1 Complex Numbers

The basic idea is that we start with the real numbers $\mathbb{R}$ and we "add" the number i such that $i^2 = -1$.

We define the complex numbers to be

$$\mathbb{C} := \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}\}$$

We define addition between two complex numbers as follows

$$(\alpha + i\beta) + (\gamma + i\delta) := (\alpha + \gamma) + i(\beta + \gamma)$$

And Multiplication:

$$(\alpha + i\beta) \cdot (\gamma + i\delta) := (\alpha \cdot \gamma - \beta\delta) + i(\alpha\delta + \beta\gamma)$$

For division, we just have to find the multiplicative inverse for any $z = (\alpha + i\beta) \neq 0$. The condition for its inverse $(x + iy)$ must then be

$$(\alpha + i\beta)(x + iy) = 1 \implies \alpha x - \beta y = 1, \beta x + \alpha y = 0$$

Which just be written as a linear transformation

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To solve this, we have to find the (non-zero) determinant $\det(A) = \alpha^2 + \beta^2$ and get

$$\frac{1}{(\alpha + i\beta)} := \frac{(\alpha - i\beta)}{\alpha^2 + \beta^2}$$

One of the main reasons to introduce the complex numbers was to find the square root of -1, so can we find square roots of any complex number?

In particular, we are looking for a solution to $(x + iy)^2 = (x^2 - y^2) + i(2xy) = \alpha + i\beta$.
Squaring the factors we get

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \alpha^2 + \beta^2$$

In particular, we have

$$\begin{aligned} x^2 + y^2 = \sqrt{\alpha^2 + \beta^2}, \quad x^2 - y^2 = \alpha \\ \implies x^2 = \frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + \beta^2}\right), y^2 = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + \beta^2}\right) \end{aligned}$$

Which gives us the solutions

$$x = \pm\sqrt{\frac{1}{2}\left(\alpha + \sqrt{\alpha^2 + \beta^2}\right)}, \quad y = \pm\sqrt{\frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + \beta^2}\right)}$$

This might look like we have four solutions, but as we have $2xy = \beta$, if $\beta \neq 0$, we must have $\operatorname{sign}(xy) = \operatorname{sign}(\beta)$. So only two of them are valid.

If β is zero, then either x or y must be zero. In the case where $\alpha \geq 0$, then $x = \pm\sqrt{\alpha}$ and $y = 0$. And if $\alpha < 0$, then $x = 0$ and $y = \pm\sqrt{-\alpha}$
So this means that the root function wants to take on two branches, so we need to make a choice when deciding what value the square root of $(\alpha + i\beta)$ is.

A very useful operation is the complex conjugate, which visually flips the complex plane along the real axis:

$$\bar{\cdot} : \mathbb{C} \to \mathbb{C}, \quad (\alpha + i\beta) \mapsto (\alpha - i\beta)$$

which has the following properties:

$$\overline{a+b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{a}\overline{b}$$

Exercise: Try to show that $a\overline{a} \geq 0 \in \mathbb{R}$. From the other calculation rules of the complex conjugate, we will also have that $\overline{\left(\frac{a}{b}\right)} = \frac{\overline{a}}{\overline{b}}$. Further, if we have $x = \frac{a}{b}$ it is clear that $\overline{a} = \overline{x}\overline{b}$.
Moreover, we can generalize this for any rational function $p = \frac{p_1}{p_2}$ for p_1, p_2 polynomial.

If we consider a complex polynomial

$$p(z) = C_0 z^n + C_1 z^{n-1} + \ldots + C_n$$

if a is a root of p, then $\overline{a}$ is a root of

$$\overline{p}(z) = \overline{C_0} z^n + \overline{C_1} z^{n-1} + \ldots + \overline{C_n}$$

And if we only look at polynomials, with real coefficients, it means that the complex roots come in conjugate pairs.
Next we will define **modulus** of a complex number z. For $z = \alpha + i\beta$ we have that

$$z\overline{z} = \alpha^2 + \beta^2 \in \mathbb{R} = |z|^2$$

The complex modulus has the following properties

- $|ab| = |a||b|$
- If $b \neq 0$, then $|\frac{a}{b}| = \frac{|a|}{|b|}$
- $|a+b|^2 = |a|^2 + |b|^2 + 2\mathrm{Re}(a\overline{b})$

The proof follows from the commutativity of a and b

The properties above also give rise to the inequalities for the real and imaginary part

$$\begin{aligned} -|a| \leq \mathrm{Re}(a) && \leq |a| \\ -|a| \leq \mathrm{Im}(a) && \leq |a| \end{aligned}$$

Aswell as the triangle inequality

$$|a+b|^2 \leq |a|^2 + |b|^2 + 2|a\overline{b}| = (|a| + |b|)^2 \implies |a+b| \leq |a| + |b|$$

Following some of the steps taken in Linear Algebra, we will get the *Cauchy-Schwarz* inequality:

$$|\sum_{i=1}^{n} a_i b_i|^2 \leq \left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right)$$
$$|\sum_{i=1}^{n} a_i \overline{b_i}|^2 \leq \left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right)$$

Where the proof is more or less the same as for the real case.

1.1 Polar Representation

For many of the things we previously defined, we have some nice geometric intuition. If we visualize the complex numbers $\mathbb{C}$ as the complex plane, then addition will look like vector addition, the complex modulus will be the "length" of the vector. Conjugation is nothing but flipping the vector along the real axis etc.

But one of our operations we defined doesn't have a straightfordward interpretation. If we look at the definition $(\alpha + i\beta) \cdot (\gamma + i\delta) := (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma)$ it is not necessarily clear what happens.

If we take a look at the multiplicativity of the complex modulus, it is suggesting that instead of looking at the real and imaginary parts of the complex numbers, we should look at its complex modulus. Of course, that isn't enough to uniquely determine a complex number.
When looking at all numbers with the same modulus, they form a circle. And in order to seperate them, we will look at their *angle* along the real axis.

From this we get another representation of the complex plane, namely the **Polar coordinates** of the complex numbers.
If we know the modulus and the angle of a, we can get the real and imaginary parts by taking the cosine and sine of the angle to get

$$a = r(\cos\varphi + i\sin\varphi), \quad \text{for} \quad r := |a|, \varphi := \arg(a)$$

, we call $\varphi(a)$ the **argument** of a. It should be noted that the argument is not unique. If we want to define a single-valued function $\arg : \mathbb{C} \to I \subseteq \mathbb{R}$, then have to restrict the argument inside for example the interval $(-\pi, \pi]$.

Now with the new representation of the complex numbers, let's take a look at what muliplication does from this new angle.
For $a_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $a_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$, their product will be

$$a_1 a_2 = r_1 r_2 \left(\left(\underbrace{\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2}_{=\cos(\varphi_1+\varphi_2)}\right) + i\left(\underbrace{\cos\varphi_1\sin\varphi_2 + \cos\varphi_2\sin\varphi_1}_{=\sin(\varphi_1+\varphi_2)}\right)\right)$$

Where we used the trigonometric addition formulas. Here we can see that multiplication adds the arguments of the numbers and mutiplies their moduli. (i.e. $\arg(a_1 a_2) = \arg(a_1) + \arg(a_2)$)
One should be careful, that for $a = 0$, the modulus isn't well defined and that the argument will always

work $\mod 2\pi$
Next let's look at powers of a. We will see that for any $n \in \mathbb{N}$ we have

$$a^n = r^n(\cos(n\varphi) + i\sin(n\varphi)), \quad a^{-1} = r^{-1}(\cos(-\varphi) + i\sin(-\varphi))$$

The case for $n = -1$ allows the above to be true for $n \in \mathbb{Z}$. So what for $n \in \mathbb{Q}$? For $n = \frac{1}{m}$, this is the same as solving the equation $z^m = a$, which will have multiple solutions:

$$z = \sqrt[m]{r}\left(\cos\left(\frac{\varphi}{m} + \frac{2\pi k}{m}\right) + i\sin\left(\frac{\varphi}{m} + \frac{2\pi k}{m}\right)\right), \quad k = 0, \ldots, n-1$$

In the case where $a = 1$, we call these solutions the n **roots of unity**:

$$z = \cos\left(\frac{2\pi k}{m}\right) + i\sin\left(\frac{2\pi k}{m}\right), \quad k = 0, \ldots, n-1$$

If we were to plot these points on the complex plane, we would see that form the regular n-sided polygon centered around the origin with a vertex at 1.

1.2 Riemann Sphere

The Idea is to add a point called ∞ to $\mathbb{C}$. It turns out that we can identify the new space with the two-dimensional unit sphere $S^2 \subset \mathbb{R}^3$.
Indeed, there exists a one-to-one mapping $\Phi : S^2 \to \mathbb{C}$ given by

$$\Phi(x) = \begin{cases} \frac{x_1 + ix_2}{1 - x_3}, & \text{for} \quad x \neq (0,0,1), \\ \infty, & \text{for} \quad x = (0,0,1) \end{cases}$$

The mental image for this is that you place the unit sphere around the origin of the complex plane and for any point on the sphere, you take a line from the northpole to the point and see where the line intersects the complex plane.

2 Functions

Now let's study functions $f : \mathbb{C} \to \mathbb{C}$. A key part in real analysis was the **limit**. We define the complex limit similar to the real limit, i.e. we write $\lim_{x\to a} f(x) = A$, if

$$\forall \varepsilon > 0\ \exists \delta > 0 :\ 0 < |x - a| < \delta \implies |f(x) - A| < \varepsilon$$

We then get the following properties

- $\lim_{x\to a} \overline{f(x)} = \overline{A}$
- $\lim_{x\to a} \mathrm{Re} f(x) = \mathrm{Re}(A)$
- $\lim_{x\to a} \mathrm{Im} f(x) = \mathrm{Im}(A)$
- f is continuous at a if $\lim_{x\to a} f(x) = f(a)$

Now, the derivative will behave differently than in the real case. We define the derivative as usual:

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

but they key difference is when this limit exists. In the real case, h can approach 0 in only two ways, from the left and from the right.
However, in the complex case, h can approach 0 from any direction!

One scenario where this difference is noticable is a function, where $f(a) \in \mathbb{R}, \forall a \in \mathbb{C}$. In the real case, we are dividing real numbers $f(a+h) - f(a)$ with a real number h and get a real number. But in the complex case, h might be complex so we can get a purely imaginary result.
So for the two ways to approach 0 to be equal, we must have in this case, that the derivate $f'(a)$ be equal to 0

2.1 Cauchy-Riemann Equations

Now, more generally for functions that also have an imaginary component, we can rephrase the condition for the existence of the derivtive in terms of the derivatives of its components to recover some partial differential equations, namely the Cauchy-Riemann Equations.

Definition Holomorphic function

Let $f : D \to \mathbb{C}$, be a function, D open. If f has a derivative in every point $x \in D$, we say that f is **holmorphic** or **analytic**.

Moreover, if f, g are holmorphic, then $f + g, f \cdot g$ and f/g, for $g(x) \neq 0$ holomorphic.
And if $f'(a)$ exists, then

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \Leftrightarrow \lim_{h \to 0} f(z+h) - f(z) = \lim_{h \to 0} h f'(z) = 0$$

We consider two special ways in which h can go to 0. Once from the "right side" and once from the "left side".

We will write f as a sum of its real and complex components as

$$f(z) = u(z) + iv(z), \quad \text{for} \quad u(z), v(z) \in \mathbb{R}$$

And we will also look at the functions u and v as multivariable functions $\mathbb{R}^2 \to \mathbb{R}$:

$$z = x + iy, \quad u(x+iy) = u(x,y), \quad v(x+iy) = u(x,y)$$

So when approaching from left or right, we get

$$f'(z) = \lim_{t \to 0} \frac{f(z+t) - f(z)}{t} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

And when approaching from above or below we get

$$f'(z) = \lim_{t \to 0} \frac{f(z+it) - f(z)}{it} = \frac{1}{i} \frac{\partial f}{\partial y} = \frac{1}{i} \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right)$$

For those two to be equal we must thus have

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

which are called the **Cauchy-Riemann equations**, which will explain quite a few phenomena of holmorphic function and the behaviour of certain integrals and the connections between them.

If f is differentiable at z, we can write $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. Further if we look at $|f'(z)|^2$ we have

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

And we recovered the Jacobian determinant

If we look at the Laplacian of a function $f \in C^2$, then we have

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial \frac{\partial v}{\partial y}}{\partial x} + \frac{\partial \frac{-\partial v}{\partial x}}{\partial y} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

where we used Schwartz's theorem that indentifies the second derivaties.
So if $f = u + iv$ is holomorphic, then u and v are harmonic, i.e. their Laplacian is zero: $\Delta u = \Delta v = 0$.

Definition

Let u be a harmonic function. A harmonic function v harmonic that satisfies the Cauchy-Riemann equations with u

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

is called a **harmonic conjugate** of u.

Exercise: Show that if u and v harmonic conjugates, then $f(x + iy) := u(x, y) + iv(x, w)$ is holomorphic.
Exercise: Take $u(x, y) = x^2 - y^2$. Find it's harmonic conjugate.
We saw that differentiability implied the Cauchy-Riemann equations. It seems to be a natural question to ask, wheter the converse is also true and if not, what the weakest possible assumptions are.

Theorem

If $u(x, y), v(x, y) \in C^1$ i.e. have continuous derivative and satisfy the Cauchy-Riemann equations, then the function f defined as $f(x + iy) := u(x, y) + iv(x, y)$ is analytic.

The idea behind the proof is that to write u, v as their first derivative approximation. The continuity implies that the error of the approximation is very small, from which the proof falls out of the equations.

