Anomalous emergence of a giant component in the configuration-model scale-free networks

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The emergence of a giant connected component as one increases the number of links, i.e., the percolation transition, is one of the most studied features of networked systems, offering insights into their robustness. Under random link addition or removal preserving scale-free (SF) connection topology, the critical number of links per node, called the percolation threshold, is known to be zero when the degree exponent λ is not larger than 3 and the critical exponents may vary continuously with λ. These phenomena originate in the diverging moments of the power-law degree distribution, leading us to expect that those specific λ-dependent critical behaviors might be universal for the percolation in SF networks. Here we introduce SF networks generated with the configuration model, given a degree distribution where both the degree exponent and the number of links per node can be controlled independently. We derive exactly the size of the giant component, and find a zero percolation threshold and invariant critical exponents for $\lambda < \lambda_* = 3.81064\ldots$, in sharp contrast with the presumed universal behaviors in other models. The analytic solution reveals that these deviations originate in the functional form of the degree distribution adopted in our configuration model, which maintains a power-law trend even for low degrees. We provide a simple and general approximation to tease out the role of low-degree nodes in controlling the size of the giant component and show that it holds in empirical networks. Our study suggests that critical phenomena in complex networks may depend not only on the asymptotic behavior but also on the overall functional form of the degree distribution.

I. INTRODUCTION

Establishing interaction and communication channels in complex networks often involves limited resources. Strategies for building larger connected components with fewer links are invaluable to ensure the functioning of these networks. In addition, with the increasing availability of data on the network structure of real-world complex systems, it became highly important to understand how the sizes of their connected components depend, not only on the number of links, but also on the empirical features of these networks. In particular, the conditions for a giant component to appear—that is, for the largest connected component to reach a size comparable to the total number of nodes—have drawn wide attention. This has been considered as the generalization to complex networks of the percolation problem. One of the most important discoveries made recently in this context concerns the influence of the tail behavior of the degree distribution. It has been known that the giant component appears when the ratio of the second moment to the first moment of the degree distribution is larger than 2 [1].

Scale-free (SF) networks, with a degree distribution decaying as a power law $D(k) \sim k^{-\lambda}$, have diverging second moments unless the degree exponent λ is larger than 3 and therefore exhibit a giant component for any non-zero number of links per node [2]. Besides percolation, a zero critical point tied to the divergence of the degree distribution's second moment was also observed in a wide range of models when using SF interaction networks, such as the Ising model [3], epidemic spreading [4], synchronization [5], boolean dynamics [6], and so on [7].

Given such remarkable and universal effects of the tail behavior of the degree distribution on critical phenomena, we might guess that the low-degree behavior of the degree distribution is irrelevant to these phenomena. Yet, is it the case? The statistical properties of random walks and fluctuations on complex networks have been shown to depend on the spectral dimension and the lowest degree as well as the degree exponent [8–10]. It is an interesting and open question whether the universality of critical phenomena on SF networks depends solely on the degree exponent, or also on other relevant factors.

To examine the universality of the critical phenomena on complex networks, we investigate the formation of the giant component in SF networks generated with the configuration model [1, 13] and compare its critical behaviors with the established results described above. We illustrate known results with the static model [11], where they are analytically accessible [12]. This comparison reveals that the full shape of the degree distribution, and not only its asymptotics, can play a key role in critical phenomena, and we subsequently provide a simple and general argument for the importance of low-degree nodes in determining the giant component size.

In the static model, each pair of nodes has a pre-assigned probability of being connected, which is tuned to change the number of links per node and the degree exponent at will. The critical phenomena of the percolation transition in the static model [12] are consistent with those observed for random link removal in SF networks [2]. In the configuration model [1, 13], on the other hand, each node is assigned its degree from a given degree distribution, and randomly-selected stubs from different nodes are connected until no isolated stub is left. This model has been widely used as a prototypical model for SF networks, as we can pre-determine the degree distribution as we wish. It is therefore adopted in software packages for network study and analysis [14]. Usually the degree distribution is set to follow a power law over the entire range, $D(k) \propto k^{-\lambda}$, but

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this comes at a cost; the first moment, or equivalently twice the number of links per node, is fixed for given λ, making it impossible to separately relate the size of the giant component to the number of links per node and to the degree exponent. Hence, the percolation problem for configuration-model SF networks has been reduced to finding the critical value of the degree exponent $\lambda_c \simeq 3.47875$ at which the moment ratio $\langle k^2\rangle/\langle k\rangle = \zeta(\lambda-2)/\zeta(\lambda-1)$ becomes 2 [15].

