Topology – Lecture Notes

Han-Miru Kim

April 26, 2021

# 1 Introduction

Topology is the study of topological spaces and continous maps between them.
We will introduce topology through three aspects:

- Topology as the part of Analysis which concerns itself about general properties of continuous functions. More concretely, we know that for a continuous function $f : \mathbb{R} \to \mathbb{R}$, that on a compact interval, $f$ attains its maximum and minimum. We also know that the intermediate value theorem holds.

  In the language of topology, these theorems say that the image of *compact* subsets is again *compact*. It also says that the image of *connected* subsets is again *connected*.

  One generalisation of the intermediate theorem is Jordan's curve theorem: Let $\gamma$ be a closed simple curve in $\mathbb{R}^2$. Then $\mathbb{R}^2 \setminus \gamma$ has two connected components.

- Topology as the study, construction and classification of topological spaces. We can look at topology as the generalisation of metric spaces by replacing the notion of *distance* with the notion of *neighborhoods*. For example, we can try to use the square $Q = [0,1]^2 \subseteq \mathbb{R}$ to generate a new object by introduction an equivalence relation $\sim$, where $(1,y) \sim (0,y) \forall y \in [0,1]$. The *quotient space* is obtained by *glueing* two edges of the square which generates a cylindrical shape as a subset of $\mathbb{R}^3$. Using different kinds of equivalence relations, we can get the Moebius strip, a Torus or the Klein bottle. Another nice example is when we start with $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Consider the quotient space

  $$\mathbb{C}^* / \mathbb{R}_{>0} : \quad \text{where} \quad z \sim w, \text{ if } \frac{z}{w} \in \mathbb{R}_{>0}$$

  Which identifies two points, if they have the same argument. This space is *homeomorphic* to the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$. Note that $(\mathbb{C}^*, \cdot)$ is a group, with $(\mathbb{R}_{>0}, \cdot)$ as a normal divisor and $(\mathbb{S}^1, \cdot)$ is another subgroup of $(\mathbb{C}^*, \cdot)$. Another result is the classification of surfaces: Every *orientable* compact surface without a boundary is homeomorphic to exactly one of the following surfaces:

  $$\text{the unit ball} \mathbb{S}^2, \quad \text{higher tori of genus } n$$

  Other algebraic invariations that are used to classify topological spaces can be Numbers (*euler characateristic*), Groups (*Fundamental Groups*), fields, rings, vectorspaces etc.

- Construction and criteria for existence of continous maps. For example, let $U \subseteq \mathbb{C}^*$ be open and connected. Consider the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$, and the inclusion $\iota : U \to \mathbb{C}^*$ with the diagram $\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ U \arrow[ur, "\log"] \arrow[r, dashed, "\iota"] & \mathbb{C}^* \end{tikzcd}$ Such a map $\log : U \to \mathbb{C}$ exists if $U$ is simply connected.

## 1.1 The topological space

The central idea behind the definition of a topological space is the notion of an *open* set. In Analysis we know that a subset $U \subseteq \mathbb{R}$ is *open*, if for every $x \in U : \exists \epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$.

(a) Looking at its properties we know that the union of open sets is open and that finite intersections of open sets is open. Moreover, the empty set $\emptyset$ and $\mathbb{R}$ itself are open.

(b) We called a set $A \subseteq \mathbb{R}$ *closed*, if $\mathbb{R} \setminus A$ was open.

(c) We noted that the *closure* of the open interval was the closed interval $[0, 1]$.

(d) Sets can be neither open nor closed, or even both as the examples $[0, 1)$ and $\mathbb{R}$ show.

The following definition is a generalisation of openness on arbitrary sets. It turns out that just this alone is enough to define all the concepts described above!

**Definition 1.1.** Let $X$ be a set. A **topology** on $X$ is a collection of subsets $\tau \subseteq \mathcal{P}(X)$ such that

- The union of open sets is open: If $\{U_i\}_{i \in I}$ be a collection of open subsets $U_i \in \tau$, then $\bigcup_{i \in I} U_i \in \tau$
- Finite intersections of open subsets are open: $U, V \in \tau \implies U \cap V \in \tau$
- The empty set and $X$ itself are open: $\emptyset, X \in \tau$

Where subsetes $U \in \tau$ are called *open* and $(X, \tau)$ is called a topological space.

**Example 1.2.** It is no surprise that $\mathbb{R}$ with the Analysis-open subsets is a topological space. We call this topology the *euclidean* space. The proof is trivial.

**Definition 1.3.** Let $X$ be a topological space

- A subset $A \subseteq X$ is called **closed**, if $A^c = X \setminus A$ is open.
- A subset $U \subseteq X$ is called a **neighborhood** of $x \in X$, if there exists an open set $V \subseteq X$ such that $x \in V \subseteq U$

Let $x \in X, B \subseteq X$. We call $x$

- an **inner point** of $B$ is $B$ is a neighborhood of $x$.
- an **exterior point** of $B$, if $B^c$ is a neighborhood of $x$.
- a **boundary point** of $B$ if neither $B$ nor $B^c$ are neighborhoods of $x$.

Analagously, define the

- **interior** $B^\circ := \{x \in X | x \text{ is an inner point of } B\}$
- **closure** $\overline{B} := \{x \in X | x \text{ is } \textbf{not} \text{ an exterior point of } B\}$
- **boundary** $\partial B := \{x \in X | x \text{ is a boundary point of } B\}$

There are of course alternative ways to define a topology.

- Instead of focusing on the open sets, we could just as well have started with the closed subsets, where we swap the finiteness condition for unions and intersections.

- Haussdorff's approach was to focus on neighborhoods instead of the open sets: A topological space is a tuple $(X,\mathcal{U})$ consisting of a set $X$ and a collection of families of subsets $\mathcal{U} = \{\mathcal{U}_x\}_{x\in X}$ with $\mathcal{U}_x \in \mathcal{P}(X)$ (the neighborhoods of $x$) such that
  (a) $x \in U_x$ and $X$ is a neighborhood of every point.
  (b) If $V$ contains a neighborhood of $x$, then $V$ is also a neighborhood of $x$
  (c) The intersection of two neighborhoodsof $x$ is again a neighborhood of $x$.
  (d) Every neighborhood of $x$ contains a neighborhood of $x$ that contains all of its points.
- An approach we will take a look at in the exercise classes is using the **Hull axioms**: A topological space is a tuple $(X, )$ consisting of a set $X$ and a map $: \mathcal{P}(X) \to \mathcal{P}(X)$ that satisfies
  (a) $\overline{\emptyset} = \emptyset$
  (b) $A \subseteq \overline{\emptyset}$ for all $A \subseteq X$
  (c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all $A, B \subseteq X$

## 1.2 Metric spaces

**Definition 1.4.** A **metric space** is a tuple $(X, d)$ consisting of a set $X$ and a **metric** $d : X \times X \to \mathbb{R}$ such that

(a) $d$ is positive definite: $d(x, y) \geq 0, \forall x, y \in X$ and $d(x, y) = 0 \iff x = y$

(b) $d$ is symmetric: $d(x, y) = d(y, x), \forall x, y \in X$

(c) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X$

The euclidean metric on $\mathbb{R}^n$ given by $d(x, y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ makes $\mathbb{R}^n$ a metric space.
We can turn any metric space into a topological space as follows

**Definition 1.5.** Let $(X, d)$ be a topological space. We call the collection

$$\tau_d := \{U \subseteq X \big| \forall x \in U \exists \epsilon > 0 : B(x, \epsilon) \subseteq U\}$$

the **induced topology** on $X$

**Example 1.6.** The euclidean metric is not the only valid metric. The **discrete metric** $d$ given by

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = 0 \end{cases}$$

and its induced topology is the **discrete topology** $\tau_{\text{disk}} = \mathcal{P}(X)$..

We might ask: Is every topological space **metrisable**? That is, is there a metric $d$ on $X$ such that the induced topology $\tau_d$ on $X$ is the topology we started with?
The answer is No. Take for example the set $X = \{0, 1\}$ and take the indiscrete topology $\tau = \{\emptyset, X\}$. The positive definiteness forbids this.
We also might ask if we lose some information by turning a metric space into the induced topological space. Can we always recover the metric from an induced topology? The answer again is No.

**Example 1.7.** Let $(X, d)$ be a metric space and define $\tilde{d}$ as

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} < 1$$

We can show that the two metrices induce the same topology on $X$. This quickly follows from the fact that

$$d(x, y) < \epsilon \iff \frac{d(x, y)}{1 + d(x, y)} < \frac{\epsilon}{1 + \epsilon}$$

where we used the fact that the function $f(x) = \frac{x}{1+x}$ is strictly monotonously increasing since $f'(x) > 0$ for $x > -1$.

Even worse/better: All metrics on $\mathbb{R}^n$ that come form a norm induce the euclidean topology on $\mathbb{R}^n$. This follows form the equivalency of the norms in $\mathbb{R}^n$ that we know from Analysis II.

## 1.3 Subspaces, Sums and Products

Consider the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ as a topological space. We can use the topology on $\mathbb{R}^3$ to give a topology to $\mathbb{S}^2$.

**Definition 1.8.** Let $(X, \tau)$ be a topological space and $Y \subseteq X$. Then

$$\tau_Y := \{U \cap Y \big| U \in \tau\}$$

defines a topology on $Y$ and is called the **subspace** topology on $Y$.

For example, for $Y = [0, 1] \subseteq \mathbb{R}$, the "half open" interval $[0, \frac{1}{2})$ is open in $[0, 1]$.
Note the following. For $B \subseteq X$ we have

- $\partial B = \overline{B} \setminus B^\circ$
- $\partial\partial B \subseteq \partial B$
- If $B$ is closed, then $\partial B = \partial\partial B$. This follows trivially from the fact that $(\partial B)^\circ$ is empty.

For sets $X, Y$ let $X \sqcup Y$ denote their disjoint union $X \times \{0\} \cup Y \times \{1\}$ and $X \times Y$ their cartesian product (as sets).

**Definition 1.9.** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two topological spaces.

- Their **coproduct** is the topological space $(X \sqcup Y, \tau_{X \sqcup Y}$, where

$$\tau_{X \sqcup Y} := \{U \sqcup V \big| U \in \tau_X, V \in \tau_Y\}$$

- Their (cartesian) **product** is the topological space $(X \times Y, \tau_{X \times Y})$, where

$$\tau_{X \times Y} := \{W \subseteq X \times Y \big| \forall (x, y) \in W \exists U \in \tau_X, \exists V \in \tau_Y : (x, y) \in U \times V \subseteq W\}$$

Note that *not* every open subset $W \subseteq X \times Y$ is of the form $W = U \times Y$, for $U \subseteq X$, $V \subseteq Y$ open. For example, the product topology on $\mathbb{R}^2$ contains open balls.

## 1.4 Basis and Subbasis

Consider $\mathbb{R}^n$ with the euclidean topology. We can intuitively see that every open set $U \subseteq \mathbb{R}^n$ can be written as the union of open balls. So $U \subseteq \mathbb{R}^n$ is open if and only if there exists $(x_i, r_i)_{i\in I}$ such that $U = \bigcup_{i\in I} B_{r_i}(x_i)$.

**Definition 1.10.** Let $X$ be a topological space and $\mathcal{B}$ a collection of open sets.

- We call $\mathcal{B}$ a **basis** of the topology, if every open set can be written as a union of elements of $\mathcal{B}$.
- $\mathcal{B}$ is called a **subbasis** of the topology, if every open set can be written as a union of finite intersection of elements of $\mathcal{B}$.

**Remark 1.11.** Every basis is a subbasis. Every collection $\mathcal{B}$ of open sets is the subbasis of a unique topology, the topology **generated** by $\mathcal{B}$, which is the smallest topology that contains $\mathcal{B}$.

## 1.5 Continuous maps

Recall the definition of continuity from Analysis. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous*, if

$$\forall x \in \mathbb{R} : \forall \epsilon > 0 \exists \delta > 0 : \forall y \in \mathbb{R} : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

and compare this with the topological definition of continuity:

**Definition 1.12.** Let $X, Y$ be topological spaces. We say that a function $f : X \to Y$ is **continuous**, if the preimage of open subsets is open. So $\forall V \in \tau_Y : f^{-1}(V) \in \tau_X$
We say that $f$ is continuous at $x_0 \in X$, if for every neighborhood of $f(x_0)$ there exists a neighborhood $U$ of $x_0$ such that $f(U) \subseteq V$.

The following are pretty easy to prove:

**Remark 1.13.** Let $X, Y$ be topological spaces and $f : X \to Y$

(a) $f$ is continuous if and only if $f$ is continuous at $x \in X$ for every $x \in X$.

(b) The above notion of continuity is equivalent to the definition of the $\epsilon - \delta$ definition of continuity on a metric space..

(c) The identity $\mathrm{id}_X$ is continous the composition of continous maps is continuous.

(d) If $f$ is continuous, then the restriction $f|_A$ for a subset $A \subseteq X$ is continuous (in the subspace topology). In particular, the inclusion mapping is continuous.

(e) If $g : X \to Z$ is another function, then both $f$ and $g$ are continuous if and only if the product $(f, g) : X \to Y \times Z$ is continuous.

