STK-MAT2011
Interest Rate Models In Discrete Time

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Abstract

In this article we look at ways to implement a discrete time binomial model as a way to model the stochastic evolution of interest rates. We start in chapter 1 with a brief discussion of interest rates in relation to bond prices, and show how the notion of a riskless investment in a bank account enables us to relate amounts of currency available at different times to each other. We then move on to describing a basic discrete time binomial model and how we can use the concept of a risk neutral probability measure to ensure that the model does not allow any arbitrage opportunities. Thus the model does not allow any investors to make a profit without taking any risk using either the interest rates themselves, or any derivatives which prices depend on them. We end the chapter by looking at a specific model, the Ho and Lee model. We also presents the numerical results of an implementation of this model. More details are given in appendix A along with an example of the pricing of a derivative.

In chapter 2 we start out by defining interest rates for some future time period, the forward interest rates. We then proceed to describe some interest rate derivatives. The chapter is concluded by a discussion of the Heath, Jarrow and Morton model (HJM) which utilizes forward rates. It is shown that the Ho and Lee model can be seen as a specific case of the HJM model.

Contents

Chapter 1

Interest Rate Modeling

1.1 Introduction

Since interest rates are not traded directly we introduce interest rates in connection with assets which depend on interst rates, the so called fixed income assets. The simplest such asset on the bond market are the zero-coupon bonds (also known as T-bonds). A zero-coupon bond is a bond that pays a specified amount called face value at a specified time (maturity). At times prior to maturity the value of that asset is less than its face value, provided that the interest rate is always greater than zero. The prices that these bonds are traded at in the marketplace express the expectations of the market on the future value of money. The yield of the zero-coupon bond corresponding to a given maturity is defined as the constant interest rate that would be needed so that the time zero price of the bond invested at time zero and allowed to accumulate at this interest rate would grow to the face value of the bond at the bonds maturity. Thus given prices of zero-coupon bonds at different maturities (or terms) one can construct a yield curve also known as a term structure of interest rates.

1.2 Bonds and Interest Rates

In order to be able to relate amounts of currency at different times one must have an understanding of how interest rates might develop over time. The references for chapters 1.2 and 1.3 are [1],[8],[9].

Consider a time period $[t, T]$ where $t < T$. Define $P(t, T)$ to be the value at time t of a zero-coupon bond expiring at time T with face value 1 in a given currency and let $P(T, T) = 1$. The relationship between zero-coupon bond prices and interest rates are dependent on which type of interest rate we are considering.

Definition 1.2.1. *(Annually-compounded spot interest rate) Let $Y(t, T)$ be the annually compounded spot interest rate (also referred to as the short rate) prevailing at time t for maturity T. By making an investment of $P(t, T)$ units of currency at time t, and reinvesting the obtained amounts once a year, the investment will have grown to one unit of currency at maturity T. This gives us the following formulas:*

$$Y(t,T) := \frac{1}{(P(t,T))^{\frac{1}{T-t}}} - 1 \tag{1.2.1}$$

$$P(t,T)(1+Y(t,T))^{T-t} = 1 \tag{1.2.2}$$

Thus the bond prices can be expressed as

$$P(t,T) = \frac{1}{(1+Y(t,T))^{T-t}} \tag{1.2.3}$$

Definition 1.2.2. *(Continuously-compunded spot interest rate)*

a) Let $R(t,T)$ be the constant rate at which an investment of $P(t,T)$ units of currency at time accrues continuously to yield a unit amount of currency at maturity T.

$$R(t,T) := -\frac{lnP(t,T)}{T-t} \tag{1.2.4}$$

Since $e^{R(t,T)}P(t,T) = P(T,T) = 1$,

$$P(t,T) = e^{-R(t,T)(T-t)}. \tag{1.2.5}$$

b) Let $(r_t)_{t\in\mathbb{R}_+}$ be a time dependent function

$$P(t,T) = e^{-\int_t^T r_s ds} \tag{1.2.6}$$

Definition 1.2.3. *(Singly-compounded spot interest rate) Let $L(t,T)$ be the simply compounded spot interest rate at time t for maturity T. This is the constant rate at which an investment of $P(t,T)$ units of currency at time t accrues, proportionally to the investment time, to one unit of currency at maturity.*

$$L(t,T) := \frac{1-P(t,T)}{(T-t)P(t,T)} \tag{1.2.7}$$

$$P(t,T)(1+L(t,T)(T-t)) = 1 \tag{1.2.8}$$

$$P(t,T) = \frac{1}{1+L(t,T)(T-t)} \tag{1.2.9}$$

1.3 The Bank Account

A bank account or money-market account represents a riskless investment where profit gained over time is determined by a defined interest rate process. Let $B(t)$ be the value of a bank account at time $t \geq 0$. We assume that $B(0) = 1$. Given a discrete time interval $[0,T]$. Let t be an integer $0 \leq t \leq T$. Then $r(t)$ is the compounded fixed interest rate applied to the bank account for the time interval $[t, t+1]$. Define an interest rate process as a sequency of random variables $r(0), r(1), \cdots, r(T-1)$ where $r(0)$ is not random. Since we know the interest rate that is applied to the bank account in each period, the money in the account will increase to

$$B(t) = (1+r(0))(1+r(1))\cdots(1+r(t-1)) \tag{1.3.1}$$

during that time interval. It follows that

$$B(t+1) = B(t)(1+r(t)). \tag{1.3.2}$$

This allows us to relate amounts of currency available at different times by utilizing a discount process defined by

$$D(t,T) = \frac{B(t)}{B(T)} \tag{1.3.3}$$

Thus the discounted value at time 0 of 1 currency at time T is given as

$$D(0,T) = \frac{B(0)}{B(T)} = \frac{1}{(1+r(0))\cdots(1+r(T-1))}. \tag{1.3.4}$$

If the interest rate is the continuously compounded spot rate the bank account and discount factors are given by either

a) If the short rate is a constant $r(t)$.

$$B(t) = e^{\sum_{k=0}^{t-1} r(k)} \tag{1.3.5}$$

$$D(t,T) = e^{-\sum_{k=t}^{T-1} r(k)} \tag{1.3.6}$$

b) If the interest rate is a function of time

$$B(t) = B(0)e^{\int_0^t r_s ds} \tag{1.3.7}$$

$$D(t,T) = \frac{B(t)}{B(T)} = e^{-\int_t^T r_s ds} \tag{1.3.8}$$

1.4 The Binomial model

The basic references are [5],[7] and [9]. We will consider a multiperiod model where the following are specified as data

- $T+1$ trading dates: $t = 0, 1, \cdots, T$
- A finite sample space Ω with $K < \infty$ elements, $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$
- A probability measure $\mathbb{P}$ on Ω with $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$
- A filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$ which is a submodel describing how the information about bond prices are revealed to any investors.
- A bank account process $B = \{B(t); t = 0, 1, \cdots, T\}$, where B is a stochastic process with $B(0) = 1$ and $B(t)(\omega) > 0$ for all t and ω.
- A stochastic zero coupon bond price process $P = \{P(t,T); t = 0, 1, \cdots, T\}$ where $P(t,T)$ is the price of a zero coupon bond with maturity T at time t.

At each period in the model there are two possibilities. The price of a bond either goes up or down. Let ω be the event of a bonds price moving up or down over a period of time. We can think of ω as a sequence of coin flips $\omega = \omega_1\omega_2 \cdots \omega_N$ representing up and down movements over N trading periods. Let Ω be the set of 2^N possible outcomes of N tosses of a coin where each $\omega \in \Omega$ represents a sequence of possible coin flips.

