Game Theory: Lecture (Week 14)

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Remaining Course Schedule

Wk	Date	Online (Coursera)	Offline	Notes
14	12/1	1st-half Wk7 video	45min meeting	**Quiz 4**
15	12/8	2nd-half Wk7 video	75min meeting	
16	12/15	-	-	**Final**

Two remaining office hours

- From 3:00 to 4:30PM on Dec 1, 2023
- (Extended) From 2:00 to 5:00PM on Dec 8, 2023

Review of Quiz 4

Cooperative Game Theory

- So far, we have considered non-cooperative games. Now we switch our gears.
- Given a set of agents, a cooperative game (or coalitional game) defines how well each group (or coalition) of agents can do for itself.
- We are not concerned with:
  - how agents make individual choices within a coalition;
  - how they coordinate within a coalition.
  - Thus, we won't talk about individuals' strategies.
- Instead, we take the payoffs to a coalition as given.
- Forget about the term "game". Regard cooperative game as something like an "allocation rule" or "agreeable principle."

Coalitional game with transferable utility

Definition

A coalitional game with transferable utility *is a pair* (N, v)*, where* N *is a finite set of players, indexed by* i*; and* $v : 2^N \to \mathbb{R}$ *associates with each coalition* $S \subseteq N$ *a real-valued payoff* $v(S)$ *that the coalition's members can distribute among themselves. We assume that* $v(\emptyset) = 0$*.*

Definition

A game $G = (N, v)$ *is* **superadditive** *if for all* $S, T \subset N$*, if* $S \cap T = \emptyset$*, then* $v(S \cup T) \geq v(S) + v(T)$*.*

Coalitional game with transferable utility: Example

Suppose there are three students, $N \equiv \{1, 2, 3\}$. They can form a team to work on an assignment.

- There are eight ($= 2^3$) possible teams: $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$.
- Imagine that each subset has a corresponding value (say, team performance), for example, $v(\{1\}) = 60$ and $v(\{1, 3\}) = 75$.

A coalitional game with transferable utility is $\left(N, \begin{array}{c} v(\emptyset) \\ v(\{1\}) \\ v(\{2\}) \\ v(\{3\}) \\ v(\{1,2\}) \\ v(\{1,3\}) \\ v(\{2,3\}) \\ v(\{1,2,3\}) \end{array} \right)$.

Coalitional game with transferable utility: Example

Suppose there are three students, $N \equiv \{1, 2, 3\}$. They can form a team to work on an assignment.

Q1) If $v(\{1\}) = 60, v(\{3\}) = 25$, and $v(\{1, 3\}) = 80$, is the game superadditive?

Q2) Does $v(\{1\}) = -5$ violate the superadditivity?

Q3) Suppose $v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{1, 2\}) = 0$, $v(\{3\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = 80$. Is the game superadditive?

Superadditivity rules out the possibility of 'obviously destructive' coalitions, for example, teams with troll members.

Shapley value

- The **Shapley Value** allocates the value of a group according to marginal contribution calculations.
- Many fair ways of appreciating each member's contribution.

  (Imagine you spent 10 hours for a team project when you are free, and your teammate spent 3 hours while he's hospitalized. Professor told the project scores should be divided within a team. What's the fair way of allocating the scores?)
- $\psi(N, v) = (\psi_1(N, v), \psi_2(N, v), \ldots, \psi_n(N, v))$ is a function that maps a game to each player's value.
- Shapley value is one way of defining $\psi(N, v)$. There could be other ways of allocating value.
- Let's consider some axioms for $\psi(N, v)$.

Shapley value

Three definitions

- i and j are *interchangeable* if $v(S \cup \{i\}) = v(S \cup \{j\}) \forall S \subset N$.
- i is *a dummy player* if $v(S \cup \{i\}) = v(S)\ \forall S \subset N$.
- (N, v) is *additively separable* to (N, v_1) and (N, v_2) if $v(S) = v_1(S) + v_2(S)\ \forall S \subset N$.

Axiom (Symmetry)

For any v, if i and j are interchangeable, then $\psi_i(N, v) = \psi_j(N, v)$.

Axiom (Dummy Player)

For any v ,if i is a dummy player, then $\psi_i(N, v) = 0$.

Axiom (Additivity)

If (N, v) is additively separable to (N, v_1) and (N, v_2), then $\psi_i(N, v) = \psi_i(N, v_1) + \psi_i(N, v_2)$.

Shapley value

Given a coalitional game (N, v), the Shapley value of player i is:

$$\psi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i\}) - v(S) \right],$$

where $n = |N|$ and $s = |S|$.
It is known that the Shapley value is the unique function $\psi(N, v)$ that satisfies the Symmetry, Dummy Player and Additivity axioms.

- Don't be scared of the long mathematical notation!
- First, list all subsets that do not include i.
- Second, for each case, check how much value is added by including i. $(v(S \cup \{i\}) - v(S))$
- Third, calculated the weighted sum of additional values, whose weight is $\frac{s!(n-s-1)!}{n!}$.