(f) If $X$ is discrete, then any function $f : X \to Y$ is continuous. If $X$ is indiscrete, then only constant functions are continuous. If $Y$ is indiscrete, then any function into $Y$ is continuous.

Note that for the coproduct, the inclusions $\iota_X : X \to X \sqcup Y$ and $\iota_Y : Y \to X \sqcup Y$ are continuous.
For the product, the projection mappings $\pi_X : X \times Y \to X$ $\pi(x, y) = x$ are continuous with respect to the product topology.
We can also use this property to define the product spaces.

The product topology is the *coarsest* topology on $X \times Y$, such that the projection mappings $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are continuous. This means that any other topology for which $\pi_X$ and $\pi_Y$ are continuous is bigger than $\tau_{X\times Y}$
Similar to the notion of Group isomorphisms, isomorphisms of vectorspaces, bijections of sets etc, we get the notion of isomorphism for topological spaces.

**Definition 1.14.** A bijective map $f : X \to Y$ such that its inverse $f^{-1}$ is continuous is called a **homeomorphism** and we write $f : X \overset{\cong}{\to} Y$ or $X \cong Y$.

## 1.6 Connectedness

We intuitively know what it means for a set to be connected. To put this in the language of topology, we obtain the following definition:

**Definition 1.15.** A topological space is **connected**, if it can't be split into the disjoint union of two open, non-empty sets. Alternatively: If $X = U \cup V$ for $U, V$ open, non-empty, then $U \cap V \neq \emptyset$.

**Lemma 1.16.** *The connected subsets in $\mathbb{R}$ are exactly the intervals*

. Let $I \subseteq \mathbb{R}$ be connected. If $x, y \in I$ with $x \leq y$. Then $x \leq z \leq y \implies z \in I$ or else we could write $I$ as the disjoint union of the non-empty open sets.

$$I = I \cap (-\infty, z) \sqcup I \cap (z, \infty)$$

On the other hand, let $I \subseteq \mathbb{R}$ be an interval and $U, V$ oen and non-empty such that $I = U \cup V$. We then can show that $U \cap V \neq \emptyset$ by using the axiom of completeness for $\mathbb{R}$. To do so, let $a \in U$ and $b \in V$. Then without loss of generality $a < b$. Set

$$s := \sup\{x \in U \big| x < b\} \in I$$

Since $I = U \cup V$, at least $s \in U$ or $s \in V$ has to be true.
If $s \in U$ then since $U$ is open, there exists an $\epsilon > 0$ such that

$$(s - \epsilon, s + \epsilon) \subseteq U \implies b \in U$$

If $s \in V$ it follows analogously that $U \cap V \neq \emptyset$. ☐

The notion of connected sets gives us the generalisation of the intermediate value theorem.

**Theorem 1.17.** *The image of connected sets under continuous functions is connected.*

. Let $f : X \to Y$ be a continuous function and $A \subseteq X$ connected. Write $B = f(A) \subseteq Y$ and assume that $B = U \cup V$ for $U, V \subseteq Y$ open, non-empty. By continuity of $f$, their preimages $f^{-1}(U), f^{-1}(V)$ are open and non-empty. Moreover, since $B = f(A)$ we have that $f^{-1}(U) \cup f^{-1}(V) = A$. Since $A$ is connected, there exists an $x_0 \in f^{-1}(U) \cap f^{-1}(V)$, and therefore $f(x_0) \in U \cap V$. ☐

Usually, the definition of connectedness matches with our intuition, but there are some examples where that is not the case. A *stronger* type of connectedness is that of path-connectedness

**Definition 1.18.** A **path** on a topological space $X$ is a continuous function $\gamma : [0, 1] \to X$. If we write $a = \gamma(0)$ and $b = \gamma(1)$, we We say that $\gamma$ is an $a - b$ path and that it *connects* $a$ and $b$.
A topological space $X$ is said to be **path connected**, if for any two points $a, b \in X$ there exists an $a - b$ path.

**Remark 1.19.** The following are pretty easy to prove

(a) $X$ path connected $\implies$ $X$ connected.

(b) The image of path connected spaces under continuous maps is path connected.

*Proof.* (a) Let $X = U \sqcup V$ with $U, V \subseteq X$ non-empty and open. We can therefore chose $a \in U$ and $b \in V$. Since $X$ is path connected, there exists a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $gamma(1) = (b)$. But since $[0,1]$ is connected, its image $\gamma([0,1])$ must also be connected.

The decomposition $\gamma([0,1]) \cap U \sqcup \gamma([0,1]) \cap V$ is clearly disjoint, open and non-empty, but since $\gamma([0,1])$ is connected, their intersection is non-empty so

$$\exists c \in (\gamma([0,1]) \cap U) \cap (\gamma([0,1]) \cap V) \subseteq U \cap V$$

(b) It is trivial as the composition of contiuous maps is continuous. For $a, b \in X$ connected by $\gamma$, we can connect their images $f(a), f(b)$ using the composition $f \circ \gamma : [0,1] \to f(X)$.

□

**Example 1.20.** The converse, $X$ connected $\implies$ $X$ path connected is not always true.
Take for example the closure of the **Topologists sine curve**:

$$X := \{0\} \times [-1,1] \sqcup \left\{ \left(t, \sin\frac{1}{t}\right) \in \mathbb{R}^2 \middle| t \in (0,1] \right\} \subseteq \mathbb{R}^2$$

where we write it as the disjoint union $X_0 \sqcup X_1$.
If we let $a = (1, \sin(1))$ and $b = (0,0)$ and assume that there exists a path $\gamma : [0,1] \to X$. Since the set $\{t \in [0,1] | \gamma_1(t) = 0\}$ is non-empty and closed it attains its minimum $s$. But then

$$\gamma_1([0,s]) \subseteq (0,1] \quad \lim_{t \to s} \gamma_1(t) = 0 \quad \gamma_1(0) = 1$$
$$\gamma_1([0,s])) = (0,1] \implies \gamma([0,s) = X_1$$

By the form of the sine curve, we can get a sequence of points $(s_n)_{n \in \mathbb{N}}$ whose image of the inverse sine function are its peaks:

$$\lim_{n \to \infty} s_n = s \quad \text{and} \quad \gamma_2(s_n) = 1$$

but by the form of the sinus curve, we must get a sequcence $(t_n)_{n=1}^{\infty}$ whose image is always the valles of the sine curve.

$$\lim_{n \to \infty} t_n = s \quad \text{and} \quad \gamma_2(t_n) = -1$$

which contradicts continuity of $\gamma$.
On the other hand, we can show that $X$ is connected. Assume we had a disjoint open nonempty partition $X = U \sqcup V$. Analogously to the reasoning above, we can assume without loss of gerality that $U = X_0$ and $V = X_1$.
But since $U = X_0$ should be open in the subspace topology of $\mathbb{R}^2$, there must be an open set $\tilde{U} \subseteq \mathbb{R}^2$ such that $U = \tilde{U} \cap X_0 \ni (0,0)$. It is clear however, that any neighborhood of $(0,0)$ has non-empty intersection with $X_1$.

**Remark 1.21.**
- The integers with the co-finite topology is connected but not path connected.

- For any finite topological space $X$ it is true that $X$ connected if and only if $X$ is path connected.
- For $X, Y$ non-empty topological spaces, then
$$X, Y \text{ (path) connected} \iff X \times Y \text{ (path) connected}$$
- For subsets $A, B \subseteq X$ with $A \cap B \neq \emptyset$ we have
$$A, B \text{ (path) connected} \implies A \cup B \text{ (path) connected}$$
- $X$ is *not* path connected if and only if there exists a continuous map
$$f : X \to (\{0, 1\}, \tau_{\text{disc.}})$$
As a quick consequence, we get that $O_n(\mathbb{R})$ is not connected. We can also show that its conneceted components are those with determinant $\pm 1$ each.

It is not always true that a continuous bijective map has a continuous inverse. Take for example the set $X = \{1, 2\}$ once with the discrete and indiscrete topology and the "identity" on $X$.
We want to study when that is the case.

## 1.7 Separation axioms

**Definition 1.22.** Let $X$ be a topological space and $(x_n)_{n\in\mathbb{N}}$ a sequence in $X$. We say that $a \in X$ is a **limit point** of the sequence if for every neighborhood $U$ of $a$ there exists an $N \in \mathbb{N}$ such that $U$ is a neighborhood of $x_n$ for $n \geq N$

**Example 1.23.** Limit point(s) is not always unique:

- For an indiscrete topological space, every point $a \in X$ is a limit for any sequence in $X$.
- For $(\mathbb{Z}, \tau_{\text{cofin}}$, we have $\lim_{n\to\infty} n = k$ for all $k \in \mathbb{Z}$.

We can however require that the limit be unique. This can be done using the following axiom.

**Definition 1.24.** We say that a topological space $X$ is **Hausdorff** (or $T_2$) if distict points have neighborhoods that are disjiont. i.e
$$\forall x, y \in X, x \neq y \exists U, V \text{ open such that } x \in U, y \in V, U \cap V = \emptyset$$

There are more separation axioms, which are labelled $T_0, T_1, T_{2.5}, T_3, T_{3.5}, T_4$ which are in general independent.

**Remark 1.25.** We can easily show the following

(a) Every metric space is Hausdorff. As for $x \neq y \in X$ we can take balls of radius $\frac{d(x,y)}{2}$ around $x$ and $y$.

(b) Singletons in Hausdorff spaces are closed.

(c) Every sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ has at most one limit point.

(d) Subspaces of $T_2$ spaces are $T_2$ (with the subspace topolgy)

(e) For $X, Y$ two topological spaces
$$X, Y \text{ are } T_2 \iff X \times Y \text{ are } T_2 \iff X \sqcup Y \text{ are } T_2$$

In general, $T_2$-ness is not conserved by continuous functions.

## 1.8 Compactness

Recall Heine-Borel's theorem on compactness of subspaces of $\mathbb{R}^n$.

$$K \subseteq \mathbb{R}^n \text{ compact} \iff K \text{ closed and bounded}$$

Such a definition is obviously not compatible with the language of topology as the notion of *bounded-ness* is not well defined. We need a more "topological" definition for it

**Definition 1.26.** A topological space $X$ is **compact** if every open covering of $X$ has a finite subcovering. That is, if $(U_i)_{i\in I}$ is a collection of open subsets such that $\bigcup_{i\in I} U_i = X$ there exists a finite subset $J \subseteq I$ such that $\bigcup_{j\in J} U_j = X$.
We say that a subset $A \subseteq X$ is compact, if it is compact in the subspace topology.

**Example 1.27.** Of course, any finite topological space is compact.

- $\mathbb{R}$ is not compact. Take for example the open covering of intervals $(-n, n), n \in \mathbb{N}$.
- If a metric space is compact, then it is bounded. So there exists an $R > 0$, $x_0 \in X$ such that $d(x, x_0) < R$ for all $x \in X$. Moreover, compact metric spaces are totally bounded.
- All subsets of a cofinite topological space are compact.

Some generall properties of compact spaces are

(a) Closed subsets of compact spaces are compact.

(b) The image of compact spaces under continuous functions are compact.

(c) For $X, Y$ non-empty topological spaces:

$$X \text{ and } Y \text{ compact} \iff X \times Y \text{ compact} \iff X \sqcup Y \text{ compact}$$

*Proof.* (a) Let $K \subseteq X$ closed and $(U_i \cap K)_{i\in I}$ be an open cover of $K$. Since $K$ is closed, we get the open cover $(U_i)_{i\in I} \cup (X \setminus K)$ of $X$ and the proof follows.

(b) This is trivial: Let $f : X \to Y$ continuous. Since the preimage of open subsets of the image is open any open cover of $f(X)$ induces an open cover of $X$.

(c) This follows from the continuity of the projection/inclusion mappings.

□

**Lemma 1.28.** *Let $X$ be $T_2$ and $K \subseteq X$ compact. Then $K$ is closed.*

*Proof.* Let $p \in K^c$. Since $X$ is Hausdorff, for all $x \in K$ we obtain open subsets $U_x, V_x$ which are disjoint neighborhoods of $x$ and $p$. This generates the open covering $(U_x)_{x\in K}$ of $K$. Since $K$ is compact, this gives us a finite subcovering $(U_j)_{j\in J\subseteq K}$. Define $V := \bigcap_{j\in J} V_j$ as the finite intersection of open sets. This is an open neighborhood of $p$ that does not intersect $K$. So $K^c$ is open. □

We now are able to state a criterion for the existence of homeomorphisms.