Definition 1.4.1. *A finite probability space consists of a sample space Ω and a probability measure $\mathbb{P}$. The sample space Ω is a nonempty finite set and the probability measure $\mathbb{P}$ is a function that assigns to each $\omega \in \Omega$ a number in $[0, 1]$ so that $\sum_{\omega\in\Omega} \mathbb{P}(\omega) = 1$*

Given a finite probability space $(\Omega, \mathbb{P})$, a random variable X is defined to be a real-valued function defined on Ω. The expected value of X is defined to be

$$E[X] = \sum_{\omega\in\Omega} X(\omega)\mathbb{P}(\omega) \tag{1.4.1}$$

Information about how the prices of bonds change in the model can be shown as subsets of the sample space Ω. A collection $\mathcal{F}$ of subsets of Ω is called an algebra on Ω if

a) $\Omega \in \mathcal{F}$

b) $F \in \mathcal{F} \Rightarrow F^c = \Omega \setminus F \in \mathcal{F}$

c) F and G $\in \mathcal{F} \Rightarrow$ F$\cup$ G$\in \mathcal{F}$

Given an algebra on Ω, denoted $\mathcal{F}_t$, one can always find a partition of Ω such that there is a unique collection $\{F_n\}$ of subsets F_n, such that each $F_n \in \mathcal{F}_t$. Since there is a one to one correspondence between partitions of Ω and algebras on Ω, the model of the information structure can be organized as a sequence $\{\mathcal{F}_t\}$ of algebras. The filtration $\mathbb{F}$ is defined as

$$\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\} \tag{1.4.2}$$

If the function $\omega \to X(\omega)$ is constant on any subsets in the partition corresponding to $\mathcal{F}_t$ then the random variable X is measurable with respect to the algebra $\mathcal{F}$. A stochastic process P is adapted to the filtration $\mathbb{F}$ if the random variable $P(t, T)$ is measurable with respect to $\mathcal{F}_t$ for every $t = 0, 1, \cdots, T$.

Definition 1.4.2. *(Arbitrage opportunity) An arbitrage opportunity is an asset (or portfolio of assets) whose value today is zero and whose value in all possible states at the future time is never negative, but in some state at the future time the asset has a strictly positive value.*

$$V(0) = 0$$

$$V(T, \omega) \geq 0 \; for \; all \; states \; \omega$$

$$V(T, \omega) > 0 \; for \; some \; state \; \omega$$

The absence of arbitrage opportunities ensure that there are no risk free ways to make a profit.

Assume we have a discrete time model given on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$, where the interest rate r and P($\cdot$,T) are adapted processes. Let $\widetilde{P}(t, T) = \frac{P(t,T)}{B(t)}$ be the discounted value of a zero-coupon bond.

Definition 1.4.3. *A martingale measure is a probability measure Q equivalent to $\mathbb{P}$, with respect to which the discounted price process of the zero-coupon bonds are martingales, that is it must hold that*

$$\widetilde{P}(t,T) = E^Q[\widetilde{P}(t+1,T)|\mathcal{F}_t] \tag{1.4.3}$$

A probability measure Q on Ω is said to be a risk neutral probability measure if

a) $Q(\omega) > 0$ for all $\omega \in \Omega$

b) The discounted price process $\widetilde{P}(t,T)$ is a martingale under Q for every $t = 0, \cdots, T$.

Proposition 1. *There are no arbitrage opportunities if and only if there exist a risk neutral probability measure Q.*

Definition (1.4.3) is equivalent to

Lemma 1.4.4.

$$P(t,T) = E^Q[D(t,T)|\mathcal{F}_t] \tag{1.4.4}$$

where $D(t,T)$ is the discount factor.

Proof. From Definition (1.4.3) it follows that

$$\widetilde{P}(t,T) = E^Q[\widetilde{P}(T,T)|\mathcal{F}_t], \;\; t \leq T$$

since $p(T,T) = 1$ we can write

$$\frac{P(t,T)}{B(t)} = E^Q\left[\frac{1}{B(T)}|\mathcal{F}_t\right], \;\; t \leq T.$$

as the interest rate r is an adapted process we get

$$P(t,T) = E^Q\left[\frac{B(t)}{B(T)}|\mathcal{F}_t\right], \;\; t \leq T.$$

from which (1.4.4) follows.

□

The relations between the prices of the zero-coupon bonds and the martingale measure can be characterized by

Proposition 2. *Let Q be a measure equivalent to $\mathbb{P}$. Then Q is a martingale measure if and only if*

$$\frac{P(t,T)}{P(t,t+1)} = E^Q[P(t+1,T)|\mathcal{F}_t] \tag{1.4.5}$$

Proof. Since $r(t)$ is $\mathcal{F}_t$ measurable (1.4.4) is equivalent to

$$\begin{aligned} P(t,T) &= e^{-r(t)}E^Q[D(t+1,T)|\mathcal{F}_t] \\ &= P(t,t+1)E^Q[E^Q[D(t+1,T)|\mathcal{F}_{t+1}]|\mathcal{F}_t] \\ &= P(t,t+1)E^Q[P(t+1,T)|\mathcal{F}_t] \end{aligned}$$

□

A contingent claim $X(t)$ is a random variable representing a payoff at some future time t.

Corollary 1.4.5. *Under a martingale measure Q, if a process $X = (X_t)$ where $t \leq T$ verifies*

$$X_t = E^Q[D(t,T)X_T|\mathcal{F}_t],\ t \leq T \tag{1.4.6}$$

then $\widetilde{X} = (\frac{X_t}{B(t)})$ is a Q-martingale. In particular it holds that

$$X_k = E^Q[D(k,t)X_t|\mathcal{F}_k],\ k \leq t \leq T \tag{1.4.7}$$

Proof. Since the interest rate process r is adapted, for each $k \leq n$

$$\begin{aligned} E^Q[\widetilde{X}_n|\mathcal{F}_k] &= \frac{1}{B(k)}E^Q\left[E^Q\left[\frac{D(n,N)B(k)}{B(n)}X_N|\mathcal{F}_n\right]|\mathcal{F}_k\right] \\ &= \frac{1}{B(k)}E^Q[D(k,N)X_N|\mathcal{F}_k] = \widetilde{X}_k. \end{aligned}$$

(1.4.7) then follows from the martingale property and the fact that r is adapted. □

Thus any interest rate derivatives that we are pricing using a binomial model can be defined as Q-martingales.

The bonds we have considered so far have only paid an amount at maturity. We can model a coupon paying bond as a sequence of constant quantities $C_0, C_1, \cdots, C_t$, where $C_0, C_1, \cdots, C_{t-1}$ are the coupon payment made at each time and C_t is the final payment at time t which includes both the pricipal amount as well as any coupon due at that time. A coupon paying bond can be regarded as a sum of C_1 zero-coupon bonds maturing at time 1, C_2 zero-coupon bonds maturing at time 2, up to C_t zero-coupon bonds maturing at time t. Thus the price at time 0 of a coupon paying bond may be written as

$$\sum_{k=0}^{t} C_k P(0,k) = E^Q\left[\sum_{k=0}^{t} D(0,k)C_k\right] \tag{1.4.8}$$

At a time $s < t$ after the payments $C_0, C_1, \cdots, C_{s-1}$ has been made, the price of the coupon paying bond is

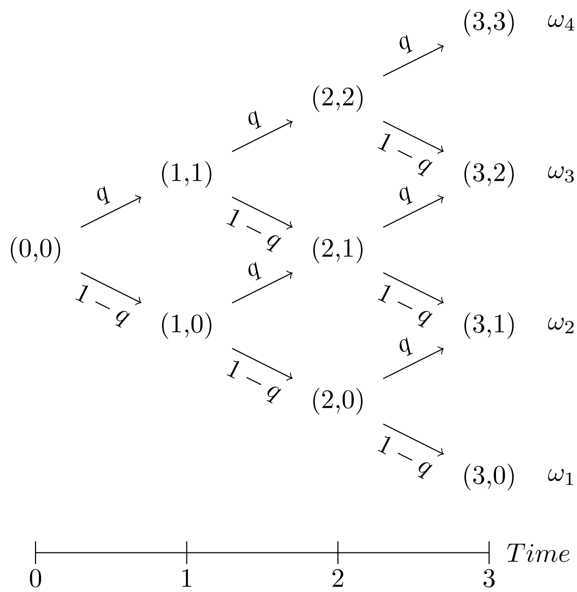
$$\sum_{k=s}^{t} C_k P(s,k) = E^Q\left[\sum_{k=s}^{t} D(s,k)C_k|\mathcal{F}_s\right] \tag{1.4.9}$$

1.5 Interest Rate Models

1.5.1 The Ho and Lee model

The Ho and Lee model first appeared in 1986 and is the first term structure model which allows the matching of the initial term structure. This enables the model to provide theoretical zero-coupon bond prices that match the market prices at the initial date. The model assumes there are no transaction costs or taxes and that all assets are perfectly divisible. It is also assumed that trading takes place at discrete time steps and that for every maturity time T there exists a bond with the respective maturity. The model takes the term structure as given and describes the feasible subsequent term structure movements using a binomial lattice. Provided that the movements described cannot permit arbitrage opportunities we can then use the model to price contingent claims. The references used are [3],[4],[6].

