Recognition without Replacement in Legislative Bargaining*

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Abstract

This paper studies infinite-horizon sequential bargaining among $n \geq 3$ players in which the proposer—a player who proposes a distribution of an economic surplus—is randomly selected from the pool of potential proposers. If the proposal is rejected, the current and previous proposers are excluded from the pool of potential proposers, and the game moves on to the next round until every player has had the same number of opportunities to be the proposer. To analyze the infinite-horizon model with a particular time dependency within each sequence of n rounds (a cycle,) I characterize the stationary equilibrium of a stochastic game, which I call *cycle-stationary subgame perfect* (CSSP) equilibrium. The CSSP equilibrium is unique in payoffs, and it is analogous to the subgame perfect equilibrium of some specific forms of finite-horizon bargaining. Even when every player is fully patient, or there is no penalty for delay, the proposer's share in the CSSP equilibrium is strictly smaller than that predicted by the stationary equilibrium of the Baron–Ferejohn legislative bargaining model under any voting rule except unanimity.

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1 Introduction

Multilateral bargaining is a political process in which many agents with conflicting preferences try to divide an economic surplus ("pie") in a democratic way. The process can be summarized as follows: One member of the group proposes a division of the pie, and the proposal is voted on. If the proposal is agreed to by a predetermined number of members, the division is implemented. Otherwise, the procedure is repeated. The main contribution of this paper is to characterize an equilibrium where the proposer selection rule is different to what it is assumed in the Baron–Ferejohn (Baron and Ferejohn, 1989, henceforth BF) legislative bargaining model.

The BF legislative bargaining model is the most renowned prototype study in the literature on multilateral bargaining. One theoretical feature of the BF model is that in their infinite-horizon game, virtually any distribution of feasible payoffs can be supported in an equilibrium.[1] Consequently, it was natural to restrict attention to a particular set of strategies, that is, stationary (history- and time-independent) strategies. In order for a stationary strategy to constitute an equilibrium, all rounds in the infinite-horizon game should be structurally equivalent. The "random recognition process," where in each round the proposer is recognized at random by a chairperson, implies structural equivalence. It is known that under the random recognition process, the uniqueness of stationary equilibrium payoffs is guaranteed for a wider set of generalized models (Eraslan, 2002; Eraslan and McLennan, 2013). The literature on legislative bargaining has followed the tradition of adopting the random recognition process,[2] not only because it is a necessary condition to enjoy the theoretical luxury of yielding predictions regarding the stationary equilibrium, but also because the random recognition process can be understood and naturally interpreted as follows: Since all legislators value the chance to be a proposer and seek to be recognized by the chair in each session of the legislature, the fair-minded chair may want to employ the random recognition rule, as it guarantees ex-ante fairness to some degree.[3]

[1]See Proposition 2 of Baron and Ferejohn (1989), which can be understood as an example of a class of results known as "folk theorems" (Austen-Smith and Banks, 2005).

[2]Banks and Duggan (2000), Diermeier and Merlo (2000), Jackson and Moselle (2002), Norman (2002), Eraslan (2002), Snyder et al. (2005), Battaglini and Coate (2007), and Volden and Wiseman (2007) are theoretical studies that adopt the random recognition process to address different issues in regard to political decisions. This list is not complete, and it would be even longer if it included experimental studies whose main purpose is to test theoretical predictions made under the assumption of random recognition.

[3]In the general case, not all members have the same probability of being recognized by a chairperson. For example, senior members are more likely than junior members to be recognized as a proposer. Even with unequal recognition probabilities, randomness plays an essential role in capturing the tensions between the proposer and nonproposers. If, on the other hand, the order of proposers is determined prior to the bargaining (Breitmoser, 2011) or some members who will not be the proposer in the next round of bargaining are indicated (Ali et al., forthcoming), then the outcome of the bargaining depends

In this paper, I modify the BF model by considering random recognition *without* replacement as a proposer selection process. If the proposer selection process can be understood as sampling from a population, the protocol used in the BF model is random sampling with replacement. Admittedly, there has been an apparent gap between what multilateral bargaining theory assumes and what is known to occur in practice: Previous proposers are not likely to be a proposer again. The random recognition process (with replacement) allows the current proposer to be recognized again in the next round. However, if theoretical analysis aims to help us understand actual multilateral bargaining within social groups, then it would make more sense to assume that the chair does not allow one legislator to propose consecutively, or at least does not allow a legislator to make his/her next proposal until all members have been granted the same number of opportunities to make proposals. Moreover, the idea of random recognition without replacement as the proposer selection process is closely related to the "one bite at the apple" principle that is often explicitly considered in legislative and judiciary processes. This principle means that each individual/agent/party has only one chance to take advantage of an opportunity. The rules of the House of the Representatives[4] state that

> it is an important tool that ensures that the minority gets at least one chance, one bite at the apple, so that 100 million Americans represented by Members of the minority here can be heard.

Politicians are well aware of this notion. For example, in a speech on the Senate floor on August 1, 2001, Mr. Bond said, "Under current law, you only get one incentive period, one bite at the apple. That's it."[5] to emphasize his (possibly only) opportunity of passing a bill on regulating medicines for children. At the first session of the 110th Congress, Mr. Dreier said, "what happened last November, when we lost the majority, we got ourselves in a position where we figured, gosh, we will have only one bite at the apple, only one opportunity to allow the majority of the House to come together and address these issues."[6] before yielding the floor. Although the expression "one bite at the apple" may be a jargon used only in particular situations, the expression appears in the Congressional Records as frequently as some widely-used three-word jargon such as "third-party candidate," and "pork-barrel politics."

In the case of an opportunity to be a proposer, the random recognition process without replacement accurately captures the "one bite at the apple" principle: Members who are recognized as proposers (members who have already "bitten at the apple") cannot again

largely on such deterministic information.

[4]Congressional Record—House, January 6, 2009, page H18

[5]Congressional Record—Senate, August 1, 2001, page S8598.

[6]Congressional Record—House, April 19, 2007, page H3571.

be a proposer (cannot take another bite) until everyone has bitten at the apple once. Specifically, I consider a sequential multilateral bargaining model in which, within a given session of the legislature, if a randomly selected member proposes in the first round and her proposal is not accepted, she will not have another chance to propose until everyone else has proposed once. If the legislature has n members, then in the second round the other $n-1$ members have an equal chance to be recognized, in the third round (if the proposal in the second round fails) the remaining $n-2$ members have an equal chance to be recognized, and so on. I call a sequence of n rounds of proposals a *cycle*, as it expresses that the opportunity of being a proposer is equally distributed among all the members, or that everyone bites at the apple exactly once in every n rounds. Every member is reinstated to the pool of potential proposers at the end of each cycle. This modification of the sequential bargaining model can bridge the gap between theory and practice while maintaining the property of ex-ante fairness.

I characterize the stationary subgame perfect equilibrium of legislative bargaining under random recognition without replacement for a q-quota voting rule. For infinite-cycle bargaining where the state-dependency of stationary strategies is repeated per cycle, I introduce cycle stationarity, characterize the cycle-stationary subgame perfect (CSSP) equilibrium, and show the CSSP equilibrium is unique in payoffs. That equilibrium has the following three notable properties: (1) The first proposer's equilibrium strategy is to offer x to each of $q-1$ randomly selected players, and to keep the remaining share, $1-(q-1)x$, for herself, where q is the qualified number of votes for acceptance of the proposal. Such x is between $\frac{\delta}{n}$, the stationary subgame perfect equilibrium offer in the BF model, and $\frac{\delta}{n-1}$, the subgame perfect equilibrium offer in one-cycle (n-round) finite-horizon bargaining in which random recognition without replacement is adopted as the proposer selection rule, where n is the size of the legislature, and δ is the common discount factor. (2) Though the out-of-the-equilibrium behavior is distinct, the cycle-stationary subgame perfect equilibrium also predicts that delay does not occur, and many theoretical predictions are the same as, or at least quantitatively similar to, those for the Baron–Ferejohn equilibrium. (3) Even if the players are entirely patient, or there is no penalty for delay, that is, if $\delta = 1$, the difference in the proposer's share still exists. The proposer will have the largest share in the BF model and the smallest share in one-cycle bargaining under random recognition without replacement.

The rest of the paper is organized as follows. In the following subsection, I discuss the related literature. Section 2 describes the model. Section 3 characterizes the subgame perfect equilibrium for a finite-horizon (n-round) legislative bargaining process as an illustration of infinite-cycle bargaining. In Section 4 I describe the cycle-stationary subgame perfect equilibrium for an infinite horizon and establish the uniqueness of the equilibrium payoffs. In Section 5 I compare the equilibrium strategy of my model with that of the BF model. Section 6 concludes. Proofs of the lemmas and propositions are provided in the Appendix.

