PROBLEM-SOLVING TUTORIAL

4.1 Solving Equations: Tutorial

Introduction

In this chapter, we will explore how Mathcad can be used to help you manipulate and solve equations. You will learn that Mathcad can solve an equation for any variable, substitute an expression into an equation, and even solve multiple equations for several variables simultaneously. In addition, we will give advice on how these features can help you solve the tough, multiple-step problems in some of your math and science classes.

Solving for a Variable

We now consider the **Variable Ë Solve** command, available in the **Symbolics** menu.

Formulas in physics usually come in a standard format. Some famous examples are shown below.

$$F = m \cdot a \qquad K = \frac{1}{2} \cdot m \cdot v^2 \qquad F_e = k \cdot \frac{q_1 \cdot q_2}{R^2}$$

Although the formulas are frequently written and recalled in a certain way, you often have to solve for one of the other variables. For example, manipulating the above equation can yield the following equations:

$$a = \frac{F}{m} \qquad v = \sqrt{\frac{2 \cdot K}{m}} \qquad R = \sqrt{\frac{k \cdot q_1 \cdot q_2}{F_e}}$$

Mathcad can perform the task of solving for one of the variables in an equation. Let's see how the **Variable Ë Solve** command works by doing an example. In order to better understand this example, though, the **symbolic equal sign** must be introduced.

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Symbolic Equals Sign

The **Variable Ë Solve** command is an example of a **symbolic** computation. The command is “symbolic” because Mathcad manipulates an equation using algebra and does not calculate any numbers, so the variables are seen only as different *symbols*.

For symbolic computations, all the equations must be written with the **symbolic equal sign =** to tell Mathcad that we are going to manipulate the variables as symbols, instead of doing a calculation with numbers or defined variables.

There are now **three** different “equal signs” in your Mathcad tool kit:

The **definition symbol** or **let equal**	:=	Typed with the **colon** [:]
The plain old **equal sign** for results	=	Typed with the **equal** [=]
The **symbolic equal sign**	**=**	Typed with [Ctrl] =

The **symbolic equal sign**, typed as [Crtl] =, is different from the other equals because Mathcad views the variables in the equations where this sign is present as pure symbols and not numbers. For example, look at the equation below:

$$y = y_o + v_y \cdot t$$

Even though the variables $y_0, v_y ..$,and t have not been previously defined with the definition symbol, there is no error message. The symbolic equal sign is **not** trying to define y in terms of $y_o + v_y \cdot t$.

Important!

In fact, until you perform a symbolic computation Mathcad essentially takes no notice of the equation. New users of Mathcad often make the mistake of thinking they have made an assignment with this equal sign when they have not. The symbolic equal sign is used just for symbolic computations (and the "solve block" discussed later).

Symbolic equal sign

Aside from the [Crtl]= keystrokes, you can also find the symbolic equal sign in the **Symbolics button** from the **Math tab .**

For practice, put a symbolic equal sign in the space to the right using both methods. $\blacksquare = \blacksquare$

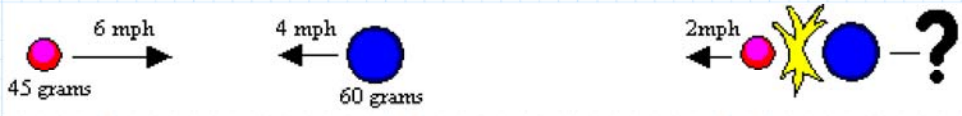
And now let's look at an example:

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Question 1

A **45 gram** ball with an initial velocity of **6 mph** to the **right** collides head-on with a **60 gram** ball that was headed to the **left** at **4 mph**. After the collision, the 45 gram ball is traveling to the **left** at **2 mph**. What is the **velocity** of the 60 gram ball?

Before   After



Information Given

While the question is still in view, let's assign variables to the known variables in this problem:

Mass of ball one:	$m1 := 45\ \mathbf{gm}$
Mass of ball two:	$m2 := 60\ \mathbf{gm}$
Initial and final velocity of ball one:	$v1_o := 6\ \mathbf{mph}$
	$v1_f := -2\ \mathbf{mph}$
Initial velocity of ball two:	$v2_o := -4\ \mathbf{mph}$

Literal Subscripts

In the definitions above, **positive** velocities are to the right and **negative** velocities are to the left. Also notice how we combined numbers (1,2) and literal subscripts (o,f) to label the variables.

