Electrodynamics – Summary

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1 Magnetostatics

In Magnetostatics, we consider systems where the current is steady. This means in particular that

$$\vec{J} = \text{const}, \implies \rho = \text{const}, \quad \vec{E} = \text{const}, \quad \vec{B} = \text{const}$$

Under the **Coulomb-Eichung**

Electrostatics	Magnetostatics
$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\frac{\rho(\vec{y})}{\lvert\vec{x}-\vec{y}\rvert}$	$A(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2}\int d^3\vec{y}\frac{\vec{J}(\vec{y})}{\lvert\vec{x}-\vec{y}\rvert}$
$\vec{E} = -\nabla\Phi$	$\vec{B} = \vec{\nabla}\times\vec{A}$
$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$	$\vec{\nabla}\times\vec{B} = \frac{\vec{J}}{\epsilon_0 c^2}$
$\vec{\nabla}\times\vec{E} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
$\int_{\partial V} d\vec{S}\cdot\vec{E} = \frac{Q_{\text{inside}}}{\epsilon_0}$	$\oint_{\partial S}\vec{B}\cdot d\vec{\ell} = \frac{I_{\text{inside}}}{\epsilon_0 c^2}$

Table 1: Analogies between Electrostatics and Magnetostatics

2 Time dependent electromagnetic fields

If the $\vec{E}$ and $\vec{B}$ are time dependent, then the Maxwell equations are as follows

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad c^2\vec{\nabla}\times\vec{B} = \frac{\vec{J}}{\epsilon_0} + \frac{\partial\vec{E}}{\partial t}$$

From the scalar and vector potential $\Phi, \vec{A}$ we can find the $\vec{E}, \vec{B}$ fields with.

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}$$

Since the elecric and magnetic field remain invariant under **gauge transformations**

$$\Phi \mapsto \tilde{\Phi} = \frac{\partial}{\partial t}f(\vec{x},t)$$
$$\vec{A} \mapsto \tilde{\vec{A}} = \vec{A} - \nabla f(\vec{x},t)$$

for some $f : \mathbb{R}^4 \to \mathbb{R}$, the potentials are not uniquely determined.
By introducing the **d'Alembert Operator**

$$\boxed{\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2}$$

we get the equations

$$\Box\vec{A} = \frac{\vec{J}}{\epsilon_0 c^2}, \quad \Box\Phi = \frac{\rho}{\epsilon_0}$$

To solve them, we can instead search for the Green's function (or more generally, the Fundamental solution) $G(\vec{x},t,\vec{y},t')$ that satisfies

$$\Box_{x,t}G(\vec{x},t,\vec{y},t') = \delta(\vec{x}-\vec{y})\delta(t-t')$$

After some Fourier transformation and Complex Analysis shenanigans, we obtain the solution

$$G(\vec{x},t,\vec{y},t') = \frac{1}{4\pi|\vec{x}-\vec{y}|}\delta\left(t - t' - \frac{\vec{x}-\vec{y}}{c}\right)\Theta(t-t')$$

or equivalently, we can write

$$\boxed{G(\Delta\vec{x},\Delta t) = \frac{1}{2\pi}\delta\left((t-t')^2 - \frac{|\vec{x}-\vec{y}|^2}{c^2}\right)\Theta(t-t')}$$

By defining the **time retardation**

$$t_{\text{ret}} := t' - \frac{|\vec{x}-\vec{y}|}{c}$$

we obtain the **retarded scalar potential**

$$\boxed{\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{y}\frac{\rho(\vec{y},t_{\text{ret}})}{|\vec{x}-\vec{y}|}}$$

aswell as the **retarded vector potential**

$$\boxed{\vec{A}(\vec{x},t) = \frac{1}{4\pi\epsilon_0 c^2}\int d^3\vec{y}\frac{\vec{J}(\vec{y},t_{\text{ret}})}{|\vec{x}-\vec{y}|}}$$

