Measure Theory– Lecture Notes

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About

Lecture notes taken from the Measure and Integration lecture given by Dr. Francesca Da Lio during Spring Semester 2021 at ETH Zürich.
Wit a focus on the theorems and their proofs, these notes have fewer examples than given in the lecture, but the proofs will be more explicit.
The exam will be a 20 minute **oral exam**. It will consist of two or three questions where we have to prove some results.
The lecture slides and a script is provided at the professors home page `https://people.math.ethz.ch/~fdalio/MASSundINTEGRALFS21`.

1 Measure spaces

If we naively try to define a notion of measure that has some intuitive properties, we can run into some problems that give paradoxical results. The **Riemann Integral** we saw in Analysis I/II also had some drawbacks of not being general enough. We can use measure theory to define a better definition of the integral.

1.1 Algebras and σ-Algebras

From now on, let X denote a non-empty set.

Definition 1.1.1. For a sequence of subsets $(A_n)_{n=1}^\infty$ in $\mathcal{P}(X)$. We define

$$\limsup_{n\to\infty} A_n := \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m$$
$$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty A_m$$

And if they are equal, we say that the sequence $(A_n)_{n=1}^\infty$ converges to its limit $\lim_{n\to\infty} A_n$.

Informally, the $\limsup$ consists of elements of X that occur in infinitely many A_n, whereas the $\liminf$ consists of elements that occur for all but finitely many A_n.

Remark 1.1.2.

(a) $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$

(b) If $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

(c) If $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

The similarity in names with the lim sup and lim inf from Analysis can be seen using the characteristic function

$$\mathbb{1}_A : X \to \{0,1\}$$
$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

It holds that

$$\limsup_{n\to\infty} A_n = A \iff \limsup_{n\to\infty} \mathbb{1}_{A_n} = \mathbb{1}_A$$
$$\liminf_{n\to\infty} A_n = A \iff \liminf_{n\to\infty} \mathbb{1}_{A_n} = \mathbb{1}_A$$

where the lim sup and lim inf on the left are as in Definition 1.1.1 and the ones on the right are the ones from Analysis.

Definition 1.1.3 (Algebras of sets)**.** A collection of subsets $\mathcal{A} \subseteq \mathcal{P}(X)$ is called an **algebra in** X if

(a) $X \in \mathcal{A}$

(b) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

(c) $A \in \mathcal{A} \implies A^c \in \mathcal{A}$

An algebra $\mathcal{E}$ is called a **σ-algebra**, if for any sequence $(A_n)_{n=1}^{\infty}$ in $\mathcal{E}$ we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{E}$

Note that using the De Morgan's identity

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$$

we can see that algebras (σ-algebras) are stable under finite (infinite) intersections aswell.

Definition 1.1.4. For a collection of sets $\mathcal{K} \subseteq \mathcal{P}(X)$, the intersection of all σ-algebras containing K forms again a σ-algebra.
We call this the σ-algebra **generated by** K and it its the smallest σ-algebra that contains K.
The algebra generated by the open sets of a topology is called the **Borel σ-Algebra** of X, denoted $\mathcal{B}(X)$.

1.2 Measures

Definition 1.2.1. Let $\mathcal{A}$ be an Algebra on X and $\mu : \mathcal{A} \to [0, \infty]$. We say that μ is

- **additive**, if for any *finite* family of disjoint sets $A_1, \dots, A_n \in \mathcal{A}$

$$\mu\left(\bigsqcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k)$$

- **σ-additive**, if for any *countable* family of disjoint sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{A}$

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

- A **pre-measure**, if it is σ-additive and satisfies $\mu(\emptyset) = 0$.

Remark 1.2.2. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathcal{A}$ such that their union is again in $\mathcal{A}$.

(a) If μ is additive, then it is monotone with respect to incusion, i.e. $A \subseteq B \implies \mu(A) \leq \mu(B)$.

(b) If μ is additive and the sets A_k are mutually disjoint, then

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} \mu(A_k)$$

(c) If μ is σ-additive, then it is also **σ-subadditive**, which means that for any sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ with $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

To see this, we can define the mutually disjoint sets

$$B_1 = A_1, \quad B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k \in \mathcal{A}$$

Since $\bigsqcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$ and $\mu(B_k) \leq \mu(A_k)$ we have

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigsqcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

It follows immediately from (b) and (c) that

$$\boxed{\mu \text{ is additive and } \sigma\text{-subadditive} \iff \mu \text{ is } \sigma\text{-additive}}$$

Example 1.2.3. Not all additive functions are σ-additive. For $X = \mathbb{N}$ and

$$\mathcal{A} = \{A \in \mathcal{P}(X) \big| A \text{ is finite or } A^c \text{ is finite}\}$$

the function $\nu : \mathcal{A} \to [0, \infty]$ with $\nu(\emptyset) = 0$ and

$$\nu(A) = \begin{cases} \sum\limits_{n \in A} \frac{1}{2^n} & \text{if } A \text{ is finite} \\ \infty & \text{if } A^c \text{ is finite} \end{cases}$$

is additive but not σ-additive because we can take the sequence

$$A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}, \dots, A_n = \{n\}, \dots$$

which is a sequence of mutually disjoint sets satsfying

$$\nu(A_1) = \frac{1}{2}, \nu(A_2) = \frac{1}{4}, \dots, \nu(A_n) = \frac{1}{2^n}$$
$$\implies \nu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \nu(\mathbb{N}) = \infty \not\leq \sum_{n=1}^{\infty} \nu(A_n) = 1$$

Definition 1.2.4. A σ-additive function $\mu : \mathcal{A} \to [0, \infty]$ is called

- **finite**, if $\mu(X) < \infty$
- **σ-finite**, if there exists a sequence $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ such that

$$\bigcup_{n=1}^{\infty} A_n = X \quad \text{and} \quad \mu(A_n) < \infty \quad \forall n \in \mathbb{N}$$

Clearly, μ finite $\implies$ μ σ-finite.
While pre-measures are only defined on algebras $\mathcal{A} \subseteq \mathcal{P}(X)$, we would like to extend the domain of such functions to $\mathcal{P}(X)$ without losing too many of its nice properties. In particular, we want to keep monotonicity and σ-subadditivity:

Definition 1.2.5. A function $\mu : \mathcal{P}(X) \to [0, \infty]$ is called a **measure**[1] on X, if

(a) $\mu(\emptyset) = 0$

(b) μ is σ-subadditive: If $A \subseteq \bigcup_{k=1}^{\infty} A_k$, then $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$

Note that subadditivity implies monotonicity with respect to inclusion, i.e. $A \subseteq B \implies \mu(A) \leq \mu(B)$.

Definition 1.2.6. Let μ be a measure on X and $A \subseteq X$. We can *restrict* μ to A (written $\mu \llcorner A$) defined by

$$(\mu \llcorner A)(B) := \mu(A \cap B) \quad \forall B \subseteq X$$

Definition 1.2.7 (Carathéodory criterion)**.** A subset $A \subseteq X$ is called **μ-measurable** if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A), \quad \forall B \subseteq X$$

Remark 1.2.8. (a) By subadditivity of the measure, the definition is equivalent to

$$\boxed{\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A), \quad \forall B \subseteq X}$$

[1] sometimes also called outer measure

(b) If $\mu(A) = 0$, then A is μ-measurable.

Theorem 1.2.9. Let $\mu : \mathcal{P}(X) \to [0, \infty]$ be a measure. Then the collection of measurable sets

$$\Sigma = \{A \subseteq X \big| A \text{ is } \mu\text{-measurable}\}$$

forms a σ-algebra.

Proof.

- $X \in \Sigma$: Let $B \subseteq X$. It's trivial to see that

$$\mu(B \cap X) + \mu(B \setminus X) = \mu(B) + \mu(\emptyset) = \mu(B)$$

- $A \in \Sigma \implies A^c \in \Sigma$: With the equalities

$$B \cap A^c = B \setminus A, \quad \text{and} \quad B \setminus A^c = B \cap A$$

we get

$$\mu(B \cap A^c) + \mu(B \setminus A^c) = \mu(B \setminus A) + \mu(B \cap A) \overset{A \in \Sigma}{=} \mu(B)$$

- $A_1, A_2 \in \Sigma \implies A_1 \cup A_2 \in \Sigma$:
Let $B \subseteq X$. From the previous remark, it is sufficient to just show the inequality

$$\mu(B) \geq \mu(B \cap (A_1 \cup A_2)) + \mu(B \setminus (A_1 \cup A_2))$$

Using μ-measurability for A_1 on the test set $B \setminus A_2$, we see

$$\begin{aligned} \mu(B \setminus A_2) &= \mu\big((B \setminus A_2) \cap A_1\big) + \mu\big((B \setminus A_2) \setminus A_1\big) \\ &= \mu\big((B \setminus A_2) \cap A_1\big) + \mu\big(B \setminus (A_2 \cup A_1)\big) \end{aligned}$$

so with the decomposition

$$(B \cap A_2) \cup \big((B \setminus A_2) \cap A_1\big) = B \cap (A_1 \cup A_2)$$

and subadditivity of the measure, we get

$$\begin{aligned} \mu(B) &= \mu(B \cap A_2) + \mu(B \setminus A_2) \\ &= \mu(B \cap A_2) + \mu\big((B \setminus A_2) \cap A_1\big) + \mu\big(B \setminus (A_2 \cup A_1)\big) \\ &\geq \mu\big(B \cap (A_2 \cup A_1)\big) + \mu\big(B \setminus (A_2 \cup A_1)\big) \end{aligned}$$

- $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma \implies A = \bigcup_{n=1}^{\infty} A_n \in \Sigma$:
We can assume without loss of generality that the sets are mutually disjoint. Otherwise, consider the sequence $\left(\tilde{A}_n\right)_{n \in \mathbb{N}} \subseteq \Sigma$ given by

$$\tilde{A}_1 := A_1, \tilde{A}_n := A_n \setminus \bigcup_{k=1}^{n-1} A_k \quad \text{which satisfy} \quad \bigsqcup_{k=1}^{\infty} \tilde{A}_k = \bigcup_{k=1}^{\infty} A_k.$$

We can use μ-measureability of A_m with the test set $B \cap \bigsqcup_{k=1}^m A_k$ to find that by induction on m

$$\begin{aligned}
\mu\left(B \cap \bigsqcup_{k=1}^m A_k\right) &= \mu\left(\left(B \cap \bigsqcup_{k=1}^m A_k\right) \cap A_m\right) + \mu\left(\left(B \cap \bigsqcup_{k=1}^m A_k\right) \setminus A_m\right) \\
&= \mu(B \cap A_m) + \mu\left(B \cap \bigsqcup_{k=1}^{m-1} A_k\right) \\
&= \sum_{k=1}^m \mu(B \cap A_k)
\end{aligned}$$

and using monotonicity of μ on the inclusion $\bigsqcup_{k=1}^m A_k \subseteq A$ it follows that

$$\begin{aligned}
\mu(B) &= \mu\left(B \cap \bigsqcup_{k=1}^m A_k\right) + \mu\left(B \setminus \bigsqcup_{k=1}^m A_k\right) \\
&\geq \sum_{k=1}^m \mu(B \cap A_k) + \mu(B \setminus A)
\end{aligned}$$

for all $m \in \mathbb{N}$. Taking the limit $m \to \infty$, we get

$$\begin{aligned}
\mu(B) &\geq \sum_{k=1}^\infty \mu(B \cap A_k) + \mu(B \setminus A) \\
&\geq \mu(B \cap A) + \mu(B \setminus A)
\end{aligned}$$

which shows μ-measurability of A.

