

# Electrodynamics – Summary

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May 19, 2021

## 1 Magnetostatics

In Magnetostatics, we consider systems where the current is steady. This means in particular that

$$\vec{J} = \text{const}, \implies \rho = \text{const}, \quad \vec{E} = \text{const}, \quad \vec{B} = \text{const}$$

Under the **Coulomb-Eichung**

<p>Electrostatics</p> $\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{y} \frac{\rho(\vec{y})}{ \vec{x}-\vec{y} }$ $\vec{E} = -\nabla\Phi$ $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ $\vec{\nabla} \times \vec{E} = 0$ $\oint_{\partial V} d\vec{S} \cdot \vec{E} = \frac{Q_{\text{inside}}}{\epsilon_0}$	<p>Magnetostatics</p> $A(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\vec{y} \frac{\vec{J}(\vec{y})}{ \vec{x}-\vec{y} }$ $\vec{B} = \vec{\nabla} \times \vec{A}$ $\vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\epsilon_0 c^2}$ $\vec{\nabla} \cdot \vec{B} = 0$ $\oint_{\partial S} \vec{B} \cdot d\vec{\ell} = \frac{I_{\text{inside}}}{\epsilon_0 c^2}$
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Table 1: Analogies between Electrostatics and Magnetostatics

$$\Phi \mapsto \tilde{\Phi} = \frac{\partial}{\partial t} f(\vec{x}, t)$$

$$\vec{A} \mapsto \vec{\tilde{A}} = \vec{A} - \nabla f(\vec{x}, t)$$

for some  $f : \mathbb{R}^4 \rightarrow \mathbb{R}$ , the potentials are not uniquely determined.

By introducing the **d'Alembert Operator**

$$\square := \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

we get the equations

$$\square \vec{A} = \frac{\vec{J}}{\epsilon_0 c^2}, \quad \square \Phi = \frac{\rho}{\epsilon_0}$$

To solve them, we can instead search for the Green's function (or more generally, the Fundamental solution)  $G(\vec{x}, t, \vec{y}, t')$  that satisfies

$$\square_{x,t} G(\vec{x}, t, \vec{y}, t') = \delta(\vec{x} - \vec{y}) \delta(t - t')$$

## 2 Time dependent electromagnetic fields

If the  $\vec{E}$  and  $\vec{B}$  are time dependent, then the Maxwell equations are as follows

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad c^2 \vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t}$$

From the scalar and vector potential  $\Phi, \vec{A}$  we can find the  $\vec{E}, \vec{B}$  fields with.

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

Since the electric and magnetic field remain invariant under **gauge transformations**

After some Fourier transformation and Complex Analysis shenanigans, we obtain the solution

$$G(\vec{x}, t, \vec{y}, t') = \frac{1}{4\pi|\vec{x} - \vec{y}|} \delta\left(t - t' - \frac{|\vec{x} - \vec{y}|}{c}\right) \Theta(t - t')$$

or equivalently, we can write

$$G(\Delta\vec{x}, \Delta t) = \frac{1}{2\pi} \delta\left((t - t')^2 - \frac{|\vec{x} - \vec{y}|^2}{c^2}\right) \Theta(t - t')$$

By defining the **time retardation**

$$t_{\text{ret}} := t' - \frac{|\vec{x} - \vec{y}|}{c}$$

we obtain the **retarded scalar potential**

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{y} \frac{\rho(\vec{y}, t_{\text{ret}})}{|\vec{x} - \vec{y}|}$$

aswell as the **retarded vector potential**

$$\vec{A}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\vec{y} \frac{\vec{J}(\vec{y}, t_{\text{ret}})}{|\vec{x} - \vec{y}|}$$

### 3 Special Relativity

Whereas classical mechanics played in three dimensional space  $\mathbb{R}^3$  with the time dimension  $t \in \mathbb{R}$  separated, special relativity plays in the **Minkowsky Space**.

Elements of the Minkowsky space are four-vectors  $x^\mu = (ct, \vec{x})$ .

