The Cycle-Stationary Subgame Perfect Equilibrium in Legislative Bargaining without Replacement*

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Abstract

This paper studies infinite-horizon sequential bargaining among $n \geq 3$ players in which the proposer—a player who proposes a distribution of an economic surplus—is randomly selected from the pool of potential proposers. If the proposal is rejected, the current and previous proposers are excluded from the pool of potential proposers and the game moves on to the next round, until every player has had the same number of opportunities to be the proposer. To analyze the infinite-horizon model with a particular time dependency within each sequence of n rounds (a cycle,) I address the equilibrium characterization with an extended notion of stationarity, which I call *cycle-stationary subgame perfect* (CSSP) equilibrium. The CSSP equilibrium is unique, and it is analogous to the subgame perfect equilibrium of some specific forms of finite-horizon bargaining. Even when every player is fully patient or there is no penalty for delay, the proposer's share in the CSSP equilibrium is strictly smaller than that predicted by the stationary equilibrium of the Baron–Ferejohn legislative bargaining model under any voting rule except unanimity.

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1 Introduction

Multilateral bargaining is a political process in which many agents with conflicting preferences try to divide an economic surplus ("pie") in a democratic way. The process can be summarized as follows: One member of the group proposes a division of the pie, and the proposal is voted on. If the proposal is agreed to by a predetermined number of members, the division is implemented. Otherwise, the procedure is repeated.

The Baron–Ferejohn (Baron and Ferejohn (1989), henceforth BF) legislative bargaining model is the most renowned prototype study in the literature on multilateral bargaining. It extends the Romer–Rosenthal agenda setter model, where the game ends in the status quo outcome if the agenda setter's proposal is voted down, to allow the process of proposing and voting to continue indefinitely. It also extends the Rubinstein–Stahl alternating bargaining model, where two players can alternately propose offers—infinitely many times if necessary—in order to reach unanimity, to a more general n-person approach to bargaining over political decisions and to the use of voting rules other than unanimity. One theoretical feature of the BF model is that in their infinite-horizon game, virtually any distribution of feasible payoffs can be supported in an equilibrium.[1] Consequently, it was natural to restrict attention to stationary (history- and time-independent) strategies. In order for a stationary strategy to constitute an equilibrium, all rounds in the infinite-horizon game should be structurally equivalent. The "random recognition process," where in each round the proposer is recognized at random by a chairperson, implies structural equivalence. It is known that under the random recognition process, uniqueness of stationary equilibrium payoffs is guaranteed for a wider set of generalized models (Eraslan (2002) and Eraslan and McLennan (2013)). The literature on legislative bargaining has followed the tradition of adopting the random recognition process,[2] not only because it is needed in order to enjoy the theoretical luxury of yielding predictions regarding the stationary equilibrium, but also because the random recognition process can be understood and interpreted in a natural way as follows: Since all legislators value the chance to be a proposer and seek to be recognized by the chair in each session of the legislature, the chair may want to employ the random recognition rule, as it

[1] See Theorem 6.1. of Austen-Smith and Banks (2005), which can be understood as an example of a class of results known as "folk theorems."

[2] Banks and Duggan (2000), Diermeier and Merlo (2000), Jackson and Moselle (2002), Norman (2002), Eraslan (2002), Snyder, Ting, and Ansolabehere (2005), Battaglini and Coate (2007), and Volden and Wiseman (2007) are theoretical studies that adopt the random recognition process to address different issues in regard to political decisions. This list is not complete, and it would be even longer if it included experimental studies whose main purpose is to test theoretical predictions made under the assumption of random recognition.

guarantees ex-ante fairness to some degree.[3]

In this paper I modify the BF model by considering random recognition *without* replacement as a proposer selection process. If the proposer selection process can be understood as sampling from a population, the protocol used in the BF model is random sampling with replacement. Admittedly, there has been a clear gap between what multilateral bargaining theory assumes and what is known to occur in practice: Previous proposers are not likely to be a proposer again. The random recognition process (with replacement) allows the current proposer to be recognized again in the next round. However, if theoretical analysis aims to help us understand actual multilateral bargaining within social groups, then it could make more sense to assume that the chair does not allow one legislator to propose consecutively, or at least does not allow a legislator to make his/her next proposal until all members have been granted the same number of opportunities to make proposals. Moreover, the idea of random recognition without replacement as the proposer selection process is closely related to the "one bite at the apple" principle that is often explicitly considered in legislative and judiciary processes. Broadly speaking, this principle means that each individual/agent/party has only one chance to take advantage of an opportunity.[4] In the case of an opportunity to be a proposer, the random recognition process without replacement exactly captures the "one bite at the apple" principle: Members who are recognized as proposers (members who have already "bitten at the apple") cannot again be a proposer (cannot take another bite) until everyone has bitten at the apple once. Specifically, I consider a sequential multilateral bargaining model in which, within a given session of the legislature, if a randomly selected member proposes in the first round and her proposal is not accepted, she will not have another chance to propose until everyone else has proposed once. If the legislature has n members, then in the second round the other $n-1$ members have an equal chance to be recognized, in the third round (if the proposal in the second round fails) the remaining $n-2$ members have an equal chance to be recognized, and so on. I call a sequence of n rounds of proposals a *cycle*, as it expresses that the opportunity of being a proposer is equally dis-

[3]In the general case, not all members have the same probability of being recognized by a chairperson. For example, senior members are more likely than junior members to be recognized as a proposer. Even with unequal recognition probabilities, randomness plays an important role in capturing the tensions between the proposer and nonproposers. If, on the other hand, the order of proposers is determined prior to the bargaining (Breitmoser (2011)) or some members who won't be the proposer in the next round of bargaining are indicated (Ali, Bernheim, and Fan (2014)), then the outcome of the bargaining depends largely on such deterministic information.

[4]For example, in a speech on the Senate floor on August 1, 2001, Mr. Bond said, "Under current law, you only get one incentive period, one bite at the apple. That's it." https://www.congress.gov/crec/2001/08/01/CREC-2001-08-01-pt2-PgS8579.pdf, page S8598.

tributed among all the members, or that everyone bites at the apple exactly once in every n rounds. Every member is reinstated to the pool of potential proposers at the end of each cycle. This modification of the sequential bargaining model can bridge the gap between theory and practice while maintaining the property of ex-ante fairness.

I characterize the subgame perfect equilibrium of legislative bargaining under random recognition without replacement for a q-quota voting rule. For infinite-horizon bargaining where the idea of random recognition without replacement is maintained within every cycle, I introduce cycle stationarity, a new notion of stationarity, and show the uniqueness of the cycle-stationary subgame perfect equilibrium. That equilibrium has the following two notable properties: (1) The first proposer's equilibrium strategy is to offer x to each of $q-1$ randomly selected players, and to keep the remaining share, $1-(q-1)x$, for herself, where n is the size of the legislature. Such x is between $\frac{\delta}{n}$, the stationary subgame perfect equilibrium offer in the BF model, and $\frac{\delta}{n-1}$, the subgame perfect equilibrium offer in one-cycle (n-round) finite-horizon bargaining in which random recognition without replacement is adopted as the proposer selection rule, where δ is the common discount factor. (2) Though the out-of-the-path equilibrium behavior is distinct, the cycle-stationary subgame perfect equilibrium also predicts that delay does not occur, and many theoretical predictions are the same as, or at least quantitatively similar to, those for the Baron–Ferejohn equilibrium. The proposer will have the largest share in the BF model and the smallest share in one-cycle bargaining under random recognition without replacement. Even if the players are fully patient, that is, if $\delta = 1$, the difference in the proposer's share still exists.

The rest of the paper is organized as follows. In the following subsection I discuss the related literature. Section 2 describes the model. Section 3 characterizes the subgame perfect equilibrium for a finite-horizon (n-round) legislative bargaining process as an illustration of infinite-cycle bargaining. In Section 4 I describe the cycle-stationary subgame perfect equilibrium for an infinite horizon and establish the uniqueness of the equilibrium payoffs. In Section 5 I compare the equilibrium strategy of my model with that of the BF model. Section 6 concludes. Proofs of some of the lemmas and propositions that appear in the main text, as well as statements and proofs of additional lemmas, are provided in the Appendix.

1.1 Related Literature

This paper contributes to the legislative bargaining literature theoretically. Modification of the random recognition process was considered by Yildirim (2007), who studies a sequential bargaining approach in which the probability that a given agent will be recognized as a proposer is proportional to the ratio of that agent's level of effort to the aggregate effort of all agents;[5] Breitmoser (2011), who considers a model allowing for priority recognition of some committee members; Bernheim, Rangel, and Rayo (2006), who focus on recognition orders where no individual is recognized twice in succession for pork barrel policies; and Ali, Bernheim, and Fan (2014), who assume that some players can be ruled out as the next proposer. It is worth comparing this study with Ali, Bernheim, and Fan (2014). In the case of the unanimity rule with no penalty for delay, in my model of finite-horizon (n-round) legislative bargaining without replacement the first proposer gets none of the economic surplus in equilibrium, which is the completely opposite prediction for another extreme case addressed by Ali, Bernheim, and Fan (2014), where in equilibrium the first proposer takes the entire economic surplus if the recognition procedure permits legislators to rule out some minimum number of proposers in the next round. My study has a common concern with Ali, Bernheim, and Fan (2014) in regard to the random recognition rule adopted in the BF model, and both studies illustrate that the proposer recognition procedure significantly affects equilibrium outcomes. I view their study as being complementary to mine. I consider a recognition rule in which the current proposer is not allowed to be the next proposer, while they consider a recognition rule in which there are d players that are not allowed to be the next proposer. In the former case the current proposer has to win over the nonproposers who have a higher continuation value than she does, while in the latter case the current proposer exploits those who have a "cheaper" vote. Since random recognition without replacement is implicitly concerned about ex-ante fairness toward other legislators in terms of proposer opportunities, this study goes in the direction opposite that of studies which considered a persistent agenda setter, such as Diermeier and Fong (2011) and Jeon (2016).