□

Definition 1.2.10. A **measure space** is a tuple (X, Σ, μ) consisting of measure μ on a set X and the σ-algebra of μ-measurable sets Σ.

Example 1.2.11. The following are measure spaces:

- For every $x \in X, A \subseteq X$, define the **Dirac measure at** x

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

  Every A is δ_x-measurable.

- For every $A \in \mathcal{P}$, the **counting measure** is a measure, where every subset is μ-measurable:

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

  Every A is μ-measurable.

- The **indiscrete measure** given by

$$\mu(A) = \begin{cases} 1 & A \neq \emptyset \\ 0 & A = \emptyset \end{cases}$$

  only has $\emptyset, X$ as μ-measurable sets.

The Carathéodory criterion of μ-measurable sets and the σ-subadditivity of the measure give us some nice properties back.

Theorem 1.2.12. Let (X, Σ, μ) be a measure space and $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$. Then the following are true

(a) μ is σ-additive.

(b) Continuity from below:

$$A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq A_{n+1} \subseteq \ldots \implies \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

(c) Continuity from above:

$$\mu(A_1) < \infty, \quad A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq A_{n+1} \supseteq \ldots \implies \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

Proof. (a) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of mutually disjoint sets. In the proof of the previous theorem, we already saw

$$\mu\left(B \cap \bigsqcup_{k=1}^{m} A_k\right) = \sum_{k=1}^{m} \mu(B \cap A_k)$$

so in particular, for $B = X$, we see

$$\mu\left(\bigsqcup_{k=1}^{m} A_k\right) = \sum_{k=1}^{m} \mu(A_k)$$

By monotonicity of μ, we have

$$\mu\left(\bigsqcup_{k=1}^{\infty} A_k\right) \geq \lim_{m\to\infty} \mu\left(\bigsqcup_{k=1}^{m} A_k\right) = \lim_{m\to\infty} \sum_{k=1}^{m} \mu(A_k) = \sum_{k=1}^{\infty} \mu(A_k)$$

The other inequality (and thus equality) follow from σ-subadditivity of the measure.

(b) Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence. Define the pairwise disjoint family

$$\tilde{A}_1 := A_1, \quad \tilde{A}_k := A_k \setminus A_{k-1} \implies \mu(\tilde{A}_k) = \mu(A_k) - \mu(A_{k-1}), \quad \bigsqcup_{k=1}^{\infty} \tilde{A}_k = \bigcup_{k=1}^{\infty} A_k,$$

from σ-additivity, summation into a telescoping sum

$$\begin{aligned}\mu\left(\bigcup_{k=1}^{\infty} A_k\right) &= \mu\left(\bigsqcup_{k=1}^{\infty} \tilde{A}_k\right) = \sum_{k=1}^{\infty} \mu(\tilde{A}_k) \\ &= \mu(\tilde{A}_1) + \lim_{m\to\infty} \sum_{k=2} \mu(A_k) - \mu(A_{k-1}) \\ &= \lim_{m\to\infty} \mu(A_m)\end{aligned}$$

(c) Let $(A_n)_{n\in\mathbb{N}}$ be a decreasing sequence. Consider instead the increasing sequence $\tilde{A}_1 \subseteq \tilde{A}_2 \subseteq \dots$ given by

$$\tilde{A}_1 := \emptyset, \quad \tilde{A}_k := A_1 \setminus A_k \implies \mu(A_1) = \mu(A_k) + \mu(\tilde{A}_k), \quad \bigcup_{k=1}^{\infty} \tilde{A}_k = A_1 \setminus \bigcap_{k=1}^{\infty} A_k$$

by (b), we find

$$\begin{aligned} \mu(A_1) - \lim_{k\to\infty} \mu(A_k) &= \lim_{k\to\infty} \mu(\tilde{A}_k) \\ &\overset{(b)}{=} \mu\left(\bigcup_{k=1}^{\infty} \tilde{A}_k\right) = \mu\left(A_1 \setminus \bigcap_{k=1}^{\infty} A_k\right) \\ &= \mu(A_1) - \mu\left(\bigcap_{k=1}^{\infty} A_k\right) \end{aligned}$$

□

The condition $\mu(A_1)$ in (c) is necessary. Consider the example $X = \mathbb{N}$ with the counting-measure and the sequence $A_n := \{m \in \mathbb{N} | m \geq n\}$. The intersections converge to the emtpy set, but the $\mu(A_k)$ is always ∞.

1.3 Construction of Measures

Let X be non-empty set.

Definition 1.3.1. A collection of subsets $\mathcal{K} \subseteq \mathcal{P}(X)$ is called a **covering** of X if

$$\emptyset \in \mathcal{K} \quad \text{and} \quad \exists (K_j)_{j\in\mathbb{N}} \subseteq \mathcal{K} : \quad X = \bigcup_{j=1}^{\infty} K_j$$

Example 1.3.2. The collection of higher-dimensional open intervals

$$\{\prod_{k=1}^{n} (a_k, b_k) | a_k \leq b_k \in \mathbb{R}\}$$

are a covering of $\mathbb{R}^n$.
It is easy to see that every Algebra $\mathcal{A}$ of X is a covering since $\emptyset, X \in \mathcal{A}$.

Theorem 1.3.3. Let $\mathcal{K}$ be a covering of X and $\lambda : \mathcal{K} \to [0, \infty]$ and any function with $\lambda(\emptyset) = 0$. Then this induces a measure μ on X given by

$$\mu(A) = \inf \left\{ \sum_{j=1}^{\infty} \lambda(K_j) | K_j \in \mathcal{K}, A \subseteq \bigcup_{j=1}^{\infty} K_j \right\}$$

Proof. Let $A \subseteq \bigcup_{k=1}^{\infty} A_k$. We show σ-subadditivity of μ, i.e $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$.
If the right-hand side is infinite, then the inequality is trivial, so assume it is finite.
By definition of μ, for all $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists a sequence $(K_{j,k})_{j\in\mathbb{N}}$ in $\mathcal{K}$ such that

$$A_k \subseteq \bigcup_{j=1}^{\infty} K_{j,k} \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda(K_{j,k}) \leq \mu(A_k) + \frac{\varepsilon}{2^k}$$

Taking the union over all sequences for each k, we get

$$A \subseteq \bigcup_{j,k=1}^{\infty} K_{j,k} \quad \text{and} \quad \mu(A) \leq \sum_{j,k} \lambda(K_{j,k}) \leq \varepsilon + \sum_{k=1}^{\infty} \mu(A_k)$$

Since $\varepsilon > 0$ was arbitrary, subadditivity follows. □

Example 1.3.4. Set $\mathcal{K} = \{\emptyset, X\}$ and define $\lambda(\emptyset) = 0$, $\lambda(X) = 1$.
The induced measure is defined by $\mu(A) = 0$ if $A = \emptyset$ and $\mu(A) = 1$ if $A \neq \emptyset$.

The function λ in the previous theorem only had minimal restrictions ($\mathcal{K}$ had to be a covering and $\lambda : \mathcal{K} \to [0, \infty]$ with $\lambda(\emptyset) = \emptyset$).
It turns out that if λ and $\mathcal{K}$ are *nice enough*, then the induced measure is a σ-additive extension of λ. Nice-enough here means that $\mathcal{K}$ is an algebra and λ is a pre-measure.
Recall that given an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$, a function $\lambda : \mathcal{A} \to [0, \infty]$ is called a **pre-measure** if it is σ-additive and satisifes $\lambda(\emptyset) = 0$.
Given a pre-measure λ on $\mathcal{A}$, we can obtain a measure μ on $\mathcal{P}(X)$ that coincides with λ on $\mathcal{A}$, i.e. μ extends λ.

Theorem 1.3.5 (Carathéodory-Hahn extension)**.** Let $\lambda : \mathcal{A} \to [0, \infty]$ be a pre-measure on X. Then for

$$\mu : \mathcal{P}(X) \to [0, \infty], \quad \mu(A) := \inf \left\{ \sum_{k=1}^{\infty} \lambda(A_k) | A \subseteq \bigcup_{k=1}^{\infty} A_k, A_k \in \mathcal{A} \right\}$$

it holds that

(a) $\mu : \mathcal{P} \to [0, \infty]$ is a measure.

(b) $\mu(A) = \lambda(A), \forall A \in \mathcal{A}$

(c) All $A \in \mathcal{A}$ are μ-measurable, i.e. satisfy $\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A), \forall B \subseteq X$.

Proof. (a) Because algebras are also coverings, we can just use the previous theorem.