The **metric tensor** is the matrix

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

which lets us define the (quasi)**inner product** on the minkowski space as

$$\langle x, y \rangle = g_{\mu\nu} x^\mu y^\nu \implies \langle x, x \rangle = c^2 t^2 - \vec{x}^2$$

#### 3.1 Lorentz Transformations

A **Lorentz transformation** is any affine linear transformation of the form

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu + \rho^\mu$$

such that it satisfies the relation

$$\Lambda^\mu_\nu \Lambda^\nu_\sigma g_{\mu\nu} = g_{\rho\sigma}$$

such transformation has the inner product as an invariant since

$$\begin{aligned} \langle \tilde{x}, \tilde{x} \rangle &= g_{\rho\nu} (\Lambda^\mu_\sigma x^\sigma) (\Lambda^\nu_\rho x^\rho) \\ &= g_{\mu\nu} \Lambda^\mu_\sigma \Lambda^\nu_\rho x^\sigma x^\rho = g_{\sigma\rho} x^\sigma x^\rho = \langle x, x \rangle \end{aligned}$$

The set of all Lorentz transformations forms a group, called the **Poincare group**.

We are especially interested in the subgroup known as the **proper Lorentz transformations**, which are all such transformations that satisfy

$$\det \Lambda = 1, \quad \Lambda^0_0 \geq 1$$

Another invariant is the **proper time**

$$d\tau^2 = dx^2 = c^2 dt^2 - d\vec{x}^2$$

Given a velocity  $\vec{v}$ , and  $dt$  we get

$$d\tau = c dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{c}{\gamma} dt$$

When a stationary observer  $O$  sees a reference frame  $\tilde{O}$  passing by with velocity  $\vec{v} = \vec{\beta}c$  along the  $x$ -axis, then the corresponding boost.

$$\Lambda_x^\mu{}_\nu(\vec{\beta}) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If the velocity is at an angle  $\theta$  with the  $x$ -axis in the  $xy$ -plane, then we break the boost into three steps

$$x \mapsto \tilde{x} = R(\theta)x \mapsto \tilde{\tilde{x}} = \Lambda(\beta)\tilde{x} \mapsto x'^\mu = R^\mu_\rho(-\theta)\Lambda_x^\rho{}_\sigma R^\sigma_\nu(\theta)x^\nu$$

The inverse of a Lorentz transformation  $\Lambda^\mu_\nu$  is given by

$$\Lambda_\mu{}^\nu = g_{\mu\nu} g^{\nu\sigma} \Lambda^\rho{}_\sigma$$

**Example 3.1** (Relativistic effects). Consider a moving particle that has a life span of  $t_0$ . By defining the events for the Start  $A$  and End  $B$ , we get

$$A = (0, \vec{0}) \quad \text{and} \quad B = (ct_0, \vec{0})$$

we see that after the Lorentz transformation we measure its lifespan to be at

$$\tilde{A} = \Lambda A = (0, \vec{0}), \quad \text{and} \quad \tilde{B} = \Lambda B = (\gamma ct_0, \beta \gamma ct_0)$$

so the outside observer sees a **time dilation** in its lifespan  $\tilde{t}_0 = \gamma t_0$ .

On the contrary, the outside observer notices a **length contraction**

$$s = vt_0 = \frac{1}{\gamma} \tilde{s}$$

#### 3.2 Tensors

In a change of reference under a Lorentz Transformation

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu + \rho^\mu$$

a **contra-variant tensor** is any object  $U$  that transforms as follows

$$T^\mu \mapsto \tilde{T}^\mu = \Lambda^\mu_\nu T^\nu$$

examples are the momentum  $p^\mu$ , the force  $f^\mu$  or the differential form  $dx^\mu$ . Contra-variant tensors are denoted with upstairs indices.

A **covariant tensor** is any object that transforms like

$$T_\mu \mapsto \tilde{T}_\mu = \Lambda_\mu^\nu T_\nu$$

examples are the covariant derivative  $\frac{\partial}{\partial x^\mu}$ . Such tensors are denoted with downstairs indices.

There are also **mixed tensors**, which have both up- and downstairs indices. They transform according to the rules

$$\tilde{T}^{\mu_1 \mu_2 \dots \mu_m}_{\nu_1 \nu_2 \dots \nu_n} = \Lambda^{\mu_1}_{\sigma_1} \dots \Lambda^{\mu_m}_{\sigma_m} \Lambda_{\nu_1}^{\rho_1} \dots \Lambda_{\nu_n}^{\rho_n} T^{\mu_1 \mu_2 \dots \mu_m}_{\rho_1 \rho_2 \dots \rho_n}$$

We can "raise/lower indices" with the metric tensor

$$T^\mu \mapsto T_\mu = g_{\mu\nu} T^\nu \quad \text{and} \quad T_\mu \mapsto T^\mu = g^{\mu\nu} T_\nu$$

what actually means is that the metric tensor transforms contravariant tensors to covariant ones and vice versa.

A **Lorentz-Scalar** is any object that stays invariant under Lorentz transformations.

