□ Analysis of Sum-Product Decoding of Low-Density Parity-Check Codes Using a Gaussian Approximation

[1] S. Chung, T. Richardson, and R. Urbanke, "Analysis of Sum-Product Decoding of Low-Density Parity-Check Codes Using a Gaussian Approximation," *IEEE Trans, Inf. Theory*, vol. 47, no. 2, Feb. 2001.

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☐ Introduction

- For many channels and iterative decoders of interest, LDPC codes exhibit a threshold phenomenon
  - As the block length tends to infinity, an arbitrarily small bit-error probability can be achieved if the noise level is smaller than a certain threshold.

- This paper presents a Gaussian approximation for message densities under density evolution to estimate the threshold on memoryless binary-input continuous-output AWGN channels with sum-product decoding.

- This approximation not only allows us to calculate the threshold quickly and to understand the behavior of the decoder better, but also makes it easier to design good irregular LDPC codes for AWGN channels.

- Local tree assumption
  - The girth of the graph is large enough so that the subgraph forms a tree.
  - We can analyze the decoding algorithm straightforwardly because incoming messages to every node are independent.

- Concentration theorem
  - For almost all randomly constructed codes and for almost all inputs, the decoder performance will be close to the decoder performance under the local tree assumption with very high probability, if the length of the code is long enough.

- Log-likelihood ratios (LLRs) as messages, The all-0 codeword was sent.
- The message update rule
  - The output message of a variable node (summation)

$$v = \sum_{i=0}^{d_v - 1} u_i$$

  - The output message of a check node (tanh rule)

$$\tanh \frac{u}{2} = \prod_{j=1}^{d_c - 1} \tanh \frac{v_j}{2}$$

- The generating functions of the degree distributions

$$\lambda(x) = \sum_{i=2}^{d_{v_{\max}}} \lambda_i x^{i-1} \text{ and } \rho(x) = \sum_{i=2}^{d_{c_{\max}}} \rho_i x^{i-1}$$

where λ_i and ρ_i are the fractions of edges belonging to degree-i variable and check nodes, respectively.

□ Gaussian approximation for regular LDPC codes

- The LLR message u_0 from the channel is Gaussian with mean $2/\sigma_n^2$ and variance $4/\sigma_n^2$, where σ_n^2 is the variance of the channel.
- If all $\{u_{i,} i \geqq 1\}$ are Gaussian, then the resulting sum is also Gaussian.
- From this point on, for regular LDPC codes, we assume that the variables $u, v, u_{i,} v_j{}'s$ are gaussian.
- *The symmetry condition for every iteration.*
  - $f(x) = f(-x)e^x$
  - If $f(x)$ is the density of gaussian, then this condition reduces to $\sigma^2 = 2m$.

$$e^{\frac{-(x-m)^2}{2\sigma^2}} = e^{\frac{-(x+m)^2}{2\sigma^2}} \bullet e^x \quad \Rightarrow \sigma^2 = 2m.$$

=> It suffices to keep only the mean.

- The means of u and v at lth iteration

  Consider an ensemble of random (d_v, d_c)-regular LDPC codes.

  - Variable node : $v = \sum_{i=0}^{d_v-1} u_i$ => $m_v^{(l)} = m_{u_0} + (d_v - 1) m_u^{(l-1)}$.

  - Check node : $\tanh \frac{u}{2} = \prod_{j=1}^{d_c-1} \tanh \frac{v_j}{2}$ => $E\left[\tanh \frac{u^{(l)}}{2}\right] = E\left[\tanh \frac{v^{(l)}}{2}\right]^{d_c-1}$.

$$E\left[\tanh \frac{u}{2}\right] = \frac{1}{\sqrt{4\pi m_u}} \int_R \tanh \frac{u}{2} e^{-\frac{(u-m_u)^2}{4m_u}} du$$

<u>Definition 1</u>:

$$\phi(x) = \begin{cases} 1 - \frac{1}{\sqrt{4\pi x}} \int_R \tanh \frac{u}{2} e^{-\frac{(u-x)^2}{4x}} du, & \text{if } x > 0 \\ 1, & \text{if } x = 0. \end{cases}$$

It is easy to check that $\phi(x)$ is monotonically decreasing and continuous on $[0, \infty)$, with $\phi(0) = 1$ and $\phi(\infty) = 0$.

