

Measure Theory– Lecture Notes

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About

Lecture notes taken from the Mass und Integral lecture given by Dr. Francesca Da Lio during Spring Semester 2021 at ETH Zürich.

The exam will be a 20 minute **oral exam**. It will consist of two or three questions where we have to prove some results.

The lecture slides and a script is provided at the professors home page <https://people.math.ethz.ch/~fdalio/MASSundINTEGRALFS21>.

1 Measure spaces

If we naively try to define a notion of measure that has some intuitive properties, we can run into some problems that give paradoxical results. The **Riemann Integral** we saw in Analysis I/II also had some drawbacks of not being general enough. We can use measure theory to define a better definition of the integral.

1.1 Algebras and σ -Algebras

From now on, let X denote a non-empty set.

Definition 1.1. For a sequence of subsets $(A_n)_{n=1}^{\infty}$ in $\mathcal{P}(X)$. We define

$$\begin{aligned}\limsup_{n \rightarrow \infty} A_n &:= \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \\ \liminf_{n \rightarrow \infty} A_n &:= \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m\end{aligned}$$

And if they are equal, we say that the sequence $(A_n)_{n=1}^{\infty}$ converges to its limit $\lim_{n \rightarrow \infty} A_n$.

Informally, the \limsup consists of elements of X that occur in infinitely many A_n , whereas the \liminf consists of elements that occur for all but finitely many A_n .

Remark 1.2.

- (a) $\liminf_{n \rightarrow \infty} A_n \subseteq \limsup_{n \rightarrow \infty} A_n$
- (b) If $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

(c) If $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

The similarity in names with the lim sup and lim inf from Analysis can be seen using the characteristic function

$$\mathbb{1}_A : X \rightarrow \{0, 1\}$$

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

It holds that

$$\limsup_{n \rightarrow \infty} A_n = A \iff \limsup_{n \rightarrow \infty} \mathbb{1}_{A_n} = \mathbb{1}_A$$

$$\liminf_{n \rightarrow \infty} A_n = A \iff \liminf_{n \rightarrow \infty} \mathbb{1}_{A_n} = \mathbb{1}_A$$

where the lim sup and lim inf on the left are as in Definition 1.1 and the ones on the right are the ones from Analysis.

Definition 1.3 (Algebras of sets). A collection of subsets $\mathcal{A} \subseteq \mathcal{P}(X)$ is called an **algebra in X** if

- (a) $X \in \mathcal{A}$
- (b) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$
- (c) $A \in \mathcal{A} \implies A^c \in \mathcal{A}$

An algebra \mathcal{E} is called a **σ -algebra**, if for any sequence $(A_n)_{n=1}^{\infty}$ in \mathcal{E} we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{E}$

Note that using the De Morgan's identity

$$\left(\bigcup_{n=1}^{\infty} A_n \right)^c = \bigcap_{n=1}^{\infty} A_n^c$$

we can see that algebras (σ -algebras) are stable under finite (infinite) intersections as well.

Definition 1.4. For a collection of sets $\mathcal{K} \subseteq \mathcal{P}(X)$, the intersection of all σ -algebras containing \mathcal{K} forms again a σ -algebra.

We call this the σ -algebra **generated by \mathcal{K}** and it is the smallest σ -algebra that contains \mathcal{K} .

The algebra generated by the open sets of a topology is called the **Borel σ -Algebra** of X , denoted $\mathcal{B}(X)$.

1.2 Measures

Definition 1.5. Let \mathcal{A} be an Algebra on X and $\mu : \mathcal{A} \rightarrow [0, \infty]$. We say that μ is

- **additive**, if for any *finite* family of disjoint sets $A_1, \dots, A_n \in \mathcal{A}$

$$\mu \left(\bigsqcup_{k=1}^n A_k \right) = \sum_{k=1}^n \mu(A_k)$$

- **σ -additive**, if for any *countable* family of disjoint sets $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ such that $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{A}$

$$\mu \left(\bigsqcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \mu(A_k)$$

- A **pre-measure**, if it is σ -additive and satisfies $\mu(\emptyset) = 0$.

Remark 1.6. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of sets in \mathcal{A} such that their union is again in \mathcal{A} .