Here, we propose a degree distribution for which the number of links per node can be controlled while the asymptotic power-law decay is preserved, and use it to generate configuration-model SF networks. The form of the degree distribution is different from that of the static model in the low-degree region, but the same in the high-degree region. The giant component in this configuration model is formed with any non-zero number of links per node for $\lambda < \lambda_*$ with $\lambda_* \simeq 3.81$ while it is only for $\lambda < 3$ in the static model. This difference originates in the difference of the degree distribution in the low-degree region. Near the zero critical point for $\lambda < \lambda_*$, the giant component grows linearly with the number of links per node and the cluster size distribution behaves as in the supercritical (percolating) phase, which also contrast with the static-model SF networks.

These results suggest that the whole functional form of the degree distribution is relevant to the critical phenomena of the percolation transition. Even for the same degree exponent and the same number of links per node, the giant component may have different sizes due to different functional form of the degree distribution, which should be accounted for when designing or studying the structure and stability of complex systems.

In Sec. II, our configuration-model SF network is introduced along with the adopted degree distribution and its basic properties. The critical behaviors of the giant component's size and the cluster-size distribution are derived analytically in Sec. III. We further investigate the effects of low-degree nodes on the giant component's size and its application to real-world networks in Sec. IV. Our findings are summarized and discussed in Sec. V.

II. MODEL

To construct a network with N nodes, $L = NK$ links, and a degree exponent λ in the configuration model, we first determine the degree k_i of each node i by taking the integer part of a real-valued random number x from the distribution

$$p_{x_0,\lambda}(x) = \begin{cases} \frac{\lambda-1}{x_0}\left(\frac{x}{x_0}\right)^{-\lambda} & x_0 \le x < \infty, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

as

$$k_i = \lfloor x \rfloor \tag{2}$$

with $\lfloor x \rfloor$ the largest integer not larger than x. Here x_0 is a parameter allowing to control the number of links per node K, and the lowest degree is $k_0 = \lfloor x_0 \rfloor$. This two-step process in Eqs. (1) and (2) can be easily implemented numerically, compared with the direct generation of integer-valued random numbers following a strict power-law distribution [12]. In the widely-used python libraries *random* and *NetworkX* [14], the degree sequence is generated in a similar way with $x_0 = 1$.

Now, each node i has k_i stubs. As in all configuration-model networks, randomly selected pairs of stubs from distinct nodes are connected, avoiding multiple links, until no isolated stub is left. In simulations, we further restrict the maximum degree, but we do not discuss this restriction in the following, as its effects [13] are not relevant to the results presented in this paper [See Appendix A].

The degree distribution $D(k)$ is then equal to the probability that the real-valued random number x is between k and $k+1$, evaluated as

$$\begin{aligned} D(k) &= \int_k^{k+1} dx' p_{x_0,\lambda}(x') \\ &= \begin{cases} 1 - {x_0}^{\lambda-1}(k_0+1)^{1-\lambda} & \text{for } k = k_0, \\ {x_0}^{\lambda-1}\left[k^{1-\lambda} - (k+1)^{1-\lambda}\right] & \text{for } k > k_0. \end{cases} \end{aligned} \tag{3}$$

Notice that the degree distribution has a power law tail

$$D(k) \simeq (\lambda-1){x_0}^{\lambda-1}k^{-\lambda} \ \text{ for } k \gg 1. \tag{4}$$

The number of links per node K is related to the parameter x_0 by

$$2K = \langle k\rangle = \sum_k k\, D(k) = k_0 + {x_0}^{\lambda-1}\zeta_\infty(\lambda-1, k_0+1), \tag{5}$$

where $\zeta_\infty(\lambda, a) \equiv \sum_{k=a}^{\infty} k^{-\lambda}$ is identical to the Riemann zeta function $\zeta(s,a)$ for $s \ge 1$; $\zeta_\infty(s,a)$ diverges for $s < 1$ but $\zeta(s,a)$ is finite for $s < 1$ by analytic continuation. We will restrict ourselves to the range $\lambda > 2$ to avoid the case of diverging K. Also we denote $\zeta_\infty(s,1)$ by $\zeta(s)$ if $s \ge 1$.