**Theorem 1.29.** *Let $f : X \to Y$ be continuous and bijective. If $X$ is compact and $Y$ is $T_2$, then $f$ is a homeomorphism.*

*Proof.* To show that the inverse $f^{-1}$ is continuous let $A \subseteq X$ be closed. Since $X$ is compact, $A$ is also compact. Then $f(A)$ is again compact as it is the image of a compact set under a continuous function. Our previous lemma says that because $f(A)$ is a compact subset of a $T_2$ space it is also closed. $\square$

This theorem is very nice. For example, it directly shows that taking the third root is continuous because

$$f : [0, 1] \to [0, 1], \quad x \mapsto x^3$$

is continuous and bijective. A counter example would be the map

$$f : [0, 1) \to \mathbb{S}^1, \quad t \mapsto e^{2\pi it}$$

which is continuous and bijective, but its inverse is not continuous at $z = 1$.
The theorem is also a bit stronger than its analogue in the category of continuosly differentiable manifolds, where we require that the Derivativebe locally invertible. So, for $F : U \to V$ continously differentiable bijective, the existence of a continously differentiable inverse depends on whether $D_xF$ is invertible for all $x$.
Compactness has the implication that some local properties can be extended to the entire space if the property is stable under finite unions. For example, if $f : X \to \mathbb{R}$ is locally bounded and $X$ is comact, then $f$ is bounded.

# 2 The Quotient Topology

## 2.1 Definitions

Given an equivalence relation $\sim$ on a set $X$, we define the the set of equivalence classes with

$$X/\sim := \{[x] | x \in X\} = \{\{y \in X | y \sim x\} | x \in X\}$$

and the canonical projection $\pi$ given by

$$\pi : X \to X/\sim, \quad x \mapsto [x]$$

When $X$ is a topological space, how can we define a reasonable topology on $X/\sim$?
We know for vector spaces that in order for an equivalence class to give rise to another vector space, we must require that

$$u \sim v \iff u - v \in W \text{ for some vector space } W$$

For topological spaces, we don't need that.

**Definition 2.1.** Let $X$ be a topological space and $\sigma$ an equivalence relation on $X$. We call a subset $U \subseteq X/\sim$ open in the **quotient topology**, if $\pi^{-1}(U)$ is open in $X$.

**Example 2.2.** (a) For $X = [0, 1]$ let $x \sim y \iff (x < 1, y < 1)$ or $x = y = 1$. This consists of only two equivalence classes. That of $[0]$ and $[1]$. The induced topology is then the **Sierpinsky topology**

$$\tau_{X/\sim} = \{\emptyset, \{[0], [1]\}, \{[0]\}\}$$

(b) For $X = [0, 1]$ we *glue* together the endpoints with the equivalence class

$$x \sim y \iff x = y \text{ or } (x, y) = (0, 1)$$

We will later see that the resulting space is homeomorphic to the circle space $\mathbb{S}^1$

How does this compare to other possible topologies on $X/\sim$? We will show in exercise sheet 4 that the quotient topology is the *finest* topology on $X/\sim$ such that the projection maping $\pi$ is continuous.
We know that compactness and connectedness are preserved under continous maps, so it follows that the quotient space of a compact/connected space is again compact/connected.
A more "topological" way to define the quotient topology is not to think of equivalence classes of an equivalence relation, but rather look at it as the *image of a surjective map* $f : X \to Y$.

## 2.2 Quotients and Maps

**Lemma 2.3.** *Let $X, Y$ be topological spaces and $\sim$ and equivalence relation on $X$ and $f : X/\sim \to Y$ a map. Then $f$ is continuous if and only if $f \circ \pi$ is continous.*
*This can be visualized in the following diagram*

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "f\circ\pi"] & \\ X/\sim \arrow[r, "f"] & Y \end{tikzcd}$$

*Proof.* Well, if $f$ is continous, then $f \circ \pi$ is the composition of continuous maps.
On the other hand if $f \circ \pi$ is continous, then let $V \subseteq Y$ be open. Then

$$(f \circ \pi)^{-1}(V) = \pi^{-1}(w^{-1}(V)) \subseteq X \text{ is open}$$

but by definition of the quotient topology this just says that $f^{-1}(V)$ is open. $\square$

In the previous example (b), we know that for

$$f : [0,1]/\sim \, :\to \mathbb{S}^1, \quad [t] \mapsto e^{2\pi i t}$$

the continuity of $f \circ \pi : [0,1] \to \mathbb{S}^1$ implies continuity of $f$. Furthermore, since $f$ is continous and bijective, $[0,1]/\sim$ is compact and $\mathbb{S}^1$ is Hausdorff, $f$ is a homeomorphism.

**Example 2.4.** For $\mathbb{D}^2 = \{v \in \mathbb{R}^2 | |v| \leq 1\}$ we set $\sim$ on $\mathbb{D}^2$ to

$$v \sim w \iff v = w \text{ or } |v| = |w| = 1$$

Then resulting space is homeomorphic to the sphere $\mathbb{S}^2$
To prove this we give a mapping

$$g : \mathbb{D}^2 \to \mathbb{S}^2, \quad v \mapsto \begin{cases} (0,0,-1) & \text{if } v = (0,0) \\ \left(\frac{\sqrt{1-(2|v|-1)^2}}{|v|} v, 2|v|-1\right) & \text{otherwise} \end{cases}$$

which is continuous. From the previous lemma, it follows that the induced map $f : \mathbb{D}^2/\sim \to \mathbb{S}^2$. And since it is also bijective and $\mathbb{D}^2/\sim$ is compact, $f$ is a homeomorphism.

While the previous lemma can be used to show that a function from the quotient space is continous, we can also look at functions into the quotient space and ask if they are continuous.

**Lemma 2.5.** *Let $X$ and $Y$ be topological spaces, $\sigma$ an equivalence relation on $X$ and $\varphi : Y \to X/\sim$ a map.*
*If there exists a continuous map $\Phi : Y \to X$ such that $\varphi = \pi \circ \Phi$, then $\varphi$ is continuous.*

$$
\begin{tikzcd}
 & X \arrow[d, "\pi"] \\
Y \arrow[ur, dashed, "\Phi"] \arrow[r, "\varphi"] & X/\sim
\end{tikzcd}
$$

The proof is obvious since the composition of continous maps is continuous.

**Example 2.6.** Let $n \in \mathbb{N}, n \geq 2$. Set $X = \mathbb{R}^n$ with the equivalence relation

$$v \sim w \iff v_i = w_i \quad \forall i \leq n-1$$

The quotient map can be thought of compressing the $n$-th dimension on $\mathbb{R}^n$ onto the remaining $n-1$ ones. From what we just showed, the mapping

$$\varphi : \mathbb{R}^{n-1} \to \mathbb{R}^n / \sim, \quad u \mapsto [(u, 0)]$$

is continuous and is a homeomorphism where the inverse map is given by

$$\varphi^{-1} : \mathbb{R}^n / \sim \to \mathbb{R}^{n-1}, \quad [v] \mapsto (v_1, \ldots, v_{n-1})$$

## 2.3 Properties of Quotient spaces

It is trivial to see that if $X$ is a topological space with equivalence relation $\sim$, then

(a) $X$ is compact/connected/path $\implies X/\sim$ is compact/connected/path connected. The converse is not always true.

(b) $X/\sim$ is $T_1$ (singletons are closed) if and only if the equivalence classes are closed in $X$.

**Example 2.7** (Cool examples)**.** Consider $X = \mathbb{R}^2$ and two equivalence classes given by

$$[x]_1 := \begin{cases} \{(x,y) | y \in \mathbb{R}\} & \text{for } |x| \geq \frac{\pi}{2} \\ \{(\arctan(y + \tan(x)), y) | y \in \mathbb{R}\} & \text{for } |x| < \frac{\pi}{2} \end{cases}$$

$$[x]_2 := \begin{cases} \{(x,y) | y \in \mathbb{R}\} & \text{for } |x| \geq \frac{\pi}{2} \\ \{(x, -(\tan(x))^2 + y) | y \in \mathbb{R}\} & \text{for } |x| < \frac{\pi}{2} \end{cases}$$

We can show that $\mathbb{R}^2/\sim_1$ is homeomorphic to $\mathbb{R}$ and $\mathbb{R}^2/\sim_2$ is not $T_2$

## 2.4 Homogenous spaces

Some of the important topological spaces such as $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{S}^1$ etc. carry a group structure. Not only this, their group multiplication (or addition) is *continuous* with respect to the product topology. Same goes for the inverse operation. A generalisation is as follows

**Definition 2.8.** A topological space $G$ equipped with a group operation $\cdot$ is called a **topological group** if the multiplication and the inverse map

$$G \times G \to G, \quad (a,b) \mapsto ab, \quad \text{and} \quad G \to G, \quad a \mapsto a^{-1}$$

are continuous.

Note that the continuity condition has to be viewed in terms of the product topology on $G \times G$.

**Example 2.9.** Since the product topology of discrete spaces is discrete, any group can be turned into a topological space with the discrete topology.
A special case of topological groups are **Lie Groups**, where instead of just requiring continuity, we also want the operations to be *smooth*.
$\mathrm{SO}(n, \mathbb{K}) \subseteq \mathrm{SL}(n, \mathbb{K}) \subseteq \mathrm{GL}(n, \mathbb{K})$ for $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ with the euclidean topology. The reason that the group operations are continuous is that they are polynomial.

**Definition 2.10.** Let $G$ be a topological group and $H \subseteq G$ a subgroup. Then the set of equivalence classes $G/H$ with the quotient topology is called a **homogenous space**.

Note that since $\{e\} \subseteq G$ is also a subgroup, every topoogical space is also a homogenous space.

**Example 2.11.** Let $G = \mathbb{C} \setminus \{0\}$ be equipped with complex multiplication. Here, $H = \mathbb{R}_{>0}$ is a (normal) subgroup and the mapping

$$f : G/H \to \mathbb{S}^1, \quad z \mapsto \frac{z}{|z|}$$

is a homeomorphism and even a group isomorphism. An isomorphism in the category of topological groups.
The reason is because $f \circ \pi : \mathbb{C}^* \to \mathbb{S}^1$ is continuous and because $H$ its kernel of it.

But why study homogenous spaces? It turns out that they have some nice properties:

**Lemma 2.12.** *Let $G$ be a topological group and $H \subseteq G$ a subgroup. Then the quotient space $G/H$ is $T_2$ if and only if $H$ is closed in $G$*

**Corollary 2.12.1.** *Let $G$ be a topological group. Then $G$ is $T_2$ if and only if $\{e\} \subseteq G$ is closed.*

In particular, the distiction between $T_2$ and $T_1$ spaces drops.
In exercise sheet 4, we will prove the **$T_2$ criterion**:

**Proposition 2.12.1** ($T_2$ criterion)**.** *A topological space $X$ is $T_2$ if and only if the diagonal*

$$\Delta_X = \{(x, x) | x \in X\} \subseteq X \times X$$

*is closed. Moreover, if $\sim$ is an equivalence relation on $X$ such that the projection mapping $\pi : X \to X/\sim$ is open. Then $X/\sim$ is $T_2$ if and only if the set*

$$\{(x, y) | x \sim y\} \subseteq X \times X$$

*is closed.*

*Proof Lemma.* Since $T_2 \implies T_1$, $\{e\}$ is closed in $G/H$. But by continuity of the quotient map, so is it's inverse image, which is $H \subseteq G$
Ond the other hand, if $H$ is closed, then by the $T_2$ criterion, so is $R = \{(a, b) | a \sim b\} \subseteq G \times G$. But $R$ is just the inverse image $H$ of the continuous map

$$m : G \times G \to G, \quad (a, b) \mapsto a^{-1}b$$

and the proof follows. □

**Example 2.13.** Consider the space of basis of $\mathbb{R}^n$ up to isometry. Since a basis of $\mathbb{R}^n$ consists of $n$ vectors, we can think of basis as a matrix $B \in \mathrm{GL}(n, \mathbb{R})$. The equivalence relation is then given by

$$B \sim A \iff \exists U \in O(n) \text{ such that } AU = B$$

the resulting space is then $\mathrm{GL}(n,\mathbb{R}) \diagup O(n)$

**Remark 2.14.** The reason why homogenous spaces are called homogenous is that the space looks the same everywhere. (Think of $\mathbb{R}/\mathbb{Z}$).
Since the multiplication $m : G \times G \to G$ is continous, multiplication with a fixed $a \in G$ is also continous since the maps

$$l_a : G \to G, \quad g \mapsto ag$$

can be written as the composition of $m$ and the map $\iota_a$ given by

$$\iota_a : G \to G \times G, \quad g \mapsto (a, g)$$

same is true for right multiplication with $a$.
This has the consequence that if $U$ is a neighborhood of $e$, then $aU$ (or $Ua$) is a neighborhood of $a$.
In particular, if $H \subseteq G$ is a subgroup, then for all $x, y \in G/H$ there exists a homeomorphism $f : G/H \to G/H$ such that $f(x) = y$. Such a homeomorphism is given by

$$f : G \diagup H \to G \diagup H, \quad gH \mapsto ba^{-1}gH$$

this mapping is indeed a homeomorphism since $f \circ \pi = \pi \circ l_{b^{-1}a}$ is continuous which shows continuity of $f$ (and similarly, of $f^{-1}$).

## 2.5 Orbit spaces

We can think of groups as symmetries of spaces. Are there any special properties of such spaces?
We of course want our group actions to be continous.

**Definition 2.15.** Let $G$ be a topological grouop and $X$ a topological space.
An **operation/continuous action** of $G$ on $X$ is a continuous group action of $G$ on $X$, i.e. a continuous map $\cdot : G \times X \to X$ such that

(a) $1x = x$ for all $x \in X$

(b) $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G, x \in X$

We call such a topological space $X$ a $G$-space.

