Methods of mathematical Physics – Lecture Notes

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Organisation

Professor: Dr. Thomas Willwacher
The lecture home page will be on moodle.
Exercise lessons will start in the second week.
Solving the weekly exercises will be rewarded with up a 0.25 bonus in the final grade if over 40% of the exercises are reasonably done.
The main resource will be the MMP II script from G.Felder, with some additions to representation theory, lie algebras and tensors.

Introduction

The main topics for this course will the the study of Algebraic methods and symmetry. We will see later that we can describe symmetries with groups and representations.
Here are some examples of symmetry in physics

- In mechanics, we could have coupled harmonic oscillators displaying mirror symmetry.
- The kepler problem will use $O(3)$-symmetry
- In solid state physics, we can look at crystal grids displaying translational $\mathbb{Z}^3$-symmetries
- In Quantum mechanics from last semester, we saw the hydrogen atom.
- In nuclear and particle physics, representation theory is central and gives us Gauge theory.
- In relativity, we see Lorentz symmetry.

We want too find out how to describe and use symmetry.

1 Groups

1.1 Basic definitions and examples

For completeness, we quickly repeat some definitions used.

Definition 1.1 (Basics)**.** A group is a set G with a binary operation

$$\cdot : G \times G \to G, \quad (g,h) \mapsto g \cdot h = gh$$

that satisfies the following

(a) (Associativity): $(gh)k = g(hk)$

(b) (Neutral element) $\exists 1 \in G$ such that $1 \cdot g = g \cdot 1 = g$ for all $g \in G$

(c) (Inverse elements) $\forall g \in G, \exists g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = 1$

It follows from the group axioms that the unit 1 and the inverse elements are uniquely determined.
We define the **order** of G to be its cardinality $|G|$, where we just write ∞ if G is infinite.
A group G is **abelian**, if for all $g, h \in G$ we have $gh = hg$. For abelian groups, the additive notation ($g + h = h + g$) is often used, where we write the unit as 0 and inverses as $-g$.
A **subgroup** of a group G is a non-empty subset $H \subseteq G$ that is closed under the group multiplication, that is

(a) $h_1, h_2 \in H \implies h_1 h_2 \in H$

(b) $h \in H \implies h^{-1} \in H$

The **direct product** is the cartesian product $G_1 \times G_2$ with component-wise multiplication (i.e $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$ with $(1_G, 1_H)$ as unit)

Example 1.2.

- $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ or vector spaces are abelian groups under addition.
- The cyclic group $\mathbb{Z}/n\mathbb{Z}$ with addition is abelian and has order n
- The symmetric group S_n, consisting of bijective maps $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ called *permutations*. It has order $|S_n| = n!$
- The Dihedral group D_n for $n \geq 2$, consisting of rotations $R \in O(2)$ that keep a regular n-gon invariant. The elements are

  $$1, R, R^2, \ldots, R^{n-1}, S, RS, \ldots, R^{n-1}S$$

  where R is multiplication with $e^{\frac{2\pi i}{n}}$ and S is complex conjucation, if one corner of the n-gon is situated at $1 \in \mathbb{C}$. In particular, $|D_n| = 2n$.

  To prove this, we need to show that each element listed above is different and that they are all possible symmetries of the n-gon. If we let $v_0, \ldots v_{n-1}$ be the corners of the n-gon, then we see that every element above maps the two corners (v_0, v_1) to different coner-pairs. Now let $X \in D_n$, then set j such that $Xv_0 = v_j$. Then it must be that Xv_1 is either v_{j+1} or v_{j-1}. In the first case, we see that $R^{-j}Xv_0 = v_0$ and $R^{-j}Xv_1 = v_1$, so $X = R^j$. In the second case we see that $SR^{-j}Xv_0 = v_0$ and $SR^{-j}Xv_1 = v_1$, so $X = R^jS$. Additionally we can show that $SRS = R^{-1}$.
- The general linear groups $\mathrm{GL}(n, K)$ for $n \in \mathbb{N}$ and K a field consisting of invertible $n \times n$ matrices.
- Similarly the orthogonal groups $O(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ consisting of orthogoal $n \times n$ matrices (i.e. $A^T A = 1$). Or equivalently, matrices such that $(Ax, Ay) = (x, y)$ with the standard dot product $(-, -)$. More generally we can define the symmetric bilinear form on $\mathbb{R}^{p+q}$ defined as

  $$(x, y)_{p,q} = \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{p+q} x_i y_i$$

we can use this to define the group $O(p,q) \subseteq \mathrm{GL}(n,\mathbb{R})$ that preserve this bilinear form. Notice that this bilinear form is no longer positive/definite. Ofspecial interest is $(-,-)_{1,3}$ called the *Minkowski*-“metric” with the **Lorentz group** $O(1,3)$

- The **unitary group** $U(n) \subseteq \mathrm{GL}(n,\mathbb{C})$ consisting of complex $n \times n$ such that $A^\dagger A = 1$, or that preserve the complex inner product $(-,-)$ where we use the physics convention in the inner product

$$(x,y) = \sum_{j=1}^{n} \overline{x_j} y_j$$

- The **symplectic group**: Consider the antisymmetric bilinear form on $\mathbb{R}^{2n}$ given by

$$\omega(X,Y) = \sum_{i=1}^{n} (X_{2i-1}Y_{2i} - X_{2i}Y_{2i-1}$$

  the symplectic group is then the subset $\mathrm{Sp}(2n) \subseteq \mathrm{GL}(n,\mathbb{R})$ that preserve this bilinear form: $\omega(X,Y) = \omega(AX,AY)$. This group can useful to describe hamiltonian systems, where the first n coordinates represent space, and the other n coordinates correspond to impulse.

- For any subgroup $G \subseteq \mathrm{GL}(n,\mathbb{R})$ or $G \subseteq \mathrm{GL}(n,\mathbb{C})$ we define the subgroup

$$SG := \{A \in G \,|\, \det A = 1\} \subseteq G$$

  Some special cases include the **special linear group** $\mathrm{SL}(n,K)$, the special orthogonal/unitary group $SO(n)$ and $SU(n)$.

Definition 1.3. A **group action** of a group G on a set M, is a map

$$G \times M \to M, \quad (g,x) \mapsto gx$$

that is compatible with the group operation, i.e. $g(hx) = (gh)x$
Equivalently, a group action is a group homomorphism $G \to \mathrm{Bij}(M)$.

Every group G can act on G itself in three ways:

(a) Left-action: $(g,x) \mapsto gx$

(b) right-action: $(g,x) \mapsto xg^1$

(c) conjugation: $(g,x) \mapsto gxg^{-1}$

Definition 1.4. A **group homomorphism** between two groups G, S is a map $\varphi : G \to S$ that is compatible with the group operations.

$$\varphi(gh) = \varphi(g)\varphi(h) \forall g,h \in G$$

If φ is bijective, we call φ an isomorphism.