- (a) If μ is additive, then it is monotone with respect to inclusion, i.e. $A \subseteq B \implies \mu(A) \leq \mu(B)$.
 (b) If μ is additive and the sets A_k are mutually disjoint, then

$$\mu \left(\bigsqcup_{k=1}^{\infty} A_k \right) \geq \sum_{k=1}^{\infty} \mu(A_k)$$

- (c) If μ is σ -additive, then it is also **σ -subadditive**, which means that for any sequence $(A_n)_{n \in \mathbb{N}}$ in \mathcal{A} with $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

$$\mu \left(\bigcup_{k=1}^{\infty} A_k \right) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

To see this, we can define the mutually disjoint sets

$$B_1 = A_1, \quad B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k \in \mathcal{A}$$

Since $\bigsqcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$ and $\mu(B_k) \leq \mu(A_k)$ we have

$$\mu \left(\bigcup_{k=1}^{\infty} A_k \right) = \mu \left(\bigsqcup_{k=1}^{\infty} B_k \right) = \sum_{k=1}^{\infty} \mu(B_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

It follows immediately from (b) and (c) that

$$\mu \text{ is additive and } \sigma\text{-subadditive} \iff \mu \text{ is } \sigma\text{-additive}$$

Example 1.7. Not all additive functions are σ -additive. For $X = \mathbb{N}$ and

$$\mathcal{A} = \{A \in \mathcal{P}(X) \mid A \text{ is finite or } A^c \text{ is finite}\}$$

the function $\nu : \mathcal{A} \rightarrow [0, \infty]$ with $\nu(\emptyset) = 0$ and

$$\nu(A) = \begin{cases} \sum_{n \in A} \frac{1}{2^n} & \text{if } A \text{ is finite} \\ \infty & \text{if } A^c \text{ is finite} \end{cases}$$

is additive but not σ -additive because we can take the sequence

$$A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}, \dots, A_n = \{n\}, \dots$$

which is a sequence of mutually disjoint sets satisfying

$$\begin{aligned} \nu(A_1) &= \frac{1}{2}, \nu(A_2) = \frac{1}{4}, \dots, \nu(A_n) = \frac{1}{2^n} \\ \implies \nu \left(\bigsqcup_{n=1}^{\infty} A_n \right) &= \nu(\mathbb{N}) = \infty \not\leq \sum_{n=1}^{\infty} \nu(A_n) = 1 \end{aligned}$$

Definition 1.8. A σ -additive function $\mu : \mathcal{A} \rightarrow [0, \infty]$ is called

- **finite**, if $\mu(X) < \infty$
- **σ -finite**, if there exists a sequence $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ such that

$$\bigcup_{n=1}^{\infty} A_n = X \quad \text{and} \quad \mu(A_n) < \infty \quad \forall n \in \mathbb{N}$$

Clearly, μ finite $\implies \mu$ σ -finite.

While pre-measures are only defined on algebras $\mathcal{A} \subseteq \mathcal{P}(X)$, we would like to extend the domain of such functions to $\mathcal{P}(X)$ without losing too many of its nice properties. In particular, we want to keep monotonicity and σ -subadditivity:

Definition 1.9. A function $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ is called a **measure**¹ on X , if

- (a) $\mu(\emptyset) = 0$
- (b) μ is monotonous and σ -subadditive: If $A \subseteq \bigcup_{k=1}^{\infty} A_k$, then $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$

Definition 1.10. Let μ be a measure on X and $A \subseteq X$. We can *restrict* μ to A (written $\mu \llcorner A$) defined by

$$(\mu \llcorner A)(B) := \mu(A \cap B) \quad \forall B \subseteq X$$

Definition 1.11 (Carathéodory criterion). A subset $A \subseteq X$ is called **μ -measurable** if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A), \quad \forall B \subseteq X$$

Remark 1.12. (a) By subadditivity of the measure, the definition is equivalent to

$$\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A), \quad \forall B \subseteq X$$

- (b) If $\mu(A) = 0$, then A is μ -measurable.

Theorem 1.13. Let $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ be a measure. Then the collection of measurable sets

$$\Sigma = \{A \subseteq X \mid A \text{ is } \mu\text{-measurable}\}$$

forms a σ -algebra.

Proof.

