Interest Rate Models In Discrete Time

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We introduce interest rates in connection with assets which depends on interest rates. The simplest such asset are zero coupon bonds, which is a bond that pays a specified amount called face value at a certain time (maturity). Consider a time period $[t, T]$ where $t < T$. Define $P(t, T)$ to be the value at time t of a zero-coupon bond expiring at time T with face value 1 in a given currency.

Annually compounded interest rate:

$$Y(t, T) := \frac{1}{(P(t, T))^{\frac{1}{T-t}}} - 1 \tag{1}$$

$$P(t, T) = \frac{1}{(1 + Y(t, T))^{T-t}} \tag{2}$$

Continuously compounded interest rate

$$R(t, T) := -\frac{lnP(t, T)}{T - t} \tag{3}$$

$$P(t, T) = e^{-R(t,T)(T-t)}. \tag{4}$$

We want to be able to relate amounts of currency available at different times to each other. This is most easily done using a bank account to represent a riskless investment where the profit gained is determined by an interest rate process. Let $B(t)$ be the value of an in a bank account at time $t \geq 0$. Given a discrete time interval $[0, T]$, and let $0 \leq t \leq T$. Let $r(t)$ be the interest rate defined for the interval $[t, t+1]$. Interest rate process: $r(0), r(1), \cdots, r(T-1)$.
Annually compounded interest rates

$$B(t) = (1 + r(0))(1 + r(1)) \cdots (1 + r(t-1)) \tag{5}$$

Continuously compunded interest rates

$$B(t) = e^{\sum_{k=0}^{t-1} r(k)} \tag{6}$$

Discount process:

$$D(t, T) = \frac{B(t)}{B(T)} \tag{7}$$

Annually compounded interest rates

$$D(t, T) = \frac{1}{(1 + r(t)) \cdots (1 + r(T - 1))} \tag{8}$$

Continuously compounded interest rates

$$D(t, T) = e^{-\sum_{k=t}^{T-1} r(k)} \tag{9}$$

We are considering a multiperiod model where the following are specified as data

- $T+1$ trading dates: $t = 0, 1, \cdots, T$
- A finite sample space Ω with $K < \infty$ elements, $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$
- A probability measure $\mathbb{P}$ on Ω with $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$
- A filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$ which is a submodel describing how the information about bond prices are revealed to any investors.
- A bank account process $B = \{B(t); t = 0, 1, \cdots, T\}$, where B is a stochastic process with $B(0) = 1$ and $B(t)(\omega) > 0$ for all t and ω.
- A stochastic zero coupon bond price process $P = \{P(t, T); t = 0, 1, \cdots, T\}$ where $P(t, T)$ is the price of a zero coupon bond with maturity T at time t.

An arbitrage opportunity is an asset (or portfolio of assets) whose value today is zero and whose value in all possible states at the future time is never negative, but in some state at the future time the asset has a strictly positive value.

$$V(0) = 0$$

$$V(T, \omega) \geq 0 \text{ for all states } \omega$$

$$V(T, \omega) > 0 \text{ for some state } \omega$$

Let $\widetilde{P}(t, T)$ be the discounted value of a zero coupon bond. A martingale measure is a probability measure Q equivalent to $\mathbb{P}$, with respect to which the discounted price process of the zero-coupon bonds are martingales, that is it must hold that

$$\widetilde{P}(t, T) = E^Q[\widetilde{P}(t+1, T)|\mathcal{F}_t] \tag{10}$$

A probability measure Q on Ω is said to be a risk neutral probability measure if

- $Q(\omega) > 0$ for all $\omega \in \Omega$
- The discounted price process $\widetilde{P}(t, T)$ is a martingale under Q for every $t = 0, \cdots, T$.

There are no arbitrage opportunities if and only if there exist a risk neutral probability measure Q.