We can show easily that if $\varphi : G \to S$ is a group homomorhpism, then

(a) $\varphi(1) = 1$, $\varphi(g)^{-1} = \varphi(g^{-1})$

(b) $\mathrm{Ker}(\varphi) \subseteq G$ and $\mathrm{Im}(\varphi) \subseteq S$ are subgroups

(c) φ is injective if and only if $\operatorname{Ker}(\varphi) = \{1\}$

(d) The composition of group homomorphisms is again a group homomorphism.

Definition 1.5. Let $H \subseteq G$ be a subgroup. We define the (Left-)**cosets** G/H the set of equivalence classes with respect to the equivalence relation $\sim$ on G:

$$g_1 \sim g_2 \iff \exists h \in H : g_1 h = g_2$$

In general, G/H does not have a group structure that is compatible with multiplication in G. That is possible if $ghg^{-1} \in H$ for all $g \in G, h \in H$.
If that is the case, we call H a **normal divisor** of G. And the mapping

$$G/H \times G/H \to G/H, \quad (g_1 H, g_2 H) \mapsto (g_1 g_2)H$$

is a well defined group operation. We then call G/H the **factor group** of G modulo H. We can see this because

$$g_1 h_1 g_2 h_2 = g_1 g_2 \underbrace{(g_2^{-1} h_1 g_2)}_{\in H} h_2 = (g_1 g_2) h h_2$$

Theorem 1.6. *Let G be a group and $H \subseteq G$ a subgroup. Then the following are equivalent*

- *H is a normal divisor of G*
- *There exists a group S and a group homomorphism $\varphi : G \to S$ such that $H = \operatorname{Ker}\varphi$*
- *$(xH)(yH) = (xy)H$ for all $x, y \in G$*
- *G/H has a group structure such that the projection $\pi : G \to G/H$, $\pi(g) = gH$ is a group homomorphism.*

Example 1.7.

- $n\mathbb{Z} \subseteq \mathbb{Z}$ is a normal divisor.
- Quotient vector spaces with addition.
- The kernel of a group homomorphism and quotient is isomorphic to its image.
- $\{+1, -1\} \subseteq \mathrm{SL}(n, \mathbb{C})$ is a normal divisor and the factor group is called the **projective special linear group** $PSL(2, \mathbb{C}) = {SL(2, \mathbb{C})}/{\{\pm 1\}}$ which is isomorphic to the group of Moebious transformations of the Riemann-sphere.

$$z \mapsto \frac{az + b}{cz + d}$$

where for a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, both A an $-A$ determine the same Moebius transformation.

Definition 1.8. Let G, H be groups and $\rho : G \to \operatorname{Aut}(H)$ a group homomorphism. If we write $\rho_g = \rho(g) \in \operatorname{Aut}(H)$, then $G \times H$ with multiplication

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1, g_2, h_1 \rho_{g_1}(h_2))$$

is called the **semi-direct** product $G \ltimes_\rho H$. Note that the unit is simply $(1, 1)$ and the inverse is

$$(g, h)^{-1} = (g^{-1}, \rho_{g^{-1}}(h^{-1}))$$

Example 1.9. The affine transformation

$$x \mapsto Ax + b, \quad \text{for} \quad A \in O(3)$$

defines a semidirect product $IO(3) := O(3) \ltimes \mathbb{R}^3$ with multiplication

$$(A_1, b_1) \cdot (A_2, b_2) = (A_1 A_2, b_1 + A_1 b_2)$$

where $\rho_A(b) = Ab$ A special case is the **Poincare** group $IO(1,3) = O(1,3) \ltimes \mathbb{R}^4$

1.2 Lie Groups

Definition 1.10. Named after Norwegian mathematician Sophus Lie, a **Lie Group** is a smooth manifold such that the multiplication and inverse maps are smooth. Note that the multiplication must be smooth with respect to the product manifold $G \times G$, not G itself.

Examples are $\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C}), O(p,q), (\mathbb{R}^n, +), U(p,q)$.
We can easily see that for example, $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$ is open (the determinant is polynomial and smooth and $\mathbb{R} \setminus \{0\}$ is open) The multiplication is polynomial in the coefficients and the inverse $A^{-1} = \frac{A^\#}{\det A}$ is a rational function without poles. So they are both smooth.
For $O(n)$, note that $O(n)$ is the zero locus of the smooth operation $A \mapsto A^T A - 1$. We know from Analysis II that such sets define a smooth manifold.

Definition 1.11. A **path** in a metric space is a continuous map $\gamma : [0,1] \to X$. We say that γ *connects* $\gamma(0)$ and $\gamma(1)$. If for all points $x, y \in G$ there exists such a path connecting x and y, we say that X is **path connected**. The (path)**connected components** of X are the equivalence classes under the equivalence relation

$$x \sim y \iff \exists \text{ path } \gamma : \gamma(0) = x, \gamma(1) = y$$

Theorem 1.12. *Let G be a group and $G_0 \subseteq G$ be path connected component containing the identity. Then G_0 is a normal divisor of G and ${}^{G}\!/_{G_0}$ is isomorhpic to the group of path connected components.*

Proof: Exercise.
Let's consider the connected components of some Lie groups.

(a) $\mathrm{SO}(n), \mathrm{SU}(n), U(n)$ are connected.

To see this, let $A \in U(n)$. Then there exists $U \in U(n)$ and

$$D = \begin{pmatrix} e^{i\varphi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\varphi_n} \end{pmatrix}$$

such that $A = UDU^\dagger$. Define a path $\gamma(t)$ by

$$\gamma(t) = U \begin{pmatrix} e^{it\varphi_1} & & 0 \\ & \ddots & \\ 0 & & e^{it\varphi_n} \end{pmatrix} U^\dagger$$

which connects the unit matrix with A as $\gamma(1) = A$ and $\gamma(0) = UU^\dagger = 1$.

The same proof works for $\mathrm{SU}(n)$: Since $\det(A) = 1$, D must be such that

$$e^{i\varphi_1 + \ldots + \varphi_n} = 0 \implies \varphi_1 + \ldots + \varphi_n = 0$$

which shows that $\gamma(t) \in \mathrm{SU}(n)$.

(b) $O(n)$ has two connected components. Those with determinant $+1$ and those with determinant -1.

We know from Linear Algebra II that there exists $O \in O(n)$ such that A can be written as

$$A = O \begin{pmatrix} R(\varphi_1) & & & & & & & 0 \\ & \ddots & & & & & & \\ & & R(\varphi_j) & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & -1 & \\ & & & & & & & \ddots \\ & & & & & & & -1 \end{pmatrix} O^T \quad \text{where} \quad R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in \mathrm{SO}(2)$$

If $A \in \mathrm{SO}(n)$ the number of -1 is even and we can write the 2×2 submatrices as $R(\pi)$. Then we can define the path

$$\gamma(t) = O \begin{pmatrix} R(t\varphi_1) & & & & & \\ & \ddots & & & & \\ & & R(t\varphi_j) & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} O^T$$

which connects A with 1.

