ECON4002 Midterm, Suggested Sol (sketch)
Fall 2024

1. [10 points] Consider an economy with two goods, x and y. Utility function is given by $u(x,y) = x^2y^4$. Income is denoted as m, the price vector is (p_x, p_y).

(a) Check whether the utility function is concave, and it is quasiconcave.

⇒ It is convex. x^2 is convex, y^4 is convex, and its multiplication is convex. (Some wrote, "the Hessian matrix is positive semidefinite." How did you show it?)

It is quasiconcave. We want to show that for any (x_1, y_1) and (x_2, y_2), $\lambda x_1^2y_1^4 + (1-\lambda)x_2^2y_2^4 \geq \min\{x_1^2y_1^4, x_2^2y_2^4\}$. Without loss of generality, assume $x_1^2y_1^4 \geq x_2^2y_2^4$. Let $\Delta = x_1^2y_1^4 - x_2^2y_2^4 \geq 0$. $\lambda x_1^2y_1^4 + (1-\lambda)x_2^2y_2^4 = \lambda(x_2^2y_2^4 + \Delta) + (1-\lambda)x_2^2y_2^4 = \lambda\Delta + x_2^2y_2^4 \geq x_2^2y_2^4 = \min\{x_1^2y_1^4, x_2^2y_2^4\}$.

(b) Find the Marshallian demand $(x(p_x, p_y, m), y(p_x, p_y, m))$ and the indirect utility function $v(p_x, p_y, m)$.

⇒ $x = \frac{1}{3}\frac{m}{p_x}$. $y = \frac{2}{3}\frac{m}{p_y}$. $v(p,m) = \frac{2^4}{3^6}\frac{m^6}{p_x^2p_y^4}$.

(c) Verify Roy's identity in this economy.

⇒ $\frac{\partial v(p,m)}{\partial m} = \frac{6}{m}\frac{2^4}{3^6}\frac{m^6}{p_x^2p_y^4}$. $\frac{\partial v(p,m)}{\partial p_x} = -\frac{2}{p_x}\frac{2^4}{3^6}\frac{m^6}{p_x^2p_y^4}$. $-\frac{\partial v(p,m)/\partial p_x}{\partial v(p,m)/\partial m} = \frac{2/p_x}{6/m} = \frac{m}{3p_x}$. (y is analogous.)

2. [20 points] Suppose that a consumer has the following expenditure function: $e(p_1, p_2, u) = \frac{2p_1p_2}{p_1+p_2}u$.

(a) Find an indirect utility function and Marshallian demand function of this consumer.

⇒ $e(p, v(p,m)) = m = \frac{2p_1p_2}{p_1+p_2}v(p,m) \Rightarrow v(p,m) = \frac{p_1+p_2}{2p_1p_2}m = \frac{m}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)$.

By Roy's identity, $x_1(p,m) = -\frac{\partial v(p,m)/\partial p_1}{\partial v(p,m)/\partial m}$ =(After some algebra)= $\frac{p_2m}{p_1(p_1+p_2)}$.

Similarly, $x_2(p,m) = \frac{p_1m}{p_2(p_1+p_2)}$.

(b) Recover a utility function $u(x_1, x_2)$ that rationalizes this consumer's demand behavior.

⇒ Let $m = 2$. $u(x_1, x_2) = \min_p v(p, 2)$ subject to $p_1x_1 + p_2x_2 = 2$. The Lagrangian is $\frac{1}{p_1} + \frac{1}{p_2} + \lambda(p_1x_1 + p_2x_2 - 2)$. The first order conditions with respect to p_1 and p_2 are $p_1^* = \frac{1}{\sqrt{\lambda x_1}}$ and $p_2^* = \frac{1}{\sqrt{\lambda x_2}}$. After some algebra, you get $\frac{p_1^*}{p_2^*} = \frac{\sqrt{x_2}}{\sqrt{x_1}}$, or $p_2^* = \sqrt{\frac{x_1}{x_2}}p_1^*$. Plugging it to $p_1^*x_1 + p_2^*x_2 = 2$, (after some algebra) $p_1^* = \frac{2}{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})}$. Plugging this result into $p_2^* = \sqrt{\frac{x_1}{x_2}}p_1^*$, $p_2^* = \frac{2}{\sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})}$.

