Numerical Analysis II – Lecture Notes

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Definition 0.1. An Ordinary differential equation (**ODE**) is an equation that contains one or more derivatives of un an unknown function $x(t)$. The equation may contain x itself and constants. We saythat and ODE **of order** n if $x^{(n)}(t)$ is the highest order derivative in the equation.

Missing 10 min

1 Methods of resolution

1.1 Separation of variables

Let I, J be two open variables and $f \in C^0(I), g \in C^0(J)$. When doing separation of variables, we look for solutions to the equation

$$\frac{dx}{dt} = f(t)g(x)$$

with some initial conditions $t_0 \in I, x_0 = x(t_0) \in J$. In the special case where $g(x_0) = 0$, then the constant function $x(t) = x_0$ is a solution to this equation.
Suppose that $g(x_0) \neq 0$ is non-zeo, then in a neighborhood of x_0, g is non-zero so we can divide by $g(x)$ to separate the variables. We find

$$\frac{dx}{g(x)} = f(t)dt$$

If we let $F(t)$ and $G(x)$ be primites of $f(t)$ and $\frac{1}{g(x)}$, we get by integration that

$$\int \frac{dx}{g(x)} = \int f(t)dt + c \implies x(t) = G^{-1}(F(t) + c)$$

1.2 Change of variables

Consider the following homogeneous ODE:

$$\frac{dx}{dt} = f\left(\frac{x(t)}{t}\right)$$

, where $f : I \to \mathbb{R}$ is a continuous function on an open interval $I \subseteq \mathbb{R}$. We change the variable to $x(t) = ty(t)$
Any ODE can be turned into an autonomous ODE.

We can see that from

$$g'^T J g' = J$$

if we take the determinant on both sides, we see that $\det g'$ must be ± 1, so g' must be invertible. Given a hamiltonian system

$$\frac{dx}{dt} = J^{-1} \nabla H(x)$$

a symplectic change of variables $y = g(x)$ preserves the hamiltonian sytem

$$\frac{dy}{dt} = J^{-1} \nabla_y G(y)$$

Theorem 1.1 (Cauchy-Lipschitz theorem)**.** *The first order initial value problem*

$$\begin{cases} \frac{dx}{dt} = f(t,x) & t \in [0,T] \\ x(0) = x_0 & x_0 \in \mathbb{R}^d \end{cases}$$

If $f \in C^0(I \times \mathbb{R}^d)$ satisfies the Lipschitz condition, then there exists a unique solution $x \in C^1(I)$ on $[0,T]$

For the proof, we define the norm on $C^0([0,T], \mathbb{R}^d)$ by

$$\|y\| := \sup_{t \in [0,T]} \{ |y(t)| e^{-C_f t} \}$$

where C_f is a Lipschitz constant for f.

Theorem 1.2 (Arzela-Ascoli)**.** *If $(f_n)_{n \in \mathbb{N}}$ is a sequence of functions on $[a,b]$ that is uniformly bounded and equicontinuous, then there exists a uniformly convergent subsequence.*

Theorem 1.3 (Cauchy-Peano existence theorem)**.** *If f is continuous, then the first order initial value problem admits a soution defined for small t.*

Theorem 1.4. *Assume that f is continuous and satisifies the Lipschitz condition in the closed domain $|x| \leq k$ and $t \in [0,T]$. Then the first oder initial value problem has a* unique *solution for*

$$t \in [0, \min\{T, \frac{k}{M}\}] \quad \textit{for} \quad M := \sup_{|x| \leq k, t \in [0,T]} |f(t,x)|$$

Theorem 1.5 (Continuity with respect to initial data)**.** *If x_1, x_2 are solutions to the initial data $x_1(0), x_2(0)$, then*

$$|x_1(t) - x_2(t)| \leq e^{C_f t} |x_1(0) - x_2(0)|$$

for all $t \in [0,T]$

Theorem 1.6. *Suppose f is of class C^1. Then $x_0 \mapsto x(t)$ is differentiable and $\frac{\partial x(t)}{\partial x_0}$ is the unique solution to the linear equation*

$$\begin{cases} \frac{d}{dt} \frac{\partial x(t)}{\partial x_0} = \frac{\partial f}{\partial x}(t, x(t)) \frac{\partial x(t)}{\partial x_0} \\ \frac{\partial x(t)}{\partial x_0} = 1 \end{cases}$$

So the linear equation is differentiable and its derivative is the unique solution to the first order ODE.

1.3 Stability

If two ODEs are very similar, then the error of their solutions can be bounded.

Theorem 1.7 (Strong continuity theorem). *For two ODEs on* $[0, T]$

$$\frac{dx}{dt} = f(t, x) \quad \text{and} \quad \frac{dy}{dt} = g(t, y)$$

If f *satisfies the Lipschitz condition and there exist* $\epsilon > 0$ *such that*

$$|f(t, x) - g(t, x)| \leq \epsilon \quad \text{for all} \quad x, t$$

then the the following inequality holds

$$|x(t) - y(t)| \leq |x(0) - y(0)|e^{C_f t} + \frac{\epsilon}{C_f}(e^{C_f t} - 1), t \in [0, T]$$

1.4 Regularity

If $f \in C^n$ for $n \geq 0$, then the solution x of the first order IVP is of class C^{n+1}.
We can let the paramerers α, β from Gronwall's inequality depend on time.

Theorem 1.8 (Generalized Gronwall inequality). *Suppose* $\varphi(t)$ *satisfies*

$$\varphi(t) \leq \alpha(t) + \int_0^t \beta(s)\varphi(s)ds, \quad \text{for all} \quad t \in [0, T]$$

for $\alpha(t) \in \mathbb{R}$ *and* $\beta(t) \geq 0$.

(a) Then

$$\varphi(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \beta(\tau)d\tau}ds$$

(b) If additionally $\alpha(s) \leq \alpha(t)$ *for* $s \leq t$*, then*

$$\varphi(t) \leq \alpha(t)e^{\int_0^t \beta(s)ds} \quad \text{for all} \quad t \in [0, T]$$

$$\Phi_{\Delta t}^{(1)} : \quad \begin{cases} p^{k+1} = p^k \\ q^{k+1} = q^k + \Delta t p^k \end{cases}$$

$$\Phi_{\Delta t}^{(2)} : \quad \begin{cases} p^{k+1} = p^k - \Delta t q^k \\ q^{k+1} = q^k \end{cases}$$

whose adjoints can be obtained by exchanging the indices k,$k + \frac{1}{2}$ with $k + \frac{1}{2}, k + 1$, respectively and changing the sign of Δt.

$$(\Phi_{\Delta t}^{(1)})^* : \quad \begin{cases} p^{k+1} = p^k \\ q^{k+1} = q^k + \Delta t p^{k+1} \end{cases}$$

$$(\Phi_{\Delta t}^{(2)})^* : \quad \begin{cases} p^{k+1} = p^k - \Delta t q^{k+1} \\ q^{k+1} = q^k \end{cases}$$

Since $(\Phi_{\Delta t}^{(2)} \circ \Phi_{\Delta t}^{(1)})^* = (\Phi_{\Delta t}^{(1)})^* \circ (\Phi_{\Delta t}^{(2)})^*$.