If however $A \in O(n) \setminus \mathrm{SO}(n)$, then we connect A with the matrix

$$S = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

using a similar path.

1.3 Orbit formula nad Applications

Definition 1.13. Let G be a group acting on a set X. For each $x \in X$ we associate to it

- the **Orbit** of x

$$\mathcal{O}_x := \{gx | g \in G\} \subseteq X$$

- the **stabiliser** of x

$$\mathrm{Stab}_x := \{g \in G | gx = x\} \subseteq G$$

Note that Stab_x is a subgroup of G

Theorem 1.14 (Orbit-Stabiliser theorem)**.** *For every $x \in X$ the mapping*

$$G/\mathrm{Stab}_x \to \mathcal{O}_x, \quad [g] \mapsto gx$$

is bijective. In particular, we have $|G| = |\mathrm{Stab}_x||Gx|$

Proof. The mapping is well defined, since for any $h \in \mathrm{Stab}_x$ we have

$$[gh] \mapsto (gh)x = g(hx) = gx$$

Surjectivity is quite obvious as removing stabilisers doesn't affect the orbit. For injectivity, assume that for two equivalence classes $[g], [g']$ we have $gx = g'x$. After multiplying by g^{-1} we get htat

$$x = g^{-1}g'x \implies g^{-1}g' \in \mathrm{Stab}_x \implies [g] = [gg^{-1}g'] = [g']$$

We will see in the exercise classes that for any subgroup $H \subseteq G$ we have $|G/H| = \frac{|G|}{|H|}$, so the formula follows trivially. $\square$

The theorem has quite a few applications.

Example 1.15. Consider the 3-dimensional cube in $\mathbb{R}^3$ and let $O \subseteq O(3)$ be its symmetry group.
To calculate $|O|$, we let $X = \{(\pm 1, \pm 1, \pm 1)\}$ be the set of its corners $v_1, \ldots, v_8$ on which O acts.
Notice that every corner can be mapped to any other corner, so $\mathcal{O}_{v_1} = X$.
The stabiliser, which consists of all symmetries that fix one corner, must consist of rotations through the corner or mirror images, so $\mathrm{Stab}_{v_1} \cong D_3$. It quickly follows that

$$|O| = |Ov_1| \cdot |D_3| = 8 \cdot 6 = 48$$

Example 1.16. Crystalline salt grid consists of Na^+ and Cl^- ions, where Na^+ ions lie at

$$\Gamma_{fcc} := \{(i,j,k) | i+j+k \text{ even}\} \subseteq \mathbb{Z}^3$$

and the Cl^- at

$$\{(i,j,k) | i+j+k \text{ odd}\}$$

and we want to study the group

$$G_{NaCl} \subseteq IO(3) = O(3) \ltimes \mathbb{R}^3$$

We see immediately that

$$\Gamma_{fcc} \subseteq G_{NaCl}, \quad O \subseteq G_{NaCl} \implies O \ltimes \Gamma_{fcc} \subseteq G_{NaClNaCl}$$

Not only that, we can show that $G_{NaCl} = O \ltimes \Gamma_{fcc}$.

Proof. As injectivity is clear, we just need to show surjectivity. For this, let $X = \Gamma_{fcc}$ be the set of Na^+ ions in the grid and let $x = 0 \in X$ be the origin. By just translating the origin to any other point in the grid, it's easy to see that it's orbit $\mathcal{O}_x$ is Γ_{fcc}.
The stabilizer Stab_x is isomorphic to the symmetry group of the 3D cube O, because any element of the stabilizer must map the cube of neighboring atoms to itself.
By the orbit-stabilizer theorem we get an isomorphism $G_{NaCl}/O \xrightarrow{\cong} \Gamma_{fcc}$ so the following commutes

$$\begin{tikzcd}
G_{NaCl}/O \arrow[r, "\cong"] & \Gamma_{fcc} \\
O \ltimes \Gamma_{fc}/O \arrow[u] \arrow[ur, "\cong"] &
\end{tikzcd}$$

which is only possible if $O \ltimes \Gamma_{fcc} \to G_{NaCl}$ is surjective. $\square$

2 Representation Theory

2.1 Definition and Examples

Definition 2.1. A **representation** of a group G on a vector space $V \neq 0$ is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. We say that V is the **representation space** of the representation ρ. The dimension of the representation is the dimension of V.

So a representation ρ associates to each $g \in G$ an invertible linear map $\rho(g) : V \to V$ that is compatible with the group multiplication, i.e.

$$\rho(gh) = \rho(g)\rho(h)$$

Example 2.2.

- Every group G can act on any vector space V by $\rho(g) = \mathrm{id}_V$, which is the *trivial representation.*
- For $G = S_n$ and $V = \mathbb{C}^n$ with basis $e_1, \dots, e_n$, we define the representation as

$$\rho(\sigma)e_i = e_{\sigma(i)}$$

- For $G = O(3)$ and $V = C^\infty(\mathbb{R}^3)$, we map $g \mapsto \rho g$ defined by $(\rho(g)f)(x) = f(g^{-1}(x)$

Definition 2.3. The **regular representation** of a finite group G is the representation on $\mathbb{C}(G)$ of all functions $f : G \to \mathbb{C}$

$$(\rho_{\mathrm{reg}}(G)f)(h) = f(g^{-1}h), \quad f \in \mathbb{C}(G), g, h \in G$$

Any function $f : G \to \mathbb{C}$ can also be described as an infinite sum of "delta distributions". In other words, $\mathbb{C}(G)$ has a basis $\{\delta_g\}_{g \in G}$ and φ_{reg} is the representation such that $\rho_{\mathrm{reg}}(g)\delta_h = \delta_{gh}$

Definition 2.4. A homomorphism of representations $(\rho_1, V_1) \to (\rho_2, V_2)$ is a linear map $\varphi : V_1 \to V_2$ such that

$$\varphi \circ \rho_1(g) = \rho_2(g) \circ \varphi$$

we say that two representations are **equivalent** (or isomorhpic), if there exists a representation homomorphism $\varphi : V_1 \to V_2$ that is also a vector space isomorphism.
We denote the vector space of all representation homomorphisms as $\mathrm{Hom}_G((\rho_1, V_1), (\rho_2, V_2))$ or sometimes just $\mathrm{Hom}_G(V_1, V_2)$

Equivalently, $\mathrm{Hom}_G(V_1, V_2)$ consists of linear maps $\varphi : V_1 \to V_2$ such that the following diagram commutes for all $g \in G$.

$$\begin{tikzcd} V_1 \arrow[r, "\varphi"] \arrow[d, "\rho_1(g)"'] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "\varphi"] & V_2 \end{tikzcd}$$

Note that if $\rho : G \to \mathrm{GL}(V)$ is a representation, then ρ can be extended to any vector space $W \supseteq V$ by taking $\rho \oplus \mathrm{id}_{V^\perp}$. The resulting representation does not "make use" of the additional dimension in W and we could come to the conclusion that we only need to study the smallest dimensional representations of a group.