Proof:

$$u(x + t, y + w) = t\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \varepsilon_1, \quad \text{where} \quad \text{where} \quad \frac{\varepsilon_1}{t + iw} \to 0, \text{ as } t, w \to 0$$
$$u(x + t, y + w) = t\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \varepsilon_2, \quad \text{where} \quad \frac{\varepsilon_2}{t + iw} \to 0, \text{ as } t, w \to 0$$

We then can write

$$\begin{aligned}
f(z+t+iw)-f(z) &= u(x+t,y+w)-u(x,y)+i\left(v(x+t,y+w)-v(x,y)\right) \\
&= t\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial y}+it\frac{\partial v}{\partial x}+iw\frac{\partial v}{\partial y}+\varepsilon_1+\varepsilon_2 \\
&= t\frac{\partial u}{\partial x}-w\frac{\partial v}{\partial x}+it\frac{\partial v}{\partial w}+iw\frac{\partial u}{\partial x}+\varepsilon_1+\varepsilon_2 \\
&= \left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(t+iw)+\varepsilon_1+\varepsilon_2
\end{aligned}$$

Which allows us to show, that the following limit exists (i.e f is complex differentiable)

$$\lim_{t+iw\to 0}\frac{f(z+t+iw)-f(z)}{t+iw}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+\frac{\varepsilon_1+i\varepsilon_2}{t+iw}$$

Now, let's look at some examples of differentiable functions. $f(z)=1$ is analytic with derivative $f'(z)=1$. The identity map $f(z)=z$ is also holomorphic, as

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{z+h-h}{h}=1$$

Polynomials $p_n(z)=a_0+a_1z+a_2z^2+\ldots+a_nz^n$ are analytic, and their derivates are as we expect from real analysis.
As we are going to show later, we will find that we can approximate every analytic function with polynomials, so we will study them next.
A very useful theorem, which we will prove later is the Fundamental theorem of Algebra, which states that every non-constant polynomial has a root in $\mathbb{C}$.
This theorem also proves that every polynomial of degree $n\geq 1$ has exactly n roots, if we count roots with their multiplicity.
We do this by *factoring out* the roots to get

$$p(z)=a_n(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)$$

We will also see that if we have roots $\alpha_j,\ldots,\alpha_n$ are roots of our polynomial, then there will be a root of its derivative inside the polygon $(\alpha_j,\ldots,\alpha_n)$. This is analogous to the Rolle's theorem in the real case.

Luca's Theorem

If all rots of a polynomial $p(z)$ are in a halfspace H, then so are roots of it's derivative $p'(z)$.

Where we define halfspaces as

$$H:\left\{z\in\mathbb{C}\middle|\operatorname{Im}\frac{z-a}{b}<0\right\}$$

Proof: If decompose the polynomial into its factors (using the fundamental theorem of algebra), we have

$$p(z)=a_n(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)$$

$$\implies\frac{p'(z)}{p(z)}=\frac{1}{z-\alpha_1}+\frac{1}{z-\alpha_2}+\ldots+\frac{1}{z-\alpha_n}$$

Since all roots $\alpha_1, \dots, \alpha_n \in H = \{z \in \mathbb{C} | \mathrm{Im}\frac{z-a}{b} < 0\}$ we know that for $z \notin H$ we have

$$\mathrm{Im}\frac{z-\alpha_k}{b} = \mathrm{Im}\frac{z-a}{b} - \mathrm{Im}\frac{\alpha_k - a}{b} > 0$$

Since inversion changes the sign of the argument (because their arguments should add to zero) we have that $\mathrm{Im}\frac{b}{z-\alpha_k} < 0$, which would imply that if z were a root of the derivate, we wowould have

$$0 = \mathrm{Im}\left(b\frac{P'(z)}{P(z)}\right) = \sum_{k=1}^{n} \mathrm{Im}\frac{b}{z-\alpha_k} < 0 \lightning$$

2.2 Rational Functions

The next set of functions we will consider are the **rational functions**, which are objects of the Form

$$R(z) = \frac{P(z)}{Q(z)}, \quad \text{where} \quad P, Q \text{ are polynomials without common factors}$$

If $Q(z) \neq 0$, we get the *quotient rule* of the derivative

$$R'(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q(z)^2}$$

But if $Q(z) = 0$ for some $z \in \mathbb{C}$, then $R(z) = \infty$ and we call z a **pole** of R.
It would also be useful to find out the *order* of a pole at infinity. And the rational functions allow us to talk a bit easier about the behaviour of the function.
For example, if we want to find out $\lim_{z\to\infty} R(z)$ for

$$R(z) = \frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m}$$

We define the auxiliary function

$$\tilde{R}(z) := R(\frac{1}{z}) = \frac{z^m}{z^n}\frac{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}{b_0 z^m + b_1 z^{m-1} + \ldots + b_m}$$

And instead look at $\lim_{z\to 0} \tilde{R}(z)$. For the case $m > n$, $R(z)$ has a zero of order $m - n$ at infinity. If however $n > m$, then $R(z)$ has a pole of order $n - m$ at infinity. And if $n = m$, we get $R(\infty) = \frac{a_n}{b_m} \neq 0, \infty$.
We can use this knowledge to find out about how rational functions behave outside of infinity. Be counting roots and poles, we know that we have n roots and m poles outside of infinity.
But since the difference between m and n is "accounted for" at infinity, we know that the number of poles and roots in $\mathbb{C} \setminus \infty$ are the same and equal the maximimum of n and m.
Since they are the same, we can define the **order** of a rational Function $R = \frac{P}{Q}$, where P is a polynomial of degree n and Q one of degree m as the number

$$\mathrm{order}(R) := \max\{n, m\}$$

When we consider the special case of a rational function $S(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ of order 1, we can sometimes recover its inverse function if $\alpha\gamma - \beta\gamma \neq 0$ to get

$$z = S^{-1}(w) = \frac{\gamma w - \beta}{-\delta w + \alpha}$$

We later also ask ourselves what we can find out about its fixpoints for functions $S(z) = z + a$ or inverse operations $S(z) = \frac{1}{z}$

2.3 Sequences and Series

In analysis we want to talk about the limits of functions. But for this we need to look at sequences first. If $(a_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{C}$, we say that $A \in \mathbb{C}$ is the **limit** of the sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \quad \text{such that} \quad \forall n \geq N |a_n - A| < \varepsilon$$

The problem with this definition is that in order to find out whether the limit exists, we need to find the limit first and then argue that the equation above holds.
One way to remedy this is to define *Cauchy-Sequence*. A sequence $(a_n)_{n=1}^{\infty}$ is called a **Cauchy-sequence**, if

$$\forall > 0 \exists N \in \mathbb{N} \quad \text{such that} \quad \forall n, m \neq N : |a_n - a_m| < \varepsilon$$

We have proven in real analysis that convergence and being a Cauchy sequence is equivalent.
The nice thing here is that we do not need to find the limit A to argue about whether the sequence converges. In particular, if we have two sequences of which we know that the first one converges, if the secnd sequence has $|b_{i+1} - b_i| \leq |a_{i+1} - a_i| \forall i \in \mathbb{N}$, then we know that the second sequence also converges.

Now consider the sum of a sequence. We define the partial sums of a sequence $(a_n)_{n=1}^{\infty}$ to be the sequence $(s_n)_{n=1}^{\infty}$ given by

$$s_n := a_1 + \ldots + a_n$$

Then we can find out that $(s_n)_{n=1}^{\infty}$ is a Cauchy-sequence if and only if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \quad \text{such that} \quad \forall n \geq N, \forall p \in \mathbb{N} : |a_n + \ldots + a_{n+p}| < \varepsilon$$

Using the triangle inequality we have that

$$|a_n + \ldots + a_{n+p}| \leq |a_n| + \ldots + |a_{n+p}|$$

So if the sequence of the absolute values $|a_1| + |a_2| + \ldots$ converges, we get something *stronger* than just convergence. We say that $(s_n)_{n=1}^{\infty}$ is **absolutely convergent**.

2.4 Sequences of functions.

For a sequence of functions $(f_n(x))_{n=1}^{\infty}$ we can have multiple notions of convergence.
If we look at **pointwise-convergence**, i.e. when

$$\forall \varepsilon > 0 \forall x, \exists N \quad \text{such that} \quad \forall n > N : |f_n(x) - f(x)| y \varepsilon$$

The problem with this is that the function can converge at different speeds to the function f, for example when the choice of N is dependent on x. It can also happen that the pointwise limit of continuous functions can be discontinuous (for example the function $f : [0, 1] \to \mathbb{R}, f(x) = x^n$)

Therefore the definition of **uniform convergence** is important:

$$\forall \varepsilon > 0 \exists N \quad \text{such that} \quad \forall x \forall n \geq N : |f_n(x) - f(x)| < \varepsilon$$

Proposition

If f_n is continous for each n and $f_n \to f$ uniformly, then f is continous.

Next we can consider a series of functions, i.e. a function given by

$$g(x) = f_1(x) + \dots f_n(x) + \dots$$

We say that the $a_1 + a_2 + \dots + a_n + \dots$ is a **majorant** if

$$|f_n(x)| < M a_n \quad \text{for some } M \in \mathbb{R} \text{ and } n \text{ large enough}$$

Then, if $a_1 + \dots + a_n + \dots$ converges, then $f_1(x) + \dots + f_n(x) + \dots$ is uniformly and absolutely convergent.

Next we want to look at **Power series**, which are objects of the form

$$a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

And of course we can center the polynomial somewhere else and instead write

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

To better understand them, we can look at the simples interesting example $1 + z + z^2 + \dots + z^n + \dots$ which has

$$1 + z + \dots + z^n = \frac{1 - z^n}{1 - z}$$

And for $|z| < 1$ the series converges and we have

$$1 + z + z^2 + \dots + z^n + \dots = \frac{1}{1 - z}$$

But for $|z| \geq 1$ this series will diverge. But this isn't really clear. If we chose $z = -1$ or $z = -i$, then it might look like the partial sums will stay bounded, but that isn't enough to fully converge.

We will have some nice criteria for the convergence of the sequences

Theorem (Abel)

Let $\sum a_n z^n$ be a power series. There exists a number $0 \leq R \leq \infty$, we call the **Radius of convergence**, which satisfies

(a) The series converges absolutely for $|z| < R$ and for $\rho < R$ the series converges absolutely uniformly in $|z| < \rho$

(b) The series diverges for $|z| > R$

(c) If $|z| < R$, then the series is analytic and the derivative is given can be done term-wise and has the same radius of convergence.

The idea of the proof is that we calculate

$$\frac{1}{R} := \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

and do what we did for the simple case $1 + z + z^2 + \ldots + z^n + \ldots$.
This theorem is very nice, since it tells us that the places where the power series converges is quite nice. It also allows to build new analytic functions. Let's say that we have a function given by a power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^{n-1} + \ldots$$

then we can find out its derivative by term-wise calculation

$$f'(z) = a_1 + 2a_2 z + \ldots + n a_n z^{n-1} + \ldots$$

$$f''(z) = 2a_2 + 2\cdot 3a_3 z + \ldots + (n-1)n a_n z^{n-2} + \ldots f^{(k)}(z) \quad = k!a_k + \frac{(k+1)!}{1!}a_{k+1}z + \frac{(k+2)!}{2!}a_{k+2}z^2 + \ldots$$

In particular, if we are given some analytic function and want to detrmine its power series, we see that

$$f^{(k)}(0) = k!a_k \quad \Longrightarrow \quad a_k = \frac{1}{k!}f^{(k)}(0)$$

which allows us to write us the **Taylor-Maclaurin series** for analytic functions

$$f(z) = f(0) + \frac{1}{1!}f^{(1)}(z) + \frac{1}{2!}f^{(2)}z^2 + \ldots + \frac{1}{k!}f^{(k)}(0)z^k + \ldots$$

2.5 Exponential and trigonometric functions

The exponential function and the trigonometric functions have the connection for $x \in \mathbb{R}$ that

$$e^{ix} = \cos x + i \sin x$$

Since the exponential function has so many nice properties which all follow from another, we have many options on how we want to define the fucntion. The version which we will take will be through it's property for its derivatives

Definition Exponential functions

The exponential function is the function $\exp : \mathbb{C} \to \mathbb{C}$ which is uniquely determined by the following differential equation

$$\exp'(z) = \exp(z), \forall z \in \mathbb{C}, \quad \exp(0) = 1$$

Using the Taylor-Maclaurin series expansion, we can get

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^2 + \ldots$$
$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \ldots + n a_n z^{n-1} + \ldots$$

From this we find that $a_n = \frac{1}{n!}$ so

$$e^z := \exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

Using Abel's convergence theorem, we see that (using Stirling's Formula) the power series has an inifite Radius of convergence, i.e. that the function is well-defined for all $z \in \mathbb{C}$.

We will see that the exponential function has the following properties

(a) It has the additive property $e^{a+b} = e^a \cdot e^b$.

(b) It respects the complex conjugate: $\exp(\overline{z}) = \overline{\exp(z)}$

(c) For $x \in \mathbb{R}$ we have $|e^{ix}| = 1$

Proof:

(a) Using the differential equation, it follows from the product rule that

$$D(e^{c-z}e^{z}) = D(e^{c-z})e^{z} + e^{c-z}D(e^{z}) = -e^{c-z}e^{z} + e^{c-z}e^{z} = 0$$

(b) Since all coefficients in the Taylor series are real, it follows immediately that $e^{\overline{z}} = \overline{e^{z}}$

(c) From property (a) and (b) we find that

$$|e^{ix}|^2 = e^{ix}\overline{e^{ix}} = e^{ix}e^{-ix} = e^{i(x-x)} = 1$$

From (a) it follows immediately that the exponential is never zero, i.e. has a multiplicative inverse

$$1 = e^0 = e^{z-z} = e^{z}e^{-z}$$

It also follows from property (c) that for $x, y \in \mathbb{R}$ we have

$$|e^{x+iy}| = |e^{x}| \cdot |e^{iy}| = e^{x}$$

As we said before, there are many ways to define the complex exponential and thus also many ways to define the trigonometric functions. We also could have started with the trigonometric functions and define the complex exponential from there, but here we do it like this:

Definition Sine and Cosine

The complex Sine and Cosine functions are the functions $\sin, \cos : \mathbb{C} \to \mathbb{C}$ defined by

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

We then can find the Taylor series of these functions to be

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \dots$$
$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots$$

From these definitions, we wee the formula stated in the beginning

$$e^{ix} = \cos(x) + i\sin(x)$$

We also see that for $x \in \mathbb{R}$ we recover the triangular identity

$$1 = |e^{iy}|^2 = |\cos(x) + i\sin(x)|^2 = \cos^2(x) + \sin^2(x)$$

Aswell als the addition formula for $x_1, x_2 \in \mathbb{R}$:

$$\cos(x_1 + x_2) + i\sin(x_1 + x_2) = e^{i(x_1+x_2)} = e^{ix_1}e^{ix_2} = (\cos(x_1) + i\sin(x_1))(\cos(x_2) + i\sin(x_2))$$

Definition Periodicity

We say that a function has a period c, if

$$f(z+c) = f(z), \forall z \in \mathbb{C}$$

For example, sine and cosine have Period 2π and the exponential function has period $2\pi i$.