Several studies have taken a different view of Baron and Ferejohn (1989) and qualitatively supported the stationary subgame perfect equilibrium of the BF model. Baron and Kalai (1993) show that the Baron–Ferejohn equilibrium is the simplest one in the sense that it minimizes the number of automaton states needed to implement a subgame perfect equilibrium. Chatterjee and Sabourian (2000) show that the unique stationary subgame perfect equilibrium allocation is sustained only by a "noisy Nash equilibrium" of a game with complexity costs in which the strategies make errors with arbitrarily small probability. In the sense that my study considers a more reasonable assumption regarding the proposer selection rule and yields qualitatively the same or similar predictions as the BF equilibrium, I claim that it

[5] Evans (1997) also assumes that recognition probabilities depend on the players' effort levels but considers a different game, where the members of the coalition that accepts a proposal leave the game and the remaining members continue to the next round.

indirectly supports the stationary subgame perfect equilibrium and theoretical studies built upon the assumption of random recognition.

2 The Model

Consider a legislature consisting of n members indexed by $i \in \{1, 2, \ldots, n\} \equiv N$, where n is an odd number greater than or equal to 3. The legislature decides how to allocate a fixed economic surplus (normalized to 1) among themselves. In round 1, one of the members is randomly selected to make a proposal, and all the members have equal probability of being selected. The proposal is immediately voted on. If the proposal is supported by the predetermined q-quota voting rule (i.e., at least q members vote for it),[6] the game ends and payoffs accrue according to the proposal. If, on the other hand, the proposal is not supported by at least q members, the process is repeated in round 2, but the new proposer is selected at random from all the members except the first proposer. The recognition probability is proportionally updated, that is, the recognition probability of every player who is in the running in round 2 is $\frac{1}{n-1}$. Delay is costly: In each round the utility is discounted by a common factor $\delta \in [0, 1]$. Formally, in round $t \in \{1, 2, \ldots\}$, a randomly recognized player makes a proposal p^t, where p^t is a distribution plan $(p_1^t, \ldots, p_n^t)$ such that $\sum_{i=1}^n p_i^t = 1$ and $p_i^t \geq 0$ for all $i \in N$. If the proposal is supported by at least q players, including the proposer in round t, then the game ends and player i receives $\delta^{t-1} U^i(p^t)$, where $U^i(p^t)$ is member i's undiscounted utility from the approved proposal p^t. Players are assumed to be risk neutral and self-interested, so $U^i(p^t) = p_i^t$. If the proposal in round t is not approved and $t \bmod n$ (the remainder in the division of t by n) is strictly greater than 0, then the proposer is excluded from the pool of potential proposers, and the game goes on to round $t + 1$. If the proposal is not approved and $t \bmod n = 0$, then every member is reinstated to the pool of potential proposers, and the game goes on to round $t + 1$. This process continues until a proposal is eventually supported by at least q members. If no allocation is ever accepted, each player receives a payoff of 0. In the sense that all the members propose once in rounds 1 through n, once in rounds $n + 1$ through $2n$, and so on, I call such a sequence of n rounds a *cycle*. Since this process could continue for infinitely many rounds, I call it *infinite-cycle bargaining*.

Let h^t denote the history at round t that includes the identities of the previous proposers

[6] The simple-majority rule is the $\frac{n+1}{2}$-quota rule, and the unanimity rule is the n-quota rule.

and the current proposer.[7] Let $\{a_i^t(h^t), x_i^t(h^t)\}$ denote a feasible action for player i in round t, where $a_i^t(h^t) \in \Delta(P)$ is the (possibly mixed) proposal offered by player i as the proposer in round t and $x_i^t(h^t)$ is the voting decision threshold of player i as a nonproposer in round t; $P = \{(p_1, p_2, \ldots, p_n) | \sum_{i=1}^n p_i = 1, \text{ and } p_i \geq 0 \forall i\}$ is the set of feasible proposals and $\Delta(P)$ is the set of probability measures on P. A strategy s_i is a sequence of actions $\{a_i^t(h^t), x_i^t(h^t)\}_{t=1}^{\infty}$, and a strategy profile s is an n-tuple of strategies, one for each player.

One of the typical solution concepts adopted for the infinite-horizon multilateral bargaining game is the stationary[8] subgame perfect[9] (SSP) equilibrium. However, we cannot restrict our attention to stationary strategies, because the subgame from round t is not structurally equivalent to that from round $t+1$ for any t, that is, any strategy profile is time-dependent by its very nature. One notable feature of infinite-cycle bargaining, however, is that the subgame in round t is structurally equivalent to the subgame in round $t+n$ for any t. I will use this feature to introduce an extended notion of stationarity, where a strategy profile is time-dependent within a cycle but the pattern of the time dependency is uniform from one cycle to another.

3 An Illustration: One-Cycle Bargaining

Before examining infinite-cycle bargaining, I consider a simpler game to illustrate the basic structure of the model. Specifically, I consider one-cycle bargaining governed by a q-quota rule, where the game ends either when a proposal is accepted before round n or when no proposal wins by the end of round n. Payoffs are 0 if no proposal wins by the end of round n. This would correspond to the case where benefits expire after a fixed time.

The solution concept for this finite-horizon game is a symmetric subgame perfect equilibrium. Backward induction is applied. Player i's pure symmetric strategy $a_i^t(h^t)$ for every t is described by the distribution plan $p^t = (p_1^t(h^t), \ldots, p_n^t(h^t))$ that player i will propose if selected in round t and the cut-off $x^t(h^t)$ such that player i will vote to accept any proposal that gives her at least x^t. To figure out what a symmetric equilibrium looks like, consider the problem of the player selected to be the proposer in round t. She obviously wants to get her proposal passed but to do so in a way that gives her district the largest share of

[7]The past history may also contain the previous proposers' proposals and the number of players who voted for each of them; however, the only way in which the identities of the previous proposers matter is in the characterization of the equilibria.

[8]A strategy profile is stationary if $a_i^t(h^t) = a_i$ for any t and h^t.

[9]A strategy profile is subgame perfect if and only if a single player's deviation at any t does not make him better off.

the budget. She therefore needs to form a minimum winning coalition (MWC) consisting of herself and $q-1$ other players. As is typical in the literature, I assume that a player votes for a proposal when she is indifferent between voting for it and voting against it. Therefore, one immediate prediction is that if the game is in round q or later, the proposer can keep all the resources for herself.

Lemma 1. *Consider one-cycle bargaining under a q-quota rule. In the subgame perfect equilibrium, the randomly selected proposer, player i, will propose $p_i^t = 1$ and $p_j^t = 0$ for all $j \neq i$ and all $t \geq q$.*

Proof: The fact that a game has reached round q implies that there are $q-1$ previous proposers, who cannot be the proposer again and thus have lost their bargaining power. This implies that there are at least $q-1$ players who will vote for a payoff of 0 in round q or later. □

For terminological clarity, I divide the set of players other than the current proposer into two groups: The previous proposers comprise the *trivial coalition pool*, because they would accept any offer. The *nontrivial coalition pool* consists of the players who have not yet been selected as a proposer. In and after round q, the trivial coalition pool (plus the proposer) constitutes an MWC. Therefore, Lemma 1 shows that in and after round q, it is perfectly safe for the recognized member to propose keeping the entire economic surplus for herself. Also by Lemma 1, we can use backward induction from round q. The proposer's strategy from round 1 to round q in the symmetric subgame perfect equilibrium is described in Lemma 2.

Lemma 2. *Consider one-cycle bargaining under a q-quota rule. In the symmetric subgame perfect equilibrium, the randomly selected proposer for round $q-l$ ($l = 0, 1, \ldots, q-1$) offers $\frac{\delta}{n-(q-l)}$ to l randomly selected players from the nontrivial coalition pool.*

Proof: See Appendix.

Lemmas 1 and 2 state jointly that if some round later than the first round were reached, the randomly recognized proposer in round t would offer a positive share of the resources to $\max\{0, q-1-\#\text{trivial coalition pool}\}$ players randomly selected from the nontrivial coalition pool to form an MWC, and the offered amount is δ divided by the number of players who have not yet proposed. The symmetric subgame perfect equilibrium of one-cycle bargaining under a q-quota rule is described in Proposition 1.

Proposition 1 (q-Quota Rule, One-Cycle Legislative Bargaining)**.** *Consider one-cycle bargaining under a q-quota rule. Each player's symmetric equilibrium strategy is summarized by* $\{x^k, \max\{q-k, 0\}\}_{k=1}^{n}$*, where the randomly recognized proposer for round k offers* $x^k = \frac{\delta}{n-k}$ *to* $\max\{q-k, 0\}$ *players randomly selected from those who have not yet proposed. In round k, previous proposers accept any offer, and the* $n-k$ *players who have not yet proposed accept offers of at least* x^k*.*

Therefore, the randomly selected proposer for round 1 offers $\frac{\delta}{n-1}$ *to* $q-1$ *players selected at random, and she gets* $1 - \frac{q-1}{n-1}\delta$*.*

Proof: By Lemma 1, in round $t = q$ or later, the randomly selected proposer i will propose $p_i^t = 1$ and $p_j^t = 0$ for all $j \neq i$. By Lemma 2, the randomly selected proposer for round $q-l$ ($l = 0, 1, \ldots, q-1$) offers $\frac{\delta}{n-(q-l)}$ to l randomly selected players from the nontrivial coalition pool. □

It is notable that the amount offered by the randomly selected first-round proposer to the members of the minimum winning coalition does not depend on q. Corollary 1 describes the equilibrium strategy profile when the voting rule is either simple majority or unanimity.