1.5.2 The arbitrage free model



Let each node in a binomial tree have a label (n, j). The label n refers to time $n = 0, 1, 2, \cdots$, and j numbers the states $j = 0, 1, 2, \cdots, n$. Thus there are $n + 1$ states at time n. Define $P_j^n(x)$ to be the price at time n in state j of a zero coupon bond maturing at time $x+n$ (x periods after n). It is also required that $P_j^n(0) = 1$ for all j, n.

If we at time 0 are in node $(0, 0)$, then at time $n = 1$ we are in either node $(1, 1)$ or $(1, 0)$. Thus we either move up or down from the starting node. This means that the price of a zero coupon bond is either $P_1^1(x)$ or $P_0^1(x)$ depending on whether we moved up or down from the previous state. If we in the first period went up then we will after the second period from time 1 to time 2, have the price $P_2^2(x)$ or $P_1^2(x)$, alternatively $P_1^2(x)$ or $P_0^2(x)$ if we went down during the first time period. The recombining tree model requires that the price attained by an upstate followed by a downstate to be equal to the price reached by a downstate followed by an upstate. That is the price in a given node are independent on the path taken to reach it.

At any n'th period and j'th state we have a zero-coupon bond price $P_j^n(x)$. If there is no interest rate risk over the next period, the bond price in the upstate must equal the price in the down state. The forward price of a zero-coupon bond when in state (n, j), can be written as

$$F_j^n(x) = P_j^{n+1}(x) = P_{j+1}^{n+1}(x) = \frac{P_j^n(x+1)}{P_j^n(1)} \tag{1.5.1}$$

If the price in the next period differs from the forward price then there will exist arbitrage opportunities. In order to model uncertainty in the term structure we introduce perturbation functions $h(x)$ and $h^*(x)$ as a way to show how the next period prices deviates from the forward price, that is they specify the difference between the upstate and downstate prices over the next time period.

$$P_{j+1}^{n+1}(x) = \frac{P_j^n(x+1)}{P_j^n(1)} h(x) \tag{1.5.2}$$

$$P_j^{n+1}(x) = \frac{P_j^n(x+1)}{P_j^n(1)} h^*(x) \tag{1.5.3}$$

Since $P_j^n(0) = 1$ if follows that

$$h(0) = h^*(0) = 1 \tag{1.5.4}$$

Let the risk neutral probability q be the probability of going up in the next time period. It can be shown that

$$qh(x) + (1-q)h^*(x) = 1 \;\; for\, n, j > 0 \tag{1.5.5}$$

This means that in the Ho and Lee model the probabilities of going up or down in the next time step is a constant independent of time and state.

In constructing the binomial lattice we are imposing a constraint on the perturbation functions $h(x)$ and $h^*(x)$ and the probability q such that at any time n and state j, an upward move followed by a downward move of a bond price equals a downward move followed by an upward move of its price. By using equations (1.5.2) and (1.5.3) we can show that from an upward move followed by a downward move we get

$$P_{j+1}^{n+2}(x) = \frac{P_j^n(x+2)h(x+1)h^*(x)}{P_j^n(2)h(1)} \tag{1.5.6}$$

and similarly for a downward move followed by an upward move we get

$$P_{j+1}^{n+2}(x) = \frac{P_j^n(x+2)h^*(x+1)h(x)}{P_j^n(2)h^*(1)} \tag{1.5.7}$$

Then the path independence condition implies that

$$h(x+1)h^*(x)h^*(1) = h^*(x+1)h(x)h(1) \tag{1.5.8}$$

We can eliminate h* using (1.5.5) and get

$$h(x+1)[1-qh(x)][1-qh(1)] = (1-q)h(1)h(x)[1-qh(x+1)]. \tag{1.5.9}$$

By simplifying this equation we get for $T \geq 1$,

$$\frac{1}{h(x+1)} = \frac{\delta}{h(x)} + \gamma \tag{1.5.10}$$

where δ is some constant such that

$$h(1) = \frac{1}{q + (1-q)\delta} \tag{1.5.11}$$

and

$$\gamma = \frac{q(h(1) - 1)}{(1-q)h(1)} \tag{1.5.12}$$

Equation (1.5.10) can be solved as a first order linear difference equation using the initial condition $h(0) = 1$ to get the solution

$$h(x) = \frac{1}{q + (1-q)\delta^x} \quad for \;\; x \geq 0. \tag{1.5.13}$$

From (1.5.5) we get

$$h^*(x) = \frac{\delta^x}{q + (1-q)\delta^x}. \tag{1.5.14}$$

The parameter δ determines the spread between the functions h and h^*. Thus a smaller value for δ represents a greater interest variability. Using backwards recursion the price of a zero coupon bond can be described at any time n and state j in terms of the initial bond prices. Let the the original term structure prices $P(0,n)$ be written as $P(n)$

$$P_j^n(x) = \frac{P(n+x)}{P(n)} \frac{h(x)h(x+1)\cdots h(x+n-1)}{h(0)h(1)\cdots h(n-1)} \delta^{(n-j)x} \tag{1.5.15}$$

This gives in the case of a one period bond (x=1) the price as

$$P_j^n(1) = \frac{P(n+1)}{P(n)} h(n)\delta^{n-j} \tag{1.5.16}$$

The model is then fully described by the initial term structure prices given as $P(0,n)$ and by the parameters $0 < q < 1$ and $0 < \delta < 1$.

The interest rates at each node in the model can then be calculated from the formulas

a) If the interest rate is the annually compounded spot rate $r(n,j)$.

$$r(n,j) = \frac{1}{P_j^n(1)} - 1 \tag{1.5.17}$$

b) If the interest rate is the continuously compounded spot rate $r_j^n(1)$

$$r_j^n(1) = -ln\, P_j^n(1) \tag{1.5.18}$$

The Ho and Lee model will sometimes have problems with negative interest rates, so that q and δ must be carefully chosen in order to prevent this from happening. In order to ensure that $r(n,j) \geq 0$ for all (n,j), then $P_j^n(1) \leq 1$. This only holds if $P(n+1)h(n) \leq P(n)$ for all n, which can be achieved by selecting $\delta \geq \delta_1$ where

$$\delta_1 = \max_{1 \leq n \leq T-1} \left[\frac{\frac{P(n+1)}{P(n)} - q}{1-q} \right]^{\frac{1}{n}} \tag{1.5.19}$$

Since $P(n+1) < P(n)$ for all n, $\delta_1 < 1$ we can choose $\delta_1 \leq \delta < 1$.

1.5.3 Pricing Contingent Claims

There are two ways we can use the model to price contingent claims. Let $V(n,j)$ be the value of a claim in a given state and its assumed that we are given $V(N,j)$ for $j = 0, 1, 2, \cdots, N$.

a) The value of the claim is given by

$$V(n,j) = P_j^n(1)[qV(n+1,j+1) + (1-q)V(n+1,j)] \tag{1.5.20}$$

b) Let $\lambda(n,j)$ be the value at $t = 0$ of an Arrow-Debreu security. That is a security that pays 1 at $t = n$ in state j and 0 in any other state. We can use the following forward induction formula to compute $\lambda(n,j)$ for all the nodes (n,j). Let the state prices $\lambda(n,j)$ be given by

$$\lambda(n,j) = \frac{P(n)q^j(1-q)^{n-j}\psi(j,n-j)}{\prod_{k=1}^{n}[q + (1-q)\delta^{n-k}]} \tag{1.5.21}$$

where $\psi(k,m)$ is calculated from

$$\psi(k,m) = \delta^m\psi(k-1,m) + \delta^{m-1}\psi(k,m-1) \tag{1.5.22}$$

$$\psi(0,m) = \delta^{\frac{m(m-1)}{2}} \tag{1.5.23}$$

$$\psi(k,0) = 1 \tag{1.5.24}$$

for $k \geq 0$, $m \geq 1$ in (1.5.20) and (1.5.21).