If $z = e^w$, then we want to define the logarithm to be $\log(z) := w$
Since the exponential function is periodic, there are multiple solutions for w also, since we know that e^z is never equal to zero, there is no solution for $z = 0$
Using what we found for the complex exponentials we know that for $w = x + iy$ we have

$$|z| = e^x, e^{iy} = \frac{z}{|z|}$$
$$\implies \log(z) = \log(x) + i\arg(z)$$

where $\log(x)$ is the real logarithm and where the argument arg can be determined up to $\mod 2\pi$.

But if we force to define the argument to be always in $[0, 2\pi)$, then the argument isn't a continuous function anymore.

3 Topology

For us to get some nice properties of the *smoothness* of a funciton, we need to be able to *zoom in* and talk about points being *close* from another.
In order to have a notion of closeness we will need a *distance* function, or more accurately a *metric*.

Definition Metric Space

Let M be a set, a function $d : M \times M \to \mathbb{R}$ is called a **metric** if it satisifes a following properties:

(a) Positive definiteness: $d(x, y)00 \iff x = y$

(b) Symmetry: $d(x, y) = d(y, x)$

(c) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

It follows from these properties, that the metric must also be ≥ 0.
We next define the open Ball centered around x with radius δ to be the set

$$B(x, \delta) := \{y \in M : d(x, y) < \delta\}$$

We call a set N a **neighborhood** of some $x \in M$ if

$$\exists \delta > 0 : B(x, \delta) \subseteq N$$

We call a set S **open**, if S is a neighborhoods of all its points $x \in S$. We call it **closed**, if its complement $M \setminus S$ is open.

The **interior** of some set X will be the *largest* open set $S \subseteq X$

$$\text{int}X := \bigcup_{\substack{S\subseteq X \\ S \text{ open}}} S$$

The **closure** is the *smallest* closed set containing X:

$$\overline{X} := \bigcup_{\substack{S\supseteq X \\ S \text{ closed}}} S$$

And the **boundary** is the difference between the two:

$$\partial X := \overline{X} \setminus \text{int}X$$

When we talk about functions, that have some nice properties, we need to talk about when these properties make sense. Openness of a region is very important, as we then are able to "zoom in" enough to make our statements wihtin the region X. There are also some more properties we want to look at.
We call a set X **connected**, if it cannot be written as the union of two disjoint, non-empty and open subsets. $X = U \sqcup V$. In other words, if there exist U, V open subsets of X such that

$$S = U \cup V, \text{ and } U \cap V = \emptyset \implies U = \emptyset \vee V = \emptyset$$

Proposition

Intervals are connected subsets of $\mathbb{R}$.

Proof: Let $I \subseteq \mathbb{R}$ be an interval and assume that $U, V \neq \emptyset$ with $I = U \sqcup V$. Then take $u_1 \in U$ and $v_1 \in V$ and assume without loss of generality that $u_1 < v_1$. Take the midpoint $m_1 := \frac{u_1+v_1}{2}$ which must either be in U or V. If $m_1 \in U$, define $u_2 := m_1, v_2 := v_1$ and take the midpoint m_2 again. Since the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are cauchy, the must converge, but also U and V are closed, so $u := \lim_{n\to\infty} u_n \in U$ and $v := \lim_{n\to\infty} v_n \in V$, and $\lim_{n\to\infty}|a_n - b_n| \to 0 \implies u = v$ ↯.

Theorem

In the plane, a non-empty open set is connected if and only if and only if every two points can be reached by a polygon line.

Totally rigorous definition and proof: Let A be open and connected and for $a \in A$ let A_1 be the set of points that can be reached from a by a polygon line and A_2 be thet set of points that can't.

These sets are obviously disjoint and have $A = A_1 \sqcup A_2$. Let $x \in A_1$. Since A is open, there exists $\delta > 0$ such that $B(x, \delta) \subseteq A$. And for any $y \in B(x, \delta)$ there exists a line connecting x and y. Therefore $B(x, \delta) \subseteq A_1$, which shows that A_1 is open. If on the other hand $x \in A_2$, then the Ball $B(x, \delta)$ must also be in A_2, or else we could extend the line $a - y$ to $a - y - x$.

Now assume that A is such that every two points of A can be reached by a polygon line and assume that $A = U \sqcup V$, of non-epty open sets U, V and let $u \in U$, $v \in V$ be linked by a polygon line. Somewhere along the line, there must be a switch, from U to V. And the polygon must locally look like an Interval. From the previous proposition, it follows that some point must exists that is both in U and V.

The next property of sets we want is *compactness*. We call a set S **compact**, if for any open cover $S = \bigcup_{x \in X} O_x$, there exists a finite subcover $S = \bigcup_{y \in Y} O_y$ for $Y \subseteq X$ finite.

Theorem

A subset $S \subseteq \mathbb{R}^n$ is compact if and only if S is closed and bounded and if S is compact, then every sequence in S has an accumulation-point and a convergent subsequence.

Proof: Let's say $(x_n)_{n=1}^{\infty}$ has no accumulation point, i.e. for all $y \in S$ there exists a $\delta_y > 0$ such that

$$B(y, \delta_y) \cap (x_n)_{n=1}^{\infty} \text{ is finite}$$

Then we have an open cover using these intersections (which are open!). But since S is compact, we must have a finite subcover of S, which would imply that S itself only has finitely many elements. This can't be the case since we have an infinite sequence in S without accumulation points.
Let E be topological space. We call a subset $\Omega \subseteq E$ a **region**, if it is open and connected

The point is that regions are a natural place to define cuntions on.
In order to define continuity on arbitrary topological spaces (without a metric) we note the following theorem:
A function $f : \Omega \to \mathbb{C}$ is continuous (in the metric space sense) if and only if the inverse image of open sets are open.
So we can define continuity of functions between topological spaces as just that:
A function $f : \Omega \to \mathbb{C}$ is called **open** (in the topological sense) if the inverse image of open sets is open in Ω.

A nice consequence of this theorem is that we can sometimes prove results for continuouty in a simpler fashion. For example the proof that the composition of continuous functions is continuous is trivial using this new definition, whereas we formerly had to work with lots of epsilons and deltas.

Another example is that we can show that the image of compact sets under continuous functions is again compact. The proof here becomes also quite simple, since any open covering of the image $f(S)$ can be used to create an open covering of S. Then using compactness of S we obtain a finite sub-covering of S and "push" it onto $f(S)$.

As a consequence, if $f : \Omega \to \mathbb{R}$ is continous and $S \subseteq \Omega$ is compact, then $f|_S$ attains its maximum and minimum.
Another one: A continuous function takes connected sets to connected sets.

However, for metric spaces, we can define something stronger than just continuity. We saw that in the ε, δ definition, we could have a different δ for every $x \in E$. We can restrict the condition to require such a δ for all points:

We call a function $f : E \to \mathbb{C}$ **uniformly continuous**, if

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \forall x, x' \in E : d(x, x') < \delta \implies d(f(x), f(x')) < \varepsilon$$

Which leads to the following theorem that on compact sets, continous functions are automatically uniformly continuous.

Eventually, we want to integrate over curves in $\mathbb{C}$. To be able to do this, we define first what a curve is: An **arc** γ on the complex plane is the image of a closed and bounded interval by a continuous function $z[\alpha, \beta] \to \mathbb{C}$

$$z(t) = x(t) + iy(t), \quad \alpha \le t \le \beta$$

If $\varphi : [\alpha', \beta'] \to [\alpha, \beta]$ is continuous, then we call $(z \circ \varphi) : [\alpha', \beta']$ a **reparametrisation** of z if $\varphi(\alpha') = \alpha$ and $\varphi(\beta') = \beta$ and φ is monotone.
We call an arc **differentiable**, if $z'(t)$ exists *and is continuous!*.
We call a differentiable arc **regular**, if $z'(t) \neq 0, \forall t$
A **path** is a continuous function $\gamma : [a, b] \to \mathbb{C}$ An arc is **Jordan**, if it doesn't intersect itself, i.e if z is injective.

The next thing is that we might want to "add" or "subtract" paths. This can make sense if two paths overlap or have matching endpoints.

Another nice thing paths allow us is that we can parametrize certain objects. For example we have that for a circle

$$\{z \in \mathbb{C} \big| |z - c| = r\}$$

can be seen as the image of the function

$$z(t) = c + re^{it}, \quad t \in [0, 2\pi)$$

Definition

A function $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is **analytic** in an open set Ω if f has a derivative in every point of Ω. If $A \subseteq \Omega$, then we also say that f is analytic in A.

One thing that often happens, is that many functions *want* to be multi-valued. For example the square root function $f(z) = \sqrt{z}$ wants to have solutions $\pm\sqrt{z}$, but if we allowed that f wouldn't be a function anymore so we have to create a **branch**, which is a subset of the co-domain. For example the half-plane

$$H := \{z \in \mathbb{C} \big| \mathrm{Re}(z) > 0\}$$

can be used to obtain a single-valued function $\sqrt{\cdot} : \mathbb{C} \to H$.
Restricting the function alllows us the say that the function is continuous in $\mathbb{C}/\{z \le 0\}$

If $f : \Omega \to \mathbb{C}$ is analytic on a region (open + connected) Ω and $f'(z) = 0, \forall z \in \Omega$, then f is constant. The same is true for the $\mathrm{Re}(f), \mathrm{Im}(f), \arg(f), |f|$.

This shows that the definition of region is a nice one to see.
If we think of a region Ω and its geometric properties, we can think of how a function transforms the geometric properties of Ω. For example, if we take an arc γ with parametrised with the function $z(t)$ under a continuous function, what can we say about the new arc that results of the new function $\omega := f \circ z$.

If f is analytic, then using the chain rule we get that

$$\omega'(t) = f'(z(t)) \cdot z'(t)$$

Let's set t_0 such that $f'(z(t_0)) \neq 0$. If z is regular ($z'(t) \neq 0 \forall t$), then we have

$$\arg(\omega'(t_0)) = \arg(f'(z(t_0)) + \arg(z(t_0))$$

We can think of $\arg(\omega'(t_0))$ as the direction the arc is moving. It says that locally, the original arc is turned according to the derivative $f'(z(t_0))$.
In particular if two arcs are tangent before the mapping, then the arcs are tangent after the mapping under analytic functions with nonzero derivative.
Furthermore, if two arc cross with some angle α, then their image will also cross with the same angle. It means that it locally preserves angles.

So with this we get the following theorem:

Theorem

Let $f : \Omega \to \mathbb{C}$. Be an analytic function from a region $\Omega \subseteq \mathbb{C}$ (open + connected). And $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Then

(a) f takes tangent arcs into tangent arcs

(b) f takes two arcs intersecting at angle α into arcs with the same relative angle.

Which gives us the definittion

Definition Conformal Map

An analytic function $f : \Omega \to \mathbb{C}$ is called a **conformal map**, if $f'(z) \neq 0, \forall z \in \Omega$

For example, let's look at the complex exponential, maps the real line to the positive numbers and the imaginary axis to the unit circle. So from this we see that it maps lines into circles (with possibly infinite radius). Moreover, it maps the the half plane with negative real part into the area inside the unit circle.

If $f : \Omega \to \mathbb{C}$ is analytic then its derivative tells us how much the space gets shrinked locally. So $|f'(z_0)|$ tells us how much the space gets expanded independent of direction.

4 Complex Integration

We define the integral of complex valued functions using the real-valued Integral. So let $f : [a, b] \to \mathbb{C}$ be a continuous function with $f(t) = u(t) + iv(t)$, then we define it's integral to be

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt$$

The Integral will have the following properties:

(a) $\int_a^b cf(t)dt = c\int_a^b f(t)dt$

(b) Triangle Inequality:

$$|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$$

Proof: If $\int_a^b f(t)dt = 0$, then there is nothing to show. If it is nonzero, then we can talk about its argument $\theta = \arg \int_a^b f(t)dt$. Since $|z| = \mathrm{Re}(\exp(-i\theta)z)$, we can write

$$|\int_a^b f(t)dt| = \mathrm{Re}\left[\exp(-i\theta)\int_a^b f(t)dt\right] = \int_a^b \mathrm{Re}\left[\exp(-i\theta)f(t)\right]dt \leq \int_a^b \underbrace{|\exp(-i\theta)|}_{=1}|f(t)|dt$$

Where we used the fact that that the Real part is always smaller than the absolute value and tha the absolute value is multiplicative.
Now that we know how to integrate over Intervals $[a,b] \subseteq \mathbb{R}$, we want to know how to integrate over paths/arcs.