**Definition 2.16.** Let $X$ be a $G$-set and $x \in X$. The **orbit** of $x$ is the set

$$G_x := \{gx | g \in G\}$$

These orbits are equivalence classes of an equivalence relation $\sim$ on $X$ given by

$$x \sim y \iff \exists g \in G : y = gx$$

and as such, we call $X/G := X/\sim$ the **orbit space** of the group action.

**Example 2.17.**

- For $SO(n)_v = \{Av | A \in SO(n)\} = \{w \in \mathbb{R}^n | |w| = |n|\}$. We can visualize the orbits as $n-1$ spheres of any radius. It is easy to see how the orbit space $\mathbb{R}^n / SO(n)$ is isomorphic to $[0, \infty)$.
- For $G = \mathbb{R}^* = (\mathbb{R} \setminus \{0\}, \cdot)$ and $X = \mathbb{R}^{n+1} \setminus \{0\}$, the map

$$\mathbb{R}^* \times X \to X, \quad (\lambda, x) \mapsto \lambda x$$

is a continuous action and the orbit space is called the $n$-dimensional **real projective space** $\mathbb{RP}^n = X/\mathbb{R}^*$. We can view this as consisting of straight lines through the origin of $\mathbb{R}^{n+1}$. One can also find an isomorphism ${S^{n-1}}/{\{\pm 1\}} \cong \mathbb{RP}^n$.

**Definition 2.18.** Let $X$ be a $G$-space, $x \in X$. We call $G_x := \mathrm{Stab}_G(x) = \{g \in G | gx = x\} \subseteq G$ the **stabilizer** of $x$.

Clearly, the stabilizer is a subgroup.

**Example 2.19.** Consider the group action of $SO(n)$ with the point $x = e_1 \in \mathbb{S}^{n-1}$.
Its stabilizer $\mathrm{Stab}_{SO(n)}(x)$ is isomorphic to $SO(n-1)$, as a matrix $A \in SO(n)$ satisfying $Ae_1 = e_1$ must be of the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}, \quad \text{where} \quad A^T A = 1 \implies B^T B = 1 \implies B \in SO(n-1)$$

It is also a homoemorphism as $S^{n-1}$ is Hausdorff and $SO(n)/\mathrm{Stab}_G(x)$ is compact (closed and bounded).

A simple, yet powerful theorem is the **topological orbit theorem**.

**Theorem 2.20** (Topological Orbit theorem)**.** *Let $X$ be a $G$ space and $x \in X$. Then the map*

$$F : {G}/{Stab_G(x)} \to \mathcal{O}_G(x), \quad gStab_G(x) \mapsto gx$$

*is a continuous bijection.*

*Proof.* This mapping is indeed well defined, as

$$a\mathrm{Stab}_G(x) = b\mathrm{Stab}_G(x) \implies \exists h \in \mathrm{Stab}_G(x) \text{ with } a = bh \implies ax = (bh)x = b(hx) = bx$$

By definition of the orbit, it's clearly surjective. For injectivity, we have

$$ax = bx \implies a^{-1}(bx) = a^{-1}(ax) = x \implies a^{-1}b \in \mathrm{Stab}_G(x)$$

and for continuity, we see that the mapping

$$f \circ \pi : G \to \mathcal{O}x, \quad g \mapsto gx$$

is continuous and can use Lemma 2.3 to show continuity of $f$. □

## 2.6 Collapsing of subspaces to a point

**Definition 2.21.** Let $X$ be a topological space and $A \subseteq X$ a non-empty subset. We write $X/A := X/\sim$ for

$$x \sim y \iff (x = y) \text{ or } x, y \in A$$

We have already seen this type of spaces when discussing quotient topologies. For example for $X = \mathbb{D}^2$ and $A = \partial X = \mathbb{S}^1$, we got $X/A \cong \mathbb{S}^2$

**Definition 2.22.** Let $X$ be a topolgical space and $A_1, \ldots, A_n \subseteq X$ be non-empty, pairwise disjoint subsets. We write $X/(A_1, \ldots, A_n) := X/\sim$, where

$$x \sim y \iff x = y \text{ or } \exists i : x, y \in A_i$$

Note that if $X$ is metrizable (or $T_2, T_4$ ) and the $A_i$ are closed, then the quotient space is $T_2$. (We will prove this later when discussing $T_4$ spaces).

**Example 2.23** (The cone)**.** Given a topological space $X$, the quotient space

$$C(X) := {}^{X \times [0,1]}\!\big/_{X \times \{1\}}$$

is called the **cone** over $X$. The name becomes clear if we look at the case where $X = [0,1]$ or $X = \mathbb{S}^1$.

**Example 2.24** (Suspension)**.** For a topological space $X$, the **suspension** of $X$ is the space

$$\Sigma(X) := {}^{X \times [-1,1]}\!\big/_{X \times \{-1\}, X \times \{1\}}$$

which can also be obtained by taking two cones and "glueing" them together at $X \times \{0\}$. (What glueing is will be defined later but it should make intuitive sense).

**Example 2.25.** For a subset $A \subseteq X$ of a topological space, the **cone over** $A$ is the space

$$C_A(X) := {}^{X \times [0,1]}\!\big/_{A \times \{1\}}$$

**Definition 2.26** (Wedge & Smash)**.** For $X, Y$ toplogical spaces with basepoints $x_0 \in X, y_0 \in Y$, we define

The **wedge** $X \vee Y := X \times \{y_0\} \cup \{x_0\} \times Y \subseteq X \times Y$

The **smash** $X \wedge Y := {}^{X \times Y}\!\big/_{X \vee Y}$

**Example 2.27.** For $X = \mathbb{S}^n$ and $Y = \mathbb{S}^m$ for $n, m \geq 1$ we have that $X \wedge Y = \mathbb{S}^{n+m}$.
To prove this, we see $\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$ as the one-point-compactification of $\mathbb{R}^n$ and define the mapping

$$g : \mathbb{R}^n \cup \{\infty\} \times \mathbb{R}^m \cup \{\infty\} \to \mathbb{R}^{n+m} \cup \{\infty\}$$

given by

$$g(x,y) = \begin{cases} (x,y) & \text{if } x \in \mathbb{R}^n, y \in \mathbb{R}^m \\ \infty & \text{otherwise} \end{cases}$$

and showing that $g^{-1}(\infty) = X \vee Y$.

## 2.7 Glueing of topological spaces

What glueing means intuitively should be clear, let's see if we can define it in a topological setting.

**Definition 2.28.** Let $X, Y$ be topological spaces, $X_0 \subseteq X$ and $\varphi : X_0 \to Y$ continuous. For the equivalence relation $\sim$ on $X \sqcup Y$ generated by $x \sim \varphi(x)$, we write

$$Y \cup_\varphi X := {}^{X \sqcup Y}\!\diagup_{\sim}$$

for the **glueing** of $X$ onto $Y$ by $\varphi$.

**Remark 2.29.** Note that the embedding $X \to Y \cup_\varphi X, x \mapsto [x]$ is continuous as it is equal to $\pi \circ \iota_X$, for $\iota_X$ the inclusion and $\pi$ the projection mapping.
The glueing is a generalisation of the collapsing of subspaces, as for $A \subseteq X$, we have a homeomorphism

$$f : X/A \to \{*\} \cup_\varphi X, \quad [x] \mapsto [x]$$

for $\varphi : A \to \{*\}$, because $x, y \in A \iff \varphi(x) = \varphi(y)$.
The mapping $Y \to Y \cup_\varphi X, \quad y \mapsto [y]$ is also inective and a homeomorphism to its image as the map is not only continuous, but open. (The same might not be true for $X$)

**Example 2.30** (Mapping torus)**.** Let $\alpha : X \to X$ be a homeomorphism. We call

$$X \times [0,1]\!\diagup_{\alpha} := {}^{X \times [0,1]}\!\diagup_{\sim}, \quad \text{with} \quad (x,0) \sim (\alpha(x), 1)$$

the **mapping torus** of $\alpha$.

- The mapping torus of $\alpha = \mathrm{id}_{\mathbb{S}^1}$ is the ordinary **Torus**.
- The **Moebius strip** $M$ is the mapping torus of $\alpha : [-, 1] \to [-1, 1], x \mapsto -x$.
- The **Klein bottle** $K$ is the mapping torus of $\alpha : \mathbb{S}^1 \to \mathbb{S}^{-1}, z \mapsto \overline{z}$.

If we glue two Moebius strips together with the mapping

$$\varphi : \underbrace{\pi(\{-1, 1\} \times [0, 1])}_{=\partial M} \to M, \quad m \to m$$

we obtain the Klein bottle $K = M \cup_\varphi M$. To show this, we use the mapping

$$\begin{aligned} g : [-1, 1] \times [0, 1] \sqcup [-1, 1] \times [0, 1] &\to \mathbb{S}^1 \times [0, 1] \\ (x, t) \cup (y, s) &\mapsto (e^{i\pi x/2}, t), (-e^{-i\pi y/2}, s) \end{aligned}$$

# 3 Homotopy

How do we measure "holes" in a topological space? One idea is to look at a pre-defined object with a hole and see how this object maps to an arbitrary space by a continuous deformation. We will have to define what continuous deformation means.

## 3.1 Homotopy of maps

**Definition 3.1.** Let $f, g : X \to Y$ be continuous maps.

- A continuous map $h : X \times [0,1] \to Y$ such that
$$h(x,0) = f(x) \quad \text{and} \quad h(x,1) = g(x) \quad \forall x \in X$$
is called a **homotopy** between $f$ and $g$ and write $f \sim_h g$.
- We say that $f$ and $g$ are **homotopic** to each other, if such a homotopy exists and we write $f \sim g$.

The notation and naming are highly suggestive, and not without good reason.

**Remark 3.2.** It is easy to prove that homotopy of maps forms an equivalence relation $\sim$ on $\mathrm{Hom}(X,Y)$:

- Reflexivity is rather obvious, as
$$f \sim f \text{ via } h : X \times [0,1] \to Y, \quad (x,t) \mapsto f(x)$$
- For symmetry we reverse the homotpy. That is, if $f \sim_h g$ for some $h : X \times [0,1] \to Y$, then we get that for
$$\tilde{h} : X \times [0,1] \to Y, \quad h(x, 1-t)$$
we have $g_{\tilde{h}} f$, as
$$\tilde{h}(x,0) = h(x,1) = g(x) \quad \text{and} \quad \tilde{h}(x,1) = h(x,0) = f(x) \quad \forall x \in X$$
- Transitivity is obtained by "glueing" together two homotopies. So if $f \sim_k g$ and $g \sim_l h$, then we can set
$$H : X \times [0,1] \to Y, \quad H(x,t) := \begin{cases} k(x,2t) & t \leq \frac{1}{2} \\ l(x,2t-1) & t \geq \frac{1}{2} \end{cases}$$
this map is well defined as for $t = \frac{1}{2}$ we have
$$k(x,1) = g(x) = l(x,0)$$
for continuity we can use the homeomorphism
$$\Phi : X \times [0,1] X \times [0,1] \cup_\varphi X \times [1,2] =: Q$$
given by
$$\varphi : X \times \{1\} \to X\{1\}(x,1) \mapsto (x,1) \quad \text{and} \quad \Phi : (x,t) \mapsto [(x,t)]$$
and use the continuous map
$$r : X \times [0,1] \sqcup X \times [1,2] \to Y, r = h(x,t) \sqcup k(x,t-1)$$
to define $H$ in another way. $H = R \circ \Phi$ and use Lemma 2.3 for
$$R : Q \to Y, \quad [(x,t)] \mapsto r(x,t)$$

We denote the set of equivalence classes with respect to homotopy of maps as

$$[X,Y] := {}^{\textstyle \mathrm{Hom}(X,Y)}\!\big/_{\textstyle \text{homotopy}}$$

**Example 3.3.** Homotopy gives us another way of showing that certain special spaces are special. The space $\mathbb{R}^n$ is special in that for any space $X$, there is exactly one homotopy class on $\mathrm{Hom}(X,\mathbb{R}^n)$. In other words, every two continuous maps $f,g : X \to \mathbb{R}^n$ are homotopic, as a homotopy $f \sim_h g$ can be given by

$$h : X \times [0,1] \to \mathbb{R}^n, \quad (x,t) \mapsto (1-t)f(x) + tg(x)$$

The singleton space is special in that two maps $f,g : \{*\} \to Y$ are homotopic if and only if their images $f(*), g(*)$ are path connected in $Y$.