- For (d_v, d_c)-regular LDPC codes,

$$m_u^{(l)} = \phi^{-1}\left(1 - \left[1 - \phi\left(m_{u_0} + (d_v - 1)m_u^{(l-1)}\right)\right]^{d_c - 1}\right), \text{ where } m_u^{(0)} = 0.$$

Lemma 1:

$$\sqrt{\frac{\pi}{x}}\, e^{-\frac{x}{4}}\left(1 - \frac{3}{x}\right) < \phi(x) < \sqrt{\frac{\pi}{x}}\, e^{-\frac{x}{4}}\left(1 + \frac{1}{7x}\right), \quad x > 0.$$

Proof:

$$\frac{1}{1+e^u} = \begin{cases} \sum_{k=0}^{\infty} (-1)^k e^{ku}, & \text{if } u < 0 \\ \sum_{k=1}^{\infty} (-1)^{k-1} e^{-ku}, & \text{if } u > 0 \end{cases}, \qquad \phi(x) = \frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} \frac{1}{e^u + 1} e^{-(u-x)^2/4x} du$$

we obtain

$$\phi(x) = \frac{1}{\sqrt{\pi x}} \int_{-\infty}^{0} \sum_{k=0}^{\infty} (-1)^k e^{ku} e^{-(u-x)^2/4x} du + \frac{1}{\sqrt{\pi x}} \int_{0}^{\infty} \sum_{k=1}^{\infty} (-1)^{k-1} e^{-ku} e^{-(u-x)^2/4x} du$$

$$= \frac{2}{\sqrt{\pi x}} e^{-x/4} \int_0^\infty \sum_{k=0}^\infty (-1)^k e^{-(\frac{1}{2}+k)u} e^{-u^2/4x} du$$
$$= 4 \sum_{k=0}^\infty (-1)^k e^{x(k^2+k)} Q\left(\sqrt{\frac{x}{2}}\,(1+2k)\right) .$$

Using

$$\left(\frac{1}{x} - \frac{1}{x^3}\right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} < Q(x) < \left(\frac{1}{x}\right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

we get

$$\phi(x) < \frac{4e^{-x/4}}{\sqrt{\pi x}} \left[\sum_{k=0}^\infty \frac{(-1)^k}{1+2k} + \sum_{j=1}^\infty \frac{2}{x(4j-1)^3} \right]$$
$$< \sqrt{\frac{\pi}{x}}\, e^{-x/4} \left(1 + \frac{1}{7x}\right).$$

where we use $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}=\frac{\pi}{4}$ and $\displaystyle\sum_{j=1}^{\infty}\frac{1}{(4j-1)^3}<\frac{\pi}{56}$.

Similarly, we get

$$\phi(x) > \frac{4e^{-x/4}}{\sqrt{\pi x}}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{1+2k}-\sum_{j=1}^{\infty}\frac{2}{x(4j-3)^3}\right]$$

$$> \sqrt{\frac{\pi}{x}}\,e^{-x/4}\left(1-\frac{3}{x}\right).$$

where we use $\displaystyle\sum_{j=1}^{\infty}\frac{1}{(4j-3)^3}<\frac{3\pi}{8}$.

- For $x > 10$, we use the average of the upper and lower bounds of Lemma 1.

Otherwise, we use $\phi(x) \sim e^{\alpha x^{r} + \beta}$.

$\alpha = -0.4527$, $\beta = 0.0218$, and $\gamma = 0.86$

were optimized by numerical simulation.

j	k	rate	σ_{GA}	σ_{exact}	error[dB]
3	6	0.5	0.8747	0.8809	0.06
4	8	0.5	0.8323	0.8376	0.06
5	10	0.5	0.7910	0.7936	0.03
3	5	0.4	1.0003	1.0093	0.08
4	6	1/3	1.0035	1.0109	0.06
3	4	0.25	1.2517	1.2667	0.10
4	10	0.6	0.7440	0.7481	0.05
3	9	2/3	0.7051	0.7082	0.04
3	12	0.75	0.6297	0.6320	0.03

□ Gaussian approximation for irregular LDPC codes

- Consider an ensemble of random codes with degree distribution $\lambda(x)$ and $\rho(x)$.
- Assume first that the individual output of a variable or a check node is Gaussian.
- The mean of the Gaussian mixture output message of a variable node at the lth iteration.

