Midterm Review

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1. [20 points] Suppose that a consumer has expenditure function $e(p_1, p_2, u) = \frac{p_1 p_2}{p_1+p_2}u$.

(a) Find an indirect utility function and Marshallian demand function of this consumer.

(b) Recover a utility function $u(x_1, x_2)$ that rationalizes this consumer's demand behavior.

(a) $e(p, v(p,m)) = m = \frac{P_1P_2}{P_1+P_2}V(P_1,P_2,m)$ $\quad \therefore V(P_1,P_2,m) = \frac{P_1+P_2}{P_1P_2}m = \frac{m}{P_1}+\frac{m}{P_2}$

By Roy's identity. $\quad x_1(P_1,P_2,M) = -\frac{\partial V/\partial P_1}{\partial V/\partial M} = -\frac{-M/P_1^2}{\frac{P_1+P_2}{P_1P_2}} = \frac{P_2M}{P_1(P_1+P_2)}$

$x_2(P_1,P_2,M) = (\text{similarly} \cdots) = \frac{P_1M}{P_2(P_1+P_2)}$

(b) $u(x_1,x_2) = \min_{P_1,P_2} V(P_1,P_2,1)$ s.t. $P_1x_1+P_2x_2=1$ (lecture note page 18)

$$\mathcal{L} = \frac{P_1+P_2}{P_1P_2} + \lambda(P_1x_1+P_2x_2-1)$$

$$\frac{\partial\mathcal{L}}{\partial P_1}: \quad -\frac{1}{P_1^2}+\lambda x_1 = 0 \iff \frac{1}{\sqrt{\lambda x_1}} = P_1^* \quad ①$$

$$\frac{\partial\mathcal{L}}{\partial P_2}: \quad -\frac{1}{P_2^2}+\lambda x_2 = 0 \iff \frac{1}{\sqrt{\lambda x_2}} = P_2^* \quad ②$$

$$\frac{\partial\mathcal{L}}{\partial \lambda}: \quad P_1^*x_1+P_2^*x_2 = 1. \quad ③$$

$$\frac{①}{②} \Rightarrow \frac{\sqrt{x_2}}{\sqrt{x_1}} = \frac{P_1^*}{P_2^*} \Rightarrow P_2^* = \frac{\sqrt{x_1}}{\sqrt{x_2}}P_1^*. \quad ④$$

Plug ④ to ① $\quad P_1^*x_1 + \frac{\sqrt{x_1}}{\sqrt{x_2}}P_1^*x_2 = P_1^*(x_1+\sqrt{x_1}\sqrt{x_2})$

$= P_1^*\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2}) = 1.$

$$\therefore P_1^* = \frac{1}{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})}$$

$$P_2^* = \frac{1}{\sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})}$$

$$\Rightarrow V(P_1^*,P_2^*,1) = \frac{1}{P_1^*}+\frac{1}{P_2^*} = \sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2}) + \sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})$$

$$= (\sqrt{x_1}+\sqrt{x_2})^2 \quad \blacksquare$$

2. [20 points] A consumer's preferences $\succeq$ on $\mathbb{R}^L_+$ can be represented by the utility function $u : \mathbb{R}^L_+ \to \mathbb{R}_+$ with the property that for any $x \in \mathbb{R}^L_+$ and $\alpha > 0$, $u(\alpha x) = \alpha u(x)$.

(a) Show that this consumer has homothetic preferences, that is, for any $x, y \in \mathbb{R}^L_+$, $x \succeq y$ if and only if $\alpha x \succeq \alpha y$ for any $\alpha > 0$.

(b) Show that this consumer's expenditure function is such that $e(p, u) = ue(p, 1)$ for any $u > 0$ and prices p. [Hint: First show $e(p, u) = e(up, 1)$, and then use the property of the expenditure function.]

(c) Is this consumer's indirect utility function linear in wealth? Explain.

(a) $x \succsim y \Leftrightarrow u(x) \geq u(y) \Leftrightarrow \alpha u(x) \geq \alpha u(y)$ for $\alpha > 0$

$\Leftrightarrow u(\alpha x) \geq u(\alpha y)$ by the given property

$\Leftrightarrow \alpha x \succsim \alpha y$.

(b). $e(p, u) = \min_x \; p \cdot x \quad$ s.t. $u(x) = u$

$= \min_x \; p x \quad$ s.t. $u(\frac{1}{u} x) = 1$

relabel $\frac{1}{u} x$ as y. thus $x = uy$.

$= \min_y \; p \cdot uy \quad$ s.t. $u(y) = 1. \quad = e(up, 1)$

Since $e(p, u)$ is homogeneous of degree 1 in price.

$e(up, 1) = u \cdot e(p, 1)$

$\therefore e(p, u) = u \cdot e(p, 1)$

(c) $m = e(p, v(p, m)) = v(p, m) e(p, 1) \Rightarrow v(p, m) = \frac{1}{e(p, 1)} m$.

