# Electrodynamics – Lecture Notes

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## 1 Introduction

Starting from the **Maxwell Equations**, we can derive that the electric and magnetic field in vacuum propagate at a constant speed $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ which we know as the speed of light.
Why are we expecting that a ray of light travelling billions of years will conform to the same equations that we use to describe light travelling today?
In mathematics, we define vectors to be an element of a vector space, but that definition doesn't give us much insight. Physically, it is nice to think of vectors as something that have a **direction** and a **magnitude**. What we can do is to add another component to this definition, ascribing to a vector a **position** that might change with time.
Suppose now, that we take our space and rotate it by some angle $\theta$. What happens is that as each position changes, the orientation of each vector also changes by the same amount as we observe the same physics before and after the rotation.
This gives us a nice definition of a vector. A vector is something that *transforms* in exactly the same way as the space transforms.
Mathematically we can define it as follows.
Given a rotation $R$, the position will change as follows:

$$\vec{r} \to \vec{r}' = R(\theta' s)\vec{r}$$
$$\vec{v} \to \vec{v}' = R(\theta' s)\vec{v}$$

where we assert that both $\vec{v}$ and $\vec{r}$ use the same roation matrix $R$.
What makes Electrodynamics powerful is that it appliesto a large range of phenomena. When studying still, moving or accelerating systems, we will discover a Portal to relativistic phenomena and inevitably to quantum physics.
A tough part of the lecture is that in order to describe more and more complicated phenomena, we will need more and more mathematics, so it is important to strenghten those.
Luckily, we can sometimes substitute in computational strengh over analytical methods. Using computers, complicated systems can be solved numerically to give us insight.
To give a quick outline of the topics: In a first phase we will study steady currents and steady charge densities and solve the maxwell equations.
We will look at Green's functions to solve PDEs with boundary conditions. Inevitably, we will make use of symmetries to solve them.
In the second phase, we want to think about the structure of the Maxwell equations and adopt different points of views, which will lead us to Special Relativity.
Once we have done that we will study the physics at hight velocities. In particular we will study the radiation that is emitted.

Important references are **Jackson** Electrodynamics and the Feynman lectures Volume II.
To give a little teaser: We can start from electrodynamics and obtain special relativity.
We first write down the maxwell equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = \frac{\vec{J}}{\epsilon c^2} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

How do they change when then looked at from different frames of references. In particular, we look at at translation and rotation and movement

(a) $\vec{r} \to \vec{r}' = \vec{r} + \vec{a}$

(b) $\vec{r} \to \vec{r}' = R\vec{r}$

(c) $\vec{r} =\to \vec{r}' = \vec{r} + \vec{r_0} + \vec{v}t$

It turns out that using gallileian transformation, the maxwell equations stay the same under fixed translation and rotation. But for a moving reference frame, the maxwell equations don't hold.
We can resolve this by either changing the equation or changing how we defined our coordinate transform. We will discard the Gallileian transform in favour of the Lorentz transform which preserve the maxwell equations.

(a) **Translation** Under the translation $\vec{r} \to \vec{r}' = \vec{r} + \vec{a}$, we notice that the direction and magnitude of vectors are preserved. In particular, we have

$$\vec{E}'(\vec{r}') = \vec{E}(\vec{r}), \quad \vec{B}'(\vec{r}') = \vec{B}(\vec{r})$$
$$\vec{J}'(\vec{r}') = \vec{J}(\vec{r}), \quad \vec{\rho}'(\vec{r}') = \vec{\rho}(\vec{r})$$

and since derivatives stay the same, the nabla operator is preserved.

$$\vec{\nabla}' = (\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}) = \vec{\nabla}$$

It follows immediately that the Maxwell equations are invariant under translations.

(b) **Rotation** We can describe rotation $\vec{r} \to \vec{r}' = R(\theta'(s)\vec{r}$ in terms of a rotation matrix $R$. In components, we can write this as $r_i = r_i' = \sum_{k=1}^{3} R_{ik} r_k$.

However, this notation is very tedious and it is easier to write the summation in an *implicit* way. The **Einstein summation covention** is shorthand notation, where: if two indices that are on "opposite sides" (one subscript, one superscript) we will write

$$R_i^k r_k := \sum_{k=1}^{3} R_{ik} r_k$$

Now we note that rotations *preserve length*. This can be stated as follows:

$$\|\vec{r}\| = \sum_{i=1}^{3} r_i \cdot r_i = r^i r_i = r^{i'} r_i' = \|\vec{r}'\|$$

We know from linear algebra that such matrices $R$ are called **orthogonal** and they satisfy $R^T R = 1$, which means

$$\|\vec{r}'\| = r^{i'} r'_i = (R^k_i R^l_i) r^k r_l = (R^i_k R^l_i) r^k r_l = (R^T R)^l_k r^k r_l = r^i r_i$$

Using the fact that $\det(A^T) = \det(A)$ it follows that $\det(R)$ must either of $\pm 1$. An additional requirement is that we want $\det(R) = 1$

We also saw that the Maxwell equations are consistent in different references frames under translationan and rotation, but not with Gallileian transformation.
Last lecture, we defined rotations as matrices $R$ such that $R^T R = 1$. Physically, we additionally require that rotations should have determinant 1, because the identity matrix should be included. Under rotation, we also know that the inner product is invariant, i.e.

$$(\vec{A}, \vec{B}) \to (\vec{A}', \vec{B}') = \vec{A} R^T R \vec{B} = (\vec{A}, \vec{B})$$

Also, the outer product transforms in the following manner

$$\vec{A} \times \vec{B} \to \vec{A}' \times \vec{B}' = R\vec{A} \times \vec{R}$$

Notice how the maxwell equations where we see the inner product is *invariant* and the maxwell equations where we see the outer product, they are *covariant* under rotations. Because the left hand side and the right hand side also transform in the same way, the maxwell equations stay consistent.

## 1.1 Electrostatics

We will study the Maxwell equations where we have no electrical currents with steady densities of charges $(\varphi(\vec{x}, t) = \rho(\vec{x})$
In particular, we will have no magnetic field $(\vec{B} = 0)$ and we want to ask if we can calculate the electricfield if we know the charge density.
The maxwell equations will then be

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \vec{\nabla} \times \vec{E} = 0$$

A naive attempt would be to just use Gauss's law to solve

$$\int_V d^3\vec{x} \vec{\nabla} \cdot \vec{E} = \int_V d^3\vec{x} \frac{\rho(\vec{x})}{\epsilon_0}$$
$$\implies \int_{\partial V} d\vec{S} \cdot \vec{E} = \int_V d^3\vec{x} \frac{\rho(x)}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

This is saying that the total flux through the surface bounding the volume quals the total charge inside the volume.
Also notice how the equation is linear for $\vec{E}$, so if we can decompose $\rho = \rho_1 + \rho_2$ we can solve them independently and put the solutions $\vec{E} = \vec{E}_1 + \vec{E}_2$ together.
This makes solving complex systems much easier as just solving for isolated charges lets us calculate a variety of problems.
Consider a single charge $q$ at position $\vec{y}$ and draw a sphere of radius $R$ around it. The spherical symmetry of the system means that it is sufficient to solve the electric field as a function of the radius $\vec{E}(\vec{x}) = \vec{E}(R)$. With $\vec{r}$ the radial vector perpendicular to the sphere, we see that $d\vec{S} = dS\vec{r}$. The electric field ist just

$$\vec{E} = E\vec{r} = E\frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{x} - \vec{y}|} = E\vec{r}\frac{1}{R}$$

Because the surface area of the sphere is $4\pi R^2$ we get that

$$\int_{\partial V} \vec{E} d\vec{S} = q \implies E(R) = \frac{1}{4\pi\epsilon_0} q \frac{1}{R^2} \hat{r}$$

where $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ is the unit vector in radial direction.
Using the linearity described earlier, we can easily generalize this for any finite amount of isolated charges $q_i$. And we get a solution $\vec{E}(\vec{x})$ that solves the Maxwell equations for $\vec{x} \neq \vec{y}_i$.
This covers the discrete charge distributions, but not continuous ones. Then we have

$$qdV = dq$$

instead of taking a sum, we take the integral

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{y} \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|^3} (\vec{x} - \vec{y}))$$

Notice that the integrand diverges as $\vec{y}$ approaches $\vec{x}$, so we need to assume that we can separate the small distance effects from the large distance effects.
Let's consider the special case where the charge distribution is not quite discrete but is localized via a guassian distribution

$$\rho(\vec{y}) = N \exp\left(-a(\vec{y} - \vec{y}_0)^2\right)$$

We expect that from far away, the charge distribution can be approximated to a discrete charge $q = \int d^3\vec{y}\rho(\vec{y})$ because main contributors to the charge is confined in a very small space around $\vec{y}_0$.
So when we solve for the electric field, we assume that how the charge $q$ is distributed in a small region does not matter much. So

$$< F >= \int d^3\vec{y} F(\vec{y})\rho(\vec{y}) \simeq \int d^3\vec{y} F(\vec{y}_0)\rho(\vec{y})$$

Theoretical physicist Paul Dirac came up with the **Dirac $\delta$-function**

$$\delta(y - y_0) = \begin{cases} \infty & y = y_0 \\ 0 & y \neq y_0 \end{cases}$$

Which has the property that when integrating it with a function

$$\int_{-\infty}^{\infty} dy \delta(y - y_0) F(y) = F(y_0)$$

we obtain the function value at the point $y_0$. We can therefore use the dirac delta function to describe the charge density as

$$\rho(y) = q\delta(y - y_0)$$

We now can verify that the maxwell equations are fulfilled.

$$\vec{E} = -\vec{\nabla}\Phi$$

$$\implies (\vec{\nabla} \times \vec{E})_i = -(\vec{\nabla} \times \vec{\nabla}\Phi)_i = -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial \Phi}{\partial x_k}$$

$$= -\epsilon_{ijk} \frac{\partial^2 \Phi}{\partial x_j \partial x_k} = 0$$

where $\epsilon_{ijk}$ is the total antisymmetric tensor.
The summation is zero as we are multiplying a symmetric tensor $\frac{\partial^2}{\partial x_j \partial x_k}$ with an antisymmetric one $\epsilon_{ijk}$.
By defining the **scalar potential**

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{y} \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|}$$

we can compute the divergence of the electric field

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 \Phi = \frac{1}{4\pi\epsilon_0} \int d^3\vec{y} \rho(\vec{y}) \left( -\nabla^2_{\vec{x}} \frac{1}{|\vec{x} - \vec{y}|} \right)$$

we have seen in MMP I that

$$\nabla^2_{\vec{x}} \frac{1}{|\vec{x} - \vec{y}|} = -4\pi\delta(\vec{x} - \vec{y})$$

so the divergence will be

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 \Phi = \frac{\rho}{\epsilon_0}$$

Which is the differential form of Gauss's Law.
What is the physical meaning of the scalar potential? Because the force acting on a particle with charge $q$ is

$$F = q\vec{E} = -q\vec{\nabla}\Phi$$

we can see that the Work done when moving from point $A$ to point $B$ is given by

$$|W_{A \to B}| = \int_{\vec{x}_A}^{\vec{x}_B} F \, \mathrm{det} = |\Phi(\vec{x}_B) - \Phi(\vec{x}_A)|$$

Let's say that we have $N$ charges $q_i$ at positions $\vec{y}_i$ and we want to know how much energy they have?
What we can do is to think of all charges starting infinitely far away. Since they are each infinitely far apart, we cansay that they have zero potential energy.
What we then do is to bring the charges into their position, one-by-one and measure the work needed to do that.
For the first charge, we won't need any work as the eletric field is zero everywhere. This then generates the first potential

$$\Phi_1(\vec{x}) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\vec{x} - \vec{y}_1|}$$