- $X \in \Sigma$: Let $B \subseteq X$. It's trivial to see that

$$\mu(B \cap X) + \mu(B \setminus X) = \mu(B) + \mu(\emptyset) = \mu(B)$$

- $A \in \Sigma \implies A^c \in \Sigma$: With the equalities

$$B \cap A^c = B \setminus A, \quad \text{and} \quad B \setminus A^c = B \cap A$$

we get

$$\mu(B \cap A^c) + \mu(B \setminus A^c) = \mu(B \setminus A) + \mu(B \cap A) \stackrel{A \in \Sigma}{=} \mu(B)$$

¹sometimes also called outer measure

- $A_1, A_2 \in \Sigma \implies A_1 \cup A_2 \in \Sigma$:

Let $B \subseteq X$. From the previous remark, it is sufficient to just show the inequality

$$\mu(B) \geq \mu(B \cap (A_1 \cup A_2)) + \mu(B \setminus (A_1 \cup A_2))$$

Using μ -measurability for A_1 on the test set $B \setminus A_2$, we see

$$\begin{aligned} \mu(B \setminus A_2) &= \mu((B \setminus A_2) \cap A_1) + \mu((B \setminus A_2) \setminus A_1) \\ &= \mu((B \setminus A_2) \cap A_1) + \mu(B \setminus (A_2 \cup A_1)) \end{aligned}$$

so with the decomposition

$$(B \cap A_2) \cup ((B \setminus A_2) \cap A_1) = B \cap (A_1 \cup A_2)$$

and subadditivity of the measure, we get

$$\begin{aligned} \mu(B) &= \mu(B \cap A_2) + \mu(B \setminus A_2) \\ &= \mu(B \cap A_2) + \mu((B \setminus A_2) \cap A_1) + \mu(B \setminus (A_2 \cup A_1)) \\ &\geq \mu(B \cap (A_2 \cup A_1)) + \mu(B \setminus (A_2 \cup A_1)) \end{aligned}$$

- $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma \implies A = \bigcup_{n=1}^{\infty} A_n \in \Sigma$:

We can assume without loss of generality that the sets are mutually disjoint. Otherwise, consider the sequence $(\tilde{A}_n)_{n \in \mathbb{N}} \subseteq \Sigma$ given by

$$\tilde{A}_1 := A_1, \tilde{A}_n := A_n \setminus \bigcup_{k=1}^{n-1} A_k \quad \text{which satisfy} \quad \bigsqcup_{k=1}^{\infty} \tilde{A}_k = \bigcup_{k=1}^{\infty} A_k.$$

We can use μ -measureability of A_m with the test set $B \cap \bigcup_{k=1}^m A_k$ to find that by induction on m

$$\begin{aligned} \mu\left(B \cap \bigsqcup_{k=1}^m A_k\right) &= \mu\left(\left(B \cap \bigsqcup_{k=1}^m A_k\right) \cap A_m\right) + \mu\left(\left(B \cap \bigsqcup_{k=1}^m A_k\right) \setminus A_m\right) \\ &= \mu(B \cap A_m) + \mu\left(B \cap \bigsqcup_{k=1}^{m-1} A_k\right) \\ &= \sum_{k=1}^m \mu(B \cap A_k) \end{aligned}$$

and using monotonicity of μ on the inclusion $\bigsqcup_{k=1}^m A_k \subseteq A$ it follows that

$$\begin{aligned} \mu(B) &= \mu\left(B \cap \bigsqcup_{k=1}^m A_k\right) + \mu\left(B \setminus \bigsqcup_{k=1}^m A_k\right) \\ &\geq \sum_{k=1}^m \mu(B \cap A_k) + \mu(B \setminus A) \end{aligned}$$

for all $m \in \mathbb{N}$. Taking the limit $m \rightarrow \infty$, we get

$$\begin{aligned} \mu(B) &\geq \sum_{k=1}^{\infty} \mu(B \cap A_k) + \mu(B \setminus A) \\ &\geq \mu(B \cap A) + \mu(B \setminus A) \end{aligned}$$

which shows μ -measurability of A .

□

Definition 1.14. A **measure space** is a tuple (X, Σ, μ) consisting of measure μ on a set X and the σ -algebra of μ -measurable sets Σ .

Example 1.15. The following are measure spaces:

- For every $x \in X, A \subseteq X$, define the **Dirac measure at x**

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Every A is δ_x -measurable.

- For every $A \in \mathcal{P}$, the **counting measure** is a measure, where every subset is μ -measurable:

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

Every A is μ -measurable.

- The **indiscrete measure** given by

$$\mu(A) = \begin{cases} 1 & A \neq \emptyset \\ 0 & A = \emptyset \end{cases}$$

only has \emptyset, X as μ -measurable sets.

The Carathéodory criterion of μ -measurable sets and the σ -subadditivity of the measure give us some nice properties back.

Theorem 1.16. Let (X, Σ, μ) be a measure space and $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$. Then the following are true

(a) μ is σ -additive.