1.1 Related Literature

This paper contributes to the legislative bargaining literature theoretically. Modification of the random recognition process was considered by Yildirim (2007), who studies a sequential bargaining approach in which the probability that a given agent will be recognized as a proposer is proportional to the ratio of that agent's level of effort to the aggregate effort of all agents;[7] Breitmoser (2011), who considers a model allowing for priority recognition of some committee members; Bernheim et al. (2006), who focus on recognition orders where no individual is recognized twice in succession for pork barrel policies; and (Ali et al., forthcoming, henceforth ABF), who assume that some players can be ruled out as the next proposer. It is worth comparing this study with ABF, because both ABF and this paper assume a specific form of predictability about the future proposers, while ABF predicts a much stronger power of the current proposer than what I claim in this paper. In my model, the current proposer has the most bargaining power now, but she has the *least* bargaining power in the next round because she has to wait at least n periods before having another opportunity. Being the weakest player in the next round implies that she has to make a more attractive offer to coalition members because the nonproposers have a higher continuation value than she does. By contrast, the predictability condition in ABF ensures that the proposer is never weaker than at least $q-1$ other players for any round, and this leads to the equilibrium where the first proposer captures the entire surplus. I view their study as being complementary to mine. Both ABF and this paper illustrate that the proposer recognition procedure significantly affects equilibrium outcomes.

Since random recognition without replacement is implicitly concerned about ex-ante fairness toward other legislators in terms of proposer opportunities, this study goes in the direction opposite that of studies which considered a persistent agenda setter, such as Diermeier and Fong (2011) and Jeon (2016).

[7] Evans (1997) also assumes that recognition probabilities depend on the players' effort levels but considers a different game, where the members of the coalition that accepts a proposal leave the game and the remaining members continue to the next round.

2 The Model

Consider a legislature consisting of n members indexed by $i \in \{1, 2, \ldots, n\} \equiv N$, where n is an odd number greater than or equal to 3. The legislature decides how to allocate a fixed economic surplus (normalized to 1) among themselves. In round 1, one of the members is randomly selected to make a proposal, and all the members have equal probability of being selected. The proposal is immediately voted on. If the proposal is supported by the predetermined q-quota voting rule (i.e., at least q members vote for it),[8] the game ends and payoffs accrue according to the proposal. If, on the other hand, the proposal is not supported by at least q members, the process is repeated in round 2, but the new proposer is selected at random from all the members except the first proposer. The recognition probability is proportionally updated, that is, the recognition probability of every player who is in the running in round 2 is $\frac{1}{n-1}$. Delay is costly: In each round the utility is discounted by a common factor $\delta \in [0, 1]$. Formally, in round $t \in \{1, 2, \ldots\}$, a randomly recognized player makes a proposal p^t, where p^t is a distribution plan $(p_1^t, \ldots, p_n^t)$ such that $\sum_{i=1}^n p_i^t = 1$ and $p_i^t \geq 0$ for all $i \in N$. If the proposal is supported by at least q players, including the proposer in round t, then the game ends and player i receives $\delta^{t-1} U^i(p^t)$, where $U^i(p^t)$ is member i's undiscounted utility from the approved proposal p^t. Players are assumed to be risk neutral and self-interested, so $U^i(p^t) = p_i^t$. If the proposal in round t is not approved and $(t \bmod n)$ (the remainder in the division of t by n) is strictly greater than 0, then the proposer is excluded from the pool of potential proposers, and the game goes on to round $t + 1$. If the proposal is not approved and $(t \bmod n) = 0$, then every member is reinstated to the pool of potential proposers, and the game goes on to round $t + 1$. This process continues until a proposal is eventually supported by at least q members. If no allocation is ever accepted, each player receives a payoff of 0. In the sense that all the members propose once in rounds 1 through n, once in rounds $n + 1$ through $2n$, and so on, I call such a sequence of n rounds a *cycle*. Since this process could continue for infinitely many rounds, I call it *infinite-cycle bargaining*.

Let h^t denote the history at round t that includes the identities of the previous proposers and the current proposer.[9] Let $\mathcal{S}$ denote the set of all nonempty subsets of N, and $s^t \in \mathcal{S}$ denote the set of eligible players for recognition in round t. Let $\{a_i^t(h^t, s^t), x_i^t(h^t, s^t)\}$ denote a feasible action for player i in round t, where $a_i^t(h^t, s^t) \in \Delta(P)$ is the (possibly mixed) proposal offered by player i as the proposer in round t

[8] The simple-majority rule is the $\frac{n+1}{2}$-quota rule, and the unanimity rule is the n-quota rule.

[9] The past history may also contain the previous proposers' proposals and the number of players who voted for each of them; however, for the characterization of the equilibria only the identities of the previous proposers matter.

and $x_i^t(h^t, s^t)$ is the voting decision threshold of player i as a nonproposer in round t; $P = \{(p_1, p_2, \ldots, p_n) \in [0, 1]^n | \sum_{i=1}^n p_i = 1\}$ is the set of feasible proposals and $\Delta(P)$ is the set of probability measures on P. A strategy σ_i is a sequence of actions $\{a_i^t(h^t, s^t), x_i^t(h^t, s^t)\}_{t=1}^{\infty}$, and a strategy profile σ is an n-tuple of strategies, one for each player. Throughout this paper, I consider symmetric strategies: Given (h^t, s^t), $\sigma_i = \sigma_j$ for all i and j.

3 An Illustration: One-Cycle Bargaining

Before examining infinite-cycle bargaining, I consider a simpler game to illustrate the basic features of the model. Specifically, I consider 3-person 3-round bargaining under a simple majority rule and a unanimity rule, where the game ends either when a proposal is accepted before round 3 or when no proposal wins by the end of round 3. In the latter case payoffs are 0.

First, consider a simple majority rule. Under the BF model, that is, with the assumption of random recognition with replacement, the symmetric[10] subgame-perfect equilibrium outcome is that the first-round proposer offers $\frac{\delta}{3}$ to one randomly selected member, and keeps the remainder, $1 - \frac{\delta}{3}$ for herself, which coincides with the stationary subgame-perfect equilibrium in infinite-horizon bargaining.

With assuming random recognition without replacement, the symmetric subgame-perfect equilibrium outcome is that the first-round proposer offers $\frac{\delta}{2}$ to one randomly selected member, and keeps the remainder, $1 - \frac{\delta}{2}$ for herself. Backward induction is applied. As is typical in the literature, I assume that a player votes for a proposal when she is indifferent between voting for it and voting against it.

- The last-round proposer keeps everything because it is the last round.
- The second-round proposer keeps everything because the first-round proposer, who lost her bargaining power in the previous round, would accept every offer. The first-round proposer does not have an incentive to reject the current offer of zero, because she will get zero in the last round too.

[10] Norman (2002) shows that there are asymmetric equilibria in a finite-horizon version of the BF model and any interior distribution is supportable as an equilibrium with sufficiently patient players and a sufficiently long horizon. The key insight behind the result is that players coincidentally agree on a pre-specified asymmetric coalition formation pattern: One example with three players would be that player 1 always selects player 2, player 2 always selects player 1, and player 3 always selects player 1 as her coalition partner. This asymmetric coalition formation would not be rationalized under random recognition without replacement, because when player 3 lost her bargaining power in a previous round, both players 1 and 2 have an incentive to deviate the coalition formation pattern and select player 3 who is in the trivial coalition pool. For this reason, I focus on symmetric strategies.

- The first-round proposer, who knows that the other two members' expected payoff of moving on to the next round is $\delta\frac{1}{2}$, randomly picks one member and offers $\frac{\delta}{2}$.

Next, consider a unanimity rule. Under the BF model, the symmetric subgame-perfect equilibrium outcome is that the first-round proposer offers $\frac{\delta}{3}$ to both of the other members, and keeps $1 - \frac{2\delta}{3}$ for herself.

With assuming random recognition without replacement, the symmetric subgame-perfect equilibrium outcome is that the first-round proposer offers $\frac{\delta}{2}$ to both of the other members, and keeps the remainder, $1 - \delta$ for herself. Backward induction is applied as follows:

- The last-round proposer keeps everything because it is the last round.

- The second-round proposer, who knows that the member who would end up being the proposer in the third round will reject any offer less than δ in round 2, offers δ to him and keeps $1 - \delta$ for herself.

- The first-round proposer, who knows that the other two members' expected payoff of moving on to the second round is $\frac{\delta}{2} = \delta\left(\frac{1}{2}\delta + \frac{1}{2}(1-\delta)\right)$, offers $\frac{\delta}{2}$ to both members and keeps $1 - \delta$ for herself.

Although the game considered in this section is simple, several important observations can be made. These observations shed light on the properties of the equilibrium I characterize in infinite-cycle bargaining.

First, in equilibrium, there is no delay in reaching an agreement. In the sense that there is no loss due to delay, the equilibrium achieves the utilitarian efficiency. The equilibrium is unique in payoffs.

Second, the first-round proposer forms a minimum winning coalition (MWC) consisting of herself and $q - 1$ other players. The first two observations also hold in the finite-horizon BF model.

