Homework I

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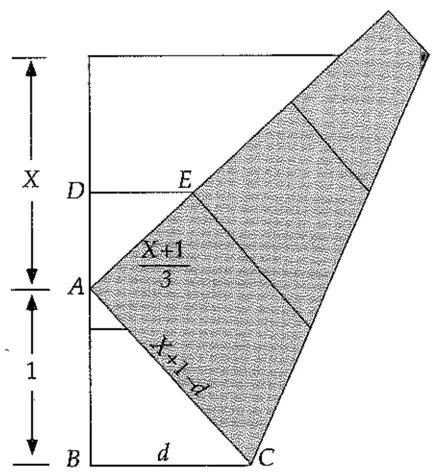
1. (20 points) In the following diagram, a piece of paper is first folded into thirds. By preforming the origami move of folding two points onto two lines, we obtain the picture below. Prove that $x = \sqrt[3]{2}$



First, we can use the Pythagorean theorem and ΔBAC to solve for d.

$$\overline{\mathrm{AB}}^2 + \overline{\mathrm{BC}}^2 = \overline{\mathrm{AC}}^2 \Rightarrow 1^2 + d^2 = (x+1-d)^2 \Rightarrow d = \frac{x^2+2x}{2x-2}$$

Next, we can find $\overline{\mathrm{DA}}$.

$$x = \overline{\mathrm{DA}} + \frac{x+1}{3} \Rightarrow \overline{\mathrm{DA}} = x - \frac{x+1}{3} \Rightarrow \overline{\mathrm{DA}} = \frac{2x-1}{3}$$

We know $\overline{\mathrm{AD}} \perp \overline{\mathrm{DE}}$ and $\overline{\mathrm{AC}} \perp \overline{\mathrm{AE}} \Rightarrow \angle DEA \cong \angle BAC \Rightarrow \Delta DEA \cong \Delta BAC$.

$$\Rightarrow \frac{\overline{\mathrm{BC}}}{\overline{\mathrm{AC}}} = \frac{\overline{\mathrm{DA}}}{\overline{\mathrm{EA}}} \Rightarrow \frac{d}{x+1-d} = \frac{2x-1}{x+1} \Rightarrow \frac{x^2+2x}{x^2+2x+2} = \frac{2x-1}{x+1}$$

$$\Rightarrow x^3 + 3x^2 + 2x = 2x^3 + 3x^2 + 2x - 2 \Rightarrow x^3 = 2 \Rightarrow x = \sqrt[3]{2}.$$

2. (20 points) Write a proof to solve the equation $ax^2 + bx + c = 0$ for x. Explain each step, starting with the following equation

$$ax^2 + bx + c = 0$$

Subtract c from both sides

$$ax^2 + bx = -c$$

Divide both sides by a

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

Add $\frac{b^2}{4a^2}$ to both sides

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$$

Completing the square we get

$$(x + \frac{b}{2a})^2 = \frac{-c}{a} + \frac{b^2}{4a^2}$$

Find a common denominator on the right hand side by multiplying $\frac{-c}{a}$ by $4a$.

$$(x + \frac{b}{2a})^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$$

Now get rid of the power on the left by square rooting both sides.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{-4ac}{4a^2} + \frac{b^2}{4a^2}}$$

Subtract off $-\frac{b}{2a}$ from both sides.

$$x = \pm\frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$$

Rearranging we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. (20 points) Show that the following complex numbers are algebraic over $\mathbb{Q}$.

Defn 1 (Algebraic)**.** *Given a number b, b is said to be algebraic over the set $\mathbb{Q}$ if $\exists$ a polynomial of the form*

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with non-zero coefficients $a_i \in \mathbb{Q}$ where b is the solution (i.e. $f(b) = 0$).

(a) $\sqrt{2}$
Let $x = \sqrt{2}$
$x^2 = 2$
$x^2 - 2 = 0$
To verify, we plug in $f(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0$.
By Defn 1, since $x = \sqrt{2}$ solves the function $f(x) = x^2 - 2$, $\sqrt{2}$ is algebraic.

(b) $\sqrt{n}$ for $n \in \mathbb{Z}$
Let $x = \sqrt{n}$
$x^2 = n$
$x^2 - n = 0$
By Defn 1, since $x = \sqrt{n}$ solves the function $f(x) = x^2 - n$, $\sqrt{n}$ is algebraic.

(c) $\sqrt{3} + \sqrt{5}$
Let $x = \sqrt{3} + \sqrt{5}$
Then $x^2 = (\sqrt{3} + \sqrt{5})^2 = 8 + 2\sqrt{15}$
and $x^4 = (\sqrt{3} + \sqrt{5})^4 = 124 + 32\sqrt{15}$
We can write each x^n in terms of linear combinations of its coefficients.

$$x^4 = 124 + 32x_1$$

$$x^2 = 8 + 2x_1$$

$$x^4 - 16x^2 = -4 + 0x_1$$

$$x^4 - 16x^2 + 4 = 0$$

Since $x = \sqrt{3} + \sqrt{5}$ solves the polynomial $x^4 - 16x^2 + 4$, $x = \sqrt{3} + \sqrt{5}$ is algebraic.

(d) $\sqrt[3]{2} + \sqrt{2}$
Let $x = \sqrt[3]{2} + \sqrt{2}$. We can write each x^n as a linear combination of its roots. The following equations gives the corresponding coefficients for each subsequent power of x.

$$x^6 = 12 + 24X_1 + 60x_2 + 80x_3 + 60x_4 + 24x_5$$

$$x^4 = 4 + 2x_2 + 8x_3 + 12x_4 + 8x_5$$

$$x^3 = 2 + 6X_1 + 6x_2 + 2x_3$$

$$x^2 = 2 + 1x_4 + 2x_5$$

The work is left to the reader, but we should get our $f(x) = x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4$. Since $\exists f(x)$ where $f(\sqrt{3} + \sqrt{5})$ is equal to zero, $x = \sqrt[3]{2} + \sqrt{2}$ is algebraic.