3 Special Relativity

Whereas classical mechanics played in three dimensional space $\mathbb{R}^3$ with the time dimension $t \in \mathbb{R}$ separated, special relativity plays in the **Minkowsky Space**.
Elements of the Minkowsky space are four-vectors $x^\mu = (ct, \vec{x})$.
The **metric tensor** is the matrix

$$g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$$

which lets us define the (quasi)**inner product** on the minkowski space as

$$\langle x,y\rangle = g_{\mu\nu}x^\mu y^\nu \implies \langle x,x\rangle = c^2t^2 - \vec{x}^2$$

3.1 Lorentz Transformations

A **Lorentz transformation** is any affine linear transformation of the form

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + \rho^\mu$$

such that it satisfies the relation

$$\boxed{\Lambda^\mu{}_\nu \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}}$$

such transformation has the inner product as an invariant since

$$\begin{aligned}\langle \tilde{x},\tilde{x}\rangle &= g_{\rho\nu}\left(\Lambda^\mu{}_\sigma x^\sigma\right)\left(\Lambda^\nu{}_\rho x^\rho\right)\\ &= g_{\mu\nu}\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho x^\sigma x^\rho \qquad &= g_{\sigma\rho}x^\sigma x^\rho = \langle x,x\rangle\end{aligned}$$

The set of all Lorentz transformations forms a group, called the **Poincare group**.
We are especially interesed in the subgroup known as the **proper Lorentz transformations**, which are all such transformations that satisfy

$$\det\Lambda = 1, \quad \Lambda^0{}_0 \geq 1$$

Another invariant is the **proper time**

$$\boxed{d\tau^2 = dx^2 = c^2dt^2 - d\vec{x}^2}$$

Given a velocity $\vec{v}$, and dt we get

$$d\tau = cdt\sqrt{1-\frac{v^2}{c^2}} = \frac{c}{\gamma}dt$$

When a stationary observer O sees a reference frame $\tilde{O}$ passing by with velocity $\vec{v} = \vec{\beta}c$ along the x-axis, then the corresponding boost.

$$\Lambda_x{}^\mu{}_\nu(\vec{\beta}) = \begin{pmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

If the velocity is at an angle θ with the x-axis in the xy-plane, then we break the boost into three steps

$$x \mapsto \tilde{x} = R(\theta)x \mapsto \tilde{\tilde{x}} = \Lambda(\beta)\tilde{x} \mapsto x'^\mu = R^\mu{}_\rho(-\theta)\Lambda_x{}^\rho{}_\sigma R^\sigma{}_\nu(\theta)x^\nu$$

The inverse of a Lorentz transformation $\Lambda^\mu{}_\nu$ is given by

$$\boxed{\Lambda_\mu{}^\nu = g_{\mu\nu}g^{\nu\sigma}\Lambda^\rho{}_\sigma}$$

Example 3.1 (Relativistic effects)**.** Consider a moving particle that has a life span of t_0. By defining the events for the Start A and End B, we get

$$A = (0,\vec{0}) \quad \text{and} \quad B = (ct_0, \vec{0})$$

we see that after the Lorentz transformation we measure its lifespan to be at

$$\tilde{A} = \Lambda A = (0,\vec{0}), \quad \text{and} \quad \tilde{B} = \Lambda B = (\gamma ct_0, \beta\gamma ct_0)$$

so the outside observes sees a **time dilation** in its lifespan $\tilde{t}_0 = \gamma t_0$.
On the contrary, the outside observer notices a **length contraction**

$$s = vt_0 = \frac{1}{\gamma}\tilde{s}$$

3.2 Tensors

In a change of reference under a Lorentz Transformation

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + \rho^\mu$$

a **contra-variant tensor** is any object U that transforms as follows

$$T^\mu \mapsto \tilde{T}^\mu = \Lambda^\mu{}_\nu T^\nu$$

examples are the momentum p^μ, the force f^μ or the differential form dx^μ. Contra-variant tensors are denoted with upstairs indices.