Third, if the game moves on to the second round or beyond, the previous proposers are maltreated: This is not because I model a behavioral retaliation for them to make the efficiency loss, but because the previous proposers are cheap enough to buy their votes for free. For terminological clarity, I divide the set of players other than the current proposer into two groups: The previous proposers comprise the *trivial coalition pool* because they would accept any offer. The *nontrivial coalition pool* consists of the players who have not yet been selected as a proposer.

Lastly, the proposer's equilibrium share is strictly smaller than that under the assumption of random recognition with replacement. Under unanimity, for example, the first proposer gets $1-\delta$. If $\delta > \frac{n-1}{n}$, the proposer's share is strictly *smaller* than that of the nonproposers. When $\delta = 1$, she gets *nothing* in equilibrium. This "proposer disadvantage" has not been predicted in the BF model under any circumstances: In the BF model, the first proposer gets $1 - \frac{n-1}{n}\delta$ under unanimity in which the proposer's power is the weakest. She still gets a larger share than the nonproposers for any δ. The intuition behind the observations can be explained by the combination of the previous proposers' decreased negotiating power and nonproposers' increased power. This is in contrast to many existing studies, including Ansolabehere et al. (2005) and Ali et al. (forthcoming), which report a formateur's significant negotiating power. In the infinite-horizon game, the random recognition process with replacement allows a proposer to maintain her negotiating power. However, in the finite-horizon game without replacement, nonproposers, especially the members of the nontrivial coalition pool, share negotiating power, because if they reject the current proposal, they benefit from both a higher chance of being the proposer in a later round and a larger number of players in the trivial coalition pool in that later round.

4 Infinite-Cycle Bargaining

The idea of random recognition without replacement can also be applied to infinite-cycle legislative bargaining, where every member's chance of being a proposer is reinstated at the beginning of each cycle. By interpreting a pair of offers (an offer and a counteroffer) in the Rubinstein–Stahl model as a single cycle in two-player bargaining, this infinite-cycle legislative bargaining can be understood as a more relevant extension of the Rubinstein–Stahl bargaining model[11] than of the BF model.

One of the typical solution concepts adopted for the infinite-horizon multilateral bargaining game is the stationary subgame perfect (SSP) equilibrium. In the BF model, the stationary strategy is defined as a time- and history-independent strategy on a singleton state, that is, a strategy is called stationary if $\{a_i^t(h^t, s^t), x_i^t(h^t, s^t)\} = \{a_i, x_i\}$ for any t and h^t. Note that $s^t = N$ for any t, that is, the state space is a singleton. This particular type of stationary strategies, with the assumption of random recognition with replacement, does not cause any issue to characterize the stationary equilibrium because each subgame is structurally equivalent to its supergame. Such a structural equivalence is guaranteed in the BF model.

I also focus on stationary strategies. Unlike the BF model, however, the infinite-cycle

[11]In the sense that the proposer in infinite-cycle bargaining is probabilistically determined, Merlo and Wilson (1995)'s stochastic model extending the Rubinstein–Stahl model is closer to the model in this paper.

game is a stochastic game with a non-singleton state space. A strategy is stationary if $\{a_i^t(h^t, s^t), x_i^t(h^t, s^t)\}_{t=1}^{\infty} = \{a_i(s^t), x_i(s^t)\}_{t=1}^{\infty}$, that is, the stationary strategies are state-dependent. I call the state-dependent stationary strategies as *cycle-stationary* strategies because $s^t = s^{t'} = s^\tau$ for any t and t' such that $(t \bmod n) = (t' \bmod n) = \tau$. In words, the stationary strategies are state-dependent, but the state-dependency is repeated per cycle. A strategy is said to be *cycle-stationary subgame perfect* (CSSP) if it is both cycle stationary and subgame perfect.

Similarly to my analysis of one-cycle bargaining, in round t I subdivide the set of players other than the current proposer into two groups: The *trivial coalition pool*, $T \subset N$, consists of the players who have proposed within the current cycle but prior to round t, that is, the players who have proposed in one of the rounds $t - \tau + 1, t - \tau + 2, \ldots, t - 1$. The *nontrivial coalition pool*, $NT \subset N$, consists of the players who have not been selected as a proposer in the current round or in an earlier round of the current cycle. In round 8 for a legislature with 5 members, for example, the trivial coalition pool consists of the proposers in rounds 6 and 7, and the nontrivial coalition pool consists of all players except those two former proposers and the current proposer. As we will see shortly, the players in the trivial coalition pool are more likely to be included as coalition partners than those in the nontrivial coalition pool. Let $x_{NT}(s^t)$ and $x_T(s^t)$ denote the shares of the economic surplus offered to some players in the nontrivial coalition pool and to some players in the trivial coalition pool in round t, respectively. For notational convenience, I write x^t for $x(s^t)$ from now on. The CSSP equilibrium is characterized as follows:

Proposition 1 (The Cycle-Stationary Subgame Perfect Equilibrium)**.** *Consider infinite-cycle legislative bargaining, with $n \geq 3$ players (n odd), a q-quota rule, and a common discount factor $\delta \in [0, 1]$. A strategy profile is a cycle-stationary subgame perfect equilibrium if and only if it has the following form:*

- *In round t with $(t \bmod n) \equiv \tau$,*
  - *If $\tau \geq q$ or $\tau = 0$, the recognized proposer offers x_T^τ to $q - 1$ players selected at random from the trivial coalition pool.*
  - *If $\tau \in [2, q)$, the recognized proposer offers x_T^τ to all $\tau - 1$ players in the trivial coalition pool, and x_{NT}^τ to $q - \tau$ players selected at random from the nontrivial coalition pool.*
  - *If $\tau = 1$, the recognized proposer offers x_{NT}^τ to $q - 1$ players selected at random from the nontrivial coalition pool.*
- *Players in the trivial coalition pool accept any offer of at least x_T^τ, and players in the nontrivial coalition pool accept any offer of at least x_{NT}^τ.*

Therefore, in equilibrium, the recognized proposer in round 1 proposes x^1_{NT} *to* $q-1$ *players selected at random, and the game ends, where* $x^1_{NT} = \frac{\delta}{n-1}(1 - x^2_T)$ *and* x^τ_T *is recursively determined by* $x^{q-l}_T = \delta x^{q-l+1}_T$ *for* $l = 1, 2, \ldots, q-1$, *and* $x^q_T = \frac{(q-1)!}{n!}(q-1)^{n-q}\delta^{n-q+1}$. *Also,* $x^1_{NT} \in [\frac{\delta}{n}, \frac{\delta}{n-1}]$.

Proof: See Appendix.

A key result that figures in the characterization of the CSSP equilibrium is that in the final round of each cycle, the equilibrium strategy will be identical to that from the BF model, which is described in Lemma 1.

Lemma 1. *In any cycle-stationary subgame perfect equilibrium, if some round* t *with* $(t \bmod n) = 0$ *is reached, the proposer offers* $\frac{\delta}{n}$ *to* $q-1$ *players selected at random, and keeps* $1 - (q-1)\frac{\delta}{n}$ *for herself.*

Proof: See Appendix.

Thus, we can use backward induction from round n to round 1, or from the last round within any cycle to the first round within that cycle. Once we characterize the unique sequence of proposal plans and voting decision rules for one proposal cycle, we will verify that such a sequence indeed characterizes a unique cycle-stationary subgame-perfect equilibrium in payoffs. Another finding that makes the equilibrium characterization simpler is that players in the trivial coalition pool, that is, the previous proposers within the current cycle, would be the first players to be considered as coalition partners, because they are, in a sense, cheaper.

Lemma 2. *Let* τ *denote* $(t \bmod n)$. *Then* $x^\tau_T < x^\tau_{NT}$ *for* $\tau \in \{2, \ldots, n-1\}$. *Therefore, a proposer includes* $\max\{\tau - 1, q-1\}$ *players randomly selected from the trivial coalition pool as coalition partners.*

Proof: See Appendix.

Except in the final round of a cycle, the players in the trivial coalition pool do not have a chance of being the proposer in the following round. Thus, their continuation value is always smaller than those who have not yet been recognized in the current cycle, and the latter ones re more likely to be recognized in the near future. By Lemma 2, it makes sense to call the set of previous proposers in the current cycle the trivial coalition pool. Though all the players have a common discount factor, the players in the trivial coalition pool can be regarded as having a smaller discount factor, and hence the proposer will try to win them over first.[12]

The remaining proofs for the characterization of the CSSP are given in the Appendix, but the logic is straightforward from the lemmas above: The equilibrium proposal strategy and voting decision rule in the last round within a cycle mimic those of BF (Lemma 1). From the last round within a cycle, backward induction is applied, and the proposer distinguishes those who have proposed within a cycle from those who have not (Lemma 2).

The uniqueness in payoffs of the SPE within a cycle, combined with the uniqueness in payoffs of the SSPE in the BF model (Eraslan, 2002) naturally leads to the uniqueness in payoffs of the CSSP equilibrium in infinite-cycle bargaining.

Corollary 1. *The CSSP equilibrium is unique in payoffs.*

Proof: See Appendix.

