Midterm Review

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Q1. (a) x^2y^4 is NOT concave. (Second derivatives are all positive. Formally, the Hessian matrix is positive semidefinite.)

x^2y^4 is quasiconcave. WTS/ For (x_1,y_1) and (x_2,y_2) $\lambda x_1^2y_1^4+(1-\lambda)x_2^2y_2^4 \ge \min\{x_1^2y_1^4, x_2^2y_2^4\}$

WLOG, set $x_1^2y_1^4 \ge x_2^2y_2^4$ and let $\Delta = x_1^2y_1^4 - x_2^2y_2^4 \ge 0$.

$\lambda x_1^2y_1^4+(1-\lambda)x_2^2y_2^4 = \lambda(x_2^2y_2^4+\Delta)+(1-\lambda)x_2^2y_2^4 = \lambda\Delta + x_2^2y_2^4 \geqslant x_2^2y_2^4$. ■

(b) monotone ~~transformation~~ of $u \Rightarrow x^{\frac{1}{3}}y^{\frac{2}{3}}$ (Cobb-Douglas) $x^* = \frac{W}{3p_x}$ $y^* = \frac{2W}{3p_y}$. $V(p_x,p_y,W) = \left(\frac{W}{3p_x}\right)^2\left(\frac{2W}{3p_y}\right)^4$

(c) $$x(p_x,p_y,W) = -\frac{\partial V/\partial p_x}{\partial V/\partial W} = -\frac{-2V/p_x}{6V/W} = \frac{\cancel{2}\cancel{V}\cdot W}{\cancel{6}\cancel{V}\cdot p_x} = \frac{W}{3p_x}.$$

$$y(p_x,p_y,W) = -\frac{\partial V/\partial p_y}{\partial V/\partial W} = -\frac{-4V/p_y}{6V/W} = \frac{\cancel{4}\cancel{V}\cdot W}{\cancel{6}\cancel{V}\cdot p_y} = \frac{W}{3p_y}$$ ■

Q2 (a) $e(p, V(p,m)) = m = \frac{2p_1p_2}{p_1+p_2}V(p,m) \Rightarrow V(p,m) = \frac{p_1+p_2}{2p_1p_2}m = \frac{m}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)$

$x_1(p,m) = -\frac{\partial V/\partial p_1}{\partial V/\partial m} = -\frac{-\frac{m}{2}\cdot\frac{1}{p_1^2}}{\frac{p_1+p_2}{2p_1p_2}} = \frac{p_1p_2m}{p_1^2(p_1+p_2)} = \frac{p_2m}{p_1(p_1+p_2)}$ $\quad x_2(p,m)$ = similarly = $\frac{p_1m}{p_2(p_1+p_2)}$

(b) Let $m=1$. $u(x_1,x_2) = \min_{p_1,p_2} V(p_1,p_2,1)$ s.t. $p_1x_1+p_2x_2=1$.

$\mathcal{L} = \frac{1}{2p_1}+\frac{1}{2p_2}+\lambda(p_1x_1+p_2x_2-1)$. $\quad \frac{\partial\mathcal{L}}{\partial p_1} = -\frac{1}{2p_1^2}+\lambda x_1 = 0 \Rightarrow p_1^* = \frac{1}{\sqrt{2\lambda x_1}}$ ①

$\frac{\partial\mathcal{L}}{\partial p_2} = -\frac{1}{2p_2^2}+\lambda x_2 = 0 \Rightarrow p_2^* = \frac{1}{\sqrt{2\lambda x_2}}$ ②

①② $\Rightarrow \frac{p_1^*}{p_2^*} = \frac{\sqrt{x_2}}{\sqrt{x_1}} \Rightarrow p_2^* = \sqrt{\frac{x_1}{x_2}}\,p_1^*$ ③.

Plug ③ to $p_1^*x_1+p_2^*x_2=1$, $p_1^*x_1+\sqrt{\frac{x_1}{x_2}}p_1^*x_2 = 1 = p_1^*(x_1+\sqrt{x_1x_2}) = 1 \Rightarrow p_1^* = \frac{1}{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})}$

By ③. $p_2^* = \sqrt{\frac{x_1}{x_2}}\cdot\frac{1}{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})} = \frac{1}{\sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})}$

$\therefore V(p_1^*,p_2^*,1) = \frac{1}{2}\left(\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})+\sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})\right) = \frac{1}{2}(\sqrt{x_1}+\sqrt{x_2})^2$. ■

(Simpler algebra if you set $m=2$.)

Q3. (a) $x \succsim y \Leftrightarrow u(x) \geqslant u(y) \Leftrightarrow \alpha u(x) \geqslant \alpha u(y)$ for $\alpha > 0$.

$\Leftrightarrow u(\alpha x) \geqslant u(\alpha y)$ by the given property

$\Leftrightarrow \alpha x \succsim \alpha y$.

(b) $e(p,u) = \min p\cdot x$ s.t. $u(x) = \bar{u}$ $= \min p\cdot x$ s.t. $\frac{1}{\bar{u}}u(x) = u\left(\frac{1}{\bar{u}}x\right) = 1$.

relabel $\frac{1}{\bar{u}}x$ as y. thus $x = \bar{u}y$.

$= \min p\bar{u}y$ s.t. $u(y) = 1$ $= e(p\bar{u}, 1)$.