Useful Tensor relations are

$$\Lambda^\mu_\sigma \Lambda_\mu^\rho = \delta^\rho_\sigma$$

### 3.3 Energy and Momentum

We define the four-momentum as

$$p^\mu := mc \frac{dx^\mu}{d\tau}$$

whose components can be given by

$$p^0 = m\gamma c = mc + \frac{1}{2c}mv^2 + \mathcal{O}\left(\frac{v^4}{c^3}\right), \quad p^i = m\gamma v^i$$

this gives us the definition for **relativistic energy** as

$$E := cp^0 = m\gamma c^2 = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}\left(\frac{v^4}{c^2}\right)$$

From the relation

$$\vec{p}^2 = m^2 \gamma^2 \vec{v}^2$$

we obtain the very useful **energy-momentum relation**

$$E^2 = c^2 \vec{p}^2 + m^2 c^4$$

## 4 Relativistic Electrodynamics

From now on, we use the convention  $c = \epsilon_0 = 1$ .

We combine current and charge density into a contravariant four-vector

$$j^\mu := (\rho, \vec{j})$$

The antisymmetric **electrodynamical field tensor**  $F^{\mu\nu}$  is defined using the  $\vec{E}$  and  $\vec{B}$  fields

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\vec{E}^T \\ \vec{E} & B^\times \end{pmatrix}$$

where  $B^\times$  is the dual tensor defined as

$$(B^\times)_{ij} = -\epsilon_{ijk} B^k \implies B^\times = \begin{pmatrix} 0 & -B^3 & B^2 \\ B^3 & 0 & -B^1 \\ -B^2 & B^1 & 0 \end{pmatrix}$$

from which the electric and magnetic field can be obtained using the relations

$$E^i = F^{i0}, \quad B^i = -\frac{1}{2}\epsilon_{ijk} F^{jk}, \quad \text{or} \quad F^{ij} = -\epsilon_{ijk} B^k$$

and the Maxwell equations become

$$\partial_\mu F^{\mu\nu} = j^\nu$$

In covariant Form, we get

$$F_{\mu\nu} = \begin{pmatrix} 0 & \vec{E}^T \\ -\vec{E} & B^\times \end{pmatrix}$$

**Example 4.1** (Transformation of EM tensor). In a change of reference along the  $x$ -axis with boost  $\beta$ , the EM tensor transforms doubly contravariant, i.e

$$F^{\mu\nu} \mapsto \tilde{F}^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\rho F^{\sigma\rho}$$

The summation over  $\rho$  corresponds to the matrix multi-

plication

$$\Lambda^\nu{}_\rho F^{\sigma\rho} = \begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & B^\times \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -E^1\beta\gamma & -E^1\gamma & -E^2 & -E^3 \\ E^1\gamma & E^1\beta\gamma & -B^3 & B^2 \\ \gamma(E^2 + B^3\beta) & \gamma(E^2\beta + B^3) & 0 & -B^1 \\ \gamma(E^3 - B^2\beta) & \gamma(E^3\beta - B^2) & B^1 & 0 \end{pmatrix}$$

and in the end, we get

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -\gamma(E^2 + \beta B^3) & -\gamma(E^3 - \beta B^2) \\ E^1 & 0 & -\gamma(B^3 + \frac{v}{c^2}E^2) & \gamma(B^2 - \frac{v}{c^2}E^3) \\ & & 0 & -B^1 \\ & & B^1 & 0 \end{pmatrix}$$

This result generalizes to arbitrary boosts, and we get

$$\begin{aligned} \vec{E}_\parallel &\mapsto \vec{E}_\parallel = \vec{E}_\parallel, & \vec{E}_\perp &\mapsto \vec{E}_\perp = \gamma(\vec{E}_\perp - \vec{v} \times \vec{B}) \\ \vec{B}_\parallel &\mapsto \vec{B}_\parallel = \vec{B}_\parallel, & \vec{B}_\perp &\mapsto \vec{B}_\perp = \gamma(\vec{B}_\perp + \frac{1}{c^2}\vec{v} \times \vec{E}) \end{aligned}$$

**Example 4.2** (Transformation of Lorentz Force). Two particles with charge  $q$  are distance  $d$  apart and are moving perpendicular to  $\vec{d}$  with velocity  $\vec{v}$ . In the system  $S$  where the particles are stationary, the Lorentz Force is simply:

$$S : \vec{F} = q \cdot \vec{E} = \frac{q^2}{4\pi\epsilon_0} \frac{1}{d^2} \hat{d}$$