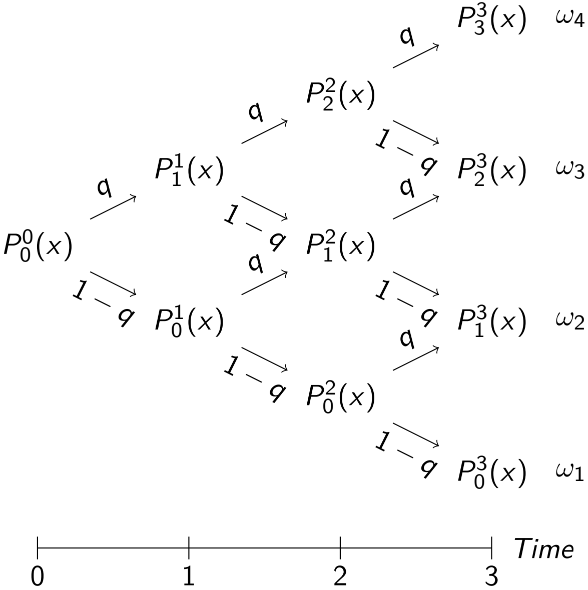
A few results:

$$P(t, T) = E^Q[D(t, T)|\mathcal{F}_t] \tag{11}$$

A contingent claim $X(t)$ is a random variable representing a payoff at some future time t. Under a martingale measure Q, if a process $X = (X_t)$ where $t \leq T$ verifies

$$X_t = E^Q[D(t, T)X_T|\mathcal{F}_t], \; t \leq T \tag{12}$$

then $\widetilde{X} = (\frac{X_t}{B(t)})$ is a Q-martingale. Thus any interest rate derivatives that we are pricing using a binomial model can be defined as Q-martingales.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$P_0^3(x)$ ω_1
q
1 − q
0
1
2
3
Time

Define $P_j^n(x)$ to be the price at time n in state j of a zero coupon bond maturing at time $x+n$ (x periods after n). The forward price of a zero-coupon bond when in state (n,j), can be written as

$$F_j^n(x) = P_j^{n+1}(x) = P_{j+1}^{n+1}(x) = \frac{P_j^n(x+1)}{P_j^n(1)} \tag{13}$$

$$P_{j+1}^{n+1}(x) = \frac{P_j^n(x+1)}{P_j^n(1)} h(x) \tag{14}$$

$$P_j^{n+1}(x) = \frac{P_j^n(x+1)}{P_j^n(1)} h^*(x) \tag{15}$$

Let the risk neutral probability q be the probability of going up in the next time period. It can be shown that

$$qh(x) + (1-q)h^*(x) = 1 \;\; for\ n, j > 0 \tag{16}$$

By using (14),(15),(16) and the constraint that at any time n and state j, an upward move followed by a downward move of a bond price equals a downward move followed by an upward move of its price we can after some calculation get an expression for $h(x)$ and $h^*(x)$.

$$h(x) = \frac{1}{q + (1-q)\delta^x} \quad for \quad x \geq 0. \tag{17}$$

$$h^*(x) = \frac{\delta^x}{q + (1-q)\delta^x}. \tag{18}$$

Where the parameter δ is some constant such that

$$h(1) = \frac{1}{q + (1-q)\delta} \tag{19}$$

and it determines the spread between the functions h and h^*.

By using (14),(15) recursively backwards and simplifying using (18) we can get an expression for the prices of a zero coupon bond at any time n and state j.

$$P_j^n(x) = \frac{P(n+x)}{P(n)} \frac{h(x)h(x+1)\cdots h(x+n-1)}{h(0)h(1)\cdots h(n-1)} \delta^{(n-j)x} \tag{20}$$

This gives in the case of a one period bond (x=1) the price as

$$P_j^n(1) = \frac{P(n+1)}{P(n)} h(n)\delta^{n-j} \tag{21}$$

The model is then fully described by the initial term structure prices given as $P(0,n)$ and by the parameters $0 < q < 1$ and $0 < \delta < 1$. The interest rates at each node in the model can then be calculated from the formulas

- If the interest rate is the annually compounded spot rate $r(n,j)$.

$$r(n,j) = \frac{1}{P_j^n(1)} - 1 \tag{22}$$

- If the interest rate is the continuously compounded spot rate $r_j^n(1)$

$$r_j^n(1) = -ln\, P_j^n(1) \tag{23}$$

An example of a Ho and Lee model where the term structure is given as $P = [0.9826, 0.9651, 0.9474, 0.9296, 0.9119]$ with $q = 0.5$ and $\delta = 0.997$.