Corollary 1 (Majority/Unanimity, One-Cycle Legislative Bargaining)**.** *When the predetermined voting rule is simple majority, in equilibrium the players other than the randomly selected first-round proposer accept offers of at least* $\frac{\delta}{n-1}$ *in round 1. Thus the randomly selected first proposer offers* $\frac{\delta}{n-1}$ *to* $\frac{n-1}{2}$ *players selected at random, and she gets* $1 - \frac{\delta}{2}$*.*

When the predetermined voting rule is unanimity, in equilibrium the players other than the randomly selected first-round proposer accept offers of at least $\frac{\delta}{n-1}$ *in round 1. Thus the randomly selected first-round proposer offers* $\frac{\delta}{n-1}$ *to all* $n-1$ *players except herself, and she gets* $1-\delta$*.*

Proof: Set $q = \frac{n+1}{2}$ and $q = n$ in Proposition 1 for the simple-majority rule and the unanimity rule, respectively. □

Several important observations in regard to one-cycle legislative bargaining can be made. First, both under simple majority and under unanimity the first-round proposer's share does not depend on n, the size of the legislature, in the symmetric subgame perfect equilibrium. Second, in the stationary subgame perfect equilibrium of the BF model the initial proposer keeps $1 - \frac{n-1}{2n}\delta$ for herself under simple majority. As n goes to infinity, that amount converges to $1 - \frac{\delta}{2}$. Therefore, although one-cycle bargaining is a finite-horizon game, it can be understood as a way to achieve the limiting outcome of the stationary equilibrium of the BF model. Third, under unanimity, that is, if $q = n$, the first proposer gets $1 - \delta$. If $\delta > \frac{n-1}{n}$, the proposer's share is strictly *smaller* than that of the nonproposers. When $\delta = 1$, she gets *nothing*. This "proposer disadvantage" is not observed in the BF model, where under unanimity the first proposer gets $1 - \frac{n-1}{n}\delta$. In that model, she gets a strictly larger share than the nonproposers if $\delta \in [0, 1)$, and an equal share if $\delta = 1$.

The theoretical prediction for the finite-horizon, unanimity, no-discount bargaining game (i.e., the one in which the first proposer gets nothing) is somewhat unintuitive, but this is the only symmetric subgame perfect equilibrium. To verify this, consider $n = 3$ and $\delta = 1$. For notational simplicity, a proposal is arranged in such a way that the kth proposer's share is the value of the kth entity. In the third (last) round, the proposer offers (0,0,1). All the previous proposers accept this proposal, because it is the final round. Knowing that the player who would end up being the proposer in the third round will reject any offer less than 1 in round 2, the second-round proposer offers (0,0,1), which is approved by all the players. The first-round proposer, who knows that one of the other two players (the one who would end up not being selected as the proposer in the second round) will get the entire economic surplus in round 2, offers $(0, 1/2, 1/2)$, so that the nonproposers' continuation value is the same as the amount being offered.

The intuition behind the observations derived from Proposition 1 and Corollary 1 can be explained by the nonproposers' increased negotiating power. This is in contrast to many existing studies, including Ansolabehere, Snyder, Strauss, and Ting (2005) and Ali, Bernheim, and Fan (2014), that report a formateur's significant negotiating power. In the infinite-horizon game, the random recognition process with replacement endows a proposer with negotiating power. However, in the finite-horizon game without replacement, nonproposers, especially the members of the nontrivial coalition pool, share negotiating power, because if they reject the current proposal, they benefit from both a higher chance of being the proposer in a later round and a larger number of players in the trivial coalition pool in that later round.

4 Infinite-Cycle Bargaining

The idea of random recognition without replacement which is used in one-cycle bargaining can be applied to infinite-cycle legislative bargaining, where every member's chance of being a proposer is reinstated at the beginning of each cycle. By interpreting a pair of offers (an offer and a counteroffer) in the Rubinstein–Stahl model as a single cycle in two-player bargaining, this infinite-cycle legislative bargaining can be understood as a more relevant extension of the Rubinstein–Stahl bargaining model than of the BF model.

In the infinite-cycle game, each cycle consists of n consecutive rounds. The first cycle consists of rounds 1 through n, the second cycle consists of rounds $n+1$ through $2n$, and so on. No player can be the proposer in two rounds of the same cycle. The recognized proposer in round t is chosen randomly from all the players who have not yet been the proposer during the cycle in which round t occurs, and all of those players have an equal probability of being recognized. The number of players from whom the recognized proposer is chosen in round t is $n - \tau + 1$ if $\tau > 0$, and 1 if $\tau = 0$, where $\tau = t \bmod n$. If the proposal made by the recognized proposer in round t is not approved, then that proposer is excluded from the pool of potential proposers during the remaining rounds of that cycle and the game goes on to round $t+1$. In rounds of the form $kn+1$ where k is a nonnegative integer, all n players are equally likely to be recognized as the proposer. In rounds of the form kn where k is a positive integer, there is only one player who has not proposed within the current cycle, so she will automatically be the proposer. This process continues until a proposal is passed. Since the number of potential proposers decreases in each round within a cycle, but jumps to n when a new cycle starts, I represent a round as a function of $t \bmod n \equiv \tau$.

The legislative bargaining literature has focused on the SSP equilibria as a solution concept, which is a natural refinement for the infinite-horizon game in which virtually any allocation can be supported as an equilibrium. I also adopt the idea of SSP equilibria here, but stationarity should be defined in a different way, since the proposal strategies and voting decision rules are ($t \bmod n$)-dependent. That is, the equilibrium proposal strategies and voting decision rules are time- and history-dependent within a cycle, but uniform from one cycle to another. Formally, a strategy profile is *cycle stationary* if $a_i^t(h^t) = a_i^{t'}(h^{t'}) = a_i^{\tau}(h^{\tau})$ for any t, t', h^t, and $h^{t'}$ such that $t \bmod n = t' \bmod n = \tau$. A strategy profile is said to be *cycle-stationary subgame perfect* (CSSP) if it is both cycle stationary and subgame perfect. A CSSP outcome and payoff are the outcome and payoff generated by a CSSP strategy profile.

Similarly to my analysis of one-cycle bargaining, in round t I subdivide the set of players other than the current proposer into two groups: The *trivial coalition pool*, $T \subset N$, consists of the players who have proposed within the current cycle but prior to round t, that is, the players who have proposed in one of the rounds $t-\tau+1, t-\tau+2, t-\tau+3, \ldots, t-1$. The *nontrivial coalition pool*, $NT \subset N$, consists of the players who have not been selected as a proposer in the current round or in an earlier round of the current cycle. In round 8 for a legislature with 5 members, for example, the trivial coalition pool consists of the proposers in rounds 6 and 7, and the nontrivial coalition pool consists of all players except those two former proposers and the current proposer. As we will see shortly, the players in the trivial coalition pool are more likely to be included as coalition partners than those in the nontrivial coalition pool. Let x^t_{NT} and x^t_T denote the shares of the economic surplus which in round t are offered to some players in the nontrivial coalition pool and to some players in the trivial coalition pool, respectively.

The CSSP equilibrium is characterized as follows:

Proposition 2 (The Cycle-Stationary Subgame Perfect Equilibrium)**.** *Consider infinite-horizon legislative bargaining without replacement within a cycle, with $n \geq 3$ players (n odd), a q-quota rule, and a common discount factor $\delta \in [0, 1]$. A strategy profile is a cycle-stationary subgame perfect equilibrium if and only if it has the following form:*

- *In round t with $t \bmod n \equiv \tau$,*
  - *If $\tau \geq q$ or $\tau = 0$, the recognized proposer offers x^τ_T to $q-1$ players selected at random from the trivial coalition pool.*
  - *If $\tau \in [2, q)$, the recognized proposer offers x^τ_T to all $\tau - 1$ players in the trivial coalition pool, and x^τ_{NT} to $q - \tau$ players selected at random from the nontrivial coalition pool.*
  - *If $\tau = 1$, the recognized proposer offers x^τ_{NT} to $q-1$ players selected at random from the nontrivial coalition pool.*
- *Players in the trivial coalition pool accept any offer of at least x^τ_T, and players in the nontrivial coalition pool accept any offer of at least x^τ_{NT}.*

Therefore, in equilibrium, the recognized proposer in round 1 proposes x^1_{NT} to $q-1$ players selected at random, and the game ends, where $x^1_{NT} = \frac{\delta}{n-1}(1 - x^2_T)$ and x^τ_T is recursively determined by $x^{q-l}_T = \delta x^{q-l+1}_T$ for $l = 1, 2, \ldots, q-1$, and $x^q_T = \frac{(q-1)!}{n!}(q-1)^{n-q}\delta^{n-q+1}$. Also, $x^1_{NT} \in [\frac{\delta}{n}, \frac{\delta}{n-1}]$.

Proof: See Appendix.

A key result that figures in the characterization of the CSSP equilibrium is that in the final round of each cycle, the equilibrium strategy will be identical to that from the Baron–Ferejohn model, which is described in Lemma 3.

