PROBLEM-SOLVING TUTORIAL

7.1 Advanced Problem Solving: Tutorial

Introduction

As we have said before, the point of working through math and science problems is to learn the concepts. After you have solved a problem for the required answer, you still may have a few questions. In fact, you may not even understand fully what you have done or why. Unless you take the time to put the solution in context and generalize your result to other situations, you are not building any real understanding. Such understanding will help you solve the next problem more quickly (and help you do well on exams!).

In this chapter, we are not going to teach many new features of Mathcad. You have learned quite a bit so far and now we want to show you how Mathcad can be used to understand the concepts of a problem more completely.

"What If . . .?"

We have taken one of the best features of Mathcad for granted: If you change the value of a variable in a problem, Mathcad automatically recalculates the answer!

Why is this such a valuable aspect of Mathcad?

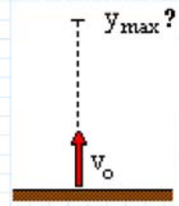
1. Often you will solve a problem, only to find at the end of your efforts that you have made a simple numerical or algebraic mistake. When you use Mathcad, it is easy to correct the mistake, once you have found it, and then let Mathcad automatically recalculate the answer.Since you may have already practiced finding errors, we will emphasize another advantage of Mathcad's ability to quickly recalculate answers:
2. After you have worked out a problem, you can go back and find out what happens if you alter the value of a certain variable (increase the *mass* for example).

By playing around with a problem, you will often learn more than you did in getting the answer. Instead of finishing a problem and wondering "what was that all about?" you can easily find out by experimenting. Mathcad lets you ask "**what if. . . ?**"

The following problem has already been solved, but you can quickly answer a few questions of your own to understand the underlying physics of the problem better.

Question 1

If an object has an initial velocity of **15 mph** upward, how long does it take to reach its maximum height? What maximum height does it reach?



Variable Definitions

Information Given

Initial upward velocity: $v_o := 15 \cdot \mathbf{mph}$

Assuming that the object starts from the ground: $y_o := 0 \cdot \mathbf{m}$

Solution

The initial velocity is slowed by gravity until the object comes to rest momentarily at the maximum height.

$$v_{max} = v_o - g \cdot t$$

When $v_{max} = 0$ the object is at the maximum height, so

$$0 = v_o - g \cdot t \qquad \text{and solving for time:} \qquad t_{max} := \frac{v_o}{g}$$

$$t_{max} = 0.68 \ \mathbf{s}$$

The height at the time t_{max} :

$$y_{max} := y_o + v_o \cdot t_{max} - \frac{1}{2} \cdot g \cdot \left(t_{max}\right)^2$$

$$y_{max} = 2.29 \ \mathbf{m}$$

Since Mathcad will recalculate the answer, you can change variable definitions and see how that affects the answer.

Generalize Results

Experiment with the initial upward velocity v_o until the height is roughly what you can jump (feel free to convert units and/or practice jumping).

I can jump 24 inches. This corresponds to an initial velocity of 7.8 mph.

Professional basketball players can reach heights of **70 cm**. How large are their initial velocities? How long do they spend in the air?

To reach a height of 70 cm, the player needs an initial velocity of 8.3 mph.

What about other objects that reach higher maximum heights?

The point of doing physics problems is to learn more about physics. After you have gone through the trouble of working out a solution, you should take some time to experiment with the dynamics of the problem.

You may find that you are interested in a range of different values. You can always make a graph of your solution, which we talk about next.

Graphing Solutions

If you are interested in how your solution depends on a certain variable, then a graph is often the best way to understand the relationship. A graph can allow you see the full behavior of a function at a glance.

In order to graph your solutions, you will have to adapt your work slightly:

1. Change the variable definition into a range of values (a range variable).
2. Alter the equation that defined the solution into a function of the range variable.

Using our previous work as an example, we could plot the maximum heights reached for different initial upward velocities. Let's look at the solution again:

Variable Definitions

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Solution

The initial velocity is slowed by gravity until the object comes to rest momentarily at the maximum height.