(b) Let $A \in \mathcal{A}$. Since A itself contains A, the term $\lambda(A)$ is present in the right hand side, so $\mu(A) \leq \lambda(A)$.
Now assume there is some other collection $\bigcup_{k=1}^{\infty} A_k$ that contains A with $A_k \in \mathcal{A}$. By inductively defining the mutually disjoint sequence

$$B_1 = A_1, \quad B_k := A_k \setminus \bigcup_{i=1}^{k-1} B_i$$

we see $\sum_{k=1}^{\infty} \lambda(B_k) \leq \sum_{k=1}^{\infty} \lambda(A_k)$, so since we're taking the infimum, we can assume that WLOG the A_k are mutually disjoint.
Setting $\tilde{A}_k := A_k \cap A \in \mathcal{A}$, we see that they are also mutually disjoint and their union contains A.
By σ-additivity of the pre-measure λ, we get

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(\tilde{A}_k) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

since the collection $(A_k)_{k \in \mathbb{N}}$ was arbitrary, the inequality $\lambda(A) \leq \mu(A)$ follows.

(c) Let $A \in \mathcal{A}$ and $B \subseteq X$ be any test set. By definition of μ, for every $\varepsilon > 0$ we can chose a collection $(B_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ that contains B and

$$\sum_{k=1}^{\infty} \lambda(B_k) \leq \mu(B) + \varepsilon$$

By (σ)-additivity of λ and $A, B_k \in \mathcal{A}$ we have

$$\lambda(B_k) = \lambda(B_k \cap A) + \lambda(B_k \setminus A) \quad \forall k$$

so since the $(B_k \cap A)_{k\in\mathbb{N}}$ and $(B_k \setminus A)_{k\in\mathbb{N}}$ contain $B \cap A$ and $B \setminus A$ each, we get

$$\begin{aligned} \mu(B \cap A) + \mu(B \setminus A) &\leq \sum_{k=1}^{\infty} \lambda(B_k \cap A) + \sum_{k=1}^{\infty} \lambda(B_k \setminus A) \\ &= \sum_{k=1}^{\infty} \lambda(B_k) \leq \mu(B) + \varepsilon \end{aligned}$$

and in the limit $\varepsilon \to 0$ the inequality follows.

□

Not only does such an extension exist, we can show that under certain assumptions it is unique:

Definition 1.3.6. A pre-measure λ is called **σ-finite** if there exists a covering $X = \bigcup_{k=1}^{\infty} S_k$, $S_k \in \mathcal{A}$ such that $\lambda(S_k) < \infty, \forall k$.

Theorem 1.3.7 (Uniqueness of Carathéodory-Hahn extension)**.** Let $\lambda : \mathcal{A} \to [0, \infty]$ be a σ-finite pre-measure on X and μ the Carathéodory-Hahn extension of λ and let Σ be the σ-algera of μ-measurable sets.
If $\tilde{\mu} : \mathcal{P}(X) \to [0, \infty]$ is another measure with $\tilde{\mu}|_{\mathcal{A}} = \lambda$, then $\tilde{\mu}|_{\Sigma} = \mu$

Proof. Let $\tilde{\mu} : \mathcal{P}(X) \to [0, \infty]$ be a measure extending λ. We show

(i) $\forall A \in \mathcal{P}(X)$: $\tilde{\mu}(A) \leq \mu(A)$.

(ii) $\forall A \in \Sigma$: $\tilde{\mu}(A) \geq \mu(A)$.

For the first claim, let $A \subseteq \bigcup_{k=1}^{\infty} A_k$ with $A_k \in \mathcal{A}$. By σ-subadditivity of $\tilde{\mu}$ it follows that

$$\tilde{\mu}(A) \leq \sum_{k=1}^{\infty} \tilde{\mu}(A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

So by taking the infimum over all such coverings $(A_k)_{k\in\mathbb{N}}$ as in the definition of μ, the inequality still holds: $\tilde{\mu}(A) \leq \mu(A)$. Note that we didn't have to use σ-finiteness of λ for this inquality.
For the second claim let $A \in \Sigma$ be μ-measurable. We then consider the simple case where there exists an $S \in \mathcal{A}$ such that

$$A \subseteq S \quad \text{and} \quad \lambda(S) < \infty$$

Then, using the first claim on $S \setminus A$ and monotonicity of μ, it follows that

$$\tilde{\mu}(S \setminus A) \leq \mu(S \setminus A) \leq \mu(S) = \lambda(S)$$

Since $S \in \mathcal{A}$ is μ-measurable and $A = S \cap A$ we get with $\mu|_{\mathcal{A}} = \lambda = \tilde{\mu}|_{\mathcal{A}}$ that

$$\begin{aligned}\tilde{\mu}(A) + \tilde{\mu}(S \setminus A) &\leq \mu(S \cap A) + \mu(S \setminus A) = \mu(S) \\ &= \lambda(S) = \tilde{\mu}(S) \\ &\leq \tilde{\mu}(A) + \tilde{\mu}(S \setminus A)\end{aligned}$$

where we used sub-additivity of $\tilde{\mu}$ in the last step. It follow sthat $\tilde{\mu}(A) = \mu(A) \leq \tilde{\mu}(A)$.
In the more general case, we can use σ-finiteness to get a covering

$$X = \bigcup_{k=1}^{\infty} S_k, S_k \in \mathcal{A}, \lambda(S_k) < \infty$$

As remarked in the proof of the last theorem, we can assume without loss of generality that the S_k are mutually disjoint.
Defining $A_k = A \cap S_k$ we get $A = \bigcup_{k=1}^{\infty} A_k$. Because $\mathcal{A}$ is closed under finite unions and $\tilde{\mu}|_{\mathcal{A}} = \mu|_{\mathcal{A}}$, we have that for all $m \in \mathbb{N}$:

$$\bigcup_{k=1}^{m} A_k \in \mathcal{A} \implies \tilde{\mu}\left(\bigcup_{k=1}^{m} A_k\right) = \mu\left(\bigcup_{k=1}^{m} A_k\right)$$

and by using monotonicity on the inclusion $A \supseteq \bigcup_{k=1}^{m} A_k$ and taking the limit $m \to \infty$, we get

$$\tilde{\mu}(A) \geq \lim_{m\to\infty} \tilde{\mu}\left(\bigcup_{k=1}^{m} A_k\right) = \lim_{m\to\infty} \mu\left(\bigcup_{k=1}^{m} A_k\right) = \mu(A)$$

□

If we denote $\tilde{\Sigma}$ to be the σ-agebra of $\tilde{\mu}$-measurable sets, the theorem doesn't tell us if $\tilde{\Sigma} = \Sigma$. Moreover, it doesn't tell us anything about the behaviour of $\tilde{\mu}$ outside of Σ.

Example 1.3.8. Let $X = [0, 1], \mathcal{A} = \{\emptyset, X\}$ and set $\lambda(\emptyset) = 0, \lambda(X) = 1$.
The Carathéodory extension of λ has $\mu(A)$ to be 0 or 1, depending on if A is empty or not. The μ-measurable sets are $\Sigma = \{\emptyset, X\}$.
However, as we will see in the next section, the Lebesuge measure Ł^1 is also an extension of λ with Ł$^1|_{\Sigma} = \mu|_{\Sigma}$, but they differ when measuring the interval $[0, \frac{1}{2}]$.

1.4 Lebesgue Measure

The Lebesgue measure is the Carethéodory-Hahn extension of the pre-measure that corresponds to the "physical" notion of what a volume of simple objects such as n-dimensional hypercubes like $[0, 1]^n$ is.
We want to give a precise definition of what these "simple objects" are and define the pre-measure.

Definition 1.4.1. For $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathbb{R}^d$ we define the d-dimensional **interval**

$$(a, b) := \begin{cases} \prod_{i=1}^{d} (a_i, b_i) & \text{if } a_i < b_i \quad \forall i \\ \emptyset & \text{otherwise} \end{cases} \subseteq \mathbb{R}^d$$

in an analogous way, we define the closed and half-open boxes $[a, b], [a, b)$ or $(a, b]$. Like on the real line, we also allow the open ends to be $\pm\infty$.

To each d-dimensional interval I (whether open, closed or half-open), we define it's **volume** to be

$$\mathrm{vol}(I) := \begin{cases} \prod_{i=1}^{d}(b_i - a_i) \in [0, +\infty] & \text{if } a_i < b_i, \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

An **elementary set** is the finite disjiont union of intervals and we define its volume to be

$$\mathrm{vol}\left(\bigsqcup_{k=1}^{d} I_k\right) := \sum_{k=1}^{d} \mathrm{vol}(I_k) \in [0, \infty]$$

Remark 1.4.2. We can check easily that the volume function is well defined. For example, the decomposition $[0,2] = [0,1) \sqcup [1,2] = [0,1) \sqcup [1,1.5) \sqcup [1.5,2]$ should all give the same volume.
More generally, if $I = \bigsqcup_{k=1}^{n} I_k = \bigsqcup_{j=1}^{m} J_j$ where I_k, J_j are Intervals, then

$$\sum_{k=1}^{n} \mathrm{vol}(I_k) = \sum_{j=1}^{m} \mathrm{vol}(J_j)$$

Proof. Let $(I_k)_{k\in\mathbb{N}}$ and $(J_j)_{j\in\mathbb{N}}$ be as above. Then

$$I_k = I \cap I_k = \bigcup_{j=1}^{m} J_j \cap I_k$$

taking the volume on both sides and summing over all k, we get

$$\sum_{k=1}^{n} \mathrm{vol}(I_k) = \sum_{k=1}^{n}\sum_{j=1}^{m} \mathrm{vol}(J_j \cap I_k)$$

flipping the roles of I_k and J_j, e also get

$$\sum_{j=1}^{m} \mathrm{vol}(J_j) = \sum_{j=1}^{m}\sum_{k=1}^{n} \mathrm{vol}(J_j \cap I_k)$$

which equals what we got before. $\square$

We of course have to show that our attempt to use the Carathéodory-Hahn Extension of vol on the elementary sets is well defined. But it should be easy to see how the class of elmentary sets forms an algebra and that the vol function is a pre-measure on it. In our example above, we used half-open intervals of length $1, 2^{-1}$ to decompose the interval $[0,2] \subseteq \mathbb{R}$.
A direct generalisation for this in higher dimensions is to introduce finer and finer hypercubes that cover $\mathbb{R}^d$. For $k \in \mathbb{N}$ let $\mathcal{D}_k$ the collection of half open cubes

$$\mathcal{D}_k := \left\{ \prod_{i=1}^{d} [\frac{a_i}{2^k}, \frac{a_i+1}{2^k}) | a_i \in \mathbb{Z} \right\}$$

In particular, $\mathcal{D}_0$ is the collection of hpyercubes of edge length 1 and vertices in $\mathbb{Z}^d$.
We call the cubes of the collection

$$\{Q | Q \in \mathcal{D}_k, k = 0, 1, 2, \dots\}$$

the **dyadic cubes**.