$\therefore$ yes $v(p, m)$ is linear in m.

3. [15 points] A person consumes two indivisible goods. (He cannot buy a fraction of a good. For example, he can buy 3 units of good 1 but cannot buy 2.4 units.) He chooses (1, 1) when prices are (3, 6) but he could have afforded the bundle (a, b). At prices (4, 4), he chooses (a, b). Find the (set of) choices (a, b) such that he doesn't violate WARP.

(a, b) was affordable at prices (3, 6). $\Rightarrow 3a + 6b \leq 3 \cdot 1 + 6 \cdot 1 = 9$

$\Leftrightarrow a + 2b \leq 3$

(a, b) is revealed preferred to (1, 1) at (4, 4) $\Rightarrow 4a + 4b < 4 + 4 = 8$

$\iff a+b<2$

discrete (a,b) such that $a+2b\leq 3$ & $a+b>2$
$\Rightarrow \{(1,0),(1,1)\}$

4. [15 points] Consider an economy in which there are L goods. Dennis, an economic agent in this economy, has a strictly increasing and strictly concave utility unction, $u:\mathbb{R}^L_+ \to \mathbb{R}$.

(a) Show that Dennis must have the indirect utility function $v(p,m)$ being strictly concave with respect to the income m.

(b) Dennis has two choices for his career, doctor and data scientist. While both careers seem equally promising in terms of the expected income, his income from being a data scientist will be more volatile than that from being a doctor. Explain why he should choose to be a doctor by using the fact in (a). Try to be as rigorous as you can.

(a) $x' = \arg\max_x u(x)$ s.t. $p\cdot x \leq m'$ $\quad V(p,m') = u(x')$

$x'' = \arg\max_x u(x)$ s.t. $p\cdot x \leq m''$ $\quad V(p,m'') = u(x'')$

Since $u(x)$ is strictly increasing, $x' \neq x''$

$\because$ strict concavity

For $\lambda \in (0,1)$, $\lambda V(p,m') + (1-\lambda)V(p,m'') = \lambda u(x') + (1-\lambda)u(x'') < u(\lambda x' + (1-\lambda)x'')$
$\overset{(*)}{\leq} V(p, \lambda m' + (1-\lambda)m'')$

(*) since $p\cdot(\lambda x' + (1-\lambda)x'') = \lambda p x' + (1-\lambda)p x'' = \lambda m' + (1-\lambda)m''$,
$\lambda x' + (1-\lambda)x''$ is feasible with budget $\lambda m' + (1-\lambda)m''$.

$V(p, \lambda m' + (1-\lambda)m'')$ is the maximized utility with x^* among all feasible goods.
it must not be smaller than $u(\lambda x' + (1-\lambda)x'')$

(b) Since $V(p,m)$ is strictly concave in m,
$E(V(p,M)) < V(p, E(M))$ due to Jensons inequality.

(For illustration. Imagine that a doctor's income is m
& a data scientist's income is $2m$ with 50% prob.
0 otherwise.
$\frac{1}{2}V(p,0) + \frac{1}{2}V(p,2m) < V(p,m)$.)

5. [20 points] Suppose there are two goods, a positional good x and a

5. [20 points] Suppose there are two goods, a positional good x and a nonpositional good y. The utility function of a consumer, $U : \mathbb{R}^2_+ \to \mathbb{R}$ is given as follows:

$x^{0.5}y^{0.5} + \mu F(x)$ (monotone transformation scaled by $\frac{1}{1-\mu}$)

$$U(x, y; F(x)) = \cancel{(1-\mu)x^{0.5}y^{0.5} + \mu F(x)}$$

where $\mu \in [0, 1]$ is the relative weight on positional concern, and $F(x) = \int_{x_{min}}^{x} f(z)dz$ is the cumulative distribution function value at x, capturing the percentile ranking of x in the population of x values, $f(x)$. Let p_x and p_y denote the prices of goods x and y. A consumer's income is m.

(a) Suppose $\mu = 0$. Find the Marshallian demand for x and y.

$$\max_{x,y} \ x^{0.5}y^{0.5} \quad \text{s.t} \quad p_x x + p_y y \leq m.$$

$$\Rightarrow \ x = \frac{m}{2p_x} \qquad y = \frac{m}{2p_y}$$

From now on, suppose $\mu > 0$.

(b) Assume that other consumers' consumption decisions are exogenously given. (That is, the consumer can "choose" the relative rank of x consumption by choosing the level.) Explain how the demand for x and y change when μ gets larger.

(c) Now suppose the entire population collectively agree to stick to the relative rank based on each consumer's income level, so $F(x)$ is now exogenously given as $G(m) = \int_{m_{min}}^{m} g(z)dz$, where $g(z)$ represents the density function of income values. Compare the Marshallian demand for x and y with your answer in (a).