**Lemma 3.4.** *Homotopy not only defines an equivalence class structure inside a given* $\mathrm{Hom}(X,Y)$*, but that structure is also compatible with certain operations in the category of topological spaces such as composition and taking products.*

*(a) Homotopies of maps is conserved under composition. If* $f \sim_h g$ *in* $\mathrm{Hom}(X,Y)$ *and* $f' \sim_{h'} g'$ *in* $\mathrm{Hom}(Y,Z)$*, then* $f' \circ f$ *and* $g' \circ g$ *are homotopic in* $\mathrm{Hom}(X,Z)$ *Such a homotopy* $H$ *can be given by*

$$H : X \times [0,1] \to Z : (x,t) \mapsto h'(h(x,t),t)$$

*(b) Homotopies of maps can be extended to products. If* $f_i \sim_{h^{(i)}} g_i$ *in* $\mathrm{Hom}(X_i,Y_i)$ *for* $i \in I$*, then*

$$H_t := \prod_{i\in I} h_t^{(i)} : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$$

*defines a homotopy in* $\mathrm{Hom}\left(\prod_{i\in I} X_i, \prod_{i\in I} Y_i\right)$*.*

## 3.2 Homotopy equivalence

Now we have all the necessary tools to define what it means for two spaces to be continuously deformable into eachother.

**Definition 3.5.** A continuous map $f : X \to Y$ is a **homotopy equivalence** between $X$ and $Y$, if there exists an **homotopy inverse**. That is, a map $g : Y \to X$ such that their compositions are homotoptic to the identity maps:

$$g \circ f \sim \mathrm{id}_X \quad \text{and} \quad f \circ g \sim \mathrm{id}_Y$$

The two spaces $X,Y$ are then called **homotopy equivalent** if such a pair $f,g$ exists and we write $X \sim Y$. (Not to be confused with $X \cong Y$).

Note that a homotopy equivalence is much weaker than a homeomorphism. For example $\mathbb{R}^n$ is homotopy equivalent to the single ton space $\{*\}$ as all continuous maps $\mathbb{R}^n \to \mathbb{R}^n$ are homotopic.
More generally, a topological space $X$ is called **contractible**, if it is homotopy equivalent to $\{*\}$.

**Example 3.6.** $\mathbb{S}^{n-1}$ and $\mathbb{R}^n \setminus \{0\}$ are homotopy equivalent, since we can use the inclusion map $\iota : \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}, \mapsto x$ aswell as the projection mapping $\rho : \mathbb{R}^n \to \mathbb{S}^{n-1}, v \mapsto \frac{x}{|x|}$ since their compositions are

$$\rho \circ \iota = \mathrm{id}_{\mathbb{S}^{n-1}}, \quad \iota \circ \rho \sim_h \mathrm{id}_{\mathbb{R}^n \setminus \{0\}} \quad \text{for} \quad h(x,t) = tx + (1-t)\frac{x}{|x|}$$

**Remark 3.7.** It is easy to show that (see exercise sheet 6) if $X$ and $Y$ are homotopy equivalent, then

(a) $X$ path connected $\iff$ $Y$ path connected

(b) $X$ connected $\iff$ $Y$ connected

(c) Homotopy equivalence forms an "equivalence relation" in the category of topological spaces:

$$X \sim Y, Y \sim Z \implies X \sim Z, \quad X \sim X, \quad X \sim Y \implies Y \sim X$$

We would like to find some criteria when spaces can be homotopy equivalent to each other. The following definition will aide us in doing so.

**Definition 3.8.** Let $X$ be a topological space and $A \subseteq X$ with the inclusion map $\iota : A \hookrightarrow X$. We say that $A$ is a **retract** of $X$, if there exists a continuous map $\rho : X \to A$ such that $\rho \circ \iota = \mathrm{id}_A$. We call $\rho$ a **retraction** of $X$ unto $A$.

$$A \xhookrightarrow{\iota} X \xrightarrow{\rho} A$$

**Example 3.9.**

- The set $A = [0,1] \subseteq X = [0,1] \cup [2,3]$ is a rectract of $X$.
- $A = \{a, b\} \subseteq X = [0,1]$ is *not* a retract.
- More generally, we can show that $\mathbb{S}^{n-1} \subseteq \mathbb{D}^n$ is not a rectract, but for higher $n$ we need some algebraic topology.

**Definition 3.10.** Let $X$ be a topological space, $A \subseteq X$.

- A retraction $\rho : X \to A$ is called a **deformation retraction** if additionally $\iota \circ \rho$ is homotopic to $\mathrm{id}_X$.

$$X \xrightarrow{\rho} A \xhookrightarrow{\iota} X$$

  If the homotopy to the identity map can be chosen such that $h(t, a) = a$ for all $a \in A, t \in [0,1]$, the deformation retraction is called **strong**.

- $A$ is called a (strong) **deformation retract**, if such a (strong) deformation retraction exists.

**Example 3.11.** The subset $A = \mathbb{S}^{n-1} \subseteq X = \mathbb{R}^n \setminus \{0\}$ is a strong deformation retract of $X$ as we can use the same mapping $\rho$ from the example shown earlier.

**Lemma 3.12.** *Every space $X$ is homotopy equivalent to its deformation retract $A \subseteq X$*

*Proof.* If $\rho$ is the deformation retract and $\iota : A \to X$ is the inclusion mapping, then

$$\iota \circ \rho \sim \mathrm{id}_X \quad \text{and} \quad \rho \circ \iota = \mathrm{id}_A$$

□

**Example 3.13.** Let $\varphi : \mathbb{S}^{n-1} \to Y$ be continuous. Then for

$$\iota(Y) \subseteq Y \cup_\varphi (\mathbb{D}^n \setminus \{0\}) =: Q$$

where $\iota : Y \to Q, y \mapsto [y]$ is the inclusion map, has as inverse a strong deformation retract.

Let $0 < k \leq n \in \mathbb{N}$. Then

$$A := (\mathbb{S}^{k-1} \times \mathbb{S}^{n-k}) \subseteq \mathbb{S}^n \setminus \left(\mathbb{S}^{k-1} \times \{0\} \cup \{0\} \times \mathbb{S}^{n-k}\right) =: X \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k+1}$$

is a strong deformation retract.
The mapping can be given by

$$\rho : X \to A, \quad (v, w) \mapsto \left(\frac{v}{|v|}, \frac{w}{|w|}\right)$$

where the homotopy is

$$h((v,w),t) = \left(\frac{tv + (1-t)\frac{v}{|v|}}{|tv + (1-t)\frac{v}{|v|}|}, \frac{tw + (1-t)\frac{w}{|w|}}{|tw + (1-t)\frac{w}{|w|}|}\right)$$

up to normalisation.

**Lemma 3.14.** *Let $X$ be a topological space. Then $X$ is contractible if and only if there exists a $x_0 \in X$ such that $\{x_0\} \subseteq X$ is a deformation retract.*

*Proof.* The proof is trivial and is left as an exercise to the reader. □

But why would we study Homotopy and Homotopy equivalence?

- Many topological properties of topological spaces are the same for homotopy equivalent spaces.
- Many classical algebraic invariants are the same for homotpy equivalent spaces.

One such example in this lecture the **Fundamental group**. $\pi_1$ The idea behind is to that to each topological space $X$ we associate a group $\pi_1(X)$ and to every continuous map $f : X \to Y$ we want to find a group homomorphism $\pi_1(X) \to \pi_2(X)$ that is a group isomorphism if $f$ is a homotopy equivalence. In other words, we want to create a **functor** $\pi_1 : \mathsf{Top} \to \mathsf{Grp}$.

# 4 Category Theory

One of the main goals of category theory is to obtain a language with which we can talk about mathematical objects (such as sets, spaces, groups etc.) in various contexts.[1]
Some of the notions used in the language of Category theory are not compatible with the restrictions in set theoretical frameworks such as plain ZFC.
For example, talking about the *Category of Categories* may get a little awkward.
To remedy this, category theorists usually work in an extension of ZFC with new axioms to let us distinguish between sets and proper classes[2].
Covering these axioms systematically is rather tedious, so we will largely skip this step and focus more on using the language of Category theory without worrying about the low-level stuff going on.
For example, we will write things like *collections* of objects and talk about them using our intuitive understanding of what such a phrase might mean, just like most of the things we do in Topology anyways.
The main goal of this section is to give a basic understanding of category theory with a focus on Topology until we have sufficient vocabulary to understand some of the tools used in the rest of the lectures.
A resource I found extremely useful to understanding category theory was the **nlab** `https://ncatlab.org/nlab/show/HomePage` which is a wiki that provides definitions and motivations for many category theoretic concepts. Another great resource is Emily Riehl's **Category Theory in Context** `https://math.jhu.edu/~eriehl/context.pdf` which provides a nice guide to category theory and its applications together with some helpful exercises.

## 4.1 The Notion of a Category

A **category** $\mathsf{C}$ consists of the following data:

- A class (collection) $\mathrm{Ob}(\mathsf{C})$ of mathematical objects. The objects can be anything we want them to be and don't need to be defined using sets or anything thelike.
- A **Hom-Set** $\mathrm{Hom}(X, Y)$ (or sometimes written $\mathsf{C}(X, Y)$ for every pair of objects $X, Y \in \mathrm{Ob}(\mathsf{C})$. The Hom-set consists of **Morphisms** between these objects, that is: every morphism must have well-defined object as its **domain** and **codomain**. If $X$ and $Y$ are objects in a category $\mathsf{C}$, we write $f : X \to Y$ (or $X \xrightarrow{f} Y$) to denote a morphism $f$ with domain $X$ and codomain $Y$.

For the data to from a valid category they must fulfill the following requirements:

- Morphisms can be composed: If $f : X \to Y$ and $g : Y \to Z$ are morphisms, then there exists a morphism $g \circ f : X \to Z$. More formally, for every triple $(X, Y, Z)$ there exists a map
$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z), \quad (f, g) \mapsto g \circ f$$
- Composition is associative: For morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ we have the equality $(f \circ g) \circ h = f \circ (g \circ h)$.
- For every object $X$ in a category $\mathsf{C}$, there exists an **identity morphism** $\mathrm{id}_X : X \to X$ that satisfies the left and right unit laws: For any object $Y \in \mathrm{Ob}(\mathsf{C})$
$$\forall f \in \mathrm{Hom}(X, Y) : f \circ \mathrm{id}_X = f \quad \text{and} \quad \forall g \in \mathrm{Hom}(Y, X) : \mathrm{id}_X \circ g = g$$

[1] I am deviating from what is given in the lecture by Prof. Feller, with mostly my own notes„ but still using the definitions from the lecture.
[2] Check out Michael Shulman's "Set theory for category theory" that explores this in more detail.

Let's take a look at some categories. When we say that something is a category, we usually specify what the objects are and what the morphisms (and composition) look like.

**Example 4.1.** Make sure to verify that these do indeed form categories by checking the properties above.

- The category of sets, denoted $\mathsf{Set}$, where the objects are sets and morphisms are functions.
- The category of topological spaces, written $\mathsf{Top}$ where the objects are topological spaces (with their topology) and where the morphisms are *continuous* functions.
- The category $\mathsf{Grp}$, where the objects are groups and morphisms are group homomorphisms.
- The category $\mathsf{Top}_*$, with Topological spaces with a basepoint $(X, \tau, x_0)$ as objects and where the morphisms are continous, base-point preserving functions.
- The category $\mathsf{Ring}$ with Rings as objects and ring homomorphisms as morphisms.
- $\mathsf{Graph}$ has graphs as objects and graph homomorphisms as morphisms.
- For a fixed field $k$, the category $\mathsf{Vect}_k$ has $k$-vector spaces as objects with linear maps as morphisms.
- One rather weird category is the category of fields with field homomomorphisms.

Some of these categories will be of special interest later (in particular $\mathsf{Top}_*$ and $\mathsf{Grp}$)

It is traditional to name categories after their objects and imply what the morphisms are. But just like how topological spaces should always be considered with their topology, we have to think of the objects and morphisms together when taking about categories.

**Example 4.2.** The examples above are categories were objects were *sets with structure* and the morphisms were *structure preserving* maps. But the objects need not be sets, and morphisms need not be functions!

- The trivial category, which consists of a single object $*$ and its identity morphism $* \xrightarrow{\text{id}} *$.
- A group (or monoid) $(G, \cdot, e)$ can be viewed as a category with a single object $*$, where the morphisms $g : * \to *$ are the elements $g \in G$ and where composition of morphisms is defined as the multiplication in $G$: $g \circ h := g \cdot h$.
- A poset $(P, \leq)$ is a category where the objects are the elements of $P$ and there exist unique morphisms $x \to y$ if and only if $x \leq y$.
- $\mathsf{Htpy}$ has the same objects as $\mathsf{Top}$, but morphisms are homotopy classes of continuous maps.
- Given a category $\mathsf{C}$, we can talk about its **arrow category**, where the objects are morphisms $f : X \to Y$ from $\mathsf{C}$ and a morphism between objects $X_0 \xrightarrow{f} Y_0, X_1 \xrightarrow{g} Y_1$ are pairs of morphisms $\alpha : X_0 \to X_1$ and $\beta : Y_0 \to Y_1$ such that the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[r, "\alpha"] \arrow[d, "f"'] & X_1 \arrow[d, "g"] \\ Y_0 \arrow[r, "\beta"] & Y_1 \end{tikzcd}$$

  i.e. such that $g \circ \alpha = \beta \circ f$.