For the second charge, the energy needed is going to be

$$W_2 = q_2(\Phi_1(\vec{y}_2) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{y}_2 - \vec{y}_1|}$$

and this generates a potential $\Phi_2$, which is just the superposition of the first potential and the potential generated by the second isolated particle.
More generally, after $n$-charges are set into postion, bringing the $n+1$-th particle from infinity will take

$$W_{n+1} = \frac{1}{4\pi\epsilon_0} q_{n+1} \sum_{i=1}^{n} \frac{q_i}{|\vec{y}_{n+1}| - \vec{y}_i}$$

so the total energy will be

$$\begin{aligned} W_{\text{tot}} &= W_1 + W_2 + \ldots + W_N \\ &= \sum_{n=2}^{N} \sum_{1 \le i < n} \frac{q_i q_n}{|\vec{y}_n| - \vec{y}_i} \\ &= \frac{1}{8\pi\epsilon_0} \sum_{\substack{1 \le i,j \le N \\ i \ne j}} \frac{q_i q_j}{|\vec{y}_i - \vec{y}_j|} \end{aligned}$$

where we use the symmetry to sum over all $1 \le i \ne j \le N$ and divide by two to account for double counting.
In the continous limit we assume that $q_i = dV\rho(\vec{y}_i)$, so

$$\begin{aligned} W &= \int \frac{d^3\vec{x}d^3\vec{y}}{8\pi\epsilon_0} \frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x} - \vec{y}|} \\ &= \int d^3\vec{x}\rho(\vec{x})\Phi(\vec{x}) \\ &= -\frac{\epsilon_0}{2} \int d^3\vec{x}(\nabla^2\Phi)\Phi(\vec{x}) \\ &= -\frac{\epsilon_0}{2} \int d^3\vec{x} \left[\vec{\nabla}(\Phi\vec{\nabla}\Phi) - (\vec{\nabla}\Phi)(\vec{\nabla}\Phi)\right] \end{aligned}$$

where we can have rewritten the charge density in terms of the poisson equation. This is nice because we only need to know the scalar potential in the last reformulation.
Notice that the integral goes from the origin to infinity. But there, the surface term $\nabla^2\Phi$ vanishes at infinity, so we can write

$$\begin{aligned} W = \frac{\epsilon_0}{2} \int d^3\vec{x}(\vec{\nabla}\Phi)^2 \\ = \frac{\epsilon_0}{2} \int d^3\vec{x}\vec{E}^2(\vec{x}) \end{aligned}$$

which is always a positive quantitiy.
Let's compare the discrete sum with the continuous limit in the case of a single charge $q$ at position $\vec{y}_0$. Its charge density can be described using a delta function.

$$\rho(\vec{x}) = q\delta(\vec{x} - \vec{y}_0)$$

Notice how the continuous expression will give us an answer that we don't expect:

$$\begin{aligned} W &= \int d^3\vec{x}d^3\vec{y} \frac{q^2}{8\pi\epsilon_0} \delta(\vec{x} - \vec{y}_0)\delta(\vec{y} - \vec{y}_0) \frac{1}{|\vec{x} - \vec{y}|} \\ &= \frac{q^2}{8\pi\epsilon_0} \frac{1}{|\vec{y}_0 - \vec{y}_0|} = \infty \end{aligned}$$

But recall that bringing a single charge from infinity to an empty field should take no energy. We call this infinity **self-energy**.
One might argue that this formula is non-sense, but it nontheless insightful. The problem orginiates from our attempt model what a charge at a point position looks like. We can't simply put a charge into an infinitesimal point and claim that the physics works as usual.

So instead of of assuming that the charge distribution is a simple delta function, let's write spread it out a little bit

$$\rho_\delta(\vec{x}) = \frac{q}{\pi^3}\frac{\delta^3}{(x_1^2+\delta^2)(x_2^2+\delta^2)(x_3^2+\delta^2)}$$

where $\delta \ll 1$ is a small parameter. Notice that as $\delta \to 0$, the $\rho_\delta(\vec{x})$ approaches the delta function.
So from far away, the charge density looks like the delta function and we can show that if we integrate over the charge density, we obtain the total charge:

$$\int d^3\vec{x}\rho_\delta(\vec{x}) = q$$

We can also show that for finite $\delta$, the integral

$$\int d^3\vec{x}d^3\vec{y}\frac{\rho_\delta(\vec{x})\rho_\delta(\vec{x})}{|\vec{x}-\vec{y}|} =: g(\delta)$$

is finite, and we can associate to this integral the *finite* self-energy $E^{(\delta)}_{\text{self}}$
At infinity, the potential energy goes to zero, but the self-energy does not vanish. So the thing that stays consistent between the discrete and continuous formulation is not $W$ itself, but

$$\Delta W := W - W_{\text{self}}$$

In quantum field theory, the analogue of self-energy is when a chage exhibits self-interaction by way of emitting a photon and absorbing it again. The charge-photon-photon interaction turn out to give us Coulomb's Law.
So the self-energy emerges when we count self-interactions of particles.

## 1.2 Static distributions and Energy

We found in the last lecture that depending on whether we used the discrete or continuous formula for the Energy, we got an additional *self-energy* term.
By introducing the **energy density** $\omega$ given by

$$W = \frac{\epsilon_0}{2}\int_V d^3\vec{x}\vec{E}^2(\vec{x}) = \int_V d^3\vec{x}\omega(\vec{x}) \implies \omega := \frac{\epsilon_0}{2}\vec{E}^2$$

If instead of just calculating the total energy, we can just calculate the *change* in energy between to states. Then the self-energy terms will drop out and we get a coherent result between the continuous and discrete measurements.

**Example 1.1** (Pressure on a conductor)**.** By definition, a **conductor** is a piece of material, where electrons (or charges) can move around freely. In reality, ideal conductors don't exist as moving charges radiate energy and lose energy there will be some friction.
Conductors exhibit the property, that if we introduce some exess charge on a conductor, then the electrons will move around until they don't want to anymore. This state will be reached when the electron distribution minimizes their total energy. We can formalize this by saying that when the electrons don't move around anymore, the electric field on the inside of the conductor is zero.

$$\vec{E}_{\text{inside}} = 0$$

or else it would exert a force on a charge, moving it around.
Since $\vec{E} = -\vec{\nabla}\Phi$, it follows that $\Phi_{\text{inside}}$ =constant.
Another property is that on the surface, the electric field must be *perpendicular to the surface*. If it had some horizontal component, it would mean that the electrons on the surface would move around. Also, using Gauss's law on the surface, if the conductor has non-zero exess charge, the perpendicular component is non-zero.

$$\vec{E}_{\parallel} = 0, \vec{E}_{\perp} \neq 0$$

moreover, if we let $\sigma$ be the charge density on the surface, then

$$\begin{aligned} \Delta S \cdot E &= \frac{\sigma}{\epsilon_0} \cdot \Delta S \\ \implies \vec{E} &= \frac{\sigma}{\epsilon_0}\vec{n} \end{aligned}$$

, where $\vec{n}$ is the surface normal. This means that the energy density $\omega$ is given by

$$\omega = \frac{\epsilon_0}{2}\vec{E}^2 = \frac{\sigma^2}{2\epsilon_0}$$

With this, we can calculate the pressure, or energy difference.
Now image that we take our conductor deforms slightly such that it has a part where the sufrace bulges by a small amount with additional volume $\Delta S \cdot \Delta x$.
This will change the energy as follows

$$\begin{aligned} E_{\text{deformed}} &= E_{\text{original}} + \int_{\Delta S \cdot \Delta x} d^3\vec{x}\omega \\ &= E_{\text{original}} + \Delta S \cdot \Delta x \cdot \frac{\sigma^2}{2\epsilon_0} \\ \implies \frac{E_{\text{deformed}} - E_{\text{original}}}{\Delta x} &= \Delta S \frac{\sigma^2}{2\epsilon_0} \end{aligned}$$

where we assume that the integrand is very small, so the integrand $\omega$ is practically constant.
So the difference in the energy is the work done by pushing theelectrons around. The work is force times distance, so

$$\begin{aligned} \frac{\text{Force} \cdot \Delta x}{\Delta x} &= \Delta S \frac{\sigma^2}{2\epsilon_0} \\ \text{Pressure} = \frac{\text{Force}}{\Delta S} &= \frac{\sigma^2}{2\epsilon_0} \end{aligned}$$

So the pressure rises quadratically with the charge.

Up to now, we were able to solve the Maxwell equations in integral form

$$\Phi = \frac{1}{4\pi\epsilon_0}\int d^3y \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|}$$

where the integral goes over all of space.
The problem is that this requires knowlege of all charges throughout all of space. What we want to do is to find an alternative way to solve the problem which only requires local information. What is the minimal amount of information that we need on the boundary $\partial V$ in order to compute $\vec{E}$ inside $V$.

The main tool will be the poisson equation

$$\vec{\nabla}^2\Phi = -\frac{\rho}{\epsilon_0}$$

First note that unless we have a boundary condition, the solution is not unique. Assume we have two solutions $\Phi_1, \Phi_2$ inside the volume. Then their difference $\Phi_1 - \Phi_2$ satisfies $\Delta(\Phi_1 - \Phi_2) = 0$. In particular, we can write

$$\begin{aligned} (\Phi_1 - \Phi_2)\Delta(\Phi_1 - \Phi_2) &= 0 \\ \implies |\nabla(\Phi_1 - \Phi_2)| - \vec{\nabla}\left((\Phi_1 - \Phi_2)\vec{\nabla}(\Phi_1 - \Phi_2)\right) &= 0 \end{aligned}$$

So using $\vec{E} = -\vec{\nabla}\Phi$ we get that

$$(\vec{E}_1 - \vec{E}_2)^2 + \vec{\nabla}\left((\Phi_1 - \Phi_2)(\vec{E}_1 - \vec{E}_2)\right) = 0$$

so by integrating over a volume $V$, we get using Gauss's Law

$$\begin{aligned} \int_V d^3\vec{x}(\vec{E}_1 - \vec{E}_2)^2 &= -\int_V d^3\vec{x}\vec{\nabla}\cdot\left((\Phi_1 - \Phi_2)(\vec{E}_1 - \vec{E}_2)\right) \\ &= -\int_{\partial V} d\vec{S}\cdot(\vec{E}_1 - \vec{E}_2)(\Phi_1 - \Phi_2) \end{aligned}$$

Assume that we know the potential everywhere on the boundary. If the boundary conditions give us a unique solutions for $\Phi$, i.e. $\Phi_1 - \Phi_2 = 0$, then the right hand side becomes zero.
The only way that the positive definite integral on the left hand side is zero, if its integrand is zero, so

$$\int_V d^3\vec{x}\underbrace{(\vec{E}_1 - \vec{E}_2)^2}_{\geq 0} = 0 \implies \vec{E}_1(\vec{x}) = \vec{E}_2(\vec{x}) \quad \forall\vec{x} \in V$$

If $\Phi$ is known uniquely on the boundary, then $\vec{E}$ is uniquely determined inside the volume. Similarly, if $\vec{E}$ is known uniquely on the boundary, then $\vec{E}$ is uniquely determined inside the volume.