Let γ be an arc that is parametrized by a differentiable function $z(t)$ for $a \leq t \leq b$. Say f is continuous on the arc γ. Then we define the Integral over the arc to be

$$\int_\gamma f(z)dz := \int_a^b (f \circ z)(t) \cdot \dot{z}(t)dt$$

And if γ is only piece-wise differentiable, then we define the integral piecewise, i.e. summing over all the differentiable parts.
Now this definition better be well defined, i.e. invariant under reparametrisations. If $t(\tau)[\alpha,\beta] \to [a,b]$ with $t(\alpha) = a, t(\beta) = b$ is a reparametrisation of $[a,b]$, then using the chain rule we have

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)dz &= \int_\alpha^\beta (f \circ z \circ t)(\tau) \cdot \frac{d}{d\tau}(z \circ t)(\tau)d\tau \\ &= \int_\alpha^\beta (f \circ (z \circ t))(\tau) \cdot \dot{z}(t(\tau))\dot{t}(\tau)d\tau \\ &= \int_a^b (f \circ z)(t)\dot{z}(t)dt = \int_\gamma f(z)dt\end{aligned}$$

It is clear that if we traverse an arc the other way around with $w : [-b,-a] \to \mathbb{C}$ for $w(t) = z(-t)$, then

$$\begin{aligned}\int_{-\gamma} f(z)dz = \int_{-b}^{-a} (f \circ w)(t)\dot{w}(t)dt &= \int_{-b}^{-a} (f \circ z)(-t)(-\dot{z}(-t))dt \\ = \int_a^b (f \circ z)(t)\dot{z}(t)dt &= -\int_\gamma f(z)dt\end{aligned}$$

Now if we have multiple arcs that intersect in their endpoints, we cann add the arcs and the integrals then

$$\int_{\gamma_1+\ldots+\gamma_n}^{\infty} fdz = \int_{\gamma_1} fdz + \ldots + \int_{\gamma_n} fdz$$

Now what if a single harc has the same start end endpoints? It will create a loop and we can say that the Integral over closed curves is independent on the starting point. So if $\gamma = \gamma_1 + \gamma_2$, then

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} = \int_{\gamma_2} + \int_{\gamma_1}$$

Missing text.

$$\int_\gamma f(z)\overline{dz} := \overline{\int_\gamma \overline{f(z)}dz}$$
$$\int_\gamma f dx := \frac{1}{2}\left(\int_\gamma f dz + \int_\gamma f\overline{dz}\right)$$
$$\int_\gamma f dy := \frac{1}{2i}\left(\int_\gamma f dz - \int_\gamma f\overline{dz}\right)$$

So if $f(z) = u(z) + iv(z)$ we can write with $dz = dx + dy$ that

$$\int_\gamma f(z)dz = \int_\gamma udx - vdy + i\int_\gamma vdx + udy$$

And we can also integrate with respect to length:

$$\int_\gamma f ds := \int_\gamma f|dz| = \int_a^b (f \circ z)(t) \cdot |\dot{z}(t)|dt$$

So in this case it doesn't matter what direction our arc is parametrized.

This allows us to define the line-integral as a function of the arcs. If Ω is a region (open and connected), and we wan to find out $\int_\gamma p(x,y)dx + q(x,y)dy$, then we will sometimes see that this will only depend on the endpoints of γ and not on the arc itself. This might help us even integrate functions of the reals. We then can integrate along a different path over the complex numbers, such that the integration will be slightly easier to compute.

In the next chapters, we will focus on finding out when exactly the Integral only depends on the endpoints. If $pdx + qdy$ are fixed, the the Integral depends only on the end points of γ if and only if $\int_\gamma = 0$ for closed gamma.
Proof: $\implies$: Becuse γ and $-\gamma$ have the same endpoints we must have $\int_\gamma = \int_{-\gamma} = -\int_\gamma = 0$.
$\impliedby$: If two curves have the same endpoints, then consider closed curve $\gamma_1 + (-\gamma_2)$. Whose integral must be zero, so then the integrals are the same.
In the real case, we often substituted the integral with the primive F of a function f. Now we can write it like this:

Theorem

The line integral $\int_\gamma pdx + qdy$ defined in a region Ω depends only on the end points of $\gamma \subseteq \Omega$ if and only if

$$\exists U(x,y) \text{ on } \Omega \text{ such that } \frac{\partial U}{\partial x} = p, \frac{\partial U}{\partial y} = q$$

Since its derivatives are fixed, it is clear that the primitive U is unique up to a constant.

Proof: If the primitive U exists, then

$$\begin{aligned}\int_\gamma pdx + qdy &= \int_a^b \frac{\partial U}{\partial x}\dot{x}(t) + \frac{\partial U}{\partial y}\dot{y}(t)dt \\ &= \int_a^b \frac{d}{dt}U(x(t), y(t))dt = U(x(b), y(b)) - U(x(a), y(a))\end{aligned}$$

If on the other hands the integrals only depend the endpoints, then chose any $(x_0, y_0) \in \Omega$. For any other point in Ω, since it is connected, we have an arc going from our startpoint to that point. So we can just set

$$U(x, y) = \int_\gamma p dx + q dy$$

which is well defined, since the choice of γ doesn't matter and we just have to check that it indeed is a primitve, but that is trivial:

$$U(X; Y) = \text{const. } + \int_\gamma p dx + q dy \implies \frac{\partial U}{\partial x} = P$$

So this gives us the central question, when is it that

$$f(z)dz = f(z)dx + if(z)dy$$

is an exact differential? If we decompose our path into real and imaginary components,

$$z(t) = x(t) + iy(t)$$

we would get

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b (f \circ z)(t)\dot{z}(t)dt = \int_a^b f\left(x(t) + iy(t)\right)\left(x'(t) + iy'(t)\right)dt \\
&= \int_\gamma f(x + iy)(dx + idy)
\end{aligned}$$

which can be viewed as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. So if we write $f(x, y) = u(x, y) + iv(x, y)$ we get

$$\int_\gamma f(z)dz = \int_\gamma f(x, y)(dx + idy) = u(x, y) + iv(x, y)(dx + idy) = \int_\gamma u(x, y)dx - v(x, y)dy + i\int_\gamma u(x, y)dy + v(x, y)dx$$

So we also could have defined the complex integral this way.

Now let's see when $f(z)dz$ is an exact differential. Does there exist an $F(x, y)$ such that $dF(z) = f(z)dz$. By decomposing we can write

$$F(x, y) = U(x, y) + iV(x, y) = F(x + iy)$$

Then we can write the differential as

$$\begin{aligned}
dF(x, y) &= \frac{\partial U}{\partial x}(x, y)dx + \frac{\partial U}{\partial y}(x, y)dy + i\frac{\partial V}{\partial x}dx + i\frac{\partial V}{\partial y}dy \\
&= \left(\frac{\partial U}{\partial x}(x, y) + i\frac{\partial V}{\partial x}(x, y)\right)dx + \left(\frac{\partial U}{\partial y}(x, y) + i\frac{\partial V}{\partial y}(x, y)\right)dy \\
&= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial x}dy
\end{aligned}$$

so when is it equal to $f(z)dz$?. This is then the case if we have

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \left(\frac{\partial U}{\partial x}(x, y) + i\frac{\partial V}{\partial x}(x, y)\right) = (u(x, y) + iv(x, y)) \\
\frac{\partial F}{\partial y} &= \left(\frac{\partial U}{\partial y}(x, y) + i\frac{\partial V}{\partial y}(x, y)\right) = i\left(u(x, y) + iv(x, y)\right)
\end{aligned}$$

So this is the case when

$$\boxed{\frac{\partial F}{\partial y} = i\frac{\partial F}{\partial x} \iff \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}}$$

which means F satisfies the Cauchy-riemann equations. So F is analytic and we can f in terms of the derivative.

$$f(z) = \frac{\partial F}{\partial x}(z) = F'(z)$$

So we end up with the following theorem.

Theorem

Let $\Omega \subseteq C$ be a region, f continous in Ω, then $\int_\gamma f(z)dz$ depends only on the endpoints of γ if and only if f has a primitive $F : \Omega \to \mathbb{C}$ with $F'(z) = f(z)$

For example: for the function $f(z) = z$, is $f(z)dz$ an exact differential? Homework!

Let's say we have a closed curve $\gamma \subseteq \mathbb{C}$ and let $n \geq 0$ and consider the function $f(z) = (z-a)^n$. What can we say about the integral $\int_\gamma f(z)dz = \int_\gamma (z-a)^n dz$?

Since f is analytic and has primitive $\frac{(z-a)^{n+1}}{n+1}$, which is analytic in $\mathbb{C}$ the integral must be zero.
But if $n = -2$, then $(z-a)^{-2}$ is the derivative of $\frac{-1}{z-a}$, which is analytic except for $\mathbb{C} \setminus \{a\}$, but for $n = -1$, it has a primitive, but which isn't analytic. And then take the circle C around a. Calculate as an exercise.

$$\int_C \frac{1}{z-a} dz = ?$$

The Problem with this function is that the $\log(z-a)$ is not single valued. So if we take the integral over $C_r := \{z \in \mathbb{C} | |z-a| = r\}$ we get that for $z(t) = a + re^{it}$

$$\int_{C_r} \frac{1}{z-a} dz = \int_0^{2\pi} \frac{1}{a + re^{it}} z'(t)dt = \int_0^{2\pi} \frac{rie^{it}}{re^{it}} dt = \int_0^{2\pi} idt = 2\pi i$$

which can be seen a as saying that the log, when we go move along C_r, takes on a different value at the end that in the beginning.
But if Ω is a half space not containing a, then $\log(z-a) = \log(|z-a|) + i\arg(z-a)$ and if we never go around a, then $\arg(z-a)$ will always stay between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and we won't have a problem like before.
This gives us a really nice outlook on finding out what the space looks like by calculating curve integrals. This algebraic struture that fits on topological spaces will give rise to the powerful tools of Algebraic Topology.

Informally, the Cauchy's theorem states that if $\Omega \subseteq \mathbb{C}$ is *nice enough* and f is analytic in Ω, then $\int_\gamma f(z)dz = 0$ for all closed curves $\gamma \subseteq \Omega$.

Note that the almost nice set $\Omega = \mathbb{C} \setminus \{0\}$ is connected, open, path connected, but for Cauchy's Theorem it is not nice enough.
Recall that we looked at $pdx + qdy = dU$ and we argued that if we had to paths connecting $z_0 = (x_0, y_0)$ and $z_1 = (x_1, y_1)$ we could calculate the path integral independent of whether we would take the path

$(x_0, y_0) - -(x_0, y_1) - -(x_1, y_1)$ or the path $(x_0, y_0) - -(x_1, y_0) - -(x_1, y_1)$. So all we need is if we show it for the rectangle, we can generalize it for more and more subsets of $\mathbb{C}$.
We begin with the rectangle. Let $R = [a, b] \times i[c, d] \subseteq \mathbb{C}$. And let $f : R \to \mathbb{C}$ be analytic in R. Then $\int_R f(z)dz = 0$.

One approach to prove this is to try to do this over the real numbers. But that is quite tiresome. The nice way to do this is to use that f is analytic and zoom in in a small region of the rectangle and approximate f, controlling the error.
Let $\eta(R) = \int_{\partial R} f(z)dz$. We will keep dividing the rectangle the rectangle into smaller rectangles by bisecting int on each side to get four rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$ and parametrize them such that on each smaller rectangle, we go along clockwise.

Since on each inner side, we integrate twice, but in opposite directions, the Integrals cancel out. So we get

$$\eta(R) = \eta(R^{(1)} + \eta(R^{(2)} + \eta(R^{(3)} + \eta(R^{(4)}$$

Using a pidgeon-hole like argument, there must be an i such that

$$|\eta(R^(i)| \geq |\eta(R)|$$

and call that one R_1. We repeat this and get a sequence of rectangles

$$R \supseteq R_1 \supseteq R_2 \supseteq R_3 \dots \quad \text{with} \quad |\eta(R_n)| \geq \frac{1}{4^n}|\eta(R)|$$

Not let $z^* \in R$ such that $\forall \varepsilon > 0$ there exists an n with $R_n \in B(z^*, \delta)$. For some δ small enough that f is analytic in $B(z^*, \delta)$. Then we can write

$$\forall \varepsilon > 0 : \exists \delta > 0 \text{ such that } |\frac{f(z) - f(z^*)}{z - z^*} - f'(z)| < \varepsilon \quad \text{for all} \quad z \in B(z^*, \delta)$$

And by multiplying with $|z - z^*|$ we get that

$$|f(z) - f(z^*) - f'(z^*)(z - z^*)| < \varepsilon|z - z^*|$$

Next we fix some $\varepsilon > 0$ and pick δ such that we get a linear approximation of our function f. Then consider the integrals

$$\int_\gamma 1dz = 0 \quad \text{and} \quad \int_\gamma zdz = 0$$

since both 1 and z have primitives. Therefore, the linear approximation of the Integrals

$$\begin{aligned}\eta(R_n) = \int_{\partial R_n} f(z)dz &= \int_{\partial R_n} f(z)dz - \int_{\partial R_n} f(z^*) + f'(z^*)(z - z^*)dz \\ &= \int_{\partial R_n} f(z) - f(z^*) - f'(z^*)(z - z^*)dz\end{aligned}$$

so since R_n is inside the Ball $B(z^*, \delta)$ we can use the previous approximations and get using the triangle inequality that

$$|\eta(R_n)| \leq \int_{\partial R_n} \varepsilon|z - z^*||dz|$$

Now let d_n be the diagonal length of R_n and l_n be its perimeter. So since $d_n = 2^{-n}d$ and $L_n = 2^{-n}L$, we can write

$$|\eta(R_n)| \leq \varepsilon \int_{\partial R_n} d_n |dz| = \varepsilon d_n L_n = \varepsilon 4^{-n} dL$$
$$\implies |\eta(r)| \leq 4^h |\eta(R_n)| \leq \varepsilon dL$$

So we find that the closed integral $\eta(R)$ is zero. So this gives us the proposition

Proposition Cauchy's Theorem on a rectangle

If $f : \Omega \to \mathbb{C}$ is analytic in a rectangle $R \subseteq \Omega$, then the contour integral is zero.

$$\int_{\partial R} f(z)dz = 0$$

But we can make this proposition a little bit stronger. The proposition is true, even if we remove a finite number of points from the interior of the rectangle:

Theorem

If $f : \Omega \to \mathbb{C}$ is analytic in R', where R' is obtained by removing a finite number of points $(\xi_k)_{k=1}^K$ from the interior of a rectangle R', such that for all k

$$\lim_{z \to \xi_k} (z - \xi_k) f(z) = 0 \quad \text{then} \quad \int_{\partial R} f(z)dz = 0$$

Since we can subdivide the rectangle R into smaller rectangles that only contain one point, we can assume without loss of generality that $K = 1$. By subdividing the rectangle even further, we can make the rectangle containing the point ξ arbitrarily small, which we call R_0. Since the other rectangles have no points missing, the only integral that remains is the one ove R_0.

$$\int_{\partial R} f(z)dz = \int_{\partial R_0} f(z)dz$$

Using the fact that the limit $\lim_{z \to \xi}(z - \xi)f(z) = 0$, in particular we have that for all $z \in R_0' := R_0 \setminus \{\xi\}$

$$|f(z)| \leq \frac{\varepsilon}{|z - \xi|}$$
$$\implies |\int_{\partial R_0} f(z)dz| \leq \int_{\partial R_0} |f(z)||dz| \leq \int_{\partial R_0} \frac{\varepsilon}{|z - \xi|} dz \leq 8\varepsilon$$

so the integral over R_0 and thus over R vanishes.

