ABEL-JACOBI MAP

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1. Differential forms

Definition 1.1. A *holomorphic* 1-*form* (*resp.meromorphic*) on an open set $V \subset \mathbb{C}$ is an expression ω of the form

$$\omega = f(z)dz$$

where f is a holomorphic(resp.meromorphic) function on V. We say that ω is a holomorphic(resp.meromorphic) 1-form *in the coordinate* z.

Clearly we need some compatibility condintion to define on a Riemann surface. But we don't give exact definition of a differential form of Riemann surface and adopt this definition as local expression of a differentail form of Riemann surface.

We employ the following notation.

$\mathcal{O}(U) = \{$ holomorphic functions $f : U \to \mathbb{C}$ $\}$.
$\Omega^1(U) = \{$ holomorphic 1-forms defined on U $\}$.

2. periods

Definition 2.1. A linear functional $\lambda : \Omega^1(X) \to \mathbb{C}$ is a *period* if it is $\int_{[c]}$ for some homology class $[c] \in H_1(X, \mathbb{Z})$.

Definition 2.2. Let X be a compact Riemann surface. Let Λ be the set of periods. Let $\Omega^1(X)^*$ be dual of the holomorphic 1-form space. The *Jacobian* of X, denoted by Jac(X), is the quotient group

$$\text{Jac}(X)= \frac{\Omega^1(X)^*}{\Lambda}$$

In[3], B. Riemann showed $y^m = h(x)$ ($h(x)$ is an algebraic function or elementary function) can be extended to complex curve by cutting a rays along a branch point. So he conceptually viewed algebraic functions as geometrical sufraces. He compatified the surface and identified $\mathbb{C} \cup \{ \infty \}$'s at branch points:

Theorem 2.3. *Suppose $f : X \to \mathbb{P}^1$ is an n-sheeted holomorphic covering mapping between compact Riemann surface X and $\mathbb{P}^1$. Let g be the topological genus of Riemann surface. Then*

$$g=\frac{b}{2} - n + 1$$

where b is equal to the total mulitipliciity of f minus the number of branch point.

If we choose a basis for $\Omega^1(X)$ and the rank is g, we may consider the Abel-Jacobi map A as mapping to $\mathbb{C}^g/\Lambda$.

Definition 2.4. Fix a base point p_0 in X. The *Abel-Jacobi map* is a map $A : X \to \mathrm{Jac}(X)$. For every point $p \in X$, choose a curve c from p_0 to p; define the map A as follows:

$$A(p) = \left(\int_{p_0}^{p} \omega_1, \int_{p_0}^{p} \omega_2, ..., \int_{p_0}^{p} \omega_g \right)$$

Note that the choice of curve does not affect the value of $A(p)$. Riemann studied more general Abel-Jacobi map. The following map is called canonical map : $\Psi_\alpha \to \mathrm{Jac}(X)$,

$$\Psi_\alpha \{x_1, ..., x_\alpha\} = \left(\sum_{i=1}^{\alpha} \psi_1 x_i, ..., \sum_{i=1}^{\alpha} \psi_g x_i \right).$$

Here $\{\psi_i\}$ denote $\left\{ \sum_{i=1}^{\alpha} \int_{x_0}^{x_i} \omega_1 \right\}$. Suppose g is not minimal in the sense of Abel's addition theorem in the second article. Then we can find rational functions of $x'_1, .., x'_{g-1}$ of $x_1, ..., x_g$ such that $\psi_i x_1 + .. + \psi_i x_g \neq \psi_i x'_1 + .. + \psi_i x'_{g-1} + v_i$ for all i. Then $\Psi_g X^g \subseteq \Psi_{g-1} X^{g-1} + (v_1, .., v_g)$. So If Ψ_g is surjective, Ψ_{g-1} is also surjective. But X^{g-1} is of complex dimension $g-1$, but $\mathrm{Jac}(X)$ is of complex dimension g. In short, the g in the Abel's theorem is the genus of Riemann surface made by $w^2 = P(x)$

References

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[2] Olav Afrnfinn Laudal, Ragni Piene *The legacy of Niels Henrik Abel.* Springer-Verlag, New York, 2004
[3] B. Riemann, *Theorie der Abelschen Functionen.* J. für Math. 54, 1857