## 1.3 Solving the poisson equation

Given boundary conitions, we want to solve $\vec{\nabla}^2\Phi = -\frac{\rho}{\epsilon_0}$ using the **method of Green's functions**.
A **Green's function** is a function, whose laplacian is a delta function:

$$\vec{\nabla}^2_{\vec{x}}G(\vec{x},\vec{y}) = -4\pi\delta(\vec{x} - \vec{y})$$

one such example is the inverse distance

$$G(\vec{x},\vec{y}) = \frac{1}{|\vec{x} - \vec{y}|}$$

or more generally, we can add to it any harmonic function $F$. (i.e. $\vec{\nabla}^2F(\vec{x},\vec{y}) = 0$.
Consider a function $\vec{F}$ given by

$$\vec{F} = \Psi\vec{\nabla}\Phi - \Phi\vec{\nabla}\Psi$$

where $\Phi$ is a potential and $\Psi = G(\vec{x}, \vec{y})$ is a Green's function.

$$\vec{\nabla}^2\Phi = -\frac{\rho}{\epsilon_0}, \quad \vec{\nabla}^2_{\vec{x}}\Psi(\vec{x}, \vec{y}) = -4\pi\delta(\vec{x} - \vec{y})$$

Taking the divergence of $\vec{F}$ we are left with

$$\begin{aligned}\vec{\nabla} \cdot \vec{F} &= \Phi\vec{\nabla}^2\Phi - \Phi\vec{\nabla}^2\Psi \\ &= \Psi(\vec{x}, \vec{y})\left(-\frac{\rho(x)}{\epsilon_0}\right) - \Phi(\vec{x})(-4\pi\Phi(\vec{x})\delta(\vec{x} - \vec{y})\end{aligned}$$

by integrating over a Volume $V$ we obtain

$$\begin{aligned}\int_V d^3\vec{x}\vec{\nabla} \cdot \vec{F} &= 4\pi\int_V d^3\vec{x}\Phi(\vec{x})\delta(\vec{x} - \vec{y}) - \frac{1}{\epsilon_0}\int_V d^3\vec{x}\rho(\vec{x})\Psi(\vec{x}, \vec{y}) \\ &= 4\pi\Phi(\vec{y}) - \frac{1}{\epsilon_0}\int_V d^3\vec{x}\rho(\vec{x})\Psi(\vec{x}, \vec{y})\end{aligned}$$

We can solve this for $\Phi(\vec{y})$ and use Gauss's Law for the divergence $\vec{\nabla} \cdot \vec{F}$.

$$\Phi(\vec{y}) = \frac{1}{4\pi\epsilon_0}\int_V d^3\rho(\vec{x})\Psi(\vec{x}, \vec{y}) + \frac{1}{4\pi}\int_{\partial V} d\vec{S} \cdot \vec{F}$$

Recall our definition of $\vec{F}$. This gives us

$$\begin{aligned}\int_{\partial V} d\vec{S} \cdot \vec{F} &= \int_{\partial V} d\vec{S} \cdot \left(\Psi\vec{\nabla}\Phi - \Phi\vec{\nabla}\Psi\right) \\ &= \int_{\partial V} d\vec{S}\left(G(\vec{x}, \vec{y})(-\vec{E}(\vec{x}) - \Phi(\vec{x})\vec{\nabla}_{\vec{x}}G(\vec{x}, \vec{y})\right) \\ &= -\int_{\partial V} d\vec{S} \cdot \left(\vec{E}(\vec{x})G(\vec{x}, \vec{y}) + \Phi(\vec{x})\vec{\nabla}_{\vec{x}}G(\vec{x}, \vec{y})\right)\end{aligned}$$

So if $\vec{E}$ and $\Phi$ are given on the boundary $\partial V$ and construct the Green function$G$, then we can solve for the the field inside the volume $V$.
But notice that our current technique requires knowledge of both $\Phi$ and $\vec{E}$ on the bondary. We want to only need one of them. Let's suppose that we only know $\Phi(\vec{x})$ on the boundary and chose our Green's function

$$G(\vec{x}, \vec{y}) = \frac{1}{|\vec{x} - \vec{y}|} + F(\vec{x}, \vec{y})$$

for a harmonic $F$ such that the green's function vanishes on boundary.

$$G_D(\vec{x}, \vec{y}) = 0, \forall\vec{x} \in \partial V, \vec{y} \in V$$

We call this the **Dirichelet Green's function**. Since it vanishes on the surface $\partial V$, we can simplify the previous equation to

$$\Phi(\vec{y}) = \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{x}\rho(\vec{x})G_D(\vec{x}, \vec{y}) - \frac{1}{4\pi}\int_{\partial V} d\vec{S}\Phi(\vec{x})\vec{\nabla}_{\vec{x}}G(\vec{x}, \vec{y})$$

to solve for $\vec{E}$ and $\Phi$ everywhere inside the volume $V$.

With the **Neumann boundary condition**, we want to find a Green's function whose gradient has a non-zero, but *constant* normal component on the surface, i.e.

$$\vec{\nabla}_{\vec{x}} G_n(\vec{x}, \vec{y}) = -\frac{4\pi}{S}\hat{n}$$

, where $S$ is the total surface of the boundary $\partial V$.
In this case, the original equation yields

$$\Phi(\vec{y}) = \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{x} G_N(\vec{x}, \vec{y})\rho(\vec{x}) + \frac{1}{4\pi}\int_{\partial V} d\vec{S}\cdot G_n(\vec{x}, \vec{y})\vec{\nabla}_{\vec{x}}\Phi(\vec{x}) + \langle\Phi\rangle_{\partial V}$$

, where $\langle\Phi\rangle_{\partial V}$ is the average of the potential on the boundary.

$$\langle\Phi\rangle_{\partial V} = \frac{1}{S}\int_{\partial V} d\vec{S}\cdot\Phi\vec{n}$$

**Example 1.2** (Half-Volume)**.** Suppose the world is split in half, where in one half we have free space $V$ to measure freely and the other half is inaccessible to us. We are however allowed to make measurements on the wall $\partial V$. Now we want to find a single Green's function $G_D(\vec{x}, \vec{y})$ which vanishes on the boundary, i.e. the wall.
We chose a coordinate system such that the wall is at $(x_1 = 0, x_2, x_3)$ and we want to measure the potential inside the accessible free space.
Starting with the Green's function $G$ of the form

$$G(\vec{x}, \vec{y}) = \frac{1}{|\vec{x} - \vec{y}|} + F(\vec{x}, \vec{y}), \quad \text{for} \quad \vec{\nabla}^2_{\vec{x}} F(\vec{x}, \vec{y}) = 0$$

With some guessing, we find that by defining for $\vec{y}$ its *dual vector* $\vec{y}^*$ to be its reflection along the wall, the Green's function

$$G(\vec{x}, \vec{y}) := \frac{1}{|\vec{x} - \vec{y}|} - \frac{1}{|\vec{x} - \vec{y}^*|}$$

satisfies all the criteria: It vanishes on the boundary and since $\vec{x}$ is always in the free space, we have

$$\vec{\nabla}^2_{\vec{x}} F(\vec{x}, \vec{y}) = 4\pi\partial(\vec{x} - \vec{y}^*) = 0$$

Now observe that since reflection is isometric, we have

$$|\vec{x} - \vec{y}^*| = |\vec{y} - \vec{x}^*|$$

so for our potential $\Phi$ we obtain

$$\begin{aligned}\Phi(\vec{y}) = &\frac{1}{4\pi\epsilon_0}\int_V d^3\vec{x}\frac{\rho(\vec{x})}{|\vec{x} - \vec{y}|} \\ &+ \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{x}\frac{\rho(\vec{x})}{|\vec{x}^* - \vec{y}|} \\ &- \frac{1}{4\pi}\int_{\partial V} d\vec{S}\cdot\Phi(\vec{x})\vec{\nabla}\left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{1}{|\vec{x}^* - \vec{y}|}\right]\end{aligned}$$

If we make tha assumption that the potential on the wall is a constant $\Phi(\vec{x}) = C$. Then we can show that the surface integral becomes

$$-\frac{1}{4\pi}\int_S d\vec{S}\cdot\Phi(\vec{x})\vec{\nabla}\left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{1}{|\vec{x}^*\vec{y}|}\right] = C$$

In the special case of a discrete charge distribution of charges $q_i$ at position $\vec{x}_i$ we would find that

$$\Phi(\vec{y}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\vec{y}-\vec{x}_i|} + \frac{-q_i}{\vec{y}-\vec{x}_i^*}$$

which is the solution to the electrostatic problem in free space, except that now it looks like there are **mirror charges** $-q_i$ at positions $\vec{x}_i^*$.

Often, we can replace special boundary conditions on some volume $V$ by introducing extra charges and forgetting about the boundary conditions.

**Example 1.3** (Spherical conductor)**.** Consider a conductor sphere with radius $R$ placed at the origin next to a charge $q$ at position $\vec{d}$. If we define the **dual position** and **dual charge** as

$$q^* = -\frac{R}{d}q \quad \text{and} \quad \vec{d^*} = \frac{R^2}{d^2}\vec{d}$$

Then the potential of the two charges will be given by

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-\vec{d}|} - \frac{\frac{R}{d}}{|\vec{r}-\frac{R^2}{d^2}\vec{d}|}\right)$$

and in particular, on the surface of the conductor ($|\vec{r}| = R$), the potential vanishes as it should:

$$\Phi(\vec{R}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{R}-\vec{d}|} - \frac{\frac{R}{d}}{\left(R^2+\frac{R^4}{d^2}-2\frac{R^2}{d^2}\vec{R}\cdot\vec{d}\right)^{1/2}}\right) = 0$$

So it seems like we can exchange boundary conditions for a modified charge distribution.

$$\rho(\vec{x}), \rho^*(\vec{x}^*) \longleftrightarrow \text{ Boundary Conditions } + \rho(\vec{x})$$

The question is, can we get the Green's function? We would have to find $G(\vec{r}, \vec{r'})$ such that

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon}\int_V d^3\vec{r}'\rho(\vec{r}')G(\vec{r}'-\vec{r}) - \frac{1}{4\pi}\int_{\partial V} d\vec{S}\cdot\Phi(\vec{r}')\vec{\nabla}_{\vec{r}'}G(\vec{r}'-\vec{r})$$

that agrees with the result for the potential we found earlier.
By setting the potential to zero at the boundary (because it is a conductor), we can instead write

$$\begin{aligned}\Phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{r}' q\delta(\vec{r}'-\vec{r})G(\vec{r}',\vec{r}) \\ &\overset{!}{=} \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\vec{r}-\vec{d}} - \frac{\frac{R}{d}}{|\vec{r}-\frac{R^2}{d^2}\vec{r}'|}\right)\end{aligned}$$

so we can read off the solution for $G$:

$$G(\vec{x},\vec{r}) = \frac{1}{\vec{r}-\vec{x}} - \frac{\frac{R}{|\vec{x}|}}{|\vec{r}-\frac{R^2}{|\vec{x}|}\vec{x}|}$$

## 1.4 Complete basis of Hilbert spaces

Recall from Quantum Mechanics that we represented momentum in terms of the derivative. In particular, this meant that the kinetic energy could be written in terms of the laplacian

$$\hat{p} = -i\hbar\vec{\nabla}, \quad \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$$

Since the eigenfunctions of hermitian operators form a basis of the solution space, we could always decompose a quantum state $|\Psi\rangle$ into Energy Eigenstates

$$|\Psi\rangle = \sum_n c_n |\Psi_n\rangle$$

The fact that this was possible was because the $|\Psi_n\rangle$ formed a complete basis of a Hilbert space.
If we can understand the Eigenstates of the Laplacian, then we can try to represent the Green's functions using these eigenstates

$$G(\vec{x}, \vec{y}) = \sum_n c_n(\vec{y})\Psi_n(\vec{x}) \quad \text{for} \quad \nabla^2\Psi_n(x) = \lambda_n\Psi_n(\vec{x})$$

The fact that the eigenvalues of hermitian operators are real corresponds to the fact that Energy measurments are real.
To see this, let's construct the vector $\vec{F}_{nm}$ given by

$$\begin{aligned} \vec{F}_{nm} &= \Psi_m^*\vec{\nabla}\Psi_n - \Psi_n\vec{\nabla}\Psi_m^* \\ \implies \vec{\nabla}\cdot\vec{F}_{nm} &= \Psi_m^*\nabla^2\Psi_n - \Psi_n\nabla^2\Psi_m^* \\ &= (\lambda_n - \lambda_m^*)\Psi_m^*\Psi_n \end{aligned}$$

Integrating this over a Volume, we can use Gauss's theorem and see that

$$\int_V d^3\vec{x}\vec{\nabla}\cdot\vec{F}_{nm} = \int_{\partial V} d\vec{S}\cdot\vec{F}_{nm}$$

where we want to calculate $\Psi_n(\vec{x})$ such that it vanishes on the boundary making the right hand side go to zero.
For $n = m$ we would find that

$$(\lambda_n - \lambda_m^*)\int_V d^3\vec{x}|\Psi_n(\vec{x})|^2 = 0$$

but since the integrand is positive definite, it follows that the Eigenvalues are real (surprise surprise ...)
But for $n \neq m$ let's assume that there is no degeneracy. That is, we assume that the eigenvalues for different eigenstates are different.
If that is the case, then $(\lambda_m - \lambda_n) \neq 0$ so it requires now that the integral vanishes.

$$\int_V d^3\vec{x}\Psi_m^*(\vec{x})\Psi_n(\vec{x}) = 0$$

By linearity of the integral, we can see this relation as an inner product on the space of functions.