  - $m_{v,i}^{(l)} = m_{u_0} + (i-1)m_u^{(l-1)}$, where u is a Gaussian mixture in general.

  - $f_v^{(l)} = \sum_{i=2}^{d_{v_{\max}}} \lambda_i N(m_{v,i}^{(l)},\, 2m_{v,i}^{(l)})$

  - $$E\left[\tanh\frac{v^{(l)}}{2}\right] = \int_R \tanh\frac{v^{(l)}}{2} \sum_{i=2}^{d_{v_{\max}}} \lambda_i N(m_{v,i}^{(l)},\, 2m_{v,i}^{(l)})\, dv^{(l)}$$
    $$= \sum_{i=2}^{d_{v_{\max}}} \lambda_i \int_R \tanh\frac{v^{(l)}}{2} N(m_{v,i}^{(l)},\, 2m_{v,i}^{(l)})\, dv^{(l)}$$
    $$= 1 - \sum_{i=2}^{d_{v_{\max}}} \lambda_i \phi(m_{v,i}^{(l)})$$

- The mean of the Gaussian mixture output message of a check node at the lth iteration

  - $E\left[\tanh\frac{u^{(l)}}{2}\right] = E\left[\tanh\frac{v^{(l)}}{2}\right]^{d_c - 1}$ => $m_{u,j}^{(l)} = \phi^{-1}\left(1 - \left[1 - \sum_{i=2}^{d_{v_{\max}}}\lambda_i\phi(m_{v,i}^{(l)})\right]^{j-1}\right).$

  - $m_u^{(l)} = \sum_{j=2}^{d_{c_{\max}}}\rho_j\phi^{-1}\left(1 - \left[1 - \sum_{i=2}^{d_{v_{\max}}}\lambda_i\phi(m_{u_0} + (i-1)m_u^{(l-1)})\right]^{j-1}\right).$

- For $0 < s < \infty$ and $0 \leqq t < \infty$, $f_j(s,j)$ and $f(s,j)$ are defined as

$$f_j(s,t) = \phi^{-1}\left(1 - \left[1 - \sum_{i=2}^{d_{v_{\max}}}\lambda_i\phi(s + (i-1)t)\right]^{j-1}\right)$$

$$f(s,t) = \sum_{j=2}^{d_{c_{\max}}}\rho_j f_j(s,t).$$

- An alternative expression to $m_u^{(l)}$

$$t_l = f(s, t_{l-1})$$

where $s = m_{u_0}$ and $t_l = m_u^{(l)}$.

  - The initial value t_0 is 0. Since $t_1 = f(s,0) > 0$ for $s > 0$, the iteration will always start.
  - Since $\phi(x)$ is monotonically decreasing on $[0, \infty)$, we conclude that $f(s,t)$ is monotonically increasing on both $0 < s < \infty$ and $0 \leqq t < \infty$.

- *Definition 2:* The threshold s^* is the infimum of all s in R^+ such that

$$t_l(s) \text{ converges to } \infty \text{ as } l \rightarrow \infty.$$

- *Lemma 2:* $t_l(s)$ will converge to ∞ iff $t < f(s,t)$ for all $t \in R^+$.

- As an alternative expression to $f(s,t)$,

  for $0 < s < \infty$ and $0 < r \leqq 1$, we define $h_i(s,r)$ and $h(s,r)$ as

$$h_i(s,r) = \phi\left(s + (i-1)\sum_{j=2}^{d_r}\rho_j\phi^{-1}(1-(1-r)^{j-1})\right)$$

$$h(s,r) = \sum_{i=2}^{d_l}\lambda_i h_i(s,r).$$

- $t_l = f(s,t_{l-1})$ becomes equivalent to $r_l = h(s,r_{l-1})$ where $s = m_{u_0}$ and $r_l = \phi(m_u^l)$.
- The convergence condition $r_l(s) \to 0$ is satisfied iff