Then the price of the contingent claim is given by

$$V(0,0) = \sum_{j=0}^{N} \lambda(N,j)V(N,j). \tag{1.5.25}$$

1.5.4 An example of a Ho and Lee model

An example of a Ho and Lee model is given in Appendix A. The term structure is given as $P = [0.9826, 0.9651, 0.9474, 0.9296, 0.9119]$ with $q = 0.5$ and $\delta = 0.997$.

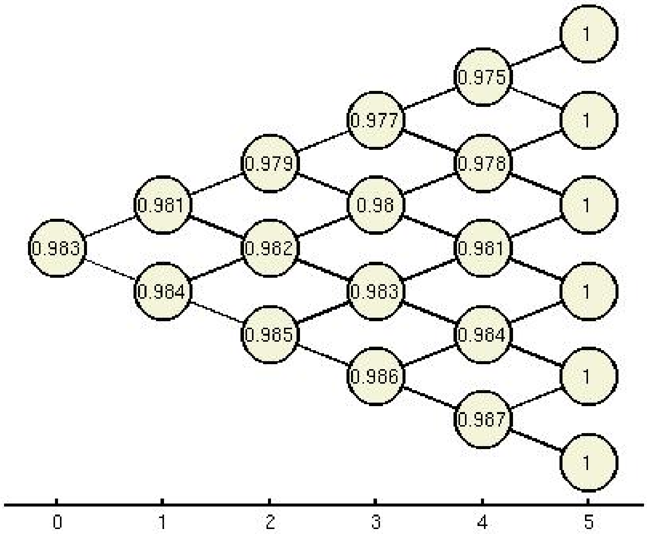


Figure 1.1: Ho and Lee zero coupon bond prices where the value in each node is $P_j^n(1)$.

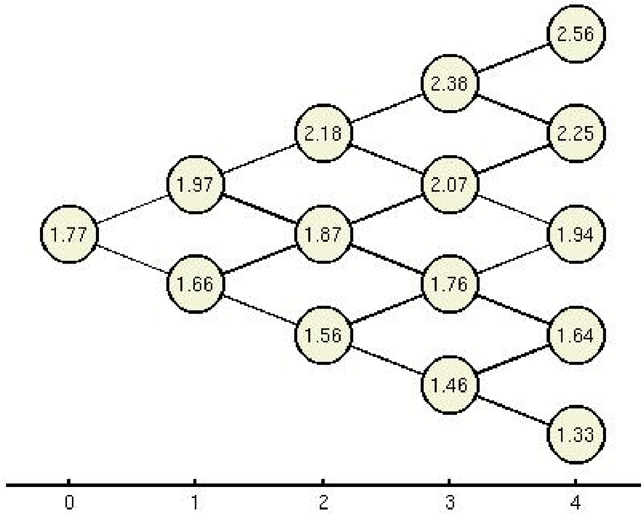


Figure 1.2: Ho and Lee annually compunded interest rates computed from (1.5.17).

Chapter 2

Forward Interest Rate Modeling

2.1 Forward Interest Rates

The references used are [1],[8]. Let $t \leq T \leq S$. There will often be a need to agree at present time t for a loan for some future time period $[T, S]$. The interest rate applied to this loan is called a forward rate and is denoted by $f(t, T, S)$.

Definition 2.1.1. *The continuously compounded forward rate f(t,T,S) at time t for a loan on $[T, S]$ is given by*

$$f(t, T, S) = -\frac{ln\, P(T, S) - ln\, P(t, T)}{S - T} \tag{2.1.1}$$

where the continuously compounded spot forward rate F(t,T) is given by

$$F(t, T) := f(t, t, T) = -\frac{ln\, P(t, T)}{T - t} \tag{2.1.2}$$

which is identical to the continuously compounded spot rate R(t,T).

By taking the limit of the forward rate $f(t, T, S)$ as $S \to T^+$ we get the instantaneous forward rate $f(t, T)$

$$f(t, T) := -\lim_{S \to T^+} \frac{ln\, P(t, S) - ln\, P(t, T)}{S - T} = -\frac{\partial ln\, P(t, T)}{\partial T} = -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T}. \tag{2.1.3}$$

Solving this as a differential equation using the initial condition $P(T, T) = 1$ gives

$$ln\, P(t, T) = ln\, P(t, T) - ln\, P(t, t) = \int_t^T \frac{\partial ln\, P(t, s)}{\partial s} ds = -\int_t^T f(t, s)\, ds \tag{2.1.4}$$

$$P(t, T) = exp\Big(-\int_t^T f(t, s)\, ds \Big) \quad 0 \leq t \leq T \tag{2.1.5}$$

The forward rate $f(t,T,S)$ can be recovered from the instantaneous forward rate

$$f(t,T,S) = \frac{1}{S-T}\int_T^S f(t,s)\ ds\ \ 0 \leq t \leq T < S \tag{2.1.6}$$

When the short rate $(r_t)_{t\in\mathbb{R}^+}$ is a deterministic function we have

$$P(t,T) = exp\Big(-\int_t^T f(t,s)ds\Big) = exp\Big(-\int_t^T r_s ds\Big) \tag{2.1.7}$$

The relationship of the forward rate and the short rate can be seen as follows

$$P(t,T) = E\Big[exp\Big(-\int_t^T r_s ds\Big)|r_t\Big] \tag{2.1.8}$$

so we have

$$\begin{aligned}\frac{\partial P}{\partial T}(t,T) &= \frac{\partial}{\partial T}E\Big[exp\Big(-\int_t^T r_s ds\Big)|r_t\Big]\\ &= E\Big[\frac{\partial}{\partial T}exp\Big(-\int_t^T r_s ds\Big)|r_t\Big]\\ &= E\Big[r_T\, exp\Big(-\int_t^T r_s ds\Big)|r_t\Big].\end{aligned}$$

it follows that the limit

$$\lim_{T\to t^+}\frac{\partial P}{\partial T}(t,T) = -E[r_t|r_t] = -r_t \tag{2.1.9}$$

the limit as $T \to t^+$ of instantanous forward rate equals the short rate r_t

$$\lim_{T\to t^+} f(t,T) = -\lim_{T\to t^+}\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} = r_t \tag{2.1.10}$$

2.2 Fixed-Income Derivatives

The reference used for chapters 2.2-2.4 is [9]. A derivative security is a security whose payoff depends on one or more underlying securities or variables. A well known example of a derivative security is a call option on a stock, which gives the buyer of the call the right to buy the stock on a specific date at a specified price. We consider a binomial model over a time period $[0,\tau]$, where the dates n, m are specified as $0 \leq n \leq m \leq \tau$. Let S_n be the price of an asset at time n. If the price S_n depends on the first n coin tosses, then the discounted asset price will be a martingale under the risk neutral measure Q.

$$D(0,n)S_n = E^Q[D(0,n+1)S_{n+1}|\mathcal{F}_n],\ \ n = 0,1,\cdots,\tau-1 \tag{2.2.1}$$

Definition 2.2.1. *A forward contract is an agrement to pay a delivery price K at a delivery date m for the asset whose price at time m is S_m. The m-forward price of this asset at time n is the value of K that makes the forward contract have no arbitrage price zero at time n.*

If we consider an asset whose price process is $S_0, S_1, \cdots, S_\tau$ in the binomial model, then the m-forward price at time n is

$$For(n,m) = \frac{S_n}{P(n,m)} \tag{2.2.2}$$

For the time period $0 \le n \le m \le \tau - 1$, the annually compounded forward interest rate can be defined as

$$F_r(n,m) = \frac{P(n,m)}{P(n,m+1)} - 1 = \frac{P(n,m) - P(n,m+1)}{P(n,m+1)} \tag{2.2.3}$$

which is the forward interest rate at time n for investing at time m.

2.2.1 Swaps

An interest rate swap is an agreement between two parties who each make a contract to pay the other on some specified dates in the future. One agent receives constant payments at each time, while the other receives a variable amount whose size depend on the future rates of interest. Thus an agent can use swaps to convert a fixed interest rate loan into a variable interest rate loan or vice versa.