The following example with $n = 3$ under the simple-majority rule illustrates the procedure. For notational simplicity, proposal p is arranged in such a way that the proposer's share in round k is the kth entity, and the other nonzero entity is the share offered to the member (randomly selected if necessary) of the pertinent coalition. This does not mean that the proposer's MWC has to include player $k-1$ or player $k+1$.

- Round 3 proposer: By Lemma 1, she proposes $(0, \delta/3, 1-\delta/3)$, and the player who receives the offer of $\delta/3$ (in this example player 2) will accept it since his continuation value does not exceed the utility yielded by the amount which he is offered.

- Round 2 proposer: By Lemma 2, she proposes $(x_T^2, 1-x_T^2, 0)$. The sole player in the trivial coalition pool (in this example player 1), who receives the offer of x_T^2 in round 2, knows that if he rejects the offer he will earn $\delta/3$ with probability 1/2 in round 3, so $x_T^2 = \delta^2/6$. Thus the proposal made in round 2 is $(\frac{\delta^2}{6}, 1-\frac{\delta^2}{6}, 0)$, and the player who receives the offer of $\frac{\delta^2}{6}$ accepts it.

- Round 1 proposer: She proposes $(1-x_{NT}^1, x_{NT}^1, 0)$. The continuation value for the player in the nontrivial coalition pool who receives the offer of x_{NT}^1 in round 1 (in this example player 2) is $\frac{\delta}{2}(1-\frac{\delta^2}{6}) = \frac{6\delta-\delta^3}{12}$, that is, with probability $\frac{1}{2}$ he will be

[12] However, the result in Lemma 2 cannot be applied in a more general model with heterogeneous discount factors and unequal recognition probabilities. Suppose, for example, that player i has proposed in one of the previous rounds of the current cycle, and player j has not. Then the continuation value of player j is smaller than that of player i if δ_j is sufficiently smaller than δ_i or if the recognition probability of player j is sufficiently smaller than that of player i.

the proposer in the second round and will keep $1-\frac{\delta^2}{6}$ for himself, and the expected payoff is discounted by δ. Thus the proposal is $(1-\frac{6\delta-\delta^3}{12}, \frac{6\delta-\delta^3}{12}, 0)$, and the player who is offered $\frac{6\delta-\delta^3}{12}$ accepts it.

Note that $\frac{\delta}{3} < \frac{6\delta-\delta^3}{12} < \frac{\delta}{2}$ for $\delta \in (0,1]$, where $\frac{\delta}{3}$ is the share offered in the stationary subgame perfect equilibrium of infinite-horizon bargaining in which $n=3$ and random recognition is the proposer selection rule, and $\frac{\delta}{2}$ is the share offered in the subgame perfect equilibrium of one-cycle bargaining in which $n=3$ and random recognition without replacement is the proposer selection rule. In general, the initial proposer's offer in the CSSP equilibrium is always between $\frac{\delta}{n-1}$ and $\frac{\delta}{n}$, where $\frac{\delta}{n-1}$ constructs the subgame perfect equilibrium proposal for the game of n rounds without replacement, and $\frac{\delta}{n}$ constructs the stationary subgame perfect equilibrium proposal for the game of infinite-horizon bargaining in which the random recognition process allows for replacement.

In the following subsection, I show that the theoretical prediction for the cycle-stationary subgame perfect equilibrium coincides with that of some possible extensions of one-cycle bargaining without replacement.

4.1 Equivalence to n+1-Round Bargaining

The theoretical predictions for infinite-cycle legislative bargaining can be attained by $n+k$ rounds of legislative bargaining, for *any* positive integer k. This is because the proposal made in round n, the final round of the first cycle, is the same for any k, regardless of the proposal made in round $n+1$, as long as such a round exists.

Proposition 2. *Consider n-person $n+k$-round bargaining under random recognition without replacement. The symmetric subgame perfect equilibrium outcome is equivalent to the CSSP outcome of infinite-cycle bargaining.*

Proof: See Appendix.

For example, under the unanimity rule with $n=3$ and $\delta=1$, the theoretical prediction of the proposer's share in one-cycle bargaining is 0, while that in infinite-cycle bargaining is 1/3. Indeed, this stark difference between one-cycle bargaining and infinite-cycle bargaining can be attained by simply adding one additional round to one-cycle bargaining. Consider the case of four rounds of bargaining, wherein the fourth round all three players are equally likely to be recognized as the proposer. In the fourth round, the randomly selected proposer keeps the entire economic surplus, since it is the final round. In the third round, the proposer offers an equal share to all three players because their continuation value is 1/3. In the second round, where every player's continuation value is $\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)+\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)=\frac{1}{3}$, the proposer offers 1/3 to all three players, including herself. In the first round, with the same logic as for the second round, the proposer offers 1/3 to all three players, including herself. Though the equilibrium proposal strategy and voting decision rules for four-round bargaining are the same as for the symmetric stationary equilibrium in the BF model, the former equilibrium is unique. This clearly illustrates that both the random recognition process and the termination rule significantly affect the equilibrium allocation.

This result is positive in that the cycle-stationary equilibrium, which could be one of the equilibria in infinite-horizon bargaining, coincides with the unique subgame perfect equilibrium for $n+k$ finite-round bargaining with any $k \geq 1$. In other words, if we focus on symmetric strategies in finite-horizon bargaining, we can appreciate the cycle-stationary equilibrium as an approximation from finite-round bargaining.

5 Comparison to the BF Model Predictions

To provide a direct comparison to the BF model, I focus on the proposer's share in equilibrium. Other important properties, such as the minimum winning coalition, full rent extraction, and no delay, are shared by all the models considered. It is trivial that when $q = 1$, that is, under a dictatorship, there is no difference in the proposer's share from one model to another in equilibrium. When $q = n$, that is, under the unanimity rule, the infinite-horizon bargaining models predict a smaller amount for the proposer's share than in the stationary equilibrium of the BF model. Those two are the same only when $\delta = 1$. The proposer's share under the simple-majority rule is $1 - \frac{n-1}{2}x$, where $x = \frac{\delta}{n-1}\left(1 - \delta^{n-1}\frac{\left(\frac{n-1}{2}\right)!}{n!}\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}\right)$, which is always between $\frac{\delta}{n}$ and $\frac{\delta}{n-1}$. For all the models considered, there is no efficiency loss caused by a delay, thus the ex-ante expected value of the game is $1/n$ for each player. When it comes to the ex-ante variance of the value of the game, however, one-cycle bargaining predicts the most egalitarian division of the economic resources given the same q-quota rule.

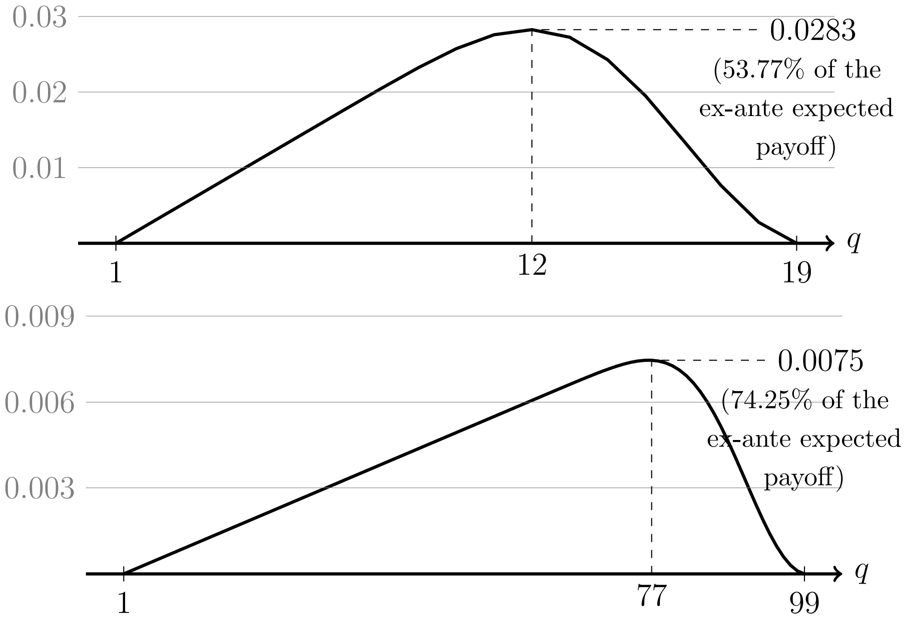
Consistent with one-cycle legislative bargaining, in the CSSP equilibrium the proposer has a weakly lesser advantage than in the protocols considered in the BF model, even when $\delta = 1$.

Proposition 3. *Suppose $\delta = 1$. The equilibrium share of the proposer in infinite-cycle bargaining is always weakly lower than that in the BF model, and strictly lower when $q \notin \{1, n\}$.*

Proof: See Appendix.

It is worth noting that the difference in terms of the proposer advantage is not negligible even when $\delta = 1$, that is, when everyone is fully patient and there is no penalty on indefinite delay. When the size of the legislature is sufficiently small, the difference is largest under the simple-majority rule. When the size of the legislature is large, however, a super-majority rule yields the largest difference. In Figure 1, the difference is plotted for $n = 19$ and $n = 99$ as a function of $q \in [1, n]$.