$$r > h(s,r) \qquad \forall\, r \in (0, \phi(s)).$$

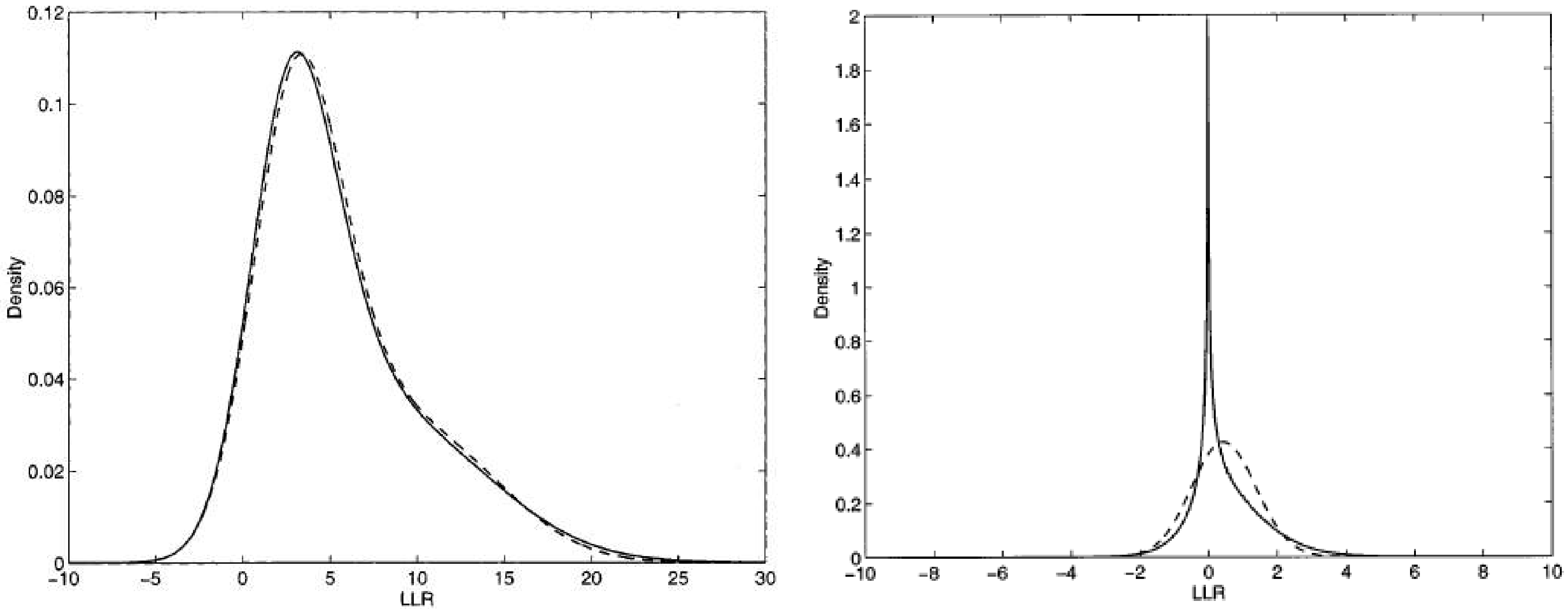


Fig. 1. (left) Approximate (- -) and exact (—) densities at the check node input.

Fig. 2. (right) Approximate (- -) and exact (—) densities at the variable node input

☐ Stability

Lemma 3: As $t \to \infty$, $\triangle t = f(s,t) - t$ becomes

$$\triangle t = s + (i_1 - 2)t - 4\log\lambda_{i_1} - 4\sum_{j=2}^{d_r}\rho_j \log(j-1) + O(t^{-1})$$

where λ_{i_1} is the first nonzero λ_i.

Proof: When t is large, $\sum_{i=2}^{d_l}\lambda_i \phi(s+(i-1)t)$ can be simplified to

$$\sum_{i=2}^{d_l}\lambda_i \phi(s+(i-1)t) = \lambda_{i_1}\phi(s+(i_1-1)t) + O(\lambda_{i_2}\phi(s+(i_{2-1})t))$$

where λ_{i_1} and λ_{i_2} are the first and the second nonzero λ_i's, respectively.