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$$y_{max} = 2.29\ m$$

1. Creating a range of initial velocities is not difficult. We can always modify the range later, if necessary.

Defining a range of initial velocities: $v_o := 0 \cdot mph, 0.1 \cdot mph .. 25 \cdot mph$

2. The real question is how to change the definition y_{max} into a function $y_{max}\left(v_o\right)$. Simply adding an argument $\left(v_o\right)$ to the definition for y_{max} will not work. The time to reach the maximum height t_{max} also depends on v_o, but that does not show up explicitly in the formula for y_{max}:

$$y_{max}\left(v_o\right) := y_o + v_o \cdot t_{max} - \frac{1}{2} \cdot g \cdot \left(t_{max}\right)^2$$

where $t_{max} = \frac{v_o}{g}$

The dependence of y_{max} on the velocity v_o must appear **explicitly** in the formula.

We must substitute the definition for t_{max} into our function. Since t_{max} occurs multiple times in the function, using the **Symbolics Ë Substitute** command on the **Math tab** is the best method. For a review of the command, use the icon in the margin.

Replace the time t_{max} in the function $y_{max}(v_o)$ with its definition in terms of v_o.

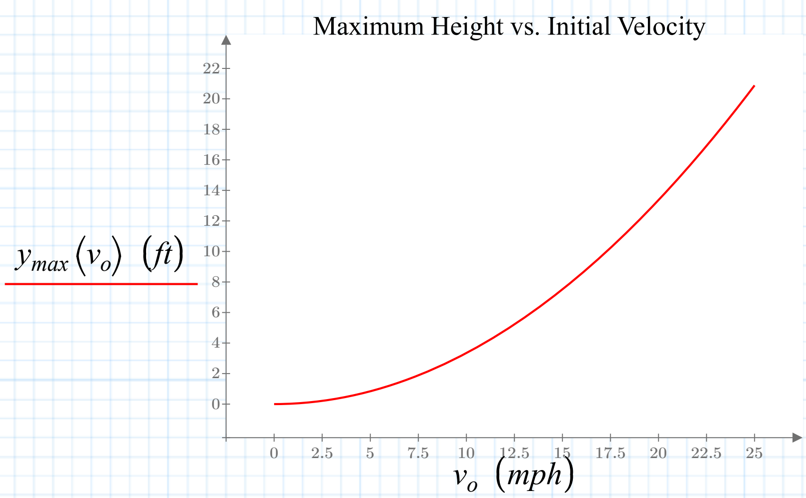
$$y_{max}(v_o) := y_o + v_o \cdot t_{max} - \frac{1}{2} \cdot g \cdot (t_{max})^2$$

$$y_{max}(v_o) := y_o + \frac{1}{2} \cdot \frac{{v_o}^2}{g}$$

Once the height's dependence on the initial velocity is clear, you can graph the function:

In the open space, graph the function $y_{max}(v_o)$.

Divide the height by ft and the velocity by mph to convert the units.



Advanced Topic

Global Definitions

Can we modify the graph and see the effects recalculated ?

Sure. Notice that the function also depends on the acceleration due to gravity g. Below the graph, we have redefined g throughout the whole document using a **global** definition (notice the three lines `).

$$g \equiv 9.81 \cdot \frac{m}{s^2}$$

Find out what would happen on different planets by changing g. The icon in the margin displays a table of the acceleration due to gravity on other planets. Can you explain why the astronauts on the moon bounce around with each step?

What if we want to compare the earth's curve directly with that of the moon? Next, we explain how to graph several related curves from the same function.

A Family of Curves

A “**family of curves**” is a plot of multiple curves of the same function, but for different values of a key variable. In our case, we are interested in the effect of different accelerations on a plot of maximum height versus initial upward velocity:

$$y_{max}(v_o) := y_o + \frac{1}{2} \cdot \frac{{v_o}^2}{g}$$

To graph several functions with different values for g, we need to change $y_{max}(v_o)$ into a function that also depends on the acceleration due to gravity:

$$y_{max}(v_o) \quad \Rightarrow \quad y_{max}(v_o, g)$$

Copy the definition above for $y_{max}(v_o)$ into the open space at right.

Change the function to depend on the acceleration due to gravity g as well as the initial velocity v_o .

$$y_{max}(v_o, g) := y_o + \frac{1}{2} \cdot \frac{{v_o}^2}{g}$$

Now, each time we choose a different value for g, we get a slightly different function. For example, using the acceleration due to gravity on **Earth**, its **moon**, and **Jupiter** (respectively from left to right), we get three curves:

$$y_{max}\left(v_o, 9.81 \cdot \frac{m}{s^2}\right)$$

or $$y_{max}\left(v_o, 1.67 \cdot \frac{m}{s^2}\right)$$

or $$y_{max}\left(v_o, 25.87 \cdot \frac{m}{s^2}\right)$$

When considering a family of curves, you will find it useful to assign variables to the different values of the parameter you are adjusting (rather than type the values over and over again for each curve).