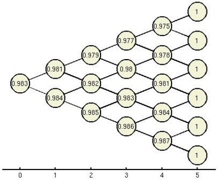


Figure: Ho and Lee zero coupon bond prices where the value in each node is $P_j^n(1)$.

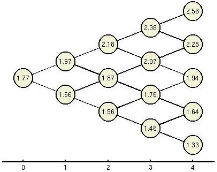


Figure: Ho and Lee annually compounded interest rates

Let $V(n,j)$ be the value of a claim in a given state and its assumed that we are given $V(N,j)$ for $j = 0, 1, 2, \cdots, N$.

- The value of the claim is given by

$$V(n,j) = P_j^n(1)[qV(n+1,j+1) + (1-q)V(n+1,j)] \tag{24}$$

- Let $\lambda(n,j)$ be the value at $t = 0$ of an Arrow-Debreu security. That is a security that pays 1 at $t = n$ in state j and 0 in any other state. Let the state prices $\lambda(n,j)$ be given by

$$\lambda(n,j) = \frac{P(n)q^j(1-q)^{n-j}\psi(j, n-j)}{\prod_{k=1}^{n}[q + (1-q)\delta^{n-k}]} \tag{25}$$

  Then the price of the contingent claim is given by

$$V(0,0) = \sum_{j=0}^{N} \lambda(N,j)V(N,j). \tag{26}$$

Let $t \leq T \leq S$. There will often be a need to agree at present time t for a loan for some future time period $[T, S]$. The interest rate applied to this loan is called a forward rate and is denoted by $f(t, T, S)$.
The continuously compounded forward rate $f(t, T, S)$ at time t for a loan on $[T, S]$ is given by

$$f(t, T, S) = -\frac{ln\,P(T, S) - ln\,P(t, T)}{S - T} \tag{27}$$

where the continuously compounded spot forward rate $F(t, T)$ is given by

$$F(t, T) := f(t, t, T) = -\frac{ln\,P(t, T)}{T - t} \tag{28}$$

which is identical to the continuously compounded spot rate $R(t, T)$.

We consider a binomial model over a time period $[0, \tau]$. For the time period $0 \leq n \leq m \leq \tau - 1$, the annually compounded forward interest rate can be defined as

$$F_r(n,m) = \frac{P(n,m)}{P(n,m+1)} - 1 = \frac{P(n,m) - P(n,m+1)}{P(n,m+1)} \qquad (29)$$

which is the forward interest rate at time n for investing at time m.
A derivative security is a security whose payoff depends on one or more underlying securities or variables. A well known example of a derivative security is a call option on a stock, which gives the buyer of the call the right to buy the stock on a specific date at a specified price.

Swaps

An interest rate swap is an agreement between two parties who each make a contract to pay the other on some specified dates in the future. One agent receives constant payments at each time, while the other receives a variable amount whose size depend on the future rates of interest. Thus an agent can use swaps to convert a fixed interest rate loan into a variable interest rate loan or vice versa. Let $1 \leq m \leq \tau$. An m-period interest rate swap is a contract that pays $S_1, \cdots, S_m$ at times $1, \cdots, m$ where

$$S_n = K - r(n-1), \quad n = 1, \cdots, m \tag{30}$$

where $r(n-1)$ is the annually compounding rate. The fixed rate K is constant.