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Identify principles

This problem is solved using **conservation of linear momentum**. Since there are no external forces acting on the system, the total momentum of the two balls is the same before and after the collision (momentum is conserved). That is

$$p1_o + p2_o = p1_f + p2_f$$

Using $p = m \cdot v$ this equation can be rewritten as

$$m1 \cdot v1_o + m2 \cdot v2_o = m1 \cdot v1_f + m2 \cdot v2_f$$

The unknown quantity here is the the final velocity $v2_f$.

Notice how the Boolean equal sign allowed us to rough out the problem in several steps. Being able to jot down ideas is another advantage of this new equal sign, either for explaining your logic or working towards the solution in the first place.

You now have two choices. You can

A. Run and get a pencil and paper to scratch out the answer, or
B. Let Mathcad handle some of the work with the **Variable Ë Solve** command.

Solve for Variable

Here is how the **Variable Ë Solve** command is used:

Notice that we have written the problem with the **symbolic equal sign** as

$$m1 \cdot v1_o + m2 \cdot v2_o = m1 \cdot v1_f + m2 \cdot v2_f$$

and **NOT** by using $\blacksquare := \blacksquare$

since we anticipated that we would need to do a **symbolic** manipulation.

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Why does everything move down?

We have repeated the equation below.

Choose **Symbolics Ë Solve operator** from the **Math tab** and specify v2f.

$$m1 \cdot v1_o + m2 \cdot v2_o = m1 \cdot v1_f + m2 \cdot v2_f$$

$$\frac{\left(m1 \cdot v1_o + m2 \cdot v2_o - m1 \cdot v1_f\right)}{m2}$$

The expression above is Mathcad's solution for $v2_f$, but Mathcad does not assume that you want to redefine this expression to the very same variable. Here is an easy way to reassign the expression above to the variable $v2_f$ with the **Copy** command.

Cut, Copy and Paste

Assigning variables to symbolic results

Select and **copy** one expression that appeared above (which one?). Then, place the crosshair at right and type:

v2.f : $v2_f := \blacksquare$

Now **paste** the solution into the empty placeholder.

$$v2_f := \frac{\left(m1 \cdot v1_o + m2 \cdot v2_o - m1 \cdot v1_f\right)}{m2}$$

Type **v2.f =** to display a result and change the units to meters per second.

$$v2_f = 0.89\ \frac{m}{s}$$

It is important to realize that you have taken an important step up in your level of problem-solving skills. Rather than just deal with numbers, you have solved the problem algebraically using symbols and then found a numerical value for the answer. In general, symbolic computations are much easier, less prone to errors, and easier to modify and troubleshoot than plugging numbers directly into a calculator.

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Here's another example if you need it.

Question 2

An **80 kg** basketball player gives himself **200 joules** of energy by jumping into the air.

How high does he lift off the ground?

Information Given

The first step in any problem is to assign variables to the information contained in the question. We make an exception here to prove a point: Equations written with the symbolic equals sign can use undefined variables.

To get an idea of our approach to solving this problem and a review of the physics involved, use the icon to the left.

Gravitational Potential Energy

Using the **symbolic equal sign**, write an equation for the potential energy U of a *mass* at a height h .

$$U = mass \cdot g \cdot h$$

The keystrokes for the symbolic equal sign are [Ctrl] = together.

Note: Even though the *mass* and h have not been defined anywhere, they are not highlighted as undefined variables. With the symbolic equal sign =, Mathcad can manipulate symbols (U,mass,g,h) without having to know their values. In this example, *mass* , g, and h were not even defined.

A key feature of the symbolic equal sign is that YOU CAN DEFINE ANYTHING WITH THE SYMBOLIC EQUAL SIGN. This new symbol lets you write down any equation you want and manipulate it with symbolic commands, like the **Variable Ë Solve** command. It is also used in the **solve block**, discussed below. But it is not at all like the definition symbol (:=), which is used to define variables and functions in terms of other variables, functions, or numbers.

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Why does everything move down?

Select and **move** the equation for the potential energy into the space to the right. To solve for the height, **VariableË Solve** from the **Symbolics** button, and specify 'h'.

$$U = mass \cdot g \cdot h$$

$$\frac{U}{mass \cdot g}$$

We have finished the **symbolic** part of our work, which did not require the calculation of any numbers. Now we need to reassign the height h to the symbolic result using the **definition symbol**. There is a convenient alternative to copying the answer like we did before.

