Definition Taylor-Series

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Its **Taylor-Series** around the point $z_0 \in \mathbb{C}$ is the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n \in \mathbb{C}$$

which converges on the disk $D_{z_0}(R)$, where R is the radius of convergence and is given by

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

0.1 Convergence theorems

Definition

We say that f_n converges locally uniform to a function f in U, if

$$\forall z \in U \exists r > 0 \text{ s.t. } D_{z_0}(r) \subseteq U \text{ and } f_n \to f \text{ uniformly}$$

For $U \subseteq \mathbb{C}$ open and $f : U \to \mathbb{C}$ holomorphic, $z_0 \in U$ with some $D_{z_0}(r) \subseteq U$ we say that f admits a Taylor expansion, if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in D_{z_0}(r) \tag{1}$$

If f is smooth and we chose

$$a_n = \frac{f^{(n)(z)}}{n!}$$

, then it converges aboslutely and locally uniform on $D_{z_0}(r)$. These coefficients are also unique, so if b_n satisfy **??**, then $b_n = a_n$.

Definition Möbius Transformation

Let $a, b, c, d, f \in \mathbb{C}$, such that $ad - bc \neq 0$ and $c \neq 0$. The **Möbius Transformation** is the function

$$f : \Omega = \mathbb{C} \setminus \{-\frac{d}{c}\} \to \mathbb{C}, \quad z \mapsto \frac{az + b}{cz + d}$$

We can extend f to the one-point compactification $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ with

$$f(-\frac{d}{c}) = \infty, f(\infty) = \frac{a}{c}, c \neq 0, f(\infty) = \infty, c = 0$$

Consider the matrix A and the map φ

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}), \quad \varphi_A = A$$

f is ijective and holomorphic form the Riemann-sphere into itself and the function

$$g = \frac{dz - b}{-cz + a}$$

Geometrically, this is the stereographic projection of $\hat{\mathbb{C}}$

Theorem Cauchy- Integral Formula

For

Higher derivatives

Let $f : \Delta \to \mathbb{C}$ be holomorphic. For $k \geq 0$ and $\gamma \subseteq \Delta$ a closed path. Then for all $z \notin \gamma$

$$\boxed{n(\gamma, z) f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{k+1}} d\xi}$$

As a corollary, we get the Cauchy estimate for a bounded function $f(z) \leq M$ we have

$$|f^{(k)}(z)| \leq \frac{k! M r}{(R - |z - a|^{k+1}} \leq \frac{k! M}{r^k}$$

As a corollary

Liouville Theorem

Every bounded entire function is constant.

Definition Convergence of power series

We say that the series $\sum_{n \geq 0} f_n(z)$ converges pointwise (uniformly) to the function $f(z)$ if the sequence $s_n(z) = \sum_{k=0}^n f_k(z)$ converges pointwise (uniformly) to f.
The series $\sum_{n=0}^\infty f_n(z)$ is **absolutely convergent**, if $\sum_{n=0}^\infty |f_n(z)|$ is convergent.
The series $\sum_{n=0}^\infty f_n(z)$ is **absolutely uniformly** convergent, if

Remark: There is no relation between uniform and absolute convergence, i.e. there exist functions which converge uniformly, but not absolutely and vice versa.

Proposition

The sequence $(f_n)_{n \in \mathbb{N}}$ converges uniformly ond D, if and only if $(f_n)_{n \in \mathbb{N}}$ is cauchy.
If a sequence of continuous functions converges *uniformly* to a function f, then f is also continuous.

Weierstrass M-test

If for all $n \geq 0$ there exists an $M_n \geq 0$ such that

$$|f_n(z)| \leq M_n, \forall z \in D \quad \text{and} \quad \sum_{n\geq 0} M_n \text{ converges}$$

Then $\sum_{n\in\mathbb{N}} f_n$ is uniformly and absolutely convergent.

Cauchy condition

The sum $\sum_{n\geq 0} f_n$ converges uniformly on D, if and only if

$$\forall \varepsilon > 0 \varepsilon \nu_\varepsilon > 0 \text{ such that } |\sum_{k=m+1}^{n} f_k(z)| < \varepsilon, \forall z \in D, n > m \geq \nu_\varepsilon$$

Definition Primitive

Let $F, f : U \to \mathbb{C}$ be functions. F is called a **primitive** of f, if F is holomorphic in U and $F'(z) = f(z), \forall z \in U$.

Theorem

For f continuous on a region Ω, the following are equivalent

1. f has ha primitive $F \in C^1(\Omega, \mathbb{C})$.
2. The path integral $\int_\gamma f(z)dz$ of any piece wise regular path $\gamma \in C^1_{\text{pw}}([0,1]), \Omega)$ depends only on its endpoints, $\gamma(0), \gamma(1)$.
3. The closed path integral is zero, for any piece wise regular path $\gamma \in C^1_{\text{pw}}([0,1]), \Omega)$.

Theorem Cauchy's Theorem on rectangle

Let $U \subseteq \mathbb{C}$ open, $z_0 \in U$, $f : U \to \mathbb{C}$ continuous and holomorphic on $U \setminus \{z_0\}$. Then for all rectangles $R \subseteq \overline{R} \subseteq U$ the contour integral vanishes. $\int_{\partial R} f(z)dz = 0$

If $f : D \to \mathbb{C}$ for D an open disk, $z_0 \in D$, f continuous on D, holomorphic on $D \setminus \{z_0\}$. Then for every path γ piecewise regular on D the closed path is zero: $\int_\gamma f(z)dz = 0$.

Trigonometric Functions

We define the complex Trigonometric functions $\sin, \cos : \mathbb{C} \to \mathbb{C}$ as follows

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

Notice that the bounds $|\cos(z)| \leq 1, |\sin(z)| \leq 1$ do <u>not</u> hold on the complex plane!