A **covariant tensor** is any object that transforms like

$$T_\mu \mapsto \tilde{T}_\mu = \Lambda_\mu{}^\nu T_\nu$$

examples are the covariant derivative $\frac{\partial}{\partial x^\mu}$. Such tensors are denoted with downstairs indices.
There are also **mixed tensors**, which have both up- and downstairs indices. They transform according to the rules

$$\tilde{T}^{\mu_1\mu_2\dots\mu_m}_{\nu_1\nu_2\dots\nu_n} = \Lambda^{\mu_1}{}_{\sigma_1}\dots\Lambda^{\mu_m}{}_{\sigma_m}\Lambda_{\nu_1}{}^{\rho_1}\dots\Lambda_{\nu_n}{}^{\rho_n}T^{\mu_1\mu_2\dots\mu_m}_{\nu_1\nu_2\dots\nu_n}$$

We can "raise/lower indices" with the metric tensor

$$\boxed{T^\mu \mapsto T_\mu = g_{\mu\nu}T^\nu \quad \text{and} \quad T_\mu \mapsto T^\mu = g^{\mu\nu}T_\nu}$$

what actally means is that the metric tensor transforms contravariant tensors to covariant ones and vice versa.
A **Lorentz-Scalar** is any object that stays invariant under Lorentz transformations.
Useful Tensor relations are

$$\Lambda^\mu{}_\sigma \Lambda_\mu{}^\rho = \delta^\rho{}_\sigma$$

3.3 Energy and Momentum

We define the four-momentum as

$$\boxed{p^\mu := mc\frac{dx^\mu}{d\tau}}$$

whose components can be given by

$$p^0 = m\gamma c = mc + \frac{1}{2c}mv^2 + \mathcal{O}(\frac{v^4}{c^3}), \quad p^i = m\gamma v^i$$

this gives us the definition for **relativistic energy** as

$$\boxed{E := cp^0 = m\gamma c^2 = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(\frac{v^4}{c^2})}$$

From the relation

$$\vec{p}^2 = m^2\gamma^2\vec{v}^2$$

we obtain the very useful **energy-momentum relation**

$$\boxed{E^2 = c^2\vec{p}^2 + m^2c^4}$$

4 Relativistic Electrodynamics

From now on, we use the convetion $c = \epsilon_0 = 1$.
We combine current and charge density into a contravariant four-vector

$$\boxed{jj^\mu := (\rho, \vec{j})}$$

The antisymmetric **electrodynamic field tensor** $F^{\mu\nu}$ is defined using the $\vec{E}$ and $\vec{B}$ fields

$$\boxed{F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\vec{E}^T \\ \vec{E} & B^\times \end{pmatrix}}$$

where $B^\times$ is the dual tensor defined as

$$(B^\times)_{ij} = -\epsilon_{ijk}B^k \implies B^\times = \begin{pmatrix} 0 & -B^3 & B^2 \\ B^3 & 0 & -B^1 \\ -B^2 & B^1 & 0 \end{pmatrix}$$

from which the electric and magnetic field can be obtained using the relations

$$\boxed{E^i = F^{i0}, \quad B^i = -\frac{1}{2}\epsilon_{ijk}F^{jk}, \quad \text{or} \quad F^{ij} = -\epsilon_{ijk}B^k}$$

and the Maxwell equations become

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu}$$

In covariant Form, we get

$$F_{\mu\nu} = \begin{pmatrix} 0 & \vec{E}^T \\ -\vec{E} & B^\times \end{pmatrix}$$

Example 4.1 (Transformation of EM tensor)**.** In a change of reference along the x-axis with boost β, the EM tensor transforms doubly contravariant, i.e

$$F^{\mu\nu} \mapsto \tilde{F}^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho F^{\sigma\rho}$$