And in the external system  $S'$  the electromagnetic field are

$$\vec{E} = (E^1, \gamma E^2, \gamma E^3), \quad \vec{B} = (B^1 = 0, -\gamma \frac{v}{c^2} E^3, \gamma \frac{v}{c^2} E^2)$$

and so we see

$$\tilde{\vec{F}} = \frac{1}{\gamma} \vec{F}$$

## 4.1 Maxwell Equations

The (inhomogenous) Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = \rho, \quad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{j}$$

become the simple equation

$$\partial_\mu F^{\mu\nu} = j^\nu$$

We can also combine the scalar and vector potential into the four vector

$$A^\mu := (\Phi, \vec{A}) = (\Phi, A^1, A^2, A^3)$$

and the equations

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} \times \vec{A}$$

get simplified with the contravariant derivative

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

For the homogenous Maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

we define the **dual field tensor**  $\tilde{F}$  as

$$\tilde{F}_{\mu\nu} := \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = 2 \begin{pmatrix} 0 & -\vec{B}^T \\ \vec{B} & E^\times \end{pmatrix}$$

and get the simple form

$$\partial^\mu \tilde{F}_{\mu\nu} = 0$$

In the Lorentz-Gauge: ( $\partial_\mu A^\mu = 0$ ) the maxwell equations become

$$\partial^2 A^\mu = j^\mu$$

**Example 4.3.** The 4-vector formulation of the Maxwell equations give a quick derivation of the continuity equation.

Because the electromagnetic field tensor is antisymmetric, we see that

$$\partial_\mu \partial_\nu F^{\mu\nu} = -\partial_\mu \partial_\nu F^{\nu\mu} = 0$$

and so we get

$$\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0$$

## 4.2 Energy Impulse Tensor

Given charges  $q_n$  at positions  $\vec{r}_n(t)$  with energies  $E_n(t)$  and momentum  $p_n(t)$  the charge and current density is given by

$$\rho(\vec{x}, t) = \sum_n q_n \delta(\vec{x} - \vec{r}_n(t)), \quad \vec{j}(\vec{x}, t) = \sum_n q_n \frac{d\vec{r}_n(t)}{dt} \delta(\vec{x} - \vec{r}_n(t))$$

So the energy and energy current densities are

$$\sum_n E_n(t) \delta(\vec{x} - \vec{r}_n), \quad \sum_n E_n(t) \frac{d\vec{r}_n(t)}{dt} \delta(\vec{x} - \vec{r}_n(t))$$

aswell as the impulse and impulse current densities

$$\sum_n p_n^i(t) \delta(\vec{x} - \vec{r}_n(t)), \quad \sum_n p_n^i(t) \frac{d\vec{r}_n(t)}{dt} \delta(\vec{x} - \vec{r}_n(t))$$

respectively.

Using tensor notation, we combine charge and current densities to a four vector

$$j^\mu := (c\rho, \vec{j})$$

define the four-vector impulse  $p^\mu = (E, \vec{p})$  and combine energy and momentum into the **Energy Momentum Tensor**

$$T^{\mu\nu} := \sum_n p_n^\mu(t) p_n^\nu \frac{dr_n^\nu}{dt} \delta(\vec{x} - \vec{r}_n(t))$$

which corresponds to a  $4 \times 4$  matrix of the Layout

$$T^{\mu\nu} = \begin{pmatrix} \text{Energy density} & \text{Energy current density} \\ \text{Impulse density} & \text{Impulse current density} \end{pmatrix}$$

where the impulse current density corresponds to a  $3 \times 3$  submatrix.

If we set  $c = 1$ , then energy and impulse are the same so we see that  $T^{\mu\nu}$  is symmetric, giving us the form

$$T^{\mu\nu} = \sum_n \frac{p_n^\mu p_n^\nu}{E_n} \delta(\vec{x} - \vec{r}_n(t))$$

### 4.3 Energy-impulse tensor in electromagnetic field

If we have charges  $q_n$  in an electromagnetic field, then the energy and impulse are *not* conserved but instead go into the electromagnetic field.