(b) Continuity from below:

$$A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots \implies \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n)$$

(c) Continuity from above:

$$\mu(A_1) < \infty, \quad A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq A_{n+1} \supseteq \dots \implies \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n)$$

Proof. (a) Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of mutually disjoint sets. In the proof of the previous theorem, we already saw

$$\mu\left(B \cap \bigsqcup_{k=1}^m A_k\right) = \sum_{k=1}^m \mu(B \cap A_k)$$

so in particular, for $B = X$, we see

$$\mu\left(\bigsqcup_{k=1}^m A_k\right) = \sum_{k=1}^m \mu(A_k)$$

By monotonicity of μ , we have

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) \geq \lim_{m \rightarrow \infty} \mu\left(\bigsqcup_{k=1}^m A_k\right) = \lim_{m \rightarrow \infty} \sum_{k=1}^m \mu(A_k) = \sum_{k=1}^{\infty} \mu(A_k)$$

The other inequality (and thus equality) follow from σ -subadditivity of the measure.

(b) Let $(A_n)_{n \in \mathbb{N}}$ be an increasing sequence. Define the pairwise disjoint family

$$\tilde{A}_1 := A_1, \quad \tilde{A}_k := A_k \setminus A_{k-1} \implies \mu(\tilde{A}_k) = \mu(A_k) - \mu(A_{k-1}), \quad \bigsqcup_{k=1}^{\infty} \tilde{A}_k = \bigcup_{k=1}^{\infty} A_k,$$

from σ -additivity, summation into a telescoping sum

$$\begin{aligned} \mu\left(\bigcup_{k=1}^{\infty} A_k\right) &= \mu\left(\bigsqcup_{k=1}^{\infty} \tilde{A}_k\right) = \sum_{k=1}^{\infty} \mu(\tilde{A}_k) \\ &= \mu(\tilde{A}_1) + \lim_{m \rightarrow \infty} \sum_{k=2}^m \mu(\tilde{A}_k) \\ &= \lim_{m \rightarrow \infty} \mu(A_m) \end{aligned}$$

(c) Let $(A_n)_{n \in \mathbb{N}}$ be a decreasing sequence. Consider instead the increasing sequence $\tilde{A}_1 \subseteq \tilde{A}_2 \subseteq \dots$ given by

$$\tilde{A}_1 := \emptyset, \quad \tilde{A}_k := A_1 \setminus A_k \implies \mu(A_1) = \mu(A_k) + \mu(\tilde{A}_k), \quad \bigcup_{k=1}^{\infty} \tilde{A}_k = A_1 \setminus \bigcap_{k=1}^{\infty} A_k$$

by (b), we find

$$\begin{aligned} \mu(A_1) - \lim_{k \rightarrow \infty} \mu(A_k) &= \lim_{k \rightarrow \infty} \mu(\tilde{A}_k) \\ &\stackrel{(b)}{=} \mu\left(\bigcup_{k=1}^{\infty} \tilde{A}_k\right) = \mu\left(A_1 \setminus \bigcap_{k=1}^{\infty} A_k\right) \\ &= \mu(A_1) - \mu\left(\bigcap_{k=1}^{\infty} A_k\right) \end{aligned}$$

□

The condition $\mu(A_1)$ in (c) is necessary. Consider the example $X = \mathbb{N}$ with the counting-measure and the sequence $A_n := \{m \in \mathbb{N} | m \geq n\}$. The intersections converge to the empty set, but the $\mu(A_k)$ is always ∞ .

1.3 Construction of Measures

Let X be non-empty set.

Definition 1.17. A collection of subsets $\mathcal{K} \subseteq \mathcal{P}(X)$ is called a **covering** of X if

$$\emptyset \in \mathcal{K} \quad \text{and} \quad \exists (K_j)_{j \in \mathbb{N}} \subseteq \mathcal{K} : \quad X = \bigcup_{j=1}^{\infty} K_j$$

Example 1.18. The collection of higher-dimensional open intervals

$$\left\{ \prod_{k=1}^n (a_k, b_k) \mid a_k \leq b_k \in \mathbb{R} \right\}$$

are a covering of \mathbb{R}^n .

It is easy to see that every Algebra \mathcal{A} of X is a covering since $\emptyset, X \in \mathcal{A}$.