Insert key

After Mathcad solves for a variable, the result is always surrounded by blue editing lines (if your result is not still surrounded by the blue lines, then do that now). Then press the **insert** key; the insert will switch to the lft, meaning that Mathcad will place the next operation to the left. Now type the **definition symbol** (with a colon **:**) and put the height h in the empty placeholder ▯ that appears to the **left**. Press [Enter].

Now the variables are undefined! The **assignment symbol** must be used to assign values to variables. You just **defined** h to have a value that depends on the values of U and $mass$, which Mathcad does not yet know.

The **symbolic equal sign** is used for symbolic computations.

The **definition symbol** is used to assign a value to a variable.

We have defined the variables of the problem in the space below:

The **mass** of the basketball player: $mass := 80 \cdot kg$

The energy provided by the basketball player is converted into **potential energy**: $U := 200 \cdot joule$

Select and move your equation for the height h into the space to the right. Now that the values of U and $mass$ are defined **above** the equation for h , Mathcad can compute the answer.

$$h := \frac{U}{(mass \cdot g)}$$

Display the answer below the equation by using the usual **equal sign** = . To judge whether the answer is feasible, you may want to change the units.

$$h = 10.04\ in$$

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Math toolbar

That covers all three signs: the **symbolic equal sign**, the **assignment symbol,** and the **results equal sign**.

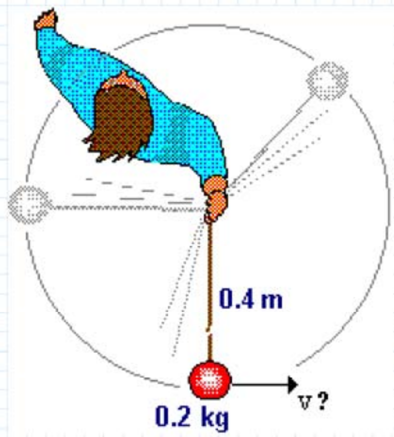
Multiple Roots

When you solve for a variable that is squared in the equation, Mathcad will display two possible answers. Remember, if you can get a positive number by squaring a positive or negative number. So when you are working backward to find the value of a squared variable, you have to choose between these positive and negative **roots** of the equation. Let's look at an example:

Question 3

A boy is twirling a **0.2 kg** ball on the end of **0.4 m** string in a circle above his head.

If the string breaks when the tension reaches **15 Newtons,** what is the speed of the ball?



Information given

We have defined the variables below:

The mass of the ball: $m_{ball} := 0.2 \cdot kg$

The length of the string: $r := 0.4 \cdot m$

The tension when the string breaks: $T := -15 \cdot N$

The key to the problem is that the force required to keep the ball in circular motion is responsible for the tension on the string (they are equal and opposite forces).

Identify principles

For **circular motion**, Newton's force law takes a special form: $F = mass \cdot \dfrac{v^2}{r}$

In this problem, the force F is equal (but opposite) to the tension T : $F := -T$

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Propose methods

Now we need to solve the force equation for the velocity v . We will use the **Symbolics Ë Solve operator** command again.

Solve problem

Notice that the velocity is squared in the force equation to the right.

$$F = mass \cdot \frac{v^2}{r}$$

Choose the **SymbolicsË Solve** command from the **Math tab**, and specify 'v' as the variable.

$$\begin{bmatrix} \frac{-1}{\sqrt{mass}} \cdot \sqrt{r} \cdot \sqrt{F} \\ \frac{1}{\sqrt{mass}} \cdot \sqrt{r} \cdot \sqrt{F} \end{bmatrix}$$

The result is explained below.

Mathcad has listed both roots for v in a column. When a result has multiple roots, you must choose the root that makes physical sense. With a velocity, the positive and negative roots would be two different directions for the **vector**. Since we are only interested in the magnitude, let's just choose the positive root.

Reassigning variables to symbolic results

Copy just the positive root to the clipboard and then reassign the expression to the velocity v .
Hint: Click on the operation that is farthest to the right in the expression you want to highlight.

$$v := \frac{1}{\sqrt{m_{ball}}} \cdot \sqrt{r} \cdot \sqrt{F}$$

Display the answer and double-check to make sure it is reasonable.

$$v = 12.25 \ mph$$

We will do more problems involving multiple roots in the Exercises and Problems.