Remark 1.4.3. The dyadic cubes have the following properties:

(a) For all $k \in \mathbb{N}$, $\mathbb{R}^n = \bigsqcup_{Q \in \mathcal{D}_k} Q$.

(b) If $Q \in \mathcal{D}_k$ and $P \in \mathcal{D}_l$, with $l \leq k$, then either $Q \subseteq P$ or $P \cap Q = \emptyset$.

(c) Every $Q \in \mathcal{D}_k$ has volume $\mathrm{vol}(Q) = 2^{-kn}$.

Definition 1.4.4. The **Lebesgue measure** $\mathcal{L}^n$ is the Carathéodory Hahn extension of the volume defined on the algebra of elementary sets[2], i.e.

$$\mathcal{L}^n(A) := \inf \left\{ \sum_{k=1}^{\infty} \mathrm{vol}(E_k) | A \subseteq \bigcup_{k=1}^{\infty} E_k, E_k \text{ is an elementary set} \right\}$$

If we want to measure open subsets $U \subseteq \mathbb{R}^n$ with the Lebesgue-measure, we want to ensure that a countable covering of U with disjoint elementary sets E_k is possible, or else taking the infimum makes it so that U is not $\mathcal{L}^n$-measurable.

Lemma 1.4.5. Every open set in $\mathbb{R}^n$ can be written as a countable union of disjoint dyadic cubes.

Proof. Let $U \subseteq \mathbb{R}^n$ be a non-empty open subset.
Let $\mathcal{S}_0$ to be the collection of all cubes in $\mathcal{D}_0$ that lie entirely in U. Let $\mathcal{S}_1$ to be the collection of all cubes in $\mathcal{D}_1$ that lie entirely in U, but are not subcubes of $\mathcal{S}_0$, etc. Let $\mathcal{S}_k$ be the collection of cubes in $\mathcal{D}_k$ which are not subcubes of any cubes in $\mathcal{S}_0, \dots, \mathcal{S}_{k-1}$. Set $\mathcal{S} := \bigcup_{k \in \mathbb{N}} \mathcal{S}_k$.
Because each $\mathcal{D}_k$ is countable, $\mathcal{S}$ is countable. By construction, the cubes in $\mathcal{S}$ are also disjoint.
Since U is open and the cubes become arbitrarily small, every $x \in U$ will be covered by some $Q \in \mathcal{S}$, so $U = \bigsqcup_{Q \in \mathcal{S}} Q$.

□

Recall that the Borel σ-algebra $\mathcal{B}(X)$ is the σ-algebra generated by open subsets of X.

Definition 1.4.6. A measure μ on $\mathbb{R}^n$ is called **Borel** (or a Borel measure), if every Borel set is μ-measurable.

Remark 1.4.7. From Lemma 1.4.5, it follows that $\mathcal{L}^n$ is a Borel measure.
The lemma says that the open sets are $\mathcal{L}^n$-measurable. Moreover, by Theorem 1.2.9 the the collection of $\mathcal{L}^n$-measurable sets form a σ-algebra. So the Borel σ-algebra is contained in the σ-algebra of $\mathcal{L}^n$-measurable subsets.

When we want to characterize $\mathcal{L}^n(A)$ for some subset $A \subseteq \mathbb{R}^n$, the definition used in the Carathéodory-Hahn extension where we consider all countable coverings using elementary sets is quite unwiedly. The following theorem gives a nicer characterisation.

Theorem 1.4.8. For every $A \subseteq \mathbb{R}^n$ it holds

$$\boxed{\mathcal{L}^n(A) = \inf_{A \subseteq U} \mathcal{L}^n(U), \quad U \text{ open}}$$

[2]Because elementary sets are finite disjoint unions of intervals, we can replace E_k with intervals I_k

Proof. By monotonicity, $\mathcal{L}^n(A) \leq \mathcal{L}^n(U)$ follows directly.
For the other inequality, suppose that $\mathcal{L}^n(A) < \infty$ (or else the inequality is trivial). By definition, for any $\varepsilon > 0$ we can find intervals $(I_k)_{k\in\mathbb{N}}$ with

$$A \subseteq \bigcup_{k=1}^{\infty} I_k, \quad \sum_{k=1}^{\infty} \operatorname{vol}(I_k) \leq \mathcal{L}^n(A) + \varepsilon$$

Since $\mathcal{L}^n(A) < \infty$, every interval I_k must have finite volume and is thus bounded. So let $\tilde{I}_k \supseteq I_k$ be open bounded intervals with $\operatorname{vol}(\tilde{I}_k) \leq \operatorname{vol}(I_k) + \frac{\varepsilon}{2^k}$.
Setting $U := \bigcup_{k=1}^{\infty} \tilde{I}_k$, we see that U is an open subset containing A and it's volume is

$$\mathcal{L}^n(U) \leq \sum_{k=1}^{\infty} \operatorname{vol}(\tilde{I}_k) \leq \sum_{k=1}^{\infty} \operatorname{vol}(I_k) + \frac{\varepsilon}{e^k} \leq \mathcal{L}^n(A) + 2\varepsilon$$

since ε was arbitrary, the result follows. $\square$

This alternative characterisation lets us find out what subsets $A \subseteq \mathbb{R}^n$ are $\mathcal{L}^n$-measurable.

Theorem 1.4.9. For any subset $A \subseteq \mathbb{R}^n$ the following are equivalent

(a) A is $\mathcal{L}^n$-measurable..

(b) $\forall \varepsilon > 0\ \exists U \supseteq A$ open with $\mathcal{L}^n(U \setminus A) < \varepsilon$.

(c) A it can be "approximated" from the inside and outside: $\forall \varepsilon > 0\ \exists F$ closed, U open with $F \subseteq A \subseteq U$ such that

$$\mathcal{L}^n(U \setminus A) + \mathcal{L}^n(A \setminus F) < \varepsilon$$

(d) $\forall \varepsilon > 0 \exists F$ closed, $\exists U$ open, such that $\mathcal{L}^n(U \setminus F) < \varepsilon$.

Proof.

(a) $\implies$ (b): Let $\varepsilon > 0$, A be $\mathcal{L}^n$-measurable.

- If $\mathcal{L}^n(A) < \infty$, by the previous theorem, we can chose a $U \supseteq A$ open such that

$$\mathcal{L}^n(U) \leq \mathcal{L}^n(A) + \varepsilon$$

Because A is $\mathcal{L}^n$-measurable we can use U as a test set and get

$$\begin{aligned}\mathcal{L}^n(U) &= \mathcal{L}^n(U \cap A) + \mathcal{L}^n(U \setminus A) \\ &= \mathcal{L}^n(A) + \mathcal{L}^n(U \setminus A)\end{aligned}$$

which gives us

$$\mathcal{L}^n(U \setminus A) = \mathcal{L}^n(U) - \mathcal{L}^n(A) < \varepsilon$$

- If $\mathcal{L}^n(A) = \infty$, we set

$$A_k = A \cap [-k, k]^n \implies A = \bigcup_{k=1}^{\infty} A_k$$

since $\mathcal{L}^n(A_k) < \infty$, we are in the first case so we can find $U_k \supseteq A_k$ open with

$$\mathcal{L}^n(U_k \setminus A_k) < \frac{\varepsilon}{2^k} \quad \forall k \in \mathbb{N}$$

Then their union $U := \bigcup_{k=1}^{\infty} U_k$ is open and contains A. Moreover, we have

$$\begin{aligned}\mathcal{L}^n(U \setminus A) &= \mathcal{L}^n\left(\bigcup_{k=1}^{\infty}(U_k \setminus A)\right) \\ &\leq \mathcal{L}^n\left(\bigcup_{k=1}^{\infty}(U_k \setminus A_k)\right) \\ &\leq \sum_{k=1}^{\infty} \mathcal{L}^n(U_k \setminus A_k) < \varepsilon\end{aligned}$$

(b) $\implies$ (a): Let $B \subseteq \mathbb{R}^n$. For $\varepsilon > 0$, chose $U \supseteq A$ open with $\mathcal{L}^n(U \setminus A) < \varepsilon$. Then

$$B \setminus A \subseteq (B \setminus U) \cup (U \setminus A)$$

Since open subsets are $\mathcal{L}^n$-measurable, we have

$$\begin{aligned}\mathcal{L}^n(B) &= \mathcal{L}^n(B \cap U) + \mathcal{L}^n(B \setminus U) \\ &\geq \mathcal{L}^n(B \cap A) + \mathcal{L}^n(B \setminus A) - \mathcal{L}^n(U \setminus A) \\ &\geq \mathcal{L}^n(B \cap A) + \mathcal{L}^n(B \setminus A) - \varepsilon\end{aligned}$$

since ε was arbitrary, $\mathcal{L}^n$-measurability of A follows.

(b) $\iff$ (c): For $\varepsilon > 0$ use (b) for A^c to get an open set $V \supseteq A^c$ with $\mathcal{L}^n(V \setminus A^c) < \varepsilon$. Then $F = V^c \subseteq A$ is closed and

$$\mathcal{L}^n(A \setminus V^c) = \mathcal{L}^n(V \setminus A^c) < \varepsilon$$

The other implication is trivial.