In the binomial model it can be shown that the risk neutral price of the swap is

$$Swap_m = E^Q\left[\sum_{n=1}^{m} D(0,n)(K - r(n-1))\right] \tag{31}$$

Caps and Floors

Caps and Floors are usually bought to insure against rises and falls in the money market interest rates. Let $1 \leq m \leq \tau$. An m-period interest rate cap is a contract that pays $C_1, \cdots, C_m$ at times $1, \cdots, m$ where

$$C_n = (r(n-1) - K)^+ \tag{32}$$

An interest rate floor is a contract that makes payments $F_1, \cdots, F_m$ at times $1, \cdots, m$ where

$$F_n = (K - r(n-1))^+ \tag{33}$$

The risk neutral price of an m-period cap is

$$Cap_m = E^Q\left[\sum_{n=1}^{m} D(0,n)(r(n-1) - K)^+\right] \tag{34}$$

and the risk neutral price of an m-period interest rate floor is

$$Floor_m = E^Q\left[\sum_{n=1}^{m} D(0,n)(K - r(n-1))^+\right] \tag{35}$$

A forward contract is an agrement to pay a delivery price K at a delivery date m for the asset whose price at time m is S_m. The m-forward price of this asset at time n is the value of K that makes the forward contract have no arbitrage price zero at time n. If we consider an asset whose price process is $S_0, S_1, \cdots, S_\tau$ in the binomial model, then the m-forward price at time n is

$$For(n,m) = \frac{S_n}{P(n,m)} \tag{36}$$

A shortcoming of forward contracts is that on any date prior to the delivery date there can be a demand for forward contracts with that delivery date. For the market to be efficient there would have to be a supply of forward contracts with different initiation dates for each delivery date. A futures contract is a contract that locks in the price of an asset before the time of purchase or sale. Thus the futures price is tied to a delivery date, but not an initiation date.

Let $[0, \tau]$ be the length of a discrete trading economy, where $\tau > 0$ is fixed. The interval between each trade is given by $\Delta > 0$, where N intervals of size Δ compose a unit in time. Given an arbitrary trading time $t \in [0, \tau]$, we write $\bar{t} = tN = t/\Delta =$ the number of trading periods of length Δ prior to and including time t.

$$\begin{aligned} f(t, T) &= -\left[log\left(P(t, T+\Delta)/P(t, T)\right)\right]/\Delta \\ &\quad \textit{for all } T \in [\Delta, \cdots, \tau] \textit{ and } t \in [0, \Delta, \cdots, T-\Delta]. \end{aligned} \tag{37}$$

$$\begin{aligned} P(t, T) &= exp\left(-\sum_{j=\bar{t}}^{\bar{T}-1} f(t, j\Delta)\Delta\right) \\ &\quad \textit{for all } T \in [\Delta, \cdots, \Delta\bar{\tau}] \textit{ and } t \in [0, \Delta, \cdots, \Delta(\bar{T}-1)]. \end{aligned} \tag{38}$$

The spot rate at time t (over $[t, t + \Delta]$), $r(t)$ is defined to be the forward rate at time t

$$r(t) = f(t, t) \tag{39}$$

Let the bank account be defined by

$$B(t) = exp\left(\sum_{j=0}^{\bar{t}-1} r(j\Delta)\Delta\right) \text{ for all } t \in [\Delta, \cdots, \Delta(\bar{\tau} - 1)] \tag{40}$$

The HJM model incorporates multiple sources of uncertainty. In the reference article they suggest the forward rate $f(t,T)$ satisfy a stochastic process:

$$\begin{aligned} f(t,T) = f(0,T) &+ \sum_{j=1}^{\bar{t}} a_j[u_1(j\Delta,T) - v_1(j\Delta,T)] \\ &+ \sum_{j=1}^{\bar{t}} v_1(j\Delta,T) + \sum_{j=1}^{\bar{t}} b_j[u_2(j\Delta,T) - v_2(j\Delta,T)] + \sum_{j=1}^{\bar{t}} v_2(j\Delta,T) \end{aligned} \tag{41}$$

for all $t \in [\Delta, \cdots, T]$, where $\{f(0,t) : T \in [0, \cdots, \Delta(\bar{\tau} - 1)]\}$ is a fixed initial forward rate curve. This forward rate process has two random shocks represented by the random variables $\{a_j, b_j\}$.

