Measure Theory– Lecture Notes

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1 Measure spaces

1.1 Algebras and σ-Algebras

From now on, let X denote a non-empty set.

Definition 1.1. For a sequence of subsets $(A_n)_{n=1}^\infty$ in $\mathcal{P}(X)$. We define

$$\limsup_{n\to\infty} A_n := \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m$$
$$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty A_m$$

And if they are equal, we say that the sequence $(A_n)_{n=1}^\infty$ converges to its limit $\lim_{n\to\infty} A_n$.

Informally, the $\limsup$ consists of elements of X that occur in infinitely many A_n, whereas the $\liminf$ consists of elements that occur for all but finitely many A_n.

Remark 1.2. (a) $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$

(b) If $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^\infty A_n$$

(c) If $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^\infty A_n$$

The similarity in names with the $\limsup$ and $\liminf$ from Analysis can be seen using the characteristic function

$$\mathbb{1}_A : X \to \{0,1\}$$
$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

It holds that

$$\limsup_{n\to\infty} A_n = A \iff \limsup_{n\to\infty} \mathbb{1}_{A_n} = \mathbb{1}_A$$
$$\liminf_{n\to\infty} A_n = A \iff \liminf_{n\to\infty} \mathbb{1}_{A_n} = \mathbb{1}_A$$

Definition 1.3 (Algebras of sets)**.** A collection of subsets $\mathcal{A} \subseteq \mathcal{P}(X)$ is called an **algebra in** X if

(a) $X \in \mathcal{A}$

(b) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

(c) $A \in \mathcal{A} \implies A^c \in \mathcal{A}$

An algebra $\mathcal{E}$ is called a **σ-algebra**, if for any sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{E}$ we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{E}$

Note that using the De Morgan's identity

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$$

we can see that algebras (σ-algebras) are stable under finite (infinite) intersections aswell.
For a collection of sets $\mathcal{K} \subseteq \mathcal{P}(X)$, the intersection of all Algebras containing K forms an algebra. We call this the algebra **generated by** K.
The algebra generated by the open sets of a topology is called the **Borel σ-Algebra** of X.

1.2 Measures

Definition 1.4. Let $\mathcal{A}$ be an Algebra on X and $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$. We say that μ is

- **additive**, if for any *finite* family of disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$

$$\mu\left(\bigsqcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k)$$

- **σ-additive**, if for any *countable* family of disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

Remark 1.5. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathcal{A}$ such that their union is again in $\mathcal{A}$.

(a) If μ is additive, then it is monotone with respect to incusion, i.e. $A \subseteq B \implies \mu(A) \leq \mu(B)$.

(b) If μ is additive and the sets A_k are mutually disjoint, then

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) \geq \mu\left(\bigsqcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k) \quad \forall n \in \mathbb{N}$$

so μ is **σ-superadditive**

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} \mu(A_k)$$

(c) If μ is σ-additive, then it is also **σ-subadditive**. To see this, we can define the mutually disjiont sets

$$B_1 = A_1, \quad B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k \in \mathcal{A}$$

Since $\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$ and $\mu(B_k) \leq \mu(A_k)$ we have

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigsqcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

It follows immediately from (b) and (c) that an additive function is σ-additive, if and only if it is σ-subadditive.

Definition 1.6. A σ-additive function $\mu : \mathcal{A} \to [0, \infty]$ is called

- **finite**, if $\mu(X) < \infty$
- **σ-finite**,if there exists a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that

$$\bigcup_{n=1}^{\infty} A_n = X \quad \text{and} \quad \mu(A_n) < \infty \quad \forall n \in \mathbb{N}$$

Definition 1.7. A function $\mu : \mathcal{P}(X) \to [0, \infty]$ is called a **measure** on X, if

(a) $\mu(\emptyset) = 0$ For $A \subseteq \bigcup_{k=1}^{\infty} A_k$, we have $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$

A measure is automatically monotonous and σ-subadditive.

Definition 1.8. Let μ be a measure on X and $A \subseteq X$. We can *restrict* μ to A (written $\mu \llcorner A$) defined by

$$(\mu \llcorner A)(B) := \mu(A \cap B) \quad \forall B \subseteq X$$

Definition 1.9 (Carathéodory criterion)**.** A subset $A \subseteq X$ is called **μ-measurable** if for all $B \subseteq X$

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$

Remark 1.10. (a) By subadditivity of the measure, the definition is equivalent to

$$\boxed{\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A), \quad \forall B \subseteq X}$$

(b) If $\mu(A) = 0$, then A is μ-measurable.