Using $(1+x)^n = 1 + nx + O(x^2)$, we get

$$\phi(f_j(s,t)) = (j-1)\lambda_{i_1}\phi(s+(i_1-1)t)[1 + O(\phi(s+(i-1)t))]$$

$$= (1+O(t^{-1}))\frac{1}{\sqrt{s+(t_1-1)t}}e^{-(s+(i_1-1)t-4\log(j-1)\lambda_{i_1})/4}$$

where we used $\phi(s+(i_1-1)t) \ll t^{-1}$ and Lemma 1. By using Lemma 1 again, we get

$$\phi(f_j(s,t)) = \phi(s+(i_1-1)t - 4\log(j-1)\lambda_{i_1} + O(t^{-1})).$$

Finally, by taking $\phi^{-1}(.)$, we get the result.

- If $i_1 > 2$, then t_l always converges to infinity regardless of s and $\rho(x)$ as l tends to infinity if t_1 is large enough.

- Otherwise, $\triangle t$ becomes a constant as t becomes large.

 - For this case, using $s = 2/\sigma_n^2$, we get

$$\triangle t = 4\log\frac{\lambda_2^*}{\lambda_2} + O(t^{-1})$$

where

$$\lambda_2^* = e^{1/2\sigma_n^2} / \prod_{j=2}^{d_r} (j-1)^{\rho_j}.$$

Theorem 1: If $\lambda_2 < \lambda_2^*$, then t_l will converge to infinity as l tends to infinity if t_1 is large enough. If $\lambda_2 > \lambda_2^*$, then t_l cannot converge to infinity regardless of its initial value.

- The probability of error P_l at the lth iteration is given by

$$P_l = \sum_{i=1}^{d_l} \lambda_i' Q\left(\sqrt{\frac{s+it_l}{2}}\right)$$

where

$$\lambda_i' = \frac{\lambda_i / i}{\sum_{j=2}^{d_r} \lambda_j / j}$$

Lemma 4: When t_l is large, P_l is approximated by

$$P_l \approx \begin{cases} \dfrac{a}{\sqrt{b+l}} \left(\dfrac{\lambda_2}{\lambda_2^*} \right)^{2l}, & \text{if } 0 < \lambda_2 < \lambda_2^* \\ \dfrac{d}{(i_1 - 1)^{l/2}} e^{-f(i_1-1)^l}, & \text{if } \lambda_2 = 0 \end{cases}$$

This implies that when $0 < \lambda_2 < \lambda_2{}^*$ and P_l is small, the error probability P_l will decrease by a factor $(\lambda_2/\lambda_2^*)^2$ at each iteration.

□ Optimization of degree distributions

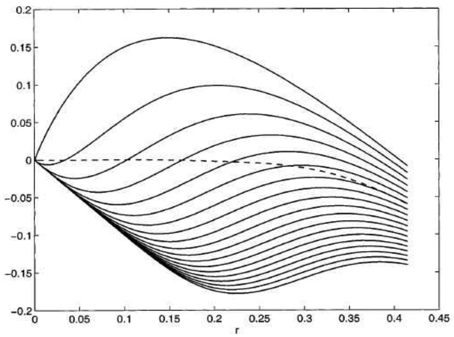
- For given $\lambda(x)$ and σ_n, we can optimize $\rho(x)$ by maximizing the rate of the code.
- The constraints: 1) $\rho(1)=1$; 2) $t<f(s,t)$ for all $t\in R^+$

Theorem 2: A concentrated degree distribution

$$\rho(x)=\rho x^{k-1}+(1-\rho)x^k$$

for some $k\geqq 2$ and $0<\rho\leqq 1$, minimizes $\sum_{j=2}^{d_r}\rho_j\log(j-1)$ subject to $\rho(1)=1$ and

$\int_0^1\rho(x)dx=c$, where c is a constant.



