Electrodynamics – Summary

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1 Magnetostatics

In Magnetostatics, we consider systems where the current is steady. This means in particular that

$$\vec{J} = \text{const}, \implies \rho = \text{const}, \quad \vec{E} = \text{const}, \quad \vec{B} = \text{const}$$

Under the **Coulomb-Eichung**

Electrostatics	Magnetostatics
$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\frac{\rho(\vec{y})}{\lvert\vec{x}-\vec{y}\rvert}$	$A(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2}\int d^3\vec{y}\frac{\vec{J}(\vec{y})}{\lvert\vec{x}-\vec{y}\rvert}$
$\vec{E} = -\nabla\Phi$	$\vec{B} = \vec{\nabla}\times\vec{A}$
$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$	$\vec{\nabla}\times\vec{B} = \frac{\vec{J}}{\epsilon_0 c^2}$
$\vec{\nabla}\times\vec{E} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
$\int_{\partial V} d\vec{S}\cdot\vec{E} = \frac{Q_{\text{inside}}}{\epsilon_0}$	$\oint_{\partial S}\vec{B}\cdot d\vec{\ell} = \frac{I_{\text{inside}}}{\epsilon_0 c^2}$

Table 1: Analogies between Electrostatics and Magnetostatics

2 Time dependent electromagnetic fields

If the $\vec{E}$ and $\vec{B}$ are time dependent, then the Maxwell equations are as follows

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad c^2\vec{\nabla}\times\vec{B} = \frac{\vec{J}}{\epsilon_0} + \frac{\partial\vec{E}}{\partial t}$$

From the scalar and vector potential $\Phi, \vec{A}$ we can find the $\vec{E}, \vec{B}$ fields with.

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$

Since the elecric and magnetic field remain invariant under **gauge transformations**

$$\Phi \mapsto \tilde{\Phi} = \frac{\partial}{\partial t} f(\vec{x}, t)$$
$$\vec{A} \mapsto \tilde{\vec{A}} = \vec{A} - \nabla f(\vec{x}, t)$$

for some $f : \mathbb{R}^4 \to \mathbb{R}$, the potentials are not uniquely determined.
By introducing the **d'Alembert Operator**

$$\boxed{\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2}$$

we get the equations

$$\Box\vec{A} = \frac{\vec{J}}{\epsilon_0 c^2}, \quad \Box\Phi = \frac{\rho}{\epsilon_0}$$

To solve them, we can instead search for the Green's function (or more generally, the Fundamental solution) $G(\vec{x}, t, \vec{y}, t')$ that satisfies

$$\Box_{x,t} G(\vec{x}, t, \vec{y}, t') = \delta(\vec{x} - \vec{y})\delta(t - t')$$

After some Fourier transformation and Complex Analysis shenanigans, we obtain the solution

$$G(\vec{x}, t, \vec{y}, t') = \frac{1}{4\pi\lvert\vec{x} - \vec{y}\rvert}\delta\left(t - t' - \frac{\vec{x} - \vec{y}}{c}\right)\Theta(t - t')$$

or equivalently, we can write

$$\boxed{G(\Delta\vec{x}, \Delta t) = \frac{1}{2\pi}\delta\left((t - t')^2 - \frac{\lvert\vec{x} - \vec{y}\rvert^2}{c^2}\right)\Theta(t - t')}$$

By defining the **time retardation**

$$t_{\text{ret}} := t' - \frac{\lvert\vec{x} - \vec{y}\rvert}{c}$$

we obtain the **retarded scalar potential**

$$\boxed{\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\frac{\rho(\vec{y}, t_{\text{ret}})}{\lvert\vec{x} - \vec{y}\rvert}}$$

aswell as the **retarded vector potential**

$$\boxed{\vec{A}(\vec{x},t) = \frac{1}{4\pi\epsilon_0 c^2}\int d^3\vec{y}\frac{\vec{J}(\vec{y},t_{\text{ret}})}{|\vec{x}-\vec{y}|}}$$

3 Maths

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

For $f(x)$ with roots $(x_i)_{i\in I}$

$$\delta(f(x)) = \sum_{i\in I}\frac{1}{|f'(x)|}\delta(x-x_i)$$