Substitute for Variable

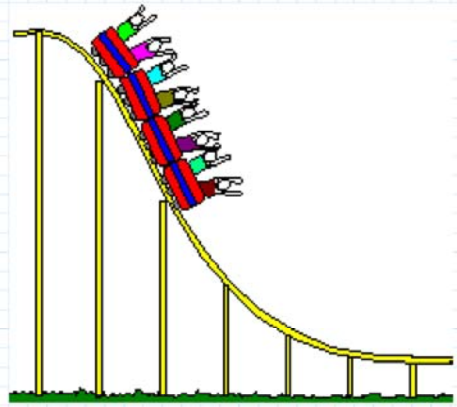
Up to this point, you have been limited to solving problems involving only **one equation**. Often you will be faced with problems that involve **two or more equations** that must be used together to solve for a single variable. Typically, you need to solve the first equation for some intermediate result and then substitute this result into the next equation. Another symbolic command, **Variable Ë Substitute**, allows you to tackle this sort of problem easily and *in symbolic form*. Consider the following example:

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Question

As a roller coaster car moves down the track, it exchanges its total energy between potential energy and kinetic energy.

Use the principle of conservation of energy to find an expression for velocity of the car in terms of its height.



Substitution

In the open space below, there is an equation for the total energy E_{total} in terms of kinetic and potential energies and then specific equation for the potential energy U in terms of height and the kinetic energy KE in terms of velocity.

Let's replace the U and KE in the total energy equation with their specific expressions by substitution. Then we can use the Solve for Variable command to find an expression for v in terms of the height h .

Mathcad's **Symbolics Ë Substitute** command replaces a variable you chose with the last expression you copied or cut to the clipboard.

Surround just the $mass \cdot g \cdot h$ with a blue editing box.
Hint: Click on the operation that is farthest to the right in the expression you want.

Once you have chosen the expression, **copy** it to the clipboard. Then click on the variable that you want to replace with the expression (the U in the total energy) and chose **Variable Ë Substitute** from the **Symbolics** menu.

Also substitute the expression for KE into your new equation for the total energy.

Now solve the total energy expression for velocity. (You should get two roots.)

$$E_{total} = U + KE$$

$$U = mass \cdot g \cdot h$$

$$KE = \frac{1}{2} \cdot mass \cdot v^2$$

$$E_{total} = mass \cdot g \cdot h + KE$$

$$E_{total} = mass \cdot g \cdot h + \frac{1}{2} \cdot mass \cdot v^2$$

$$\begin{bmatrix} \frac{-1}{\sqrt{mass}} \cdot \sqrt{2} \cdot \sqrt{E_{total} - mass \cdot g \cdot h} \\ \frac{1}{\sqrt{mass}} \cdot \sqrt{2} \cdot \sqrt{E_{total} - mass \cdot g \cdot h} \end{bmatrix}$$

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Using Copy and Paste, you can achieve the same result. However, the **Symbolics Ë Substitute** command is especially useful if the variable that you want to replace occurs several times in the same equation. You only need to choose the variable once, and Mathcad will replace **every occurrence** of that variable with the expression that was copied to the clipboard.

When an equation is written with the symbolic equal sign, we can perform a number of tasks involving algebra. **SymbolicsË Solve** command is by far the most useful for doing physics problems, but Mathcad can also do other symbolic operations.

Other symbolic commands

Using other symbolic commands follows the same pattern that we have shown above. Consult the **Getting Started tab** to find more details about these commands.

The Solve Block

Sometimes you have to use two or more equations together and solve them to find values for more than one variable. When you need to find more than one variable, and there are multiple equations that involve each variable mixed together, you are dealing with a **system of equations**. Here is a quick example you may remember seeing from algebra class:

Given: $3\ x + 8\ y = 34$

$14\ y - 9\ x = 8$

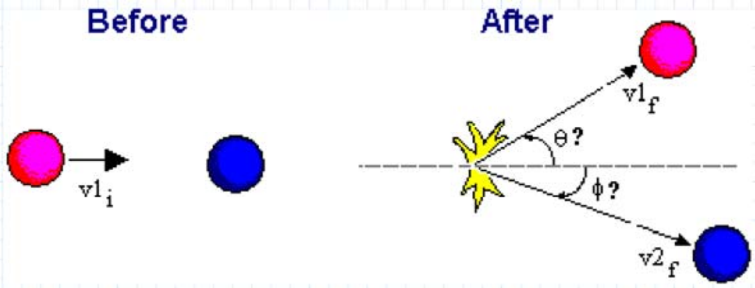
Find x and y .