One Random Shock:

$$f(t,T) \;=\; f(0,T) + \sum_{j=1}^{\bar{t}} a_j[u(j\Delta,T) - v(j\Delta,T)] + \sum_{j=1}^{\bar{t}} v(j\Delta,T). \quad (42)$$

$$(1-\pi(\bar{t}))exp\Bigg\{-\sum_{j=\bar{t}}^{\bar{T}-1} v(t,j\Delta)\Delta\Bigg\} + \pi(\bar{t}))exp\Bigg\{-\sum_{j=\bar{t}}^{\bar{T}-1} u(t,j\Delta)\Delta\Bigg\} \;=\; 1. \quad (43)$$

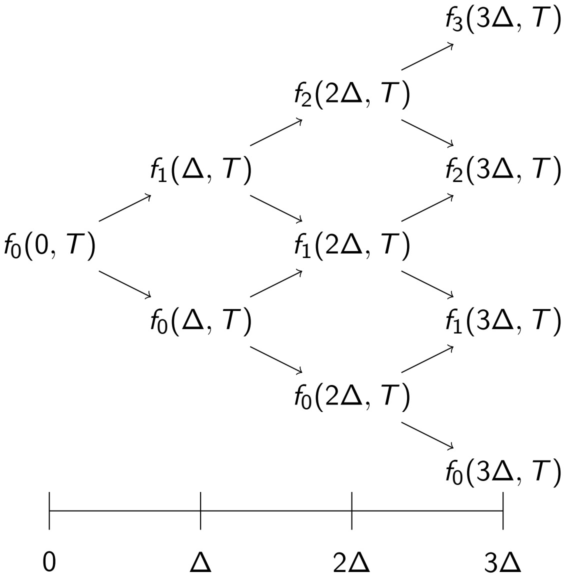
This is equal to the no arbitrage condition in the Ho and Lee model where

$$h^*(x) = exp\Bigg(-\sum_{j=1}^{\bar{x}-1} u(\Delta,j\Delta)\Delta\Bigg)$$

and

$$h(x) = exp\Bigg(-\sum_{j=1}^{\bar{x}-1} v(\Delta,j\Delta)\Delta\Bigg)$$

and this models π equals Ho and Lee's $(1-q)$.


$f_3(3\Delta, T)$
$f_2(2\Delta, T)$
$f_1(\Delta, T)$
$f_2(3\Delta, T)$
$f_0(0, T)$
$f_1(2\Delta, T)$
$f_0(\Delta, T)$
$f_1(3\Delta, T)$
$f_0(2\Delta, T)$
$f_0(3\Delta, T)$
0
Δ
2Δ
3Δ

The Ho and Lee model: By making restrictions on the variance and let $\sigma(t, T) \equiv \sigma > 0$ where σ is a constant. Also let $q(\bar{t}) \equiv q > 0$ and $\pi(\bar{t}) \equiv \pi > 0$, the forward rate process can be simplified to:

$$f(t, T) = f(0, T) + \sum_{j=1}^{\bar{t}} a_j \psi \sqrt{\Delta}$$
$$+ \left[log\left(1 + \pi\left(e^{-\bar{T}\psi\Delta^{3/2}} - 1\right)\right) - log\left(1 + \pi\left(e^{-(\bar{T}-\bar{t})\psi\Delta^{3/2}} - 1\right)\right)\right] \Big/ \Delta. \tag{44}$$

where $\psi = \sigma(q(1-q))^{-1/2}$. σ^2 can be estimated from historical data. The probabilities π and q can be specified to obtain the best fit.