Let $1 \le m \le \tau$. An m-period interest rate swap is a contract that pays $S_1, \cdots, S_m$ at times $1, \cdots, m$ where

$$S_n = K - r(n-1), \;\; n = 1, \cdots, m \tag{2.2.4}$$

where $r(n-1)$ is the annually compounding rate. The fixed rate K is constant. The value of K that makes the time zero no arbitrage price of the interest rate swap equal to zero is denoted SR_m. The time zero no arbitrage price of the m-period interest rate swap is

$$Swap_m = \sum_{n=1}^{m} P(0,n)(K - F_r(0,n-1) = K\sum_{n=1}^{m} P(0,n) - (1 - P(0,m)) \tag{2.2.5}$$

$$SR_m = \frac{\sum_{n=1}^{m} P(0,n)F_r(0,n-1)}{\sum_{n=1}^{m} P(0,n)} = \frac{1 - P(0,m)}{\sum_{n=1}^{m} P(0,n)} \tag{2.2.6}$$

In the binomial model it can be shown that the risk neutral price of the swap is

$$Swap_m = E^Q\left[\sum_{n=1}^{m} D(0,n)(K - r(n-1))\right] \tag{2.2.7}$$

2.2.2 Caps and Floors

Caps and Floors are usually bought to insure against rises and falls in the money market interest rates. Let $1 \le m \le \tau$. An m-period interest rate cap is a contract that pays $C_1, \cdots, C_m$ at times $1, \cdots, m$ where

$$C_n = (r(n-1) - K)^+ \tag{2.2.8}$$

An interest rate floor is a contract that makes payments $F_1, \cdots, F_m$ at times $1, \cdots, m$ where

$$F_n = (K - r(n-1))^+ \tag{2.2.9}$$

If the contract only makes a single payment at time n it is called a caplet or floorlet respectively. Thus a cap or floor contract will pay the difference if the interest rates raises above or falls below a certain level. The risk neutral price of an m-period cap is

$$Cap_m = E^Q\left[\sum_{n=1}^{m} D(0,n)(r(n-1)-K)^+\right] \tag{2.2.10}$$

and the risk neutral price of an m-period interest rate floor is

$$Floor_m = E^Q\left[\sum_{n=1}^{m} D(0,n)(K-r(n-1))^+\right] \tag{2.2.11}$$

2.3 Forward Measures

Denote the payoff at time m of some contract as V_m. The risk neutral price of this security at time n is

$$V_n = \frac{1}{D(0,n)} E^Q[D_m V_m | \mathcal{F}_n] \tag{2.3.1}$$

It is difficult to compute the conditional expectation when D_m is random as one would have to know the joint conditional distribution of D_m and V_m under the risk neutral measure. By using D_m as a Radon-Nikodym derivative one can change the probability measure from a risk neutral measure to a forward measure. Since the term D_m no longer appears in the formula we can then compute expectations more easily using the new measure.

Definition 2.3.1. *In a τ period binomial model let P be the actual probability measure and Q the risk neutral probability measure. Assume that $P(\omega)$>0 and $Q(\omega)$>0 for all ω. Let $Z(\omega) = \frac{Q(\omega)}{P(\omega)}$ (state price density) for every ω. The Radon-Nikodym derivative process is*

$$Z_n = E[Z|\mathcal{F}_n], \;\; n = 0, 1, \cdots, \tau \tag{2.3.2}$$

Note that Z_τ=Z and Z_0=1.

Definition 2.3.2. *Let $1 \leq m \leq \tau$. Define*

$$Z_{m,m} = \frac{D(0,m)}{P(0,m)} \tag{2.3.3}$$

the m-forward measure P^m is defined by

$$P^m(\omega) = Z_{m,m}(\omega)Q(\omega) \;\; for\, all\, \omega \in \Omega$$

We may then define the Radon-Nikodym derivative process as

$$Z_{n,m} = E^Q[Z_{m,m}|\mathcal{F}_n] \;\; n = 0, 1, \cdots, m \tag{2.3.4}$$

If V_m is a random variable only depending on the first m coin tosses then

$$E^m[V_m] = E^Q[Z_{m,m}V_m] \tag{2.3.5}$$

Also if $0 \leq n \leq m$

$$E^m[V_m|\mathcal{F}_n] = \frac{1}{Z_{n,m}} E^Q[Z_{m,m}V_m|\mathcal{F}_n] \tag{2.3.6}$$

this can be rewritten as

$$Z_{n,m} = \frac{D(0,n)P(n,m)}{P(0,m)} \tag{2.3.7}$$

Theorem 2.3.3. *Let $1 \leq m \leq \tau$, and let P^m denote the m-forward measure. If V_m is a random variable depending only on the first m coin tosses then*

$$E^m[V_m] = \frac{1}{P(0,m)} E^Q[D_m V_m] \tag{2.3.8}$$

in general

$$E^m[V_m|\mathcal{F}_n] = \frac{1}{D(0,n)P(n,m)} E^Q[D(0,m)V_m|\mathcal{F}_n] \tag{2.3.9}$$

2.4 Futures

A shortcoming of forward contracts is that on any date prior to the delivery date there can be a demand for forward contracts with that delivery date. For the market to be efficient there would have to be a supply of forward contracts with different initiation dates for each delivery date. A futures contract is a contract that locks in the price of an asset before the time of purchase or sale. Thus the futures price is tied to a delivery date, but not an initiation date.

Definition 2.4.1. *Assume we are considering an asset with price process $S_0,S_1,\cdots,S_\tau$ in the binomial model. For $0 \leq m \leq \tau$, the m-futures price process $Fut_{n,m}$, where $n = 0,1,\cdots,m$ is an adapted process with the following properties*

a) $Fut_{m,m} = S_m$

b) *For each n, $0 \leq n \leq m-1$ the risk neutral value at time n of the contract that receives the payments $Fut_{k+1,m} - Fut_{k,m}$ at time $k+1$ for all $k = n,\cdots,m-1$ is zero*

$$\frac{1}{D(0,n)} E^Q\left[\sum_{k=n}^{m-1} D(0,k+1)(Fut_{k+1,m} - Fut_{k,m})|\mathcal{F}_n\right] = 0 \tag{2.4.1}$$

The unique process that satisifies the conditions in definition (2.4.1) is

$$Fut_{n,m} = E^Q[S_m|\mathcal{F}_n],\ n = 0,1,\cdots,m \tag{2.4.2}$$

2.5 The Heath, Jarrow, Morton Term Structure Model

2.5.1 General Notation

The reference used is [2]. Let $[0,\tau]$ be the length of a discrete trading economy, where $\tau > 0$ is fixed. The interval between each trade is given by $\Delta > 0$, where N intervals of size Δ compose a unit in time. So far we have looked at models where $\Delta = 1$. Given an arbitrary trading time $t \in [0, \tau]$, we write $\bar{t} = tN = t/\Delta$= the number of trading periods of length Δ prior to and including time t. We assume there is a default free zero-coupon bond trading for each trading date $T \in [0, \tau]$. The T-maturity bond pays a certain amount at date T. Let $P(t, T)$ denote the time t price of the T-maturity bond for all $T \in [0, \tau]$ and $t \in [0, T]$. Let $P(T, T) = 1$ for all $T \in [0, \tau]$, and that $P(t, T) > 0$ for all $T \in [0, \tau]$ and $t \in [0, T]$.