Lemma 3. *In the cycle-stationary subgame perfect equilibrium, if some round t with $t \bmod n = 0$ is reached, the proposer offers $\frac{\delta}{n}$ to $q-1$ players selected at random, and keeps $1-(q-1)\frac{\delta}{n}$ for herself.*

Proof: For any round t with $t \bmod n = 0$, the recognized player is the last proposer within the current cycle. As a new cycle starts in the following round, every player will have an equal chance of being recognized as the proposer in that following round. Suppose that in equilibrium the round-t proposer offers x_T^0 to $q-1$ players selected at random and keeps $1-(q-1)x_T^0$ for herself. A player who receives an offer of x_T^0 will accept it if

$$x_T^0 \geq \delta\left(\frac{1}{n}(1-(q-1)x_{NT}^1) + \frac{q-1}{n}x_{NT}^1\right),$$

where x_{NT}^1 is the amount offered to each of $q-1$ members of the nontrivial coalition pool in the first round of a new cycle. The first term on the right-hand side of the inequality above corresponds to the event that the player is recognized as the proposer in round $t+1$, and the second term corresponds to the event that the player is not recognized as the proposer in round $t+1$ but is included in the minimum winning coalition of the proposer in round $t+1$. Although we have not fully characterized what x_{NT}^1 is, the terms on the right-hand side of the inequality above that include a factor of x_{NT}^1 cancel out, thereby leaving just $\frac{\delta}{n}$. Subgame perfection implies that the proposer need be no more generous than to offer the discounted continuation value to a minimum number of members and 0 to the others. Therefore, in every round t with $t \bmod n = 0$, the proposer offers $\frac{\delta}{n}$ to $q-1$ players selected at random. □

Thus, we can use backward induction from round n to round 1, or from the last round within any cycle to the first round within that cycle. Once we characterize the unique sequence of proposal plans and voting decision rules for one proposal cycle, we will verify that such a sequence indeed characterizes a unique cycle-stationary subgame perfect equilibrium. Another finding that makes the equilibrium characterization simpler is that players in the trivial coalition pool, that is, the previous proposers within the current cycle, would be the first players to be considered as coalition partners, because they are, in a sense, cheaper.

Lemma 4. *Let τ denote $t \bmod n$. Then $x_T^\tau < x_{NT}^\tau$ for $\tau \in \{2, \ldots, n-1\}$. Therefore, a proposer includes $\max\{\tau-1, q-1\}$ players randomly selected from the trivial coalition pool as coalition partners.*

Proof: Note that x_{NT}^0 is not defined when $t \bmod n = 0$, because the nontrivial coalition pool is empty. Similarly, x_T^1 is not defined when $t \bmod n = 1$, because the trivial coalition pool is empty. We restrict our attention to nontrivial cases, that is, rounds t with $t \bmod n \in \{2, \ldots, n-1\}$. Suppose that in any round t with $t \bmod n = \tau$, the proposer treats all the other members identically, that is, $x_T^\tau = x_{NT}^\tau = x^\tau$, and she selects $q-1$ coalition partners at random. Then the continuation value of the players in the trivial coalition pool is $\delta \frac{q-1}{n-1} x^{\tau+1}$, and that of the players in the nontrivial coalition pool is $\delta \left(\frac{1}{n-\tau}(1-(q-1)x^{\tau+1}) + \frac{q-1}{n-1} x^{\tau+1} \right)$. Since the latter is strictly greater than the former, the proposer would be strictly better off by retracting an offer made to one player in the nontrivial coalition pool and offering a strictly smaller share to one player in the trivial coalition pool who wasn't included initially, which is a contradiction. Analogously, one can show that $x_T^\tau > x_{NT}^\tau$ also leads to a contradiction. Therefore, if $\tau \leq q$, all the players in the trivial coalition pool will be included as coalition partners, and if $\tau > q$, $q-1$ of the players in the trivial coalition pool will be randomly selected as the coalition partners. $\square$

Except in the final round of a cycle, the players in the trivial coalition pool do not have a chance of being the proposer in the following round. Thus, their continuation value is always smaller than those who haven't yet been recognized in the current cycle, and the latter are are more likely to be recognized in the near future. By Lemma 4, it makes sense to call the set of previous proposers in the current cycle the trivial coalition pool. Though all the players have a common discount factor, the players in the trivial coalition pool can be regarded as having a smaller discount factor, and hence the proposer will try to win them over first.[10]

The remaining proofs for the characterization of the CSSP are given in the Appendix, but the logic is straightforward: The equilibrium strategy and voting decision rule in the last round within a cycle mimic those of BF. From the last round within a cycle, backward induction is applied, and the proposer distinguishes those who have proposed within a cycle from those who have not.

The following example with $n = 3$ under the simple-majority rule illustrates the procedure. For notational simplicity, proposal p is arranged in such a way that the proposer's share in round k is the kth entity, and the other nonzero entity is the share offered to the member (randomly selected if necessary) of the pertinent coalition. This does not mean that

[10] However, the result in Lemma 4 cannot be applied in a more general model with heterogeneous discount factors and unequal recognition probabilities. Suppose, for example, that player i has proposed in one of the previous rounds of the current cycle and player j has not. Then the continuation value of player j is smaller than that of player i if δ_j is sufficiently smaller than δ_i or if the recognition probability of player j is sufficiently smaller than that of player i.

the proposer's MWC has to include player $k-1$ or player $k+1$.

- Round 3 proposer: By Lemma 3, she proposes $(0, \delta/3, 1-\delta/3)$, and the player who receives the offer of $\delta/3$ (in this example player 2) will accept it since his continuation value does not exceed the utility yielded by the amount which he is offered.

- Round 2 proposer: By Lemma 4, she proposes $(x_T^2, 1-x_T^2, 0)$. The sole player in the trivial coalition pool (in this example player 1), who receives the offer of x_T^2 in round 2, knows that if he rejects the offer he will earn $\delta/3$ with probability $1/2$ in round 3, so $x_T^2 = \delta^2/6$. Thus the proposal made in round 2 is $(\frac{\delta^2}{6}, 1-\frac{\delta^2}{6}, 0)$, and the player who receives the offer of $\frac{\delta^2}{6}$ accepts it.

- Round 1 proposer: She proposes $(1-x_{NT}^1, x_{NT}^1, 0)$. The continuation value for the player in the nontrivial coalition pool who receives the offer of x_{NT}^1 in round 1 (in this example player 2) is $\frac{\delta}{2}(1-\frac{\delta^2}{6}) = \frac{6\delta-\delta^3}{12}$, that is, with probability $\frac{1}{2}$ he will be the proposer in the second round and will keep $1-\frac{\delta^2}{6}$ for himself, and the expected payoff is discounted by δ. Thus the proposal is $(1-\frac{6\delta-\delta^3}{12}, \frac{6\delta-\delta^3}{12}, 0)$, and the player who is offered $\frac{6\delta-\delta^3}{12}$ accepts it.

Note that $\frac{\delta}{3} < \frac{6\delta-\delta^3}{12} < \frac{\delta}{2}$ for $\delta \in (0,1]$, where $\frac{\delta}{3}$ is the share offered in the stationary subgame perfect equilibrium of infinite-horizon bargaining in which $n=3$ and random recognition is the proposer selection rule, and $\frac{\delta}{2}$ is the share offered in the subgame perfect equilibrium of one-cycle bargaining in which $n=3$ and random recognition without replacement is the proposer selection rule. In general, the initial proposer's offer in the CSSP equilibrium is always between $\frac{\delta}{n-1}$ and $\frac{\delta}{n}$, where $\frac{\delta}{n-1}$ constructs the subgame perfect equilibrium proposal for the game of n rounds without replacement, and $\frac{\delta}{n}$ constructs the stationary subgame perfect equilibrium proposal for the game of infinite-horizon bargaining in which the random recognition process allows for replacement.

In the following subsections, I show that the theoretical prediction for the cycle-stationary subgame perfect equilibrium coincides with that of many other possible extensions of one-cycle bargaining without replacement.

4.1 Equivalence to $n+1$-Round Bargaining

The theoretical predictions for infinite-cycle legislative bargaining can be attained by $n+k$ rounds of legislative bargaining, for *any* positive integer k. This is because the proposal made in round n, the final round of the first cycle, is the same for any k, regardless of the proposal made in round $n+1$, as long as such a round exists. See Lemma 3.

For example, under the unanimity rule with $n = 3$ and $\delta = 1$, the theoretical prediction of the proposer's share in one-cycle bargaining is 0, while that in infinite-cycle bargaining is 1/3. Indeed, this stark difference between one-cycle bargaining and infinite-cycle bargaining can be attained by simply adding one additional round to one-cycle bargaining. Consider the case of four rounds of bargaining, where in the fourth round all three players are equally likely to be recognized as the proposer. In the fourth round, the randomly selected proposer keeps the entire economic surplus, since it is the final round. In the third round, the proposer offers an equal share to all three players because their continuation value is 1/3. In the second round, where every player's continuation value is $\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{3}$, the proposer offers 1/3 to all three players, including herself. In the first round, with the same logic as for the second round, the proposer offers 1/3 to all three players, including herself. Though the equilibrium proposal strategy and voting decision rules for four-round bargaining are the same as for the symmetric stationary equilibrium in the BF model, the former equilibrium is unique. This clearly illustrates that both the random recognition process and the termination rule significantly affect the equilibrium allocation.

This result is positive in that the cycle-stationary equilibrium, which could be one of the equilibria in infinite-horizon bargaining, coincides with the unique subgame perfect equilibrium for $n + k$ finite-round bargaining with any $k \geq 1$. In other words, we can appreciate the cycle-stationary equilibrium as the *unique approximation* from finite-round bargaining.

