

Electrodynamics – Summary

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1 Magnetostatics

In Magnetostatics, we consider systems where the current is steady. This means in particular that

$$\vec{J} = \text{const}, \implies \rho = \text{const}, \quad \vec{E} = \text{const}, \quad \vec{B} = \text{const}$$

Under the **Coulomb-Eichung**

<p>Electrostatics</p> $\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{y} \frac{\rho(\vec{y})}{ \vec{x}-\vec{y} }$ $\vec{E} = -\nabla\Phi$ $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ $\vec{\nabla} \times \vec{E} = 0$ $\oint_{\partial V} d\vec{S} \cdot \vec{E} = \frac{Q_{\text{inside}}}{\epsilon_0}$	<p>Magnetostatics</p> $A(\vec{x}) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\vec{y} \frac{\vec{J}(\vec{y})}{ \vec{x}-\vec{y} }$ $\vec{B} = \vec{\nabla} \times \vec{A}$ $\vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\epsilon_0 c^2}$ $\vec{\nabla} \cdot \vec{B} = 0$ $\oint_{\partial S} \vec{B} \cdot d\vec{\ell} = \frac{I_{\text{inside}}}{\epsilon_0 c^2}$
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Table 1: Analogies between Electrostatics and Magnetostatics

$$\Phi \mapsto \tilde{\Phi} = \frac{\partial}{\partial t} f(\vec{x}, t)$$

$$\vec{A} \mapsto \vec{\tilde{A}} = \vec{A} - \nabla f(\vec{x}, t)$$

for some $f : \mathbb{R}^4 \rightarrow \mathbb{R}$, the potentials are not uniquely determined.

By introducing the **d'Alembert Operator**

$$\square := \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

we get the equations

$$\square \vec{A} = \frac{\vec{J}}{\epsilon_0 c^2}, \quad \square \Phi = \frac{\rho}{\epsilon_0}$$

To solve them, we can instead search for the Green's function (or more generally, the Fundamental solution) $G(\vec{x}, t, \vec{y}, t')$ that satisfies

$$\square_{x,t} G(\vec{x}, t, \vec{y}, t') = \delta(\vec{x} - \vec{y}) \delta(t - t')$$

2 Time dependent electromagnetic fields

If the \vec{E} and \vec{B} are time dependent, then the Maxwell equations are as follows

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad c^2 \vec{\nabla} \times \vec{B} = \frac{\vec{J}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t}$$

From the scalar and vector potential Φ, \vec{A} we can find the \vec{E}, \vec{B} fields with.

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

Since the electric and magnetic field remain invariant under **gauge transformations**

After some Fourier transformation and Complex Analysis shenanigans, we obtain the solution

$$G(\vec{x}, t, \vec{y}, t') = \frac{1}{4\pi|\vec{x} - \vec{y}|} \delta\left(t - t' - \frac{|\vec{x} - \vec{y}|}{c}\right) \Theta(t - t')$$

or equivalently, we can write

$$G(\Delta\vec{x}, \Delta t) = \frac{1}{2\pi} \delta\left((t - t')^2 - \frac{|\vec{x} - \vec{y}|^2}{c^2}\right) \Theta(t - t')$$

By defining the **time retardation**

$$t_{\text{ret}} := t' - \frac{|\vec{x} - \vec{y}|}{c}$$

we obtain the **retarded scalar potential**

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{y} \frac{\rho(\vec{y}, t_{\text{ret}})}{|\vec{x} - \vec{y}|}$$

aswell as the **retarded vector potential**

$$\vec{A}(\vec{x}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int d^3\vec{y} \frac{\vec{J}(\vec{y}, t_{\text{ret}})}{|\vec{x} - \vec{y}|}$$

3 Special Relativity

Whereas classical mechanics played in three dimensional space \mathbb{R}^3 with the time dimension $t \in \mathbb{R}$ separated, special relativity plays in the **Minkowsky Space**.

Elements of the Minkowsky space are four-vectors $x^\mu = (ct, \vec{x})$.

The **metric tensor** is the matrix

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

which lets us define the (quasi)**inner product** on the minkowski space as

$$\langle x, y \rangle = g_{\mu\nu} x^\mu y^\nu \implies \langle x, x \rangle = c^2 t^2 - \vec{x}^2$$

3.1 Lorentz Transformations

A **Lorentz transformation** is any affine linear transformation of the form

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + \rho^\mu$$

such that it satisfies the relation

$$\Lambda^\mu{}_\nu \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}$$

such transformation has the inner product as an invariant since

$$\begin{aligned} \langle \tilde{x}, \tilde{x} \rangle &= g_{\rho\nu} (\Lambda^\mu{}_\sigma x^\sigma) (\Lambda^\nu{}_\rho x^\rho) \\ &= g_{\mu\nu} \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho x^\sigma x^\rho = g_{\sigma\rho} x^\sigma x^\rho = \langle x, x \rangle \end{aligned}$$

The set of all Lorentz transformations forms a group, called the **Poincare group**.