Given bond prices, the forward rate structure is determined (and conversely). The forward rate at time t for the time interval $[T, T + \Delta]$, $f(t, T)$ is defined by

$$\begin{aligned} f(t,T) = &- [log\,(P(t,T+\Delta)/P(t,T))]/\Delta \\ &for\,all\,T \in [\Delta, \cdots, \tau] \; and \; t \in [0, \Delta, \cdots, T - \Delta]. \end{aligned} \tag{2.5.1}$$

$$\begin{aligned} P(t,T) = exp\left(- \sum_{j=\bar{t}}^{\bar{T}-1} f(t, j\Delta)\Delta \right) \\ for\,all\,T \in [\Delta, \cdots, \Delta\bar{\tau}] \; and \; t \in [0, \Delta, \cdots, \Delta(\bar{T} - 1)]. \end{aligned} \tag{2.5.2}$$

P(T,T)=1. Date ranges from $0, \cdots, \Delta(\bar{T} - 1)$ and the forward rates are $f(t,T) \cdots \; f(t, \Delta(\bar{T} - 1))$. The spot rate at time t (over $[t, t+\Delta]$, $r(t)$ is defined to be the forward rate at time t

$$r(t) = f(t,t) \tag{2.5.3}$$

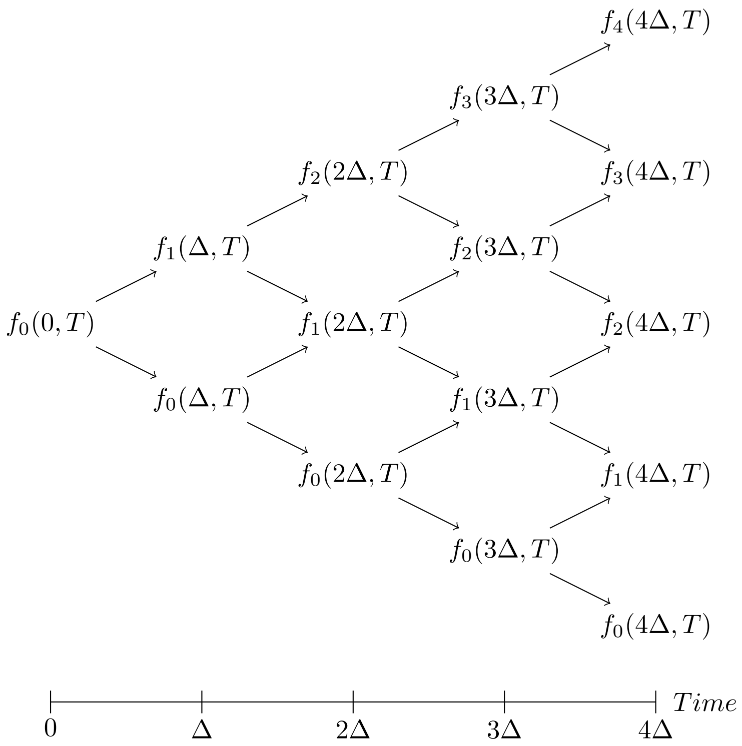
Let the bank account be defined by

$$B(0) = 1 \tag{2.5.4}$$

$$B(t) = exp\left(\sum_{j=0}^{\bar{t}-1} r(j\Delta)\Delta \right) \; for \; all \; t \in [\Delta, \cdots, \Delta(\bar{\tau} - 1)] \tag{2.5.5}$$

Notation: Note that in this section the notation used is the same as the one in the reference paper. The risk neutral probabilities are denoted by π and the variable q denotes the real world probability of a shock happening. The discounted value of a bond is given as $Z(t, T)$ and the expectation under the risk neutral measure is given as E^*. Also note that *log* refers to the natural logarithm.

2.5.2 Term Structure Movements



In the figure above the variance is assumed to be constant so that the forward rate process can be written as a binomial lattice. Let $f_i(j\Delta, T)$ be the forward rate in state i at trading date $j\Delta$.

Condition 1. *The forward rate movements are represented by the following family of forward rate processes. Let $T \in [\Delta, \cdots, \Delta(\bar{\tau}-1)]$ be fixed but arbitrary, then the forward rate $f(t, T)$ satisfy the following stochastic process:*

$$\begin{aligned} f(t,T) &= f(0,T) + \sum_{j=1}^{\bar{t}} a_j[u_1(j\Delta,T) - v_1(j\Delta,T)] \\ &\quad + \sum_{j=1}^{\bar{t}} v_1(j\Delta,T) + \sum_{j=1}^{\bar{t}} b_j[u_2(j\Delta,T) - v_2(j\Delta,T)] + \sum_{j=1}^{\bar{t}} v_2(j\Delta,T) \end{aligned} \tag{2.5.6}$$

for all $t \in [\Delta, \cdots, T]$, where $\{f(0,t) : T \in [0, \cdots, \Delta(\bar{\tau} - 1)]\}$ is a fixed initial forward rate curve; u_1, u_2, v_1, v_2 are random functions and a_j, b_j for $j \in \{1, \cdots, \bar{T} - 1\}$ are random variables taking on the values $\{0, 1\}$ with joint prob-

abilities, summing to one, given by:

$$\begin{aligned} &q_{00}(j)\ if\, a_j = 0,\ b_j = 0 \\ &q_{01}(j)\ if\, a_j = 0,\ b_j = 1 \\ &q_{10}(j)\ if\, a_j = 1,\ b_j = 0 \\ &q_{11}(j)\ if\, a_j = 1,\ b_j = 1 \end{aligned} \tag{2.5.7}$$

This forward rate process has two random shocks represented by the random variables $\{a_j, b_j\}$. Let $u_1(t,T), u_2(t,T)$ be the magnitude of the "upward" movement of a jump at time t for $a_{\bar{t}}$ and $b_{\bar{t}}$ respectively. Likewise the magnitude of a "downward" movement of a jump at time t is denoted $v_1(t,T)$,$v_2(t,T)$ for $a_{\bar{t}}$ and $b_{\bar{t}}$.

The spot rate process is determined as

$$\begin{aligned} r(t) = f(0,t) &+ \sum_{j=1}^{\bar{t}} a_j[u_1(j\Delta,t) - v_1(j\Delta,t)] + \sum_{j=1}^{\bar{t}} v_1(j\Delta,t) \\ &+ \sum_{j=1}^{\bar{t}} b_j[u_2(j\Delta,t) - v_2(j\Delta,t)] + \sum_{j=1}^{\bar{t}} v_2(j\Delta,t) \end{aligned} \tag{2.5.8}$$

Let $Z(t,T) = P(t,T)/B(t)$ for $T \in [0,\tau]$ and $t \in [0,T]$ be the relative price for a T-bond. This can be written as

$$Z(t,T) = \left\{ -\sum_{j=\bar{t}}^{\bar{T}-1} f(t,j\Delta)\Delta - \sum_{j=0}^{\bar{t}-1} f(j\Delta,j\Delta)\Delta \right\} \tag{2.5.9}$$

by substituting the forward rates into (2.5.9),

$$\begin{aligned} Z(t,T) = exp\Bigg\{ &-\sum_{j=0}^{\bar{T}-1} f(0,j\Delta)\Delta \\ &-\sum_{j=\bar{t}}^{\bar{T}-1}\sum_{i=1}^{\bar{t}}[a_i(u_1(i\Delta,j\Delta) - v_1(i\Delta,j\Delta) + v_1(i\Delta,j\Delta)]\Delta \\ &-\sum_{j=\bar{t}}^{\bar{T}-1}\sum_{i=1}^{\bar{t}}[b_i(u_2(i\Delta,j\Delta) - v_2(i\Delta,j\Delta) + v_2(i\Delta,j\Delta)]\Delta \\ &-\sum_{j=1}^{\bar{t}-1}\sum_{i=1}^{j}[a_i(u_1(i\Delta,j\Delta) - v_1(i\Delta,j\Delta) + v_1(i\Delta,j\Delta)]\Delta \\ &-\sum_{j=1}^{\bar{t}-1}\sum_{i=1}^{j}[b_i(u_2(i\Delta,j\Delta) - v_2(i\Delta,j\Delta) + v_2(i\Delta,j\Delta)]\Delta \Bigg\} \end{aligned} \tag{2.5.10}$$

By changing the order of summation and combining terms we get

$$
\begin{aligned}
Z(t,T) &= exp\Bigg\{ -\sum_{j=0}^{\bar{T}-1} f(0,j\Delta)\Delta \\
&\quad -\sum_{i=1}^{\bar{t}}\sum_{j=1}^{\bar{T}-1}[a_i(u_1(i\Delta,j\Delta)+v_1(i\Delta,j\Delta)+v_1(i\Delta,j\Delta)]\Delta \qquad (2.5.11)\\
&\quad -\sum_{i=1}^{\bar{t}}\sum_{j=1}^{\bar{T}-1}[b_i(u_2(i\Delta,j\Delta)+v_2(i\Delta,j\Delta)+v_2(i\Delta,j\Delta)]\Delta\Bigg\}
\end{aligned}
$$

for $T \in [\Delta, \cdots, \Delta\bar{\tau}]$, and $t \in [\Delta, \cdots, \Delta(\bar{T}-1)]$. This implies the following difference equation for $Z(t,T)$,

$$
\begin{aligned}
Z(t,T) &= Z(t-\Delta,T)\ exp\Bigg\{ -a_{\bar{t}}\left[\sum_{j=\bar{t}}^{\bar{T}-1}(u_1(t,j\Delta)-v_1(t,j\Delta))\Delta\right] \\
&\quad -\sum_{j=\bar{t}}^{\bar{T}-1} v_1(t,j\Delta)\Delta - b_{\bar{t}}\left[\sum_{j=\bar{t}}^{\bar{T}-1}(u_2(t,j\Delta)-v_2(t,j\Delta))\Delta\right] \qquad (2.5.12)\\
&\quad -\sum_{j=\bar{t}}^{\bar{T}-1} v_2(t,j\Delta)\Delta\Bigg\},
\end{aligned}
$$

for $T \in [\Delta, \cdots, \Delta\bar{\tau}]$, and $t \in [\Delta, \cdots, \Delta(\bar{T}-1)]$.

