Analysis II – Summary

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1 Differential Geometry

1.1 Implicit function theorem

Theorem Implicit function theorem

Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ open, $(x_0, y_0) \in U$, $F : U \to \mathbb{R}^m$ continuous that satisfies

(a) $F(x_0, y_0) = 0$

(b) For all $k = 1, \ldots, m$ the partial derivatives $\partial_{y_k} F : U \to \mathbb{R}^m$ exist and are continuous.

(c) The matrix $A = (\partial_{y_k} F_j(x_0, y_0)))_{j,k} \in \mathbb{R}^{m \times m}$ is invertible.

Then the equation $F(x, y) = 0$ can "solved for y" as a function of x in a small region around x_0.
That is, there exist $r, s > 0$ and a continuous function $f : B(x_0, r) \to B(y_0, s)$ such that for all $(x, y) \in B(x_0, r) \times B(y_0, s)$:

$$F(x, y) = 0 \iff y = f(x)$$

Additionally if $F \in C^d(U)$, then $f \in C^d(B(x_0, r))$ and has derivative

$$Df(x) = - ((D_y F)(x, f(x)))^{-1} \circ (D_x F)(x, f(x))$$

Theorem Inverse function theorem

Let $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ d-times differentiable and $x_0 \in U$ such that $Df(x_0)$ is invertible.
Then in a small region round x_0, f is invertible. That is, there exists open neighborhoods $U_0 \subseteq U$ of x_0 and $V_0 \subseteq \mathbb{R}^n$ if $y_0 = f(x_0)$. such that $f|_{U_0}$ is bijective.
Its inverse f^{-1} has derivative

$$(Df^{-1})(y) = (Df(x))^{-1}$$

for all $x \in U_0$ and $y = f(x) \in V_0$.

1.2 Submanifolds

Definition

A subset $M \subseteq \mathbb{R}^n$ is called a k-dimensional **smooth submanifold** (of $\mathbb{R}^n$), if M is locally isomorphic to $\mathbb{R}^k$.
That is, for every point $p \in M$, there exists an open neighborhood $U_p \subseteq \mathbb{R}^n$ of p and a diffeomorphism $\varphi_p : U_p \to V_p = \varphi_p(U_p) \subseteq \mathbb{R}^n$ such that

$$\varphi_p(U_p \cap M) = \{y \in V_p \big| y_i = 0 \text{ for all } i > k\}$$

We call φ a **map** of M around p and its inverse $\varphi^{-1} : V_p \to U_p$ a **parametrisation** of M around p.

Theorem Constant rank theorem

Let $U \subseteq \mathbb{R}^n$ open, $F : U \to \mathbb{R}^m$ smooth. If for all $p \in M$ the derivative $DF(p) : \mathbb{R}^n \to \mathbb{R}^m$ is surjective, then the niveau set of zeros

$$M = \{p \in U \big| F(p) = 0\}$$

is an $(n - m)$-dimensional smooth submanifold.

Definition

Let $M \subseteq \mathbb{R}^n$ be a k-dimensional submanifold.

- The **tangent space** of M at $p \in M$ is the k-dimensional vector space

$$T_p M = \{\gamma'(0) \big| \gamma : (-1, 1) \to M \text{ differentiable with } \gamma(0) = p\}$$

- The **tangent bundle** of M is the collection of tangent spaces of every point $p \in M$:

$$TM = \{(p, v) \in \mathbb{R}^n \times \mathbb{R}^m \big| p \in M, v \in T_p M\}$$

- The **canonical projection** is the map

$$\pi : TM \to M, \quad (p, v) \mapsto p$$

A map $s : M \to TM$ is called a **section** (or vector field) if $\pi \circ s = \mathrm{id}_M$.

$$M \xrightarrow{s} TM \xrightarrow{\pi} M$$

Proposition

Let $U \subseteq \mathbb{R}^n$ open, $F : U \to \mathbb{R}^m$ smooth and $M = F^{-1}(0)$. If for all $p \in M$ the derivative $DF(p) : \mathbb{R}^n \to \mathbb{R}^m$ is surjective, then

$$TM = \{(p, v) \in U \times \mathbb{R}^n \big| F(p) = 0 \text{ and } DF(p)(v) = 0\}$$

and in particular, for all $p \in M$

$$T_pM = \operatorname{Ker} DF(p)$$