Theorem 1.19. Let \mathcal{K} be a covering of X and $\lambda : \mathcal{K} \rightarrow [0, \infty]$ and any function with $\lambda(\emptyset) = 0$. Then this induces a measure μ on X given by

$$\mu(A) = \inf \left\{ \sum_{j=1}^{\infty} \lambda(K_j) \mid K_j \in \mathcal{K}, A \subseteq \bigcup_{j=1}^{\infty} K_j \right\}$$

Proof. Let $A \subseteq \bigcup_{k=1}^{\infty} A_k$. We show σ -subadditivity of μ , i.e. $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$.

If the right-hand side is infinite, then the inequality is trivial, so assume it is finite.

By definition of μ , for all $k \in \mathbb{N}$ and $\epsilon > 0$ there exists a sequence $(K_{j,k})_{j \in \mathbb{N}}$ in \mathcal{K} such that

$$A_k \subseteq \bigcup_{j=1}^{\infty} K_{j,k} \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda(K_{j,k}) \leq \mu(A_k) + \frac{\epsilon}{2^k}$$

Taking the union over all sequences for each k , we get

$$A \subseteq \bigcup_{j,k=1}^{\infty} K_{j,k} \quad \text{and} \quad \mu(A) \leq \sum_{j,k} \lambda(K_{j,k}) \leq \epsilon + \sum_{k=1}^{\infty} \mu(A_k)$$

Since $\epsilon > 0$ was arbitrary, subadditivity follows. □

Example 1.20. Set $\mathcal{K} = \{\emptyset, X\}$ and define $\lambda(\emptyset) = 0$, $\lambda(X) = 1$.

The induced measure is defined by $\mu(A) = 0$ if $A = \emptyset$ and $\mu(A) = 1$ if $A \neq \emptyset$.

The function λ in the previous theorem only had minimal restrictions (\mathcal{K} had to be a covering and $\lambda : \mathcal{K} \rightarrow [0, \infty]$ with $\lambda(\emptyset) = 0$).

It turns out that if λ and \mathcal{K} are *nice enough*, then the induced measure is a σ -additive extension of λ . Nice-enough here means that \mathcal{K} is an algebra and λ is a pre-measure.

Recall that given an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$, a function $\lambda : \mathcal{A} \rightarrow [0, \infty]$ is called a **pre-measure** if it is σ -additive and satisfies $\lambda(\emptyset) = 0$.

Definition 1.21. A pre-measure λ is called **σ -finite** if there exists a covering $X = \bigcup_{k=1}^{\infty} S_k$, $S_k \in \mathcal{A}$ such that $\lambda(S_k) < \infty$, $\forall k$.

Given a pre-measure λ on \mathcal{A} , we can obtain a measure μ on $\mathcal{P}(X)$ that coincides with λ on \mathcal{A} , i.e. μ extends λ .

Theorem 1.22 (Carathéodory-Hahn extension). *Let $\lambda : \mathcal{A} \rightarrow [0, \infty]$ be a pre-measure on X . Then for*

$$\mu : \mathcal{P}(X) \rightarrow [0, \infty], \quad \mu(A) := \inf \left\{ \sum_{k=1}^{\infty} \lambda(A_k) \mid A \subseteq \bigcup_{k=1}^{\infty} A_k, A_k \in \mathcal{A} \right\}$$

it holds that

- (a) $\mu : \mathcal{P} \rightarrow [0, \infty]$ is a measure.
- (b) $\mu(A) = \lambda(A) \forall A \in \mathcal{A}$
- (c) All $A \in \mathcal{A}$ are μ -measurable.

Proof. (a) This just follows from the previous theorem

- (b)
- (c)

□

Not only does such an extension exist always, we can show that under certain assumptions it is unique:

Theorem 1.23 (Uniqueness of Carathéodory-Hahn extension). *Let $\lambda : \mathcal{A} \rightarrow [0, \infty]$ be a σ -finite pre-measure on X and μ the Carathéodory-Hahn extension of λ and let Σ be the σ -algebra of μ -measurable sets.*

If $\tilde{\mu} : \mathcal{P}(X) \rightarrow [0, \infty]$ if another measure with $\tilde{\mu}|_{\mathcal{A}} = \lambda$, then $\tilde{\mu}|_{\Sigma} = \mu$

Proof. We show

- (a) $\forall A \in \mathcal{P}(X): \tilde{\mu}(A) \leq \mu(A)$.
- (b) $\forall A \in \Sigma: \tilde{\mu}(A) \geq \mu(A)$.