$$\langle f|g\rangle := \int_V d^3\vec{x}f^*(x)g(x)$$

The equation before then says that the eigenfunctions to different eigenvalues are **orthogonal**.

$$\langle \Psi_n | \Psi_m \rangle = \int_V d^3\vec{x} \Psi_n^*(\vec{x}) \Psi_m(\vec{x}) = \delta_{nm}$$

where by rescaling, they are also ortho**normal**.
For the expansion of a function $f$ in terms of the eigenfunctions, we can obtain the coefficients of $f$ using the integral

$$f = \sum_n c_n \Psi_n(x), \quad \text{for} \quad c_n = \int_V d^3\vec{x} \Psi_m^*(\vec{x}) f(\vec{x})$$

But plugging this expansion of the coefficients in, we get

$$f(\vec{x}) = \int_V d^3\vec{y} \left[ \sum_n \Psi_n^*(\vec{y}) \Psi_n(\vec{x}) \right] f(\vec{y})$$

which is only possible if the **completenes property** is satisfied

$$\sum_n \Psi_n^*(\vec{x}) \Psi_n(\vec{y}) = \delta(\vec{x} - \vec{y})$$

We can use the fact that every function can be written as a linear combination of the eigenfunctions for the Green's function. We then get

$$G_D(\vec{x}, \vec{y}) = \sum_n c_n(\vec{y}) \Psi_n(\vec{x})$$

so instead of solving for the Green's function conditions, we solve for the laplace equation and then calculate the coefficients.
Since we want the Green's function to satisfy

$$\nabla_{\vec{x}}^2 G_D(\vec{x}, \vec{y}) = -4\pi\delta(\vec{x} - \vec{y})$$

so plugging this into the eigenfunction expansion together with the completness, we obtain

$$\sum_n c_n(\vec{y}) \nabla_{\vec{x}}^2 \Psi(\vec{x}) = \sum_n c_n(\vec{y}) \lambda_n \Psi_n(\vec{x}) = -4\pi\psi(\vec{x} - \vec{y}) = -4\pi \sum_n \Psi_n^*(\vec{y}) \Psi_n(\vec{x})$$

or equivalently,

$$\sum_n \left[ c_n(\vec{y}) \lambda_n + 4\pi \Psi_n^*(\vec{y}) \right] \Psi_n(\vec{x}) = 0$$

which can only be zero if the coefficients are all zero, so it must be that for each $n$ we have

$$c_n(\vec{y}) = -\frac{4\pi}{\lambda_n} \Psi_n^*(\vec{y}))$$

Putting this together we get a recipe to find the Green's function:

$$G_D(\vec{x}, \vec{y}) = -4\pi \sum_n \frac{\Psi_n^*(\vec{y}) \Psi_n(\vec{x})}{\lambda_n}$$

However, this is all under the assumption that the eigenstates showed no degeneracy. But we obvioulsy know since the eigenvalues are energy, they are degenerate as multiple states can have the same energy and we *might* not have orthogonality and completeness.
There is a way around this. If instead of solving for laplace eigenfunctions , we solve for eigenfunctions of the momentum operator

$$\hat{p}\Psi_n = \vec{\nabla}\Psi_n = \vec{p}_n\vec{\Psi}$$

we see that there is no degeneracy here.

**Example 1.4** (Sanity check)**.** We want to find the Green's function $G_D(\vec{x},\vec{y})$ with no boundary conditions. We know the answer, as it is simply

$$G_D(\vec{x},\vec{y}) = \frac{1}{|\vec{x}-\vec{y}|}$$

Let's to use our new laplace eigenfunction method and see if it produces the right result. Before we solve the laplace equation, we first solve for the momentum eigenstate:

$$\vec{\nabla}\Psi(\vec{x}) = i\vec{p}\Psi(\vec{x})$$

That is very easy as we can just solve the first order ODE for each component of $\Psi$. The eigenfunction is then given by

$$\Psi_{\vec{p}}(\vec{x}) = \frac{1}{(2\pi)^{3/2}}e^{i\vec{p}\cdot\vec{x}}$$

so for the laplacian, the we get the eigenfunctions

$$\nabla^2\Psi_{\vec{p}}(\vec{x}) = -p^2\Psi_{\vec{p}}(\vec{x})$$

Now, we check for orthogonality and completness:

$$\begin{aligned}\int d^3\vec{x}\Psi^*_{\vec{p}}(\vec{x})\Psi_{\vec{k}}(\vec{x}) &= \int d^3\vec{x}\frac{1}{(2\pi)^3}e^{-i\vec{p}\cdot\vec{x}}e^{+i\vec{k}\cdot\vec{x}}\\ &= \int d^3\vec{x}\frac{1}{(2\pi)^3}e^{-i(\vec{p}-\vec{k})\cdot\vec{x}}\\ &= \delta(\vec{p}-\vec{k})\end{aligned}$$

where we used the fact from MMP-I that the fourier transform of the identity is the delta function.
For completeness, our eigenfunctions are indexed by a continuous variable $\vec{p}$, so we need to calculate

$$\int d^3\vec{k}\Psi^*_{\vec{k}}(\vec{x})\Psi_{\vec{k}}(\vec{y}) = \int d^3\vec{k}\frac{1}{(2\pi)^3}e^{-i\vec{k}\cdot(\vec{x}-\vec{y}} = \delta(\vec{x}-\vec{y})$$

So our Eigenfunctions are indeed both orthonormal and complete. Now we can use our recipe to find the Green's function:

$$\begin{aligned}G_D(\vec{x},\vec{y}) &= -4\pi\int d^3\vec{k}\frac{\Psi^*_{\vec{k}}(\vec{y})\Psi_{\vec{k}}(\vec{x})}{\lambda_n}\\ &= -4\pi\int d^3\vec{k}\frac{1}{(2\pi)^3}\frac{e^{-i\vec{k}\cdot(\vec{y}-\vec{x}}}{-|\vec{k}|^2}\\ &= \frac{1}{|\vec{x}-\vec{y}|}\end{aligned}$$

where in the last step we used the fourier transform from MMP-I.

For such a simple problem the Laplace method may seem overkill, but we can solve the electrostatic problem for various situations.

**Example 1.5** (Electrostatics in a box)**.** In this example, we have a box with sidelengths $a, b, c$ and we want a Green's function for a solution $\Psi$ that vanishes on the $\pi a \rho a \lambda \eta \lambda \epsilon \pi i \pi \epsilon \delta \sigma$
A solution we have derived in Physics III is

$$\Psi_{lmn}(\vec{x}) = \sqrt{\frac{8}{abc}} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}$$

which satisfies the boundary condtion and is an eigenfunction of the laplace equation

$$\nabla^2 \Psi_{lmn}(\vec{x}) = -\pi \left( \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right) \Psi_{lmn}(\vec{x})$$

This time, the eigenfunctions are discrete and we can also show orthogonality and completeness (but that is now shown here)
Then, the Green's function is given by

$$G_D(\vec{x}, \vec{y}) = -\pi \sum_{n,l,m} \frac{\Psi^*_{lmn}(\vec{x}) \Psi_{lmn}(\vec{y})}{\lambda_{lmn}}$$

which cannot really be simplified any further. They belong to a family called **hypergeometric functions**

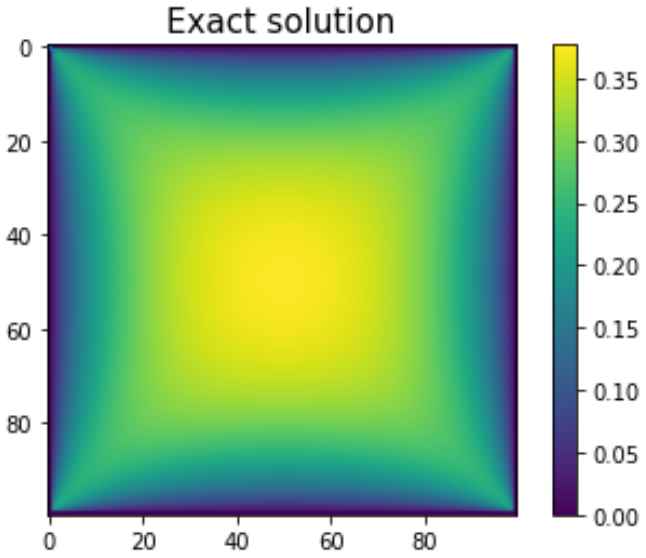


Figure 1: Plot of the Potential $\Phi$ for the 2-D square paralelepipeds

## 1.5 Laplace in spherical coordinates

When solving for Laplace eigenfunctions in spherical coordinate systems, we have seen in MMP-I and Physics III that the laplace operator can be decomposed in a **radial** and **angular** part

$$\nabla^2 = \frac{1}{r} \frac{\partial^2}{\partial r^2} + \frac{A(\theta, \varphi)}{r^2}$$

where the angular part may seem a bit obtuse:

$$A = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

but it turns out that solving it isn't as bad as it seems. Let's denote the eigenfunctions to the $\varphi$ part, with $\ell_m$, so

$$\frac{\partial^2}{\partial\varphi^2}\ell_m(\varphi) = -m^2\ell_m(\varphi)$$

which is just a harmonic oscillator. The solution can be found using the Euler ansatz and it is just given by

$$\ell_m(\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}$$

Moreover, because the angle $\varphi$ should be $2\pi$ periodic, we should have

$$\ell_m(\varphi) = \ell_m(\varphi + 2\pi) \implies e^0 = e^{2\pi im} \implies m = 0, \pm1, \pm2, \dots$$

so we can restrict $m$ to only take integer values. We can also check for orthogonality.

$$\langle \ell_m, e_n\rangle = \int_0^{2\pi} \psi\ell_m^*(\varphi)e_n(\varphi) = \delta_{mn}$$

For the angular operator $\hat{A}$, let's denote its Eigenfunctions with $Y$, and expand it it terms of the eigenfunctions $\ell_m$, where the coeffients may depend on $\theta$:

$$\hat{A}(\theta,\varphi) = Y(\theta,\varphi) = \rho Y(\theta,\varphi)$$
$$Y(\theta,\varphi) = \sum_{m\in\mathbb{Z}} c_m(\theta)\ell_m(\varphi)$$

Inserting $c_m(\theta)\ell_m(\varphi)$, we want to solve for

$$\begin{aligned}\rho c_m(\theta)\ell_m(\varphi) &= \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] c_m(\theta)\ell_m(\varphi)\\ &= \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}(-m^2)\right] c_m(\theta)\ell_m(\varphi)\end{aligned}$$

and we can divide by $\ell_m(\varphi)$ to completely remove the $\varphi$ dependence in the differential equation. Using the insights of past mathematicians, it is easier to replace $\rho$ with $-l(l+1)$, and re-label $c_m(\theta)$ with $P_{l,m}(\theta)$ so we are solving the equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{m^2}{\sin^2\theta}\right] P_{l,m}(\theta) = -l(l+1)P_{l,m}(\theta)$$

Let's first solve it for $m = 0$ to get solutions $P_l^0 = P_l$ and see if we can generalize from there. Because $\theta$ must be $\pi$-periodic, we make the Ansatz $P_l(\theta) = P_l(\cos\theta)$ and then we take the taylor expansion of $P_l$.

$$P_l(\cos\theta) = \sum_{n\in\mathbb{N}} \beta_l^n(\cos\theta)^n$$

and we will show in the exercise classes that solutions only exist for $l = 0, 1, \ldots$. The resulting functions are called the **Legendre Polynomials** are are given by

$$P_l(x) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}\, x^l}(x^2 - 1)^l$$

They also are orthogonal and are complete.
Now for arbitrary $m$, the solution is similar but not covered here. The solutions $P_{l,m}$ are called the **associated Legendre Polynomials**.

$$P_{l,m}(x) = \frac{1 - x^2)^{m/2}}{2^l l!} \frac{\mathrm{d}^{m+l}}{\mathrm{d}\, x^{m+l}}(Z^2 - 1)^l \overset{m\geq 0}{=} (1 - Z^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\, x^m} P_l(x)$$