Thus, $v(p_1^*, p_2^*, 2) = \frac{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})}{2} + \frac{\sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})}{2} = (\sqrt{x_1} + \sqrt{x_2})^2$.

3. [20 points] A consumer's preferences $\succeq$ on $\mathbb{R}_+^L$ can be represented by the utility function $u : \mathbb{R}_+^L \to \mathbb{R}_+$ with the property that for any $x \in \mathbb{R}_+^L$ and $\alpha > 0$, $u(\alpha x) = \alpha u(x)$.

(a) Show that this consumer has homothetic preference, that is, for any $x, y \in \mathbb{R}_+^L$, $x \succeq y$ if and only if $\alpha x \succeq \alpha y$ for any $\alpha > 0$.

⇒ $x \succ y \Leftrightarrow u(x) \geq u(y) \Leftrightarrow \alpha u(x) \geq \alpha u(y)$ for $\alpha > 0 \Leftrightarrow u(\alpha x) \geq u(\alpha y) \Leftrightarrow \alpha x \succ \alpha y$.

The second to the last ⇔ is by the given property.

(b) Show that this consumer's expenditure function is such that $e(p,u) = ue(p,1)$ for any $u > 0$ and prices p. [Hint: First show $e(p,u) = e(up,1)$, and then use the property of the expenditure function.]

⇒ First, we want to show $e(p,u) = e(up,1)$.

$$\begin{aligned}
e(p,\bar{u}) &= \min p \cdot x \text{ s.t. } u(x) = \bar{u} \\
&= \min p \cdot x \text{ s.t. } \frac{1}{\bar{u}}u(x) = u(\frac{x}{\bar{u}}) = 1 \\
&= \min p \cdot \bar{u}y \text{ s.t. } u(y) = 1 \quad \text{(Relabel } x/\bar{u} \text{ as, say, } y\text{, so that } x = \bar{u}y) \\
&= e(\bar{u}p, 1)
\end{aligned}$$

Since $e(p,u)$ is homogeneous of degree 1 in p, $e(p\bar{u},1) = \bar{u}e(p,1)$.

(c) Is this consumer's indirect utility function linear in wealth? Explain.

⇒ Yes, $m = e(p, v(p,m)) = v(p,m)e(p,1)$, where the last equality is from the property derived in part (b). $v(p,m) = \frac{1}{e(p,1)}m$. Thus, $v(p,m)$ is linear in m.

4. [15 points] Consider three revealed price-commodity pairs given by

$$\begin{aligned}
p^1 &= (1,1,2), \quad x^1 = (1,0,0) \\
p^2 &= (2,1,1), \quad x^2 = (0,1,0) \\
p^3 &= (1,2,1+\varepsilon), \quad x^3 = (0,0,1)
\end{aligned}$$

, where $\varepsilon > 0$ is very small.

(a) Check if it satisfies WARP.

⇒ The observations satisfy WARP if $p^0 \cdot x^1 \leq p^0 \cdot x^0$, then we must have $p^1 \cdot x^0 > p^1 \cdot x^1$. If $1 = p^1x^2 \leq p^1x^1 = 1$, then $2 = p^2x^1 > p^2x^2 = 1$. If $1 = p^2x^3 \leq p^2x^2 = 1$, then $2 = p^3x^2 > p^3x^3 = 1 + e$. If $1 = p^3x^1 \leq p^3x^3 = 1 + e$, then $2 = p^1x^3 > p^1x^1 = 1$. So it satisfies WARP.

(b) Check if it satisfies GARP.

