Q1(a) For (x_1,y_1) & (x_2,y_2), $\lambda x_1^2y_1^4 + (1-\lambda)x_2^2y_2^4 \geq \min\{x_1^2y_1^4, x_2^2y_2^4\}$

WLOG, $x_1^2y_1^4 > x_2^2y_2^4$ and $x_1^2y_1^4 - x_2^2y_2^4 = \Delta > 0$.

$\lambda x_1^2y_1^4 + (1-\lambda)x_2^2y_2^4 = \lambda(x_2^2y_2^4+\Delta) + (1-\lambda)x_2^2y_2^4 = x_2^2y_2^4 + \lambda\Delta > x_2^2y_2^4 = \min\{x_1^2y_1^4, x_2^2y_2^4\}$

(b) Note that $(u(x,y))^{\frac{1}{6}} = x^{\frac13}y^{\frac23}$ ⇒ $x = \frac{1}{3}\frac{w}{p_x}$, $y = \frac{2}{3}\frac{w}{p_y}$

⇒ $u(x,y) = \left(\frac{w}{3p_x}\right)^2\left(\frac{2w}{3p_y}\right)^4$.

(c) $\frac{\partial V}{\partial w} = \frac{6}{w}u$, $\frac{\partial V}{\partial p_x} = -\frac{2}{p_x}u$, ⇒ $x(p,w) = -\frac{\partial V/\partial p_x}{\partial V/\partial w} = \frac{2w}{6p_x} = \frac{w}{3p_x}$ ∎

Q2.(a) Duality ⇒ $e(p_1,p_2,V(p_1,p_2,m)) = \frac{2p_1p_2}{p_1+p_2}V(p_1,p_2,m) = m$ ∴ $V(p_1,p_2,m) = \frac{m}{2}\frac{p_1+p_2}{p_1p_2} = \frac{m}{2}\left(\frac{1}{p_1}+\frac{1}{p_2}\right)$

Roy's identity ⇒ $\frac{\partial V}{\partial m} = \frac12\left(\frac1{p_1}+\frac1{p_2}\right)$, $\frac{\partial V}{\partial p_1} = -\frac{m}{2p_1^2}$ ⇒ $x_1 = -\frac{\partial V/\partial p_1}{\partial V/\partial m} = \frac{p_2 m}{p_1(p_1+p_2)}$.

(b) set $m=2$. $u(x) = \min_{p_1,p_2} V(p_1,p_2)$ s.t. $p_1x_1+p_2x_2=2$

$\mathcal{L} = \frac1{p_1}+\frac1{p_2}+\lambda(2-p_1x_1-p_2x_2)$.

$\frac{\partial\mathcal L}{\partial p_1} = -\frac{1}{p_1^2}-\lambda x_1 = 0$ ⇒ $\lambda = -\frac{1}{p_1^2x_1}$ ①

$\frac{\partial\mathcal L}{\partial p_2} = -\frac{1}{p_2^2}-\lambda x_2 = 0$ ⇒ $\lambda = -\frac{1}{p_2^2x_2}$ ②

①=② ⇒ $p_2 = p_1\sqrt{x_1/x_2}$ ④ $\quad \frac{\partial\mathcal L}{\partial\lambda} = 2-p_1x_1-p_2x_2=0$. ③

plug ④ to ③ ⇒ $p_1x_1+p_1\sqrt{\frac{x_1}{x_2}}\cdot x_2 = 2$ ⇒ $p_1 = \frac{2}{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})}$ $\quad p_2 = \frac{2\sqrt{x_1}}{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})\sqrt{x_2}} = \frac{2}{\sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})}$

∴ $u(x_1,x_2) = \frac{\sqrt{x_1}(\sqrt{x_1}+\sqrt{x_2})}{2} + \frac{\sqrt{x_2}(\sqrt{x_1}+\sqrt{x_2})}{2} = (\sqrt{x_1}+\sqrt{x_2})^2$

Q3. (a) $x \succsim y \Leftrightarrow u(x)\geq u(y) \overset{\forall\alpha>0}{\Leftrightarrow} \alpha u(x)\geq\alpha u(y) \Leftrightarrow u(\alpha x)\geq u(\alpha y) \Leftrightarrow \alpha x\succsim\alpha y$

(b) $e(p,u) = \min p\cdot x$ s.t. $u(x)=\bar u \Leftrightarrow \min p\cdot x$ s.t. $u(\frac1{\bar u}x)=1$

(let $x' = \frac{1}{\bar u}x$) $\Leftrightarrow \min \bar u\, p\cdot x'$ s.t. $u(x')=1 \Leftrightarrow u\cdot e(p,1)$

(c) Duality ⇒ $e(p,v(p,m)) = v(p,m)e(p,1) = m$ ⇒ $v(p,m) = m\cdot\frac{1}{e(p,1)}$ ∴ linear in m.