The summation over ρ corresponds to the matrix multiplication

$$\Lambda^{\nu}{}_{\rho}F^{\sigma\rho} = \begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & B^{\times} \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -E^1\beta\gamma & -E^1\gamma & -E^2 & -E^3 \\ E^1\gamma & E^1\beta\gamma & -B^3 & B^2 \\ \gamma(E^2+B^3\beta) & \gamma(E^2\beta+B^3) & 0 & -B^1 \\ \gamma(E^3-B^2\beta) & \gamma(E^3\beta-B^2) & B^1 & 0 \end{pmatrix}$$

and in the end, we get

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -\gamma(E^2+\beta B^3) & -\gamma(E^3-\beta B^2) \\ E^1 & 0 & -\gamma(B^3+\frac{v}{c^2}E^2) & \gamma(B^2-\frac{v}{c^2}E^3 \\ & & 0 & -B^1 \\ & & B^1 & 0 \end{pmatrix}$$

This result generalizes to arbitrary boosts, and we get

$$\boxed{\begin{aligned} \vec{E}_{\parallel} \mapsto \tilde{\vec{E}}_{\parallel} = \vec{E}_{\parallel}, \quad \vec{E}_{\perp} \mapsto \tilde{\vec{E}}_{\perp} = \gamma(\vec{E}_{\perp} - \vec{v}\times\vec{B}) \\ \vec{B}_{\parallel} \mapsto \tilde{\vec{B}}_{\parallel} = \vec{B}_{\parallel}, \quad \vec{B}_{\perp} \mapsto \tilde{\vec{B}}_{\perp} = \gamma(\vec{B}_{\perp} + \frac{1}{c^2}\vec{v}\times\vec{B}) \end{aligned}}$$

Example 4.2 (Transformation of Lorentz Force)**.** Two particles with charge q are distance d apart and are moving perpendicular to $\vec{d}$ with velocity $\vec{v}$.
In the system S where the particles are stationary, the Lorentz Force is simply:

$$S : \vec{F} = q \cdot \vec{E} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{d^2}\hat{d}$$

And in the external system S' the electromagnetic field are

$$\tilde{E} = (E^1, \gamma E^2, \gamma E^3), \quad \tilde{B} = (B^1 = 0, -\gamma\frac{v}{c^2}E^3, \gamma\frac{v}{c^2}E^2)$$

and so we see

$$\tilde{\vec{F}} = \frac{1}{\gamma}\vec{F}$$

4.1 Maxwell Equations

The (inhomogenous) Maxwell equations

$$\vec{\nabla}\cdot\vec{E} = \rho, \quad \vec{\nabla}\times\vec{B} - \partial_t\vec{E} = \vec{j}$$

become the simple equation

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu}$$

We can also combine the scalar and vector potential into the four vector

$$\boxed{A^\mu := (\Phi, \vec{A}) = (\Phi, A^1, A^2, A^3)}$$

and the equations

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}, \quad \vec{B}\times\vec{A}$$

get simplified with the contravariant derivative

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$$

For the homogenous Maxwell equations

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

we define the **dual field tensor** $\tilde{F}$ as

$$\tilde{F}_{\mu\nu} := \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = 2\begin{pmatrix} 0 & -\vec{B}^T \\ \vec{B} & E^{\times} \end{pmatrix}$$

and get the simple form

$$\boxed{\partial^\mu\tilde{F}_{\mu\nu} = 0}$$

In the Lorentz-Gauge: $(\partial_\mu A^\mu = 0)$ the maxwell equations become

$$\partial^2 A^\mu = j^\mu$$

Example 4.3. The 4-vector formulation of the Maxwell equations give a quick derivation of the continuity equation.
Because the electromagnetic field tensor is antisymmetric, we see that

$$\partial_\mu\partial_\nu F^{\mu\nu} = -\partial_\mu\partial_\nu F^{\nu\mu} = 0$$

and so we get

$$\partial_\mu j^\mu = \partial_\mu\partial_\nu F^{\mu\nu} = 0$$

4.2 Energy Impulse Tensor

Given charges q_n at positions $\vec{r}_n(t)$ with energies $E_n(t)$ and momentum $p_n(t)$ the charge and current density is given by

$$\rho(\vec{x},t) = \sum_n q_n\delta(\vec{x}-\vec{r}_n(t)), \quad \vec{j}(\vec{x},t) = \sum_n q_n\frac{d\vec{r}_n(t)}{dt}\delta(\vec{x}-\vec{r}_n(t))$$