Shapley value: Example

Consider again the teamwork example with three students.
Suppose $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 20$,
$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 50$, and $v(\{1,2,3\}) = 70$.
What's the Shapley value of player 1?

- First, list all subsets that exclude 1: $\emptyset, \{2\}, \{3\}, \{2,3\}$.
- Check the additional value added by player 1.
  - From $\emptyset$ to $\{1\}$: 20
  - From $\{2\}$ to $\{1,2\}$: 30
  - From $\{3\}$ to $\{1,3\}$: 30
  - From $\{2,3\}$ to $\{1,2,3\}$: 20
- Now calculate the weighted sum:

$$\frac{0!(3-0-1)!}{3!}20 + \frac{1!(3-1-1)!}{3!}30 + \frac{1!(3-1-1)!}{3!}30 + \frac{2!(3-2-1)!}{3!}20$$
$$= \frac{2}{6}20 + \frac{1}{6}30 + \frac{1}{6}30 + \frac{2}{6}20 = \frac{140}{6} = \frac{70}{3}$$

Shapley value: Example 2

A game is called a *simple game* if $v(S) \in \{0, 1\}$ for all $S \subset N$.
Suppose $v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and
$v(\{1, 2\}) = v(\{2, 3\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = 1$.
(Think a majority voting.) What is the Shapley value of player 1?

- First, list all subsets that exclude 1: $\emptyset, \{2\}, \{3\}, \{2, 3\}$.
- Check the additional value added by player 1.
  - From $\emptyset$ to $\{1\}$: 0
  - From $\{2\}$ to $\{1, 2\}$: 1
  - From $\{3\}$ to $\{1, 3\}$: 1
  - From $\{2, 3\}$ to $\{1, 2, 3\}$: 0
- Now calculate the weighted sum:

$$\frac{0!(3-0-1)!}{3!}0 + \frac{1!(3-1-1)!}{3!}1 + \frac{1!(3-1-1)!}{3!}1 + \frac{2!(3-2-1)!}{3!}0 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

- Thus, in a simple game, the Shapley value equals the probability of being pivotal.

(Check the Coursera videos and problem sets for more examples.)

Core

- The Shapley value defined a fair way of dividing the grand coalition's payment among its members.
- But sometimes smaller coalitions can be more attractive for subsets of the agents, even if they lead to lower value overall.
- In the prisoner's dilemma, we learned that (C,C) won't be played even if it had greater value overall. Can we consider a concept analogous to Nash equilibrium, say, a subset of the agents that they do not want to deviate from the current coalition?

Core

Definition (Core)

A payoff vector x is in the core of a coalitional game (N, v) iff

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S) \quad \textit{and} \quad \sum_{i \in N} x_i = v(N).$$

The core may not exist. If it exists, it may not be unique. So it is easier to think about the "core set" whose size is 0 (empty), 1 (unique core), or larger.

- Consider an arbitrary value vector $x = (x_1, x_2, \ldots, x_n)$.

1. List all nonempty subsets of N.
2. List the inequality condition ($\sum_{i \in S} x_i \geq v(S)$) for each subset.
3. Check if there exists x such that all conditions are satisfied.

Core: Example

Consider again the teamwork example with three students.
Suppose $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 20$,
$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 50$, and $v(\{1,2,3\}) = 70$.
Does the core exist?

1. All nonempty subsets:
   $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$.
2. All inequality conditions:

$$
\begin{aligned}
x_1 \geq 20, \qquad & x_2 \geq 20, \qquad & x_3 \geq 20 \\
x_1 + x_2 \geq 50, \quad & x_1 + x_3 \geq 50, \quad & x_2 + x_3 \geq 50 \\
& & x_1 + x_2 + x_3 = 70
\end{aligned}
$$

3. You can easily check there is no payoff vector x that satisfies all inequality conditions.

   $x_1 + x_2 \geq 50$, $x_2 + x_3 \geq 50$, and $x_1 + x_3 \geq 50$ imply $x_1 + x_2 + x_3 \geq 75$.

(If $v(\{i,j\}) = 40$ instead of 50, the core exists.)

Core: Example 2

Consider again a simple game with three players.
Suppose $v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and
$v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\}) = v(\{1,2,3\}) = 1$.
Does the core exist?

1. All nonempty subsets:
   $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$.
2. All inequality conditions:

$$\begin{aligned} x_1 \geq 0, \quad & x_2 \geq 0, \quad & x_3 \geq 0 \\ x_1 + x_2 \geq 1, \quad & x_1 + x_3 \geq 1, \quad & x_2 + x_3 \geq 1 \\ & & x_1 + x_2 + x_3 = 1 \end{aligned}$$

3. The core doesn't exist.

Shapley value and Core

Definition (Convex game)

A game (N, v) *is* **convex** *if* $v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$ $\forall S, T \subset N$.

- It is known that every convex game has core, and the Shapley value of a convex game is in the core.
- Check that the teamwork example used in this note is not a convex game. Hint: Consider $S = \{1, 2\}$ and $T = \{2, 3\}$.
  (If $v(\{1, 2\}), v(\{2, 3\}), w(\{1, 3\})$ were to be 40, then it would've been a convex game.)