The language of category theory lets us unify similar ideas from different contexts:

**Definition 4.3.** In any category, an **isomorphism** (or **iso**, for short) is a morphism with a two-sided inverse. That is, a morphism $f : X \to Y$ is iso, if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

In $\mathsf{Set}$, the isomorphisms are bijective functions. In $\mathsf{Grp}$, they are group isomorphisms. In $\mathsf{Top}$ they are homeomorphisms. In $\mathsf{Htpy}$ an isomorphism is a homotopy equivalence (up to homotopy of course).
The possibility to use the same definition and applying it to different contexts to recover "classical" definitions is quite common and also what makes category theory so nice to use.

## 4.2 Duality

If we visualize the morphisms in a category as arrows pointing from their domain to their codomain, one might be tempted to reverse the directions of all the arrows.

**Definition 4.4.** Let $\mathsf{C}$ be a category. The **opposite category** $\mathsf{C}^{\mathrm{op}}$ has the same objects as $\mathsf{C}$, but for every morphism $f : X \to Y$ in $\mathsf{C}$, we have a morphism $f^{\mathrm{op}} : Y \to X$ in $\mathsf{C}^{op}$

Verify that this forms a valid category by defining composition and checking associativity as well as the left/right unit relations for the identity morphisms.

**Definition 4.5.** In the following, let $c$ be an object in a Category $\mathsf{C}$.

- There is a category $c/\mathsf{C}$ called the **slice category** of $\mathsf{C}$ **under** $c$, where the objects are morphisms $f : c \to X$ from $\mathsf{C}$, and a morphism between objects $c \xrightarrow{f}$ and $c \xrightarrow{g} Y$ is a morphism $h : X \to Y$ from $\mathsf{C}$ such that the following diagram commutes:

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "h"] & & Y \end{tikzcd}$$

  i.e. so that $h \circ f = g$ .

- There is a category $\mathsf{C}/c$ called the **slice category** of $\mathsf{C}$ **over** $c$, where the objects are morphisms $f : X \to c$ and a morphism between objects $X \xrightarrow{f} c$, $Y \xrightarrow{g} c$ is a morphism $h : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

Note how the two examples above are *dual* to eachother. We used one definition, reversed the arrows and obtained another definition of a category.

The definition of injective/surjective maps between sets we have seen in introductory set theory makes use of the word "element" (A function $f : X \to Y$ is injective, if for all $x, y \in X$ such that $\ldots$). In category theory however, cannot talk about "elements" or what is "inside" an object. We wish to understand objects not by analyzing its internal properties, but rather through its relation to other objects from that category.

**Definition 4.6.** A morphism $f : X \to Y$ in a category is said to be

- a **monomorphism** (or **monic**), if for any pair of parallel morphisms $g, h : W \to X$ we have the implication $f \circ g = f \circ h \implies g = h$.

$$W \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

- an **epimorphism** (or **epic**), if for any pair of parallel morphisms $g, h : Y \to Z$ we have the implication $g \circ f = h \circ f \implies g = h$.

$$Z \underset{h}{\overset{g}{\leftleftarrows}} Y \xleftarrow{f} X$$

In the categories $\mathsf{Set}$, $\mathsf{Top}$ and some others, monomorphisms/epimorphisms are excactly the injective/surjective maps. Just like how in topology, a bijective continuous function need not be a homeomorphism, a morphism in a category that is both monic and epic need not be iso.
Another counter example is the canonical embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ in the category $\mathsf{Ring}$, which is both monic and epic, but clearly not an isomorphism.
The fact that these definitions of monomorphisms and epimorphisms are dual to each other is useful because if we can prove something for monic maps, then the same holds true in the opposite category for epic maps, because the monomorphisms in $\mathsf{C}$ are excatly the epimorphisms in $\mathsf{C}^{\mathrm{op}}$.

**Lemma 4.7.**

*(a) If $f : X \rightarrowtail Y$ and $g : Y \rightarrowtail Z$ are monomorphisms, then so is $g \circ f : X \rightarrowtail Z$*

*(b) If $f : X \to Y$ and $g : X \to Z$ are morphisms such that $g \circ f$ is monic, then $f$ is monic.*

*And dually:*

*(a') If $f : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$ are epimorphisms, then so is $g \circ f : X \twoheadrightarrow Z$.*

*(b') If $f : X \to Y$ and $g : X \to Z$ are morphisms such that $g \circ f$ is epic, then $g$ is epic.*

We only need to prove (a) and (b) here. (a') and (b') follow by duality.

**Definition 4.8.** If $A \xrightarrow{s} X \xrightarrow{r} A$ are morphisms such that $r \circ s = \mathrm{id}_A$, then we call $s$ a **section** or **right inverse** to $r$, while $r$ defines a **retraction** or **left inverse** to $s$. We call $A$ a **retract** of the object $X$.

Applying this definition in the category $\mathsf{Top}$, we recover the definition of rectraction: A retraction (in the sense of Definition 3.8) is a left inverse to the inclusion map $\iota$. A deformation retract is a right inverse to the inclusion map $\iota$: $X \xrightarrow{\rho} A \xhookrightarrow{\iota} X$ in the category $\mathsf{Htpy}$.
Many objects (such as Quotient Spaces, Tensor product, Product topology, Polynomial rings etc.) we studied in Linear Algebra and Topology could be understood as a construction, where we directly defined the object through its elements and its structure. It turns out that we can understand them through their relationship with other objects. The special relationships usually came in the form of **universal properties**.

**Definition 4.9.** Let $\mathsf{C}$ be a category.

- An **initial object** of the category, if it exists, is an object $\emptyset$ with the **universal property** that, for any other object $X$ in $\mathsf{C}$, there exists a *unique* morpshism $\emptyset \to X$.

- A **terminal object**, if it exists, is an object 1 with the universal property that, for any other object $X$ in $\mathsf{C}$, there exists a unique morphism $X \to 1$.

**Example 4.10.** The notation used for $\emptyset$ and 1 are suggestive, but can be misleading depending on the category.

- In the category $\mathsf{Set}$ (and in $\mathsf{Top}$), the initial object is the empty set $\emptyset$ and the terminal object is the singleton set $\{*\}$.
- In the category $\mathsf{Grp}$, the trivial group $\{e\}$ is both an initial and terminal object. In this case, we also call it a **zero object**.
- In the category $\mathsf{Ring}$, the initial object is the ring of integers $\mathbb{Z}$. Depending on whether we use the axiom $0 \neq 1$ in the definition of a ring, the category may or may not have a terminal object $\{1\}$.

Notice that we wrote "the" initial object or "the" terminal object even though for example, there are multiple sets with a single element. That is because in category theory, if two things are isomorphic (in the sense of definition 4.3) they can be thought of as being essentially the same in that category.

**Lemma 4.11.** *In any category, the inital and terminal objects,* if they exist*, are unique up to isomorphism.*

*Proof.* Again, since the terminal object is the initial object in the opposite category, we only need to prove it for one of them. Suppose $X, Y$ are initial objects with identity morphisms $\mathrm{id}_X$ and $\mathrm{id}_Y$, respectively. By their universal properties there exist (unique) morphisms $f : X \to Y$ and $g : Y \to X$. Considering their composition, we get a morphism $g \circ f : X \to X$. But by the universal propery, there can only be one unique morphism $X \to X$, so $g \circ f = \mathrm{id}_X$. Likewise for $Y$, we get $f \circ g = \mathrm{id}_Y$. $\square$

We can use the notion of universal property in a more general way by using a meta-defintion.

**Definition 4.12.** An object is said to have a **universal property**, if it is the inital or terminal object in the category of objects with this property.

Let's see how the meta-definition can be used to describe universal properties of some known objects.

**Example 4.13** (Product)**.** In the category $\mathsf{Set}$, the (cartesian) product $X \times Y$ of two sets $X, Y$ is equipped with **projection mappings**

$$\pi_X : X \times Y \to X, \quad (x, y) \mapsto x, \quad \text{and} \quad \pi_Y : X \times Y \to Y, \quad (x, y) \mapsto y$$

The cartesian product has the universal property that, for any other set $Z$ with functions $f : Z \to X, g : Z \to Y$, there exists a *unique* function $\varphi : Z \to X \times Y$ such that the following diagram commutes.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "!\exists\varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \\
& Y &
\end{tikzcd}$$

Where the map $\varphi$ is given by $\varphi : Z \to X \times Y, z \mapsto (f(z), g(z))$.
If we view this in the category where the objects are sets with functions to $X$ and $Y$, and where a morphism between objects $\left\{Z' \xrightarrow{f'} X, Z' \xrightarrow{g'} Y\right\}$ and $\left\{Z \xrightarrow{f} X, Z \xrightarrow{g} Y\right\}$ is a function $\varphi : Z' \to Z$ such that $f \circ \varphi = f'$ and $g \circ \varphi = g'$, then the cartesian product $X \times Y$ is the terminal object of this category.

In the category $\mathsf{Top}$, the cartesian product with the product topology has the same universal property of being the terminal object of such a category.
In the category $\mathsf{k\text{-}Vect}$, the direct product $V \oplus W$ has the same universal property.

This leads to our definition of a Product in a general category:

**Definition 4.14.** Let $(X_i)_{i\in I}$ be a collection of objects in a category. The (cartesian) **product** of these objects (if it exists), is an object $Z$ with morphisms $\left\{Z \xrightarrow{\pi_i} X\right\}_{i\in I}$ such that for any other object $Z'$ with morphisms $\left\{Z' \xrightarrow{\pi_i'} X\right\}_{i\in I}$ there exists a *unique* morphism $\varphi : Z' \to Z$ such that for all $i \in I$ the following diagram commutes

$$\begin{tikzcd}
& Z' \arrow[dl, dashed, "!\exists\varphi"'] \arrow[dr, "\pi_i'"] & \\
Z \arrow[rr, "\pi_i"] & & X_i
\end{tikzcd}$$

Again, it follows from the universal proparty (and more specifically, Lemma 4.11) that the product is unique up to isomorphism.
The dual notion of a product is the *coproduct*. We could just say that the coproduct of objects $(X_i)_{i\in I}$ is the product in the opposite category and call it a day, but it's a little easier with some examples

**Example 4.15** (Coproduct)**.** In the category $\mathsf{Set}$, the **disjoint union** $X \sqcup Y$ of sets $X, Y$ is equipped with canonical **embeddings**

$$\iota_X : X \to X \sqcup Y, x \mapsto (x, 0) \quad \text{and} \quad \iota_Y : Y \to X \sqcup Y, y \mapsto (y, 1)$$

This embedding is universal in that for any other set $Z$ with maps $f : X \to Z, g : Y \to Z$, there exists a unique function $\varphi : X \sqcup Y \to Z$ such that the following diagram commutes

$$\begin{tikzcd}
Z & & \\
& X \sqcup Y \arrow[ul, dashed, "!\exists\varphi"'] & X \arrow[l, "\iota_X"'] \arrow[ull, bend right, "f"'] \\
& Y \arrow[u, "\iota_Y"] \arrow[uul, bend left, "g"] &
\end{tikzcd}$$

## 4.3 Functors

If we look at categories where objects are Sets with structure and morphisms are strucutre preserving maps we might ask what the morphisms in the "category of categories" are.
The structure of a category would the morphisms, so we can ask what "morphism-preserving" morphisms are.
This gives rise to the definition of a functor.

**Definition 4.16.** Let $\mathsf{C}, \mathsf{D}$ be categories. A **covariant Functor** $\mathcal{F}$ from $\mathsf{C}$ to $\mathsf{D}$ (written $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ does the following:

- It provides a map $F : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$ that maps an object $X$ from $\mathsf{C}$ to and object $FX$ in $\mathsf{D}$.
- It maps morphisms $f \in \mathrm{Hom}(X, Y)$ to a morphism $Ff \in \mathrm{Hom}(FX, FY)$ such that

(i) $F(\mathrm{id}_X) = \mathrm{id}_{FX}$

(ii) $F(g \circ f) = Fg \circ Ff$

the second condition is equivalent to saying that for any morphism $f : X \to Y$ in $\mathsf{C}$, the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "f"] & FX \arrow[d, "Ff"] \\ Y \arrow[r, "F"] & FY \end{tikzcd}$$

**Example 4.17.** Quite a few concepts that we encountered througout Algebra or Topology are functors.

(a) If we have a group $(G, \cdot, e)$, we can forget the group structure and only look at the underlying set $G$. Group homomorphisms turn into functions. We call this the **forgetful functor**. The same construction works for topological groups, where we forget the topology of a topological space.

(b) We can turn every field into a group, by removing the 0. Group homomorphisms then become field homomorphisms.

(c) There is a functor $\mathsf{Top} \to \mathsf{Htpy}$ that preserves the objects (topolgocial spaces), but maps continuous maps to their homotopy equivalence classes.

(d) Consider the **endofunctor** $\mathsf{F} : \mathsf{Vect}_k \to \mathsf{Vect}_k$ which maps vector spaces to their dual and linear maps to their dual maps. So for vector spaces $V, W$ and a linear map $f \in \mathrm{Hom}(V, W)$ we define

$$FV = V^*, \quad \text{and} \quad Ff =: f^* : W^* \to V^*, f^*\beta = \beta \circ f : V \to k$$

This is an example of a **contravariant functor**, as the directon of $Ff$ is opposite to that of $f$.