4.2 Non-Wasteful Allocation of the Economic Surplus as the No-Agreement Outcome of One-Cycle Bargaining

Though the assumption that agents' payoffs are 0 when they fail to reach agreement by the end of the bargaining period is typical, it may be worthwhile to investigate how modifying the 0 no-agreement outcome of one-cycle bargaining studied in Section 3 to any non-wasteful allocation of the economic surplus could affect the allocation in the subgame perfect equilibrium. Specifically, suppose that if the proposal in round n is rejected, then every member will earn a randomly assigned or predetermined share, which admits the possibility of equal shares for all the players, at the beginning of round $n+1$, and the game ends. As it turns out, the theoretical prediction for this modification is equivalent to that for infinite-cycle legislative bargaining without replacement. The proposer in round n would offer $\frac{\delta}{n}$ to $\frac{n-1}{2n}$ randomly selected members, and those selected members would accept the offer because accepting it gives the same utility as rejecting it: When the proposal is rejected, each player's expected payoff is $1/n$, which is discounted by δ. This allocation is attained in round n in infinite-cycle legislative bargaining as well, so the strategy profile for the first round in the subgame perfect equilibrium of this finite-horizon bargaining game is the same as that in the CSSP equilibrium.

5 Comparison to the BF Model Predictions

To provide a direct comparison to the BF model, I focus on the proposer's share in equilibrium. Other important properties, such as the minimum winning coalition, full rent extraction, and no delay, are shared by all the models considered. Table 1 shows the theoretical predictions of several models when different recognition processes and termination rules are applied. It is trivial that when $q = 1$, that is, under a dictatorship, there is no difference in the proposer's share from one model to another in equilibrium. When $q = n$, that is, under the unanimity rule, the infinite-horizon bargaining models predict a smaller amount for the proposer's share than in the stationary equilibrium of the BF model. Those two are the same only when $\delta = 1$. The proposer's share under the simple-majority rule is $1 - \frac{n-1}{2}x$, where $x = \frac{\delta}{n-1}\left(1 - \delta^{n-1}\frac{\left(\frac{n-1}{2}\right)!}{n!}\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}\right)$, which is always between $\frac{\delta}{n}$ and $\frac{\delta}{n-1}$. For all the models considered, there is no efficiency loss caused by a delay, thus the ex-ante expected value of the game is $1/n$ for each player. When it comes to the ex-ante variance of the value of the game, however, one-cycle bargaining predicts the most egalitarian division of the economic resources given the same q-quota rule.

To better understand the differences in the proposer's share predicted by different models, let $\mathrm{BF}_{q,n,\delta}$ denote the proposer's share in the SSP equilibrium of the BF model under a q-quota rule with n players (n odd) and a discount factor of δ, and let $\mathrm{InfC}_{q,n,\delta}$ denote the corresponding share in infinite-cycle bargaining without replacement. Also, let $q^*(n, \delta)$ denote the q that yields the largest difference $\mathrm{BF}_{q,n,\delta} - \mathrm{InfC}_{q,n,\delta}$.

Proposition 3. *$BF_{q,n,\delta} - InfC_{q,n,\delta} \geq 0$, where equality holds either when $q = n$ and $\delta = 1$ or when $q = 1$. As n increases, $BF_{q,n,\delta} - InfC_{q,n,\delta}$ decreases. $\frac{q^*(n,1)}{n}$ is monotone increasing in n. $q^*(n, 1) \geq \frac{n+1}{2}$ for all $n \geq 3$, and the strict inequality holds for $n \geq 11$.*

Proof: See Appendix.

Table 1: Theoretical Predictions of the Proposer's Share When $\delta = 0.8$

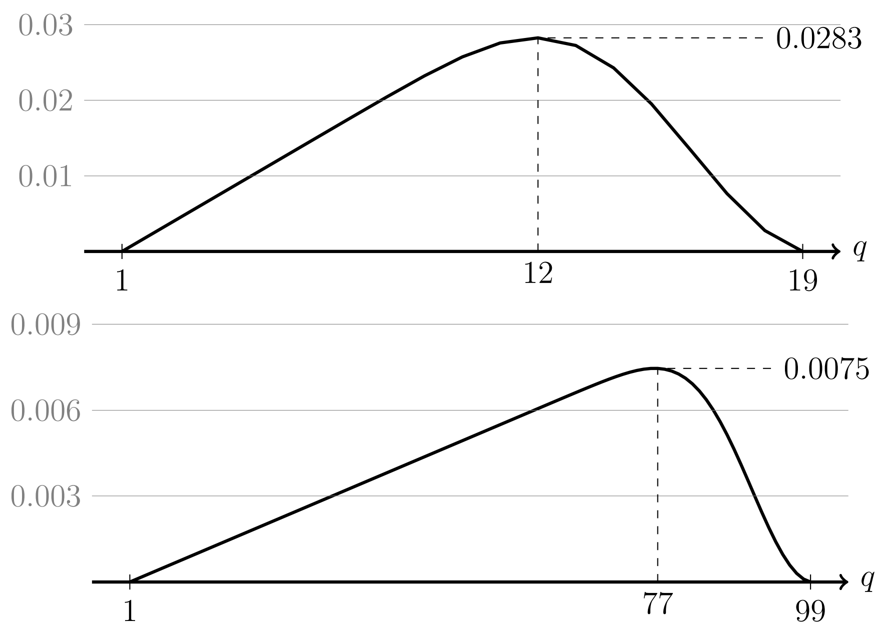
Voting rule	Protocol	$n = 3$	$n = 7$	Proposer's share for general n
Simple Majority	Baron-Ferejohn	0.73	0.66	$1 - \frac{n-1}{2}\frac{\delta}{n}$
	1-Cycle w/o Repl.	0.6	0.6	$1 - \frac{n-1}{2}\frac{\delta}{n-1}$
	∞-Cycle w/o Repl.	0.64	0.61	$1 - \frac{n-1}{2}x,\ x \in \left(\frac{\delta}{n}, \frac{\delta}{n-1}\right)$
Unanimity	Baron-Ferejohn	0.47	0.31	$1 - (n-1)\frac{\delta}{n}$
	1-Cycle w/o Repl.	0.2	0.2	$1 - (n-1)\frac{\delta}{n-1}$
	∞-Cycle w/o Repl.	0.37	0.23	$1 - (n-1)\left(\frac{\delta}{n-1} - \frac{\delta^n}{n(n-1)}\right)$

This table shows the theoretical predictions when the common discount factor, δ, is 0.8 (i.e., a penalty of 20% per delay) and the size of the legislature, n, is either 3 or 7. Under both the simple-majority rule and the unanimity rule, one-cycle bargaining without replacement predicts a smaller share for the proposer than in the BF model, and that share is constant in the size of the legislature. Under unanimity, the proposer's share under the infinite-cycle recognition process is equal to that under the protocol of the BF model. Under unanimity, a notable feature arises when the bargaining protocol is one-cycle without replacement: The proposer's share can be smaller than that of the nonproposers.

Consistent with one-cycle legislative bargaining, in the CSSP equilibrium the proposer has a weakly lesser advantage than in the protocols considered in the BF model. When the size of the legislature is sufficiently small, the difference in terms of the proposer advantage is largest under the simple-majority rule. When the size of the legislature is large, however, a super-majority rule yields the largest difference. In Figure 1, the difference is plotted for $n = 19$ and $n = 99$ as a function of $q \in [1, n]$.

Proposition 3 gives some sense of when using the BF model is innocuous as an analytical tool for legislative bargaining without replacement. Although the assumption of random recognition deviates from reality, it obviously has great theoretical merit, as it is parsimonious, it guarantees structural equivalence in every subgame, it allows many possible extensions, and it enables us to characterize the stationary equilibrium in a straightforward manner. Since all other important properties in equilibrium are shared by all the models considered here, it would be of interest to know when the BF model would be used in cases where random recognition without replacement is the proposer selection rule. Proposition 3 implies that when the size of the legislature is small and the simple-majority rule is applied, it is not a good idea to use the BF model for understanding actual legislative behaviors. Even when $\delta = 1$, that is, when all players are fully patient, the proposer's share in the CSSP equilibrium is strictly smaller than that in the SSP equilibrium except in the case of the unanimity rule. The difference in the proposer's share is substantial: For example, when $n = 19$, the largest difference between the proposer's share in the SSP equilibrium and that in the CSSP equilibrium is 0.0283 at 12-quota rule. This difference is more than a half of the ex-ante expected payoff of each player, 0.0526 (= 1/19). Even though the absolute difference in the prediction of the proposer's share is decreasing in n, the ratio of the difference to the ex-ante expected payoff is increasing in n. These observations suggest that the BF model should be used with some caveats especially when the legislature adopts a super-majority rule and maintain the idea of the "one bite at the apple" rule.

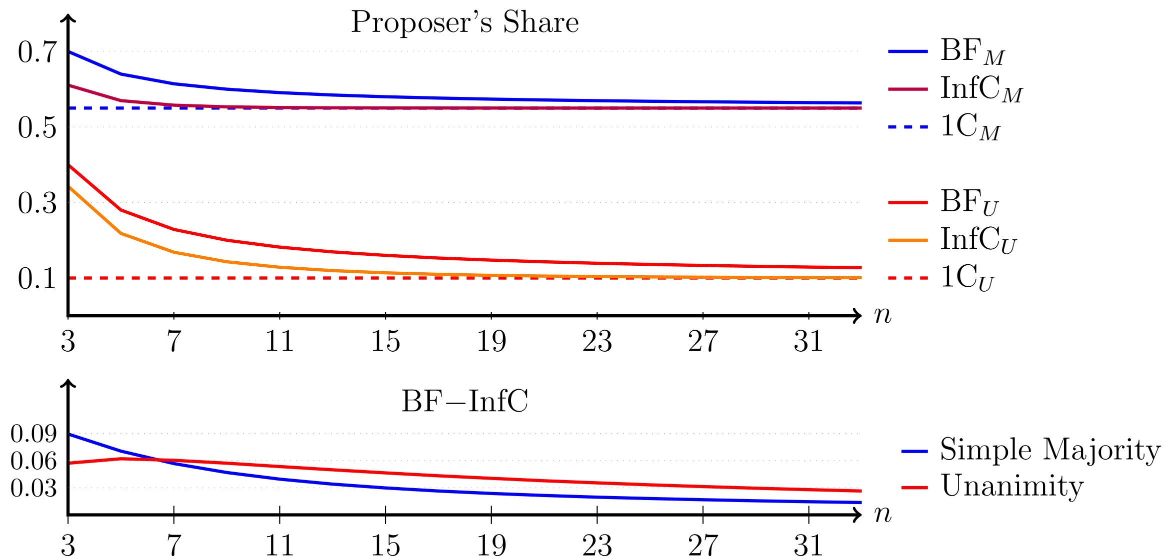
Figure 1: Super-Majority Rules Yielding the Largest Difference



Each graph is a plot of the difference (as a function of q) between the proposer's share in the SSP equilibrium of the BF model and that in the CSSP equilibrium of infinite-cycle bargaining without replacement under a q-quota rule. The size of the legislature is 19 in the upper graph, and 99 in the lower graph. As n gets larger, the maximum difference gets smaller (0.0283 when $n = 19$ and 0.0075 when $n = 99$). The difference is largest at $q^*(n)$, and $q^*(n)/n$ increases in n ($12/19 = 0.6315$ when $n = 19$, and $77/99 = 0.7778$ when $n = 99$).