With this we want to show that if f is analyti over a nice enough space Ω, then any integral over a closed curve is zero. Here we must have that our definition for nice enough must exclude disks with holes in it, since $\int_{\partial B(0,1)} \frac{1}{z} dz = 2\pi i \neq 0$.
Our next candidate will be the disk:

Proposition Cauchy's Theorem on a disk

Let Δ be an open disk in $\mathbb{C}$. If $f : \Delta \to \mathbb{C}$ is anyltic in Δ and γ is a closed ruve in Δ, then $\int_\gamma f(z)dz = 0$.

The idea behind the proof is that show that f has a primitive, i.e. that $f(z)dz = dF(z)$. We define a function $F(z)$ by

$$F(z) := \int_\gamma f dz$$

where γ consists of the horizontal line segment from the center (x_0, y_0) to (x, y_0) and the vertical component (x, y_0) to (x, y). We then complete the square by going back the other way with γ_2. And using the previous theorem, the Integral over the square vanishes.
The function $F(z)$ is also a primitive of f, since

$$F(z) = \int_{\gamma_1} f(z)dz, \quad \frac{\partial}{\partial y} F(z) = if(z)$$
$$F(z) = \int_{\gamma_2} f(z)dz, \quad \frac{\partial}{\partial x} F(z) = f(z)$$

So we see that F satisfies the Cauchy-Riemann equations with derivative $f(z)$.

We saw that in the case of the rectangle we were allowed to remove a finite number of points. The same is true for open disks.

Theorem

Let f be analytic in a region Ω' obtained from an open disk Ω and removing a finite number of points $(\xi_n)_{n=1}^K$
If $f : \Omega \to \mathbb{C}$ satisfies that for all k

$$\lim_{z \to \xi_k} (z - \xi_k) f(z) = 0$$

then path integrals over closed curves in Ω' vanish.

"Proof": Instead of taking one rectangle from (x_0, y_0) to $(x, y) \in \Omega'$ that may include removed points ξ_k on the edges, we can instead make smalller rectangles from (x_0, y_0) to (x_1, y_1) to $\ldots$ to $(x_n, y_n) = (x, y)$ that only contain removed points on the inside of the rectangles. So we can use the previous corollary inside the proof so the same proof still works.

4.1 Winding Numbers

We saw that in the example of the punctured circle

$$C := \{z \in \mathbb{C} | |z - a| = r\}$$

The closed path integral $\int_C \frac{1}{z-a} dz = 2\pi i$ did not vanish. And we saw that the reason was that we had to loop around the point a. So to further investigate this, we want to define something that tells us how often a path winds around a point. Therefore we define the **winding number** of the curve γ for the point a to be the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz =: \eta(\gamma, a)$$

Which gives us the following theorem

Theorem

Let $a \in \mathbb{C}$ and γ be a closed arc such that $a \notin \gamma$. Then

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz \in \mathbb{Z}$$

Proof: we show that the integral is an integer multiple of $2\pi i$ by taking the exponential function and showing that it equals 1.
If we take a parametrisation of the curve $z(t)$ for $\alpha \leq t \leq \beta$ then the partial curve integral up to T is

$$h(T) := \int_\alpha^T \frac{z'(t)}{z(t)-a}dt$$

we know that $h(\alpha) = 0$ and that $h(\beta)$ is the integral over the path. Taking the derivative of h we get that

$$h'(T) = \frac{z'(T)}{z(T)-a}$$

So when taking the derivative of the exponential we get

$$\frac{d}{dT}\left[\exp\left(-h(T)\right)\left(z(T)-a\right)\right] = -h'(T)\exp(-h(T))(z(T)-a) + \exp(-h(T))z'(T) = 0$$

Which shows that the function is a constant. In particular we must have that

$$\exp(-h(\alpha))(z(\alpha)-a) = \exp(-h(\beta))(z(\beta)-a)$$

And since $a \notin \gamma$, the parts $z(\alpha)-a, z(\beta)-a$ are non-zero, so we have that

$$\exp(-h(\beta)) = \exp(-h(\alpha)) = \exp(0) = 1$$

The next property of the winding number is that if the point lies “outside” of the curve, then the winding number should be zero.
So if $\Delta \subseteq \mathbb{C}$ is a disk containing γ such that $a \notin \Delta$. Then the function $\frac{1}{z-a}$ is analytic in Δ and therefore the integral over γ is zero.
In a more general statement, we get the following theorem

Theorem

Let γ be a closed curve in $\mathbb{C}$. If a, b are enclosed in the same region determined by γ, then

$$n(\gamma, a) = n(\gamma, b)$$

Proof: Since a and b can be linked by a polygon line that never crosses γ, we can assume without loss of generality, that there exists a direct line segment between a and b that doesn't intersect with γ. Consider the function $f(z) = \log\left(\frac{z-a}{z-b}\right)$. Since the line segment doesn't intersect with γ, the log is continous and

$$f'(z) = \log'\left(\frac{z-a}{z-b}\right) = \frac{1}{z-a} - \frac{1}{z-b}$$

This means that $f'(z)$ has a primivte, so the closed integral over $f'(z)$ vanishes. But that is just the difference between the winding numbers

$$n(\gamma, a) - n(\gamma, b) = \int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz = \int_\gamma f'(z)dz = 0$$

The winding number can give us insight into the regularity of analytic functions, as shown in the Cauchy Integral formula:

Theorem

Let $f : \Delta \to \mathbb{C}$ be analytic in an open disk $\Delta \subseteq \mathbb{C}$ and let γ be a closed curve in Δ. For $a \notin \gamma$, the following equation holds

$$n(\gamma, a)f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz$$

Proof: Consider the function $F(z)$ defined as

$$F(z) := \frac{f(z) - f(a)}{z-a}$$

Since f is analytic, so is F in $\Delta \setminus \{a\}$ and we have that

$$\lim_{z\to a} (z-a)F(z) = \lim_{z\to a} (f(z) - f(a)) = 0$$

So we get that using the Cauchy's Theorem the integral over F must be zero. So

$$0 = \int_\gamma F(z)dz = \int_\gamma \frac{f(z)}{z-a}dz - \underbrace{\int_\gamma \frac{f(a)}{z-a}dz}_{=f(a)n(\gamma,a)}$$

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This theorem is extremely powerful, since it allows us to represent the derivative of f in terms of an integral.
In the special case where f is analytic in an open disk $\Delta \subseteq \mathbb{C}$ and γ is a closed curve in Δ, then for Since $n(\gamma, z) = 1$, we have that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z'-z} dz'$$

and if we can take the derivative inside the integral we get

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{(z'-z)^2} dz'$$

and we can repeat this however often we want to calculate the n-th derivative of f:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(z')}{(z'-z)^{n+1}} dz'$$

So in particular, if f is analytic in an open set,

Morera's Theorem

If f is continuous in a region Ω and for any closed path in γ in Ω the path integral vanishes, then f is analytic.

Proof: Since $\int_\gamma f(z)dz = 0$ for all closed paths, f has a prmitive $F(z)$. And $F(z)$ is analytic and therefore also has a second (and third and ...) derivative. So $f'(z) = F''(z)$.

Cauchy Estimates

Let's say f is anyltic in a disk Δ and C is a circle in Δ with radius r such that

$$|f(\xi)| \leq M \forall \xi \in C$$

Then we can bound the derivatives using

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_C \frac{M}{r^{n+1}} |d\xi| = n! M r^{-n}$$

In particular if r can be arbitrarily big, we get the following theorem

Liouville's Theorem

If $f : \mathbb{C} \to \mathbb{C}$ is entire (analytic everywhere in $\mathbb{C}$), and f is bounded, then f is constant!

Proof: for $n = 1$ the derivative Cauchy's estimate gives us that $|f'(z)| \leq \frac{M}{r}$, so since r is arbitrary, $f'(z) = 0$.
This gives us the one-liner Proof for the fundamental theorem of algebra: If a polynomial f is non-zero, then the inverse $\frac{1}{f}$ is bounded and analytic.
It is bounded since for $|z| > R$ for some R large enough, $P(z)$ will not keep getting smaller.
Recall that Cauchy's estimate allows us to show that if we have an analytic function on an open disk Δ then we can describe the n-th derivative through a path integral for closed path γ in Δ

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z')}{(z'-z)^{n+1}} dz$$

Now, even if our function f is not analytic on the entire open disk, Cauchy's integral formula still holds if f has some singularities $(a_j)_{j \in J}$ and f is analytic on $\Delta \setminus (a_j)_{j \in J}$ and such that

$$\lim_{z \to a_j} (z - a_j) f(z) = 0$$

as long as our path γ doesn't go through these singularities. So we get the following theorem:

Theorem Analytic Extension

If f is analyitic in a region Ω' obtained from a region Ω by removing a point $a \in \Omega$ and if

$$\lim_{z \to a} (z-a) f(z) = 0$$

then f can be extended to an analytic in Ω be taking

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

where C is a circle around a contained in Ω

If we use our theorem with the primitive

$$F(z) = \frac{f(z) - f(a)}{z-a}$$

then if F is analytic in $\Omega \setminus \{a\}$ and if

$$\lim_{z \to a} (z-a) F(z) = 0$$

then the limit of F at the point a will be given by

$$\lim_{z \to a} F(z) = f'(a)$$

We then create a sequence of functions $f_i : \Omega \to \mathbb{C}$ given by

$$f_1(z) := \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

which is analytic and recursively define

$$f_{n+1}(z) := \begin{cases} \frac{f_n(z)-f_n(a)}{z-a} & z \neq a \\ f_n'(a) & z = a \end{cases}$$

which are all analytic. We then can write

$$\begin{aligned} f(z) &= f(a) + (z-a) f_1(z) \\ f_1(z) &= f_1(a) + (z-a) f_2(a) \\ &\vdots \\ f_n(z) &= f_n(a) + (z-a) f_{n+1}(z) \end{aligned}$$

which, if we write this out, we obtain the following theorem.

Theorem

If $f : \Omega \to \mathbb{C}$ is analytic in a region Ω, then

$$f(z) = f(a) + (z-a) f_1(a) + (z-a)^2 f_2(a) + \ldots + (z-a)^n f_n(z)$$

, where $f_n(z)$ is analytic in Ω.

Now, if we take the $n-th$ derivative, and set $z=a$, then we get that

$$f^{(n)}(a)=n!f_n(a)\implies f_n(a)=\frac{1}{n!}f^{(n)}(a)$$

which just gives us the Taylor expansion

$$f(z)=f(a)+\frac{(z-a)}{1!}f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots+\frac{(z-a)^n}{n!}f^{(n)}(a)+(z-a)^{n+1}f_{n+1}(z)$$

where $f_{n+1}(z)$ is analytic in Ω.
Our next goal is to show that if $f(a)=0$ and $f^{(n)}(a)=0$ for all $n>0$, then f must be zero in the entire region.
If we take the Cauchy integral Formula for $f_n(z)$, we get that

$$f_n(z)=\frac{1}{2\pi i}\int_C\frac{f_n(z')}{z'-z}dz'$$

which, when plugged into the taylor expansion we get

$$f_n(z)=\frac{1}{2\pi i}\int_C\frac{\frac{f(z')}{(z'-a)^n}-\frac{f^{(1)}(a)}{1!(z'-a)^{n-1}}-\frac{f^{(2)}(a)}{2!(z'-a)^{n-2}}-\ldots-\frac{f^{(n-1)}(a)}{(n-1)!(z'-a)}}{z'-z}dz'$$

Now all the blue colored terms can be written up to a constant as

$$F_k(a)=\int_C\frac{1}{(z'-a)^k(z'-z)}dz'$$

and we want to show that $F_k(a)=0$ for all $1\leq k\leq n-1$. For $k=1$ we have that

$$F_1(a)=\int_C\frac{1}{(z'-a)(z'-z)}dz'=\frac{1}{a-z}\int_C\frac{1}{z'-a}-\frac{1}{z'-z}dz'=\frac{1}{a-z}\left[n(C,a)-n(C,z)\right]$$

since z and a have the same winding number. to show it for the other k, we just differentiate $F_(a)$ and we get

$$F_1'(a)=2F_2(a)=0,\quad\text{and}\quad F_1^{(k)}(a)=k!F_k(a)=0$$

so only the orange term remains and we get

$$\boxed{f_n(z)=\frac{1}{2\pi i}\int_C\frac{f(z')}{(z'-a)^n(z'-z)}dz'}$$

For any z inside of C. The reason this is useful is if we take some $a\in\Omega$ such that $f(a)=0$ and $f^{(k)}(a)=0$ for all k, then from the Taylor expansion, we have that for any $n>0$ we must have that only the f_n term remains, so $f(z)=(z-a)^nf_n(z)$. Then take the maximum of all values inside the circle C

$$M:=\max_{z\in C}|f(z)|\implies|f_n(z)|\leq\frac{1}{2\pi}2\pi RM\frac{1}{R^n}\frac{1}{R-|z-a|}=\frac{M}{R^{n-1}}\frac{1}{R-|z-a|}$$

And from the way we chose R we have $|z-a| \leq R$, so

$$|f(z)| \leq |z-a|^n |f_n(z)| = \left(\frac{|z-a|}{R}\right)^n \frac{RM}{R-|z-a|}$$

and for $n \to \infty$, then the term goes to zero. So $f(z)$ is zero inside the circle C.
Therefore: If $f(a)$ and $f^{(n)}(a)$ are zero for all $n > 0$, then this is also true in a neighborhood of a. And define the sets

$$E_1 := \left\{ z | f(z) = 0 \text{ and } f^{(n)}(z) \forall n > 0 \right\}$$
$$E_2 := \left\{ z | f^{(n)}(z) \neq 0 \text{ for some } n \right\}$$

These sets are disjoint and by continuity of $f^{(n)}(z)$ the sets are also open. So since Ω is connected, on of them must be empty! So f is zero everywhere on Ω.
Rephrased in another way, analytic functions that are not identically zeros, then their zeros are isolated. This in turn means that if two functions agree on some open region, then they must agree everywhere.
Take $f : \Omega \to \mathbb{C}$ analytic and not identical to zero and let a be a zero of f. Then by the previous theorem there must exist some n such that $f^{(n)}(a) \neq 0$ and without loss of generality, we can take the smallest one. By the Taylor theorem we know that

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f^{(2)}(a) + \dots f_n(z)(z-a)^n$$

where f_n is analytic. Since n is the smallest number, where the derivative is non-zero, we have n particular

$$f(z) = f_n(z)(z-a)^n$$

So it must be that $f(a) \neq 0$, which means that there exists a $\delta > 0$ such that

$$f_n(z) \neq 0, \quad \forall z : |a-z| < \delta$$

Then take the punctured disk $\{z : 0 < |a-z| < \delta\}$. Then

$$f(z) = (z-a)^n f_n(z) \neq 0$$

is non-zero for a punctured disk around a. This gives us the following theorem:

Theorem

If $f : \Omega \to \mathbb{C}$ is analytic and not identically zero, then the zeros of f are isolated.