2.5.3 Arbitrage Free Pricing and Term Structure Movements

In order for there to be no arbitrage opportunities in the economy there must be restrictions on the jump magnitudes in the forward rate process.

Proposition 3. *Given a family of forward rate jump magnitudes,* $\{u_1(\cdot,T), v_1(\cdot,T), u_2(\cdot,T), v_2(\cdot,T) : T \in [\Delta, \cdots, \Delta(\bar{\tau}-1)]\}$, *satisfying Condition 1, then the following expressions are equivalent:*

(a) The forward rate process given by Condition 1 is an arbitrage free process.

(b) There exists probabilities, summing to one and denoted by $\{\pi_{00}(j), \pi_{01}(j), \pi_{10}(j), \pi_{11}(j)\}$, *with respect to* $\{a_j, b_j\}$ *for each* $j \in \{1, \cdots, \bar{\tau}-1\}$, *such that* $Z(t,T)$ *is a martingale with respect to these probabilities for all* $T \in [\Delta, \cdots, \Delta\bar{\tau}]$ *and* $t \in [0, \cdots, T-\Delta]$.

(c) There exist probabilities, summing to one and denoted by $\{\pi_{00}(j), \pi_{01}(j), \pi_{10}(j), \pi_{11}(j)\}$, *with respect to* $\{a_j, b_j\}$ *for each* $j \in \{1, \cdots, \bar{\tau}-1\}$, *such*

that

$$\begin{aligned}
&\pi_{00}(\bar{t})\ exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}[v_1(t,j\Delta)+v_2(t,j\Delta)]\Delta\right\}+\\
&\pi_{01}(\bar{t})\ exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}[v_1(t,j\Delta)+u_2(t,j\Delta)]\Delta\right\}+\\
&\pi_{10}(\bar{t})\ exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}[u_1(t,j\Delta)+v_2(t,j\Delta)]\Delta\right\}+\\
&\pi_{11}(\bar{t})\ exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}[u_1(t,j\Delta)+u_2(t,j\Delta)]\Delta\right\} \ = \ 1\,,
\end{aligned} \tag{2.5.13}$$

for all $\bar{T}\in[2,\cdots,\bar{\tau}]$ *and* $\bar{t}\in[1,\cdots,\bar{T}-1]$

Let $E_t^*(\cdot)$ be the expectation with respect to the probabilities $\{\pi_{00}(j),\pi_{01}(j),\pi_{10}(j),\pi_{11}(j)\}$, then

$$Z(t,T)=E_t^*(Z(T,T)) \tag{2.5.14}$$

$$P(t,T)=E_t^*(1/B(T))B(t) \tag{2.5.15}$$

2.5.4 One Random Shock Processes

By omitting the subscripts the forward rate process under a single random shock be written as

$$f(t,T) \ = \ f(0,T)+\sum_{j=1}^{\bar{t}}a_j[u(j\Delta,T)-v(j\Delta,T)]+\sum_{j=1}^{\bar{t}}v(j\Delta,T). \tag{2.5.16}$$

for $T\in[\Delta,\cdots,\Delta(\bar{\tau}-1)]$ and $t\in[\Delta,\cdots,T]$, where $q(j)$ is the probability that a_j equals 1. In order for there to be an absence of arbitrage opportunities the necessary conditions on the jump magnitudes means there must exist probabilities $\pi(\bar{t})$ for $\bar{t}\in[0,\cdots,\bar{\tau}-1]$ such that

$$(1-\pi(\bar{t}))exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}v(t,j\Delta)\Delta\right\}+\pi(\bar{t}))exp\left\{-\sum_{j=\bar{t}}^{\bar{T}-1}u(t,j\Delta)\Delta\right\} \ = \ 1. \tag{2.5.17}$$

for $\bar{T}\in[1,\cdots,\bar{\tau}-1]$ and $\bar{t}\in[0,\cdots,\bar{T}-1]$.

Note that this is equal to the no arbitrage condition in the Ho and Lee model in equation (1.5.5) where

$$h^*(x)=exp\left(-\sum_{j=1}^{\bar{T}-1}u(\Delta,j\Delta)\Delta\right)$$

and

$$h(x)=exp\left(-\sum_{j=1}^{\bar{T}-1}v(\Delta,j\Delta)\Delta\right)$$

and this models π equals Ho and Lee's $(1-q)$.

By imposing the following variance restriction we can verify this assertion

$$Var_{t-\Delta}(f(t,T)-f(t-\Delta,T)=\sigma^2(t,T)\Delta \tag{2.5.18}$$

This is equivalent to

$$q(\bar{t})(1-q(\bar{t}))[u(t,T)-v(t,T)]^2=\sigma^2(t,T)\Delta \tag{2.5.19}$$

or

$$u(t,T)-v(t,T)=\sigma(t,T)[\Delta/q(\bar{t})(1-q(\bar{t}))]^{\frac{1}{2}} \tag{2.5.20}$$

Which yields

$$\begin{aligned}&-\sum_{j=\bar{t}}^{\bar{T}-1}u(t,j\Delta)\Delta=\\&-\sum_{j=\bar{t}}^{\bar{T}-1}[v(t,j\Delta)\Delta+\sigma(t,j\Delta)\Delta^{\frac{3}{2}}/(q(\bar{t})(1-q(\bar{t})))^{\frac{1}{2}}]\end{aligned} \tag{2.5.21}$$

By substituting this expression into (2.5.17) we get

$$\begin{aligned}&\sum_{j=\bar{t}}^{\bar{T}-1}v(t,j\Delta)\Delta=\\&log\Bigg(1+\pi(\bar{t})\Bigg(exp\Bigg\{-\sum_{j=\bar{t}}^{\bar{T}-1}\sigma(t,j\Delta)\Delta^{\frac{3}{2}}/(q(\bar{t})(1-q(\bar{t})))^{\frac{1}{2}}\Bigg\}-1\Bigg)\Bigg).\end{aligned} \tag{2.5.22}$$

which can be solved for

$$\begin{aligned}&v(j\Delta,\bar{T}\Delta)=\\&\Bigg[log\Bigg(1+\pi(j)\Bigg(exp\Bigg\{-\sum_{i=j}^{\bar{T}}\sigma(j\Delta,i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\Bigg\}-1\Bigg)\Bigg)\\&-log\Bigg(1+\pi(j)\Bigg(exp\Bigg\{-\sum_{i=j}^{\bar{T}-1}\sigma(j\Delta,i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\Bigg\}-1\Bigg)\Bigg)\Bigg]\Bigg/\Delta\end{aligned} \tag{2.5.23}$$

Summing from $j=1,\cdots,\bar{t}$ gives

$$\begin{aligned}&\sum_{j=1}^{\bar{t}}v(j\Delta,\bar{T}\Delta)=\\&\sum_{j=1}^{\bar{t}}\Bigg[log\Bigg(1+\pi(j)\Bigg(exp\Bigg\{-\sum_{i=j}^{\bar{T}}\sigma(j\Delta,i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\Bigg\}-1\Bigg)\Bigg)\\&-log\Bigg(1+\pi(j)\Bigg(exp\Bigg\{-\sum_{i=j}^{\bar{T}-1}\sigma(j\Delta,i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\Bigg\}-1\Bigg)\Bigg)\Bigg]\Bigg/\Delta\end{aligned} \tag{2.5.24}$$