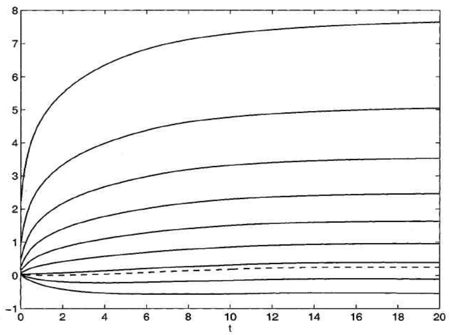
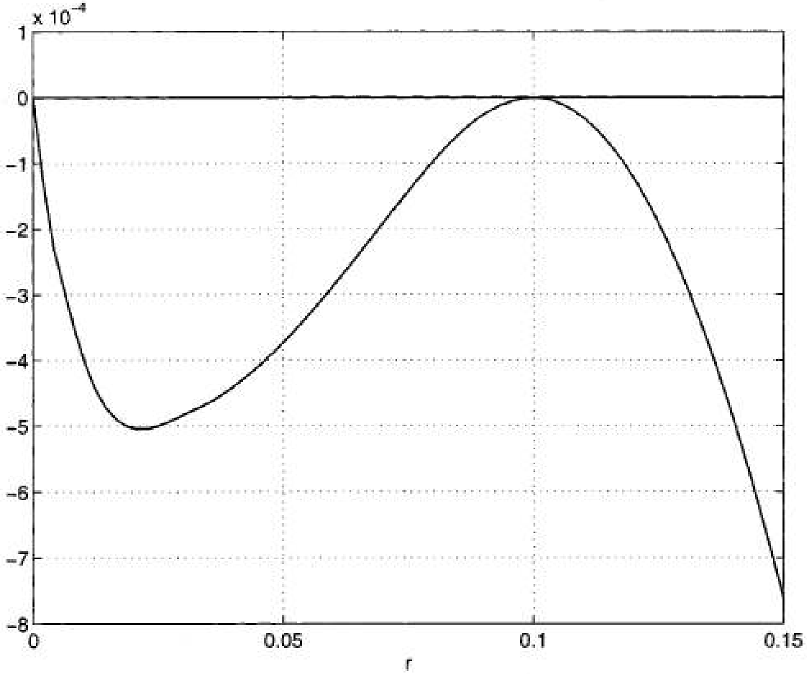


Fig. 3. (left) $\{h_i(s,r)-r\}$ for $i=2,...,20$ (top to bottom) and $\{h(s,r)-r\}$ (- -).

Fig. 4. (right) $\{f_i(s,t)-t\}$ for $i=2,...,20$ (top to bottom) and $\{f(s,t)-t\}$ (- -).

☐ Fixed points



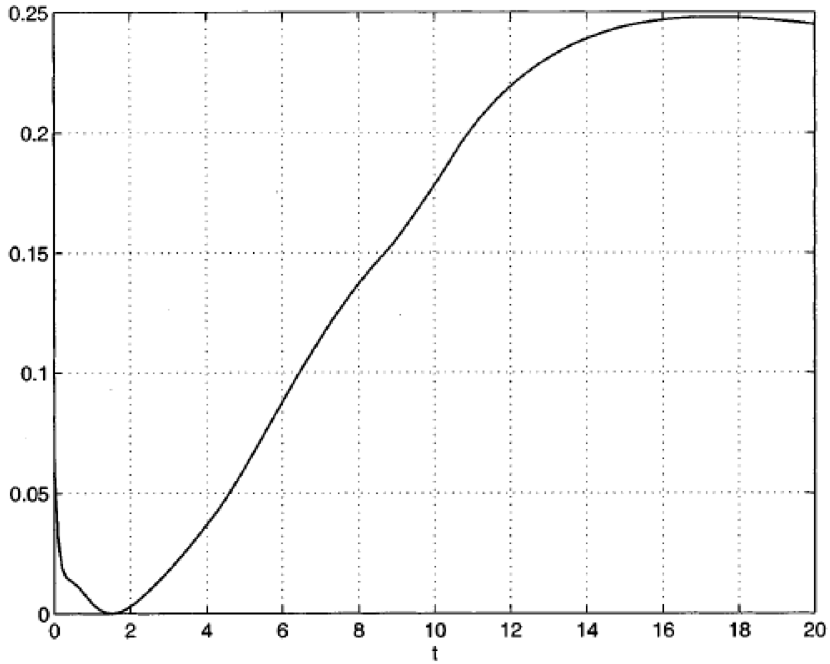
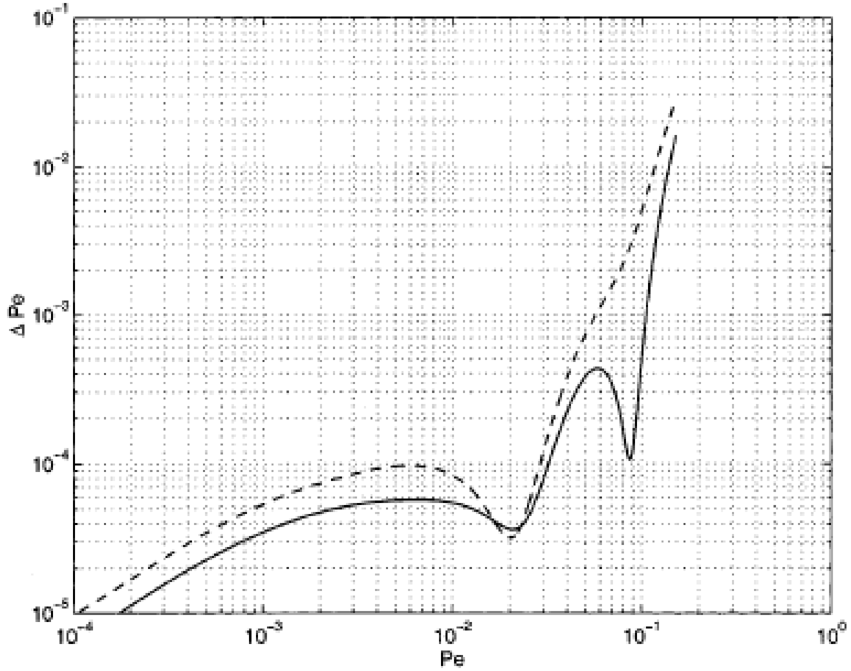