Theorem 1.11. *Let $\mu : \mathcal{P}(X) \to [0, \infty]$ be a measure. Then the collection of measurable sets*

$$\Sigma = \{A \subseteq X | A \textit{is } \mu\textit{-measurable}\}$$

is a σ-algebra.

. Let $B \subseteq X$

- $X \in \Sigma$: It's trivial to see that

$$\mu(B \cap X) + \mu(B \setminus X) = \mu(B) + \mu(\emptyset) = \mu(B)$$

- $A \in \Sigma \implies A^c \in \Sigma$: It holds that

$$B \cap A^c = B \setminus A, \quad \text{and} \quad B \setminus A^c = B \cap A$$

therefore

$$\mu(B \cap A^c) + \mu(B \setminus A^c) = \mu(B \setminus A) + \mu(B \cap A) = \mu(B)$$

- $A_1, A_2 \in \Sigma \implies A_1 \cup A_2 \in \Sigma$: Let $B \subseteq X$. From the previous remark, it is sufficient to just show the inequality

$$\mu(B) \geq \mu(B \cap (A_1 \cup A_2)) + \mu(B \setminus (A_1 \cup A_2))$$

By chosing the test set $B \setminus A_2$ and using μ-measurability for A_1, we get

$$\begin{aligned} \mu(B \setminus A_2) &= \mu((B \setminus A_2) \cap A_1) + \mu((B \setminus A_2) \setminus A_1) \\ &= \mu((B \setminus A_2) \setminus A_1) + \mu(B \setminus (A_2 \cup A_1)) \end{aligned}$$

Using the decompositions

$$\begin{aligned} B \setminus A_2 &= [(B \setminus A_2) \cap A_1] \sqcup [B \setminus (A_1 \cup A_2)] \\ B \cap (A_1 \cup A_2) &= (B \cap A) \cup [(B \setminus A_2) \cap A_1] \\ \implies \mu(B \cap (A_1 \cup A_1)) &\leq \mu(B \cap A) + \mu((B \setminus A_2) \cap A_1) \end{aligned}$$

and the property (b) of any measure, we get

$$\begin{aligned} \mu(B) &= \mu(B \cap A_2) + \mu(B \setminus A_2) \\ &= \mu(B \cap A_2) + \mu((B \setminus A_2) \cap A_1) + \mu((B \setminus A_2) \setminus A_1) \\ &\geq \mu(B \cap (A_2 \cup A_1)) + \mu(B \setminus (A_2 \cup A_1)) \end{aligned}$$

- $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma \implies A = \bigcup_{n=1}^{\infty} A_n \in \Sigma$:

  We can assume without loss of generality that the sets are mutually disjoint. Otherwise, consider the sequence $\left(\tilde{A}_n\right)_{n\in\mathbb{N}} \subseteq \Sigma$ given by

$$\tilde{A}_1 := A_1, \tilde{A}_n := A_n \setminus \bigcup_{k=1}^{n-1} A_k \quad \text{which satisfy} \quad \bigcup_{k=1}^{\infty} \tilde{A}_k = \bigcup_{k=1}^{\infty} A_k$$

  We can use μ-measureability of A_m with the test set $B \cap \bigcup_{k=1}^{m} A_k$ to find that

$$\begin{aligned} \mu\left(B \cap \bigcup_{k=1}^{m} A_k\right) &= \mu\left(\left(B \cap \bigcup_{k=1}^{m} A_k\right) \cap A_m\right) + \mu\left(\left(B \cap \bigcup_{k=1}^{m} A_k\right) \setminus A_m\right) \\ &= \mu(B \cap A_m) + \mu\left(B \cap \bigcup_{k=1}^{m-1} A_k\right) \\ &= \sum_{k=1}^{m} \mu(B \cap A_k) \end{aligned}$$

and since μ is monotonous, we get with $\bigcup_{k=1}^{m} A_k \subseteq A$ that

$$\begin{aligned}\mu(B) &= \mu\left(B \cap \bigcup_{k=1}^{m} A_k\right) + \mu\left(B \setminus \bigcup_{k=1}^{m} A_k\right) \\ &\geq \sum_{k=1}^{m} \mu(B \cap A_k) + \mu(B \setminus A)\end{aligned}$$

for all $m \in \mathbb{N}$. Taking the limit as $m \to \infty$, we ge that

$$\begin{aligned}\mu(B) &\geq \sum_{k=1}^{\infty} \mu(B \cap A_k) + \mu(B \setminus A) \\ &\geq \mu(B \cap A) + \mu(B \setminus A)\end{aligned}$$

which shows μ-measurability of A.