We are especially interested in the subgroup known as the **proper Lorentz transformations**, which are all such transformations that satisfy

$$\det \Lambda = 1, \quad \Lambda^0{}_0 \geq 1$$

Another invariant is the **proper time**

$$d\tau^2 = dx^2 = c^2 dt^2 - d\vec{x}^2$$

Given a velocity \vec{v} , and dt we get

$$d\tau = c dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{c}{\gamma} dt$$

When a stationary observer O sees a reference frame \tilde{O} passing by with velocity $\vec{v} = \vec{\beta}c$ along the x -axis, then the corresponding boost.

$$\Lambda_x{}^\mu{}_\nu(\vec{\beta}) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If the velocity is at an angle θ with the x -axis in the xy -plane, then we break the boost into three steps

$$x \mapsto \tilde{x} = R(\theta)x \mapsto \tilde{\tilde{x}} = \Lambda(\beta)\tilde{x} \mapsto x'^\mu = R^\mu{}_\rho(-\theta)\Lambda_x{}^\rho{}_\sigma R^\sigma{}_\nu(\theta)x^\nu$$

The inverse of a Lorentz transformation $\Lambda^\mu{}_\nu$ is given by

$$\Lambda_\mu{}^\nu = g_{\mu\nu} g^{\nu\sigma} \Lambda^\rho{}_\sigma$$

Example 3.1 (Relativistic effects). Consider a moving particle that has a life span of t_0 . By defining the events for the Start A and End B , we get

$$A = (0, \vec{0}) \quad \text{and} \quad B = (ct_0, \vec{0})$$

we see that after the Lorentz transformation we measure its lifespan to be at

$$\tilde{A} = \Lambda A = (0, \vec{0}), \quad \text{and} \quad \tilde{B} = \Lambda B = (\gamma ct_0, \beta\gamma ct_0)$$

so the outside observer sees a **time dilation** in its lifespan $\tilde{t}_0 = \gamma t_0$.

On the contrary, the outside observer notices a **length contraction**

$$s = vt_0 = \frac{1}{\gamma} \tilde{s}$$

3.2 Tensors

In a change of reference under a Lorentz Transformation

$$x^\mu \mapsto \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + \rho^\mu$$

a **contra-variant tensor** is any object U that transforms as follows

$$T^\mu \mapsto \tilde{T}^\mu = \Lambda^\mu{}_\nu T^\nu$$

examples are the momentum p^μ , the force f^μ or the differential form dx^μ . Contra-variant tensors are denoted with upstairs indices.

A **covariant tensor** is any object that transforms like

$$T_\mu \mapsto \tilde{T}_\mu = \Lambda_\mu^\nu T_\nu$$

examples are the covariant derivative $\frac{\partial}{\partial x^\mu}$. Such tensors are denoted with downstairs indices.

There are also **mixed tensors**, which have both up- and downstairs indices. They transform according to the rules

$$\tilde{T}^{\mu_1 \mu_2 \dots \mu_m}_{\nu_1 \nu_2 \dots \nu_n} = \Lambda^{\mu_1}_{\sigma_1} \dots \Lambda^{\mu_m}_{\sigma_m} \Lambda_{\nu_1}^{\rho_1} \dots \Lambda_{\nu_n}^{\rho_n} T^{\mu_1 \mu_2 \dots \mu_m}_{\rho_1 \rho_2 \dots \rho_n}$$

We can "raise/lower indices" with the metric tensor

$$T^\mu \mapsto T_\mu = g_{\mu\nu} T^\nu \quad \text{and} \quad T_\mu \mapsto T^\mu = g^{\mu\nu} T_\nu$$

what actually means is that the metric tensor transforms contravariant tensors to covariant ones and vice versa.

A **Lorentz-Scalar** is any object that stays invariant under Lorentz transformations.

Useful Tensor relations are

$$\Lambda^\mu_\sigma \Lambda_\mu^\rho = \delta^\rho_\sigma$$

3.3 Energy and Momentum

We define the four-momentum as

$$p^\mu := mc \frac{dx^\mu}{d\tau}$$

whose components can be given by

$$p^0 = m\gamma c = mc + \frac{1}{2c}mv^2 + \mathcal{O}\left(\frac{v^4}{c^3}\right), \quad p^i = m\gamma v^i$$

this gives us the definition for **relativistic energy** as

$$E := cp^0 = m\gamma c^2 = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}\left(\frac{v^4}{c^2}\right)$$

From the relation

$$\vec{p}^2 = m^2 \gamma^2 \vec{v}^2$$

we obtain the very useful **energy-momentum relation**

$$E^2 = c^2 \vec{p}^2 + m^2 c^4$$

4 Relativistic Electrodynamics

From now on, we use the convention $c = \epsilon_0 = 1$.