The contraposition of this theorem can be stated as follows:
Let $S \subseteq \mathbb{C}$ be a set with an accumulation point. Then if $f(z) = 0$ for all $z \in S$, then $f = 0$ in all of Ω.
If we take two functions and look at their difference, we obtain the following theorem:

Theorem

If $f, g : \Omega \to \mathbb{C}$ are analytic and coincide in a set with an accumulation point, then $f = g$ in all of Ω.

Now we want to study singularities at single points a, and how fast the f explodes at a. If f is analytic in a punctured neighborhood of a. If

$$\lim_{z\to a}(z-a)f(z) = 0$$

then the singularity at a is removable: If f blows up at a, i.e $\lim_{z\to a} f(z) = \infty$. This means that there exists a $\delta > 0$ such that

$$0|z-a| < \delta \implies f(z) \neq 0$$

if we look at its inverse $g = \frac{1}{f}$ on the puctured disk, g is also analytic and

$$\lim_{z\to a}(z-a)g(z) = \lim_{z\to a}\frac{z-a}{f(z)} = 0$$

so g has a removable singularity at a. Since z is not identically zero there exists an n such that

$$g(z) = (z-a)^n g_n(z), \quad \text{for} \quad g_n(a) \neq 0$$

then take the $\delta > 0$ such that $g_n(z)$ is non-zero in the punctured neighborhood $\{z|0 < |z-a| < \delta\}$. Then we know that $f_n(z) = \frac{1}{g_n(z)}$ is analytic and

$$f(z) = (z-a)^{-n} f_n(z)$$

which means that a is a pole of order n of f.

Definition Meromorphic functions

A function that is analytic except for poles is called **meromorphic** in Ω.

If f and g are anyltic in Ω and g is not identical to zero, then $\frac{f}{g}$ is meromorphic.
Recall that when we talked about fractions of polynomials, we also defined orders of zeros and poles. The definition of order of poles and zeros of meromorphic functions is compatible with the definition for rational functions.

Theorem

Now let $f \neq 0$ be analytic in a neighborhood of a with an isolated singularity at a.

(a) If there exists some $\alpha \in \mathbb{R}$, such that

$$\text{(A)} \quad \lim_{z\to a}|z-a|^\alpha |f(z)| = 0$$

,then $g(z) = (z-a)^\alpha f(z)$ has a removable singularity with $g(a) = 0$. And if a is a zero of g of order k, then

$$\forall\beta > \alpha - k: \quad \lim_{z\to a}(z-a)^\beta f(z) = \lim_{z\to a}(z-a)^{\beta-\alpha} g(z) = 0$$
$$\forall\gamma < \alpha - k: \quad \lim_{z\to a}(z-a)^\gamma f(z) = \infty$$

(b) If there exists some $\beta \in \mathbb{R}$ such that

$$\text{(B)} \quad \lim_{z\to a}|z-a|^\beta |f(z)| = \infty$$

Theorem

If (A) is satisfied for some $\alpha \in \mathbb{R}$. Then there exists an $h \in \mathbb{Z}$ such that

- (A) is satisfied for all $\alpha > h \in \mathbb{Z}$.
- (B) is satisfied for all $\beta < h$

Afonso plz do maths.

Theorem

Let f have an isolated singularity at a. Then one of the following must hold

(a) f is identically zero.

(b) There exists some $h \in \mathbb{Z}$ such that (A)/(B) are satisfied for $\beta < h < \alpha$

(c) Neither (A) or (B) are satisfied for any $\alpha \in \mathbb{R}$.

We want to look at the third case more closely.

Definition

An isolated singularity a of f is called an **essential singularity** if neither (A) or (B) are satisfied for any $\alpha \in \mathbb{R}$

Theorem Weierstrass

An analytic function comes arbitrarily close to any point in $\mathbb{C}$ in a neighborhoood of an essential singularity a. In other words

$$\forall \varepsilon > 0 \left\{ f(z) | z \in B(a, \varepsilon) \right\} \text{ is dense in } \mathbb{C}$$

Proof by contradiction:

$$\exists A \in \mathbb{C} \exists \delta > 0 : \forall z \in N_a : |z-a| < \delta : \lim_{z \to a} |z-a|^{-1} |f(z) - A| = \infty$$

Then a is not an essential singularty of $g(z) = |f(z) - A|$, since there would exist some $\beta \in \mathbb{R}$ such that

$$\lim_{z \to a} |z-a|^{\beta} g(z) = 0$$

Therefore we can pick $\alpha > 0$, with $\alpha > \beta$ such that

$$\lim_{z \to a} |z-a|^{\alpha} g(z) = 0$$

but then we would have

$$\begin{aligned} |z-a|^{\alpha} g(z) &\geq |z-a|^{\alpha} |f(z)| - \underbrace{|z-a|^{\alpha} |A|}_{\to 0} \\ \implies \lim_{z \to a} |z-a|^{\alpha} |f(z)| &= 0 \end{aligned}$$

so a is not an essential singularity.
Let $f \neq 0$ be analytic in a open disk Δ. such that f has only finitely many zeros $z_1, \ldots, z_n \in \Delta$ counted with multiplicity, i.e. such that if z is of order k, then it occurs k-times in the list.
THen if γ is a closed curve in Δ that doesn't contain any zeros, then we can write

$$f(z) = (z - z_1)(z - z - 2) \ldots (z - z_n) g(z)$$

where $g(z)$ does not have any zeros in Δ. If we take the logarithmic derivative

$$\frac{f'(z)}{f(z)} = \frac{(z - z_2)g(z) + (z - z_1)(z - z_3)g(z) + \ldots + (z - z_1) \ldots (z - z_n)g'(z)}{(z - z_1)(z - z_2) \ldots (z - z_n)g(z))} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \ldots + \frac{g'(z)}{g(z)}$$

Since $\frac{g'(z)}{g(z)}$ is analytic, by Cauchy's Theorem on the disk we have that

$$\int_\gamma \frac{g'(z)}{g(z)} dz = 0$$

Then we can take the path integral on both sides to get

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} \int_\gamma \frac{1}{z - z_k} dz$$

but the right hand side is basically the sum of all the winding numbers:

$$\boxed{\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i \sum_{k=1}^{n} n(\gamma, z_k)}$$

Now let's see what happends when we have a map $f : \Delta \to \mathbb{C}$ that sends a path γ to another path Γ. Then if we write $\omega = f(z)$, then

$$n(\Gamma, 0) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\omega} d\omega = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_k n(\gamma, z_k)$$

So if we want to find the solutions for $f(z) = a$ for some $a \in \mathbb{C}$ we can count the zeros of $g(z) = f(z) - a$ by looking at the image Γ and the point a.

$$n(\Gamma, a) = \frac{1}{2\pi i} \int_\Gamma \frac{d\omega}{\omega - a} = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} = \sum_k n(\gamma, z_k)$$

, where z_k are the solutions to $f(z) = a$.
So if a and b are in the same region of $\mathbb{C} \setminus \Gamma$, they both have the same winding number. In particular, they have an equal number of solutions $f(z) = a$ and $f(z) = b$. In other words, for all $a \notin \Gamma$, there exists a $\delta > 0$ such that

$$\forall b \in B(a, \delta) : \quad n(\Gamma, a) = n(\Gamma, b)$$

Theorem

For $\omega_0 \in \mathbb{C}$, if $f(z)$ is analytic at z_0, such that z_0 is a zero of $f(z) - \omega_0$ or order n, then for any $\varepsilon > 0$ there eexists a $\delta > 0$ such that

$$|\omega - \omega_0| < \delta \implies f(z) = w \text{ has } n \text{ solutions in } B(z_0, \varepsilon)$$

Proof: Since f is analytic in $B(z_0, \varepsilon)$ take γ as a circle around z_0 with radius ε and use the formula from before.

Theorem

If $f(z)$ is anaytic at z_0 and $w_0 = f(z_0)$ and $g(z) := f(z) - w_0$ has a zero of order n. Then for $\varepsilon > 0$ small enough there exists a $\delta > 0$ such that for all point $a : 0|a - w_0| < \delta$, the equation $f(z) = a$ has exactly n solutions in the open disk $|z - z_0| < \varepsilon$

In the special case where $n = 1$, i.e. if $f'(z_0) \neq 0$, it shows that f is one to one, which can be interpreted as f maps a neighborhood of z_0 into a neighborhood of w_0. So we can even show that f is locally homoeomorphic.
From this theorem, we get many corollaries for free:

Theorem

If f is a non-constant analytic function, then f maps open sets to open sets.

Proof: Let U be an open neighborhood of z_0. Then take $\varepsilon > 0$. The theorem says that there exists a $\delta > 0$ such that

$$\forall w : |w - w_0| < \delta \exists z \in Z : f(z) = w$$

so we found an open neighborhood around w_0

$$w \in f(U) \implies \{w : ||w - w_0| < \delta\} \subseteq f(U)$$

Theorem Maximum Principle

If $f : \Omega \to \mathbb{C}$ is analytic in a region Ω, and f is non-constant, then $|f(z)|$ does not attain a maximum in Ω.

Theorem

If f is continuous in a closed bounded set E and anylytic on the interior, then the maximum of f is attained in ∂E.

Another proof of the maximum principle is using Cauchy's Integral formula:
Let $z_0 \in \Omega$ and $\varepsilon > 0$ such that $\overline{B}(z_0, \varepsilon) \subseteq \Omega$. Then from the integral formula we have

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z_0} d\xi \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{z_0 + re^{i\theta} z_0} \frac{d\xi}{d\theta} d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta}}{re^{i\theta}} rie^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \end{aligned}$$

$$\implies |f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta}| d\theta$$

but if $|f(z_0)|$ were a maximum, then $|f(z_0 + re^{i\theta})| = |f(z_0)|$, so f would need to be constant in a neighborhood of z_0.
Now that we saw what Cauchy's Theorem can do for us, we might be interested in finding out for which regions Ω or curves γ Cauchy's Theorem is true.
The answer that Topology gives is that it is the case when the domain is simply connected.
We saw that there is some topology going when when we asked on which regions Ω and closed curves γ Cauchy's Theorem holds.
It is certainly not true for all open and connected regions. With the punctured disk $\Delta \setminus \{z_0\}$ being a counterexample.
In this course we won't give the real definition of topological terms[1], but one that work for the 2-dimensional case in.

Definition

We call a subset $\Omega \subseteq \mathbb{C}$ is simply connected if $\mathbb{C}_\infty \setminus \Omega$ is connected, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the one-point compactification of $\mathbb{C}$.

The addition of the point infinity is necessary, since for example the real line $\{z | \operatorname{Im}(z) = 0\}$ should be simply connected.
We have another equivalent definition

Proposition

$\Omega \subseteq \mathbb{C}$ is simply connected if and only if for all closed curves γ and any $a \in \mathbb{C} \setminus \Omega$ the winding number $n(\gamma, a) = 0$ is zero.

Proof: Let $\Omega \subseteq \mathbb{C}$ and $\gamma \subseteq \Omega$ be a closed curve. Then take the inside of the curve γ°. Since Ω^c is connected, it must be inside on the of the regions delimited by γ. So $\gamma \cap \Omega^c = \emptyset$. Therefore for $z_0 \in \Omega^c$, it must be that Ω^c is contained in the unbounded region determined by γ. So $\forall a \in \Omega^c$, $a \notin \gamma \cup \gamma^{circ}$ and $n(\gamma, a) = 0$
Now assume that Ω^c is not connected and let A and B be to closed disjoint sets such that $A \cap B = \Omega$. Without loss of generality A must be the bounded one. Since they are disjoint, let δ be the distance between A and B. Then fill Ω with a net of squares with side length $< \frac{1}{\sqrt{2}}\delta$ Then if we take all squares Q_j such tha $Q_j \cap A \neq \emptyset$, we see that the union of all of these is closed. So there exists an $a \in \Omega^c$ and a curve, such that $n(a, \gamma) \neq 0$

Definition

A cycle $\gamma \subseteq \Omega$ in a region Ω is **homologous** to zero (and we write $\gamma \sim 0 \mod \omega$), if for all $a \in \Omega^c : n(\gamma, a) = 0$.
We write $\gamma_1 \sim \gamma_2$ if $\gamma_1 - \gamma_2 \sim 0 \mod \Omega$.

