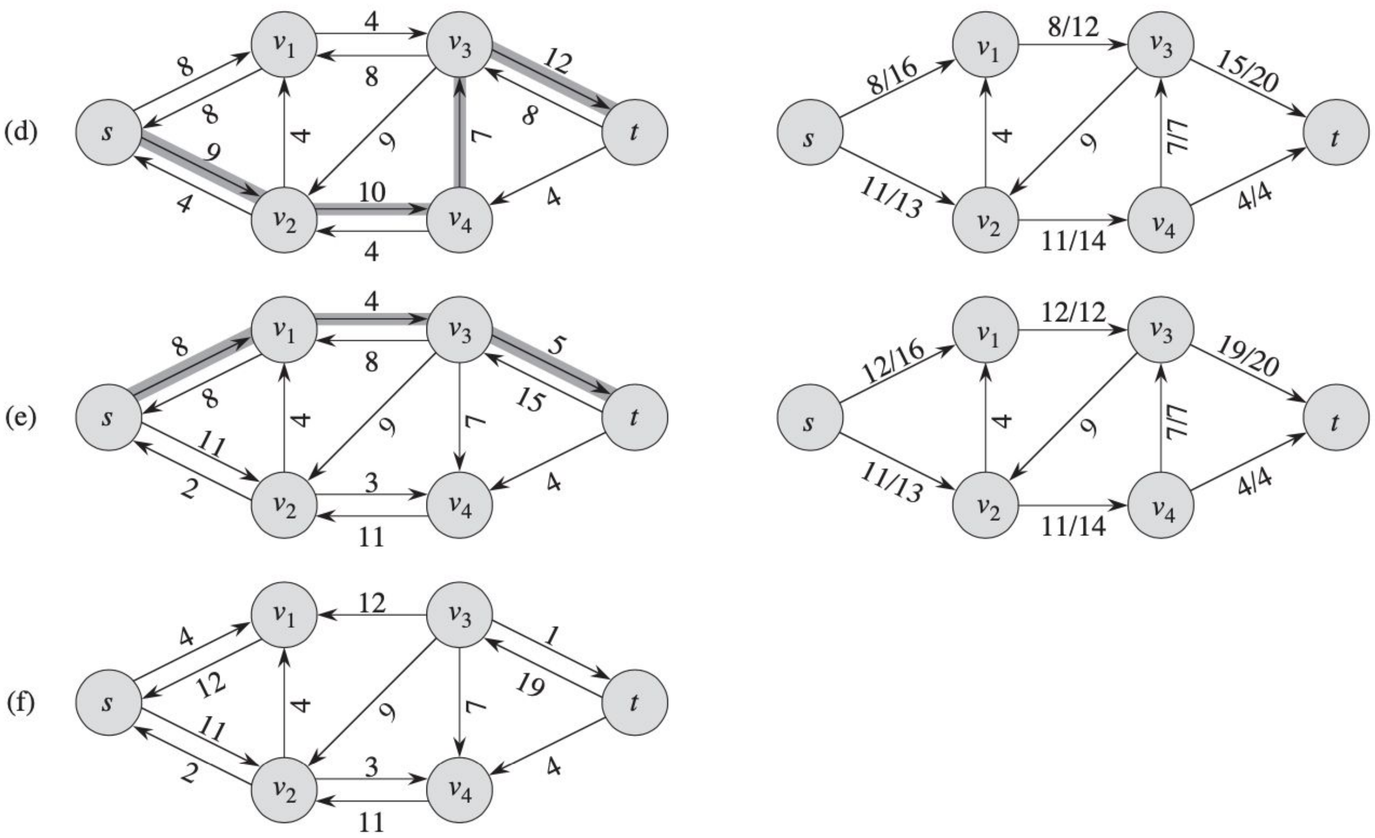
Ford-Fulkerson algorithm



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Edmonds-Karp algorithm

- Developed due to high running time on Ford-Fulkerson with large edge weights
- Improves bound by finding augmenting paths using breadth-first search
  - Augmenting path is shortest path from *s* to *t* in residual network
- Running time is $O(VE^2)$

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References

Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. Introduction to Algorithms, Third Edition (3rd. ed.). The MIT Press.

Chapter 26. Maximum Flow pgs 708 - 730

Erik Demaine, Srini Devadas, and Nancy Lynch. 6.046J Design and Analysis of Algorithms. Spring 2015. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu. License: Creative Commons BY-NC-SA.

Lecture 13. Incremental Improvement: Max Flow, Min Cut

Lecture 14. Incremental Improvement: Matching

Back To Back SWE. Network Flows: Max-Flow Min-Cut Theorem (& Ford-Fulkerson Algorithm). Oct 2019. Youtube.