We combine current and charge density into a contravariant four-vector

$$j^\mu := (\rho, \vec{j})$$

The antisymmetric **electrodynamical field tensor** $F^{\mu\nu}$ is defined using the \vec{E} and \vec{B} fields

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\vec{E}^T \\ \vec{E} & B^\times \end{pmatrix}$$

where B^\times is the dual tensor defined as

$$(B^\times)_{ij} = -\epsilon_{ijk} B^k \implies B^\times = \begin{pmatrix} 0 & -B^3 & B^2 \\ B^3 & 0 & -B^1 \\ -B^2 & B^1 & 0 \end{pmatrix}$$

from which the electric and magnetic field can be obtained using the relations

$$E^i = F^{i0}, \quad B^i = -\frac{1}{2}\epsilon_{ijk} F^{jk}, \quad \text{or} \quad F^{ij} = -\epsilon_{ijk} B^k$$

and the Maxwell equations become

$$\partial_\mu F^{\mu\nu} = j^\nu$$

In covariant Form, we get

$$F_{\mu\nu} = \begin{pmatrix} 0 & \vec{E}^T \\ -\vec{E} & B^\times \end{pmatrix}$$

Example 4.1 (Transformation of EM tensor). In a change of reference along the x -axis with boost β , the EM tensor transforms doubly contravariant, i.e

$$F^{\mu\nu} \mapsto \tilde{F}^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\rho F^{\sigma\rho}$$

The summation over ρ corresponds to the matrix multi-

plication

$$\Lambda^\nu{}_\rho F^{\sigma\rho} = \begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & B^\times \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -E^1\beta\gamma & -E^1\gamma & -E^2 & -E^3 \\ E^1\gamma & E^1\beta\gamma & -B^3 & B^2 \\ \gamma(E^2 + B^3\beta) & \gamma(E^2\beta + B^3) & 0 & -B^1 \\ \gamma(E^3 - B^2\beta) & \gamma(E^3\beta - B^2) & B^1 & 0 \end{pmatrix}$$

and in the end, we get

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -\gamma(E^2 + \beta B^3) & -\gamma(E^3 - \beta B^2) \\ E^1 & 0 & -\gamma(B^3 + \frac{v}{c^2}E^2) & \gamma(B^2 - \frac{v}{c^2}E^3) \\ & & 0 & -B^1 \\ & & B^1 & 0 \end{pmatrix}$$

This result generalizes to arbitrary boosts, and we get

$$\begin{aligned} \vec{E}_\parallel &\mapsto \vec{E}_\parallel = \vec{E}_\parallel, & \vec{E}_\perp &\mapsto \vec{E}_\perp = \gamma(\vec{E}_\perp - \vec{v} \times \vec{B}) \\ \vec{B}_\parallel &\mapsto \vec{B}_\parallel = \vec{B}_\parallel, & \vec{B}_\perp &\mapsto \vec{B}_\perp = \gamma(\vec{B}_\perp + \frac{1}{c^2}\vec{v} \times \vec{E}) \end{aligned}$$

Example 4.2 (Transformation of Lorentz Force). Two particles with charge q are distance d apart and are moving perpendicular to \vec{d} with velocity \vec{v} . In the system S where the particles are stationary, the Lorentz Force is simply:

$$S : \vec{F} = q \cdot \vec{E} = \frac{q^2}{4\pi\epsilon_0} \frac{1}{d^2} \hat{d}$$

And in the external system S' the electromagnetic field are

$$\vec{E} = (E^1, \gamma E^2, \gamma E^3), \quad \vec{B} = (B^1 = 0, -\gamma \frac{v}{c^2} E^3, \gamma \frac{v}{c^2} E^2)$$

and so we see

$$\tilde{\vec{F}} = \frac{1}{\gamma} \vec{F}$$

4.1 Maxwell Equations

The (inhomogenous) Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = \rho, \quad \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{j}$$

become the simple equation

$$\partial_\mu F^{\mu\nu} = j^\nu$$

We can also combine the scalar and vector potential into the four vector

$$A^\mu := (\Phi, \vec{A}) = (\Phi, A^1, A^2, A^3)$$

and the equations

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} \times \vec{A}$$

get simplified with the contravariant derivative

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

For the homogenous Maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

we define the **dual field tensor** \tilde{F} as

$$\tilde{F}_{\mu\nu} := \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = 2 \begin{pmatrix} 0 & -\vec{B}^T \\ \vec{B} & E^\times \end{pmatrix}$$

and get the simple form

$$\partial^\mu \tilde{F}_{\mu\nu} = 0$$

In the Lorentz-Gauge: $(\partial_\mu A^\mu = 0)$ the maxwell equations become

$$\partial^2 A^\mu = j^\mu$$

Example 4.3. The 4-vector formulation of the Maxwell equations give a quick derivation of the continuity equation.

Because the electromagnetic field tensor is antisymmetric, we see that

$$\partial_\mu \partial_\nu F^{\mu\nu} = -\partial_\mu \partial_\nu F^{\nu\mu} = 0$$

and so we get

$$\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0$$

5 Maths

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

For $f(x)$ with roots $(x_i)_{i \in I}$

$$\delta(f(x)) = \sum_{i \in I} \frac{1}{|f'(x)|} \delta(x - x_i)$$

5.1 Tensor Stuff

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$