The algebra involved in solving such problems can sometimes be quite difficult (although most of you have probably seen a system like the one above and solved it successfully). For those systems of equations that involve more complex algebra, Mathcad has a mathematical routine, called a **solve block**, that finds solutions quickly and easily.

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Question 4

A **0.5 kg** ball traveling at **5 mph** glances off a second **0.5 kg** ball that is initially at rest. After the collision, the first ball's speed has been reduced by half (to **2.5 mph**) and the resting ball has a final speed of **4.33 mph**. In what directions do the balls travel after the collision? (Measure the angles from the original direction of the first ball).



Here's how a **solve block** works:

1. Assign the variables to the information contained in the question.

Mass of each ball:	$mass := 5\ \mathbf{kg}$	
Initial and final speeds of 1st ball:	$v1_i := 5\ \mathbf{mph}$	$v1_f := 2.5\ \mathbf{mph}$
Initial and final speeds of 2nd ball:	$v2_i := 0\ \mathbf{mph}$	$v2_f := 4.33\ \mathbf{mph}$

Why two equations?

Since we are trying to find **2** variables (θ and ϕ), there must be **2** equations that apply to the situation. The icon to the left explains why we are sure there are **2** equations.

In this problem, momentum (a vector!) is conserved. Since the symbolic equals sign allows us to jot down equations without worrying about the variables, let's write out the momentum equation:

Conservation of momentum: $p_{1i} = p1_f + p2_f$

(same equation in terms of *mass* and *v*): $mass \cdot v1_i = mass \cdot v1_f + mass \cdot v2_f$

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

The two equations we anticipated come from breaking this last equation into components along the vertical and horizontal directions:

Horizontal component: $mass \cdot v1_i = mass \cdot v1_f \cdot \cos(\theta) + mass \cdot v2_f \cdot \cos(\phi)$

Vertical component: $0 = mass \cdot v1_f \cdot \sin(\theta) + mass \cdot v2_f \cdot \sin(\phi)$

We now have a system of two equations and two unknowns (the angles θ and ϕ). Now we could solve one equation for an angle and substitute back into the other equation; however, let's use the solve block for practice.

2. Guess the solutions. This is not a joke! Of course, it is not that serious either. Mathcad finds the answer by a series of well planned guesses (if only we could do this!), but needs some initial values as a starting place.

Do we have to make an ***educated*** *guess*?

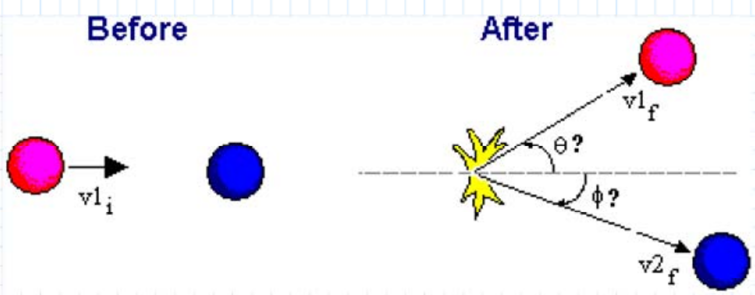
Not really; for most problems you will encounter, you can guess any reasonable number. Often, the crucial factor is guessing whether the answer is **positive** or **negative**.

So, here are some guesses:

$\theta := 1\ deg$

$\phi := -1\ deg$

Since the size of our guesses will have little effect on whether Mathcad finds the answers, you can just guess **±1** (and remember the units!!). **Note:** We chose θ to be positive and ϕ to be a negative angle, as shown in the diagram of the problem. But for more complicated non-linear systems, these guesses will help you land on the one solution among many that you may be looking for. Therefore, it's a good idea to practice your guessing on more straight-forward problems.



3. Write the solve block. Select a point on the worksheet and then go to the **Math tab** and select **Solve Block**.

List the equation(s) that will be used to derive the answers for the unknown variables.

Symbolic equal sign

Note: Inside the solve block you must write these equations with the **Boolean equal sign** ([Ctrl] =).

Solve Block

제약조건값

$$m \cdot v1_i = m \cdot v1_f \cdot \cos(\theta) + m \cdot v2_f \cdot \cos(\phi)$$

$$0 \cdot \frac{m^2}{sec} = m \cdot v1_f \cdot \sin(\theta) + m \cdot v2_f \cdot \sin(\phi)$$

The Find( ) function, which ends the solve block, has as its arguments the unknown variables that you want to find: (θ, ϕ).