**Lemma 4.18.** *Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ be a functor. If $X, Y \in Ob(\mathsf{C})$ and $f \in \mathrm{Hom}(X, Y)$ is an isomorphism, then $Ff \in \mathrm{Hom}(FX, FY)$ (or $\mathrm{Hom}(FY, FX)$ for contravariant functors) is also an isomorphism.*

*Proof.* Since $f$ is an isomorphism, there exists a morphism $g \in \mathrm{Hom}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. By the functor laws, $Ff$ and $Fg$ satisfy for covariant functors,

$$Fg \circ Ff = F(g \circ f) = F\,\mathrm{id}_X = \mathrm{id}_{FX} \quad \text{and} \quad Ff \circ Fg = F(f \circ g) = F\,\mathrm{id}_Y = \mathrm{id}_{FY}$$

and for contravariant functors:

$$Fg \circ Ff = F(f \circ g) = F\,\mathrm{id}_Y = \mathrm{id}_{FY} \quad \text{and} \quad Ff \circ Fg = F(g \circ f) = F\,\mathrm{id}_X = \mathrm{id}_{FX}$$

□

We can use this lemma to immediately prove that every homeomorphism (iso in $\mathsf{Top}$) is also a homotopy equivalence (iso in $\mathsf{Htpy}$).

# 5 The fundamental group

One central idea in category theory is the view that we can understand objects in a category by studying their Hom-sets to other objects.
Of course understanding *all* Hom-sets to other objects is too much to reasonably do. So what we can do is to instead look at "nice" spaces and study the Hom-sets to other spaces.
One such nice space is the circle space $\mathbb{S}^1$. Given a topological space $X$ and a basepoing $x_0 \in X$, we want to understand all possible maps $f \in \mathrm{Hom}(\mathbb{S}^1, X)$. This corresponds to studying **loops** with basepoint $x_0$. If we are given two loops $f, g$ that share the same basepoint $x_0$, then can move along one loop and then move along the next one to obtain a new loop with the same basepoint. This gives us a binary operation on the Hom-Set $\mathrm{Hom}_{\mathsf{Top}^*}(\mathbb{S}^1, (X, x_0))$ It turns out that if we only consider continuous functions up to homotopy, then the Hom-Set $\mathrm{Hom}_{\mathsf{Htpy}^*}(\mathbb{S}^1, X)$ carries a group structure!

## 5.1 Definition and examples

**Definition 5.1.**
- A path $\alpha : [0,1] \to X$ is called a **loop** with basepoint $x_0$ if $\alpha(0) = \alpha(1) = x_0$.
- Two paths $\alpha, \beta : [0,1] \to X$ are **homotopic rel. endpoints**, if there exists a homotopy preserving the endpoints. That is, if $\alpha \sim_h \beta$ with

$$h(0,t) = a(0) \quad \text{and} \quad h(1,t) = \alpha(1) \quad \text{for all} \quad t \in [0,1]$$

- If $\alpha$ is a path from $a$ to $b$ and $\beta$ is a path from $b$ to $c$, we define their **multiplication** as

$$\alpha\beta : [0,1] \to X, t \mapsto \begin{cases} \alpha(2s) & s \leq \frac{1}{2} \\ \beta(2s-1) & s \geq \frac{1}{2} \end{cases}$$

We write $\alpha \sim \beta$ rel endpoints and

$$[\alpha] := \{\beta | \beta \text{ is a path and } \alpha \sim \beta \text{ rel endpoints}\}$$

for the equivalence class.

**Definition 5.2.** Let $(X, x_0) \in \mathsf{Top}^*$. The **Fundamental Group** of $(X, x_0)$ is

$$\pi_1(X, x_0) := \{[\alpha] | \alpha \text{ is a loop with basepoint } x_0\} = \mathrm{Hom}_{\mathsf{Htpy}^*}(\mathbb{S}^1, (X, x_0))$$

with multiplication

$$([\alpha], [\beta]) \mapsto [\alpha\beta]$$

We have to show that the multiplication is well-defined and that it does indeed define a group structure.

*Proof.* The well-defined-ness of the multiplication is in the Exercise sheet 7, Problem 1.
The other group axioms are easily proven. The identity of the group is the constant map $x_0$ and inverse is the reverse traversal of the loop. ◻

**Example 5.3.** Let $K \subseteq \mathbb{R}^n$ be convex and $x_0 \in K$. Then the fundamental group $\pi_1(K, x_0)$ is the trivial group $\{e\}$.

**Theorem 5.4.** *(a) The fundamental group of the sphere is also the trivial group. More generally, for all $n \geq 2$, $\pi_1(\mathbb{S}^n, x_0) = \{e\}$*

(b) *The fundamental group of the circle is $\mathbb{Z}$ and representants of the group are the maps*

$$\alpha_k : [0,1] \to \mathbb{S}^1, \quad t \mapsto e^{2\pi i k t}$$

*Proof.* (a) Let $n \geq 2$ and $\alpha$ a loop with basepoint $x_0$. We show that it is homotopic to the constant map $x_0$. To do this, we first show that there exists a loop $\beta \sim \alpha$ rel $x_0$ which is not surjective. If $\operatorname{Im}\alpha = \mathbb{S}^n$, then take the hemisphere $D^n \subseteq \S^n$ with center $x_0$. Take the point $-x_0$ opposite to $x_0$ on the other hemisphere and let $(I_j)_{j\in J}$ be the maximal intervals such that $\operatorname{Im}(\alpha|_{I_J}) \subseteq D^n$. Because $D^n$ is contractible, we set

$$\beta(t) = \begin{cases} \alpha(s) & s \notin \bigcup_{j\in J} I_j \\ \beta_j(s) & s \in I_j \end{cases}$$

such that $\beta_j(s) : I_j \to \partial D^n$ satisfies

$$\beta_j|_{\partial I_j} = \alpha|_{\partial I_j}$$

From this, we take a point $x \in \mathbb{S}^n \setminus \operatorname{Im}(B)$. Then we use the isomorphism $\varphi : \mathbb{S}^n \setminus x \to \mathbb{R}^n$. Because $\mathbb{R}^n$ is contractible it follows that $\alpha \sim \beta \sim x_0$.

(b) We only provide an informal proof. We will do it more rigorously later. For surjectvitiy of the map $\Phi : \mathbb{Z} \to \pi_1(\mathbb{S}^1, 1), k \mapsto \alpha_k$, let $\alpha$ be a loop with basepoint 1.

The idea then is that we use the map

$$\psi : \mathbb{R} \to \mathbb{S}^1, \quad x \mapsto e^{2\pi i x}$$

that wraps $\mathbb{R}$ around the circle. We then show that there exists a path $\tilde{\alpha} : [0,1] \to \mathbb{R}$ such that $\alpha = \psi \circ \tilde{\alpha}$.

So if we set $k = \tilde{\alpha}(1) \in \mathbb{Z}$. then we can show that $\tilde{\alpha} \sim \tilde{a}_k$ rel endpoints. Therefore

$$\alpha = \pi \circ \tilde{\alpha} \sim \pi \circ \tilde{\alpha}_k = \alpha_k \text{ rel } 1$$

For injectivity, we prove that $\Phi$ is group homomorphism with kernel 0.

□

**Lemma 5.5.** *The fundamental group defines a covariant functor $\pi_1$ : $\mathsf{Top}^* \to \mathsf{Grp}$ that maps pointed topological spaces $(X, x_0)$ to their fundamental group $\pi_1(X, x_0)$ and that maps a basepoint preserving maps $f : (X, x_0) \to (Y, y_0)$ to group homomorphisms*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad [\alpha] \mapsto [f \circ \alpha]$$

*Proof.* We have to prove that

(a) For any topological space $X$

$$(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)} \qquad \text{for all } x_0 \in X$$

(b) For continuous, basepoint preserving maps $f : (X, x_0) \to (Y, y_0)$, $g : (Y, y_0) \to (Z, z_0)$

$$(g \circ f)_* = g_* \circ f_*$$

$\square$

**Corollary 5.5.1.** *If $f : X \to Y$ is a homeomorphism, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group isomorphism.*

*Proof.* This follows directly from Lemma 4.18 because homeomorphisms are the isomorphisms in $\mathsf{Top}^*$ and group isommorphisms are the isomorphisms in $\mathsf{Grp}$. $\square$

Two applications of the functoriality of the fundamental group are the Brower fixpoint theorem, aswell as the fundamental theorem of Algebra.

**Theorem 5.6** (Brower Fixpoint theorem)**.** *Let $n \in \mathbb{N}$ and $f : \mathbb{D}^n \to \mathbb{D}^n$ continuous. Then $f$ has a fixpoint, so $\exists x \in \mathbb{D}^n$ with $f(x) = x$.*

We will only prove it for $n \leq 2$ by using the fundamental group. For higher $n$, we will have to use another invariant.

*Proof.* Assume that $f$ has no fixpoint. We then can show that there exists a retraction $\rho : \mathbb{D}^n \to \partial\mathbb{D}^n = \mathbb{S}^{n-1}$.
The reason this is true is because if $f(x) \neq x$, then there exists a unique line from $x$ to $f(x)$. We then define $\rho(x)$ to be the intersection of that line with $\partial\mathbb{D}^n$, so

$$\{\rho(x)\} = \partial\mathbb{S}^{n-1} \cap \{f(x) + t\,(x - f(x))\,|t \in \mathbb{R}_{>0}\}$$

this function is continuous, because in a neighborhood $U$ of $x$, $f(U)$ is a neighborhood of $f(x)$, and geometrically $\rho(U)$ is a neighborhood of $\rho(x)$.
Now, we show that there cannot exist such a retraction $\rho : \mathbb{D}^n \to \mathbb{S}^n - 1$.
We already saw this for $n = 1$, because $\mathbb{D}^1$ is connected and $\{0, 1\}$ isn't.
For $n = 2$ assume there would exist such a retraction $\rho : \mathbb{D}^2 \to \mathbb{S}^1$. This means

$$\mathrm{id}_{\mathbb{S}^1} = \rho \circ \iota, \quad \text{where} \quad \iota : \mathbb{S}^1 \to \mathbb{D}^2, s \mapsto s$$

but by functoriality of the fundamental group we would have

$$\mathrm{id}_{\pi_1(\mathbb{S}^1,1)} = (\mathrm{id}_{\mathbb{S}^1})_* = (\rho \circ \iota)_* = \rho_* \circ \iota_*$$

where

$$\rho_* : \pi_1(\mathbb{D}^2, 1) \to \pi_1(\mathbb{S}^1, 1) \quad \text{and} \quad \iota_* : \pi_1(\mathbb{S}^1, 1) \to \pi_1(\mathbb{D}^2, 1)$$

But this contradicty our previous calculation of their fundamental groups, where we found $\pi_1(\mathbb{D}^2, 1) = \{e\}$ and $\pi_1(\mathbb{S}^1, 1) = \mathbb{Z}$.
In a diagram, the argument looks like this

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, "\iota"] & \mathbb{D}^2 \arrow[r, "\rho"] & \mathbb{S}^1 \arrow[d, bend left, "\pi_1"] \\
\mathbb{Z} \arrow[r, "\iota_*"] & \{e\} \arrow[r, "\rho_*"] & \mathbb{Z}
\end{tikzcd}$$

$\square$

In the proof, we showed that if $\rho : X \to A$ is a retraction and $\iota : A \to X$ is the embedding, then $\rho_*$ is surjective and $\iota_*$ is injective.
Moreover, if $\rho$ is a strong deformation retract, then $\rho_*$ and $\iota_*$ are inverse group homomorphisms.
Let's calculate some more fundamental groups.

**Example 5.7.** Because $\mathbb{S}^1 \subseteq \mathbb{C}^*$ and $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$ for $n \geq 3$ are strong deformation retracts, we have

$$\pi_1(\mathbb{R}^n \setminus \{0\}) = \begin{cases} \mathbb{Z} & n = 2 \\ \{e\} & n \geq 3 \end{cases}$$

**Theorem 5.8** (Fundamental theorem of Algebra)**.** *Let $p(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_0 \in \mathbb{C}[z]$ be a non-constant polynomial. Then $\exists x_0 \in \mathbb{C}$ with $p(z_0) = 0$*

*Proof.* Assume $p(z) \neq 0 \forall z \in \mathbb{C}$. Then chose a radius $r$ large enough, (for example $r > \sum_{j=0}^{d-1} |a_j|$) and construct he path

$$\alpha : [0,1] \to \mathbb{C}^*, \quad s \mapsto \frac{p(re^{2\pi i s})}{p(s)}$$

Then, $\alpha$ is homotopic to the constant map 1 with the homotopy

$$h(s,t) = \frac{p(tre^{2\pi i s})}{p(tr)}$$

If we move along the path $d$-times, we get the path $\alpha^d$, which is again homotopic to $\alpha$ by

$$h(s,t) = \frac{tp(re^{2\pi i s}) + (1-t)(re^{2\pi i s})^d}{tp(r) + (1-t)r^d} \in \mathbb{C}^*$$

which is well-defined because

$$\begin{aligned} |tp(re^{2\pi i s}) + (1-t)(re^{2\pi i s})^d| &\geq r^d - t\sum_{j=0}^{d-1} |a_j| r^j \\ &= r^{d-1}\left(r - t\sum_{j=0}^{d-1} |a_j| r^{j-dt}\right) \\ &\geq r^{d-1}(r - t\sum_{j=0}^{d-1} |a_j|) \geq r^{d-1}\left(r - \sum_{j=0}^{d-1} |a_j|\right) > 0 \end{aligned}$$

but by the previous example with $\pi_1(\mathbb{R}^2 \setminus \{0\}$ we would have

$$1 = [\alpha] = [\alpha_d] \neq 1 \quad \text{in } \pi_1(\mathbb{C}^*, 1) \lightning$$

$\square$

The fundamental group by definition depends on the choice of the basepoint, but how much? Can we predict when a change of the basepoint changes the fundamental group?