This gives the forward rate process as

$$
\begin{aligned}
&f(t,T) = \\
&f(0,T) + \sum_{j=1}^{\bar{t}} a_j \sigma(j\Delta, T)(\Delta/[q(j)(1-q(j))]^{1/2} \\
&+ \sum_{j=1}^{\bar{t}} \Bigg[log\Bigg(1 + \pi(j)\Bigg(exp\Bigg\{ - \sum_{i=j}^{\bar{T}} \sigma(j\Delta, i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\Bigg\} - 1\Bigg)\Bigg) \\
&- log\Bigg(1 + \pi(j)\Bigg(exp\Bigg\{ - \sum_{i=j}^{\bar{T}-1} \sigma(j\Delta, i\Delta)(q(j)(1-q(j)))^{-1/2}\Delta^{3/2}\Bigg\} - 1\Bigg)\Bigg)\Bigg] \Bigg/ \Delta
\end{aligned}
\tag{2.5.25}
$$

for all $T \in [\Delta, \cdots, \Delta(\bar{\tau} - 1)]$ and $t \in [\Delta, \cdots, \Delta\bar{T}]$.

2.5.5 Ho and Lee Model

Let $\sigma(t,T) \equiv \sigma > 0$ where σ is a constant. Also let $q(\bar{t}) \equiv q > 0$ and $\pi(\bar{t}) \equiv \pi > 0$. Then (2.5.24) can be simplified to

$$
\begin{aligned}
\sum_{j=1}^{\bar{t}} v(j\Delta, T\Delta) = &\Bigg[log\Bigg(1 + \pi\Bigg(e^{-\bar{T}\psi\Delta^{3/2}} - 1\Bigg)\Bigg) \\
&- log\Bigg(1 + \pi\Bigg(e^{-(\bar{T}-\bar{t})\psi\Delta^{3/2}} - 1\Bigg)\Bigg)\Bigg] \Bigg/ \Delta.
\end{aligned}
\tag{2.5.26}
$$

where $\psi = \sigma(q(1-q))^{-1/2}$. The forward rate process is then

$$
\begin{aligned}
f(t,T) &= f(0,T) + \sum_{j=1}^{\bar{t}} a_j \psi\sqrt{\Delta} \\
&+ \Bigg[log\Bigg(1 + \pi\Bigg(e^{-\bar{T}\psi\Delta^{3/2}} - 1\Bigg)\Bigg) - log\Bigg(1 + \pi\Bigg(e^{-(\bar{T}-\bar{t})\psi\Delta^{3/2}} - 1\Bigg)\Bigg)\Bigg] \Bigg/ \Delta.
\end{aligned}
\tag{2.5.27}
$$

for all $T \in [\Delta, \cdots, \Delta(\bar{\tau} - 1)]$ and $t \in [\Delta, \cdots, \Delta\bar{T}]$. The solution depends on π and ψ where σ^2 can be estimated from historical data. The probabilities π and q can be specified to obtain the best fit.

References

[1] Damiano Brigo, Fabio Mercurio, Interest Rate Models - Theory and Practice, Springer, 2006.

[2] D. Heath, R. Jarrow, A. Morton, Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation, Journal of Financial and Quantitive Analysis 25(1990), no 4, 419-440.

[3] Thomas S.Y. Ho, Sang B. Lee, Terms Structure Movements and Pricing Interest Rate Contingent Claims, Journal of Finance, 1986.

[4] John van der Hoek, Robert J. Elliott, Binomial Models in Finance, Springer, 2006.

[5] A. Pascucci, W.J. Runggaldier, Financial Mathematics. Theory and Problems for Multi-period Models, Springer-Verlag Italia, 2012.

[6] Hal W. Pedersen, Elias S.W. Shiu, A.E. Thorlacius, Arbitrage-Free Pricing of Interest-Rate Contingent Claims, Transactions of Society of Actuaries, vol. 41, 1989.

[7] Stanley R. Pliska, Introduction to Mathematical Finance, Blackwell Publishing Limited, 1997.

[8] Nicolas Privault, An elementary introduction to stochastic interest rate modeling, World Scientific Publishing Company, 2008.

[9] Steven E. Shreve, Stochastic Calculus for Finance I: The binomial asset pricing model, Springer, 2004.

Appendix A

Matlab code for Ho and Lee model

```
P=[0.9826 0.9651 0.9474 0.9296 0.9119];
q=0.5;
delta=0.997;
n=length(P)+1;
A=zeros(n,n);

for a=1:n
    A(a,n)=1;
end


for k=1:(n-1)
    counter1=n-k;
    h=1/(q+(1-q)*(delta^(counter1 -1)));
    if (counter1==1)
        h=1;
    end
    tempcount=counter1;
    for j=1:(counter1)
        counter2=tempcount;
        if (counter1 -1==0)
            p1=P(counter1);
            p2=1;
        else
            p1=P(counter1);
            p2=P(counter1 -1);
        end

        price=(p1/p2)*h*(delta^(counter1-counter2));
        A(counter2,counter1)=price;
        tempcount=counter2 -1;
    end
end

A

% B contains the annually compunded interest rate for each node
B=zeros(n-1,n-1);

for i=1:(n-1)
```

```
    for j=1:(n-1)
        if (A(i,j)==0)
            B(i,j)=0;
        else
            B(i,j)=(1/A(i,j))-1;
        end
    end
end

B

% Rounds off the interest rates in B to show better in lattice
% format
D=zeros(n-1,n-1);
for i=1:(n-1)
    for j=1:(n-1)
        D(i,j)=round(B(i,j)*10000)/100;
    end
end
D
%print_lattice(D);

% Calculates the price of a cap with K=1.9%
K=0.019;
C=zeros(n-1,n-1);
for i=1:(n-1)
    for j=1:(n-1)
        cap=B(i,j)-K;
        if (cap<=0)
            C(i,j)=0;
        else
            C(i,j)=cap;
        end
    end
end

C

% Calculates the price of a cap at time 0 by backwards recursion
for i=1:(n-1)
    C(i,(n-1))=C(i,(n-1))/(1+B(i,(n-1)));
end


for i=1:(n-2)
    c1=n-1-i;
    tmpc1=c1;
    for j=1:(c1)
        c2=tmpc1;
        C(c2,c1)=(C(c2,c1)/(1+B(c2,c1)))+(1/(1+B(c2,c1)))*...
                (q*C(c2,c1+1)+(1-q)*C(c2+1,c1+1));
        tmpc1=c2-1;
    end
end
C

% The price of the cap at time 0 is 0.0034. Thus for a nominal
% value of 10000$, the price is 10000*0.0034=34$ for the cap.



%Results from running the code
```

```

A =

    0.9826    0.9807    0.9787    0.9768    0.9751    1.0000
         0    0.9837    0.9817    0.9797    0.9780    1.0000
         0         0    0.9846    0.9827    0.9809    1.0000
         0         0         0    0.9856    0.9839    1.0000
         0         0         0         0    0.9869    1.0000
         0         0         0         0         0    1.0000


B =

    0.0177    0.0197    0.0218    0.0238    0.0256
         0    0.0166    0.0187    0.0207    0.0225
         0         0    0.0156    0.0176    0.0194
         0         0         0    0.0146    0.0164
         0         0         0         0    0.0133


D =

    1.7700    1.9700    2.1800    2.3800    2.5600
         0    1.6600    1.8700    2.0700    2.2500
         0         0    1.5600    1.7600    1.9400
         0         0         0    1.4600    1.6400
         0         0         0         0    1.3300


C =

         0    0.0007    0.0028    0.0048    0.0066
         0         0         0    0.0017    0.0035
         0         0         0         0    0.0004
         0         0         0         0         0
         0         0         0         0         0


C =

    0.0034    0.0060    0.0090    0.0095    0.0064
         0    0.0010    0.0018    0.0035    0.0034
         0         0    0.0001    0.0002    0.0004
         0         0         0         0         0
         0         0         0         0         0
```