(c) $\implies$ (d): Using (c), we get $F \subseteq A$ closed and $U \supseteq A$ open. Because $F \subseteq A \subseteq U$,

$$U \setminus F = (U \setminus A) \cup (A \setminus F)$$

it follows from subadditivity that

$$\mathcal{L}^n(U \setminus F) \leq \mathcal{L}^n(U \setminus A) + \mathcal{L}^n(A \setminus F) < \varepsilon$$

(d) $\implies$ (c): For $\varepsilon > 0$, use (d) to get $F \subseteq A$ closed, $U \supseteq A$ open with $\mathcal{L}^n(U \setminus F) < \varepsilon$. Because $F \subseteq A \subseteq U$

$$U \setminus A \subseteq U \setminus F, \quad A \setminus F \subseteq U \setminus F$$

so we get

$$\mathcal{L}^n(U \setminus A) + \mathcal{L}^n(A \setminus F) \leq 2\mathcal{L}^n(U \setminus F) < 2\varepsilon$$

□

1.5 Comparision between Lebesgue and Jordan Measure

Definition 1.5.1. A bounded subset $A \subseteq \mathbb{R}^n$ is **Jordan-measurable** if $\underline{\mu}(A) = \overline{\mu}(A)$, where

$$\underline{\mu}(A) := \underline{\int}_{\mathbb{R}^n} \chi_A d\mu := \sup\{\mathrm{vol}(E) \big| E \subseteq A, E \text{ elementary set}\}$$
$$\overline{\mu}(A) := \overline{\int}_{\mathbb{R}^n} \chi_A d\mu := \inf\{\mathrm{vol}(E) \big| A \subseteq E, E \text{ elementary set}\}$$

If that is the case, denote the Jordan measure of A with the common value $\mu(A)$.
We call $\underline{\mu}(A)$ the **Jordan inner measure** of A and $\overline{\mu}(A)$ the **Jordan outer measure** of A.

Example 1.5.2. For $f : I \to \mathbb{R}$ continuous, $I \subseteq \mathbb{R}^n$ compact, its graph

$$\Gamma = \{(x, f(x)) \big| x \in I\} \subseteq \mathbb{R}^{n+1}$$

is a Jordan measurable set.
The area under a function

$$G = \{(x,t) \in I \times \mathbb{R} \big| 0 \leq t \leq f(x)\}$$

is also Jordan-measurable

As the following theorem will show, the Lebesgue measure can measure more sets than the Jordan measure can.

Theorem 1.5.3. Let $A \subseteq \mathbb{R}^n$ be bounded, then

(a) $\underline{\mu}(A) \leq \mathcal{L}^n(A9 \leq \overline{\mu}(A)$

(b) If A is Jordan-measurable, then A is $\mathcal{L}^n$-measurable and $\mathcal{L}^n(A) = \mu(A)$.

Proof. (a) Because elementary sets are finite disjoint unions of intervals, we have

$$\begin{aligned}\mathcal{L}^n(A) &= \inf\left\{\sum_{k=1}^{\infty} \mathrm{vol}(I_k) \big| A \subseteq \bigcup_{k=1}^{\infty} I_k, I_k \text{ intervals}\right\}\\ &\leq \inf\left\{\sum_{k=1}^{m} \mathrm{vol}(I_k) \big| A \subseteq E = \bigsqcup_{k=1}^{m} I_k, I_k \text{ intervals}\right\}\\ &= \overline{\mu}(A)\end{aligned}$$

For the other inequality, for every elementary set $E = \bigsqcup_{k=1}^m I_k \subseteq A$ we have

$$\mathrm{vol}(E) = \mathcal{L}^n(E) \leq \mathcal{L}^n(A)$$

so when taking the sup over such E, we get

$$\underline{\mu}(A) \leq \mathcal{L}^n(A)$$

(b) If A is Jordan measurable, then it follows from (i) that

$$\underline{\mu}(A) \leq \mathcal{L}^n(A) \leq \overline{\mu}(A) = \underline{\mu}(A)$$

To show that A is $\mathcal{L}^n$-measurable, we use characterisation (b) from Theorem 1.4.9

Because A is bounded, $\mathcal{L}^n(A) < \infty$ and because it is Jordan-measurable, we can find for all $\varepsilon > 0$ elementary sets $E_\varepsilon, E^\varepsilon$ such that

$$E_\varepsilon \subseteq A \subseteq E^\varepsilon \quad \text{and} \quad \operatorname{vol}(E^\varepsilon) - \varepsilon < \mu(A) < \operatorname{vol}(E_\varepsilon) + \varepsilon$$

Because the volume doesn't depend on whether the intervals comprising the elementary set are open, half-open or closed, we can assume WLOG that E^ε is open, so

$$\begin{aligned} \mathcal{L}^n(E^\varepsilon \setminus A) &\leq \mathcal{L}^n(E^\varepsilon \setminus E_\varepsilon) = \operatorname{vol}(E^\varepsilon \setminus E_\varepsilon) \\ &= \operatorname{vol}(E^\varepsilon) - \operatorname{vol}(E_\varepsilon) < 2\varepsilon \end{aligned}$$

which shows the condition from the previous theorem.

□

One would naturally think that the "physical" volume of an object should stay invariant under translation or rotation.

Theorem 1.5.4. The Lebegue measure is invariant under isometries of $\mathbb{R}^n$, which are maps

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto x_0 + Rx, \quad R \in O(n)$$

Proof. `Missing` □

Definition 1.5.5. A Borel measure μ on $\mathbb{R}^n$ is called **Borel regular**, if for every $A \subseteq \mathbb{R}^n$ there exists a Borel set $B \supseteq A$ such that $\mu(A) = \mu(B)$.

Lemma 1.5.6. The Lebesgue measure is Borel regular.

Proof. If $\mathcal{L}^n(A) = \infty$, we can simply take $B = \mathbb{R}^n$, so assume $\mathcal{L}^n(A) < \infty$.
By the characterisation with open sets from Theorem 1.4.8, we can chose for every $k \in \mathbb{N}$ an open set $U_k \supseteq A$ open with

$$\mathcal{L}^n(U_k) < \mathcal{L}^n(A) + \frac{1}{k}, \quad k \in \mathbb{N}$$

by intersecting each U_k with the previous ones, we can also assume without loss of generality that the sequence $(U_k)_{k\in\mathbb{N}}$ is monotonously decreasing (i.e. $U_{k+1} \subseteq U_k$).
By Remark 1.4.7, the open sets U_k are in the σ-algebra of $\mathcal{L}^n$-measurable subsets. Setting $B := \bigcap_{k=1}^\infty U_k$ it follows from continuity from above (Theorem 1.2.12)

$$\mathcal{L}^n(B) \overset{\text{c.f.a}}{=} \lim_{k\to\infty} \mathcal{L}^n(U_k) = \mathcal{L}^n(A)$$

□

1.6 Special-Examples of sets

As we will see, not all subsets of $\mathbb{R}^n$ are $\mathcal{L}^n$-measurable.
To construct such a non-measurable set, we will use the Axiom of Choice, which states that for any family of non-empty disjoint sets $(A_i)_{i\in I}$, there exists a choice-function $f : I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$.
With this, we can construct the set $\{f(i) | i \in I\}$ that contains exactly one element from each set A_i.

For $x, y \in [0,1)$ we define $\oplus := (\mod 1 \circ +)$

$$x \oplus y = \begin{cases} x+y & \text{if } x+1<1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

So if we have a subset $E \subseteq [0,1)$, we can "shift" the set E by x, with $E \oplus x \subseteq [0,1)$.
Where some part $E \cap [0,1-x)$ moves naturally to the right and the set $E \cap [1-x,1)$ moves back to the left side. Set

$$E_1 := E \cap [0,1-x) \oplus x$$
$$E_2 := E \cap [1-x,1) \oplus x$$

which are disjoint.
If E is $\mathcal{L}^1$-meaurable, then the translated sets E_1, E_2 are also $\mathcal{L}^1$-measurable and

$$\begin{aligned} \mathcal{L}^1(E \oplus x) &= \mathcal{L}^1(E_1) + \mathcal{L}^1(E_2) \\ &= \mathcal{L}^1(E \cap [0,1-x)) + \mathcal{L}^1(E \cap [1-x,1)) \\ &= \mathcal{L}^1(E) \end{aligned}$$

A non-measurable set

Then we define the equivalence relation

$$x, y \in [0,1) \quad x \sim y \iff x - y \in \mathbb{Q}$$

by the axiom of choice, there exists a set $P \subseteq [0,1)$ that contains exactly one representative of each equivalence class.
By enumerating all rational points in $[0,1)$ by an index $Q \cap [0,1) = \{r_k\}_{k \in \mathbb{N}}$ with $r_0 = 0$ we define

$$P_k := P \oplus r_k$$

Then it is easy to see that

(a) The P_j are disjoint and $[0,1) = \bigsqcup_{j=0}^{\infty} P_j$.

Because if $x \in P_n \cap P_m$, then $x = p_n \oplus r_n = p_m \oplus r_m$. Since $r_n, r_m \in \mathbb{Q}$ it follows that also $p_n - p_m \in \mathbb{Q}$ so they must be of the same equivalence class.

It also covers $[0,1)$ because by construction, every $x \in [0,1)$ belongs to a unique equivalence class.

(b) If P were $\mathcal{L}^1$-measurable, then so is $P_j = P \oplus r_j$ and $\mathcal{L}^1(P) = \mathcal{L}^1(P_j)$.

We just showed this earlier.

But P cannot be $\mathcal{L}^1$-measurable, because by σ-additivity on $\mathcal{L}^1$-measurable subsets

$$1 = \mathcal{L}^1([0,1)) = \sum_{i=0}^{\infty} \mathcal{L}^1(P_j) = \sum_{i=0}^{\infty} \mathcal{L}^1(P)$$

and the right hand side is either 0 or infinite.
So since P is not $\mathcal{L}^1$-measurable there exists a set $B \subseteq \mathbb{R}$ with

$$\mathcal{L}^1(B) < \mathcal{L}^1(B \cap P) + \mathcal{L}^1(B \setminus P)$$

We also know that $\mathcal{L}^1(P)$ can't be zero, or else it would be $\mathcal{L}^1$-measurable.
Moreover, if $E \subseteq P$ is $\mathcal{L}^1$-measurable, then $\mathcal{L}^1(E) = 0$ because we can set

$$E_i := E \oplus r_i \implies F := \bigsqcup_{i=0}^{\infty} E_i \subseteq [0,1) \text{ is } \mathcal{L}^1\text{-measurable}$$

and we have

$$1 = \mathcal{L}^1([0,1)) \geq \mathcal{L}^1(F) = \sum_{i=0}^{\infty} \mathcal{L}^1(E_i) = \sum_{i=0}^{\infty} \mathcal{L}^1(E)$$

which can only be true if $\mathcal{L}^1(E) = 0$.
Not only does there exists a non-$\mathcal{L}^1$-measurable subset, we can construct more using P as a "template".

