Game Theory: In-class Midterm, Suggested Solutions Fall 2024

1. [14 points] Consider the following normal-form game:

P1 \ P2	x	y	z
A	4, 2	0, 0	5, 0
B	0, 0	2, 4	0, 3
C	1, 3	0, 2	2, 2

(a) Examine if there are strictly dominated strategies. If so, explain how such strategies are dominated.

⇒ C is strictly dominated by, for example, $(A, 0.5; B, 0.5)$. Once C is removed, z is strictly dominated by, for example, $(x, 0.2; y, 0.8)$.

(b) Is the game dominance solvable?

⇒ No.

(c) Check if there are pure strategy Nash equilibria. If so, describe them.

⇒ (A, x) and (B, y) are pure-strategy Nash equilibria.

(d) Find a mixed-strategy Nash equilibrium.

⇒ First, any mixed strategy using strictly dominated actions with some probability is also strictly dominated. Thus, probabilities assigned to C and z must be zero.

Let p be the probability for Player 2 to play x, and $1 - p$ be to play y. P1's expected payoff of playing A, $4p$, must be equal to that of playing B, $2 - 2p$. Thus, $p = 1/3$.

Let q be the probability for Player 1 to play A, and $1 - p$ be to play B. P2's expected payoff of playing x, $2q$, must be equal to that of playing y, $4 - 4q$. Thus, $q = 2/3$.

Therefore, $((A, 2/3; B, 1/3; C, 0), (x, 1/3; y, 2/3; z, 0))$ is a mixed-strategy Nash equilibrium.

2. [14 points] Consider the following game:

P1 \ P2	L	R
U	4, 4	0, 2
M	2, 2	1, 2
D	1, 2	2, 2

Find all pure-strategy and mixed-strategy Nash equilibria.

⇒ (U, L) and (D, R) are pure strategy Nash equilibria. Since M is strictly dominated by, for example, $(U, 0.4; D, 0.6)$, the Nash equilibrium strategy should assign zero probabilities for playing M.

Let p be the probability for Player 2 to play L. Player 1's expected payoff of playing U, $4p$, must be equal to that of playing D, $p + 2 - 2p = 2 - p$. Thus, $p = \frac{2}{5}$.

Let q be the probability for Player 1 to play U, and $1-q$ be to play D. Player 2's expected payoff of playing L, $4q+2-2q = 2+2q$, must be equal to that of playing R, $2q+2-2q = 2$. Thus, $q = 0$.

Therefore, $((U, 0; M, 0; D, 1), (L, \frac{2}{5}; R, \frac{3}{5}))$ is a mixed-strategy Nash equilibrium.

3. [18 points] Two firms compete by choosing quantity produced in a market. The demand function is given by $P(q_1, q_2) = 12 - q_1 - q_2$, where q_1 and q_2 are quantity produced by firm 1 and firm 2. Firm 1 has a cost function $C_1(q_1) = q_1^2$ and Firm 2 has a cost function $C_2(q_2) = \frac{q_2^2}{2}$.

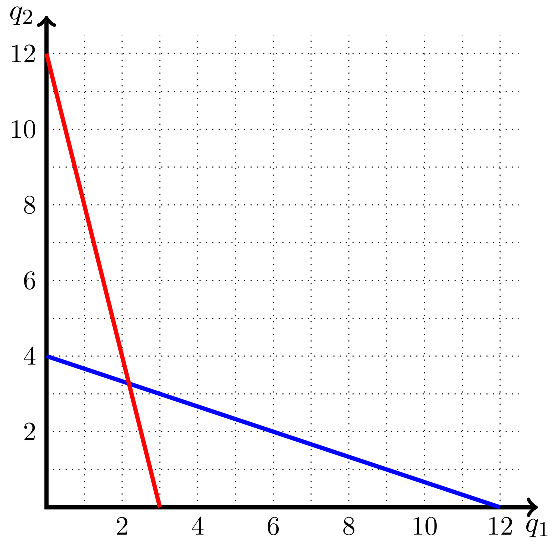
(a) Describe each firm's profit as a function of q_1 and q_2.

⇒ Firm 1 chooses q_1 to maximize $(12 - q_1 - q_2)q_1 - q_1^2$. The first order condition is $12 - q_2 - 4q_1 = 0$, so the best response function of Firm 1 is $q_1 = \frac{12 - q_2}{4} = 3 - \frac{q_2}{4}$.