As shown in Fig. 1 (a), K increases monotonically with x_0, which ensures the unique value of $x_0(K)$ for given K. Therefore, for given K and λ, one can construct the ensemble of the configuration-model SF networks by using the probability distribution $p_{x_0(K),\lambda}$ in Eq. (1). In Fig. 1 (b), the degree distribution $D(k)$ is shown for selected values of K and λ, illustrating its power law tail, and compared to the degree distribution of the static model for the same parameter values.

III. PERCOLATION THRESHOLD AND CRITICAL BEHAVIORS: NUMERICS AND EXACT SOLUTIONS

When the largest-connected component (LCC) of size S is so large that $m = \lim_{N\to\infty}\frac{S}{N}$ is non-zero, it is called the giant component. We investigate the behavior of the (normalized) size m of the giant component as a function of the number of links per node K for a given number of nodes N and degree exponent λ in the configuration-model networks introduced in the previous section. This investigation was not possible in previous works on the configuration model which used the degree distribution $D(k) = k^{-\lambda}/\zeta(\lambda)$ for $k \ge 1$, as the number of links per node is then fixed by λ.

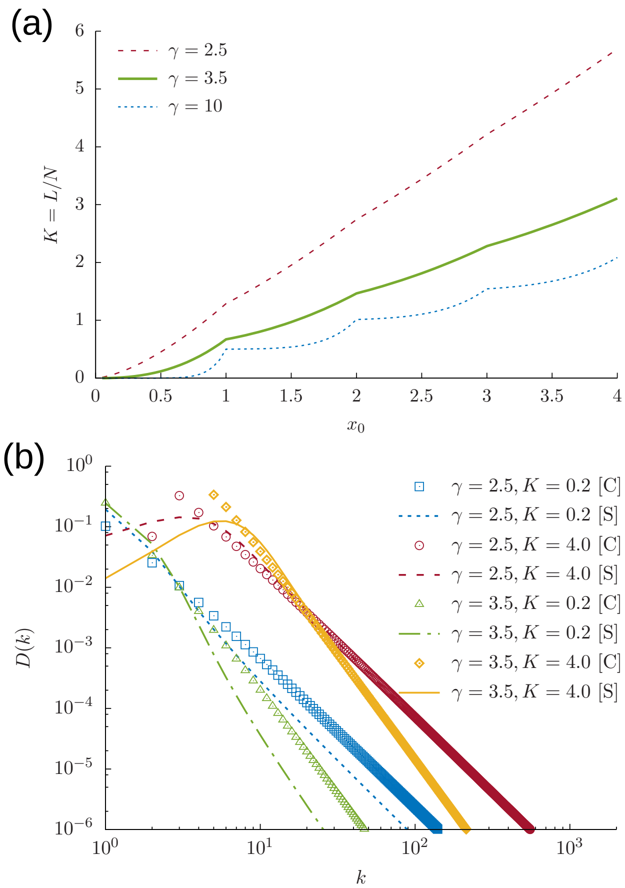

FIG. 1. The number of links per node and the degree distribution in the configuration model. (a) The number of links per node K as a function of the parameter x_0 as given in Eq. (5) for different degree exponents λ. (b) The degree distribution $D(k)$ of the configuration-model network ([C]) as in Eq. (3) for different λ and K (points). For comparison, the degree distribution of the static model ([S]) from Eq. (B1) for the same sets of λ and K are also shown (lines).