Figure 1: Super-Majority Rules Yielding the Largest Difference



Each graph is a plot of the difference (as a function of q) between the proposer's share in the SSP equilibrium of the BF model and that in the CSSP equilibrium of infinite-cycle bargaining without replacement under a q-quota rule. The size of the legislature is 19 in the upper graph, and 99 in the lower graph. As n gets larger, the maximum difference gets smaller (0.0283 when $n = 19$ and 0.0075 when $n = 99$) in level. Compared with the ex-ante expected payoff, the differences are substantial. The difference is largest at $q^*(n)$, and $q^*(n)/n$ increases in n ($12/19 = 0.6315$ when $n = 19$, and $77/99 = 0.7778$ when $n = 99$).

Proposition 3 gives some sense of when using the BF model is innocuous as an analytical tool for legislative bargaining without replacement. Although the assumption of random recognition deviates from reality, it obviously has great theoretical merit, as it is parsimonious, it guarantees structural equivalence in every subgame, it allows many possible extensions, and it enables us to characterize the stationary equilibrium in a straightforward manner. Since all other important properties in equilibrium are shared by all the models considered here, it would be of interest to know when the BF model would be used in cases where random recognition without replacement is the proposer selection rule. Proposition 3 implies that when the size of the legislature is large and the simple-to-super-majority rule is applied, it is not a good idea to use the BF model for understanding actual legislative behaviors. Even when $\delta = 1$, the proposer's share in the CSSP equilibrium is strictly smaller than that in the SSP equilibrium except in the case of the unanimity rule and a dictatorship. The difference in the proposer's share is substantial: For example, when $n = 19$, the largest difference between the proposer's share in the SSP equilibrium and that in the CSSP equilibrium is 0.0283 at 12-quota rule. This difference is more than a half (53.77%) of the ex-ante expected payoff of each player, 0.0526 (= 1/19). Even though the absolute difference in the prediction of the proposer's share is decreasing in n, the ratio of the maximum difference to the ex-ante expected payoff is increasing in n. When $n = 99$, the largest difference is nearly three quarters (74.25%) of the ex-ante expected payoff. Indeed, the ratio is increasing in n.

Proposition 4. *The ratio of the maximum difference between the equilibrium proposer share in infinite-cycle bargaining and that in the BF model to the ex-ante expected payoff of each player is increasing in n.*

Proof: See Appendix.

When the budget is normalized to 1, it is natural for the difference to decrease as n gets larger. However, Proposition 4 implies that if the surplus being divided gets larger as the size of the legislature increases, the difference in terms of the proposer advantage would be much of concern as the ratio of the maximum difference to the ex-ante expected payoff increases. These observations suggest that the BF model should be used with some caveats especially when the legislature with a large number of members adopts a super-majority rule and maintains the idea of the "one bite at the apple" rule.

Another interesting observation is that a super-majority rule yields the largest difference when the size of the legislature is large. Since the difference is strictly positive except either when $q = n$ and $\delta = 1$ or when $q = 1$, casual analysis and an expectation of symmetry may lead to the conclusion that the largest difference occurs under the simple-majority rule, and hence that understanding actual legislative behaviors through the lens of the BF model is reasonable for situations where a super-majority rule is applied. However, this is not the case. For example, when the size of the legislature is around 30, the largest difference in terms of the proposer's share occurs under the 2/3 majority rule.[13] Considering that the average size of the upper house in the legislatures of U.S. states and U.S. territories is 37 (39 if the five U.S. territories are excluded), and a supermajority of the state legislature is required to approve tax increases in fifteen states,[14] this difference should not be overlooked.

[13] $q^*(29, 1) = 19$, and $q^*(31, 1) = 21$. Thus $\frac{q^*(n,1)}{n} \approx \frac{2}{3}$ when n is either 29 or 31.

[14] Source: National Conference of State Legislators [online] Last access: 10/22/2016.

Another noticeable difference, which is not captured by the proposer's share in equilibrium, is the off-the-path equilibrium behavior. Unlike the BF model, which treats the previous proposers in an equal manner, legislative bargaining in which random recognition without replacement is the proposer selection rule predicts asymmetric responses between those who had proposed in the previous rounds and those who had not. However, even though the infinite-cycle bargaining model adopts a more realistic assumption regarding the proposer selection rule and yields more intuitive predictions about the out-of-the-path equilibrium behavior, the predictions for the SSP strategy profile are not qualitatively different from those for the CSSP strategy profile.

6 Concluding Remarks

This paper examines how the equilibrium characterization of a sequential, multilateral bargaining process is affected by the random recognition rule. In the existing legislative bargaining literature, random recognition allows the current proposer to be recognized again in the following rounds, while the model considered here prohibits recognition of any player as the proposer in more than one round until everyone has had the same number of chances to be the proposer. Since this infinite-horizon bargaining game has a specific cyclical pattern, I introduce an extended notion of stationarity, which I call cycle stationarity, and characterize the cycle-stationary subgame perfect equilibrium. Some forms of finite-horizon bargaining have the strategy profile in the CSSP equilibrium as the unique subgame perfect equilibrium. Legislative bargaining under the random recognition process without replacement yields a smaller proposer advantage, but all other features in equilibrium are the same as, or qualitatively similar to, those of the Baron–Ferejohn model. However, the difference in the proposer advantage is quite substantial under super-majority rules, so the Baron–Ferejohn model should be used with some caveats especially when the proposer share is the subject of study.

There are many potential directions for extension of this legislative bargaining model that adopts random recognition without replacement as the proposer selection rule. For example, there would be theoretical merit in extending the model presented in this paper by allowing for individual discount factors and asymmetric recognition probabilities. Further investigation of one-cycle bargaining would be required to determine its welfare implications: Since the simple-majority rule predicts a proposer advantage, and the unanimity rule predicts a proposer disadvantage for some values of the parameters, one can characterize the optimal voting rule that yields the most egalitarian distribution of the economic resources. Conducting lab experiments would help us to gain a better understanding of multilateral bargaining behavior. In particular, out-of-equilibrium-path observations in the laboratory would shed some light on other factors that could or should be accounted for in the model.

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Appendix A: Omitted Proofs

Proof of Lemma 1: For any round t with $(t \bmod n) = 0$, the recognized player is the last proposer within the current cycle. As a new cycle starts in the following round, $s^{t+1} = N$, that is, every player will have an equal chance of being recognized as the proposer in that following round. Suppose that in equilibrium the round-t proposer offers x_T^0 to $q-1$ players selected at random and keeps $1-(q-1)x_T^0$ for herself. A player who receives an offer of x_T^0 will accept it if

$$\delta^{t-1} x_T^0 \geq \delta^t \left(\frac{1}{n}(1-(q-1)x_{NT}^1) + \frac{q-1}{n} x_{NT}^1 \right),$$

where x_{NT}^1 is the amount offered to each of $q-1$ members of the nontrivial coalition pool in the first round of a new cycle. The first term on the right-hand side of the inequality above corresponds to the event that the player is recognized as the proposer in round $t+1$, and the second term corresponds to the event that the player is not recognized as the proposer in round $t+1$ but is included in the minimum winning coalition of the proposer in round $t+1$. Although we have not fully characterized what x^1_{NT} is, the terms on the right-hand side of the inequality above that include a factor of x^1_{NT} cancel out, thereby leaving just $\frac{\delta}{n}$ after dividing both sides by δ^{t-1}. Subgame perfection implies that the proposer need be no more generous than to offer the discounted continuation value to a minimum number of members and 0 to the others. Therefore, in every round t with $(t \bmod n) = 0$, the proposer offers $\frac{\delta}{n}$ to $q-1$ players selected at random. $\square$

Proof of Lemma 2: Note that x^0_{NT} is not defined when $(t \bmod n) = 0$, because the nontrivial coalition pool is empty. Similarly, x^1_T is not defined when $(t \bmod n) = 1$, because the trivial coalition pool is empty. We restrict our attention to nontrivial cases, that is, rounds t with $(t \bmod n) \in \{2, \ldots, n-1\}$. Suppose that in any round t with $(t \bmod n) = \tau$, the proposer treats all the other members identically, that is, $x^\tau_T = x^\tau_{NT} = x^\tau$, and she selects $q-1$ coalition partners at random. Then the continuation value of the players in the trivial coalition pool is $\delta \frac{q-1}{n-1} x^{\tau+1}$, and that of the players in the nontrivial coalition pool is $\delta \left(\frac{1}{n-\tau}(1-(q-1)x^{\tau+1}) + \frac{q-1}{n-1} x^{\tau+1} \right)$. Since the latter is strictly greater than the former, the proposer would be strictly better off by retracting an offer made to one player in the nontrivial coalition pool and offering a strictly smaller share to one player in the trivial coalition pool who wasn't included initially, which is a contradiction. Analogously, one can show that $x^\tau_T > x^\tau_{NT}$ also leads to a contradiction. Therefore, if $\tau \leq q$, all the players in the trivial coalition pool will be included as coalition partners, and if $\tau > q$, $q-1$ of the players in the trivial coalition pool will be randomly selected as the coalition partners. $\square$

Proof of Proposition 1: This proof is followed by Lemma 1, which describes the equilibrium strategy profile in the final round of each cycle, and Lemma 2, which states that whenever possible members in the trivial coalition pool are considered to form a minimum winning coalition first. Now we can assure that the finite subgame (a cycle of n rounds) has a unique subgame perfect equilibrium (with allowing a mixed strategy in terms of selecting some of identical players at random,) because the equilibrium strategy at the final node is characterized by Lemma 1, and how to choose coalition partners in each subgame is described by Lemma 2. Though we consider an infinite-horizon game, we only need to implement backward induction for n times. The exact characterization of the equilibrium is, however, nontrivial because the backward induction may be, though doable, computationally heavy.