Cauchy's Theorem (General)

Let $\Omega \subseteq \Omega$ be a region, $f : \Omega \to \mathbb{C}$ be analytic. Then for all closed curves $\gamma \sim 0 \mod \Omega$

$$\int_\gamma f(z)dz = 0$$

As an immediate corollary we get the following

[1] Because why the fuck not?

Theorem

Let $\Omega \subseteq \mathbb{C}$ be simply connected, then Cauchy's Theorem holds.

If Ω is simply connected, then $f : \Omega \to \mathbb{C}$ has a primitive.

Proof of Cauchy's Theorem: Let $\Omega \subseteq \mathbb{C}$ be a region, $\gamma \subseteq \Omega$ a closed curve and $f : \Omega \to \mathbb{C}$ analytic.
Since γ is a closed curve, we can assume that Ω is bounded. Then take $\delta > 0$ and fill the space with squares Q_j of side length $\delta > 0$. Then the set

$$J := \{j | Q_J \subseteq \Omega\}$$

is finite. We then approximate the boundary of Ω with the boundaries of the squares

$$\Gamma_\delta := \sum_{j \in J} \partial Q_J \quad \text{and} \quad \Omega_\delta := \operatorname{int} \bigcup_{j \in J} Q_j$$

Since $\gamma \subseteq \Omega$, there exists a $\delta > 0$ such that $\gamma \subseteq \Omega_\delta$.
Then take some $\xi \in \Omega \setminus \Omega_\delta$.Which belogs to a Q_k for some $k \notin J$, i.e $Q_k \nsubseteq \Omega$. Therefore there exists some $\xi_0 \neq \Omega$ such that $\xi_0 \in Q_k$.
If we connect ξ and ξ_0 with a line segment that doesn't intersect Ω_δ and also not with γ so we have

$$n(\gamma, \xi) = n(\gamma, \xi) = 0$$

So for all $\xi \in \Gamma_\delta : n(\gamma, \xi) = 0$. And for $z \in \operatorname{int}(Q_{j_0})$ we can use Cauchy's integral formula in ∂Q_j

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} f(z) & j = j_0 \\ 0 & \end{cases}$$

If Ω were unbounded, it suffices to take a bounded Ω_R large enough such that γ is contained.
And what if Ω is not simply connected? If it has multiple connected regions, then we can say Ω^c has n components.
Let's write $A_1, \ldots, A_n$ for those components. Then for some curve γ, the winding number stay contstant for each of those components, i.e

$$a_1, a_2 \in A_k \implies n(\gamma, a_1) = n(\gamma, a_2)$$

and if $a \in A_n$ is in the unbounded component we also have $n(\gamma, a) = 0$, since it is "outside" of the curve γ.
And for each k there exists a curve γ_k such that if $a \in A_k$ the winding number is $n(\gamma_k, a) = 1$ and zero in all the other A_js.
Then

$$\gamma \sim c_1\gamma_1 + c_2\gamma_2 + \ldots + c_n\gamma_n \mod \Omega$$

So for $f : \Omega \to \mathbb{C}$ we can write

$$\int_\gamma f(z)dz = \sum_{k=1}^{n-1} c_k \int_{\gamma_k} f(z)dz$$

where A_n is the unbounded one which doesn't contribute anything to the integral

4.2 Calculus of residues

Let's say $f : \Omega' \to \mathbb{C}$ is anyltic in a simply connected region Ω except in isloated singularties $a_1, \dots, a_n$. Then we want to create a homology basis. Because Ω' is open

$$\forall j \exists \delta_j \text{ such that } \overline{B(a_j, \delta_j)} \setminus \{a_j\} \subseteq \Omega'$$

then define $C_j := \partial B(a_j, \delta_j)$ and

$$P_j := \int_{C_j} f(z) dz$$

The goal is we want some way to compute the P_js.
If $f(z) = \frac{1}{z-a_j}$, then

$$\int_{C_j} \frac{1}{z-a_j} dz = 2\pi i$$

Then define $R_j := \frac{P_j}{2\pi i}$ then we know that

$$\int_{C_j} f(z) - \frac{R_j}{z-a_j} dz = 0$$

Definition

Let $f : \Omega \setminus \{a_j\}_{j \in J} \to \mathbb{C}$ be holomorphic.
The **residue** of f at the isolated singularity a_j is the complex number R_j such that the function

$$g_j(z) := f(z) - \frac{R}{z - z_j}$$

is the derivative of a function that is analytic in $B(a_j, \varepsilon)$ for some $\varepsilon > 0$.

Often, the residues will be really easy to compute and once we know the residues then we can compute the integrals.
We also write

$$R_j = \operatorname{Res}_{z=a_j} f(z)$$

If γ is a circle around a_j with a radius small enough, then

$$R_j = \frac{1}{2\pi i} \int_\gamma f(z) dz$$

Now let's say that $\gamma \subseteq \Omega'$ and $\gamma \sim 0 \mod \Omega$ (not necessary $\mod \Omega'$).
And for each a_j, let C_j be a circle around a_j small enough to only contain a_j and no other a_k.
then we can write

$$\frac{1}{2\pi i} \int_\gamma f(z) = \sum_j n(\gamma, a_j) R_j$$

which is very useful for poles.

Let's say f has a pole at $z = a$ of order n. Then $(z-a)^n f(z)$ is analytic in a neighborhood of a and so

$$f(z) = B_n(z-a)^{-n1} + B_{n-1}(z-a)^{-(n-1)} + \ldots + B_1(z-a)^{-1}\varphi(a)$$

for some φ analytic in a neighborhood of a.
Then B_1 is the residue and $f(z) - \frac{B_1}{z-a}$ is the derivative of a function in a punctured neighborhood of a
For example consider the function

$$f(z) = \frac{e^{ab}}{(z-a)(z-b)}$$

for $a \neq b$. It clearly has two poles at a and b. Then

$$\mathrm{Res}_{z=a} f(z) = \frac{e^{ab}}{z-a}|_{z=a} = \frac{e^{ab}}{a-b} \quad \text{and} \quad \mathrm{Res}_{z=b} f(z) = \frac{e^{ab}}{b-a}$$

And let's say we have a curve γ with winding numbers $n(\gamma, a) = 2$ and $n(\gamma, b) = -1$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{e^{ab}}{(z-a)(z-b)}dz = 2\frac{e^{ab}}{a-b} + (-1)\frac{e^{ab}}{b-a}$$

Now that we get Cauchy's integal formula for free. If $f : \Omega \to \mathbb{C}$ is analytic and $\gamma \sim 0 \mod \Omega$. Then $\frac{f(z)}{z-a}$ has a simple pole at a. Then simply

$$\mathrm{Res}_{z=a}\frac{f(z)}{z-a} = f(a)$$

and what this says is that

$$2\pi i \int_\gamma \frac{f(z)}{z-a}dz = n(\gamma, a)\mathrm{Res}_{z=a}\frac{f(z)}{z-a} = n(\gamma, a)f(a)$$

which shows Cauchy's integral formula. For some closed γ.
When we counted zeros of a function, we were doing calculus of residues. And if $f : \Omega \to \mathbb{C}$ was analytic except for isolated points where f had a zero of order h and $z = a$, then $f(z) = (z-a)^h f_h(z)$ and

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{\left((z-a)^h f_h(z)\right)'}{(z-a)^h f_h(z)} = \frac{h(z-a)^{h-1}f_h(z) + (z-a)^h f_h'(z)}{(z-a)^h f_h(z)} \\ &= \frac{h}{(z-a)} + \frac{f_h'(z)}{f_h(z)}\end{aligned}$$

which means

$$\mathrm{Res}_{z=a}\frac{f'(z)}{f(z)} = h$$

This also holds if $f(z)$ has a pole of order $-h$. If f has a pole of order k and $z = a$ then

$$\mathrm{Res}_{z=a}\frac{f'(z)}{f(z)} = -k$$

Theorem

If f is meromorphic in Ω with zeros $\{a_j\}$ and poles $\{b_k\}$ counted with multiplicity then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{\omega\in\{a_j,b_k\}} n(\gamma,\omega)\mathrm{Res}_{z=w}\frac{f'(z)}{f(z)} = \sum_j n(\gamma,a_j) - \sum_k n(\gamma,b_k)$$

For all $\gamma \subseteq \Omega$ that don't intersect the poles/zeros

This theorem is most useful, when $n(\gamma, z) = 0, 1$.
This gives us the **Argument Principle**.
For any function $g : \Omega \to \mathbb{C}$ analytic.

$$g(z)\frac{f'(z)}{f(z)} \text{ has residue } \begin{cases} g(a)h & \text{if } a \text{ is a zero of order } h \text{ of } f \\ -g(a)h & \text{if } a \text{ is a pole of order } h \text{ of } f \end{cases}$$

Therefore, if $f(z_0) = \omega_0$ with $f'(z_0) \neq 0$ then there exists some δ such that for $|\omega - \omega_0| < \delta$ there exists only one solution for $f(z) = \omega$ in $|z - z_0| < \varepsilon$ for some $\varepsilon > 0$.
Therefore we can use this for a function $f(z) - \omega$ to get

$$\frac{1}{2\pi i}\int_{C(z_0,\varepsilon)} g(z)\frac{f'(z)}{f(z)-\omega}dz = g(z)$$

so in a neighborhood of $f^{-1}(\omega_0)$ we get the **integral formula for the inverse function**

$$f^{-1}(\omega) = \frac{1}{2\pi i}\int_{C(z_0,\varepsilon)} \xi\frac{f'(\xi)}{f(\xi)-\omega}d\xi$$

But what happens when $f'(z_0) \neq 0$?. Then we get multiple branches. See exercise classes.

5 Series

This section wants to show that every analytic function has a Taylor series.

5.1 Sequences of analytic functions

Let's say we have a sequcene of analytic functions $f_n(z)$ that converge to $f(z)$.
When is it the case that the limit fucntion f is also analytic.
We know that point-wise convergence isn't enough, as the limit function doesn't even have to be continuous. (For example $\lim_{n\to\infty} x^n$ on $[0, 1]$)
And we want to find the optimal condition that still implies that the limit function is analytic, and is general enough and easy to check.
We will see the following theorem.

Theorem Weierstrass

Let $f_n : \Omega_n \to \mathbb{C}$ be an analytic sequence of functions that converge uniformly in each compact subset of Ω. Then f is analytic is Ω and f'_n converges uniformly on each compact subset of Ω

The proof follows from Morerea's theorem and Cauchy's Integral Formula.
Let $a \in \Omega$ and take a compact disk $\overline{\Delta}$ around a and inside Ω. Then

$$\exists n_0 \text{ such that } \forall n > n_0, \Omega_n \supseteq \overline{\Omega}$$

and for any $\gamma \subseteq \overline{\Delta}$ Cauch'y's Integral formula says

$$\int_\gamma f_n z) dz = 0$$

Because of uniform convergence in $\overline{\Omega}$,we can take the into the integral

$$0 = \lim_{n \to \infty} \int_\gamma f_n(z) dz = \int_\gamma f(z) dz$$

which shows that f is analytic in Δ. We will show that $f_n' \to f'$ converges in the exercise class.
This theorem is very useful vor series, for if we have

$$f(z) = f_1(z) + f_2(z) + \ldots + f_n(z) + \ldots$$

so if the series converges uniformly in every compact subset of Ω, then f is analytic and the derivative can be taken termwise.
For the compact set uniform convergence on boundary implies uniform convergece on the set because $|f_n(z) - f_m(z)|$ is maximized in the boundary by the Maximum Principle.

Theorem Hurwitz

Let f_n be a sequence of analytic functions and $\neq 0$ in a region Ω that converges uniformly on every compact subset of Ω. Then either

(a) $f = 0$

(b) $f(z) \neq 0, \forall z \in \Omega$

Proof: Homework.
Recall that if f is analytic in Ω, then for $z_0 \in \Omega$ we can write the taylor series

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f^{(2)(z_0)}}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}$$

for some analytic function f_{n+1} in Ω. Our goal is to control the f_{n+1}. We know that we can write

$$f_{n+1}(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}(\xi - z)} d\xi$$

for any circle C such that the corresponding disk $\overline{\Delta}$ is contained in Ω. If $\rho > 0$ is the radius of the circle, then let

$$M := \max_{z \in C} |f(z)|$$

then we can bound the f_{n+1} by giving upper bounds to the numerator and lower bounds for the denominator

$$\begin{aligned}|f_{n+1}(z)(z-z_0)^{n+1}| &\leq |z-z_0|^{n+1}\cdot\frac{1}{2\pi}\int_C\frac{|f(\xi)|}{|\xi-z_0|^{n+1}|\xi-z|}d\xi\\ &\leq |z-z_0|^{n+1}\rho\frac{M}{\rho^{n+1}(\rho-|z-z_0|}\\ &=\rho M\left(\frac{|z-z_0|}{\rho}\right)^{n+1}\frac{1}{\rho-|z-z_0|}\end{aligned}$$

Then for any $0<r<\rho$, we take the smaller closed disk.

$$\overline{\Delta_r}:=\{z||z-z_0|\leq r\}$$

then inside this closed disk $\overline{\Delta_r}$ we get the stronger bound

$$|f_{n+1}(z)(z-z_0)^{n+1}|\leq\rho M\left(\frac{r}{\rho}\right)^{n+1}\frac{1}{\rho-r}$$

which shows that the error converges to zero for $n\to\infty$ and since the estimate is the same for all $z\in\overline{\Delta_r}$, the convergence is uniform.
In other words, for any region Ω, and any point $z_0\in\Omega$, then we can say that the taylor series converges uniformly in the largest disk Δ contained in Ω.
The issues arise when z_0 is close to the boundary. Then the radius of convergence is very small. For example for the function

$$f(z)=\frac{1}{1+z^2}\implies\Omega=\mathbb{C}\setminus\{\pm i\}$$

over the reals, we can't really see why the taylor series at $z_0=0$ only has such a small radius of convergence. But now over the complex numbers we immediately see why the taylor series only converges for $|z|<1$. So if Ω is not circle-shaped, then the taylor series only converges for a small subset of Ω. The key idea is that we take power series of the form

$$\sum_n a_n z^{-n}$$

which is called the **Laurent Series**. This will converge *outside* of a circle for $|z|>R$.
Then we can define a double series of the form

$$\sum_{n=-\infty}^{\infty}a_n z^n$$

and we say that this series converges if both the Taylor- and Laurent subseries converge.
And so the double series will converge in an annulus

$$R_1<|z|<|R_2|$$

in particular the series

$$\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$$

will converge in some annulus $R_1 < |z - z_0| < R_2$. Moreover we will show that any function f that is analytic in some annulus can be written in terms of this double series