Q4. (a) (WARP) If $p^i\cdot x^j \le p^i\cdot x^i$ then $p^j\cdot x^i > p^j\cdot x^j$.

$p^1\cdot x^1=1$, $p^1\cdot x^2=1$, $p^1\cdot x^3=2$, $p^2\cdot x^1=2$, $p^2\cdot x^2=1$, $p^2\cdot x^3=1$, $p^3\cdot x^1=1$, $p^3\cdot x^2=2$, $p^3\cdot x^3=1+\varepsilon$

- $p^1\cdot x^2 \le p^1\cdot x^1$ and $p^2\cdot x^1=2>p^2\cdot x^2=1$ (✓)
- $p^2\cdot x^3\le p^2\cdot x^2$ and $p^3\cdot x^2=2>p^3\cdot x^3=1+\varepsilon$ (✓)
- $p^3\cdot x^1\le p^3\cdot x^3$ and $p^1\cdot x^3=2>p^1\cdot x^1=1$. (✓)

⇐ WARP is satisfied.

(b) (GARP) If x^1 is revealed preferred to x^2 & x^2 is revealed preferred to x^3, then $p^3\cdot x^1\geq p^3\cdot x^3$.

- $p^3\cdot x^1 = 1 \not\geq p^3\cdot x^3 = 1+\varepsilon$. Thus, GARP is not satisfied.

(c) Since GARP is violated, symmetry of the Slutsky matrix is not guaranteed. Thus, can't recover the preference.

Q5. (a) Consider m and m', $\boxed{\text{WTS}}$ $V(p,\lambda m+(1-\lambda)m') > \lambda V(p,m)+(1-\lambda)V(p,m')$.

$x(m) = \arg\max u(x)$ s.t. $p\cdot x = m$. $\quad V(p,m) = u(x(m))$ and $V(p,m') = u(x(m'))$

Note that $\lambda x(m)+(1-\lambda)x(m')$ is feasible with the budget $\lambda m+(1-\lambda)m'$. (*)

⇒ $\lambda u(x(m))+(1-\lambda)u(x(m')) < u(\lambda x(m)+(1-\lambda)x(m')) \underset{(*)}{\le} V(p,\lambda m+(1-\lambda)m')$

(b) S is a MPS of D. Due to strict concavity of $V(p,m)$ in m, $\int V(p,m)dS < \int V(p,m)dD$ ∎

Q6. (a) For $\lambda>1$, $Q(\lambda L,\lambda K) = ((\lambda L)^\alpha+(\lambda K)^\alpha)^{\frac1\beta} = \lambda^{\frac\alpha\beta}(L^\alpha+K^\alpha)^{\frac1\beta} = \lambda^{\frac\alpha\beta}Q(L,K)$

⇒ Q is IRTS/CRTS/DRTS if $\alpha>\beta$ / $\alpha=\beta$ / $\alpha<\beta$.

(b) $\min 2L+K$ s.t. $(L^\alpha+K^\alpha)^{1/\beta}=Q$. $\quad \mathcal L = 2L+K+\lambda(Q-(L^\alpha+K^\alpha)^{\frac1\beta})$ $\quad \frac{\partial\mathcal L}{\partial L} = 2-\lambda\frac1\beta\frac{Q}{L^\alpha+K^\alpha}\alpha L^{\alpha-1}=0$ ①

①/② $\Leftrightarrow 2 = \left(\frac{L}{K}\right)^{\alpha-1} = \left(\frac{K}{L}\right)^{1-\alpha}$ ⇒ $K = 2^{\frac{1}{1-\alpha}}L$ ③ $\quad \frac{\partial\mathcal L}{\partial K} = 1-\lambda\frac1\beta\frac{Q}{L^\alpha+K^\alpha}\alpha K^{\alpha-1}=0$ ②

plug ③ into $(L^\alpha+K^\alpha)^{\frac1\beta}=Q$, $(1+2^{\frac{\alpha}{1-\alpha}})L^\alpha = Q^\beta$ ⇒ $L = (1+2^{\frac{\alpha}{1-\alpha}})^{-\frac1\alpha}Q^{\frac\beta\alpha}$; $K = 2^{\frac{1}{1-\alpha}}(1+2^{\frac{\alpha}{1-\alpha}})^{-\frac1\alpha}Q^{\frac\beta\alpha}$.

⇒ $2L+K = 2(1+2^{\frac{\alpha}{1-\alpha}})^{-\frac1\alpha}Q^{\frac\beta\alpha} + 2^{\frac{1}{1-\alpha}}(1+2^{\frac{\alpha}{1-\alpha}})^{-\frac1\alpha}Q^{\frac\beta\alpha} = C\cdot Q^{\frac\beta\alpha}$, where C is the chunk.

(c) cost function is convex/constant/concave is $\beta>\alpha$ / $\beta=\alpha$ / $\beta<\alpha$. (d) IRTS ⇔ concave cost fn. ∴ economies of scale