For the first claim, let $A \subseteq \bigcup_{k=1}^{\infty} A_k$ with $A_k \in \mathcal{A}$. By σ -subadditivity of μ it follows that

$$\tilde{\mu} \leq \sum_{k=1}^{\infty} \tilde{\mu}(A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

So by taking the infimum over all such coverings $(A_k)_{k \in \mathbb{N}}$ as in the definition of μ , the inequality still holds: $\tilde{\mu}(A) \leq \mu(A)$.

For the second claim we consider two cases: One where there exists an $S \in \mathcal{A}$ such that

$$A \subseteq S \quad \text{and} \quad \lambda(S) < \infty$$

Then, using the first claim on $S \setminus A$ and monotonicity of μ , it follows that

$$\tilde{\mu}(S \setminus A) \leq \mu(S \setminus A) \leq \mu(S) = \lambda(S)$$

Since $S \in \mathcal{A}$ is μ -measurable and $S \cap A = A$ we get

$$\begin{aligned} \tilde{\mu}(A) + \tilde{\mu}(S \setminus A) &\leq \mu(A \cap S) + \mu(S \setminus A) = \mu(S) \\ &= \lambda(S) = \tilde{\mu}(S) \\ &\leq \tilde{\mu}(A) + \tilde{\mu}(S \setminus A) \end{aligned}$$

where we used sub-additivity of $\tilde{\mu}$ in the last step. □

The condition for the uniqueness cannot be weakened as the following counterexample shows:

1.4 Lebesgue Measure

The Lebesgue measure is the Carathéodory-Hahn extension of the pre-measure of “volumes”, that assigns products of intervals to their products of lengths.

We want to give a definition of what these “volumes” and what their measure is going to be.

Definition 1.24. For $a = (a_1, \dots, a_d), b = (b_1, \dots, b_d) \in \mathbb{R}^d$ we define the d -dimensional **interval**

$$(a, b) = \begin{cases} \prod_{i=1}^d (a_i, b_i) & \text{if } a_i < b_i \quad \forall i \\ \emptyset & \text{otherwise} \end{cases}$$

in an analogous way, we define the closed and half-open boxes $[a, b], [a, b)$ or $(a, b]$. Like on the real line, we also allow the open ends to be $\pm\infty$.

To each d -dimensional interval I (whether open, closed or half-open) to be

$$\text{vol}(I) := \begin{cases} \prod_{i=1}^d (b_i - a_i) \in [0, +\infty] & \text{if } a_i < b_i, \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

An **elementary set** is the finite disjoint union of intervals and we define its volume to be

$$\text{vol}\left(\bigsqcup_{k=1}^d I_k\right) := \sum_{k=1}^d \text{vol}(I_k)$$

Remark 1.25. We can check easily that the volume function is well defined. For example, the decomposition $[0, 2] = [0, 1) \sqcup [1, 2] = [0, 1) \sqcup [1, 1.5) \sqcup [1.5, 2]$ should all give the same volume.

More generally, if $I = \bigsqcup_{k=1}^n I_k = \bigsqcup_{j=1}^m J_j$ where I_k, J_j are Intervals, then

$$\sum_{k=1}^n \text{vol}(I_k) = \sum_{j=1}^m \text{vol}(J_j)$$

We of course have to show that our attempt to use the Carathéodory-Hahn Extension of vol on the elementary sets is well defined. But it should be easy to see how the class of elementary sets forms an algebra and that the vol function is a pre-measure on it.

In our example above, we used half-open intervals of length 2^{-k} to decompose an interval. A direct generalisation is to introduce the dyadic cubes to obtain a basic decomposition of sets in \mathbb{R}^d . For every $k \in \mathbb{N}$, let \mathcal{D}_k the collection of half open cubes

$$\mathcal{D}_k := \left\{ \prod_{i=1}^d \left[\frac{a_i}{2^k}, \frac{a_i+1}{2^k} \right) \mid a_i \in \mathbb{Z} \right\}$$

In particular, \mathcal{D}_0 is the collection of hypercubes of edge length 1 and vertices in \mathbb{Z}^d .

Definition 1.26. A measure μ on \mathbb{R}^n is called **Borel**, if every Borel set is μ -measurable.

The Lebesgue measure \mathcal{L}^n on \mathbb{R}^n is Borel and the σ -algebra of \mathcal{L}^n -measurable sets contains the Borel σ -algebra.