Content TIP

"g" on Other Planets

Define variables for the accelerations due to gravity on the earth g_E, its moon g_m , and on Jupiter g_J.

You can either refer to the values in the example above or the pop-up in the margin.

$$g_E := 9.81 \cdot \frac{m}{s^2}$$

$$g_m := 1.67 \cdot \frac{m}{s^2}$$

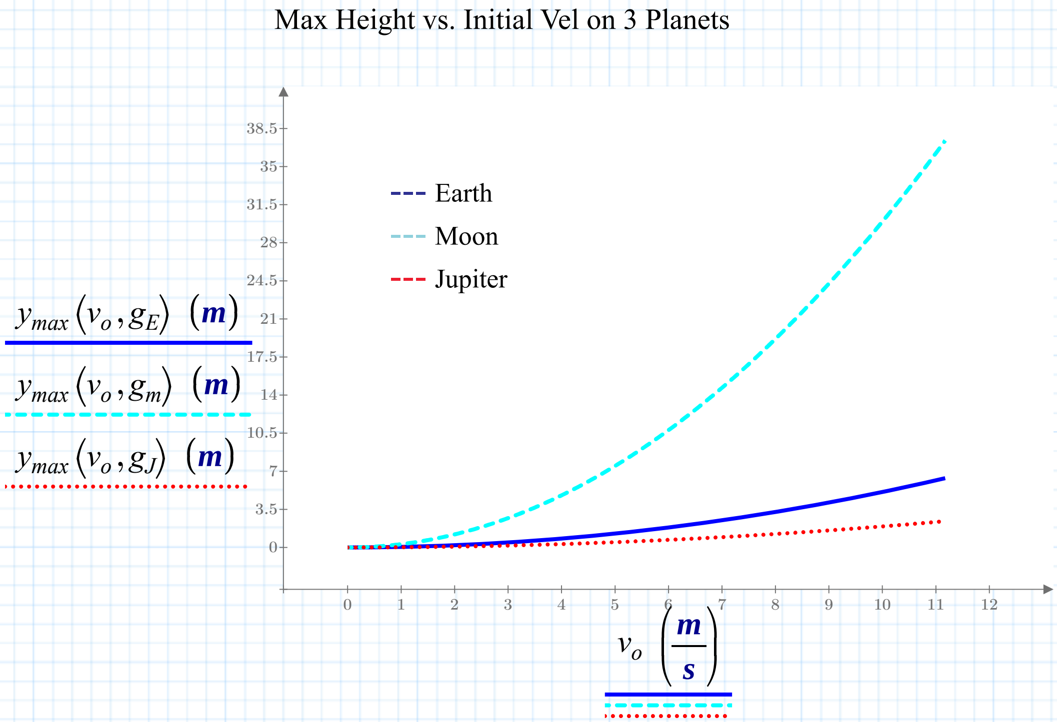
$$g_J := 25.87 \cdot \frac{m}{s^2}$$

Although we have chosen three specific values for g, these functions can still be plotted over a range of initial velocities as we had done before.

? HUH

HELP!!! (with family of curves)

In the space below, plot the **three** curves you get by evaluating the function for the maximum height $y_{max}(v_o, g)$ at **each** of the **three** accelerations (g_E, g_m and g_J) against the range of initial velocities v_o. You will have three functions (separated by commas) on the **y**-axis. Use the pop-up for additional help.



Plotting your solution is a natural extension of asking "what if...?" and adjusting variables to investigate their role in your solution. By doing so, problems become more tangible and you can learn a lot of physics quickly. Plotting a "family of curves" is a much more sophisticated technique, but the basic idea is the same.

We next discuss another useful technique to explore the physics of a problem. This technique, which we term "limiting cases," provides an excellent check on the correctness of your general solution to any particular problem. The technique also helps to develop intuition about and provides insight into physics problems.

Limiting Cases

"Do I have the right answer ?" One way of double-checking this is to consider what should happen to your answer when a variable in the equation takes on large or small values. For example:

Problem SOLVING
Analyze Results

Should the equation "blow up"?
Does the equation simply go to zero?
Is there a certain value to the equation that you would expect?