□

Definition 1.12. A **measure space** is a tuple (X, Σ, μ) consisting of measure μ on a set and the σ-algebra of μ-measurable sets Σ.

Example 1.13.

- For every $x \in X, A \subseteq X$, define the **Dirac measure at** x

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

- For every $A \in \mathcal{P}$, the **counting measure** is a measure, where every subset is μ-measurable:

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \texttt{otherwise} \end{cases}$$

- The **indiscrete measure** given by

$$\mu(A) = \begin{cases} 1 & A \neq \emptyset \\ 0 & A = \emptyset \end{cases}$$

only has $\emptyset, X$ as μ-measurable sets.

Theorem 1.14. *Let (X, Σ, μ) be a measure space and $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$. Then the following are true*

(a) μ is σ-additive.

(b) If $A_n \subseteq A_{n+1} \forall n \in \mathbb{N}$, then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$

(c) If $A_{n+1} \subseteq A_n \forall n \in N$ and $\mu(A_1) < \infty$, then

1.3 Construction of Measures

Let X be non-empty set.

Definition 1.15. A collection of subsets $\mathcal{K} \subseteq \mathcal{P}(X)$ is called a **covering** of X if

$$\emptyset \in \mathcal{K} \quad \text{and} \quad \exists (K_j)_{j\in\mathbb{N}} \subseteq \mathcal{K}: \quad X = \bigcup_{j=1}^{\infty} K_j$$

It is easy to see that every Algebra $\mathcal{A}$ of X is a covering since $\emptyset, X \in \mathcal{A}$.

Theorem 1.16. *Let $\mathcal{K}$ be a covering of X and $\lambda : \mathcal{K} \to [0,\infty]$ with $\lambda(\emptyset) = 0$. Then we can define a measure μ on X given by*

$$\mu(A) = \inf \left\{ \sum_{j=1}^{\infty} \lambda(K_j) \middle| K_j \in \mathcal{K}, A \subseteq \bigcup_{j=1}^{\infty} K_j \right\}$$

Proof. We show σ-subadditivity of μ. □

Definition 1.17. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. A mapping $\lambda : \mathcal{A} \to [0,\infty]$ is called a **pre-measure** if

(a) $\lambda(\emptyset) = 0$

(b) For every $A \in \mathcal{A}$ such that $A = \bigsqcup_{k=1}^{\infty} A_k$, $A_k \in \mathcal{A}$

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$$

λ is called **σ-finite** if there exists a covery $X = \bigcup_{k=1}^{\infty} S_k$, $S_k \in \mathcal{A}$ such that $\lambda(S_k) < \infty, \forall k$.

Given a pre-measure λ on $\mathcal{A}$, we can obtain a measure μ on $\mathcal{P}(X)$ that coincides with λ on $\mathcal{A}$, i.e. μ extends λ.

Theorem 1.18 (Carathéodory-Hahn extension)**.** *Let $\lambda : \mathcal{A} \to [0,\infty]$ be a pre-measure on X. Then for*

$$\mu : \mathcal{P}(X) \to [0,\infty], \quad \mu(A) := \inf \left\{ \sum_{k=1}^{\infty} \lambda(A_k) \middle| A \subseteq \bigcup_{k=1}^{\infty} A_k, A_k \in \mathcal{A} \right\}$$

it holds that

(a) $\mu : \mathcal{P} \to [0,\infty]$ is a measure.

(b) $\mu(A) = \lambda(A) \forall A \in \mathcal{A}$

(c) All $A \in \mathcal{A}$ are μ-measurable.

Proof. (a) This just follows from the previous theorem

(b)

(c)

□

Not only does such an extension exist always, we can show that under certain assumptions it is unique:

Theorem 1.19 (Uniqueness of Carathéodory-Hahn extension)**.** *Let $\lambda : \mathcal{A} \to [0, \infty]$ be a σ-finite pre-measure on X and μ the Carathéodory-Hahn extension of λ and let Σ be the σ-algera of μ-measurable sets.*
If $\tilde{\mu} : \mathcal{P}(X) \to [0, \infty]$ if another measure with $\tilde{\mu}|_{\mathcal{A}} = \lambda$, then $\tilde{\mu}|_{\Sigma} = \mu$

Proof. We show

(a) $\forall A \in \mathcal{P}(X)$: $\tilde{\mu}(A) \leq \mu(A)$.