Theorem 1.27. For any subset $A \subseteq \mathbb{R}^n$ the following are equivalent

- (a) A is \mathcal{L}^n -measurable..
- (b) $\forall \epsilon > 0 \exists G \supseteq A$ with $\mathcal{L}^n(G \setminus A) < \epsilon$.
- (c) A it can be “approximated” from the inside and outside: $\forall \epsilon > 0 \exists F, G$ with $F \subseteq A \subseteq G$ such that
$$\mathcal{L}^n(G \setminus A) + \mathcal{L}^n(A \setminus F) \leq \epsilon$$
- (d) $\forall \epsilon > 0 \exists F$ closed, $\exists G$ open, such that $\mathcal{L}^n(G \setminus F) < \epsilon$.

1.5 Comparison between Lebesgue and Jordan Measure

Definition 1.28. A bounded subset $A \subseteq \mathbb{R}^n$ is **Jordan-measurable** if

$$\underline{\mu}(A) := \sup\{\text{vol}(E) \mid E \subseteq A, E \text{ elementary set}\} = \inf\{\text{vol}(E) \mid A \subseteq E, E \text{ elementary set}\} =: \bar{\mu}(A)$$

As the following proof will show, The Lebesgue measure can measure more sets than the Jordan measure can.

Theorem 1.29. Let $A \subseteq \mathbb{R}^n$ be bounded, then

$$(a) \quad \underline{\mu}(A) \leq \mathcal{L}^n(A) \leq \bar{\mu}(A)$$

$$(b) \quad \text{If } A \text{ is Jordan-measurable, then } A \text{ is } \mathcal{L}^n\text{-measurable and } \mathcal{L}^n(A) = \mu(A).$$

One would naturally think that the volume of a cube should stay the same whether we move it by a fixed point or rotate it.

Theorem 1.30. The Lebesgue measure is invariant under isometries $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto x_0 + Rx$, for $R \in O(n)$.

Definition 1.31. A Borel measure μ on \mathbb{R}^n is called **Borel regular**, if for every $A \subseteq \mathbb{R}^n$ there exists a borel set $B \supseteq A$ such that $\mu(A) = \mu(B)$.

The Lebesgue measure is Borel regular.

1.6 Special-Examples of sets

Although (or rather, because) the Lebesgue measure has all these nice properties, not all sets are measurable. To construct such a non-measurable set, we will use the Axiom of Choice, which states that for any family of non-empty disjoint sets $(A_i)_{i \in I}$, there exists a choice-function $f : I \rightarrow \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$.

With this, we can construct the set $\{f(i) \mid i \in I\}$ that contains exactly one element from each set A_i .

For $x, y \in [0, 1)$ we define $\oplus := \text{mod } 1 \circ +$

$$x \oplus y = \begin{cases} x + y & \text{if } x + y < 1 \\ x + y - 1 & \text{if } x + y \geq 1 \end{cases}$$

Example 1.32. For every $A \subseteq \mathbb{R}$ with $\mathcal{L}(A) > 0$, there exists a $B \subseteq A$ such that B is not \mathcal{L} -measurable.

The Cantor tridadic set

The real numbers can be defined as the set of Cauchy-sequences in \mathbb{Q} up to limits.

The cantor set is the set of numbers whose triadic digits don't contain a 1.

$$C = \{x \in [0, 1] \mid d_i(x) \in \{0, 2\} \forall i\}$$

Then C is uncountable and $\mathcal{L}(C) = 0$.

1.7 The Lebesgue-Stieltjes Measure

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing and left continuous:

$$F(x_0) = \lim_{x \rightarrow x_0^-} F(x) \quad \forall x_0 \in \mathbb{R}$$

For a, b we set

$$\lambda_F[a, b) = \begin{cases} F(b) - F(a) & \text{if } a < b \\ 0 & \text{otherwise} \end{cases}$$

1.8 Hausdorff Measures

The Lebesgue measure of a subset of \mathbb{R}^n of dimension less than n is always zero. It also has the weakness of failing to properly measure fractal sets. The Hausdorff measure tries to solve this.

We start by introducing an intermediate measure, where instead of covering a subset A with quaders, we do this using open balls of radius smaller than some given $\delta > 0$.

Definition 1.33. For $s \geq 0, \delta > 0$ and $A \subseteq \mathbb{R}^n$ non-empty, we set

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{k \in I} r_k^s, A \subseteq \bigcup_{k \in I} B(x_k, r_k), 0 < r_k \delta \right\}$$

where the set I is at most countable and we set $\mathcal{H}_\delta^0(\emptyset) = 0$.

