THETA FUNTION

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1. Riemann's bilinear relation

Riemann surface with genus g has $2g$ periods. It's intuitively clear. For example, Complex Torus has two cycles spanning the first homology group of complex torus. Let $\{A_i, B_i\}$ be the basis of $H_1(X, \mathbb{Z})$ such that $A_i \cap A_j = 0$, $B_i \cap B_j$=0, $A_i \cap B_j = \delta_{i,j}$. Let X_0 be $X - \cup(A_i \cup B_i)$.

Lemma 1.1. *Let $f(p) = \int_c^p \omega$.(ω is a meromorphic form). Then* $\int_{\partial X_0} f\tilde{\omega} = \sum_{i=1}^g \int_{A_i} \omega \int_{B_i} \tilde{\omega} - \int_{B_i} \omega \int_{A_i} \tilde{\omega}$.

Proof. $f(p) - f(p') = -\int_{A_i} \omega$ for proper p, p' on B_i. Similarly $f(q) - f(q') = -\int_{B_i} \omega$ for proper q, q' on A_i. Then

$$\begin{aligned}
&\int_{\partial X_0} f\tilde{\omega} \\
&= \sum_{i=1}^g \left(\int_{A_i} - \int_{A_i'} + \int_{B_i} - \int_{B_i'}\right) f\tilde{\omega} \\
&= \sum_{i=1}^g \left(\int_{p\in A_i} [f(p) - f(p')]\tilde{\omega} + \int_{q\in B_i} [f(q) - f(q')]\tilde{\omega}\right) \\
&= \sum_{i=1}^g \left(\int_{B_i} \omega \int_{A_i} \tilde{\omega} - \int_{A_i} \omega \int_{B_i} \tilde{\omega}\right)
\end{aligned}$$

Lemma 1.2. *If ω is a nonzero holomorphic 1-form on X. Then*

$$Im \sum_{i=1}^g \left(\int_{A_i} \omega\right)\left(\overline{\int_{B_i} \omega}\right) < 0.$$

Proof. If we write $w = f(z)dz$, $\bar{\omega} = f\bar{(z)}d\bar{z}$. Then $\omega \wedge \bar{\omega} = -2i\,|f|^2\, dx \wedge dy$. So by lemma 1.1 we obtain the result. Given a basis of cycles, there exists a unique basis of holomorphic 1-forms space such that $\int_{A_i} \omega_j = \delta_{i,j}$. If we let $\epsilon_i = \int_{A_i} \omega$, the by the above lemma the vector $(\epsilon_1, .., \epsilon_g)$ can't vanish. So we can pick the basis of holomorphic 1-form space.

Definition 1.3. The $g \times g$ matrix

$$\tau_{i,j} = \int_{B_i} \omega_j$$

is called the Riemann matrix of periods.

Riemannian matrix is symmetric and imaginary part is positive definite. Let $\tilde{\omega} = \omega_j$, $\omega = \omega_i$ in Lemma1.1 Then we can easily obtain the results. Note that $\{\omega_i\}$ is normalized basis.

Definition 1.4. If τ belongs to the g-dimensional Siegel space(symmetric complex matrices whose imaginary part is positive definite) the the map

$$\begin{aligned}\Theta : \mathbb{C}^g &\to \mathbb{C}\\ u \mapsto \Theta(u,\tau) &:= \sum_{N\in\mathbb{Z}^g} e^{2\pi i(N,u)}e^{\pi i(N,\tau N)}.\end{aligned}$$

is called Riemann Theta function. This map is anlaytic. Here (,) denote the standard inner product.

It satisfies following:

$$\begin{aligned}\Theta(-u) =&\Theta(u)\\ \Theta(u+N) =&\Theta(u)\\ \Theta(u+\tau N) =&\Theta(u)e^{-2\pi i(N,u)}e^{-\pi i(N,\tau N)}\\ \Theta(\frac{1}{2}a+\frac{1}{2}\tau b) =&0 \quad a,b\in\mathbb{Z}^g,(a,b)\in 2\mathbb{Z}+1\end{aligned}$$

Definition 1.5. Let q a point of X_0, and $\zeta \in \mathbb{C}^g$. Let u be the Abel-Jacobi map. Define the map $X_0 \to \mathbb{C}$:

$$p \mapsto g_{\zeta,q}(p) = \Theta(u(p) - u(q) + \zeta).$$

Lemma 1.6. *Let ζ a zero of the Θ. If all partial derivatie is zero at ζ, then ζ is called a singular zero.*
If ζ is a singular zero, then the functions $g_{\zeta,q}$ vanishes identically on X_0 for any q. If not, then the map $X_0 \to \mathbb{C}$, $p \mapsto g_{\zeta,q}$ has g zeros on X_0. One of these zeros is q, and let $P_1,..,P_{g-1}$ the other zeros. The Abel map of the divisor $P_1+...+P_{g-1}$ is

$$u(P_1)+..+u(P_{g-1}) = K-\zeta$$

where K is the Riemnna's constant:

$$K_k = \frac{-\tau_{k,k}}{2} - \sum_i \int_{u_i} A_i du_k$$

Proof $\log g_{\zeta,q}(p)_{A_i} - \log g_{\zeta,q}(p)_{A_i{}'} = 2\pi i(u_i(p) - u_i(q) + \zeta_i + \frac{1}{2}\tau_{i,i})$. Taking the differential, $d\log g_{\zeta,q}(p)_{A_i} - d\log g_{\zeta,q}(p)_{A_i{}'} = 2\pi i du_i(p)$. It follows that $\sharp$zeros of $g_{\zeta,q} = \sum_i^g \int_{A_i} du_i = g$.
Since $\Theta(\zeta) = 0$ we have $g_{\zeta,q}(q) = 0$. We call $P_g = q$ and $P_1,....,P_{g-1}$ the other zeros. we have

$$\frac{1}{2\pi i}\int_{\partial X_0} u_k(p)d\log g_{\zeta,q}(p) = u_k(q) + u_k(\sum_{j=1}^{g-1} P_j)$$

From

$$\frac{1}{2\pi i}\int_{\partial X_0} u_k(p)d\log g_{\zeta,q}(p) = -\frac{1}{2\pi i}\int_{\partial X_0} \log g_{\zeta,q}(p)du_k(p).$$

we obtain

$$
\begin{aligned}
u_k(\sum_{j=1}^{g} P_j) =& -\sum_i \int_{A_i} du_k(p)(u_i(p) - u_i(q) + \zeta_i + \frac{1}{2}\tau_{i,i} \\
=& u_k(q) - \zeta_k - \frac{1}{2}\tau_{k,k} - \sum_i \int_{A_i} u_i du_k \\
=& u_k(q) + K_k - \zeta_k.
\end{aligned}
$$

we call the set (Θ) of zeros of Θ the Theta divisor. Let W_{g-1} be the set of divisors of $g-1$ points. The above lemma shows the map

$$
\begin{aligned}
W_{g-1} &\to (\Theta) \\
D &\mapsto K - u(D)
\end{aligned}
$$

is an isomorphism.

Next article, we shall complete the proof of Riemann's addition theorem.

References

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