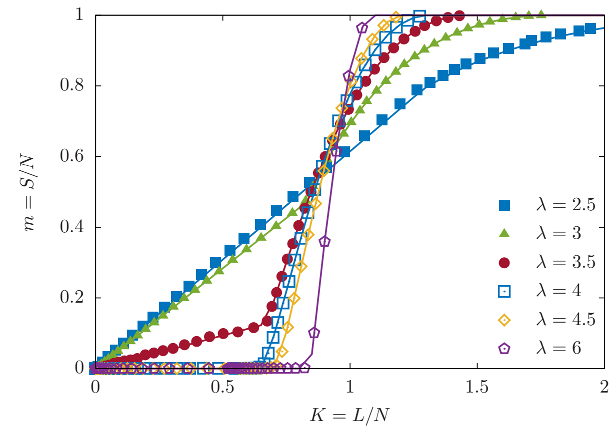

FIG. 2. The relative size of the LCC $m = S/N$ as a function of the number of links per node $K = L/N$ in the configuration-model SF networks with the degree distribution in Eq. (3) for $N = 10^6$ and selected values of the degree exponent λ. The lines are from the exact solution to Eq. (8), which agree very well with simulation results (points).

A. Simulation results for the giant component's size

The size of the giant component is given for various degree exponents in Fig. 2. The most remarkable feature is that up to λ as large as 3.5, the critical point K_c at which the giant component begins to form is zero—that is, the giant component is formed for any non-zero value of K. By contrast, the static model only displays a vanishing threshold for $\lambda \le 3$ [See Eq. (B2) in Appendix B]. Furthermore, when $\lambda \lesssim 3.5$, the giant component grows *linearly* with K for small K then abruptly changes to a concave increasing function at a certain value of K. In contrast, for $\lambda = 4.5$ or 6, m shows a transition behavior at a threshold K_c, as in the static model. Yet the critical point K_c is between ~ 0.6 and 1 in our configuration-model networks, while it is between 0 and 1/2 in the static model.

These simulation results cast many questions regarding the behavior of m as a function of K and λ in the configuration-model SF networks. What is the origin of the linear growth of the giant component's size with K and in what range of λ is that behavior observed? What is the critical point K_c for large λ? How does m behave near the critical point? Most of all, one may wonder whether these critical behaviors are different from those previously known for random SF networks or in the static-model SF networks. It does not seem that these questions can be answered merely by examining simulation results. In the remaining part of this paper, we will present the analytic solutions which can answer these questions. We will show that the anomalous behaviors of the giant component in the configuration model originate in specific properties of the degree distribution in Eq. (3), which deviates from that of the static model particularly in the small-k region [Fig. 1]. The size of the giant component for small K or K near the critical point K_c can be obtained by assuming that the giant component is of tree structure, which allows a mapping to branching processes [12, 16]. The obtained analytic solution will allow us to rigorously understand the behavior of the giant component in our configuration-model networks.

B. Mapping to branching processes

While the branching process approach [17] for the study of cluster formation in networks is well known and has been widely used [12, 16, 18], we review the method here to provide a self-contained analysis.

It can be assumed and self-consistently verified that finite clusters have a low fraction of loops and are almost tree-like in structure [12]. For given K and λ, the ensemble of connected components in realizations of these networks can therefore be approximated by the ensemble of trees generated by a branching process, whose branching probability is given by the degree distribution of the networks. The probability that a root node generates k daughters is set to be equal to $D(k)$ and the probability that a node other than the root generates k daughters is given by $(k+1)D(k+1)/\langle k\rangle$. This mapping holds when finite connected components have a tree structure and the degrees of neighboring nodes are not correlated. Then, the cluster-size distribution $P(s)$, the probability that a node belongs to a size-s cluster, corresponds to the probability of a node to be the root of a size-s tree. $P(s)$ depends on the probability $R(s)$ that a link leads to a size-s tree.