Then, for the eigenfunctions $Y(\theta, \varphi)$ of $\hat{A}(\theta, \varphi)$ we take the Ansatz that a complete basis of solutions can be obtained by taking a product of the $P_{l,m}$ and the $\ell_m$. This gives us the **spherical harmonics** $Y_{l,m}(\theta, \varphi)$ given by

$$\boxed{Y_{l,m}(\theta, \varphi) = \frac{(-1)^m}{\sqrt{2\pi}} \sqrt{\frac{2l+1}{2} \frac{(l-m)!}{(l+m)!}} P_{l,m}(\cos\theta) e^{im\varphi}}$$

We argued that the $e^{im\varphi}$ are complete and that $P_{l,m}$ are also complete in their respective spaces. We have proved in MMP-I that the $Y_{l,m}$ for $m = -l, \ldots, l$ form an Orthonormal Basis of $L^2(\mathbb{S}^2)$, that is

$$\langle Y_{l'm'}, Y_{lm} \rangle = \int_{\mathbb{S}^2} d\Omega Y^*_{l'm'} Y_{lm} = \delta_{l'l} \delta_{m'm}$$

Completeness in particular means that

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{l,m}(\theta, \varphi) Y_{l,m}(\theta', \varphi') = \delta(\varphi - \varphi') \delta(\cos\theta' - \cos\theta)$$

Now that we have got the angular part down, we now want to solve for the radial part $R(r)$. Using the Ansatz

$$\Psi(r, \theta, \varphi) = \sum_{l,m} \frac{R_{lm}(r)}{r} Y_{lm}(\theta, \varphi)$$

And checking the differential equation $\nabla^2 \Psi = \lambda \Psi$, we get that $R_{lm}$ has to satisfy

$$\frac{\mathrm{d}^2}{\mathrm{d}\, r^2} \frac{R_{lm}(r)}{r} - \frac{l(l+1)}{r^3} = \lambda \frac{R_{lm}(r)}{r}$$

In the special case of a zero eigenvalue $\lambda = 0$, the differential equation becomes

$$\frac{1}{R} \frac{\mathrm{d}^2 R}{\mathrm{d}\, r^2} = \frac{l(l+1)}{r^2}$$

and with the ansatz $R(r) = r^\alpha$, we obtain two solutions

$$R_l(r) = A_l r^{-l} + B_l r^{l+1}$$

which means the the general solution of the **Laplace** differential equation

$$\nabla^2 \Psi(r, \theta, \varphi) = 0$$

is given by

$$\Psi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a_{l,m}r^{-l-1} + b_{lm}r^{l}\right)Y_{lm}(\theta,\varphi)$$

where we can solve for the coefficients $a_{lm}, b_{lm}$ with boundary conditions. (See MMP-I).

## 1.6 Multipole expansion

When we're given a charge distribution $\rho(\vec{x})$ contained to a small volume $V$, we want to find out the potential outside of the region $V'$.
Inutitively, the charge distribution approximately "looks like" a point-charge from a point $\vec{r}$ far away from $V'$.
Starting with the potential

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{V'} d^3\vec{x}\frac{\rho(\vec{x})}{|\vec{x}-\vec{r}|}$$

we expand the inverse distance $\frac{1}{|\vec{x}-\vec{r}|}$ in terms of the angle $\gamma$ and distance $\frac{x}{r}$ to get

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int_{V'} d^3\vec{x}\rho(\vec{x})x^l P_l(\cos\gamma)$$

Writing the vectors $\vec{r}, \vec{x}$ in spherical coordinates

$$\vec{r} =: (r,\theta,\varphi), \quad \vec{x} =: (x,\theta_x,\varphi_x)$$

we can show that

$$\cos\gamma = \frac{\vec{x}\cdot\vec{r}}{xr} = \cos\theta\cos\theta_x + \sin\theta\sin\theta_x\cos(\varphi-\varphi_x)$$

to do the integration, we use the identity

$$P_l(\cos\gamma) = \frac{4\pi}{1+2l}\sum_{m=-l}^{l} Y^*_{lm}(\theta_x,\varphi_x)Y_{lm}(\theta,\varphi)$$

so the Potential can be expressed as

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{1+2l}\frac{1}{r^{l+1}}\sum_{m=-l}^{l} q_{lm}Y_{lm}(\theta,\varphi)$$

, where we define the **multipole moments** $q_{lm}$ as

$$q_{lm} := \int_{V'} d^3\vec{x}Y^*_{lm}(\theta_x,\varphi_x)\rho(\vec{x})x^l$$

which characterisze the geometry of the charge distribution. For example

$$q_{00} = \int_{V'} d^3\vec{x}\rho(\vec{x})Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}\int_{V'} d^3\vec{x}\rho(\vec{x}) = \frac{Q}{\sqrt{4\pi}}$$

is proportional to the total charge. For $l = 1$, we have

$$q_{11} = -\sqrt{\frac{3}{8\pi}}(p_1 - ip_2), \quad q_{10} = \sqrt{\frac{3}{4\pi}}p_3, \quad q_{1-1} = \sqrt{\frac{3}{8\pi}}(p_1 + ip_2)$$

where

$$\vec{p} = (p_1, p_2, p_3) = \int d^3\vec{x}\vec{x}\rho(\vec{x})$$

is the **dipole moment**.
So the third order aproximation of the Potential is then given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \dots\right]$$

For $l = 2$ we will see the **quadrupole tensor**

$$Q_{ij} = \int d^3\vec{x}(x_i x_j - x^2\delta_{ij})\rho(\vec{x})$$

to express $q_{20}, q_{21}, q_{22}$. For higher $l$ the $q_{lm}$ have higher powers of $1/r$ in them and therefore contribute less and less to the potential.

# 2 Magnetostatics

In magentostatics, we have that the charges do not move and that the $\vec{E}$ and $\vec{B}$ field do not change over time

$$\frac{\partial\rho}{\partial t} = \frac{\partial\vec{J}}{\partial t} = \frac{\partial E}{\partial t} = \frac{\partial\vec{B}}{\partial t} = 0$$

so the Maxwell equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{B} = \frac{\vec{J}}{\epsilon_0 c^2} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

turn into the decoupled form

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{B} = \frac{\vec{J}}{\epsilon_0 c^2}, \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla}\times\vec{E} = 0$$

where the $\vec{E}$ and $\vec{B}$ field do not have an interdependence.
The main goals of this chapter will be the following

- What exactly is the current density $\vec{J}$?
- Look at the integral form of the above equations.
- Introduce the **vector potential** $\vec{A}$, leading to **gauge symmetry**
- Study the universality of magnetostatics and its applications
- Understand the duality of $\vec{E}$ and $\vec{B}$. In particular, we will see that magntic fields turn into electric fields when changing the reference frame and vice versa.

To give a definition of a current density $\vec{J}$, we consider a stream of charges moving in some direction $\vec{v}$. We then place a surface $d\vec{S}$ perpendicular to the direction and measure how much charge moves trough the surface

$$\boxed{\frac{\mathrm{d}\,Q}{\mathrm{d}\,t} =: \vec{J} \cdot d\vec{S}}$$

Now assume we have a volume $V$ and we we measure how many charges are escaping the volume. Well, that measurement should match what we measure for the current trough its surface $\partial V$, which is exactly $\vec{J} \cdot d\vec{S}$. Using the divergence theorem, we get

$$\int_V d^3x \frac{\partial \rho}{\partial t} = \int_{\partial V} \vec{J} \cdot d\vec{S} = -\int_V dV \vec{\nabla} \cdot \vec{J}$$

therefore, we get the **continuity equation**

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

This gives us the relation

$$\boxed{\vec{J} = \rho \vec{v}}$$

that suffices as an alternate definition of the charge density
The integral form of the maxwell equation

$$c^2 \vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\epsilon_0}$$

is the following

$$c^2 \int_{\partial V} d\vec{S} \cdot \left(\vec{\nabla} \times \vec{B}\right) = \int_{\partial} ...$$

Since the magnetic field has no divergence

$$\vec{\nabla} \cdot \vec{B} = 0$$

there exists a **vector potential** $\vec{A}$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

which satisfies

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = \frac{\partial}{\partial x_i} (\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \frac{\partial^2}{\partial_i \partial_j} A_k = 0$$

where $\epsilon_{ijk}$ is the total antisymmetric **Levi-Civita** tensor and we have used the fact the the contraction of an antisymmetric with a symmetric tensor is always zero.
Using hte Maxwell equations, we also end up with

$$\vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\epsilon_0 c^2} \implies \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \frac{\vec{J}}{\epsilon_0 c^2}$$

Let's see how the vector potential transforms under

$$\vec{A}(\vec{x}) \to \vec{A}'(\vec{x}') := \vec{A}(\vec{x}) + \vec{\nabla} f(\vec{x})$$

Since $\vec{F} = q\vec{v} \times \vec{B}$ stays invariant, we have that

$$\vec{B}' := \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times \vec{\nabla} f}_{=0} = \vec{B}$$

so the vector potential is only unique up to the gradient of a potential. This symmetry is called **gauge symmetry** and is important in the full standard model of particle physics.
This means that if we don't like how our $\vec{A}$ looks we can chose $f$ such that $\vec{A}'$ is nice.
We can for example require $\vec{\nabla} \cdot \vec{A}' = 0$, we we can chose $f$ such that

$$\vec{\nabla}^2 f = -\vec{\nabla} \cdot \vec{A} = \rho(x)$$

which is just a poission equation and is solvable. This then gives us the solution for $\vec{A}$:

$$\vec{A}(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\vec{y} \frac{\vec{J}(\vec{y})}{|\vec{x} - \vec{y}|}$$

The vector potential can be used to describe a magnetic dipole.

$$\vec{A}(\vec{r}) = \frac{1}{4\pi\epsilon_0 c^2} \left[ \frac{1}{|\vec{r}|} \int d^3\vec{x} \vec{J}(\vec{x}) + \frac{1}{|\vec{r}|^3} \int d^3\vec{x} \vec{J}(\vec{x})(\vec{x} \cdot \vec{r}) + \dots \right]$$

but in magnetostatics, there is no total current and so the first integral vanishes. This is also called **dipole aproximation**, so we then have

$$\vec{A}(\vec{r}) = \frac{1}{4\pi\epsilon_0 c^2} \left( \int d^3\vec{x} \frac{1}{2} \vec{x} \times \vec{J}(\vec{x}) \right) \times \frac{\vec{r}}{|\vec{r}|^3}$$

If we define the **magnetic moment** $\vec{\mu}$ to be

$$\boxed{\vec{\mu} := \frac{1}{2} \int d^3\vec{x} \left( \vec{x} \times \vec{J}(\vec{x}) \right)}$$

then we can simply find out the vector potential as

$$\boxed{\vec{A}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \vec{\mu} \times \frac{\vec{r}}{r^3}}$$

With some calculation, one can also derive the formula

$$\vec{B} = (\vec{\mu} \cdot \vec{\nabla}) \vec{\nabla} \frac{1}{r}$$

and up to first order, the force acting on an object $\Omega$ with current density $\vec{J}$ can be described as

$$\boxed{\vec{F} = \vec{\nabla}_x (\vec{\mu} \cdot \vec{B})|_{x=0}}$$

## 2.1 Relativity of $\vec{E}$ and $\vec{B}$ fields

Consider a wire in which electrons move with velocity $\vec{v}$ and assume that the wire is electrically neutral ($\rho_- = -\rho_+$).
Then we put an observer with charge $q$ at some distance $r$ from the wire that also moves with the same velocity $\vec{v}$ along the wire.
From the outside, we see current generating a magnetic field and an charge $q$ moving inside the magnetic field. So we see the Lorentz force

$$F = qvB = \frac{1}{2\pi\epsilon_0}\frac{qv}{c^2}\frac{I}{r} = \frac{qS}{2\pi\epsilon_0}\frac{\rho_-}{r}\frac{v^2}{c^2}$$

where $\rho_-$ is the charge density of the moving electrons and $S$ is the cross section of the wire.
Now we change the reference frame to that of the observing particle $q$. It doesn't see any moving charges so no magnetic fields. But then it would no longer experience the Lorentz force. How can this happen?
What happens is that the positive charges, which in the static reference frame cancel out the electrons now are moving in the opposite direction.
So because of the Lorentz contraction from special relativity, the charge density of the electrons $\rho'_-$ and the positive ones $\rho'_+$ no longer cancel each other out.
This means that the particle $q$ sees that the wire is charged. Let's calculate what charge the particle $q$ expects.
Given a section of the wire of volume $\Delta V$ we have that in the resting reference frame:

$$Nq = Q = \rho_{\text{rest}}\Delta V$$

but in the moving reference frame, we have

$$Q' = \rho_{\text{moving}}\Delta V_{\text{moving}}$$

Because total charge is invariant under Lorentz transformation, we have $Q = Q'$, so by the Lorentz contraction factor $\gamma$:

$$\rho_{\text{moving}} = \rho_{\text{rest}}\frac{\Delta V_{\text{rest}}}{\Delta V_{\text{moving}}} = \gamma\rho_{\text{rest}}$$