From the last round of the cycle to the qth round, the backward induction can be applied in a rather simple manner, since the number of players in the trivial coalition is greater than the number of coalition partners needed. Lemma 3 characterizes an equilibrium strategy at a subgame in the qth node of the cycle or after.

Lemma 3. *For any odd $n \geq 3$, in round $\tau = (t \bmod n) \in \{q, q+1, \ldots, n-1\}$, a proposer's equilibrium strategy is to offer $x_T^\tau = \frac{(\tau-1)!}{n!}(q-1)^{n-\tau}\delta^{n-\tau+1}$ to $q-1$ players randomly selected from the trivial coalition pool.*

Proof: By Lemma 2, from round q to round n, proposers always offer only to players in the trivial coalition pool because there is a sufficient number of 'cheap' players. By Lemma 1, in round n, the proposer offers $x_T^n = \frac{\delta}{n}$ to $q-1$ players randomly selected among all the other players. In round $n-1$, the proposer offers x_T^{n-1} to $q-1$ players randomly selected among $n-2$ players in the trivial coalition pool. The player who got offered x_T^{n-1} knows that if he rejects the offer he will be offered x_T^n with probability $\frac{q-1}{n-1}$. Thus a nonproposer will accept any offer larger than or equal to $\delta x_T^n \frac{q-1}{n-1} = \frac{\delta^2(q-1)}{n(n-1)}$. Since the proposer in round $n-1$ wants to maximize her utility, she offers $x_T^{n-1} = \frac{\delta^2(q-1)}{n(n-1)}$. In round $n-2$, a player offered x_T^{n-2} knows that if he rejects the offer he will be offered x_T^{n-1} with probability $\frac{q-1}{n-2}$. Analogously, the proposer in round $n-2$ offers $x_T^{n-2} = \frac{\delta^3(q-1)^2}{n(n-1)(n-2)}$. This backward induction is analogously applied to the round q, and the proposer in round q offers $x_T^q = \frac{\delta^{n-q+1}(q-1)^{n-q}}{n(n-1)\cdots(n-q+1)} = \frac{(q-1)!}{n!}(q-1)^{n-q}\delta^{n-q+1}$ to all players in the trivial coalition. $\square$

Now we need to characterize an equilibrium strategy at the subgame of round $q-1, q-2, \ldots, 1$.

Lemma 4. *For any $n \geq 3$, in the first round, a proposer's equilibrium strategy is to offer $\frac{\delta}{n-1}(1-x_{NT}^2)$ to q players randomly selected from the nontrivial coalition pool, where x_{NT}^2 is recursively determined by $x_{NT}^{q-l} = \delta x_T^{q-l+1}, l = 1, 2, \ldots, q-1$.*

Proof: In round $q-l$, $l = 1, 2, \ldots, q-2$, there are $q-l-1$ players in the trivial coalition, so a proposer must offer a nonnegative share to l players randomly selected from the nontrivial coalition pool. In round $q-1$, the continuation value of the player in the trivial coalition is δx_T^q, because when the game continues to the next round, he will receive x_T^q with probability 1. Thus $x_T^{q-1} = \delta x_T^q$. On the other hand, the player being offered x_{NT}^{q-1} knows that if he rejects the offer, then with probability $\frac{1}{n-q}$ he will be the proposer in the next round and earn $1-(q-1)x_T^q$, nothing otherwise. Thus $x_{NT}^{q-1} = \frac{\delta}{n-q}(1-(q-1)x_T^q)$. The proposer in round $q-1$, therefore, keeps $1-(q-2)x_T^{q-1} - x_{NT}^{q-1}$ for herself. In round $q-2$, the continuation value of the player in the trivial coalition is, analogously, δx_T^{q-1}, thus $x_T^{q-2} = \delta x_T^{q-1}$. Each of the players offered x_{NT}^{q-2} knows that if he continues to play in the next round, then with probability $\frac{1}{n-q+1}$, he will be the proposer and keeps $1-(q-2)x_T^{q-1} - x_{NT}^{q-1}$ for himself, and with another probability $\frac{1}{n-q+1}$, he will be a coalition partner and receive x_{NT}^{q-1}. Thus, the current proposer offers x_T^{q-2} to all the players in the trivial coalition, offers $x_{NT}^{q-2} = \frac{\delta}{n-q+1}\left(1-(q-2)x_T^{q-1}\right)$ to randomly selected two from the nontrivial coalition pool, and keeps $1-(q-3)x_T^{q-2} - 2x_{NT}^{q-2}$ for herself. This backward induction is analogously applied to the second round. In round 2, the continuation value of the player in the trivial coalition is δx_T^3, thus $x_T^2 = \delta x_T^3$. Each of the players offered x_{NT}^2 knows that if he continues to play in the next round, then with probability $\frac{1}{n-2}$, he will be the proposer and keeps $1-2x_T^3-(q-3)x_{NT}^3$ for himself, and with probability $\frac{q-3}{n-2}$ he will be included as a coalition partner and receives x_{NT}^3. Thus, the current proposer offers $x_{NT}^2 = \delta\left(\frac{1}{n-2}\left(1-2x_T^3-(q-3)x_{NT}^3\right)+\frac{q-3}{n-2}x_{NT}^3\right) = \frac{\delta}{n-2}\left(1-2x_T^3\right)$ to randomly selected $q-2$ members from the nontrivial coalition pool, and keeps $1-x_T^2-(q-2)x_{NT}^2$ for herself. Finally, in round 1, where there is no trivial coalition, a proposer offers x_{NT}^1 to $q-1$ players selected at random, where x_{NT}^1 is equal to the coalition partner's continuation value, that is, $x_{NT}^1 = \delta\left(\frac{1}{n-1}(1-x_T^2-(q-2)x_{NT}^2)+\frac{q-2}{n-1}x_{NT}^2\right) = \frac{\delta}{n-1}(1-x_T^2)$. □

In the course of proving Lemmas 1 to 4, the cycle-stationary subgame perfect equilibrium is completely characterized. The remaining task is to show that x_{NT}^1 is always between $\frac{\delta}{n}$ and $\frac{\delta}{n-1}$. To see this, we need to fully expand x_{NT}^1.

$$\begin{aligned}
x_{NT}^1 &= \frac{\delta}{n-1}(1-x_T^2) = \frac{\delta}{n-1} - \frac{\delta^2}{n-1}x_T^3 = \ldots = \frac{\delta}{n-1} - \frac{\delta^{q-1}}{n-1}x_T^q \\
&= \frac{\delta}{n-1} - \frac{\delta^{q-1}}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q}\delta^{n-q+1} \\
&= \frac{\delta}{n-1} - \frac{\delta^{n}}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q}
\end{aligned}$$

When $q=1$, x_{NT}^1 is $\frac{\delta}{n-1}$. (However, it will be offered to $q-1$, that is, 0 members.) When $q=n$, $x_{NT}^1 = \frac{\delta}{n-1} - \frac{\delta^n}{n(n-1)}$, which is strictly smaller than $\frac{\delta}{n-1}$, and larger than $\frac{\delta}{n}$ with equality hold when $\delta = 1$. Now I consider $q \in \{2,\ldots,n-1\}$. Since $\frac{\delta^n}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q}$ is positive, x_{NT}^1 is smaller than $\frac{\delta}{n-1}$. Also, except when $\delta=0$, x_{NT}^1 is strictly greater than $\frac{\delta}{n}$. I assume $\delta >$ so that the terms in the following inequalities can be divided by δ.

$$\begin{aligned}
&\frac{\delta}{n} \le x_{NT}^1 = \frac{\delta}{n-1} - \frac{\delta^n}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q} \\
\Leftrightarrow &\frac{\delta^n}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q} \le \frac{\delta}{n-1} - \frac{\delta}{n} = \frac{\delta}{n(n-1)} \\
\Leftrightarrow &\delta^{n-1}\frac{(q-1)!}{(n-1)!}(q-1)^{n-q} \le 1 \\
\Leftrightarrow &\delta^{n-1}\frac{q-1}{n-1}\frac{q-1}{n-2}\cdots\frac{q-1}{q} \le 1
\end{aligned}$$

Since each of $\frac{q-1}{n-1}, \ldots, \frac{q-1}{q}$ is smaller than 1, so is the product of them. □