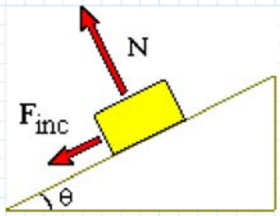
These are called **limiting cases** of the equation. If the answer behaves just as you predicted, then you have checked your answer and have gained confidence in your understanding of the problem.

Here is an good example of how this technique can be used in solving problems:

Question

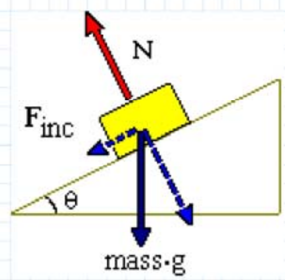
A **120 gram** block is on an inclined plane that makes an angle of **20 deg** with the ground.

Find the normal force and the force along the incline.



Checking Angles

The force of gravity $(mass \cdot g)$ pulls straight down on the block. The question asks for the component of this weight $mass \cdot g$ parallel to the incline and the normal force, which opposes the component of the weight $mass \cdot g$ that is perpendicular to the plane.



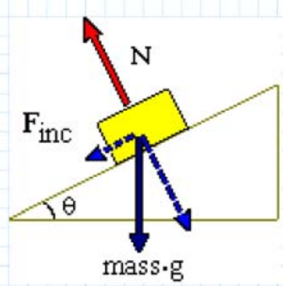
Ultimately, we are interested in the components of the weight $mass \cdot g$ (shown as dotted blue lines).

Having worked with vectors, you may remember that decomposing a vector into its components involves $sin(\theta)$ and $cos(\theta)$. The confusing part about inclined planes is deciding how the angle of incline relates to the components of the force of gravity.

A student has attempted to solve the problem below:

Mass of the block: $mass := 120 \cdot \mathbf{gm}$

Angle of the incline: $\theta := 20 \cdot \mathbf{deg}$



The force of gravity $(mass \cdot g)$ must be decomposed into components, but which pair of these equations are correct?

$$N = mass \cdot g \cdot cos(\theta) \qquad N = mass \cdot g \cdot sin(\theta)$$

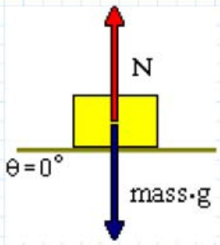
OR **?**

$$F_{inc} = mass \cdot g \cdot sin(\theta) \qquad F_{inc} = mass \cdot g \cdot cos(\theta)$$

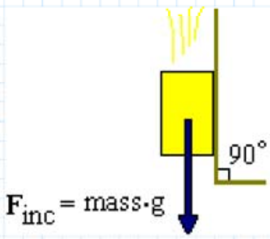
You do not have to memorize the specific formulas for inclined planes or run off to look up "complementary angles." By considering limiting cases, you can deduce the correct equations for the components: N and F_{inc}.

The point is to ask yourself, "*when does the problem take on more predictable (even perhaps ridiculous) behavior* ?" You are trying to simplify the problem by imagining the behavior of the functions in situations that you can easily predict (the limiting cases).

Imagine what would happen to the components of the force if the block was not inclined, but on flat ground.



Now imagine what would happen if the incline was so steep the mass was practically in free fall.



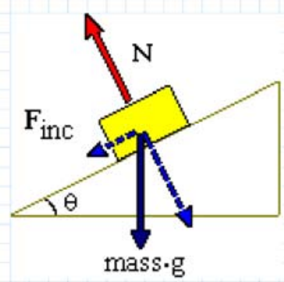
These situations are limiting cases of the inclined plane problem, but are more easily understood. You now have an idea of how the components of the force of gravity should behave as the angle of incline goes from $0 \cdot deg$ (flat ground) to $90 \cdot deg$ (vertical).

With Mathcad, we can easily test our theory by changing the angle θ and watching the behavior of the equations. In the space below, a student has finished the problem and is trying to decide whether the solution is correct.

Mass of the block: $mass := 120 \cdot \boldsymbol{gm}$

Angle of the incline: $\theta := 20 \cdot \boldsymbol{deg}$

The force of gravity: $mass \cdot g = 1.18\ \boldsymbol{N}$



The components perpendicular (normal N) and parallel to (F_{inc}) the inclined plane:

$N := mass \cdot g \cdot \sin(\theta)$ $\quad$ $F_{inc} := mass \cdot g \cdot \cos(\theta)$

$N = 0.4\ \boldsymbol{N}$ $\quad$ $F_{inc} = 1.11\ \boldsymbol{N}$

Limiting cases for angles

By considering limiting cases of the angle θ, decide whether the student has the correct equations for the components of the force of gravity. For help, the pop-up in the margin reviews the discussion of the two cases we considered before.