Another interesting observation is that a super-majority rule yields the largest difference when the size of the legislature is large. Since the difference is strictly positive except either when $q = n$ and $\delta = 1$ or when $q = 1$, casual analysis and an expectation of symmetry may lead to the conclusion that the largest difference occurs under the simple-majority rule, and hence that understanding actual legislative behaviors through the lens of the BF model is reasonable for situations where a super-majority rule is applied. However, this is not the case. For example, when the size of the legislature is around 30, the largest difference in terms of the proposer's share occurs under the 2/3 majority rule.[11] Considering that the average size of the upper house in the legislatures of U.S. states and U.S. territories is 37 (39 if the five U.S. territories are excluded), and a supermajority of the state legislature is required to approve tax increases in fifteen states,[12] this difference should not be overlooked.

Figure 2: Proposer's Share, and Differences between Models, $\delta = 0.9$



The upper graph shows the proposer's share in equilibrium under different proposer recognition rules (random with replacement and random without replacement,) voting rules (simple majority and unanimity,) and termination rules (finite and infinite). BF, InfC, and 1C refer to the BF model, infinite-cycle bargaining, and one-cycle bargaining, respectively. M and U refer to the simple-majority rule and the unanimity rule, respectively. For all n, the proposer's share in the CSSP equilibrium is smaller than that in the SSP equilibrium. The lower graph shows that the difference between these two equilibrium shares is generally decreasing in n but that monotonicity is not guaranteed for small n.

Thus far I have focused on cases with $\delta = 1$. When $\delta = 1$, $\text{BF}_U - \text{InfC}_U = 0$, but this doesn't necessarily mean that for all n and δ, the difference is smallest under unanimity. Though the difference is monotone decreasing in n for any q-quota rule if $\delta = 1$, monotonicity may not be guaranteed for other values of δ. Figure 2 illustrates the differences between models with different proposer selection rules and termination rules when $\delta = 0.9$. The

[11] $q^*(29,1) = 19$, and $q^*(31,1) = 21$. Thus $\frac{q^*(n,1)}{n} \approx \frac{2}{3}$ when n is either 29 or 31.

[12] Source: National Conference of State Legislators (http://www.ncsl.org/issues-research/budget/state-tax-and-expenditure-limits-2010.aspx) Last access: 10/22/2016.

figure shows that when the size of the legislature is larger than 7, the difference is larger under unanimity than under simple majority. In addition, it shows that when the size of the legislature is small, there are substantial differences between the two models. When $n = 3$ and $\delta = 0.9$, under the simple-majority rule the proposer's share in the CSSP equilibrium, 0.6108, is 12.74% smaller than that in the SSP equilibrium, 0.7.

Another noticeable difference, which is not captured by the proposer's share in equilibrium, is the off-the-path equilibrium behavior. Unlike the BF model, which treats the previous proposers in an equal manner, legislative bargaining in which random recognition without replacement is the proposer selection rule predicts asymmetric responses between those who had proposed in the previous rounds and those who had not. However, even though the infinite-cycle bargaining model adopts a more realistic assumption regarding the proposer selection rule and yields more intuitive predictions about the out-of-the-path equilibrium behavior, the predictions for the SSP strategy profile are not qualitatively different from those for the CSSP strategy profile.

6 Concluding Remarks

This paper examines how the equilibrium characterization of a sequential, multilateral bargaining process is affected by the random recognition rule. In the existing legislative bargaining literature, random recognition allows the current proposer to be recognized again in the following rounds, while the model considered here prohibits recognition of any player as the proposer in more than one round until everyone has had the same number of chances to be the proposer. Since this infinite-horizon bargaining game has a specific cyclical pattern, I introduce an extended notion of stationarity, which I call cycle stationarity, and characterize the cycle-stationary subgame perfect equilibrium. Many interesting forms of finite-horizon bargaining have the strategy profile in the CSSP equilibrium as the unique subgame perfect equilibrium. Legislative bargaining under the random recognition process without replacement yields a smaller proposer advantage, but all other features in equilibrium are the same as, or qualitatively similar to, those of the Baron–Ferejohn model. However, the difference in the proposer advantage is quite substantial under super-majority rules, so the Baron–Ferejohn model should be used with some caveats especially when the proposer share is the subject of study.

There are many potential directions for extension of this legislative bargaining model that adopts random recognition without replacement as the proposer selection rule. For example, there would be theoretical merit in extending the model presented in this paper by allowing for individual discount factors and asymmetric recognition probabilities. Further investigation of one-cycle bargaining would be required to determine its welfare implications: Since the simple-majority rule predicts a proposer advantage, and the unanimity rule predicts a proposer disadvantage for some values of the parameters, one can characterize the optimal voting rule that yields the most egalitarian distribution of the economic resources. Conducting lab experiments would help us to gain a better understanding of multilateral bargaining behavior. In particular, out-of-equilibrium-path observations in the laboratory would shed some light on other factors that could or should be accounted for in the model.

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Appendix A: Omitted Proofs

Proof of Lemma 2: Let's first consider trivial cases. When $l = 0$, that is, when qth round is reached, the proposer will keep the entire budget by Lemma 1. When $q = 1$, that is, under dictatorship, the proposer will keep the entire resource. When $n = 3$ and $q = 2$, the randomly selected proposer in round 2 will get the entire budget. Therefore, the proposer selected in the first round would offer x to one of the two other players, and one who received the offer will accept it only when x is greater than $\delta/2$, which is the expected gain when rejecting the offer: He will obtain 1 in the next round with probability $1/2$ being recognized as a proposer, and 0 with another probability $1/2$ being not recognized, and the next round is discounted by δ. Thus if the first proposer offers $\frac{\delta}{2}$ to one of the two other players, the proposal is approved by 2 players and the game ends. Similarly for any $n \geq 5$, if $q = 2$, the first proposer offers $\frac{\delta}{n-1}$ to one randomly selected members. Now I consider $n \geq 5$ and $q \geq 3$ so that l and $l + 1 \leq q - 1$ are well defined.

Since the equilibrium strategy isn't stationary, backward induction has to be adopted. First, let's check if the $(q - l)$th proposer offers $\frac{\delta}{n-q+l}$ to one player when $l = 1$. By the fact that there are $q - 2$ previous proposers in the trivial coalition, she wants to offer some nonnegative payoff, x, to only one additional player to form a MWC. The player received an offer x would accept it only when his continuation value is not as great as accepting x. If he rejects the offer, he would have 1 in the next round with probability $\frac{1}{n-(q-1)}$ being a proposer, and zero otherwise by Lemma 1. His expected payoff in the next round, $\frac{1}{n-(q-1)}$ is discounted by δ, so he will accept x if it is greater or equal to $\frac{\delta}{n-(q-1)}$. Now suppose the claim holds for some $l = 1, \ldots, q-2$. That is, the $(q - l)$th proposer offers $\frac{\delta}{n-q+l}$ to l randomly selected players from the nontrivial coalition pool. I want to show this will also hold for $l = q - 1$. The $(q - l)$th proposer, or the first proposer, needs to offer some nonnegative payoff, x, to l players from the nontrivial coalition pool. Each of players who received the offer x would accept if it is greater than the continuation value. When one offered player rejects the offer, he would expect to earn $1 - (q - 2)\frac{\delta}{n-2}$ with probability $\frac{1}{n-1}$ being a proposer, and earn $\frac{\delta}{n-2}$ with probability $\frac{q-2}{n-1}$ being in a nontrivial MWC. Thus the expected payoff in the next round is $\frac{1}{n-1}\left(1 - \frac{q-2}{n-2}\delta\right) + \frac{q-2}{n-1}\frac{\delta}{n-2} = \frac{1}{n-1}$. Since the continuation value for the next round is discounted by δ, the nonproposers will accept if $x = \frac{\delta}{n-1}$. □

Proof of Proposition 2: This proof is followed by Lemmas 3, which describes the equilibrium strategy profile in the final round of each cycle, and Lemma 4, which states that whenever possible members in the trivial coalition pool are considered to form a minimum winning coalition first. Now we can assure that the finite subgame (a cycle of n rounds) has a unique subgame perfect equilibrium (with allowing a mixed strategy in terms of selecting some of identical players at random,) because the equilibrium strategy at the final node is characterized by Lemma 3, and how to choose coalition partners in each subgame is described by Lemma 4. Though we consider an infinite-horizon game, we only need to implement backward induction for n times. The exact characterization of the equilibrium is, however, nontrivial because the backward induction may be, though doable, computationally heavy.