Definition 2.5. An **invariant subspace** of a representation (ρ, V) is a subspace $W \subseteq V$ such that $\rho(g)W \subseteq W$ for all $g \in G$. A representation is said to be **irreducible**, if the only invariant subspaces are V and $\{0\}$.
If $W \neq \{0\}$ an invariant subspace, then the representation induces a representation by restrction of $\rho(g)$ on W. We call this the **subrepresentation**

$$\rho|_W : G \to \mathrm{GL}(W), g \mapsto \rho(g)|_W$$

Definition 2.6. A representation (ρ, V) is called **completely reducible**, if there exist invariant subspaces $V_1, \dots, V_n$ such that $V = V_1 \oplus \ldots \oplus V_n$ such that the subprepresentations $(\rho|_{V_i}, V_i)$ are irreducible.

Such a decomposition is also called a decomposition into irreducible representations.
Altough most reducible representations we want to study are completely reducible, there are some counterexamples.

Example 2.7. The representation $\rho : \mathbb{Z} \to \mathrm{GL}(2, \mathbb{C})$ given by

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$$

is reducible, but *not* complete reducible. The only non-trival invariant subspace is $\mathrm{span} e_1$.

We can ask, *when* are representations completely reducible?

Lemma 2.8. *Set (ρ, V) be a finite dimensional representation of G such that to every subpsace $W \subseteq V$ there exists an invariant subspace $W' \subseteq V$ such that $V = W \oplus W'$. Then (ρ, V) is completely reducible.*

Proof. We use induction on $d = \dim V$: For $d = 1$, V is irreducible so $V = V$ is such a decomposition.
Let $\dim V = d + 1$. If V is reducible, there exists an invariant subspace $W \subseteq V$ with $1 \leq \dim W \leq d$ and an invariant subspace $W' \subseteq V$ such that $V = W \oplus W'$.
We now show that W fulfills the conditions for the lemma. Then by induction the proof follows.
So let $U \subseteq W$ be invariant. Then there exists an invariant subspae $U' \subseteq V$ such that $V = U \oplus U'$, then $W = U \oplus (W \cap U')$, since every $w \in W$ can be uniquely written as a sum $w = u + u'$ for $u \in U, u' \in U'$. But since $U \subseteq W$, $u' = w - u \in W$. $\square$

2.2 Examples

Consider the representation ρ of the group S_3 on $\mathbb{C}^3$ with basis e_1, e_2, e_3 with $\rho(\sigma)e_i = e_{\sigma(i)}$.
Tis representation has two invariant subspaces

$$V_1 = \mathrm{span} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad V_2 = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 0\}$$

and $V = V_1 \oplus V_2$, since

$$(1, 1, 1)^T, (1, -1, 0)^T, (0, 1, -1)^T$$

are a basis of $\mathbb{C}^3$. Furthermore, since $\dim V_1 = 1$ it is irreducible. We can also show that V_2 is irreducible, because if $W \subseteq V_2$ were a 1-dimensional subspace, we could write

$$W = \mathrm{span} w = \mathrm{span}(x_1, x_2, x_3)^T \neq \{0\}, \quad x_1 + x_2 + x_3 = 0$$

but since W should be $\rho(g)$ invariant, we would have that

$$\rho(\tau_{1,2})(x_1, x_2, x_3)^T = (x_2, x_1, x_3)^T \in W \implies (x_2, x_1, x_3)^T = \lambda(x_1, x_2, x_3)^T$$

doing this for the other permutations, it follows that $x_1 = x_2 = x_3$ which is a contradiction.
This decomposition can easily be generalized for $\mathbb{C}^n$

2.3 Unitary Representations

In this section, we want to show that all representations of finite groups are completely reducible.

Definition 2.9. A representation ρ on V with inner product is called **unitary**, if $\rho(g)$ is unitary for all $g \in G$, i.e.

$$(\rho(g)u, \rho(g)v) = (u, v) \forall g \in G, \forall u, v \in V$$

In particular $\rho(g)^* = \rho(g)^{-1}$.

Theorem 2.10. *Unitary finite dimensional representations are completely reducible.*

Proof. Let W be a invariant subspace of a unitary representation (ρ, V) and let $W^\perp$ be its orthogonal complement. Then $W^\perp$ is again invariant, since

$$v \in W^\perp, w \in W, g \in G \implies (\rho(g)v, w) = (\rho(g)^{-1}\rho(g)v, \rho(g)^{-1}w) = 0$$

which shows that $\rho(g)v \in W^\perp$. Therefore we can use the previous lemma on the decomposition $V = W \oplus W^\perp$ to show that V is completely reducible. □

We might ask when there exists such an inner product $(\cdot, \cdot)$ on V such that the representation is unitary. The answer is always.

Theorem 2.11. *Let (ρ, V) be a finite dimensional representaion of a finite group G. Then there exists an inner product $(\cdot, \cdot)$ on V such that (ρ, V) is unitary.*

Proof. Let $(\cdot, \cdot)_0$ be any inner product on V and construct the inner product $(\cdot, \cdot)$ defined as

$$(v, w) := \sum_{g \in G} (\rho(g)v, \rho(g)w)_0$$

which is an inner product on V as it is the sum of symmetric positve definite matrices.
To show that it is unitary, let $g \in G$. Then

$$(\rho(g)v, \rho(g)h) = \sum_{h \in G} (\rho(hg)v, \rho(hg)w) = \sum_{h'=hg \in G} (\rho(h')v), \rho(h')w) = (v, w)$$

□

Corollary 2.11.1. *Every finite dimensional representation of finite groups is completely reducible.*

2.4 Schur's Lemma

How can we use the result from the previous section in our everyday use?
Let (ρ_1, V_1) and (ρ_2, V_2) be complex finite dimensional representations of a group G and we want to study

$$\mathrm{Hom}_G(V_1, V_2) = \{\varphi : V_1 \to V_2 \text{ linear} | \varphi\rho_1(g) = \rho_2(g)\rho, \forall g \in G\}$$

Theorem 2.12 (Schur's Lemma)**.** *Let (ρ_1, V_1) and (ρ_2, V_2) be finite dimensional irreducible representations of a group G. Then*

(a) $\varphi \in \mathrm{Hom}_G(V_1, V_2) \implies \varphi = 0$ or φ is an isomorphism.