So because the electrons were resting and are moving in the frame of $q$ and the positive charges were moving and are now resting, we have

$$\rho' = \rho'_+ + \rho'_- = \gamma\rho_+ + \frac{1}{\gamma}\rho_- = \frac{\rho_-}{\sqrt{1-\frac{v^2}{c^2}}}\frac{v^2}{c^2}$$

So the new force that acts on the charge $q$ is now an electric force, instead of a magnetic one. And so we can explain the same physical effect with an electric field or a magnetic field.
This shows how the electric and magnetic field are really dual to each other under Lorentz transformation.
Now, using our vector potential $\vec{\nabla}\times\vec{A} = \vec{B}$, we can re-write the maxwell equations to read

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$

The maxwell equations also imply charge conservation. Given that

$$\epsilon_0 c^2\vec{\nabla}\times\vec{B} = \vec{J} + \frac{\partial\vec{E}}{\partial t}\epsilon_0$$
$$\implies 0 = \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

which when we integrate over the universe, we get

$$\begin{aligned} 0 &= \int_{\text{universe}} d^3\vec{x}\vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}\int_{\text{universe}} d^3\vec{x}\rho(\vec{x} \\ &= \underbrace{\int_{\partial\text{universe}} d\vec{S}\cdot\vec{J}}_{=0} + \frac{\partial Q_{\text{universe}}}{\partial t} \end{aligned}$$

where we say that there is no charge flowing "out of the universe" meaning twhich says that the total charge in the universe is constant.
Now, with the bac-cab identity for the curl of the curl,we can re-write

$$\begin{aligned} c^2\vec{\nabla}\times\vec{B} &= \frac{\vec{J}}{\epsilon_0} + \frac{\partial\vec{E}}{\partial t} \\ \implies c^2\left(\vec{\nabla}(\nabla\cdot\vec{A}) - \nabla^2\vec{A}\right) &= \frac{\vec{J}}{\epsilon_0} + \frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right) \end{aligned}$$

which gives us the relation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{A} = \frac{\vec{J}}{\epsilon_0 c^2} - \vec{\nabla}\left(\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\vec{A}\right)$$

which is not so nice as we simultanously need the scalar potential and the vector potential. We will see later how we can only write it in terms of the vector potential.
Using our previous equation $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$ and introducing the **d'Alembert Operator**

$$\boxed{\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2}$$

we will later use symmetry to get

$$\Box\vec{A} = \frac{\vec{J}}{\epsilon_0 c^2} \qquad \Box\Phi = \frac{\rho}{\epsilon_0}$$

## 2.2 Gauge invariance

The electric and magnetic field remain invariant under a simultanoues transformation known as **gauge transformation** given by

$$\begin{aligned} \Phi &\mapsto \Phi' = \Phi + \frac{\partial}{\partial t}f(\vec{x},t) \\ \vec{A} &\mapsto \vec{A}' = \vec{A} - \nabla f(\vec{x},t) \end{aligned}$$

and after the substitution we get

$$\Box\Phi' = \frac{\rho}{\epsilon_0} + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\Phi'}{\partial t} + \vec{\nabla}\cdot\vec{A}'\right) \Box\vec{A}' = \frac{\vec{J}}{\epsilon_0 c^2} - \vec{\nabla}\left(\frac{1}{c^2}\frac{\partial\Phi'}{\partial t} + \vec{\nabla}\cdot\vec{A}\right)$$

The idea then is that we choose $f(\vec{x},t)$ such that

$$\frac{1}{c^2}\frac{\partial\Phi'}{\partial t} + \vec{\nabla}\cdot\vec{A}' = 0$$

We will then find that the d'Alembert operator is consistent with the Lorentz transformation. So under the transformation

$$\vec{x} \mapsto \vec{x}' = \gamma(\vec{x} + \vec{v}t)$$
$$t \mapsto t' = \gamma(t - \frac{\vec{x} \cdot \vec{v}}{c^2}$$

To find such an $f$, we first look for wave solutions

$$f = f(\vec{n} \cdot \vec{x} + ct)$$

then if we set $u := \vec{n} \cdot \vec{x} + ct$, we see

$$\vec{\nabla} f = \frac{\partial f(u)}{\partial u} \cdot \vec{n} \quad \text{and} \quad \frac{1}{c}\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} \cdot 1$$

which means that its d'Alembert Operator evaluates to zero.

$$\Box f = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f - \nabla^2 f = \frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial u^2} \underbrace{u^2}_{=1} = 0$$

which is just saying that $f$ is indeed a wave.
If we write the electomagnetic fields as waves

$$\vec{E} = \vec{e} f(\vec{n} \cdot \vec{x} + ct), \quad \vec{B} = \vec{b} f(\vec{n} \cdot \vec{x} + ct)$$

then the maxwell equations show that they are perpendicular to each other, as

$$0 = \vec{\nabla} \cdot \vec{E} = \hat{e}_i \hat{n}_i \partial_u E = 0, \quad \vec{\nabla} \cdot \vec{B} = 0$$

each imply that

$$\vec{e} \cdot \vec{n} = 0 = \vec{b} \cdot \vec{n}$$

and by $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ we get that

$$|\vec{E}| = c|\vec{B}|$$

The maxwell equations tell us that electromagnetic waves propagate at the speed of light.
Another example of wave functions are spherical waves. There, we do not have have any angular dependence so

$$\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \underbrace{\frac{\vec{A}(\theta, \varphi)}{r^2}}_{=0} = \nabla^2$$

and if we assume then $\Box f = 0$, we get

$$\Box f = \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2} r\right] f(r,t) = 0$$

If we look for solutions of the form $f(r,t) = \frac{\psi(r,t)}{r}$ then the general solution is of the form

$$\psi(r,t) = A(r+ct) + B(r-ct) \implies f(r,t) = \frac{A(r+ct) + B(r-ct)}{r}$$

which we can interpret as ingoing and outgoing waves that decrease inamplitude as $r$ increases.
In many physical situations however is that we only have outgoing waves ($A = 0$).
Suppose we are given a charge distribution $\rho$ and we want to find out the potential that solves

$$\Box\Phi_{\text{sol}} = \frac{\rho}{\epsilon_0}$$

Note that by adding a potential with $\Box\Phi_{\text{free}} = 0$ we obtain another solution $\Phi' = \Phi_{\text{sol}} + \Phi_{\text{free}}$.
So the uniqueness of $\Phi$ must be found using the boundary conditions.
Given a green's function $G$ we can set

$$\Phi(\vec{x}, t) = \int_{-\infty}^{\infty} dt' \int d^3\vec{y} G(\vec{x} - \vec{y}, t - t') \frac{\rho(\vec{y}, t'}{\epsilon_0}$$

and indeed, $\Box\Phi = \frac{\rho}{\epsilon_0}$ But looking at the integral, we have to integrate the time variable over all time, so we need to "see into the future" to compute the integral. How can it be that the Green's function in the future affects the Potential in the past? This is very unphysical in that it violates causality.
To remedy this problem, we need to extend Maxwell's theorey such that it incorporates causality explicitly.
The way we do this is to take write

$$\frac{1}{c}\frac{\partial}{\partial t} = \lim_{\delta \to 0} \frac{1}{c}\frac{\partial}{\partial t} + \delta$$

and deform the d'Alembert operator $\Box$ to a modified operator

$$\Box_\delta = \left(\frac{1}{c}\frac{\partial}{\partial t} + \delta\right)^2 - \nabla^2$$

and find a solution $\vec{A}_\delta(\vec{x}, t)$ for the equation

$$\Box_\delta \vec{A}_\delta(\vec{x}, t) = \frac{1}{\epsilon_0 c^2}\vec{J}$$

and then take the limit

$$\vec{A}(\vec{x}, t) := \lim_{\delta \to 0} \vec{A}_\delta(\vec{x}, t)$$

It turns out that taking this limit will eliminate the need to "see into the future" to solve for the potential
With the Fourier transform

$$\tilde{f}(\vec{k}) = \frac{1}{\sqrt{2\pi}} \int d\vec{x} e^{-i\vec{k}\cdot\vec{x}} f(\vec{x})$$

and its inverse transform, the simple Green's function can be written as

$$G(\Delta\vec{x}, \Delta t) = \frac{1}{(2\pi)^4} \int d^3\vec{k} dE e^{-1(Ec\Delta t, -\vec{k}\cdot\Delta\vec{x}} \tilde{G}(\vec{k}, E)$$

which, gives us

$$\tilde{G}(\vec{k}, E) = -\frac{c}{E^2 - k^2}$$

and when we enter the $\delta$ corrections we obtain

$$\boxed{\tilde{G}(\vec{k},E) = \frac{-c}{(E+i\delta)^2 - k} = \frac{c}{2k}\left[\frac{1}{E+k+i\delta} - \frac{1}{E-k+i\delta}\right]}$$

so given the fourier transform, we can recover the Green's function as

$$G(\Delta\vec{x},\Delta t) = \frac{c}{2(2\pi)^4}\int d^3\vec{k}dE\frac{1}{k}e^{-i\left((E+i\delta)c\Delta t - \vec{k}\cdot\Delta\vec{x}\right)}\left[\frac{1}{E+k+i\delta} - \frac{1}{E-k+i\delta}\right]$$

to calculate the integral, we use spherical coordinates and end up with

$$G(\Delta\vec{x},\Delta t) = \frac{ic}{2(2\pi)^3\Delta x}\int dk\left[e^{-ik\Delta x} - e^{+ik\Delta x}\right]\int dE e^{-i(E+i\delta)c\Delta t}\left[\frac{1}{E+k+i\delta} - \frac{1}{E-k+i\delta}\right]$$

We will do the integral

$$\int_{-\infty}^{\infty} dE\frac{e^{-ic\Delta tE}}{(E+i\delta)^2 - k^2}$$

not over $(-\infty,\infty)$, but over a closed semi-circle. For $\Delta t < 0$ we use the upper complex half-plane, and for $\Delta t > 0$ we use the lower half-plane to make the integrand go to zero.
On the upper half-plane, the integrand has no poles and therefore the integral is zero for negative $\Delta t$, i.e

$$G(\Delta\vec{x},\Delta t) = 0 \quad \text{for} \quad \Delta t < 0$$

But on the upper complex half-plane. We have two poles at $E = \pm k - i\delta$. With the residue theorem, we obtain

$$\begin{aligned} G(\Delta\vec{x},\Delta t) &= \frac{c}{2(2\pi)^2\Delta x}\int_{-\infty}^{\infty} dk e^{-ik(c\Delta t - \Delta x)} - e^{+ik(c\Delta t + \Delta x} \\ &= [...] \\ &= \frac{c}{4\pi|\vec{x}-\vec{y}|}\delta(c\delta t - \Delta x) \quad \text{for} \quad \Delta t > 0 \end{aligned}$$

So combining the two cases, we arrive at the compact formula

$$\boxed{G(\vec{x}-\vec{y},t-t') = \frac{1}{4\pi|\vec{x}-\vec{y}|}\delta\left(t - t' - \frac{\vec{x}-\vec{y}}{c}\right)\Theta(t-t')}$$

so, doing the integral for $\Phi$, we get

$$\begin{aligned} \Phi(\vec{x},t) &= \frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\int_{-\infty}^{\infty} t'\rho(\vec{y},t')\frac{1}{|\vec{x}-\vec{y}|}\delta\left(t - t' - \frac{\vec{x}-\vec{y}}{c}\right)\Theta(t-t') \\ &= \frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\frac{1}{|\vec{x}-\vec{y}|}\rho\left(\vec{y}, t - \tfrac{|\vec{x}-\vec{y}|}{c}\right) \end{aligned}$$