From the last round of the cycle to the qth round, the backward induction can be applied in a rather simple manner, since the number of players in the trivial coalition is greater than the number of coalition partners needed. Lemma 5 characterizes an equilibrium strategy at a subgame in the qth node of the cycle or after.

Lemma 5. *For any odd $n \geq 3$, in round $\tau = t \bmod n \in \{q, q+1, \ldots, n-1\}$, a proposer's equilibrium strategy is to offer $x_T^{\tau} = \frac{(\tau-1)!}{n!}(q-1)^{n-\tau}\delta^{n-\tau+1}$ to $q-1$ players randomly selected from the trivial coalition pool.*

Proof: By Lemma 4, from round q to round n, proposers always offer only to players in the trivial coalition pool because there is a sufficient number of 'cheap' players. By Lemma 3, in round n, the proposer offers $x_T^n = \frac{\delta}{n}$ to $q-1$ players randomly selected among all the other players. In round $n-1$, the proposer offers x_T^{n-1} to $q-1$ players randomly selected among $n-2$ players in the trivial coalition pool. The player who got offered x_T^{n-1} knows that if he rejects the offer he will be offered x_T^n with probability $\frac{q-1}{n-1}$. Thus a nonproposer will accept any offer larger than or equal to $\delta x_T^n \frac{q-1}{n-1} = \frac{\delta^2(q-1)}{n(n-1)}$. Since the proposer in round $n-1$ wants to maximize her utility, she offers $x_T^{n-1} = \frac{\delta^2(q-1)}{n(n-1)}$. In round $n-2$, a player offered x_T^{n-2} knows that if he rejects the offer he will be offered x_T^{n-1} with probability $\frac{q-1}{n-2}$. Analogously, the proposer in round $n-2$ offers $x_T^{n-2} = \frac{\delta^3(q-1)^2}{n(n-1)(n-2)}$. This backward induction is analogously applied to the round q, and the proposer in round q offers $x_T^q = \frac{\delta^{n-q+1}(q-1)^{n-q}}{n(n-1)\cdots(n-q+1)} = \frac{(q-1)!}{n!}(q-1)^{n-q}\delta^{n-q+1}$ to all players in the trivial coalition. □

Now we need to characterize an equilibrium strategy at the subgame of round $q-1, q-2, \ldots, 1$.

Lemma 6. *For any $n \geq 3$, in the first round, a proposer's equilibrium strategy is to offer $\frac{\delta}{n-1}(1 - x_{NT}^2)$ to q players randomly selected from the nontrivial coalition pool, where x_{NT}^2 is recursively determined by $x_{NT}^{q-l} = \delta x_T^{q-l+1}, l = 1, 2, \ldots, q-1$.*

Proof: In round $q-l$, $l = 1, 2, \ldots, q-2$, there are $q-l-1$ players in the trivial coalition, so a proposer must offer a nonnegative share to l players randomly selected from the nontrivial coalition pool. In round $q-1$, the continuation value of the player in the trivial coalition is δx_T^q, because when the game continues to the next round, he will receive x_T^q with probability 1. Thus $x_T^{q-1} = \delta x_T^q$. On the other hand, the player being offered x_{NT}^{q-1} knows that if he rejects the offer, then with probability $\frac{1}{n-q}$ he will be the proposer in the next round and earn $1-(q-1)x_T^q$, nothing otherwise. Thus $x_{NT}^{q-1} = \frac{\delta}{n-q}(1-(q-1)x_T^q)$. The proposer in round $q-1$, therefore, keeps $1-(q-2)x_T^{q-1} - x_{NT}^{q-1}$ for herself. In round $q-2$, the continuation value of the player in the trivial coalition is, analogously, δx_T^{q-1}, thus $x_T^{q-2} = \delta x_T^{q-1}$. Each of the players offered x_{NT}^{q-2} knows that if he continues to play in the next round, then with probability $\frac{1}{n-q+1}$, he will be the proposer and keeps $1-(q-2)x_T^{q-1} - x_{NT}^{q-1}$ for himself, and with another probability $\frac{1}{n-q+1}$, he will be a coalition partner and receive x_{NT}^{q-1}. Thus, the current proposer offers x_T^{q-2} to all the players in the trivial coalition, offers $x_{NT}^{q-2} = \frac{\delta}{n-q+1}\left(1-(q-2)x_T^{q-1}\right)$ to randomly selected two from the nontrivial coalition pool, and keeps $1-(q-3)x_T^{q-2} - 2x_{NT}^{q-2}$ for herself. This backward induction is analogously applied to the second round. In round 2, the continuation value of the player in the trivial coalition is δx_T^3, thus $x_T^2 = \delta x_T^3$. Each of the players offered x_{NT}^2 knows that if he continues to play in the next round, then with probability $\frac{1}{n-2}$, he will be the proposer and keeps $1-2x_T^3-(q-3)x_{NT}^3$ for himself, and with probability $\frac{q-3}{n-2}$ he will be included as a coalition partner and receives x_{NT}^3. Thus, the current proposer offers $x_{NT}^2 = \delta\left(\frac{1}{n-2}\left(1-2x_T^3-(q-3)x_{NT}^3\right) + \frac{q-3}{n-2}x_{NT}^3\right) = \frac{\delta}{n-2}\left(1-2x_T^3\right)$ to randomly selected $q-2$ members from the nontrivial coalition pool, and keeps $1-x_T^2-(q-2)x_{NT}^2$ for herself. Finally, in round 1, where there is no trivial coalition, a proposer offers x_{NT}^1 to $q-1$ players selected at random, where x_{NT}^1 is equal to the coalition partner's continuation value, that is, $x_{NT}^1 = \delta\left(\frac{1}{n-1}(1-x_T^2-(q-2)x_{NT}^2) + \frac{q-2}{n-1}x_{NT}^2\right) = \frac{\delta}{n-1}(1-x_T^2)$. $\square$

In the course of proving Lemmas 3 to 6, the cycle-stationary subgame perfect equilibrium is completely characterized. The remaining task is to show that x_{NT}^1 is always between $\frac{\delta}{n}$ and $\frac{\delta}{n-1}$. To see this, we need to fully expand x_{NT}^1.

$$\begin{aligned}
x_{NT}^1 &= \frac{\delta}{n-1}(1-x_T^2) = \frac{\delta}{n-1} - \frac{\delta^2}{n-1}x_T^3 = \ldots = \frac{\delta}{n-1} - \frac{\delta^{q-1}}{n-1}x_T^q \\
&= \frac{\delta}{n-1} - \frac{\delta^{q-1}}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q}\delta^{n-q+1} \\
&= \frac{\delta}{n-1} - \frac{\delta^{n}}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q}
\end{aligned}$$

When $q = 1$, x^1_{NT} is $\frac{\delta}{n-1}$. (However, it will be offered to $q-1$, that is, 0 members.) When $q = n$, $x^1_{NT} = \frac{\delta}{n-1} - \frac{\delta^n}{n(n-1)}$, which is strictly smaller than $\frac{\delta}{n-1}$, and larger than $\frac{\delta}{n}$ with equality hold when $\delta = 1$. Now I consider $q \in \{2, \dots, n-1\}$. Since $\frac{\delta^n}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q}$ is positive, x^1_{NT} is smaller than $\frac{\delta}{n-1}$. Also, except when $\delta = 0$, x^1_{NT} is strictly greater than $\frac{\delta}{n}$. I assume $\delta >$ so that the terms in the following inequalities can be divided by δ.

$$\begin{aligned}
&\frac{\delta}{n} \leq x^1_{NT} = \frac{\delta}{n-1} - \frac{\delta^n}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q} \\
\Leftrightarrow &\frac{\delta^n}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q} \leq \frac{\delta}{n-1} - \frac{\delta}{n} = \frac{\delta}{n(n-1)} \\
\Leftrightarrow &\delta^{n-1}\frac{(q-1)!}{(n-1)!}(q-1)^{n-q} \leq 1 \\
\Leftrightarrow &\delta^{n-1}\frac{q-1}{n-1}\frac{q-1}{n-2}\cdots\frac{q-1}{q} \leq 1
\end{aligned}$$

Since each of $\frac{q-1}{n-1}, \dots, \frac{q-1}{q}$ is smaller than 1, so is the product of them. □

Proof of Proposition 3: From the proof of Proposition 2, we know that x^1_{NT} is equal to $\frac{\delta}{n}$ only when $\delta = 1$ and $q = n$. It is trivial that the equality holds when $q = 1$, that is, under a dictatorship. Other than that, x^1_{NT} is strictly greater than $\frac{\delta}{n}$ for any δ. Thus the proposer's share in the CSSP equilibrium is strictly smaller than at in the SSP equilibrium. We focus on the cases where $\delta = 1$. My first goal is to find $q^*(n)$, the argument that maximizes the difference between the proposer's share in the SSP equilibrium of the Baron–Ferejohn model (denoted by BF_q) and that in the CSSP equilibrium of infinite-cycle bargaining without replacement (InfC_q). Since

$$\begin{aligned}
\mathrm{BF}_q - \mathrm{InfC}_q &= \left(1 - \frac{q-1}{n}\right) - \left(1 - (q-1)\left(\frac{1}{n-1} - \frac{1}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q}\right)\right) \\
&= -\frac{q-1}{n} + \frac{q-1}{n-1} - \frac{1}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q+1} \\
&= \frac{1}{n(n-1)}\underbrace{\left(q - 1 - \frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1}\right)}_{=Q(n,q)},
\end{aligned}$$

$$q^*(n) \in \arg\max_{q\in\{1,\dots,n\}} Q(n,q) = q - 1 - \frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1}.$$

I want to show the followings: (1) $Q(n,q)$ is single-peaked in n so that $q^*(n)$ is unique.