(b) $\varphi \in \mathrm{Hom}_G(V_1, V_1) \implies \varphi = \lambda\,\mathrm{id}_V$ for some $\lambda \in \mathbb{C}$

Proof. (a) We can show that $\operatorname{Ker}\varphi \subseteq V_1$ and $\operatorname{Im}\varphi \subseteq V_2$ are invariant subspaces since

$$v \in \operatorname{Ker}\varphi \implies \varphi(\rho_1(g)v) = \rho_2(g)\varphi(v) = 0(v) = 0 \implies \rho_1(g)v \in \operatorname{Ker}\varphi$$
$$\rho_2(g)\varphi(v') = \varphi(\rho_1(g)v') \implies \rho_2(g)\varphi(v') \in \operatorname{Im}\varphi$$

But since they are both invariant subspaces and V_1, V_2 are irreducible it must that either

$$\operatorname{Ker}\varphi = \{0\} \quad \text{or} \quad \operatorname{Ker}\varphi = V_1$$
$$\operatorname{Im}\varphi = \{0\} \quad \text{or} \quad \operatorname{Im}\varphi = V_2$$

so either $\varphi = 0$ or bijective. (Notice how we didn't use that V_i is complex.

(b) Let $\lambda \in \mathbb{C}$ be an eigenvalue of φ[1] Consider $\varphi - \lambda\,\mathrm{id}_V \in \mathrm{Hom}_G(V, V)$. It cannot have full rank, so using (a), it follows that $\varphi = \lambda\,\mathrm{id}_V$.

□

Corollary 2.12.1. *Every irreducible finite dimensional complex representation of an abelian group is one-dimensional.*

Proof. Let (ρ, V) irreducible. Since G is abelian, $\rho(g) \in \mathrm{Hom}_G(V, V)$ for all $g \in G$. By Schur's lemma, it follows that $\rho(g) = \lambda(g)\,\mathrm{id}_V$, so every subspace is invariant. But V is irreducible, so that is only possible, if V itself is one-dimensional. □

3 Representation theory of finite groups

In the following, let G be a finite group and let all representations be finite dimensional.

3.1 Orthogonality relation for matrix elements

Theorem 3.1. *Let $\rho : G \to \mathrm{GL}(V)$, $\rho' : G \to \mathrm{GL}(V')$ irreducible and unitary representations of G and let*

$$(\rho_{ij}(g))_{ij} \quad \textit{and} \quad (\rho'_{ij}g)_{ij}$$

be the matrices of $\rho(g)$ and $\rho'(g)$ with respect to any orthonormal basis of V, V'.

[1] By the fundamental theorem of algebra, such an eigenvalue always exists, as it is the root of a polynomial with complex coefficients)

(a) If ρ, ρ' are inequivalent, then

$$\frac{1}{|G|}\sum_{g\in G}\overline{\rho_{ij}(g)}\rho'_{kl}(g) = 0 \quad \forall i,j,k,l$$

(b) If they are equivalent, then

$$\frac{1}{|G|}\sum_{g\in G}\overline{\rho_{ij}(g)}\rho_{kl}(g) = \frac{1}{\dim V}\delta_{ik}\delta_{jl}$$

Proof. Let $\Phi : V \to V'$ be some linear map. Define

$$\Psi_G = \frac{1}{|G|}\sum_{g\in G}\rho'(g^{-1})\Phi\rho(g)$$

Then we get with a change of variables ($g \mapsto gh$)

$$\begin{aligned}\rho'(h)\Phi_G &= \frac{1}{|G|}\sum_{g\in G}\rho'((gh^{-1})^{-1})\Phi\rho(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\rho'(g^{-1})\Phi\rho(gh)\\ &= \Phi_G\rho(h)\end{aligned}$$

which shows that $\Phi_G \in \mathrm{Hom}_G(V,V')$. By Schur's lemma, if ρ, ρ' are inequivalent, then $\Phi_G = 0$ and so

$$\begin{aligned}(\Phi_G)_{jl} &= \frac{1}{|G|}\sum_{g\in G}\sum_{i,k}\overline{\rho'_{ij}(g)}\Phi_{ik}\rho_{kl}(g)\\ &= \sum_{i,k}\Phi_{ik}\left(\frac{1}{|G|}\sum_{g\in G}\overline{\rho'_{ij}(g)}\rho_{kl}(g)\right) = 0\end{aligned}$$

since this holds for all Φ, (a) follows.
On the other hand, if $\rho \sim \rho'$ are equivalent, then by Schur's Lemma $\Phi_G = c\,\mathrm{id}_V$ for some $c \in \mathbb{C}$. Then

$$\begin{aligned}c &= \frac{1}{\dim V}A\,\mathrm{tr}\,\Phi_G\\ \mathrm{tr}\,\Phi_G &= \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}\left(\rho(g^{-1})\Phi\rho(g)\right)\\ &= \mathrm{tr}\,\Phi\cdot\frac{|G|}{|G|} = \sum_i \Phi_{ii}\end{aligned}$$

By comparing the coefficients of the expressions

$$\sum_{i,k}\Phi_{ik}\left(\frac{1}{|G|}\sum_{g\in G}\overline{\rho_{ij}(g)}\rho_{kl}(g)\right) = \delta_{jl}\frac{1}{\dim V}\sum_i\Phi_{ii}$$

it follows that

$$\frac{1}{|G|}\sum_{g\in G}\overline{\rho_{ij}(g)}\rho_{kl}(g) = \delta_{jl}\delta_{kl}\frac{1}{\dim V\dim V}$$

□

3.2 Characters

Definition 3.2. The **character** of a representation $\rho : G \to \mathrm{GL}(V)$ is a function $\chi : G \to \mathbb{C}$ given by

$$\chi_\rho(g) = \mathrm{tr}(\rho(g)) = \sum_{i=1}^{\dim V} \rho_{ii}(g)$$

Theorem 3.3. *(a) $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$ for all $g, h \in G$.*

(b) If ρ, ρ' are equivalent, then $\chi_\rho = \chi_{\rho'}$

Proof. This follows directly from the fact that the trace is cyclic: $\mathrm{tr}(ABC) = \mathrm{tr}(C) = \mathrm{tr}(BCA)$. □