Now, if the charge density $\rho$ is time-independent, then we recover the usual result

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}d^3\vec{y}\frac{\rho(\vec{y},t)}{|\vec{x}-\vec{y}|}$$

Looking again at

$$\Phi(\vec{x},t)\frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\frac{1}{|\vec{x}-\vec{y}|}\rho\left(\vec{y},t-\tfrac{|\vec{x}-\vec{y}|}{c}\right)$$

what this is saying is that in order to compute the potential, we do not look at whatwhat the charge density is now, but need to know what the charge distribution was back when light started to travel from the location $\vec{y}$ was.
This seems very intuitive, because the information that the charge density has changed only arrives after $\frac{|\vec{x}-\vec{y}|}{c}$ time has passed.
So by defining the **time retardation**

$$t_{\text{ret}} := t - \frac{|\vec{x}-\vec{y}|}{c}$$

we can write more compactly

$$\boxed{\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\frac{\rho(\vec{y},t_{\text{ret}})}{|\vec{x}-\vec{y}|}}$$

and likewise, we get the **retarded vector potential**

$$\boxed{\vec{A}(\vec{x},t) = \frac{1}{4\pi\epsilon_0 c^2}\int d^3\vec{y}\frac{\vec{J}(\vec{y},t_{\text{ret}}}{|\vec{x}-\vec{y}|}}$$

and the Green's function becomes

$$G(\vec{x}-\vec{y},t-t') = \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}\delta(t-t_{\text{ret}})$$

If we combine the vector and the scalar potential into a 4-vector, we can write even more compactly

$$\begin{pmatrix}\Phi(\vec{x},t)\\ \vec{A}(\vec{x},t)\end{pmatrix} = \frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\frac{1}{|\vec{x}-\vec{y}|}\begin{pmatrix}\rho(\vec{y},t_{\text{ret}}\\ \frac{1}{c}\vec{J}(\vec{y},t_{\text{ret}})\end{pmatrix}$$

By using the identity

$$2\delta(a^2-|b|^2)\Theta(a) = 2\delta((a-|b|)(a+|b|))\Theta(a) = \frac{\delta(a-|b|}{a}\Theta(a) = \frac{\delta(a-|b|}{|b|}$$

we can re-write the Green's function as

$$\boxed{G(\vec{x}-\vec{y},t-t') = \frac{1}{2\pi}\delta\left((t-t')^2-\frac{|\vec{x}-\vec{y}|^2}{c^2}\right)\Theta(t-t')}$$

So when we calculate $\Phi$ using the new formula for the Green's function, we get

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0 c}\int d^3\vec{y}dt'\delta\left((t-t')^2-\frac{|\vec{x}-\vec{y}|^2}{c^2}\right)\rho(\vec{y},t)\Theta(t-t')$$

Let's use these equations in an example

## 2.3 Potential of a moving charge with constant velocity

We consider a charge $q$ moving with constant velocity $\vec{v}$ and want to calculate $\Phi(\vec{x},t)$. Assume that at time $t'=0$, it is at positoin $\vec{y}=0$. The charge density is then given by

$$\rho(\vec{y},t') = q\delta(\vec{y}-\vec{v}t')$$

so substituting into the equation for the potential, we get

$$\begin{aligned}\Phi(\vec{x},t) &= \frac{1}{2\pi\epsilon_0 c}\int d^3\vec{y}dt'\delta\left((t-t')^2 - \frac{|\vec{x}-\vec{y}|^2}{c^2}\right)q\delta(\vec{y}-\vec{v}t')\Theta(t-t') \\ &= \frac{q}{2\pi\epsilon_0 c}\int dt'\Theta(t-t')\delta\left((t-t')^2 - \frac{|\vec{x}-\vec{v}t|^2}{c^2}\right)\end{aligned}$$

So we now need to find the zeros of the argument in the delta function. By decomposing $\vec{x}$ into it's paralell $\vec{x}_\parallel$ and perpendicular $\vec{x}_\perp$ components with respect to the velocity $\vec{v}$, we can find the solutions to

$$0 = (t-t')^2 - \frac{|\vec{x}-\vec{v}t'|^2}{c^2} = t'^2 - 2t't + t^2 - \frac{(x_\parallel - vt')^2 + x_\perp^2}{c^2}$$

by defining the **boosted variables**

$$x_b = \gamma(x_\parallel - vt), \quad t_b = \gamma\left(t - \frac{x_\parallel v}{c^2}\right) \quad \text{for} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

and introducing the quantity

$$\tau^2 = c^2t^2 - (x_\parallel^2 + x_\perp^2) = c^2t_b^2 - (x_b^2 + x_\perp^2)$$

we can re-write the equation for $t'$ from before to

$$\frac{t'^2}{\gamma^2} - 2\frac{t'}{\gamma} + \frac{\tau^2}{c^2} = 0$$

which is just a quadratic. The discriminant is

$$\Delta = \frac{4(t_b^2 - \tau^2/c^2)}{\gamma^2} = \frac{4r_b^2}{\gamma^2 c^2}, \quad \text{for} \quad r_b^2 = x_b^2 + x_{\perp^2}$$

so the solutions for $t'$ are

$$t_\pm = \gamma\left(t_b \pm \frac{r_b}{c}\right)$$

so we can now do the $t'$ for $\Phi$ to get

$$\Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\frac{\gamma}{r_b r_b}$$

similarly, for the vector potential, we obtain

$$\vec{A}(\vec{x},t) = \frac{\vec{v}}{c^2}\Phi(\vec{x},t)$$

Comparing this to the reference frame where the charge is at rest, we get

$$\Phi(\vec{x},t)|_{\text{rest}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left[x_\parallel^2 + x_\perp^2\right]^{1/2}}, \quad \vec{A}(\vec{x},t)|_{\text{rest}} = 0$$

so we see that coordinates tranform according to the Lorentz transformations we know form special relativity:

$$x_\parallel \mapsto \tilde{x}_\parallel = \gamma(x_\parallel - vt), \quad x_\perp \mapsto \tilde{x}_\perp = x_\perp, \quad t \mapsto \tilde{t} = \gamma t$$

The equations become more concise to write if we combine time and space, and scalar and vector potential to **4-vectors**.

$$A^\mu := \begin{pmatrix}\Phi \\ \vec{A}\end{pmatrix}, \quad x^\mu = \begin{pmatrix}ct \\ \vec{x}\end{pmatrix}$$

# 3 Special Relativity

One motivation for special relativity was Galileo's observation that the law's of physics (such as $\vec{F} = m\vec{a}$) stayed the same when transforming between intertial reference frames, viewed as vectors in $\mathbb{R}^3$. Even though the individual components tranform in some complex manner under galileian transformation, the transformation preserves the strucuture of the vectors in $\mathbb{R}^3$.
The invariants under Galilean transformations were the spatial distance $|\vec{x} - \vec{y}|$ and temporal distance $t - t'$.
What we will be doing is to combine the space and time components in a 4-vector and use the **spacetime distance**

$$\boxed{\Delta s^2 := c^2(t-t') - |\vec{x} - \vec{x}'|^2}$$

as our invariant of Lorentz Transformations to derive all other equations.
Defining the **metric tensor** $g_{\mu\nu}$ as

$$g_{\mu\nu} = \begin{pmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}$$

then we define Lorentz Transformations as any (affine linear) transformation of the form

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu + \rho^\mu \quad \text{for some} \quad x^\mu = (x_0 := ct, x_1, x_2, x_3)^T, \quad \rho^\mu = \text{const}$$

such that

$$\boxed{\Lambda^\mu_\rho \Lambda^\nu_\sigma g_{\mu\nu} = g_{\rho\sigma}}$$

## 3.1 Proper time

The differential form of the spacetime interval is known as the **proper time interval**

$$\boxed{d\tau^2 = d\tau^2 := c^2dt^2 - d\vec{x}^2 = g_{\mu\nu}dx^\mu dx^\nu}$$

Then under a Lorentz transformation

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu + \rho^\mu \implies d\tilde{x}^\mu = \Lambda^\mu_\nu dx^\nu$$

we can show that the proper time interval is preserved.

$$\begin{aligned} d\tilde{s}^2 &= g_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu \\ &= g_{\mu\nu} \left(\Lambda^\mu_\rho dx^\rho\right) \left(\Lambda^\nu_\sigma dx^\sigma\right) \\ &= \left(g_{\mu\nu}\Lambda^\mu_\rho \Lambda^\nu_\sigma\right) dx^\rho dx^\sigma \\ &= g_{\rho\sigma} dx^\rho dx^\sigma = d\tau^2 \end{aligned}$$

As a consequence, it follows that the speed of light is the same in all inertial reference frames because if something is moving at the speed of light, then

$$|\frac{d\vec{x}}{dt}| = c \implies d\tau^2 = c^2 dt^2 - d\vec{x}^2 = 0$$

and so under a Lorentz transformation, it follows formthe perservation of the proper time interval that

$$d\tilde{s}^2 = d\tau^2 = 0 \implies |\frac{d\vec{\tilde{x}}}{d\tilde{t}}| = c$$

Moreover, we can also show that the Lorentz transformations are the only non-singluar transformations that preserve the proper time intervals. Assume that there exists change of reference $x^\mu \mapsto \tilde{x}^\mu$ such that $d\tau^2 = d\tilde{s}^2$. Then we get

$$\begin{aligned} g_{\rho\sigma} dx^\rho dx^\sigma &= g_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu \\ \implies g_{\rho\sigma} dx^\rho dx^\sigma &= g_{\mu\nu} \frac{\partial x^\mu}{\partial x_\rho} \frac{\partial x^\nu}{\partial x_\sigma} dx^\rho dx^\sigma \\ \implies g_{\rho\sigma} &= g_{\mu\nu} \frac{\partial x^\mu}{\partial x_\rho} \frac{\partial x^\nu}{\partial x_\sigma} \end{aligned}$$

If we differentiate it with respect to $dx^\epsilon$, we get

$$0 = g_{\mu\nu} \left[ \frac{\partial^2 \tilde{x}^\mu}{\partial x^\epsilon \partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma} + \frac{\partial^2 \tilde{x}^\mu}{\partial x^\epsilon \partial x^\sigma} \frac{\partial \tilde{x}^\nu}{\partial x^\rho} \right]$$

To this we add the same equation with $\epsilon$ and $\rho$ reversed and subtract the equation with $\epsilon$ and $\sigma$ reversed, we get

$$0 = 2 g_{\mu\nu} \frac{\partial^2 \tilde{x}^\mu}{\partial x^\epsilon \partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma}$$

So from the relations

$$\frac{\partial \tilde{x}^\mu}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} = \delta_{\mu\nu} \implies \frac{\partial^2 \tilde{x}^\mu}{\partial x^\epsilon \partial x^\rho} = 0$$

the transformation must be of linear kind.
Given a transformation $\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu + \rho^\mu$ we the **velocity vector**

$$\boxed{\vec{v} = v^i := \frac{d\tilde{x}^i}{d\tilde{t}} = c\frac{d\tilde{x}^i}{d\tilde{x}^0}}$$

## 3.2 The Poincare group

The set of all Lorentz transformations

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu + \rho^\mu \quad \text{with} \quad g_{\mu\nu}\Lambda^\mu_\nu\Lambda^\nu_\sigma = g_{\rho\sigma}$$

forms a group, known as the inhomogeneous Lorentz group under composition.
This group has some non-trivial subgroups. For example, the subset of transformations

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu + \rho^\mu$$

with $\rho^\mu = 0$ is a subgroup.
Or the set of transformations with

$$\det\Lambda = 1, \quad \Lambda^0_0 \geq 1$$

is a subgroup known as the **proper Lorentz transformations** as they are the ones which are physical. The condition $\Lambda^0_0 \geq 1$ means that the transformation preserves the flow of time, since $d\tilde{t} = \Lambda^0_0 dt \geq dt$
Another important subgroup is the **subgroup of rotations**. They are the ones of the Form

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \quad \text{for some} \quad R \in O(3)$$