(2) When $q=(n+1)/2$, changing q to $q'=q+1$ yields a larger difference for any $n\geq 11$.
(3) $\frac{q^*(n+1)}{n+1}\geq\frac{q^*(n)}{n}$.

Lemma 7. *$q^*(n)$ is unique.*

Proof: Since the factorial is a discrete function, we cannot directly differentiate it with respect to q. Instead, using Stirling's approximation, I first show that the continuous function that approximates the objective function has a unique maximizing argument. Then, I show that any two adjacent integers cannot be the maximizing arguments at the same time. These two claims jointly imply the uniqueness of the maximizing argument. Since $n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$,

$$\begin{aligned} q-1-\frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1} &\approx q-1-\frac{\sqrt{2\pi(q-1)}\left(\frac{q-1}{e}\right)^{q-1}}{\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}}(q-1)^{n-q+1} \\ &= q-1-(q-1)^{n+\frac{1}{2}}e^{-q}K(n) := F(n,q), \end{aligned}$$

where $K(n)=e^n\left(\frac{1}{n-1}\right)^{n-\frac{1}{2}}$. The first order condition with respect to q is

$$\frac{\partial F(n,q)}{\partial q}=1-\left(n-q+\frac{3}{2}\right)(q-1)^{n-\frac{1}{2}}e^{-q}K(n),$$

which is $-\frac{1}{2}$ when $q=n$, and 1 when $q=1$. Since this function is continuous, by the intermediate value theorem, there exists q^* such that $1-\left(n-q^*+\frac{3}{2}\right)(q^*-1)^{n-\frac{1}{2}}e^{-q^*}K(n)=0$. The second order condition with respect to q, $\frac{\partial^2 F(n,q)}{\partial q^2}$, is

$$(q-1)^{n-\frac{1}{2}}e^{-q}K(n)-\frac{n-\frac{1}{2}}{q-1}\left(n-q+\frac{3}{2}\right)(q-1)^{n-\frac{1}{2}}e^{-q}K(n)+\left(n-q+\frac{3}{2}\right)(q-1)^{n-\frac{1}{2}}e^{-q}K(n).$$

Using the fact that $1=\left(n-q^*+\frac{3}{2}\right)(q^*-1)^{n-\frac{1}{2}}e^{-q^*}K(n)$, we know that

$$\left.\frac{\partial^2 F(n,q)}{\partial q^2}\right|_{q=q^*}=\frac{1}{n-q^*+\frac{3}{2}}-\frac{n-\frac{1}{2}}{q^*-1}+1,$$

which is strictly smaller than 0 for $q\in[1,\underline{q})$, and greater than 0 for $q\in[\underline{q},n]$, where $\underline{q}\leq n$ is such that $\underline{q}-1=\left(n+\frac{1}{2}-\underline{q}\right)\left(n+\frac{3}{2}-\underline{q}\right)$. I claim that $\left.\frac{\partial^2 F(n,q)}{\partial q^2}\right|_{q=q^*}$ is strictly smaller than 0, so that q^* is unique. Suppose for the sake of contradiction that $\left.\frac{\partial^2 F(n,q)}{\partial q^2}\right|_{q=q^*}\geq 0$. It implies that such q^* is in $[\underline{q},n]$. Two observations that $\left.\frac{\partial F(n,q)}{\partial q}\right|_{q=q^*}=0$ and $\left.\frac{\partial^2 F(n,q)}{\partial q^2}\right|_{q=q^*}>0$ jointly imply that $\left.\frac{\partial F(n,q)}{\partial q}\right|_{q=n}>0$. However, we know that it is $-1/2$, which is a contradiction.

Next, I want to show that for any q, both q and $q+1$ cannot be the maximizing arguments at the same time. Suppose for the sake of contradiction that q and $q+1$ are both maximizing arguments. Then,

$$q-1-\frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1}=q-\frac{q!}{(n-1)!}q^{n-q}$$

Rearranging terms, we have

$$q^{n-q+1}-(q-1)^{n-q+1}=\frac{(n-1)!}{(q-1)!}$$

This equality holds only when $q=n$, which contradicts existence of $q+1$. $\square$

Lemma 7 assures that the difference between the proposer's share in the CSSP equilibrium and that in the SSP equilibrium is described by a single-picked function with respect to q. Next, I show that a supermajority, not a simple majority, is the voting rule that maximizes the difference for a sufficiently large n. Specifically, I want to show that for $n \geq 11$, $Q(n, \frac{n+3}{2}) > Q(n, \frac{n+1}{2})$.

Plugging $q=\frac{n+1}{2}$ and $q+1$ into $Q(n,q)=q-1-\frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1}$, and taking the difference, we have

$$Q\left(n,\frac{n+3}{2}\right)-Q\left(n,\frac{n+1}{2}\right)=1-\frac{\left(\frac{n-1}{2}\right)!}{(n-1)!}\left(\left(\frac{n+1}{2}\right)^{\frac{n+1}{2}}-\left(\frac{n-1}{2}\right)^{\frac{n+1}{2}}\right).$$

I claim this is strictly positive when n is sufficiently large.

$$\begin{aligned}
&1-\frac{\left(\frac{n-1}{2}\right)!}{(n-1)!}\left(\left(\frac{n+1}{2}\right)^{\frac{n+1}{2}}-\left(\frac{n-1}{2}\right)^{\frac{n+1}{2}}\right)>0\\
\Leftrightarrow &\left(\frac{n+1}{2}\right)^{\frac{n+1}{2}}-\left(\frac{n-1}{2}\right)^{\frac{n+1}{2}}<\frac{(n-1)!}{\left(\frac{n-1}{2}\right)!}=\underbrace{(n-1)(n-2)\cdots\left(\frac{n+1}{2}\right)}_{\frac{n-1}{2}\text{ terms}}\\
\Leftrightarrow &\frac{\frac{n+1}{2}}{n-1}\frac{\frac{n+1}{2}}{n-2}\cdots\frac{\frac{n+1}{2}}{\frac{n+1}{2}}\frac{n+1}{2}-\frac{\frac{n-1}{2}}{n-1}\frac{\frac{n-1}{2}}{n-2}\cdots\frac{\frac{n-1}{2}}{\frac{n+1}{2}}\frac{n-1}{2}<1
\end{aligned}$$

When n approaches to infinity, the first term of the left-hand side of the inequality above converges to 0 because $\frac{\frac{n+1}{2}}{n-1}\frac{\frac{n+1}{2}}{n-2}\cdots\frac{\frac{n+1}{2}}{\frac{n+1}{2}}$ converges to 0 exponentially faster than $\frac{n+1}{2}$ approaches to infinity. Similarly, the second term also converges to 0. Therefore there should exist $\underline{n}$ such that for any $n \geq \underline{n}$, the left-hand side is close to 0. I manually find that such $\underline{\mathrm{n}}$ is 11. Lastly, I show that $\frac{q^*(n)}{n}$ is increasing in n. Let $f(n,q)$ denote $\frac{\partial F(n,q)}{\partial q}$. Note that $f(n,q^*) = 1 - \left(n - q^* + \frac{3}{2}\right)(q^* - 1)^{n-\frac{1}{2}} e^{-q^*} K(n) = 0$. Since every entity in $\left(n - q^* + \frac{3}{2}\right)(q^* - 1)^{n-\frac{1}{2}} e^{-q^*} K(n)$ is increasing in n, $f(n,q)$ is decreasing in n. Since we are interested in $\frac{d(q^*/n)}{dn}$, introduce a new variable $y = q^*/n$. Since $q^* \geq \frac{n+1}{2}$ for all n, y is in $[1/2, 1)$. Replacing q^* with ny, we have

$$f(n,y) = 1 - \left(n - ny + \frac{3}{2}\right)(ny-1)^{n-\frac{1}{2}} e^{-ny} K(n) = 0$$

The remaining tasks are to show $\partial f(n,y)/\partial y \leq 0$ because if it is so, $\frac{dy}{dn} = -\frac{\partial f(n,y)/\partial n}{\partial f(n,y)/\partial y} \geq 0$ by the implicit function theorem. $\frac{\partial f(n,y)}{\partial n} = (1-y)\frac{\partial f(n,q)}{\partial n} > 0$ since $y < 1$.

$$\begin{aligned}\frac{\partial f(n,y)}{\partial y} &= n(yn-1)^{n-\frac{1}{2}} e^{-ny} K(n) - \frac{n-\frac{1}{2}}{yn-1}(n - yn + \frac{3}{2})(yn-1)^{n-\frac{1}{2}} e^{-ny} K(n) \\ &\quad + n(n - yn + \frac{3}{2})(yn-1)^{n-\frac{1}{2}} e^{-ny} K(n) \\ &= \frac{n}{n - yn + \frac{3}{2}}(1 - f(n,y)) - \frac{n-\frac{1}{2}}{yn-1}(1 - f(n,y)) + n(1 - f(n,y)).\end{aligned}$$

Since $f(n,y) = 0$, $\frac{\partial f(n,y)}{\partial y} = \frac{n}{n-yn+\frac{3}{2}} - \frac{n-\frac{1}{2}}{yn-1} + n$. Both $\frac{n}{(1-y)n+\frac{3}{2}}$ and $-\frac{n-\frac{1}{2}}{yn-1}$ are monotone increasing in $y \in [1/2, 1)$, it is sufficient to show that $\left.\frac{\partial f(n,y)}{\partial y}\right|_{y=1/2} = \frac{2n}{n+3} - \frac{2n-1}{n-2} + n = \frac{2(n+3-3)}{n+3} - \frac{2(n-2+1)}{n-2} + n = n - \frac{6}{n+3} - \frac{2}{n-2} \geq 0$, which holds for any $n \geq 3$. $\square$