The energy impulse tensor in an electromagnetic field is

$$T_{\text{em}}^{\mu\nu} := F^\mu{}_\rho F^{\rho\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

which is symmetric and gauge invariant. Its components are

$$w := T_{\text{em}}^{00} = \frac{\vec{E}^2 + \vec{B}^2}{2} \quad \text{and} \quad T_{\text{em}}^{0i} = T_{\text{em}}^{i0} = (\vec{E} \times \vec{B})_i$$

where  $w$  is the energy density of the electromagnetic field. Defining the sum

$$\Theta^{\mu\nu} := T^{\mu\nu} + T_{\text{em}}^{\mu\nu}$$

we see that energy and momentum is conserved

$$\partial_\nu T_{\text{em}}^{\mu\nu} = -F^{\mu\nu} j_\nu \implies \partial_\nu \Theta^{\mu\nu} = 0$$

Taking the sum of the four-momentum of the charges

$$P_{\text{charges}}^\mu = \sum_n p_n^\mu$$

and the momentum of the electromagnetic field

$$P_{\text{em}}^\mu = \int d^3\vec{x} T^{\mu 0}$$

the total momentum is conserved.

$$P^\mu := \int d^3\vec{x} \Theta^{\mu 0} = P_{\text{charges}}^\mu + P_{\text{em}}^\mu = \text{const}$$

We also define the **Poynting vector** as

$$S^i = T_{\text{em}}^{0i} = (\vec{E} \times \vec{B})^i$$

so the equation

$$\partial_0 T^{00} + \partial_i T^{i0} = -F^{0i} j_i$$

turns into the well-known formula

$$\frac{\partial w}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{E} \cdot \vec{j}$$

### 4.4 Radiation

A moving charge  $q$  generates a Potential  $A^\mu(x)$  given by

$$\begin{aligned} A^\mu(x) &= \int d^4y G_{\text{ret}}(x-y) j^\mu(y) \\ &= \frac{q}{4\pi} \frac{v^\mu(t_{\text{ret}})}{\langle v(\tau_{\text{ret}}) \rangle, x - r(\tau_{\text{ret}})} \end{aligned}$$

where  $\tau_{\text{ret}}$  is the solution to the equation

$$x^0 - r^0(\tau_{\text{ret}}) = |\vec{x} - \vec{r}(\tau_{\text{ret}})|$$

this is known as the **Lienard-Wiechert** potential.

**Example 4.4.** If the charge is stationary, i.e  $v^\mu = (1, \vec{0})$  and  $r(\tau) = (r^0, \vec{r})$  for some constant  $\vec{r}$ . We get

$$A^\mu(x) = \frac{q}{4\pi} \frac{(1, \vec{0})}{|x^0 - r^0|} \Big|_{\text{ret}} = \frac{q}{4\pi} \frac{(1, \vec{0})}{|\vec{x} - \vec{r}|}$$

which is just the Coulomb potential.

**Example 4.5** (Circular motion). The charge  $q$  is now moving in a circle radius  $R$  and frequency  $\omega$  and we want to find out the potential at the center of the circle  $\vec{x} = (0, 0, 0)$ . With  $\tau = t\gamma$ , we find that the position of the charge is

$$r(\tau) = (\tau\gamma, r \cos(\omega\tau\gamma), -r \sin(\omega\tau\gamma), 0)$$

so the velocity is

$$v = \frac{dr(\tau)}{d\tau} = (\gamma, -\gamma\omega r \sin(\omega\tau\gamma), -\gamma\omega r \cos(\omega\tau\gamma), 0)$$

we find out  $\tau_{\text{ret}}$  by solving the equation

$$x^0 - r^0(\tau_{\text{ret}}) = |\vec{x} - \vec{r}(\tau_{\text{ret}})|$$

since  $|\vec{x} - \vec{r}(\tau)|$  is always constant  $R$ , we get

$$x^0 - \gamma\tau_{\text{ret}} = r \implies \tau_{\text{ret}} = \frac{x^0 - R}{\gamma}$$

so after calculating the scalar product

$$\langle v(\tau_{\text{ret}}), x - r(\tau_{\text{ret}}) \rangle = \gamma R$$

we can use the formula for the potential to find

$$\begin{aligned} A(x) &= \frac{q}{4\pi} \frac{v(\tau_{\text{ret}})}{\langle v(\tau_{\text{ret}}), x - r(\tau_{\text{ret}}) \rangle} \\ &= \frac{q}{4\pi} \frac{1}{R} \begin{pmatrix} 1 \\ -R\omega \sin(\omega(x^0 - R)) \\ -R\omega \cos(\omega(x^0 - R)) \\ 0 \end{pmatrix} \end{aligned}$$

## 4.5 Fields of moving charges

From the Lienard-Wiechert Potential, we can calculate the Electromagnetic field tensor with the formula