Proposition 1.6.1. For every $A \subseteq \mathbb{R}$ with $\mathcal{L}^1(A) > 0$, there exists a subset $B \subseteq A$ that is not $\mathcal{L}^1$-measurable.

Proof. Because we can shift and scale A or take subsets of A, we can assume without loss of generality that $A \subseteq (0,1)$.
Then set $B_i = A \cap P_i$. Then $A = \bigsqcup_{i=0}^{\infty} B_i$
As we showed earlier, if B_i were $\mathcal{L}^1$-measuralble, then $\mathcal{L}^1(B_i) = 0$, which contradicts $\mathcal{L}^1(A) = \sum_{i=0}^{\infty} \mathcal{L}^1(B_i)$. □

Remark 1.6.2. Because singletons $\{\alpha\} \in \mathbb{R}$ are contained in the arbitrarily small interval $(\alpha - \varepsilon, \alpha + \varepsilon)$ with Lebesgue measure 2ε, singletons have Lebesgue measure zero.
It follows that by subadditivity, every countable subset of $\mathbb{R}$ also has Lebesgue measure zero.

The Cantor tridadic set

The real numbers can be defined as the set of Cauchy-sequences in $\mathbb{Q}$ up to equivalence of Cauchy sequences. This gives for every $x \in \mathbb{R}$ and base $b > 2 \in \mathbb{N}$ a b-ary expansion with digits $d_i(x) \in \{0, \dots, b-1\}$.

$$x = \sum_{i=1}^{\infty} d_i(x) b^{-i}$$

Although the digit expansion is not always unique, the set of those with multiple expansions is countable and thus have measure zero.

Proposition 1.6.3. The **Cantor set** is the set of numbers whose 3-adic digits don't contain a 1.

$$C = \{x \in [0,1] | d_i(x) \in \{0,2\} \forall i\}$$

Then C is uncountable and $\mathcal{L}(C) = 0$.

Proof. We construct the Cantor set as $C := \bigcap_{n=1}^{\infty} C_n$, where

$$C_n = \{x \in [0,1] | d_i(x) \neq 1 \forall i \leq n$$

Then each C_n can be written as a finite union closed intervals. For example

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], \text{etc.}$$

They are in particular Borel and have Lebesgue measure

$$\mathcal{L}^1(C_n) = \left(\frac{2}{3}\right)^n$$

because this sequence is decreasing, by continuity from above we have

$$\mathcal{L}^1(C) = \mathcal{L}^1\left(\bigcap_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mathcal{L}^1(C_n) = 0$$

To show that C is countable, we define a function that maps elements of C surjectively to $[0, 1]$
Because C consists of numbers whose 3-ary sequence of digits don't contain a 1, i.e. only use the digits 0 and 2, we can map them to binary sequences of digits which use the digit 0 and 1, by converting every digit 2 to a 1 and look at it as a binary sequence.

$$f : C \to [0, 1], \quad \sum_{i=1}^{\infty} \frac{d_i(x)}{3^i} \mapsto \sum_{i=1}^{\infty} \frac{d_i(x)}{2} \frac{1}{2^i}$$

For example, $\frac{8}{27} = 0.022_3 \mapsto 0.011_2 = \frac{3}{8}$.
Because this lets us generate any (even infinite) binary sequence of digits, the map is surjective. □

The construction of the Cantor set can be generalised to give us the so-called **fat Cantor sets**, where we start off with the interval $I_1 = [0, 1]$, and for $n \in \mathbb{N}$, if some interval inside I_n has length ℓ, then we remove the centered subinterval of length $\beta\ell$ and let $I_{n+1} \subseteq I_n$ be the remaining pieces of this operation. The fat cantor set with parameter β is then $C_\beta := \bigcap_{n=1}^{\infty} I_n$.
We see that the "normal" Cantor set has the parameter $\beta = \frac{1}{3}$ and if $\beta < \frac{1}{3}$, then we have

$$\mathcal{L}^1(I_n \setminus I_{n+1}) = 2^{n-1}\beta^n \implies \mathcal{L}^1(I_1 \setminus C_\beta) = \sum_{n=1}^{\infty} 2^{n-1}\beta^n = \frac{\beta}{1-2\beta}$$

but this set is not Jordan-measurable as

$$\underline{\mu}(C_\beta) = 0 \quad \text{but} \quad \overline{\beta}(C_\beta) = 1 - \frac{\beta}{1-2\beta}$$

1.7 The Lebesgue-Stieltjes Measure

Let $F : \mathbb{R} \to \mathbb{R}$ be non-decreasing and continuous from the left, i.e:

$$F(x_0) = \lim_{x\to x_0^-} F(x) := \lim_{\substack{x\to x_0 \\ x<x_0}} F(x) \quad \forall x_0 \in \mathbb{R}$$

For a, b we define

$$\lambda_F[a, b) = \begin{cases} F(b) - F(a) & \text{if } a < b \\ 0 & \text{otherwise} \end{cases}$$

Because the collection of bounded half-open sets $\mathcal{K} = \{[a, b) | a, b \in \mathbb{R}\}$ does not form an Algebra (see 1.7.6), we cannot use the Carathéodory-Hahn extension theorem to produce a measure induced by λ_F.
However, $\mathcal{K}$ constitutes a covering of $\mathbb{R}$ as in Definition 1.3.1, so by Theorem 1.3.3, the function λ_F induces a measure

$$\Lambda_F(A) := \inf\left\{\sum_{k=1}^{\infty} \lambda_F[a_k, b_k), \quad A \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k)\right\}$$

called the **Lebesgue-Stieltjes Measure** generated by F.
To find out if Λ_F is nice, we will find the following definition useful.

Definition 1.7.1. A measure μ on $\mathbb{R}^n$ is called **metric,** if the measure is additive on separated sets, i.e. for all $A, B \subseteq \mathbb{R}^n$ with

$$\operatorname{dist}(A, B) := \inf\{|a - b|, a \in A, b \in B\} > 0$$

it holds

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

By subadditivity, the inequality "$\geq$" is sufficient.

Theorem 1.7.2 (Carathéodory criterion for Borel measures)**.** A metric measure μ on $\mathbb{R}^n$ is Borel.

Proof. Let μ be a metric measure on $\mathbb{R}^n$. Because the μ-measurable subsets (1.2.9) form a σ-Algebra, it is sufficient to show that closed sets are μ-measurable.
Let $F \subseteq \mathbb{R}^n$ be closed and $B \subseteq \mathbb{R}^n$ be some test set. If $\mu(B) = \infty$, then the inequality

$$\mu(B) \geq \mu(B \cap F) + \mu(B \setminus F)$$

is trivial, so assume $\mu(B) < \infty$. For $k = 1, 2, \ldots$, we define

$$F_k := \{x \in \mathbb{R}^n | 0 \leq \operatorname{dist}(x, F) \leq \frac{1}{k}\}$$

It should be clear that

$$\operatorname{dist}(B \setminus F_k, B \cap F) \geq \frac{1}{k} > 0$$

so since μ is metric and monotonous, we have

$$\mu(B \cap F) + \mu(B \setminus F_k) = \mu((B \cap F) \cup (B \setminus F_k)) \leq \mu(B) \quad \forall k$$

If we can show that $\lim_{k\to\infty} \mu(B \cap F_k) = \mu(B \setminus F)$, then we are done.
To do so, first note that the $(F_k)_{k\in\mathbb{N}}$ form a decreasing sequence $F_{k+1} \subseteq F_k$. Moreover, we have $F = \bigcap_{k=1}^{\infty} F_k$, so we can write

$$B \setminus F = B \setminus \bigcap_{l=1}^{\infty} F_l = \bigcup_{l=1}^{\infty} (B \setminus F_l)$$

We can expand the union above in telescoping fashion [3] and use the fact that the $(B \setminus F_l)_{l\in\mathbb{N}}$ form an increasing sequence to get

$$\begin{aligned}\bigcup_{l=1}^{\infty}(B \setminus F_l) &= (B \setminus F_1) \cup \bigcup_{l=1}^{\infty}(B \setminus F_{l+1}) \setminus (B \setminus F_l)\\ &= (B \setminus F_k) \cup \bigcup_{l=k}^{\infty}(F_l \setminus F_{l+1}) \cap B\end{aligned}$$

Setting

$$R_l := (F_l \setminus F_{l+1}) \cap B = \{x \in B | \frac{1}{l+1} < d(x, F) \leq \frac{1}{l}\}$$

[3]For example, for any sequence $(A_l)_{l\in\mathbb{N}}$ we can write $\bigcup_{l=1}^{\infty} A_l = A_1 \cup \bigcup_{l=1}^{\infty} A_{l+1} \setminus A_l$.

we see that the $(R_l)_{l\in\mathbb{N}}$ are pairwise disjoint, so we have

$$B \setminus F = (B \setminus F_k) \cup \bigsqcup_{l=k}^{\infty} R_l$$

Therefore, for all $k \in \mathbb{N}$ it holds

$$\mu(B \setminus F_k) \leq \mu(B \setminus F) \leq \mu(B \setminus F_k) + \sum_{l=k}^{\infty} \mu(R_l)$$

Now we only need to show that

$$\lim_{k\to\infty} \sum_{l=k}^{\infty} \mu(R_l) = 0$$

Observe that R_i only “touches” its neighbors R_{i-1}, R_{i+1}, in other words

$$\operatorname{dist}(R_i, R_j) > 0, \quad \text{if } |i-j| \geq 2$$

decomposing the sum $\sum_{l=1}^{\infty} \mu(R_l)$ into the even and odd numbers, we can use the fact that μ is metric to get

$$\sum_{l=1}^{2m+1} \mu(R_l) = \left(\sum_{k=1}^{m} \mu(R_{2k})\right) + \left(\sum_{k=1}^{m} \mu(R_{2k})\right) = \mu\left(\bigcup_{k=1}^{m} R_{2k}\right) + \mu\left(\bigcup_{k=1}^{m} R_{2k}\right) \leq 2\mu(B) < \infty$$

so even in the limit $m \to \infty$, the series converges. But in the inequality we showed earlier

$$\mu(B \setminus F_k) \leq \mu(B \setminus F) \leq \mu(B \setminus F_k) + \sum_{l=k}^{\infty} \mu(R_l)$$

we were allowed to omit any number of (non-negative) starting terms $\mu(R_l)$ for $l < k$, so in the limit we get $\lim_{k\to\infty} \mu(B \setminus F_k) = \mu(B \setminus F)$, and the result follows. □

Theorem 1.7.3. The Lebesgue-Stieltjes measure Λ_F is Borel regular.