(b) $$\max_{x,y} \ x^{0.5}y^{0.5} + \mu F(x) \quad \text{s.t.} \quad p_x x + p_y y = m.$$

$$\Leftrightarrow \max_{x} \ x^{0.5}\left(\frac{m - p_x x}{p_y}\right)^{0.5} + \mu F(x)$$

since it has increasing differences in (μ, x).

$x^*(\mu)$ is increasing by Topkis' theorem.

(c) $$\max_{x,y} \ x^{0.5}y^{0.5} + \underbrace{\mu F(m)}_{\text{constant.}} \quad \text{s.t} \quad p_x x + p_y y = m.$$

$\therefore$ same as the answer in (a)

6. [20 points] Consider the following production function:

$$Q = (L^\alpha + K^\alpha)^{1/\beta},$$

where L is the input of labor and K is the input of capital. We assume $\alpha \in (0,1)$ and $\beta \in (0,1)$. Assume that wage is $w = 2$ and rent is $r = 1$.

(a) Find the condition on α and β for increasing/constant/decreasing return to scale production function.

(b) Find the cost function (as a function of Q).

(c) Find the condition on α and β for convex/linear/concave cost function.

(d) Compare your answers for part (a) and part (c) and explain the intuition.

(a) For $\lambda > 1$, $\left((\lambda L)^\alpha + (\lambda K)^\alpha\right)^{\frac{1}{\beta}} = \lambda^{\frac{\alpha}{\beta}}(L^\alpha + K^\alpha)^{\frac{1}{\beta}} = \lambda^{\frac{\alpha}{\beta}} Q$.

$\therefore$ IRTS iff $\alpha > \beta$
CRTS iff $\alpha = \beta$
DRTS iff $\alpha < \beta$.

(b) $\min\limits_{L,K} \ 2L + K$ s.t. $(L^\alpha + K^\alpha)^{\frac{1}{\beta}} = Q$

$$2 = \lambda \frac{1}{\beta}(L^\alpha + K^\alpha)^{\frac{1}{\beta}-1} \cdot \alpha L^{\alpha-1} \quad ①$$

$$1 = \lambda \frac{1}{\beta}(L^\alpha + K^\alpha)^{\frac{1}{\beta}-1} \alpha K^{\alpha-1} \quad ②$$

$$\frac{①}{②} = 2 = \left(\frac{L}{K}\right)^{\alpha-1} = \left(\frac{L}{K}\right)^{1-\alpha} \Rightarrow 2^{\frac{1}{1-\alpha}} K = L. \quad ③$$

plug ③ into $(L^\alpha + K^\alpha)^{\frac{1}{\beta}} = Q$, $\left(2^{\frac{\alpha}{1-\alpha}} K^\alpha + K^\alpha\right)^{\frac{1}{\beta}} = Q$

$$\Leftrightarrow \left(2^{\frac{\alpha}{1-\alpha}} + 1\right) K^{\alpha/\beta} = Q$$

$$\boxed{\begin{aligned} &\Rightarrow K^* = \left(\frac{1}{2^{\frac{\alpha}{1-\alpha}}+1}\right)^{\frac{\beta}{\alpha}} Q^{\frac{\beta}{\alpha}} \\ &\Rightarrow L^* = 2^{\frac{1}{1-\alpha}}\left(\frac{1}{2^{\frac{\alpha}{1-\alpha}}+1}\right)^{\frac{\beta}{\alpha}} Q^{\frac{\beta}{\alpha}} \end{aligned}}$$

Thus
$$2L^* + K^* = 2^{\frac{1}{1-\alpha}+1}\left(\frac{1}{2^{\frac{\alpha}{1-\alpha}}+1}\right)^{\frac{\beta}{\alpha}} Q^{\frac{\beta}{\alpha}} + \left(\frac{1}{2^{\frac{\alpha}{1-\alpha}}+1}\right)^{\frac{\beta}{\alpha}} Q^{\frac{\beta}{\alpha}}$$

$$= \underbrace{\left(2^{\frac{2-\alpha}{1-\alpha}} + 1\right)\left(\frac{1}{2^{\frac{\alpha}{1-\alpha}}+1}\right)^{\frac{\beta}{\alpha}}}_{:=C} Q^{\frac{\beta}{\alpha}} = CQ^{\frac{\beta}{\alpha}}.$$

(c) $CQ^{\frac{\beta}{\alpha}}$ is convex if $\beta > \alpha$.
linear if $\beta = \alpha$
concave if $\beta > \alpha$.

(d) DRTS $\Leftrightarrow$ cost function is convex $\Leftrightarrow$ diseconomies of scale.

IRTS $\Leftrightarrow$ economies of scale.