Definition 3.4. The conjugation classes of G are the sets of the form $\{hgh^{-1} | h \in G\}$, or equivalently, the orbits of the group action of G on itself by conjugation

$$G \to \mathrm{Bij}(G), \quad h \mapsto (g \mapsto hgh^{-1})$$

or equivalently, they are the equivalence classes of the equivalence relation $\sim$ given by

$$g \sim g' \iff \exists h \in G : g' = hgh^{-1}$$

Lemma 3.5. *The following are easy to show*

(a) $\chi_\rho(1) = \dim V = \dim \rho$

(b) $\chi_{\rho \oplus \rho'}(g) = \chi_\rho(g) + \chi_{\rho'}(g)$

(c) $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$

Proof. The first item is trivial. For (b), we look at the the direct sum as a block matrix

$$(\rho \oplus \rho)(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho'(g) \end{pmatrix}$$

For (c) we use the fact that in an orthonormal basis, we can write the inverse matrix as $\rho_{ij}(g^{-1}) = \overline{\rho_{ij}(g)}$. □

Example 3.6 (Character of the regular representation)**.** The regular representation on $V = \mathbb{C}(G)$ with the basis of delta functions δ_h is given by

$$\rho_{\mathrm{reg}}(g)\delta_h := \delta_{gh}$$

so to compute the character, need to see when $\rho_{\mathrm{reg}}(g)$ maps a basis vector δ_h to itself. That is only the case when $gh = h \iff g = 1$, so

$$\chi_{\mathrm{reg}}(g) = \mathrm{tr}(\rho_{\mathrm{reg}}(g)) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1 \end{cases}$$

3.3 Orthogonality relations for characters

We define the inner product on $V = \mathbb{C}(G)$ given by

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G}\overline{f_1(g)}f_2(g)$$

Theorem 3.7. *Let ρ, ρ' be irreducible representations of G with characters $\chi_\rho, \chi_{\rho'}$. Then*

$$\langle \chi_\rho, \chi_{\rho'}\rangle = \begin{cases} 1 & \text{if } \rho, \rho' \text{ are equivalent} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If they are not equivalent, then

$$\langle \chi_\rho, \chi_{\rho'}\rangle = \sum_{i,j}\underbrace{\langle \rho_{ii}, \rho'_{jj}\rangle}_{=0} = 0$$

and if they are equivalent, then they have the same character, so by the orthogonality relation for matrix elements, we have

$$\langle \chi_\rho, \chi_{\rho'}\rangle = \sum_{i,j}\langle \rho_{ii}, \rho'_{jj}\rangle = \sum_{i,j}\delta_{ij}\delta_{ij}\frac{1}{\dim V} = \sum_i \frac{1}{\dim V} = 1$$

□

Corollary 3.7.1. *If $\rho = \rho_1 \oplus \ldots \oplus \rho_n$ is a decomposition into irreducible representation, then the number of ρ_j that are equivalent to the irreducible representation σ are given by*

$$\#\text{ of } j, \text{ such that } \rho_j = \sigma =: n_\sigma = \langle \chi_\rho, \chi_\sigma\rangle = \langle \chi_\sigma, \chi_\rho\rangle$$

Proof. This follows trivially from the fact that

$$\chi_\rho = \chi_{\rho_1} + \ldots + \chi_{\rho_n}$$

so by taking the scalar product with χ_σ on both sides, we can use the previous theorem to immediately get the result. □

Corollary 3.7.2. *A representation ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho\rangle = 1$*

Proof. This also follows trivially from the theorem. If we had a decomposition $\rho = \rho_1 \oplus \ldots \oplus \rho_n$, then we would have

$$\langle \chi_\rho, \chi_\rho\rangle = \sum_{i,j}\langle \chi_{\rho_i}, \chi_{\rho_j}\rangle \geq n$$

□

Now if we are given a group G, how many irreducible representation are there? A trick is to look at the regular representation.

3.4 Decomposition of the regular representation

The regular representation contains a lot of information about other irreducible representations.

Theorem 3.8. *Every irreducible representation σ of G appreas in $\mathbb{C}(G)$ with multiplicity* $\dim \sigma$.

Proof. From the first corollary before, we can use the fact that χ_{reg} is $|G|$ on the identity, so

$$n_\sigma = \langle \chi_\sigma, \chi_{\text{reg}} \rangle = |G| \chi_\sigma(1) \frac{1}{|G|} = \chi_\sigma(1) = \dim \sigma$$

□

Corollary 3.8.1. *A finite group only has finitely many equivalence classes of irreducible representations. Moreover, if $\rho_1, \ldots, \rho_k$ is a list of equivalent irreducible representations,then*

$$|G| = \sum_{j=k} (\dim \rho_j)^2$$

Proof. It follows from $\dim(\mathbb{C}(G)) = |G|$ because every time we have a $\dim \sigma$-dimensional representation it occurs exactly $n_\sigma = \dim \sigma$ times. □

Corollary 3.8.2 (Peter Weyl theorem for finite groups). *Let $\rho_1, \ldots, \rho_k$ be a list of the irreducible inequivalent representations of G.*
Then the matrix elements $\rho_{\alpha,ij}$ for $1 \leq \alpha \leq k$ with respect to an orthogonoal basis from an orthogonal basis of $\mathbb{C}(G)$

Proof. By the orthogonality relation for matrix elements, we konw that the $\rho_{\alpha,ij}$ already form an orthogonal family. This family is also a basis, since there are exactly $\sum_\alpha (\dim \rho_\alpha)^2 = |G| = \dim \mathbb{C}(G)$ many members in the family. □

Definition 3.9. A function $f : G \to \mathbb{C}$ is called a **class function**, if

$$f(ghg^{-1}) = f(h) \quad \forall h, g \in G$$

For example the characers of representations are class functions. Also, the class functions are a subspace of $\mathbb{C}(G)$ and as such, a Hilbert space.

Corollary 3.9.1. *The characters $\chi_1, \ldots, \chi_k$ form an ONB of the Hilbert space of class functions.*

Proof. We just saw that the $\chi_1, \ldots, \chi_k$ form an orthogonal family and now we have to show that they also span the space of class functions.
Let $f : G \to \mathbb{C}$ be a class function, then we can write

$$f(g) = \frac{1}{|G|} \sum_{h \in G} f(hgh^{-1})$$

but the matrix elements also form an orthonormal basis of all functions,so also of the space of class functions, so

$$f = \sum_{\alpha,i,j} \lambda_{\alpha,ij} \rho_{\alpha,ij}$$

which, when inserted into the formulation above, we get

$$\begin{aligned} f(g) &= \sum_{\alpha,i,j} \lambda_{\alpha,ij} \frac{1}{|G|} \sum_{h\in G} \rho_{\alpha,ij}(hgh^{-1}) \\ &= \sum_{\alpha,i,j} \lambda_{\alpha,ij} \frac{1}{|G|} \sum_{h\in G} \sum_{k,l} \rho_{\alpha,i,k}(h)\rho_{\alpha,k,l}(g)\overline{\rho_{\alpha,j,l}(h)} \\ &= \sum_{\alpha,i,j} \lambda_{\alpha,ij} \frac{1}{\dim \rho_\alpha} \sum_{k,l} \rho_{\alpha,k,l}(g)\delta_{ij}\delta_{kl} \end{aligned}$$

but the sum $\sum_{k,l} \delta_{kl}$ is exactly the trace or character $\chi_\alpha(g)$, so

$$\begin{aligned} f(g) &= \sum_{\alpha,i,j} \lambda_{\alpha,ij}\chi_\alpha(g)\delta_{ij} \frac{1}{\dim \rho_\alpha} \\ &= \sum_{\alpha,i} \lambda_{\alpha,ii}\chi_\alpha(g) \frac{1}{\dim \rho_\alpha} \end{aligned}$$

which is a linear combination of the $\chi_\alpha(g)$. □

Corollary 3.9.2. *The number of equivalence classes of irreducible representations is equal to the number of conjugacy classes in G.*