Explain your decision in the space below (and give the correct solution if you think the one above is wrong).

Changing the angle to 0 degrees: $\theta := 0 \cdot \mathbf{deg}$ we would expect the normal force to be at a maximum (in order to keep the block from going through the ground).

$N := mass \cdot g \cdot \sin(\theta)$ $\qquad$ $F_{inc} := mass \cdot g \cdot \cos(\theta)$

$N = 0\ \mathbf{N}$ $\qquad$ $F_{inc} = 1.18\ \mathbf{N}$

Since the opposite is true, then the equations must be backwards. The correct equations are:

$N := mass \cdot g \cdot \cos(\theta)$ $\qquad$ $F_{inc} := mass \cdot g \cdot \sin(\theta)$

$N = 1.18\ \mathbf{N}$ $\qquad$ $F_{inc} = 0\ \mathbf{N}$

Considering the limiting cases of a problem allows you to simplify the behavior of complicated equations, which lets you decide whether the behavior matches your understanding of the problem.

If you have more time and want to understand the full behavior of the equations, you can always change them into functions and create a graph. We have made a graph for the inclined plane problem below:

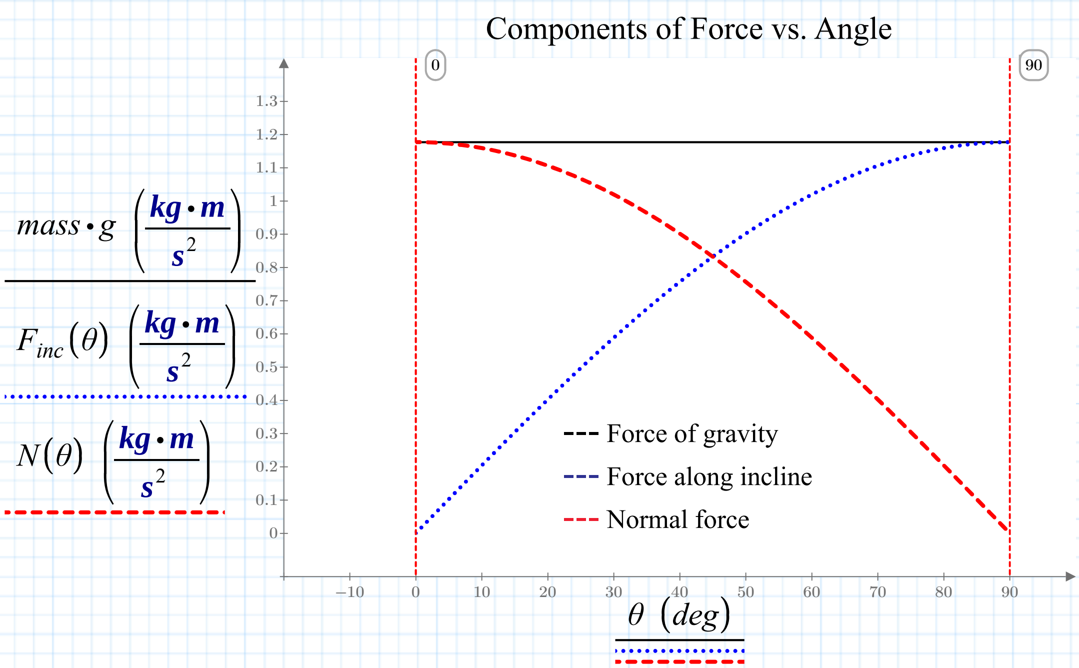
Change the single definition into a range of values:

$$\theta := 0 \cdot \mathbf{deg}, 1 \cdot \mathbf{deg} .. 90 \cdot \mathbf{deg}$$

Alter the equations to be functions of the variable:

$$F_{inc}(\theta) := mass \cdot g \cdot \sin(\theta)$$

$$N(\theta) := mass \cdot g \cdot \cos(\theta)$$



Save your changes!

Once again, make sure we have the right functions by considering the limiting cases. Correct the definitions of the functions if necessary.