Proof of Proposition 2: Consider a subgame perfect equilibrium of the subgame in round $t+1$. Since k is a positive integer, such a round exists. Suppose in the subgame perfect equilibrium of the $t+1$-round subgame, the proposer offers x^{t+1} to $q-1$ randomly selected nontrivial members, and keeps $1-(q-1)x^{t+1}$ for herself. Given this equilibrium in round $t+1$, the proposer in round t offers $q-1$ randomly selected trivial members x^t, and keeps $1-(q-1)x^t$ for herself. A player who receives an offer of x^t will accept it if

$$\delta^{t-1}x^t \geq \delta^t\left(\frac{1}{n}(1-(q-1)x^{t+1}) + \frac{q-1}{n}x^{t+1}\right),$$

which is equivalent to the inequality in the proof of Lemma 1. Thus, the subgame perfect equilibrium in t-round subgame is equivalent to the round-t strategy profile of the CSSP equilibrium. Backward induction is applied in the same way. □

Proof of Corollary 1: First, observe that the CSSP equilibrium in the last round of each cycle mimics the SSPE of the BF model from Lemma 1. Due to Eraslan (2002), we know that the SSPE of the BF model is unique in payoffs. Thus, we apply backward induction from round n to round 1, with knowing that the uniqueness in payoffs in round n is guaranteed. Suppose for the sake of contradiction that there are two subgame-perfect equilibria at a subgame in round $t \in \{1, \ldots, n\}$ which render different payoffs to coalition members. (Different payoffs to the proposer directly violate the subgame perfection.) In each equilibrium, different payoffs across the coalition members in the same (either trivial or nontrivial) coalition pool directly violate the subgame perfection. Since the number of the trivial coalition pool is the same in both equilibria, the number of the coalition members from the nontrivial coalition pool must be the same. Thus, the only possible situation is that in one equilibrium members in the nontrivial coalition pool are offered $\tilde{x}^t_{NT}$ more than what they are offered in the other equilibrium, x^t_{NT}. It contradicts that the strategy is subgame perfect: The proposer can exploit further by reducing $\tilde{x}^t_{NT}$ to x^t_{NT}. Therefore, the payoff in each subgame is unique. □

Proof of Proposition 3: Let $\mathrm{BF}_{q,n}$ denote the proposer's share in the SSP equilibrium of the BF model under a q-quota rule with n players (n odd) and a discount factor of $\delta = 1$, and let $\mathrm{InfC}_{q,n}$ denote the corresponding share in infinite-cycle bargaining. Also, let $q^*(n)$ denote the q that yields the largest difference between $\mathrm{BF}_{q,n}$ and $\mathrm{InfC}_{q,n}$. From the proof of Proposition 1, we know that x^1_{NT} is equal to $\frac{\delta}{n}$ only when $\delta = 1$ and $q = n$. It is trivial that the equality holds when $q = 1$, that is, under a dictatorship. Other than that, x^1_{NT} is strictly greater than $\frac{\delta}{n}$ for any δ. Thus the proposer's share in the

CSSP equilibrium is strictly smaller than at in the SSP equilibrium. We focus on the cases where $\delta = 1$. My first goal is to find $q^*(n)$, the argument that maximizes the difference between the proposer's share in the SSP equilibrium of the Baron–Ferejohn model (denoted by $\mathrm{BF}_{q,n}$) and that in the CSSP equilibrium of infinite-cycle bargaining without replacement ($\mathrm{InfC}_{q,n}$). Since

$$\begin{aligned}
\mathrm{BF}_{q,n} - \mathrm{InfC}_{q,n} &= \left(1 - \frac{q-1}{n}\right) - \left(1 - (q-1)\left(\frac{1}{n-1} - \frac{1}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q}\right)\right) \\
&= -\frac{q-1}{n} + \frac{q-1}{n-1} - \frac{1}{n-1}\frac{(q-1)!}{n!}(q-1)^{n-q+1} \\
&= \frac{1}{n(n-1)} \underbrace{\left(q - 1 - \frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1}\right)}_{=Q(n,q)},
\end{aligned}$$

$$q^*(n) \in \arg\max_{q\in\{1,\ldots,n\}} Q(n,q) = q - 1 - \frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1}.$$

I want to show the followings: (1) $Q(n,q)$ is single-peaked in n so that $q^*(n)$ is unique. (2) When $q = (n+1)/2$, changing q to $q' = q+1$ yields a larger difference for any $n \geq 11$. (3) $\frac{q^*(n+1)}{n+1} \geq \frac{q^*(n)}{n}$.

Lemma 5. *$q^*(n)$ is unique.*

Proof: Since the factorial is a discrete function, we cannot directly differentiate it with respect to q. Instead, using Stirling's approximation, I first show that the continuous function that approximates the objective function has a unique maximizing argument. Then, I show that any two adjacent integers cannot be the maximizing arguments at the same time. These two claims jointly imply the uniqueness of the maximizing argument. Since $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$,

$$\begin{aligned}
q - 1 - \frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1} &\approx q - 1 - \frac{\sqrt{2\pi(q-1)}\left(\frac{q-1}{e}\right)^{q-1}}{\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}}(q-1)^{n-q+1} \\
&= q - 1 - (q-1)^{n+\frac{1}{2}}e^{-q}K(n) := F(n,q),
\end{aligned}$$

where $K(n) = e^n\left(\frac{1}{n-1}\right)^{n-\frac{1}{2}}$. The first order condition with respect to q is

$$\frac{\partial F(n,q)}{\partial q} = 1 - \left(n - q + \frac{3}{2}\right)(q-1)^{n-\frac{1}{2}}e^{-q}K(n),$$

which is $-\frac{1}{2}$ when $q = n$, and 1 when $q = 1$. Since this function is continuous, by the intermediate value theorem, there exists q^* such that $1 - \left(n - q^* + \frac{3}{2}\right)(q^* - 1)^{n-\frac{1}{2}}e^{-q^*}K(n) = 0$.

The second order condition with respect to q, $\frac{\partial^2 F(n,q)}{\partial q^2}$, is

$$(q-1)^{n-\frac{1}{2}}e^{-q}K(n)-\frac{n-\frac{1}{2}}{q-1}\left(n-q+\frac{3}{2}\right)(q-1)^{n-\frac{1}{2}}e^{-q}K(n)+\left(n-q+\frac{3}{2}\right)(q-1)^{n-\frac{1}{2}}e^{-q}K(n).$$

Using the fact that $1=\left(n-q^*+\frac{3}{2}\right)(q^*-1)^{n-\frac{1}{2}}e^{-q^*}K(n)$, we know that

$$\left.\frac{\partial^2 F(n,q)}{\partial q^2}\right|_{q=q^*}=\frac{1}{n-q^*+\frac{3}{2}}-\frac{n-\frac{1}{2}}{q^*-1}+1,$$

which is strictly smaller than 0 for $q\in[1,\underline{q})$, and greater than 0 for $q\in[\underline{q},n]$, where $\underline{q}\leq n$ is such that $\underline{q}-1=\left(n+\frac{1}{2}-\underline{q}\right)\left(n+\frac{3}{2}-\underline{q}\right)$. I claim that $\left.\frac{\partial^2 F(n,q)}{\partial q^2}\right|_{q=q^*}$ is strictly smaller than 0, so that q^* is unique. Suppose for the sake of contradiction that $\left.\frac{\partial^2 F(n,q)}{\partial q^2}\right|_{q=q^*}\geq 0$. It implies that such q^* is in $[\underline{q},n]$. Two observations that $\left.\frac{\partial F(n,q)}{\partial q}\right|_{q=q^*}=0$ and $\left.\frac{\partial^2 F(n,q)}{\partial q^2}\right|_{q=q^*}>0$ jointly imply that $\left.\frac{\partial F(n,q)}{\partial q}\right|_{q=n}>0$. However, we know that it is $-1/2$, which is a contradiction.

Next, I want to show that for any q, both q and $q+1$ cannot be the maximizing arguments at the same time. Suppose for the sake of contradiction that q and $q+1$ are both maximizing arguments. Then,

$$q-1-\frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1}=q-\frac{q!}{(n-1)!}q^{n-q}$$

Rearranging terms, we have

$$q^{n-q+1}-(q-1)^{n-q+1}=\frac{(n-1)!}{(q-1)!}$$

This equality holds only when $q=n$, which contradicts existence of $q+1$. $\square$

Lemma 5 assures that the difference between the proposer's share in the CSSP equilibrium and that in the SSP equilibrium is described by a single-picked function with respect to q. Next, I show that a supermajority, not a simple majority, is the voting rule that maximizes the difference for a sufficiently large n. Specifically, I want to show that for $n\geq 11$, $Q(n,\frac{n+3}{2})>Q(n,\frac{n+1}{2})$.