Proof. The number of conjugacy classes is the dimension of the space of class functions, which as we just proved is the number of characters χ_α which equals the number of equivalency classes of irreducible representations. □

3.5 The character table

The character table is a table listing all irreducible representations and their characters on the conjugacy classes.

G	$[1]$	c	$\dots$
χ_1	$\ddots$		$\ddots$
χ_j		$\chi_j(c)$	
χ_k	$\ddots$		$\ddots$

Table 1: The character table has in the left column all irreducible representations and on the top row the conjugacy classes of the group. The entries is the character evaluated at a (representant) of the conjucacy class.

Example 3.10. The group $G = S_3$ has 6 elements, which can be written in terms of the cycle $\tau = (123)$ and the permutation $\sigma = (12)$

$$G = \{1, \sigma, \tau, \tau\sigma\tau^{-1}, \tau^2 st^{-2}, sts^{-1}\}$$

This group has three conjugacy classses:

$$[1], [s] = \{s, tst^{-1}, t^2st^{-2}\}, [t] = \{t, sts^{-1-1}\}$$

the irreducible representations are

$$\rho_1 \text{ trivial}, \rho_2 \text{ on } \mathbb{C}^2, \quad \rho_\epsilon \text{ on } V = \mathbb{C}, \rho_\epsilon(\sigma) = \operatorname{sgn}(\sigma)$$

the character table is then To fill out the character table, we can use weighted orthogonality of the caracter.

$6S_3$	$[1]$	$3[s]$	$2[t]$
χ_1	1	1	1
χ_2	2	a	b
χ_ϵ	1	-1	$+1$

Table 2: The character table of S_3. The numbers show the number of elements in the conjugacy class. Non-trivial entries can be found using the orthogonalit condition

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)} \chi_2(g) \stackrel{!}{=} 0$$

which means that we need

$$\langle \chi_1, \chi_2 \rangle = 0 \implies 2 + 3a + 2b = 0, \quad \text{and} \quad \langle \chi_2, \chi_\epsilon \rangle = 0 \implies 2 - 3a + 2b = 0$$

which has the solution $a = 0, b = -1$.
Note that if a matrix has orthonormal rows, then its transpose also has orthonormal rows, which are exactly the columns of the matrix.

$$\left(\sqrt{\frac{|C_j|}{|G|}} \chi_i(C_j) \right)_{ij} \in O(n)$$

This shows that the orthonormality of the characters (with respect to the weighted inner product) corresponds to the orthonormality (with respect to the "unweighted" inner product) of the conjugacy classes, so

$$\sum_{j=1}^{k} \overline{\chi_j(C_\alpha)} \chi_j(C_\beta) = \frac{|G|}{|C_\alpha|} \delta_{\alpha,\beta}$$

The orthonormality gives us many tricks to compute the character table.

- $|G| = \sum_{j=1}^{k} (\dim \rho_j)^2$ so if the order of G is rather small, then this gives us some restrictions on the dimensions of the representation.
- Orthogonality of rows and columns (with weights) can be used to find missing entries
- The existence of trivial representations (and if possible, the signum representation) lets us fill out the initial rows. Also note that the first column contains the dimension of the representations.

3.6 The canonical decomposition of a representation

Let $\rho_i : G \to \mathrm{GL}(V_i)$ be a list of inequivalent irreducible representations of G.
Now let ρ be some other representation of G on V. If we decompose $V = U_1 \oplus \ldots \oplus U_n$ into irreducible representations, then define W_i as the direct sum of all U_j that are equivalent to ρ_i (which means that $\rho|U_j$ is equivalent to ρ_i).
Then we write $V = W_1 \oplus + \ldots \oplus W_k$.

Theorem 3.11. *The decomposition $V = W_1 \oplus \ldots \oplus W_k$ is invariant under the choice of the decomposition $V = U_1 \oplus \ldots \oplus U_n$ and the projection*

$$p_i : V \to W_i, \quad w_1 \oplus \ldots \oplus w_k \mapsto w_i$$

is given by

$$p_i(v) = \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho(g) v$$

Proof. Since $W_i = \operatorname{Im} p_i$ and the $p_i$are independent of the decomposition into the U_i, we only need to show the second statement. We can show that $p_i \in \mathrm{Hom}_G(V, V)$ commutes with the group action, because

$$\begin{aligned} p_i(\rho(h)v) &= \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho(gh) v \\ &= \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho(hg) v \\ &= \rho(h)(p_i(v)) \end{aligned}$$

For$v \in U_j$ we have $\rho(g)v \in U_j$, so $p_i v \in U_j$. Now we can use Schur's Lemma for $p_i|_{U_j} : U_j \to U_j$. The lemma then says that

$$p_i|_{U_j} = c_{ij} 1_{U_j} \implies \mathrm{tr}(p_i|_{U_j}) = c_{ij} \dim U_j$$

and so

$$\frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \underbrace{\mathrm{tr}((\rho(g)|_{U_j})}_{=\chi_{\rho|_{U_j}}(g) =: \chi(g)} = \dim V_i \langle \chi_i, \chi \rangle = \begin{cases} 1 & \text{if } \rho|_{U_j}, \rho_i \text{ are equivalent} \\ 0 & \text{otherwise} \end{cases}$$

and so, we can find the coefficient

$$c_{ij} = \begin{cases} 1 & \text{if } \rho|_{U_j}, \rho_i \text{ are equivalent} \\ 0 & \text{otherwise} \end{cases}$$

so the mapping

$$p_i(v) = \begin{cases} v & \text{if } v \in W_i \\ 0 & v \in W_j, j \neq i \end{cases}$$

does indeed define the projector.