Fig. 5. (left) $\{h(s,r)-r\}$ as a function of r magnified.

Fig. 6. (right) $\{f(s,t)-t\}$ as a function of t magnified.



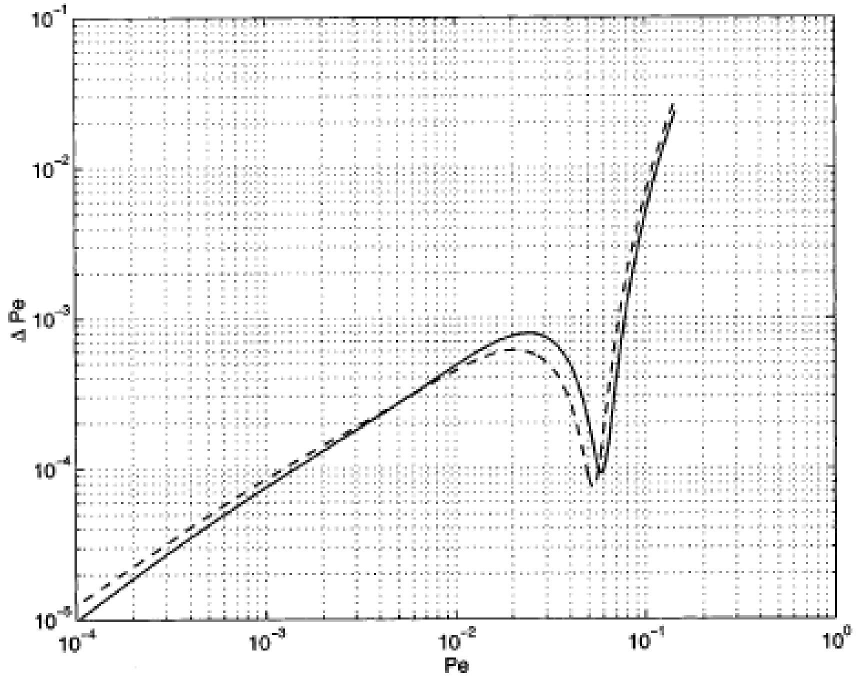


Fig. 7. (left) The probability of error versus decrease in the probability of error of the $d_l = 20$ code using density evolution (—) and the Gaussian approximation (- -).

Fig. 8. (right) The probability of error versus decrease in the probability of error for the $d_l = 7$ code using density evolution (—) and the Gaussian approximation (- -).

☐ Conclusion

- For LDPC codes under sum-product decoding, the message distributions are well approximated by Gaussians or Gaussian mixtures when memoryless AWGN channels are used.
- Because of the huge reduction in the dimension of the problem without much sacrifice in accuracy, we can find thresholds faster and optimize degree distributions faster and easier.
- Using the Gaussian approximation, we have simplified density evolution to a one-dimensional problem, and found simple expressions for describing approximate density evolution.
- It also enables us to analyze the behavior of density evolution especially near the zero probability of error.