(b) $\forall A \in \Sigma$: $\tilde{\mu}(A) \geq \mu(A)$.

For the first claim, let $A \subseteq \bigcup_{k=1}^{\infty} A_k$ with $A_k \in \mathcal{A}$. By σ-subadditivity of μ it follows that

$$\tilde{\mu} \leq \sum_{k=1}^{\infty} \tilde{\mu}(A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

So by taking the infimum over all such coverings $(A_k)_{k \in \mathbb{N}}$ as in the definition of μ, the inequality still holds: $\tilde{\mu}(A) \leq \mu(A)$.
For the second claim we consider two cases: One where there exists an $S \in \mathcal{A}$ such that

$$A \subseteq S \quad \text{and} \quad \lambda(S) < \infty$$

Then, using the first claim on $S \setminus A$ and monotonicity of μ, it follows that

$$\tilde{\mu}(S \setminus A) \leq \mu(S \setminus A) \leq \mu(S) = \lambda(S)$$

Since $S \in \mathcal{A}$ is μ-measurable and $S \cap A = A$ we get

$$\begin{aligned} \tilde{\mu}(A) + \tilde{\mu}(S \setminus A) &\leq \mu(A \cap S) + \mu(S \setminus A) = \mu(S) \\ &= \lambda(S) = \tilde{\mu}(S) \\ &\leq \tilde{\mu}(A) + \tilde{\mu}(S \setminus A) \end{aligned}$$

where we used sub-additivity of $\tilde{\mu}$ in the last step. ☐

The condition for the uniqueness cannot be weakened as the following counterexample shows:

Example 1.20.

1.4 Lebesgue Measure

The Lebesgue measure is the Carethéodory-Hahn extension of the pre-measure of “volumes”, that assigns products of intervals to their products of lengths.
We want to give a definition of what these “volumes” and what their measure is going to be.

Definition 1.21. For $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathbb{R}^d$ we define the d-dimensional **interval**

$$(a, b) = \begin{cases} \prod_{i=1}^{d} (a_i, b_i) & \text{if } a_i < b_i \quad \forall i \\ \emptyset & \text{otherwise} \end{cases}$$

in an analogous way, we define the closed and half-open boxes $[a, b], [a, b)$ or $(a, b]$. Like on the real line, we also allow the open ends to be $\pm\infty$.

To each d-dimensional interval I (whether open, closed or half-open) to be

$$\mathrm{vol}(I) := \begin{cases} \prod_{i=1}^{d}(b_i - a_i) \in [0, +\infty] & \text{if } a_i < b_i, \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

An **elementary set** is the finite disjiont union of intervals and we define its volume to be

$$\mathrm{vol}\left(\bigsqcup_{k=1}^{d} I_k\right) := \sum_{k=1}^{d} \mathrm{vol}(I_k)$$

Remark 1.22. We can check easily that the volume function is well defined. For example, the decomposition $[0,2] = [0,1) \sqcup [1,2] = [0,1) \sqcup [1,1.5) \sqcup [1.5,2]$ should all give the same volume.
More generally, if $I = \bigsqcup_{k=1}^{n} I_k = \bigsqcup_{j=1}^{m} J_j$ where I_k, J_j are Intervals, then

$$\sum_{k=1}^{n} \mathrm{vol}(I_k) = \sum_{j=1}^{m} \mathrm{vol}(J_j)$$

We of course have to show that our attempt to use the Carathéodory-Hahn Extension of vol on the elementary sets is well defined. But it should be easy to see how the class of elmentary sets forms an algebra and that the vol function is a pre-measure on it.

In our example above, we used half-open intervals of length 2^{-k} to decompose an interval. A direct generalisation is to introduce the dyadic cubes to obtain a basic decomposition of sets in $\mathbb{R}^d$. For every $k \in \mathbb{N}$, let $\mathcal{D}_k$ the collection of half open cubes

$$\mathcal{D}_k := \left\{ \prod_{i=1}^{d} [\frac{a_i}{2^k}, \frac{a_{i+1}}{2^k}) | a_i \in \mathbb{Z} \right\}$$

In particular, $\mathcal{D}_0$ is the collection of hpyercubes of edge length 1 and vertices in $\mathbb{Z}^d$.
In general we have that a Lebesgue measureable set is also Borel measureable. The opposite however is not true.

Definition 1.23. A measure μ on $\mathbb{R}^n$ is called **Borel**, if every Borel set is μ-measurable.

The Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ is Borel and the σ-algebra of $\mathcal{L}^n$-measurable sets contains the Borel σ-algebra.

Theorem 1.24. *For any subset $A \subseteq \mathbb{R}^n$ the following are equivalent*

(a) A is $\mathcal{L}^n$-measurable..

(b) $\forall \epsilon > 0 \exists G \supseteq A$ with $\mathcal{L}^n(G \setminus A) < \epsilon$.

(c) A it can be "approximated" from the inside and outside: $\forall \epsilon > 0 \exists F, G$ with $F \subseteq A \subseteq G$ such that

$$\mathcal{L}^n(G \setminus A) + \mathcal{L}^n(A \setminus F) \leq \epsilon$$

(d) $\forall \epsilon > 0 \exists F$ closed, $\exists G$ open, such that $\mathcal{L}^n(G \setminus F) < \epsilon$.

1.5 Comparision between Lebesgue and Jordan Measure

Definition 1.25. A bounded subset $A \subseteq \mathbb{R}^n$ is **Jordan-measurable** if

$$\underline{\mu}(A) := \sup\{\text{vol}(E) | E \subseteq A, E \text{ elementary set}\} = \inf\{\text{vol}(E) | A \subseteq E, E \text{ elementary set}\} =: \overline{\mu}(A)$$

As the following proof will show, The Lebesgue measure can measure more sets than the Jordan measure can.

Theorem 1.26. *Let $A \subseteq \mathbb{R}^n$ be bounded, then*

(a) $\underline{\mu}(A) \leq \mathcal{L}^n(A9 \leq \overline{\mu}(A)$

(b) If A is Jordan-measurable, then A is $\mathcal{L}^n$-measurable and $\mathcal{L}^n(A) = \mu(A)$.

One would naturally think that the volume of a cube should stay the same whether we move it by a fixed point or rotate it.

Theorem 1.27. *The Lebegue measure is invariant under isometries $\Phi : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto x_0 + Rx$, for $R \in O(n)$.*

Definition 1.28. A Borel measure μ on $\mathbb{R}^n$ is called **Borel regular**, if for every $A \subseteq \mathbb{R}^n$ there exists a borel set $B \supseteq A$ such that $\mu(A) = \mu(B)$.

The Lebesgue measure is Borel regular.

1.6 Special-Examples of sets

Although (or rather, because) the Lebesgue measure has all these nice properties, not all sets are measurable. To construct such a non-measurable set, we will use the Axiom of Choice, which states that for any family of non-empty disjoint sets $(A_i)_{i\in I}$, there exists a choice-function $f : I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$.
With this, we can construct the set $\{f(i) | i \in I\}$ that contains exactly one element from each set A_i.
For $x, y \in [0, 1)$ we define $\oplus := \mod 1 \circ +$

$$x \oplus y = \begin{cases} x + y & \text{if } x + 1 < 1 \\ x + y - 1 & \text{if } x + y \geq 1 \end{cases}$$

Example 1.29. For every $A \subseteq \mathbb{R}$ with $\mathcal{L}(A) > 0$, there exists a $B \subseteq A$ such that B is not $\mathcal{L}$-measurable.

The Cantor tridadic set

The real numbers can be defined as the set of Cauchy-sequences in $\mathbb{Q}$ up to limits.
The cantor set is the set of numbers whose triadic digits don't contain a 1.

$$C = \{x \in [0, 1] | d_i(x) \in \{0, 2\} \forall i\}$$

Then C is uncountable and $\mathcal{L}(C) = 0$.

1.7 The Lebesgue-Stieltjes Measure

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and left continuous:

$$F(x_0) = \lim_{x \to x_0^-} F(x) \quad \forall x_0 \in \mathbb{R}$$

For a, b we set

$$\lambda_F[a, b) = \begin{cases} F(b) - F(a) & \text{if } a < b \\ 0 & \text{otherwise} \end{cases}$$

1.8 Hausdorff Measures

The Lebesgue measure of a subset of $\mathbb{R}^n$ of dimension less than n is always zero. It also has the weakness of failing to properperly measure fractal sets. The Hausdorff measure tries to solve this.
We start by introducing an intermediate measure, where instead of covering a subset A with quaders, we do this using open balls of radius smaller than some given $\delta > 0$.

Definition 1.30. For $s \geq 0, \delta > 0$ and $A \subseteq \mathbb{R}^n$ non-empty, we set

$$\mathcal{H}^s_\delta(A) := \inf\left\{\sum_{k\in I} r_k^s, A \subseteq \bigcup_{k\in I} B(x_k, r_k), 0 < r_k\delta\right\}$$

where the set I is at most countable and we set $\mathcal{H}^0_\delta(\emptyset) = 0$.

