Optimal Transport

Han-Miru Kim

June 18, 2021

1 Introduction

Introduction is stolen from `https://golem.ph.utexas.edu/category/2021/06/duality_in_transport_problems.html`.
Suppose we want to transport some material from s suppliers $S_1, \ldots, S_s$ to r receivers $R_1, \ldots, R_n$, where the supply available from supplier S_i is σ_i and the demand at receiver R_j is ρ_j.
If the cost of moving one part of material from S_i to R_j is $k_{ij} \in \mathbb{R}_{\geq 0}$, then we are interested in finding a **transport plan**, which can be given by a matrix $(\alpha_{ij})_{i,j}$ where α_{ij} denotes the amount of material moved from supplier S_i to receiver R_j, such that

$$\forall j : \sum_i \alpha_{ij} \geq \rho_j, \quad \forall i : \sum_j \alpha_{ij} \leq \sigma_i$$

and the total cost of the transport plan $\sum_{i,j} k_{ij}\alpha_{ij}$ is minimized.
Because the cost of moving material is positive, it is clear that the demand constraint is an equality $\sum_i \alpha_{ij} = \rho_j$.

Example 1.0.1. Let's say that there are three supplies and three receivers, with

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) = (350, 100, 200), \quad \rho = (\rho_1, \rho_2, \rho_3) = (200, 200, 250)$$

and where the transport cost is given by

$$K = \begin{pmatrix} 39 & 44 & 47 \\ 22 & 22 & 30 \\ 14 & 25 & 29 \end{pmatrix}$$