For the proper Lorentz transformations it follows from $g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma = g_{\rho\sigma}$ that for $\rho = \sigma = 0$ we have

$$(\Lambda^0_0)^2 - (\Lambda^i_0)^2 = 1$$
$$\Longrightarrow \Lambda^0_0 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} =: \gamma$$

and thus we also find

$$\Lambda^i_0 = \gamma\frac{v^i}{c}$$

So an example of a proper Lorentz transformation is given by

$$\Lambda^\mu_\nu : \Lambda^0_0 = \gamma, \quad \Lambda^i_0 = \Lambda^0_i = \gamma\frac{v^i}{c}, \quad \Lambda^i_j = \delta^i_j + (\gamma-1)\frac{v^i v^j}{v^2}$$

## 3.3 Time dilation

Consider a clock which is at rest for an observer in reference frame $O$. Between two clock ticks, the clock will have moved in the space-timeinterval

$$d\vec{x} = 0, \quad dt = \Delta t$$

and its proper time interval is

$$d\tau^2 = (c^2dt^2 - d\vec{x}^2) = c^2(\Delta t)^2$$

A second observer $O'$ sees the clock moving with velocity $\vec{v}$. Then two ticks of the clock will be separated by a space-time interval

$$d\vec{\tilde{x}} = \vec{v}d\tilde{t}, \quad d\tilde{t} = \Delta\tilde{t}$$

the proper-time interval in the new reference frame is then

$$d\tilde{s}^2 = c^2 d\tilde{t}^2 - d\vec{x}^2 = c^2(\Delta\tilde{t})^2\left(1 - \frac{v^2}{c^2}\right)$$

and since the proper time is the same in both reference frames, we get

$$\boxed{\Delta\tilde{t} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\Delta t}$$

Now, imagine that the clock emits light with frequency

$$\omega = \frac{2\pi}{\Delta t}$$

The observer $O'$ will see the time interval as $\Delta t' = \gamma\Delta t$, and in the same time period, the distance to the light sources increases by $v_r\Delta t'$, where $v_r$ is the component of the velocity in the direction of he light. Observer $O'$ measures the time between two wave fronts $\Delta t_0$ as

$$c\Delta t_0 = c\Delta t' + v_r\Delta t' \implies \Delta t_0 = \gamma(1 + \frac{v_r}{c})\Delta t$$

so the perceived frequencey for $O'$ is

$$\omega_0 = \frac{2\pi}{\Delta t_0} = \omega\frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v_r}{c}}$$

and in the special case where the light source is moving directly away from $O'$, $v_r = v$ and so

$$\omega_0 = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}\omega$$

Consider the case, where we know that a star has a lot of hydrogen. Looking at the frequency spectrum of the emitted light, we would expect there to be a peak where the characteristic frequency for hydrogen is. Then we compare the theoretical peak with the measurements and see how much the hydrogen peak has moved. This let's us calculate how fast the star is moving away/towards us.

## 3.4 Relativistic force

The classical formula for a force exerted on a particle with mass $m$ is the well-known $\vec{F} = m\frac{d^2\vec{x}}{dt^2}$.
For the relativistic analogue, we define the **relativistic force** acting on a particle as

$$\boxed{f^\mu = mc^2\frac{d^2x^\mu}{d\tau^2} = g_{\rho\sigma}\frac{d^2x^\mu}{dx^\rho dx^\sigma}}$$

and check if we can recover Newton's equations when in a stationary reference frame. Indeed, if the particle is at rest, then the proper time interval only has it's temporal part and we get

$$d\tau = cdt \implies f^i_{\text{rest}} = m\frac{d^2x^i}{dt^2} = F^i_{\text{Newton}}$$

Now in a different frame of reference, where the particle is moving with velocity $\vec{v}$, we get

$$f^\mu = \Lambda^\mu_\nu f^\nu_{\text{rest}}$$

so in particular, the time component is

$$f^0 = \Lambda^0_\nu f^\nu_{\text{rest}} = 0 + \Lambda^0_i f^i_{\text{rest}} = \gamma \frac{\vec{v} \cdot \vec{F}_{\text{Newton}}}{c}$$

and the spatial components are

$$f^i = \Lambda^i_\nu f^\nu_{\text{rest}} = \Lambda^i_j f^j_{\text{rest}}$$

So as a whole, we get

$$\vec{f} = \vec{F}_{\text{Newton}} + (\gamma - 1)\vec{v}\frac{\vec{v} \cdot \vec{F}_{\text{Newton}}}{v^2}, \quad f^0 = \frac{\vec{v}}{c} \cdot \vec{f}$$

If Newtonian mechanics, we can calculate the trajectory $\vec{x}(t)$ as a function of time by solving

$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{m}\vec{F}(\vec{x}, t)$$

But in special relativity, if we want to find the trajectory as a function of proper time $x^\mu(\tau)$ we cannot simply solve

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{m}f^\mu$$

because we would need to know what $f^0$ is, but for that we need to already know one of the other three equations.
We resolve this by defining

$$\Omega := g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

and requiring that this be an invariant under change of references. Missing 15 mins.

## 3.5 Energy and Momentum

Two very important invariants in classical mechanics are energy and momentum. Recall that momentum is defined as

$$\vec{p} = m\frac{d\vec{x}}{dt}$$

the relativistic four-vector analogue is

$$\boxed{p^\mu := mc\frac{dx^\mu}{d\tau}}$$

and under a Lorentz transformation, the momentum transforms as follows

$$\tilde{p}^\mu = \Lambda^\mu{}_\nu p^\nu$$

Since the time interval can be written as

$$d\tau = \sqrt{c^2dt^2 - d\vec{x}^2} = cdt\sqrt{1 - \frac{\vec{v}^2}{c^2}} = \frac{cdt}{\gamma}$$

we find that the time component is given by

$$p^0 = mc\frac{dx^0}{d\tau} = \frac{mc^2dt}{\frac{c\ dt}{\gamma}} = m\gamma c$$

and the space components are

$$p^i = mc\frac{dx^i}{d\tau} = m\gamma\frac{dx^i}{dt} = m\gamma v^i$$

so for small velocities ($\gamma \sim 1$) we recover the classical definition of momentum.
In the second order approximation of $\gamma$

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}(\frac{v^4}{c^4})$$

, the time component becomes

$$p^0 = mc + \frac{1}{2c}mv^2 + \dots$$

Notice that we see the kinetic energy $\frac{1}{2}mv^2$ in the second order term.
Using this, we define the **relativistic energy** of a particle with

$$\boxed{E := cp^0 = m\gamma c^2 = mc^2 + \frac{1}{2}mv^2 + \dots}$$

where the term $mc^2$ is also called **rest energy** of the particle.
We are implicitly making a bet that this quantity does behave like we expect energy to behave, so we need to experimentally verify that this is the case.
Substituting the relation

$$\vec{p}^2 = m^2\gamma^2\vec{v}^2$$

into the equation $E = cp^0$, we obtain the **energy-momentum relation** of special relativity:

$$\boxed{E^2 = c^2\vec{p}^2 + m^2c^4}$$

from which we can derive all sorts of useful relations.

## 3.6 The inverse of a Lorentz transformation

Recall the definition of the metric tensor

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

Note that it is its own inverse. But instead of simply writing $g^{-1}_{\mu\nu} = g_{\mu\nu}$, we will instead introduce a new piece of notation, where we raise the indices to define its inverse

$$g^{\mu\nu} := g^{-1}_{\mu\nu} = \text{diag}(1, -1, -1, -1) \quad (= g_{\mu\nu})$$

this might look like it unnecessary complicates things because we have two different symbols $(g_{\mu\nu}, g^{\mu\nu})$ for the exact same matrix, but it turns out that the concept of raising and lowering indices is very useful.
Before moving on, note that the fact that $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ can be compactly written by saying that the two matrices satisfy the Kronecker relation

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\nu}$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker delta.
If we wish to find the inverse of a Lorentz transformation associated to a boost $\vec{v}$, we would physically expect that the inverse is the transformation associated to the inverse boost $-\vec{v}$, so

$$(\Lambda^{\mu}_{\nu})^{-1}(\vec{v}) = \Lambda^{\mu}_{\nu}(-\vec{v})$$

We can easily prove this, but as before, we will denote the inverse of the matrix with $\Lambda_{\mu}{}^{\nu} := (\Lambda^{-1})^{\mu}{}_{\nu}$.
It is important to regocnize that the horizontal positions of the indices is important. From now on, we will pay attention to index ordering.
One can verify that the inverse is given by

$$\Lambda_{\mu}{}^{\nu} = g_{\mu\rho} g^{\nu\sigma} \Lambda^{\rho}{}_{\sigma}$$

Indeed, this does satisfy the Kronecker relation since

$$\Lambda^{\mu}{}_{\lambda} \Lambda_{\mu}{}^{\nu} = g_{\mu\rho} g^{\nu\sigma} \Lambda^{\rho}{}_{\sigma} \Lambda^{\mu}{}_{\lambda} = g_{\sigma\lambda} g^{\nu\sigma} = \delta^{\lambda}{}_{\nu}$$

Looking at the components of the inverse, we see

$$\Lambda_0{}^0(\vec{v}) = \gamma, \quad \Lambda_i{}^0(\vec{v}) = \Lambda_0{}^i(\vec{v}) = -\gamma \frac{v^i}{c},$$
$$\Lambda_i{}^j(\vec{v}) = \delta^j_i + \frac{v^i v^j}{v^2}(\gamma - 1)$$

which does match our physical prediction $\Lambda_{\mu}{}^{\nu}(\vec{v}) = \Lambda^{\mu}{}_{\nu}(-\vec{v})$.

## 3.7 Tensors

Recall that in a change of reference

$$x^{\mu} \mapsto \tilde{x}^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + \rho^{\mu}$$

many of the quantities we defined such as $f^{\mu}, p^{\mu}, dx^{\mu}$ etc. transformed in the same way, namely

$$f^{\mu} \mapsto \tilde{f}^{\mu} = \Lambda^{\mu}{}_{\nu} f^{\nu}$$
$$p^{\mu} \mapsto \tilde{p}^{\mu} = \Lambda^{\mu}{}_{\nu} p^{\nu}$$
$$dx^{\mu} \mapsto d\tilde{x}^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu}$$

We will use this as our definition of what a vector is.

A **contra-variant 4-vector** is any set of four components that transforms according to the rule

$$V^\mu \mapsto \tilde{V}^\mu = \Lambda^\mu{}_\nu V^\nu$$

Not everything is such a contavariant vector, since for example $x^\mu$ transforms differently.
There is another important class of objects that transform not with $\Lambda^\mu{}_\nu$, but rather with $(\Lambda^{-1})^\mu_\nu$.
One example is the (covariant) derivative $\frac{\partial}{\partial x^\mu}$ which transforms as

$$\frac{\partial}{\partial x^\mu} \mapsto \frac{\partial}{\partial \tilde{x}^\mu} = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial}{\partial x^\rho} = \Lambda^\rho_\mu \frac{\partial}{\partial x^\rho}$$

so, a **covariant 4-vector** is any set four components that transform to the rule

$$U_\mu = \Lambda^\nu_\mu U_\nu = g_{\mu\rho}g^{\nu\sigma}\Lambda^\rho_\sigma U_\nu$$

and we will use lower indices to denote covariant objects.
The contra- and covariant vectors play the role of a vector space and the dual space from linear algebra.
Notably, for every contravariant vector $U^\mu$, there exists a dual vector $U_\mu$ and vice versa.
We can obtain one from another by **raising/lowering** indices:

$$\boxed{U_\mu = g_{\mu\nu}U^\nu \quad \text{and} \quad U^\mu = g^{\mu\nu}U_\nu}$$

the above transformation is an isomorphism for finite dimensional vector spaces, where we naturally identify the double dual with the vector space itself:

$$W \longrightarrow W^* \longrightarrow (W^*)^* = W$$

$$U^\nu \xrightarrow{g_{\mu\nu}} U_\mu \xrightarrow{g^{\mu\nu}} U^\nu$$

and one can verify that the dual of a covariant vector is a contra-variant one and vice versa.
Similar to how a dual vector is a linear map from the vector space to the field $k$, we define the