Remark 1.31. $\mathcal{H}^s_\delta$ defines a measure on $\mathbb{R}^n$ and for fixed s, A it is inversely monotonous with respect to δ, that is

$$\delta_2 \leq \delta_1 \implies \mathcal{H}^s_{\delta_1}(A) \leq \mathcal{H}^s_{\delta_2}(A)$$

since every δ_2 covering is also a δ_1 covering.

Since $\mathcal{H}^s_\delta$ is non-increasing in δ, the limit

$$\mathcal{H}^s(A) := \lim_{\delta\downarrow 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A)$$

exists. We now use this as for our next definition.

Definition 1.32. We call $\mathcal{H}^s$ the **s-dimensional Hausdorff measure** on $\mathbb{R}^n$

For $s = 0$, we call $\mathcal{H}^0$ the **counting measure**, which measure the cardinality of A.
Now that we call it a measure, we should of course prove some things

Theorem 1.33. *For $s \geq 0$, $\mathcal{H}^s$ is a Borel regular measure on $\mathbb{R}^n$*

Proof. Let $s \geq 0$. We first prove that it is a meausre. Clearly, $\mathcal{H}^s(\emptyset) = 0$. Let $(A_k)_{k\in\mathbb{N}}$ and $A \subseteq \bigcup_{k=1}^\infty A_k$. Since $\mathcal{H}^s_\delta$ is σ-subadditive for all $\delta > 0$, we get

$$\mathcal{H}^s_\delta(A) \leq \sum_k \mathcal{H}^s_\delta(A_k) \leq \sum_k \mathcal{H}^s(A_k)$$

by taking the limit (in the definition of $\mathcal{H}^s$) we get the σ-subadditivity of $\mathcal{H}^s$.
To show that it is borel, we just show that it is metric. Let $A, B \subseteq \mathbb{R}^n$ such that $\delta_0 := \operatorname{dist}(A, B) > 0$. We then take a covering $A \cup B$ of balls of size smaller than $\delta := \frac{\delta_0}{4}$ and claim that we can partition the covering into two non-overlapping coverings of A and B each.
Since $\mathcal{H}^s_\delta$ takes the infimum over all such coverings, suppose that $A \cup B = \bigcup_k B(x_k, r_k)$ with $r_k < \delta$. Then we set

$$\mathcal{A} = \{B(x_k, r_k) | B(x_k, r_k) \cap A \neq \emptyset\} \mathcal{B} = \{B(x_k, r_k) | B(x_k, r_k) \cap B \neq \emptyset\}$$

And it becomes obvious that these are non-overlapping coverings of A and B each (by using the triangle inequality).

Therefore, we get

$$\mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \leq \sum_k r_k^s$$

and taking the infimum of coverings of $A \cup B$, this means

$$\mathcal{H}^s_\delta(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$$

which, when taking the limit $\delta \to 0$ just states $\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B)$. By (σ)-subadditivity of $\mathcal{H}^s$, the reverse inequality holds and so $\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$ shows that $\mathcal{H}^s$ is metric and thus also Borel.
For Borel regularity, let $A \subseteq \mathbb{R}^n$ and suppose $\mathcal{H}^s(A) < \infty$ (Otherwise, just take $B = \mathbb{R}^n$). By monotonicity of $\mathcal{H}^s_\delta$, this also means that $\mathcal{H}^s_\delta(A) < \infty$ for all $\delta > 0$.
For $\delta = \frac{1}{l}$

□

2 Radon Measure

3 Measurable Functions

3.1 Basic properties

For X, Y nonempty sets and $f : X \to Y$ with $A \subseteq Y$, the inverse image is defined as

$$f^{-1}(A) = \{x \in X | f(x) \in A\}$$

which satisfies

(a) $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$

(b) If $A, B \subseteq Y$, then

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

(c) If $(A_k)_k$ is a sequence of subsets, then

$$f^{-1}\left(\bigcup_{k=1}^\infty A_k\right) = \bigcup_{k=1}^\infty f^{-1}(A_k)$$

From this, it follows that if $(Y, \mathcal{A}, \mu)$ is a measure space, then

$$\Sigma := f^{-1}(\mathcal{A}) := \{f^{-1}(A) | A \in \mathcal{A}\}$$

is a σ-algebra in X.