So the energy and energy current densities are

$$\sum_n E_n(t)\delta(\vec{x}-\vec{r}_n), \quad \sum_n E_n(t)\frac{d\vec{r}_n(t)}{dt}\delta(\vec{x}-\vec{r}_n(t))$$

aswell as the impulse and impulse current densities

$$\sum_n p_n^i(t)\delta(\vec{x}-\vec{r}_n(t), \quad \sum_n p_n^i(t)\frac{d\vec{r}_n(t)}{dt}\delta(\vec{x}-\vec{r}_n(t))$$

respectively.
Using tensor notation, we combine charge and current densities to a four vector

$$\boxed{j^\mu := (c\rho, \vec{j})}$$

define the four-vector impulse $p^\mu = (E, \vec{p})$ and combine energy and momentum into the **Energy Momentum Tensor**

$$T^{\mu\nu} := \sum_n p_n^\mu(t) p_n^\mu \frac{dr_n^\nu}{dt}\delta(\vec{x}-\vec{r}_n(t))$$

which corresponds to a 4×4 matrix of the Layout

$$T^{\mu\nu} = \begin{pmatrix} \text{Energy density} & \text{Energy current density} \\ \text{Impulse density} & \text{Impulse current density} \end{pmatrix}$$

where the imuplse current density corresponds to a 3×3 submatrix.
If we set $c=1$, then energy and impulse are the same so we see that $T^{\mu\nu}$ is symmetric, giving us the form

$$T^{\mu\nu} = \sum_n \frac{p_n^\mu p_n^\nu}{E_n}\delta(\vec{x}-\vec{r}_n(t))$$

4.3 Energy-impulse tensor in electromagnetic field

If we have charges q_n in an electromagnetic field, then the energy and impulse are *not* conserved but instead go into the electromagnetic field.
The energy impulse tensor in an electromagnetic field is

$$\boxed{T_{\text{em}}^{\mu\nu} := F^\mu{}_\rho F^{\rho\nu} + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma}F^{\rho\sigma}}$$

which is symmetric and gauge invariant. Its components are

$$w := T_{\text{em}}^{00} = \frac{\vec{E}^2+\vec{B}^2}{2} \quad \text{and} \quad T_{\text{em}}^{0i} = T_{\text{em}}^{i0} = (\vec{E}\times\vec{B})_I$$

where w is the energy density of the electromagnetic field. Defining the sum

$$\Theta^{\mu\nu} := T^{\mu\nu} + T_{\text{em}}^{\mu\nu}$$

we see that energy and momentum is conserved

$$\partial_\nu T_{\text{em}}^{\mu\nu} = -F^{\mu\nu} j_\nu \implies \partial_\nu \Theta^{\mu\nu} = 0$$

Taking the sum of the four-momentum of the charges

$$P_{\text{charges}}^\mu = \sum_n p_n^\mu$$

and the momentum of the electromangentic field

$$P_{\text{em}}^\mu = \int d^3\vec{x} T^{\mu 0}$$

the total momentum is conserved.

$$P^\mu := \int d^3\vec{x}\Theta^{\mu 0} = P_{\text{charges}}^\mu + P_{\text{em}}^\mu = \text{const}$$

We also define the **Poynting vector** as

$$\boxed{S^i = T_{\text{em}}^{0i} = (\vec{E}\times\vec{B})^i}$$

so the equation

$$\partial_0 T^{00} + \partial_i T^{i0} = -F^{0i} j_i$$

turn into the well-known formula

$$\frac{\partial w}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{E}\cdot\vec{j}$$

5 Maths

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

For $f(x)$ with roots $(x_i)_{i\in I}$

$$\delta(f(x)) = \sum_{i\in I}\frac{1}{|f'(x)|}\delta(x-x_i)$$

5.1 Tensor Stuff

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$