Plugging $q=\frac{n+1}{2}$ and $q+1$ into $Q(n,q)=q-1-\frac{(q-1)!}{(n-1)!}(q-1)^{n-q+1}$, and taking the difference, we have

$$Q\left(n,\frac{n+3}{2}\right)-Q\left(n,\frac{n+1}{2}\right)=1-\frac{\left(\frac{n-1}{2}\right)!}{(n-1)!}\left(\left(\frac{n+1}{2}\right)^{\frac{n+1}{2}}-\left(\frac{n-1}{2}\right)^{\frac{n+1}{2}}\right).$$

I claim this is strictly positive when n is sufficiently large.

$$
\begin{aligned}
&1-\frac{\left(\frac{n-1}{2}\right)!}{(n-1)!}\left(\left(\frac{n+1}{2}\right)^{\frac{n+1}{2}}-\left(\frac{n-1}{2}\right)^{\frac{n+1}{2}}\right)>0\\
\Leftrightarrow&\left(\frac{n+1}{2}\right)^{\frac{n+1}{2}}-\left(\frac{n-1}{2}\right)^{\frac{n+1}{2}}<\frac{(n-1)!}{\left(\frac{n-1}{2}\right)!}=\underbrace{(n-1)(n-2)\cdots(\frac{n+1}{2})}_{\frac{n-1}{2}\text{ terms}}\\
\Leftrightarrow&\frac{\frac{n+1}{2}}{n-1}\frac{\frac{n+1}{2}}{n-2}\cdots\frac{\frac{n+1}{2}}{\frac{n+1}{2}}\frac{n+1}{2}-\frac{\frac{n-1}{2}}{n-1}\frac{\frac{n-1}{2}}{n-2}\cdots\frac{\frac{n-1}{2}}{\frac{n+1}{2}}\frac{n-1}{2}<1
\end{aligned}
$$

When n approaches to infinity, the first term of the left-hand side of the inequality above converges to 0 because $\frac{\frac{n+1}{2}}{n-1}\frac{\frac{n+1}{2}}{n-2}\cdots\frac{\frac{n+1}{2}}{\frac{n+1}{2}}$ converges to 0 exponentially faster than $\frac{n+1}{2}$ approaches to infinity. Similarly, the second term also converges to 0. Therefore there should exist $\underline{\mathrm{n}}$ such that for any $n \geq \underline{\mathrm{n}}$, the left-hand side is close to 0. I manually find that such $\underline{\mathrm{n}}$ is 11. Lastly, I show that $\frac{q^*(n)}{n}$ is increasing in n. Let $f(n,q)$ denote $\frac{\partial F(n,q)}{\partial q}$. Note that $f(n,q^*) = 1-\left(n-q^*+\frac{3}{2}\right)(q^*-1)^{n-\frac{1}{2}}e^{-q^*}K(n) = 0$. Since every entity in $\left(n-q^*+\frac{3}{2}\right)(q^*-1)^{n-\frac{1}{2}}e^{-q^*}K(n)$ is increasing in n, $f(n,q)$ is decreasing in n. Since we are interested in $\frac{d(q^*/n)}{dn}$, introduce a new variable $y = q^*/n$. Since $q^* \geq \frac{n+1}{2}$ for all n, y is in $[1/2, 1)$. Replacing q^* with ny, we have

$$
f(n,y) = 1-\left(n-ny+\frac{3}{2}\right)(ny-1)^{n-\frac{1}{2}}e^{-ny}K(n) = 0.
$$

The remaining tasks are to show $\partial f(n,y)/\partial y \leq 0$ because if it is so, $\frac{dy}{dn} = -\frac{\partial f(n,y)/\partial n}{\partial f(n,y)/\partial y} \geq 0$ by the implicit function theorem. $\frac{\partial f(n,y)}{\partial n} = (1-y)\frac{\partial f(n,q)}{\partial n} > 0$ since $y < 1$.

$$
\begin{aligned}
\frac{\partial f(n,y)}{\partial y} &= n(yn-1)^{n-\frac{1}{2}}e^{-ny}K(n)-\frac{n-\frac{1}{2}}{yn-1}(n-yn+\frac{3}{2})(yn-1)^{n-\frac{1}{2}}e^{-ny}K(n)\\
&\quad+n(n-yn+\frac{3}{2})(yn-1)^{n-\frac{1}{2}}e^{-ny}K(n)\\
&=\frac{n}{n-yn+\frac{3}{2}}(1-f(n,y))-\frac{n-\frac{1}{2}}{yn-1}(1-f(n,y))+n(1-f(n,y)).
\end{aligned}
$$

Since $f(n,y)=0$, $\frac{\partial f(n,y)}{\partial y} = \frac{n}{n-yn+\frac{3}{2}} - \frac{n-\frac{1}{2}}{yn-1} + n$. Both $\frac{n}{(1-y)n+\frac{3}{2}}$ and $-\frac{n-\frac{1}{2}}{yn-1}$ are monotone increasing in $y \in [1/2, 1)$, it is sufficient to show that $\left.\frac{\partial f(n,y)}{\partial y}\right|_{y=1/2} = \frac{2n}{n+3} - \frac{2n-1}{n-2} + n = \frac{2(n+3-3)}{n+3} - \frac{2(n-2+1)}{n-2} + n = n - \frac{6}{n+3} - \frac{2}{n-2} \geq 0$, which holds for any $n \geq 3$. □

Proof of Proposition 4: Since the ex-ante expected payoff of each player is $1/n$, my goal is to show $(\mathrm{BF}_{q^*(n),n} - \mathrm{InfC}_{q^*(n),n})/(1/n)$, or $n(\mathrm{BF}_{q^*(n),n} - \mathrm{InfC}_{q^*(n),n})$, is increasing in n. Since we deal with integers, my goal boils down to show

$$(n+1)(\mathrm{BF}_{q^*(n+1),n+1} - \mathrm{InfC}_{q^*(n+1),n+1}) \geq n(\mathrm{BF}_{q^*(n),n} - \mathrm{InfC}_{q^*(n),n}).$$

(Because we have considered odd integers, $\mathrm{BF}_{q^*(n),n} - \mathrm{InfC}_{q^*(n),n}$ might be compared with $\mathrm{BF}_{q^*(n+2),n+2} - \mathrm{InfC}_{q^*(n+2),n+2}$, but this does not affect the result.) Since $\mathrm{BF}_{q^*(n+1),n+1} - \mathrm{InfC}_{q^*(n+1),n+1} \geq \mathrm{BF}_{q^*(n),n+1} - \mathrm{InfC}_{q^*(n),n+1}$ by definition of $q^*(\cdot)$, the inequality holds if

$$(n+1)(\mathrm{BF}_{q^*(n),n+1} - \mathrm{InfC}_{q^*(n),n+1}) \geq n(\mathrm{BF}_{q^*(n),n} - \mathrm{InfC}_{q^*(n),n}),$$

where

$$\begin{aligned}
&(n+1)(\mathrm{BF}_{q^*(n),n+1} - \mathrm{InfC}_{q^*(n),n+1}) \\
&\qquad = (n+1)\frac{1}{(n+1)n}\left(q^*(n) - 1 - \frac{(q^*(n)-1)!}{(n-1)!}(q^*(n)-1)^{n-q^*(n)+1}\frac{q^*(n)-1}{n}\right) \\
&\qquad = \frac{1}{n}\left(q^*(n) - 1 - \frac{(q^*(n)-1)!}{(n-1)!}(q^*(n)-1)^{n-q^*(n)+1}\frac{q^*(n)-1}{n}\right)
\end{aligned}$$

and

$$\begin{aligned}
n(\mathrm{BF}_{q^*(n),n} - \mathrm{InfC}_{q^*(n),n}) &= n\frac{1}{n(n-1)}\left(q^*(n) - 1 - \frac{(q^*(n)-1)!}{(n-1)!}(q^*(n)-1)^{n-q^*(n)+1}\right) \\
&= \frac{1}{n-1}\left(q^*(n) - 1 - \frac{(q^*(n)-1)!}{(n-1)!}(q^*(n)-1)^{n-q^*(n)+1}\right).
\end{aligned}$$

For notational simplicity, denote $q^*(n) - 1$ as X, and $\frac{(q^*(n)-1)!}{(n-1)!}(q^*(n)-1)^{n-q^*(n)+1}$ as C. Then, we need to check the inequality of the following:

$$\begin{aligned}
\frac{1}{n}\left(X - C\frac{X}{n}\right) &\geq \frac{1}{n-1}(X - C) \\
\Leftrightarrow \frac{n-1}{n} &\geq \frac{X-C}{X - C\frac{X}{n}}.
\end{aligned}$$

If $q = n$, $\mathrm{BF}_{q,n} - \mathrm{InfC}_{q,n} = 0$. Thus $q^*(n)$, the quota rule that yields the largest difference, must be smaller than $n-1$. Therefore, $\frac{X}{n} \leq \frac{n-1}{n}$. Also, $q = 1$ cannot be $q^*(n)$, so $q^*(n) \geq 1$, or $X \geq 1$. The denominator of the right-hand side of the inequality becomes much smaller than the numerator, while the left-hand side approaches 1. Therefore if n is sufficiently large, the inequality holds. $\square$