Firm 2 chooses q_2 to maximize $(12 - q_1 - q_2)q_2 - \frac{q_2^2}{2}$. The first order condition is $12 - q_1 - 3q_2 = 0$, so the best response function of Firm 2 is $q_2 = \frac{12 - q_2}{3} = 4 - \frac{q_1}{3}$.

(b) Draw Best Response functions below.

⇒ See below. The red line is the best response function for Firm 1, and the blue line is for Firm 2.



(c) Find a Nash equilibrium.

⇒ (q_1^*, q_2^*) is a Nash equilibrium if $q_1^* = BR_1(q_2^*)$ and $q_2^* = BR_2(q_1^*)$.

Since $q_2^* = 4 - \frac{1}{3}q_1^* = 4 - \frac{1}{3}(3 - \frac{q_2^*}{4}) = 3 + \frac{q_2^*}{12}$, $\frac{11}{12}q_2^* = 3$, or $q_2^* = \frac{36}{11}$.

Plugging $q_2^* = \frac{36}{11}$ to $q_1^* = 3 - \frac{q_2^*}{4}$, $q_1^* = 3 - \frac{9}{11} = \frac{24}{11}$.

Thus, $(q_1^*, q_2^*) = (\frac{24}{11}, \frac{36}{11})$ is the Nash equilibrium.

4. [18 points] Suppose there are N bystanders who observe an emergency. If no one calls 911, all bystanders get a payoff of 0. If at least one person calls 911, the emergency is soon resolved, and every bystander earns a payoff of 1. However, the bystanders who called 911 must spend some extra cost $c \in (0,1)$, so their payoff is $1-c$.

(a) Suppose $N=2$. Describe the game among bystanders on a payoff matrix form.

$\Rightarrow$

P1 \ P2	Not	Call
Not	$0, 0$	$1, 1-c$
Call	$1-c, 1$	$1-c, 1-c$

(b) Find a symmetric mixed-strategy Nash equilibrium, and in that equilibrium, calculate the probability that no one calls 911.

$\Rightarrow$ Let p be the probability for each player to play Not. The expected payoff of playing Not, $1*(1-p) = 1-p$, must be equal to that of playing Call, $1-c$. Thus, $p=c$. Both players playing Not with probability c is the Nash equilibrium.

The probability that no one calls 911 is c^2.

(c) Now suppose $N=3$. Find a symmetric mixed-strategy Nash equilibrium. (Hint: Don't draw a payoff matrix. By symmetry, all players will call 911 with the same probability.)

$\Rightarrow$ Let p be the probability for each player to play Not. The expected payoff of playing Not, $1*(1-p^2) = 1-p^2$, must be equal to that of playing Call, $1-c$. Thus, $p=\sqrt{c}$. All the three players playing Not with probability $\sqrt{c}$ is the Nash equilibrium.

(d) Compare the probability that no one calls 911 when $N=3$ with your answer in part (b). Is the probability that no one calls 911 increases, decreases, or stays the same when N increases from 2 to 3?

$\Rightarrow$ The probability that no one calls 911 when $N=3$ is $p^3 = c^{3/2}$, which is **greater** than c^2 since $c \in (0,1)$.

5. [16 points] An airline carrier lost bags of passengers X and Y. They do not know each other, but coincidentally, their bags (and the items inside) are identical. The airline manager tries to compensate their losses in the following manner:

- Two passengers simultaneously report the bag's value. For simplicity, assume there are only three options: \$100, \$200, and \$300.
- The claimed value will be paid. Also, if one person claims \$100 **lower than** the other person, then that person will be additionally paid \$150.
- For example, if passenger X claimed \$300, and Y claimed \$100, they will be paid as they claimed. If passenger X claimed \$300, and Y claimed \$200, then X will be paid \$300, and Y will be paid \$350 (=200 + 150).

(a) Describe the game on a payoff matrix form.

⇒

P1 \ P2	100	200	300
100	100,100	250,200	100,300
200	200,250	200,200	350,300
300	300,100	300,350	300,300

(b) Examine if there are pure strategy Nash equilibria. If so, describe them.

⇒ (200, 300) and (300, 200) are pure strategy Nash equilibria.

(c) Find a mixed-strategy Nash equilibrium.

⇒ Fist, note that playing 100 is strictly dominated by 300 for both players. After removing 100 from the set of actions, we have a typical Hawk–Dove game.

P1 \ P2	200	300
200	200,200	350,300
300	300,350	300,300

Both players playing $(100, 0; 200, \frac{1}{3}; 300, \frac{2}{3})$ is the mixed-strategy Nash equilibrium.