□

We call the decomposition $V = W_1 \oplus \ldots \oplus W_k$ the **canonical decomposition** and the W_i the **isotypical compents**.
So given a representation χ, we can write V as the direct sum

$$V = \bigoplus_{i=1}^{k} V_i \otimes \mathbb{C}^{n_i}, \quad \text{for} \quad n_i = \langle \chi_i, \chi \rangle$$

3.7 The dihedral group D_n

Recall that the dihedral group was generated by rotations and reflections:

$$D_n = \{R^a S^b | a = 0, \ldots, n-1, b = 0, 1\}$$

Every representation is then uniquely determined by $\rho(R) =: \overline{R}$ and $\rho(S) =: \overline{S} \in \mathrm{GL}(V)$. These obviously cannot be chosen freely, but must follow the following relations

$$R^n = S^2 = 1, \quad SR = R^{-1}S \implies \overline{R}^n = \overline{S}^2 = 1, \quad \overline{SR} = \overline{R}^{-1}\overline{S}$$

and it turns out that any other relation can be written using the above relations because

$$R^a S^b R^{a'} S^{b'} = R^{a-a'-ba'} S^{b+b'}$$

so every choice of $\overline{R}, \overline{S}$ defines a well-defined representation of D_n.
Now if we want to find all irreducible representations of a group, we start with the one-dimensional ones. For $V = \mathbb{C}$, the elements $\overline{R}, \overline{S}$ can be viewed as elements in $\mathbb{C} \setminus 0$. We immediately get that

$$\overline{S}^2 = 1 \implies \overline{S} = \pm 1 \implies \overline{SR} = \overline{R}^{-1}\overline{S} \iff \overline{R}^2 = 1$$

which gives us two cases. For n odd, we must have $\overline{R}^n = 1 \implies \overline{R} = +1$ and for n even, both $\overline{R} = \pm 1$ are possible.
So for n odd, tere are two irreducible 1-dimensional representations$\rho_{\pm}$, and for n even, there are four 1-dimensional irreducible representations $\rho_{\pm,\pm}$.
Now to find two-dimensional representations ($V = \mathbb{C}^2$), let $v \in V$ be an eigenvalue of $\overline{R}$ to the eigenvalue ϵ ($\overline{R}v = \epsilon v$. The relatios described above now give us

$$\epsilon \overline{S} v = \overline{SR} v = \overline{R}^{-1} \overline{S} v \iff \overline{RS} v = \frac{1}{\epsilon} \overline{v}$$

which means $\overline{S}v$ is an eigenvector of $\overline{R}$ to the eigenvalue $\frac{1}{\epsilon}$.
Now, $v, \overline{S}v$ would be linearly dependent, then $\mathrm{span}(v)$ would be an invariant subspace, so they must be independent. That means that they form a basis of $\mathbb{C}^2$ and we can write our matrices with respect to this basis:

$$\overline{R} = \begin{pmatrix} \epsilon & 0 \\ 0 & \frac{1}{\epsilon} \end{pmatrix}, \quad \text{and} \quad \overline{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now, the condition $\overline{R}^n = 1$ can only be satisfied if

$$\epsilon = e^{\frac{2\pi i}{n} j}, \quad \text{for} \quad j \in \mathbb{Z}$$

which gives us representations ρ_j for $j = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$ and we claim that they are all possible irreducible representations up to equivalence.
We prove this by looking at the characters

$$\chi_j(g) = \operatorname{tr}(\rho_j(g)) \implies \chi_j(R^{a)} = \epsilon^a + \epsilon^{-a} = e^{\frac{2\pi i}{n} j} + e^{-\frac{2\pi i}{n} j} = 2\cos\left(\frac{2\pi j}{n} a\right)$$
$$\chi_j(R^s S) = 0$$

and then use the orthogonality conditions to find the other ones:

$$\begin{aligned}\langle \chi_i, \chi_j \rangle &= \frac{1}{|G|} \sum_{a=0}^{n-1} \overline{\left(\epsilon_i^a + \epsilon_i^{-a}\right)} \left(\epsilon_j^a + \epsilon_j^{-a}\right) \\ &= \frac{1}{2n} \sum_{a=0}^{n-} 1 \left((\epsilon_i \epsilon_j)^a + \left(\epsilon_i \epsilon_j^{-1}\right)^a + \left(\epsilon_i^{-1} \epsilon_j\right)^a + \left(\epsilon_i^{-1} \epsilon_j^{-1}\right)^a \right)\end{aligned}$$

so for ξ the n-th root of unity, we get

$$\sum_{a=0}^{n-1} \xi^a = \begin{cases} \frac{1-\xi^n}{1-\chi} = 0 & \xi \neq 0 \\ n & \xi = 1 \end{cases}$$

which means that since the first term $(\epsilon_i \epsilon_j)^a$ is not 1, it drops out of the sum, aswell as the last part $\left(\epsilon_i^{-1} \epsilon_j^{-1}\right)^a$. So the only remaining terms are

$$\langle \chi_i, \chi_j \rangle = \frac{1}{2n} \left(n\delta_{ij} + n\delta_{ij}\right) = \delta_i j$$

which shows that the representations ρ_i, ρ_j are irreducible and not equivalent for $i \neq j$.
Now we need to shwow that they form all possible irreducible representations up to equivalency. To do this, we use the formula

$$|G| = \sum_{j=1}^{k} (\dim \rho_j)^2$$

to check that there aren't any more. Indeed, for n even we have

$$1^2 + 1^2 + 1^2 + 1^2 + 2^2 \frac{n-2}{2} = 2n = |G|$$

and for n odd:

$$1^2 + 1^2 + 2^2 \frac{n-1}{2} = 2n = |G|$$

which checks out.

3.8 Compact groups

We can generalize some results for finite groups to when the group is compact. The representation theory for finite groups was made possible by taking averages of the group by dividing by $\frac{1}{|G|}$ when summing over the entire group.

It turns out what we can generalize the integral over $\mathbb{R}$ to an integral over compact (or even locally compact groups)

$$\int_G f(g)d\mu \quad \text{where} \quad d\mu \text{ is the } \textbf{Haar measure}$$

which is normed $\int_G 1d\mu = 1$ and has the invariance property:

$$\int_G f(gh)d\mu = \int_G f(g)d\mu, \quad \forall h \in G$$

the construction of the Haar measure takes some measure theory and is out of the scope of the lecture. Applying the results of the previous chapters for compact groups, we the following results:
Orthogonality of matrix elements and characters given by the inner product on $L^2(G)$

$$\langle f_1, f_2 \rangle := \int_G \overline{f_1(g)} f_2(g)d\mu$$

One property that we lose is that we can't use $\mathbb{C}(G)$ for the regular representation, but have to use $L^2(G)$. The decomposition of the regular distribution gives us the (actual) Peter-Weyl theorem.
One example of an infinite, but compact topological group is $SO(n)$ where we use the term $\frac{1}{\mathrm{vol}(SO(n))} \int_{SO(n)} d\mu$ with the standard integral over $SO(n) \subset \mathbb{R}^{n^2}$ all the time.