Since $e(p,u)$ is h.d 1 in price. $e(p\bar{u},1) = \bar{u}\,e(p,1)$. $\therefore e(p,u) = u\cdot e(p,1)$

(c) $m = e(p, v(p,m)) = v(p,m)\cdot e(p,1) \Rightarrow v(p,m) = \frac{1}{e(p,1)}m$. ∴ yes. linear in m.

Q4 (a) WARP is satisfied. x^1 is revealed preferred to x^2 because $p^1\cdot x^2 \le p^1\cdot x^1$ but $p^2\cdot x^1 > p^2\cdot x^2$

x^2 // x^3 // $p^2\cdot x^3 \le p^2\cdot x^2$ but $p^3\cdot x^2 > p^3\cdot x^3$

x^3 // x^1 // $p^3\cdot x^1 \le p^3\cdot x^3$ but $p^1\cdot x^3 > p^1\cdot x^1$

(b) GARP is NOT satisfied. $x^1 \succsim^{RP} x^2$ and $x^2 \succsim^{RP} x^3$ should have implied $x^1 \succsim^{RP} x^3$ but $x^3 \succsim^{RP} x^1$.

(c) Not possible to recover preference. Symmetry of the Slutsky matrix is not guaranteed

Q5.(a) WTS) For $\lambda \in (0,1)$ and m', m'', $\lambda v(p,m') + (1-\lambda) v(p,m'') < v(p, \lambda m' + (1-\lambda) m'')$

$x' = \arg\max_x u(x)$ s.t. $p \cdot x \le m'$ $\quad v(p,m') = u(x')$

$x'' = \arg\max_x u(x)$ s.t. $p \cdot x \le m''$ $\quad v(p,m'') = u(x'')$

$\because \lambda x' + (1-\lambda) x''$ is IN the budget set with $\lambda m' + (1-\lambda) m''$.

Since $u(x)$ is strictly increasing, $x' \ne x^2$

$\because$ strict concavity

For $\lambda \in (0,1)$, $\lambda v(p,m') + (1-\lambda) v(p,m'') = \lambda u(x') + (1-\lambda) u(x'') < u(\lambda x' + (1-\lambda) x'') \le v(p, \lambda m' + (1-\lambda) m'')$

(b) Since $v(p,m)$ is strictly concave in m, and S is a MPS of D.
$E_S(v(p,m)) < E_D(v(p,m))$ Thus, recommend D.

Q6.(a) For $\lambda > 1$, $((\lambda L)^\alpha + (\lambda K)^\alpha)^{\frac{1}{\beta}} = \lambda^{\frac{\alpha}{\beta}} (L^\alpha + K^\alpha)^{\frac{1}{\beta}} = \lambda^{\frac{\alpha}{\beta}} Q$. $\Rightarrow \therefore$ IRTS if $\alpha > \beta$, CRTS if $\alpha = \beta$, DRTS if $\alpha < \beta$

(b) $\min_{L,K} 2L + K$ s.t. $(L^\alpha + K^\alpha)^{\frac{1}{\beta}} = Q$ $\quad$ FOC

$2 = \lambda \frac{1}{\beta} (L^\alpha + K^\alpha)^{\frac{1}{\beta} - 1} \alpha L^{\alpha - 1}$ ①

$1 = \lambda \frac{1}{\beta} (L^\alpha + K^\alpha)^{\frac{1}{\beta} - 1} \alpha K^{\alpha - 1}$ ②

$\frac{①}{②} = 2 = \left(\frac{L}{K}\right)^{\alpha - 1} = \left(\frac{K}{L}\right)^{1-\alpha} \Rightarrow 2^{\frac{1}{1-\alpha}} K = L$. ③

Plug ③ into $(L^\alpha + K^\alpha)^{\frac{1}{\beta}} = Q$. $\left(2^{\frac{\alpha}{1-\alpha}} K^\alpha + K^\alpha\right)^{\frac{1}{\beta}} = Q \Leftrightarrow \left(2^{\frac{\alpha}{1-\alpha}} + 1\right) K^{\alpha/\beta} = Q$

$\therefore K^* = \left(\frac{1}{2^{\frac{\alpha}{1-\alpha}} + 1}\right)^{\frac{\beta}{\alpha}} Q^{\frac{\beta}{\alpha}} := \kappa Q^{\frac{\beta}{\alpha}}$

$L^* = 2^{\frac{1}{1-\alpha}} \cdot \left(\frac{1}{2^{\frac{\alpha}{1-\alpha}} + 1}\right)^{\frac{\beta}{\alpha}} \cdot Q^{\frac{\beta}{\alpha}} := \nu Q^{\frac{\beta}{\alpha}}$

Thus $2L^* + K^* = 2\nu Q^{\frac{\beta}{\alpha}} + \kappa Q^{\frac{\beta}{\alpha}} = (2\nu + \kappa) Q^{\beta/\alpha}$
$:= \underline{\underline{C Q^{\beta/\alpha}}}$

(c) $C Q^{\beta/\alpha}$ is convex in Q if $\beta > \alpha$.
linear in Q if $\beta = \alpha$
concave in Q if $\beta < \alpha$.

(d) DRTS $\Leftrightarrow$ cost function is convex $\Leftrightarrow$ diseconomies of scale.
IRTS $\Leftrightarrow$ economies of scale.