Proof. (i) **$\boldsymbol{\Lambda_F}$ is Borel.** We show that it is metric and use the previous theorem, so let $A, B \subseteq \mathbb{R}$ with $\delta := \operatorname{dist}(A, B) > 0$. We now show that for all $\varepsilon > 0$ we have

$$\Lambda_F(A) + \Lambda_F(B) \leq \Lambda_F(A \cup B) + \varepsilon$$

Per definition of the Lebesgue-Stieltjes measure, we can find a collection of half-open intervals with

$$A \cup B \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k) \quad \text{and} \quad \sum_{k=1}^{\infty} < \Lambda_F(A \cup B) + \varepsilon$$

Because we can always subdivide any interval $[a_k, b_k)$ further, we may also assume that $|b_k - a_k| < \delta$ for all k.

Because A and B are separated, for each interval $[a_k, b_k)$ either

$$A \cap [a_k, b_k) = \emptyset \quad \text{or} \quad B \cap [a_k, b_k) = \emptyset$$

, so the covering of $A \cup B$ gives us a covering $\mathcal{A}$ of A and a covering $\mathcal{B}$ of B. Therefore

$$\begin{aligned}\Lambda_F(A) + \Lambda_F(B) &\leq \sum_{[a_k,b_k)\in\mathcal{A}} \Lambda_F([a_k,b_k)) + \sum_{[a_k,b_k)\in\mathcal{B}} \Lambda_F([a_k,b_k)) \\ &= \sum_{k\in\mathbb{N}} \lambda_F([a_k,b_k) \leq \Lambda_F(A\cup B) + \varepsilon\end{aligned}$$

This shows that Λ_F is metric and thus Borel.

(ii) **Λ_F is Borel regular.**

To show that Λ_F is Borel regular, let $A \subseteq \mathbb{R}$. Of course we can assume $\Lambda_F(A) < \infty$. Then for any $n \in \mathbb{N}$ we can find coverings

$$A \subseteq \bigcup_{k=1}^{\infty} \left[a_k^{(n)}, b_k^{(n)}\right) =: B_n \quad \text{with} \quad \sum_{k=1}^{\infty} \lambda_F\left(\left[a_k^{(n)}, b_k^{(n)}\right)\right) \leq \Lambda_F(A) + \frac{1}{n}$$

If we set $B := \bigcap_{n=1}^{\infty} B$, then B is Borel and $A \subseteq B \subseteq B_n$ and

$$\Lambda_F(A) \leq \Lambda_F(B) \leq \Lambda_F(B_n) \leq \sum_{k=1}^{\infty} \lambda_F\left(\left[a_k^{(n)}, b_k^{(n)}\right)\right) \leq \Lambda_F(A) + \frac{1}{n}$$

in the limit $n \to \infty$, we get $\Lambda_F(A) = \Lambda_F(B)$, so Λ_F is Borel regular.

□

The Carathéodory-Hahn extension had the property that it coincided with the pre-measure on the algebra, on which the pre-measure was defined. Despite not being such an extension, the Lebesgue-Stieltjes measure has a similar property.

Theorem 1.7.4. For $a < b \in \mathbb{R}$ it holds

$$\Lambda_F([a,b)) = \lambda_F([a,b)) = F(b) - F(a)$$

Proof. Let $a < b \in \mathbb{R}$. By definition of Λ_F, we already have $\Lambda_F([a,b)) \leq \lambda_F([a,b))$.
For the other inequality, let $([a_k,b_k))_{k\in\mathbb{N}}$ be a covering of $[a,b)$.
Since F is left-continuous, for every $\varepsilon > 0$ there exist $\delta, \delta_k > 0$ such that

$$F(b) - F(b-\delta) \leq \varepsilon, \quad \text{and} \quad F(a_k) - F(a_k - \delta_k) \leq 2^{-k}\varepsilon \quad \forall k \in \mathbb{N}$$

Because $[a, b-\delta]$ is compact and is covered by $\bigcup_{k=0}^{\infty}(a_k - \delta_k, b_k)$, there exists a finite subcovering

$$[a, b-\delta] \subseteq \bigcup_{k=0}^{m} (a_k - \delta_k, b_k)$$

By removing any redundant intervals, we can decrease the sum $\sum_{k=0}^{m} \lambda_F(a_k - \delta_k, b_k)$, so we can assume WLOG that they are ordered in such a way that

$$a_k - \delta_k < b_{k-1} \quad \text{for all} \quad k = 1, \ldots, m$$

Since F is increasing and $a_0 - \delta_0 < a < b < b_m$, we have

$$\begin{aligned}F(b-\delta) - F(a) &\leq F(b_m) - F(a_0 - \delta_0) \\ &\leq F(b_m) - F(a_1 - \delta_1) + F(b_0) - F(a_0 - \delta_0) \\ &\leq \ldots \leq \sum_{k=0}^{m} F(b_k) - F(a_k - \delta_k)\end{aligned}$$

so with the initial estimates, we have

$$\begin{aligned}
\lambda_F([a,b)) &= F(b) - F(a) = F(b) - F(b-\delta) + F(b-\delta) - F(a) \\
&\leq \varepsilon + \sum_{k=0}^{m} F(b_k) - F(a_k - \delta_k) \\
&= \varepsilon + \sum_{k=0}^{m} F(b_k) - F(a_k) + \sum_{k=0}^{m} F(a_k) - F(a_k - \delta_k) \\
&\leq \varepsilon + \sum_{k=0}^{\infty} F(b_k) - F(a_k) + \sum_{k=0}^{\infty} 2^{-k}\varepsilon \\
&= \sum_{k=0}^{\infty} \lambda_F([a_k, b_k)) + 3\varepsilon
\end{aligned}$$

Since this is true for all coverings $([a_k, b_k))_{k\in\mathbb{N}}$, we get in the limit $\varepsilon \to 0$

$$\lambda_F([a,b)) \leq \Lambda_F([a,b))$$

□

Example 1.7.5.

- The Lebesgue measure is the special case when $F(x) = x$, so $\Lambda_{\mathrm{id}_{\mathbb{R}}} = \mathcal{L}^1$
- The Dirac measure δ_0 from Example 1.2.11 is the Lebesgue-Stieltjes measure Λ_Θ for the Heaviside step function

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Remark 1.7.6. In the beginning of this section, we noted that the collection of bounded half-open sets $\mathcal{K} = \{[a,b) | a, b \in \mathbb{R}\}$ does not form an Algebra. If we set $\tilde{\mathcal{K}}$ to be the collection of finite disjoint unions:

$$\tilde{\mathcal{K}} = \left\{ \bigsqcup_{k=1}^{m} [a_k, b_k) | m \geq 1 \in \mathbb{N}, a_k, b_k \in \mathbb{R} \right\}$$

then the set $\tilde{\mathcal{K}}$ is stable under *intersection* and *difference*. And we say that $\tilde{\mathcal{K}}$ forms a **ring**.
That is: $\emptyset \in \tilde{\mathcal{K}}$ and $A, B \in \tilde{\mathcal{K}} \implies A \cap B, A \setminus B \in \tilde{\mathcal{K}}$
The same is not true for the collection of open and closed intervals.

1.8 Hausdorff Measures

Say we have the unit square $A := [0,1]^2$. It's $\mathcal{L}^2$-measure would of course be 1.
If we however were to embed the square into $\mathbb{R}^3$ with

$$\iota : \mathbb{R}^2 \to \mathbb{R}^3, \quad (x,y) \mapsto (x,y,0)$$

we would find that $\mathcal{L}^3(\iota(A)) = 0$. More generally, the Lebesgue measure $\mathcal{L}^n$ on subsets $A \subseteq \mathbb{R}^n$ that have "dimension" $< n$ is always going to be zero.
Moreover, the Lebesgue measure also has the weakness of failing to properply measure fractal sets (which can be though of as having non-integer "dimension")

It would be nice to define a collection of measures that are able to measure sets, regardless of whether they are embedded into a higher-dimensional space. The Hausdorff measures try to solve this.
We start by introducing an intermediate measure, where instead of covering a subset A with dyadic cubes (as was the case for the Lebesgue measure) we do this using open balls of radius smaller than some $\delta > 0$.

Definition 1.8.1. For $s \geq 0, \delta > 0$ and $A \subseteq \mathbb{R}^n$ non-empty, we set

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{k \in I} r_k^s \middle| I \text{ at most countable}, A \subseteq \bigcup_{k \in I} B(x_k, r_k), 0 < r_k < \delta \right\}$$

where we set $\mathcal{H}^0_\delta(\emptyset) = 0$.

Remark 1.8.2. $\mathcal{H}^s_\delta$ defines a measure on $\mathbb{R}^n$ and for fixed s, A, the function $\delta \mapsto \mathcal{H}^s_\delta(A)$ is non-increasing:

$$\delta_2 \leq \delta_1 \implies \mathcal{H}^s_{\delta_1}(A) \leq \mathcal{H}^s_{\delta_2}(A)$$

since every δ_2 covering is also a δ_1 covering. Therefore, the limit

$$\boxed{\mathcal{H}^s(A) := \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)}$$

exists. We now use this as for our next definition.

Definition 1.8.3. We call $\mathcal{H}^s$ the s**-dimensional Hausdorff measure** on $\mathbb{R}^n$

As hinted at earlier with the case of fractals, notice that s may take on non-integer values.
To build some intuition, let's consider an example of a "one-dimensional" set $A \subseteq \mathbb{R}^2$.