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n$$

This is particularly useful for $R_1 = 0$ to study functions with isolated singularities.
The Proof idea is to show that if $f(z)$ can be written as the double series, then we can break it down to

$$f(z) = \sum_{n>0} A_n (z - z_0)^n + \sum_{n \leq 0} A_n (z - z_0)^n =: f_1(z) + f_2(z)$$

then f_1 should be analytic for $|z| < R_2$ and f_2 is analytic for $R_1 < |z|$ and then expand each of these in power series.
Then define f_1 and f_2 as

$$f_1(z) = \frac{1}{2\pi i} \int_{|\xi - z_0| = r} \frac{f(\xi)}{\xi - z} d\xi$$
$$f_2(z) = \frac{-1}{2\pi i} \int_{|\xi - z_0| = r'} \frac{f(\xi)}{\xi - z} d\xi$$

which is well defined and analytic for $|z - z_0| < r < R_2$ then do the same for Notice that the defintion does not depend on r and r'
Then we can show using Cauchy's theorem that in the annulus $f(z) = f_1(z) + f_2 z)$ because $C_r - C_{r'} \sim 0$ mod Ω.
So by Cauchy's Integral formula

$$f(z) = \frac{1}{2\pi i} \int_{C_r - C_{r'}} \frac{f(\xi)}{\xi - z} d\xi = f_1(z) + f_2(z)$$

The next thing we want to look at is to make sense of infinite products

$$p_1 \dots p_n \dots = \prod_{n=1}^{\infty} p_n$$

to properly define these infinite products we look at the sequence of partial products

$$P_n = \prod_{k=1}^{n} p_k$$

and we say that the infinite product converges if the sequence $(P_n)_{n=1}^{\infty})$ of partial products converges.
It is easy to see that for this to converge, we must have

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = 1 \quad \text{or} \quad \lim_{k\to\infty} p_k = 1$$

Usually we will write

$$\prod_{n=1}^{\infty} (1 + a_n)$$

instead and can understand this product instead through

$$\sum_{n=1}^{\infty} \log(1 + a_n)$$

so if we write S_M of the M-th partial sum of the sum above, then $P_M = e^{S_M}$
We can clearly see that if S_M converges, then so does P_M. The converse however is not so clear to see, because it might be that $S_M = 2\pi i M$ is jumping from branch to branch.

Theorem

The finite product $\prod_{n=1}^{\infty}(1+a_n)$ with $1+a_n \neq 0$ converges if and only if $\sum_{n=1}^{\infty} \log(1+a_n)$ converges.

For the proof we can show that $\sum \log(1+a_n)$ converges absolutely if and only if $\sum a_n$ converges absoutely because if $a_n \to 0$, then n large enough

$$(1-\varepsilon)|a_n| < |\log(1+a_n)| < (1+\varepsilon)|a_n|$$

Which motivates your next definition

Definition

We say that $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely, if the series $\sum_{n=1}^{\infty} a_n$ converges abosolutely.

6 Prime Number Theorem

This section covers material that is not directly relevant for the exam but shows the application of complex analysis in other fields.
Let $\pi(x)$ count the number of primes $p \leq x$.
The prime number theorem says that

$$\pi(x) \sim \frac{x}{\log x}$$

Gauss, Legendre first conjectured this to be the case in the last 1700 and early 1800.
In the mid 1800s, Chebyshev gave lower bounds for $\pi(x)$. He showed that there exist constants C, D such that

$$C\frac{x}{\log x} \leq \pi(x) \leq D\frac{x}{\log x}$$

for x large enough.
And then in the late 1800s, the Prime number theorem was proven by Hadamard and Pussin using complex analysis.
This is striking because when we're counting primes in the natural numers, we don't see any complex numbers or even integrals.
The idea behind the proof is that we know that there always exists a prime between a number and its double. In particular, if we look at the binomial coefficient $\binom{2n}{n}$ we see that that it needs to be divisible by every prime between n and $2n$.

$$\binom{2n}{n} \geq \prod_{n<p\leq 2n} p \geq n^{\pi(2n)-\pi(n)}$$

So we only need to find an upper bound for $\binom{2n}{n}$. We can show that 2^{2n} is enough, so if we take the log we get

$$2n \geq \log_2(n)\left(\pi(2n) - \pi(n)\right) \implies \pi(2n) - \pi(n) \leq \frac{2n}{\log 2n}$$

So if we follow the recursion, we get that

$$\pi(2n) \leq \frac{2n}{\log_2 n} + \frac{n}{\log_2 \frac{n}{2}} + \ldots + \frac{\frac{n}{2^{k-1}}}{\log_2 \frac{n}{2^k}}$$

For $2^k = \sqrt{n}$. Then

$$\pi(2n) \leq \frac{2n}{\log_2 n} + \ldots + \frac{2\sqrt{n}}{\frac{1}{2}\log n} + \pi(\sqrt{n})$$

doing some analysis for this geometric series we obtain

$$\pi(2n) \leq \frac{8n}{\log_2 n} + \sqrt{n}$$

For the lower bound we can also use the binomial formula and refine it.
It turns out that if we can look at a function that is a little nicer than $\pi(x)$. We can define

$$\theta(x) = \sum_{p \leq x} \log p = \log \prod_{p \leq x} p$$

so we show $\theta(x) \sim x$ instead because it is equivalent to the prime number theorem because

$$\theta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log(x) = \pi(x) \log x$$

and conversely let $\varepsilon > 0$, then

$$\begin{aligned} \theta(x) &\geq \sum_{x^{1-\varepsilon} \leq p \leq x} \log p \geq \sum_{x^{1-\varepsilon} \leq p \leq x} \log x^{1-\varepsilon} \\ &= \left(\pi(x) - \pi(x^{1-\varepsilon}\right)(1-\varepsilon) \log x \end{aligned}$$

The key behind using complex numbers in number theory is the **Riemann Zeta Function**

$$\boxed{\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}}$$

Recall that we showed $\theta(x) = \mathcal{O}(x)$. So if we can show that $\theta(x) \sim x$, then the prime number theorem follows. We will do exactly that now.
Compare the Riemann Zeta Function with

$$\varphi(s) = \sum_p \frac{\log p}{p^s}$$

where the sum goes over all p prime.
Notice that the sum converges absolutely and locally uniformly for $\mathrm{Re}(s) > 1$, because it can be bounded by the series going over all natural numbers. So if we set $\sigma = \mathrm{Re}(s)$, then

$$\sum_p |\frac{\log p}{p^s}| \leq \sum_n \frac{\log n}{n^\sigma} \leq \sum_n \frac{\log n}{n^\sigma} \leq \sum_n \frac{1}{n^{\sigma-\varepsilon}} = \zeta(\sigma - \varepsilon)$$

For $\varepsilon > 0$ and n large enough.
Therefore both $\varphi(s)$ and $\zeta(s)$ are analytic in $\mathrm{Re}(s) > 1$.
Next we want to show that

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \implies \frac{1}{\zeta(s)} = \prod_p (1-p^{-s})$$

so this can be used to show that there are infinitely many primes.
Because of the fundamental theorem of number theorey, we can use the prime factorisation to write

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{r_1 \geq 0, \dots} (p_1^{r_1} p_2^{r_2} \dots p_k^{r_k} \dots)^{-s} = \prod_p \sum_{r \geq 0} p^{-rs} = \prod_p \frac{1}{1-p^{-s}}$$

Now that we know that the function is analytic in $\mathrm{Re}(s) > 1$, we can try to find the unique analytic continuation. (We proved that two analytic functions that agree on a region have to agree everywhere.)
For this we first look at the region with $\mathrm{Re}(s) < 0$, and then look at the special region with $0 < \mathrm{Re}(s) < 1$.
For the special case where $s_0 = 1$, we obtain the harmonic series, which diverges, so we have a pole there.
Then taking the residue $\zeta(s) - \frac{1}{s-1}$, which extends analytically to $\mathrm{Re}(s) > 0$ because then we have

$$\zeta(s) - \frac{1}{s-1} = \sum_n \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s} dx = \sum_n \int_n^{n+1} \frac{1}{n^s} \frac{1}{x^s} dx$$

And we need to show that it converges abosolutely. This can be done by writing the summand as a double integral

$$|\int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s} dx| = |\int_n^{n+1} \int_n^x \frac{s}{u^{s+1}} du dx| \leq \max_{n \leq u \leq n+1} \frac{|s|}{|u^{s+1}|} = \frac{|s|}{n^{\mathrm{Re}(s)+1}}$$

Then we will show that for s with $\mathrm{Re}(s) \geq 1$, the function $\varphi(s) - \frac{1}{s-1}$ is analytic and $\zeta(s) \neq 0$, from which we can prove the Prime number theorem.
For $\mathrm{Re}(s) > 1$ the product formula makes it clear that is has no roots. Also

$$-\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{-\left(\frac{1}{1-p^{-s}}\right)^2 (\log p p^{-s})}{(1-p^{-s})^{-1}} = \sum_p \frac{\log p p^{-s}}{(1-p^{-s})} = \sum_p \frac{1}{p^s - 1} \log p =: (\#)$$

recall that $\varphi(s) = \sum_p \frac{\log p}{p^s}$ so

$$(\#) - \varphi(s) = \sum_p \log p \left(\frac{1}{p^s - 1} - \frac{1}{p}^s \right) \frac{1}{p^s(p^s-1)}$$

which shows absolute and locally uniform convergence for $\mathrm{Re}(s) > \frac{1}{2}$ because

$$\varphi(s) = \frac{-\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s(p^s-1)}$$

so $\varphi(s)$ extends meromorphically to $\mathrm{Re}(s) > \frac{1}{2}$ with poles at $s = 1$ and zeros of $\zeta(s)$.
This means that the riemann hypothesis is equivalent to saying that the only poles of φ are at $s = 1$

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Assume that $\zeta(s)$ has a zero of order μ, then

$$\lim_{\varepsilon \to 0} \varepsilon \varphi(1+\varepsilon) = 1$$

$$\lim_{\varepsilon \to 0}$$

There is an anyltic theorem that says:
Let $f(t)$ for $t \geq 0$ be a bounded locall inttegrable function and suppose that the function

$$g(z) = \int_0^\infty f(t) e^{-zt} dt$$

defined for $\mathrm{Re}(z) > 0$ extends analytically to $\mathrm{Re}(z) \geq 0$. Then $\int_0^\infty f(t)dt$ exists, (and equals $g(0)$.
The idea behind the proof is to consider for $T > 0$

$$g_T(z) = \int_0^T f(t) e^{-zt} dt$$

which is analytic for all of $\mathbb{C}$. And then want to show that the limit exists and is given by

$$\lim_{T \to \infty} g_T(0) = g(0)$$

by drawing curves around zero.
Then we can use this theorem to prove the prime number theorem by showing that

$$\int_0^\infty \frac{v(x) - x}{x^2} dx$$

converges. For $\mathrm{Re}(s) > 1$, we integrate by parts to get

$$\Phi(s) = \sum_p \frac{\log p}{p^s} = \int_1^\infty \frac{dv(x)}{x^s} dx = -\int_1^\infty v(x) \left(\frac{-s}{x^{s+1}} dx + 0 \right)$$

$$\Phi(s) = s \int_0^\infty \frac{v(e^t)}{e^{ts+t}} e^t dt = s \int_0^\infty e^{-ts} v(e^t) dt$$

then use the anaytic theorem for

$$f(t) = v(e^t)e^{-t} - 1 \quad \text{and} \quad g(z) = \frac{\Phi(z+1)}{z+1} - \frac{1}{z}$$

To check that we can apply the analytic theorem

$$\begin{aligned} \frac{\Phi(z+1)}{z+1} - \frac{1}{z} &= \frac{1}{z+1} \left((z+1) \int_0^\infty e^{-t(t+1)} v(e^t) dt \right) - \frac{1}{z} \\ &= \int_0^\infty e^{-tz} e^{-t} v(e^t) dt - \frac{1}{z} \\ &= \int_0^\infty e^{-tz} (e^{-t} v(e^t) - 1) dt \end{aligned}$$

notice that $f(t)$ is bounded because $v(x) \leq Cx$ which means $v(e^t)e^{-t} \leq C$. Then $g(z)$ extends analyticall to $\mathrm{Re}(z) \geq 0$ and

$$\frac{\Phi(z+1)}{z+1} - \frac{1}{z}$$

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In the end, we have that

$$\int_1^\infty \frac{v(x) - x}{x^2} dx$$

exists.
The way this proves the prime number theorem is by contradiction: Assume that there exists a $\lambda > 0$ such that for arbitrarily large values of C

$$\exists x \geq C \text{ with } v(x) \geq \lambda x$$

since $v(x)$ is non-decreasing and the integral converges, we must have that

$$\lim_{x \to \infty} \int_x^{\lambda x} \frac{v(t) - t}{t^2} dt = 0$$

but it can be bounded by

$$\int_x^{\lambda x} \frac{v(t) - t}{t^2} dt \geq \int_x^{\lambda x} \frac{v(x) - t}{t^2} dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2} dt = \int_1^{\lambda} \frac{\lambda x - ux}{u^2 x^2} x du > 0$$

which can't possibly converge. By doing the same argument for $v(x) \leq \lambda x$ for $\lambda < 1$, we will have shown that $v(x) \sim x$.