Remark 1.34. \mathcal{H}_δ^s defines a measure on \mathbb{R}^n and for fixed s, A it is inversely monotonous with respect to δ , that is

$$\delta_2 \leq \delta_1 \implies \mathcal{H}_{\delta_1}^s(A) \leq \mathcal{H}_{\delta_2}^s(A)$$

since every δ_2 covering is also a δ_1 covering.

Since \mathcal{H}_δ^s is non-increasing in δ , the limit

$$\mathcal{H}^s(A) := \lim_{\delta \downarrow 0} \mathcal{H}_\delta^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A)$$

exists. We now use this as for our next definition.

Definition 1.35. We call \mathcal{H}^s the **s -dimensional Hausdorff measure** on \mathbb{R}^n

For $s = 0$, we call \mathcal{H}^0 the **counting measure**, which measure the cardinality of A .

Now that we call it a measure, we should of course prove some things

Theorem 1.36. For $s \geq 0$, \mathcal{H}^s is a Borel regular measure on \mathbb{R}^n

Proof. Let $s \geq 0$. We first prove that it is a measure. Clearly, $\mathcal{H}^s(\emptyset) = 0$. Let $(A_k)_{k \in \mathbb{N}}$ and $A \subseteq \bigcup_{k=1}^\infty A_k$. Since \mathcal{H}_δ^s is σ -subadditive for all $\delta > 0$, we get

$$\mathcal{H}_\delta^s(A) \leq \sum_k \mathcal{H}_\delta^s(A_k) \leq \sum_k \mathcal{H}^s(A_k)$$

by taking the limit (in the definition of \mathcal{H}^s) we get the σ -subadditivity of \mathcal{H}^s .

To show that it is borel, we just show that it is metric. Let $A, B \subseteq \mathbb{R}^n$ such that $\delta_0 := \text{dist}(A, B) > 0$. We then take a covering $A \cup B$ of balls of size smaller than $\delta := \frac{\delta_0}{4}$ and claim that we can partition the covering into two non-overlapping coverings of A and B each.

Since \mathcal{H}_δ^s takes the infimum over all such coverings, suppose that $A \cup B = \bigcup_k B(x_k, r_k)$ with $r_k < \delta$. Then we set

$$\mathcal{A} = \{B(x_k, r_k) \mid B(x_k, r_k) \cap A \neq \emptyset\} \quad \mathcal{B} = \{B(x_k, r_k) \mid B(x_k, r_k) \cap B \neq \emptyset\}$$

And it becomes obvious that these are non-overlapping coverings of A and B each (by using the triangle inequality).

Therefore, we get

$$\mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B) \leq \sum_k r_k^s$$

and taking the infimum of coverings of $A \cup B$, this means

$$\mathcal{H}_\delta^s(A \cup B) \geq \mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B)$$

which, when taking the limit $\delta \rightarrow 0$ just states $\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B)$. By (σ) -subadditivity of \mathcal{H}^s , the reverse inequality holds and so $\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$ shows that \mathcal{H}^s is metric and thus also Borel.

For Borel regularity, let $A \subseteq \mathbb{R}^n$ and suppose $\mathcal{H}^s(A) < \infty$ (Otherwise, just take $B = \mathbb{R}^n$). By monotonicity of \mathcal{H}_δ^s , this also means that $\mathcal{H}_\delta^s(A) < \infty$ for all $\delta > 0$.

For $\delta = \frac{1}{l}$

□

2 Measurable Functions

2.1 Basic definitions

For X, Y nonempty sets and $f : X \rightarrow Y$ with $A \subseteq Y$, the inverse image is defined as

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}$$

which satisfies

(a) $f^{-1}(A^c) = (f^{-1}(A))^c$

(b) If $A, B \subseteq Y$, then

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

(c) If $(A_k)_k$ is a sequence of subsets, then

$$f^{-1}\left(\bigcup_{k=1}^{\infty} A_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(A_k)$$

From this, it follows that if (Y, \mathcal{A}, μ) is a measure space, then

$$\Sigma := f^{-1}(\mathcal{A}) := \{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is a σ -algebra in X .

Definition 2.1. A function $f : \Omega \rightarrow [-\infty, \infty]$ is called **μ -measurable** if in the sense of definition 1.11

(a) $f^{-1}\{+\infty\}, f^{-1}\{-\infty\}$ are μ -measurable.

(b) $f^{-1}(U)$ for every $U \subseteq \mathbb{R}$ open is μ -measurable.

The composition of μ -measurable functions