Example 1.8.4. Let $A = \mathbb{S}^1 = \{x \in \mathbb{R}^2, \|x\| = 1\}$.

$s = 0$: We see that $\mathcal{H}^0_\delta(A)$ depends only on the number of balls covering A.

If $\delta > 1$, we see that A can be covered by the ball $B(0, 1 + \varepsilon)$, for ε small enough. Therefore $\mathcal{H}^0_\delta(A) = 1$ for $\delta > 1$.

On the other hand, if $\delta < 1$ then we have to cover A by using multiple balls. It should be clear that in the limit $\delta \to 0$, we have $\mathcal{H}^s(A) = \infty$.

$s = 1$: Again, for $\delta > 1$, it's easy to see that the covering with the single ball $B(0, 1+\varepsilon)$ give us $\mathcal{H}^1_\delta(A) \leq 1+\varepsilon$.

But for arbitrary $\delta > 0$, let $n \in \mathbb{N}$ such that $\delta > \frac{\pi}{n}$. We then can place n equally spaced balls along the circle each with radius $\frac{\pi}{n} + \varepsilon$, for some $\varepsilon > 0$ small enough. This covers A entirely and gives us an upper bound

$$\mathcal{H}^1_\delta(A) \leq \sum_{i=1}^n \frac{\pi}{n} = \pi$$

One can convince themself that there is no "better" covering strategy resulting in a lower upper bound, so $\mathcal{H}^1(A) = \pi$.

$s = 2$: The same covering strategy as decribed in the case $s = 1$ gives us the upper bound

$$\mathcal{H}^s_\delta(A) \leq \sum_{i=1}^n \left(\frac{\pi}{n}\right)^2 = \frac{\pi^2}{n}$$

But unlike for $s = 1$, chosing bigger and bigger n means that we can make the $\mathcal{H}^2$-measure of A arbitrarliy small. So $\mathcal{H}^2(A) = 0$.

A similar argumentation also shows that $\mathcal{H}^s(A) = 0$ for all $s > 1$.

Before we prove that $\mathcal{H}^s$ is actually a measure, let's take a look at the case $s = 0$ more closely.

Remark 1.8.5. $\mathcal{H}^0$ is the counting measure from 1.2.11.

Proof. If A is finite and has k elements $A = \{a_1, \dots, a_k\}$, let $\delta > 0$ be the minimal distance between all elements.
It easily follows from the triangle inequality, that $\mathcal{H}^0(A) \geq \mathcal{H}^0_{\frac{\delta}{2}} \geq k$. The other inequality is also trivial.
If A is infinite, then for any $k \in \mathbb{N}$ we can find a subset A_k with at least k elements. By monotonicity we have $k = \mathcal{H}^0(A_k) \leq \mathcal{H}^0(A)$ and in the limit $k \to \infty$, the proof follows.
And if A is empty, by defintion we have $\mathcal{H}^0(\emptyset) = 0$. □

Theorem 1.8.6. For $s \geq 0$, $\mathcal{H}^s$ is a Borel regular measure on $\mathbb{R}^n$

Proof. Let $s \geq 0$.

(i) **$\mathcal{H}^s$ is a measure.** Clearly, $\mathcal{H}^s(\emptyset) = 0$. Let $(A_k)_{k\in\mathbb{N}}$ and $A \subseteq \bigcup_{k=1}^\infty A_k$. Since $\mathcal{H}^s_\delta$ is σ-subadditive for all $\delta > 0$, we get

$$\mathcal{H}^s_\delta(A) \leq \sum_k \mathcal{H}^s_\delta(A_k) \leq \sum_k \mathcal{H}^s(A_k) \quad \forall \delta > 0$$

by taking the limit $\delta \to 0$ (as in the definition of $\mathcal{H}^s$) we get the σ-subadditivity of $\mathcal{H}^s$.

(ii) **$\mathcal{H}^s$ is metric and therefore Borel.** The proof is more or less the same as for the Lebesgue-Stieltjes measure. Let $A, B \subseteq \mathbb{R}^n$ such that $\delta_0 := \operatorname{dist}(A, B) > 0$. We then take a covering $A \cup B$ of balls of size smaller than $\delta := \frac{\delta_0}{4}$ and claim that we can partition the covering into two non-overlapping coverings of A and B each.

Since $\mathcal{H}^s_\delta$ takes the infimum over all such coverings, suppose that $A \cup B = \bigcup_k B(x_k, r_k)$ with $r_k < \delta$. Then we set

$$\mathcal{A} = \{B(x_k, r_k) \big| B(x_k, r_k) \cap A \neq \emptyset\} \mathcal{B} = \{B(x_k, r_k) \big| B(x_k, r_k) \cap B \neq \emptyset\}$$

And it becomes obvious that these are non-overlapping coverings of A and B each (by using the triangle inequality).

Therefore, we get

$$\mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \leq \sum_k r_k^s$$

and taking the infimum of coverings of $A \cup B$, this means

$$\mathcal{H}^s_\delta(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$$

which, when taking the limit $\delta \to 0$ just states $\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B)$. By ($\sigma$)-subadditivity of $\mathcal{H}^s$, the reverse inequality holds and so $\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B)$ shows that $\mathcal{H}^s$ is metric and thus also Borel.

(iii) **$\mathcal{H}^s$ is Borel regular.** Again, the proof follows the same structure as in the proof for the Lebesgue-Stieltjes measure. Let $A \subseteq \mathbb{R}^n$ and suppose $\mathcal{H}^s(A) < \infty$ (Otherwise, just take $B = \mathbb{R}^n$). By monotonicity of $\mathcal{H}^s_\delta$, this also means that $\mathcal{H}^s_\delta(A) < \infty$ for all $\delta > 0$.

For $\delta = \frac{1}{m}, m = 1, 2, \ldots$, this gives us a covering $\bigcup_{k\in I} B(x_{k,m}, r_{k,m}) \supseteq A$ with $r_{k,m} < \frac{1}{m}$ and

$$\sum_{k\in I} r_{k,m}^s \leq \mathcal{H}^s_{\frac{1}{m}}(A) + \frac{1}{m}$$

Then set $A_m := \bigcup_{k\in I} B(x_{k,m}, r_{k,m})$ and $B = \cap_{m=1}^{\infty} A_m$. Then B is a Borel set containing A. Which by monotonicity of $\mathcal{H}^s_{\frac{1}{m}}$ lets us sandwich

$$\begin{aligned} \mathcal{H}^s_{\frac{1}{m}}(A) \leq \mathcal{H}^s_{\frac{1}{m}}(B) \leq \mathcal{H}^s_{\frac{1}{m}}(A_m) &\leq \sum_{k\in I} r_{k,l}^s \\ &\leq \mathcal{H}^s_{\frac{1}{m}}(A) + \frac{1}{m} \end{aligned}$$

so in the limit $m \to \infty$, we get $\mathcal{H}^s(B) = \mathcal{H}^s(A)$.

□

In the example, where we calculated $\mathcal{H}^s(\mathbb{S}^1)$, we saw that

$$\mathcal{H}^0(A) = \infty, \quad \mathcal{H}^1(A) = \pi, \quad \mathcal{H}^2(A) = 0$$

The following Lemma proves the general pattern.

Lemma 1.8.7. Let $A \subseteq \mathbb{R}^n$ and $0 \leq s < t < \infty$. Then

(a) $\mathcal{H}^s(A) < \infty \implies \mathcal{H}^t(A) = 0$

(b) $\mathcal{H}^t(A) > 0 \implies \mathcal{H}^s(A) = \infty$

Proof. Since (b) is just the contraposition of (a), it's enough to prove (a).
Let $0 \leq s < t \in \mathbb{R}$ and $A \subseteq \mathbb{R}^n$ with $\mathcal{H}^s(A) < \infty$.
For any covering $A \subseteq \bigcup_{k\in I} B(x_k, r_k)$ with $r_k < \delta$, we have

$$\mathcal{H}^t_\delta(A) \leq \sum_{k\in I} r_k^t = \sum_{k\in I} r_k^{t-s} r_k^s \leq \delta^{t-s} \sum_{k\in I} r_k^s$$

Considering the infimum over all such coverings we get

$$\mathcal{H}^t_\delta(A) \leq \delta^{t-s} \mathcal{H}^s_\delta(A)$$

so as $\delta \to 0$, we get $\mathcal{H}^t(A) = 0$. □

This Lemma makes the definition of "dimension" possible.

Definition 1.8.8. The **Hausdorff dimension** of a subset $A \subseteq \mathbb{R}^n$ is defined as

$$\boxed{\dim_{\mathcal{H}}(A) := \inf\{s \geq 0 | \mathcal{H}^s(A) = 0\}}$$

Example 1.8.9. Let $Q = [-1, 1]^n \subseteq \mathbb{R}^n$. Then

$$2^{-n}\mathcal{L}^n(Q) \leq \mathcal{H}^n(Q) \leq 2^{-n} n^{\frac{n}{2}}(Q)$$

Proof. `Missing` □

2 Measurable Functions

2.1 Basic definitions

For X, Y nonempty sets and $f : X \to Y$ with $A \subseteq Y$, the inverse image is defined as

$$f^{-1}(A) = \{x \in X | f(x) \in A\}$$

which satisfies

(a) $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$

(b) If $A, B \subseteq Y$, then

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

(c) If $(A_k)_k$ is a sequence of subsets, then

$$f^{-1}\left(\bigcup_{k=1}^{\infty} A_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(A_k)$$

From this, it follows that if $(Y, \mathcal{A}, \mu)$ is a measure space, then

$$\Sigma := f^{-1}(\mathcal{A}) := \{f^{-1}(A) | A \in \mathcal{A}\}$$

is a σ-algebra in X.

Definition 2.1.1. A function $f : \Omega \to [-\infty, \infty]$ is called $\boldsymbol{\mu}$**-measurable** if in the sense of definition 1.2.7

(a) $f^{-1}\{+\infty\}, f^{-1}\{-\infty\}$ are μ-measurable.

(b) $f^{-1}(U)$ for every $U \subseteq \mathbb{R}$ open is μ-measurable.

The composition of μ-measurable functions