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

By defing the vector

$$R^\mu := x^\mu - r^\mu(\tau_{\text{ret}}) =: |\vec{R}|(1, \hat{n})$$

we can write the electric and magnetic field with

$$\begin{aligned} \vec{E} &= \frac{q}{4\pi(1 - \hat{n}\vec{v})^3} \left[ \frac{(1 - \vec{v}^2)}{|\vec{R}|^2} (\hat{n} - \vec{v}) + \frac{1}{|\vec{R}|} \hat{n} \times ((\hat{n} - \vec{v}) \times \vec{a}) \right] \\ &= \hat{n} \times \vec{E} \end{aligned}$$

To describe the radiation of an accelerated particle, we define

$$dP = \frac{dW}{dt} = d\vec{A} \cdot \vec{S}$$

where

$$\vec{S} = \vec{E} \times \vec{B}$$

is the **Poynting-Vector**.

In the non-relativistic case ( $v \ll c$ ) we have

$$\frac{dP_{\text{rad}}}{d\Omega} = \frac{q^2}{16\pi^2} |\vec{a}|^2 \sin^2 \Theta$$

and the total power is given by **Larmor's Formula**

$$P_{\text{rad}} = \frac{q^2}{4\pi} \frac{2}{3} |\vec{a}|^2 \left( \frac{1}{\epsilon_0 c} \right)$$

In the relativistic case, the power radiated per solid angle is

$$\frac{dP_{\text{rad}}}{d\Omega} = \frac{q^2}{16\pi^2} |\vec{a}|^2 \frac{\sin^2 \Theta}{(1 - v \cos \Theta)^6}$$

so the total power radiated over all angles is

$$P_{\text{rad}} = \int d\Omega \frac{dP_{\text{rad}}}{d\Omega} = \frac{q^2}{4\pi} \frac{2}{3} \gamma^6 [|\vec{a}|^2 - |\vec{v} \times \vec{a}|^2]$$

In circular motion, the  $\gamma$  dependency is of order  $\gamma^4$ , and in linear motion, it is of order  $\gamma^6$ .

## 4.6 Natural units

Recall the SI (système international) units

Quantity	Symbol	Base unit	Name
Length	$\ell$	1m	Meter
Time	$t$	1s	Second
Mass	$m$	1kg	Kilogram
Current	$I$	1A	Ampère

Table 2: SI Units

For example, Coulomb's Law is

$$F = \frac{q^2}{4\pi\epsilon_0 r^2}$$

where

$$\begin{aligned} [F] &= \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \\ [q] &= \text{A} \cdot \text{s} \\ [\epsilon_0] &= \frac{\text{A}^2 \cdot \text{s}^4}{\text{m}^3} \\ [r] &= \text{m} \end{aligned}$$

Now we define the set of natural units

$$[\ell, t, m, I] \mapsto [c, \hbar, m_e, \epsilon_0]$$

Quantity	Symbol	Base unit	Name
Speed	$v$	$1c$	Speed of light
		$1\hbar$	Plank's reduced constant
Mass	$m$	$1m_e$	Mass of electron
		$1\epsilon_0$	
Length	$1 \left[ \frac{\hbar}{m_e c} \right]$	$3.86 \cdot 10^{-13} \text{m}$	Natural unit length
Time	$1 \left[ \frac{\hbar}{m_e c^2} \right]$	$1.288 \cdot 10^{-21} \text{s}$	Second
Mass	$m$	$1m_e = 9.109 \cdot 10^{-31} \text{kg}$	Mass of Electron
Charge	$q$	$1\sqrt{\epsilon_0 \hbar c} = 5.29 \cdot 10^{-19} \text{A} \cdot \text{s}$	

Table 3: Natural units

## 5 Lagrange Formalism of Electrodynamics

A charged particle in an electromagnetic field experiences the Lorentz force

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

In special relativity, the momentum is  $m\vec{v}\gamma = \frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}$ .  
And the Lagrangian is

$$\mathcal{L}(\vec{x}, \vec{v}, t) = -\sqrt{1 - \vec{v}^2} - e(\Phi - \vec{v} \cdot \vec{A})$$

Recall that the Euler-Lagrange equations were

$$\frac{d}{dt} \frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial x_i}$$

$$\mathcal{L}(A^\mu, \partial^\nu A^\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

## 6 Maths

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

For  $f(x)$  with roots  $(x_i)_{i \in I}$

$$\delta(f(x)) = \sum_{i \in I} \frac{1}{|f'(x_i)|} \delta(x - x_i)$$

### 6.1 Tensor Stuff

$$\epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$$