풀이

$$\mathbf{Find}(\theta, \phi) = \begin{bmatrix} 60 \\ -30 \end{bmatrix} deg$$

You can remember the format because the Given command, along with the set of equations and the Find( ) function form a “block” on the screen. Note that it didn't matter above that we added extra text between the sections of the solve block.

Notice that Mathcad found θ to be positive and ϕ to be negative. Our initial guesses lead Mathcad in the right directions. The next example should help you understand when the solve block will be sensitive to initial guesses and how to think about your guesses correctly.

The main limitation of the solve block is the lack of a symbolic result (that can be used as a function, graphed, etc.). The advantage is that a solve block can obtain solutions that are difficult, or perhaps impossible, to arrive at symbolically. As an example of a more difficult problem, we solve a system of equations next.

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Question 5

Two **1 kg** balls undergo a head-on, totally **elastic** collision. The first ball had an initial velocity of **3 m/sec** to the **right**, while the second ball was going **5 m/sec** to the **left**. What are the final velocities of both balls after the collision?

1. Assign the variables to the information contained in the question.

You will define the variables for this problem, so that you can decide on your own convention for the positive and negative direction of the velocity vectors.

Problem SOLVING
Information Given

Define the mass *mass* and the velocities $v1_i$ and $v2_i$ that are given in the question. **Also**, explain which direction you chose to be positive and negative in a text region.

Mass of each ball: $mass := 1\ kg$

Initial speed of 1st ball: $v1_i := 3\ \frac{m}{s}$

Initial speed of 2nd ball: $v2_i := -5\ \frac{m}{s}$

Left is negative and right is the positive direction.

Problem SOLVING
Identify principles

i. Momentum is always conserved (unless there is some outside force in the problem).

$$mass \cdot v1_i + mass \cdot v2_i = mass \cdot v1_f + mass \cdot v2_f$$

ii. Since the collision is **elastic**, there is no energy lost in the collision.

$$KE1_i + KE2_i = KE1_f + KE2_f$$ or

$$\frac{1}{2} \cdot mass \cdot \left(v1_i\right)^2 + \frac{1}{2} \cdot mass \cdot \left(v2_i\right)^2 = \frac{1}{2} \cdot mass \cdot \left(v1_f\right)^2 + \frac{1}{2} \cdot mass \cdot \left(v2_f\right)^2$$

What is the next step in using the solve block?

2. Guess the solutions.

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.

Problem SOLVING

Propose methods

Write the word "**Guesses:**" in a text region, then define guesses for final velocities $v1_f$ and $v2_f$. Remember, the **direction** (±) is crucial, while the magnitude does not matter. Ask yourself, "Which way should the balls head after the collision?" Use the appropriate units!!

Guesses:

Final speed of 1st ball: $v1_f := 1\ \frac{m}{s}$

Final speed of 2nd ball: $v2_f := -1\ \frac{m}{s}$

Before you write the solve block, make sure that all the variables in your equations have been defined, whether the value is given in the question or just a guess.

3. Now for the solve block.

Problem SOLVING

Solve problem

In the space provided below, add the **Solve Block** from the **Math tab** and then add appropiate equations and data. **Hint:** Just **copy** the two equations from above.

제약 추측값

$$mass \cdot v1_i + mass \cdot v2_i = mass \cdot v1_f + mass \cdot v2_f$$

$$\frac{1}{2} \cdot mass \cdot (v1_i)^2 + \frac{1}{2} \cdot mass \cdot (v2_i)^2 = \frac{1}{2} \cdot mass \cdot (v1_f)^2 + \frac{1}{2} \cdot mass \cdot (v2_f)^2$$

풀이

$$\mathbf{Find}\left(v1_f, v2_f\right) = \begin{bmatrix} 3 \\ -5 \end{bmatrix} \frac{m}{s}$$

Your answer should show that the balls exchange speeds and head in opposite directions (they bounce off each other). Depending on your initial guesses for the solve block, you can get different answers.

STOP

Save your changes!

As you will see from the exercises in the next section, Mathcad's symbolic commands and the **Solve Block** expand your problem-solving abilities even further than before.

PTC Mathcad Express로 작성되었습니다. 자세한 정보는 www.mathcad.com의 내용을 참조하십시오.