**Lemma 5.9.** *The following are easy to prove*

*(a) Let $\beta$ be a path from $x_0$ to $x_1$ in $X$. Then the mapping*

$$\Psi_\beta : \pi_1(X, x_1) \to \pi_1(X, x_0), \quad [\alpha] \mapsto [\beta\alpha\beta^-]$$

*is an isomorphism (in* $\mathsf{Grp}$*).*

(b) *The fundamental group of the product is the product of the fundamental groups. So for $(X, x_0), (Y, y_0)$ in $\mathsf{Top}^*$, there exists an isomorphism*

$$\pi_1(X, x_0) \times \pi_1(Y, y_0) \xrightarrow{\sim} \pi_1(X \times Y, (x_0, y_0))$$

*In other words, for a fixed space $(X, x_0)$ in $\mathsf{Top}^*$, the following "diagram" commutes*

$$\begin{tikzcd} \mathsf{Top}^* \arrow[r, "\pi_1"] \arrow[d, "(X,x_0)\times -"'] & \mathsf{Grp} \arrow[d, "\pi_1(X,x_0)\times -"] \\ \mathsf{Top}^* \arrow[r, "\pi_1"] & \mathsf{Grp} \end{tikzcd}$$

From now on, we will write $\pi_1(X) := \pi_1(X, x_0)$ if $X$ is path connected.

**Corollary 5.9.1.** $\mathbb{R}^2 \cong \mathbb{R}^n \implies n = 2$

*Proof.* Because a homeomorphism $\varphi : \mathbb{R}^2 \setminus \{0\}$ induces a homeomorhpism

$$\tilde{\varphi} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^n \{0\}$$

it follows directly from the computation done in Example 5.7. $\square$

Now we can finally define what it means for a topological space to have no point-holes.

**Definition 5.10.** A path connected space $X$ is called **simply connected**, if $\pi_1(X) = \{e\}$

For example, $\mathbb{S}^n$ is simply connected for $n \geq 2$.

**Theorem 5.11.** *Let $f : (X, x_0) \to (Y, y_0)$ be a homotopy equivalence. Then the induced mapping $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group isomorphism.*

To prove the theorem, we will use the following lemma

**Lemma 5.12.** *Let $f_0 \sim f_1 : X \to Y$ be homotopic via $h$, $x_0 \in X$ and $\beta$ a path from $f_0(x_0) \to f_1(x_0)$. Then $(f_0)_* = \Psi_\beta \circ (f_1)_*$. The corresponding diagram is*

$$\begin{tikzcd} & \pi_1(Y, f_1(x_0)) \arrow[dd, "\Psi_\beta"] \\ \pi_1(X, x_0) \arrow[ur, "(f_1)_*"] \arrow[dr, "(f_0)_*"] & \\ & \pi_1(Y, f_0(x_0)) \end{tikzcd}$$

*Proof Lemma.* Set $\beta_t : [0, 1] \to Y$ as $\beta_t(s) = \beta(st)$. Then for any loop $\alpha$ at $x_0$ we get

$$f_0 \circ \alpha \sim \beta(f_1 \circ \alpha_)\beta^- \text{ rel } x_0$$

via the homotopy

$$H(s, t) = \beta_t(h_t \circ \alpha)\beta_t^-$$

and as such, the group homomorhpisms are equal:

$$(f_0)_*[\alpha] = [f_0 \circ \alpha] = [\beta(f_1 \circ \alpha)\beta^-)] = \Psi_\beta([f_1 \circ \alpha]) = (\Psi_\beta \circ (f_1)_*)[\alpha]$$

$\square$

*Proof theorem.* Let $f : X \to Y$ be a homotopy equivalence and $g : Y \to X$ its homotopy inverse. So let $h, k$ be the corresponding homotopies

$$g \circ f \sim_h \mathrm{id}_X \quad \text{and} \quad f \circ g \sim_k \mathrm{id}_Y$$

Let $x_0 \in X$ and consider the diagram

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, (g \circ f)(x_0)) \xrightarrow{f_*} \pi_1(X, (f \circ g \circ f)(x_0))$$

We then take the path $\beta(s) := h(x_0, s)$ that connects $(g \circ f)(x_0)$ with $x_0$, we can use the previous lemma to get

$$g_* \circ f_* = (g \circ f)_* = \Psi_\beta \circ (\mathrm{id}_X)_* = \Psi_\beta$$

which shows that $f_*$ is injective. For surjectivity, we take the path $\beta'(s) = k(f(x_0), s)$ connecting $(f \circ g \circ f)(x_0)$ with $f(x_0)$ and get similarly

$$f_* \circ g_* = \Psi_{\beta'}$$

which shows injectivity of $g_*$, so since $g_* \circ f_*$ is bijective, $f_*$ is also surjective and thus an isomorphism. □

**Example 5.13.** If $X$ is contractible, then $X$ is simply connected because the fundamental group of the singleton space is $\{e\}$.

We have called the fundamental group functor $\pi_{\mathbf{1}}$. But if there is a $\pi_1$, is there a $\pi_2$?
Because a loop $\alpha : [0, 1]$ is the same as continuous function $\alpha \in \mathrm{Hom}_{\mathsf{Top}^*}(\mathbb{S}^{\mathbf{1}}, (X, x_0))$ we can generalize the notion of the fundamental group.

**Definition 5.14.** For $(X, x_0) \in \mathsf{Top}^*$ and $n \in \mathbb{N}$ we define the **higher homotopy groups**

$$\pi_n(X, x_0) := \{[\alpha] | \alpha : \mathbb{S}^n \to X \text{ with } \alpha(e_1) = x_0\} = \mathrm{Hom}_{\mathsf{Htpy}^*}(\mathbb{S}^n, (X, x_0))$$

They are actually only a group for $n \geq 1$, but for $n \geq 2$ they are all abelian.
To give an intuition what the homotopy groups mean, let's first note that

$$\pi_0(X, x_0) = \{[\alpha] | \alpha : \{-1, 1\} \to X, \alpha(1) = x_0\} \cong \{[x] | x \in X\}$$

so:

- $\pi_0(X, x_0)$ measures the number of path connected components of $X$ (irrespective of the choice of $x_0$).
- $\pi_1(X, x_0)$ tells us how many one-dimensional "holes" there are in the path connected component of $x_0$

**Remark 5.15.** There is a theorem that says that the functor $\pi_1$ is "surjective". That is, for every group $G$ there exists a topological space $X$ such that $\pi_1(X) = G$.

## 5.2 Free groups

Consider the example

$$X = \mathbb{C} \setminus \{-1, 1\}, z_0 = 0$$

and loops $\alpha, \beta$ with basepoint $z_0$, where $\alpha$ loops around 1 and $\beta$ loops around $-1$.
If we set

$$A := \{z \in X | \mathrm{Re}(z) > -1\} \cong B := \{z \in X | \mathrm{Re}(z) < 1\} \cong \mathbb{C} \setminus \{0\}$$

then we see that the fundamental groups

$$\pi_1(A, x_0) \cong \mathbb{Z} \quad \pi_1(B, x_0) \cong \mathbb{Z}$$

are generated by $[\alpha]$ and $[\beta]$ each.
Since $[\alpha][\beta] \neq [\beta][\alpha]$, we could get the hint that $\pi_1(X) = F_2 = \mathbb{Z} * \mathbb{Z}$ is the free group generated by two elements $[\alpha], [\beta]$.
So the main idea is that we want to express the fundamental group of a union $X = A \cup B$ using the fundamental groups of $A$ and $B$.

**Definition 5.16.** Let $H, K$ be (disjoint) groups.

- A **word** in $H$ and $K$ is an expression of the form $g_1 g_2 \dots g_n$ for $n \in \mathbb{N}$ such that $g_j \in H \cup K$.
- A word is said to be **reduced**, if $g_j \notin \{e_H, e_K\}$, and $g_j, g_{j+1}$ are in alternatting groups.

**Example 5.17.** For $H = \langle a \rangle = \{a^k | k \in \mathbb{Z}\}$ and $K = \langle b \rangle = \{b^k | k \in \mathbb{Z}\}$, the following are reduced words

$$aba^{-1}b^{-1}, \quad a^2, \quad a^{-3}, \quad b^{-7}a, \quad aba$$

Note: Every word $w$ has a unique **reduction** $R(w)$. For example

$$w = aab1_H b^{-2}a^{-1} \implies R(w) = a^2 b^{-1} a^{-1}$$

**Definition 5.18.** The **free product** (or **coproduct**) of two groups $H, K$ is the group

$$H * G := \{w | w \text{ is a reduced word in } H \& K\}$$

where the multiplication is defined as

$$(H * K) \times (H * K) \to H * K, \quad (w_1, w_2) \mapsto R(w_1 w_2)$$

As the name implies, it is the coproduct in the category $\mathsf{Grp}$.
One quickly verifies that this does indeed form a group and has the universal property, where the inclusion mappings are given by

$$\iota_H : H \hookrightarrow H * K, \quad h \mapsto \begin{cases} h & \text{if } h \neq e_H \\ 1 & \text{if } h = e_h \end{cases}$$

The universal property says that for any group $G$ with morphisms $H \xrightarrow{f} G, K \xrightarrow{g} G$, there exists a unique homomorphism $\varphi : H * K \to G$ such that the following diagram commutes

$$\begin{tikzcd} & H * K \arrow[dd, dotted, "\varphi"] & \\ H \arrow[ur, "\iota_H"] \arrow[dr, "f"'] & & K \arrow[ul, "\iota_K"'] \arrow[dl, "g"] \\ & G & \end{tikzcd}$$

this morphism is given by

$$\varphi : H * K \to G, \quad h_1k_1h_2k_2 \ldots \mapsto f(h_1)g(k_1)f(h_2)g(k_2)\ldots$$

- $H * K = K * H$
- If $H = \{e\}$, then $H * K \cong K$.
- If $H, K$ are non-trivial groups, then $H * K$ is anabelian, because for $h \neq 1 \in H, k \neq 1 \in K$ the words $hk, kh$ are different words.

**Example 5.19.** Let $\langle a \rangle, \langle b \rangle$ be cyclical groups.

- Then $\langle a \rangle * \langle b \rangle$ is the free group with two generators.
- If $\langle a \rangle$ is finite of order 2 and $\langle b \rangle$ of order 3, then

  $$\langle a \rangle * \langle b \rangle = \{1, a, b, b^2, ab, ba, ab^2, ba^2, \ldots\}$$

  This group is isomorphic to the projective special linear group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$

## 5.3 Seifert - van Kampen's theorem

Let $X$ be a topological space, $A, B \subseteq X$ and $x_0 \in A \cap B$. Let

$$\begin{aligned} j_A &: \pi_1(A) \to \pi_1(X) \\ j_B &: \pi_1(B) \to \pi_1(X) \\ i_A &: \pi_1(A \cap B) \to \pi_1(A) \\ i_B &: \pi_1(A \cap B) \to \pi_1(B) \end{aligned}$$

be the group homomorphisms induced by the functor $\pi_1$ acting on the inclusion mappings. This universal property of the coproduct induces a group homomorphism $\varphi : \pi_1(A) * \pi_1(B) \to \pi_(X)$ such that the following diagram commutes

$$\begin{tikzcd} & \pi_1(A \cap B) \arrow[dl, "i_A"'] \arrow[dr, "i_B"] & \\ \pi_1(A) \arrow[r, "\iota_A"] \arrow[dr, "j_A"'] & \pi_1(A) * \pi_1(B) \arrow[d, dotted, "\varphi"] & \pi_1(B) \arrow[l, "\iota_K"'] \arrow[dl, "j_B"] \\ & \pi_1(X) & \end{tikzcd}$$

**Theorem 5.20** (Seifert - van Kampen). *If $A, B, A \cap B$ are path connected and open such that $A \cup B = X$, let*

$$\varphi : \pi_1(A) * \pi_1(B) \to \pi_1(X)$$

*be the unique group homomorhpism induced by the coproduct. Then*

*(a)* *$\varphi$ is surjective*

*(b)* $\operatorname{Ker}\varphi = \langle\langle \{ i_A(c)\,(i_B(c))^{-1} \,|\, c \in \pi_1(A \cap B) \} \rangle\rangle$

*In particuar*

$$\pi_1(A) * \pi_a(B) \big/ \operatorname{Ker}\varphi \cong \pi_1(X)$$