⇒ It does not. x^1 is revealed preferred to x^2, x^2 is revealed preferred to x^3, but x^3 is revealed preferred to x^1, which violates GARP.

(c) Discuss the possibility of recovering preference from those observations.

⇒ We can't recover preferences as the symmetry of the Slutsky Matrix is not guaranteed. In other words, because of the preference cycle, the observed choices cannot be rationalized.

5. [20 points] Suppose there are three consumers in the market. Consumers' utility functions are $u_1(x_1,x_2) = x_1^{1/2}x_2^{1/2}$, $u_2(x_1,x_2) = x_1^{1/3}x_2^{2/3}$, and $u_3(x_1,x_2) = x_1^{2/3}x_2^{1/3}$. Their incomes are 10, 20, and 30, respectively. The price of good 1 is normalized to 1 for simplicity.

(a) Derive demand functions of the three consumers.

⇒ $x_1^1 = \frac{10}{2} = 5$, $x_2^1 = \frac{10}{2p} = \frac{5}{p}$. $x_1^2 = \frac{20}{3}$, $x_2^2 = \frac{40}{3p}$. $x_1^3 = \frac{60}{3} = 20$, $x_2^3 = \frac{30}{3p} = \frac{10}{p}$.

(b) Describe the aggregate demand for goods 1 and 2.

$\Rightarrow$ $X_1 = 5 + \frac{20}{3} + 20 = \frac{95}{3}$. $X_2 = \frac{5}{p} + \frac{40}{3p} + \frac{10}{p} = \frac{15+40+30}{3p} = \frac{85}{3p}$.

(c) Examine whether a representative consumer exists. If so, describe the utility function of the representative consumer with incomes of 60. If not, explain why a representative consumer cannot exist.

$\Rightarrow$ Since everyone's indirect utility function has a Gorman form, a rep. consumer exists. The representative consumer has the utility function of $u_R(x_1, x_2) = x_1^{19/36} x_2^{17/36}$. (Check by yourself $x_1(p, 60) = \frac{19}{36}60 = \frac{19}{3}5 = \frac{95}{3}$ and $x_2(p, 60) = \frac{17}{36}\frac{60}{p} = \frac{85}{3p}$.

6. [15 points] Consider an economy with L goods. Dennis in this economy has a strictly increasing and strictly concave utility unction, $u : \mathbb{R}_+^L \to \mathbb{R}$.

(a) Show that Dennis' indirect utility function $v(p, m)$ must be strictly concave with respect to income m.

$\Rightarrow$ We want to show for $\lambda \in (0, 1)$ and two different m and m', $\lambda v(p, m) + (1 - \lambda)v(p, m') < v(p, \lambda m + (1 - \lambda)m')$. Let x and x' denote the maximizing arguments of the utility maximization problem with income m and m', respectively.

$$\begin{aligned}
\lambda v(p, m) + (1 - \lambda)v(p, m') &= \lambda u(x) + (1 - \lambda)u(x') \\
&< u(\lambda x + (1 - \lambda)x') \quad \because \text{Jensen's inequality} \\
&\leq v(p, \lambda m + (1 - \lambda)m'),
\end{aligned}$$

where the last inequality is from the fact that $\lambda x + (1 - \lambda)x'$ is affordable with income $\lambda m + (1 - \lambda)m'$. (That is, $\lambda x + (1 - \lambda)x'$ is one of the consumption bundles inside of the budget set, while $v(p, \lambda m + (1 - \lambda)m')$ takes the utility-maximizing best bundle.)

(b) Dennis has two career choices, doctor and scientist. Doctor's income is drawn from distribution D and scientist's income is from distribution S. Two distributions have the same mean. If D second-order stochastically dominates S, recommend one career to Dennis, and explain why you recommend so, by using the fact in (a).

$\Rightarrow$ Since $v(p, m)$ is strictly concave in m, and S is an MPS of D, $E_S(v(p, m)) < E_D(v(p, m))$. Thus, recommend D.