Let us define the generating functions $\mathcal{P}(z) \equiv \sum_{s<\infty} P(s) z^s$ and $\mathcal{R}(z) \equiv \sum_{s<\infty} R(s) z^s$, where the summation runs only over finite size s. The two generating functions satisfy the following relations

$$\mathcal{P}(z) = z\, g_0(\mathcal{R}(z)), \tag{6}$$
$$\mathcal{R}(z) = z g_1(\mathcal{R}(z)), \tag{7}$$

where $g_0(z) \equiv \sum_{k=0}^{\infty} D(k) z^k$ and $g_1(z) \equiv \sum_{k=0}^{\infty}(k+1)D(k+1)z^k/\langle k\rangle = \frac{g_0'(z)}{\langle k\rangle}$ are defined in terms of the degree distribution [12]. Considering Eqs. (6) and (7) at $z \to 1^-$ and denoting $\mathcal{R}(1)$ by u, we find that the giant component size is evaluated as

$$\begin{aligned} m &= 1 - g_0(u), \\ u &= g_1(u). \end{aligned} \tag{8}$$

The variable $u \equiv \mathcal{R}(1) = \sum_{s<\infty} R(s)$ represents the probability that a link leads to a finite cluster. u is obtained by solving the second line in Eq. (8). The function $g_1(u)$ increases monotonically with u from $g_1(0) = D(1)/\langle k\rangle$ to $g_1(1) = 1$. Therefore there is always a trivial solution $u = 1$. There exists a non-trivial solution $u < 1$ if

$$g_1'(1) = \frac{\langle k^2\rangle - \langle k\rangle}{\langle k\rangle} > 1, \tag{9}$$

in which case the non-trivial solution u is the true value of $\mathcal{R}(1)$, and gives a non-zero value of m by Eq. (8). Therefore the critical point K_c is determined by Eq. (9), yielding the condition that the moment ratio $\langle k^2\rangle/\langle k\rangle$ must be larger than 2 for the emergence of the giant component.

C. Critical point

Let us first use Eq. (9) to determine the critical point for the giant component formation in configuration-model SF networks with the degree distribution in Eq. (3). For given x_0 and λ, the generating function $g_0(z)$ is given by

$$g_0(z) = z^{k_0}[1 - {x_0}^{\lambda-1}(1-z)\Phi(z, \lambda-1, k_0+1)], \tag{10}$$

and $g_1(z)$ is given by

$$\begin{aligned} g_1(z) &= z^{k_0-1}\frac{k_0 + {x_0}^{\lambda-1}\Psi(z,\lambda-1,k_0+1)}{k_0 + {x_0}^{\lambda-1}\zeta_\infty(\lambda-1,k_0+1)}, \\ \Psi(z,s,a) &\equiv \Phi(z,s,a) - (1-z)\Phi(z,s-1,a), \end{aligned} \tag{11}$$

where we used Eq. (5) for $\langle k\rangle$, $k_0 = \lfloor x_0 \rfloor$, and the Lerch transcendent $\Phi(z,s,a) = \sum_{\ell=0}^{\infty}(\ell+a)^{-s} z^\ell$. Note that $\Phi(1,s,a) = \zeta_\infty(s,a)$ and $\Phi(z,s,1) = z^{-1}\mathrm{Li}_s(z)$ with $\mathrm{Li}_s(z) = \sum_{\ell=1}^{\infty} \ell^{-s} z^\ell$ the polylogarithm function.

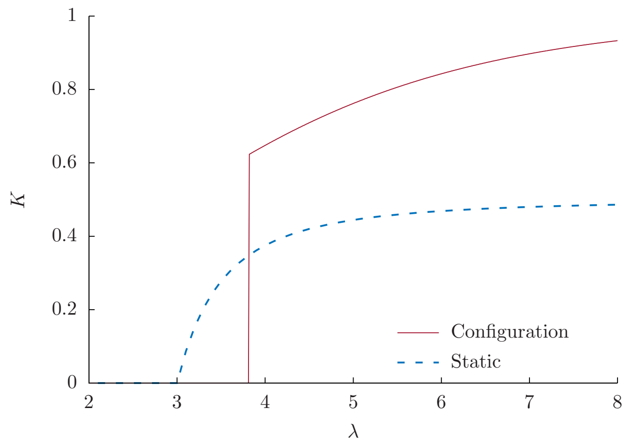

FIG. 3. Phase boundary between the percolating and non-percolating phase in the configuration model and the static model. The critical number of links per node K_c is given as a function of the degree exponent λ such that the giant component exists for $K > K_c$. Note that $K_c = 0$ for $\lambda < 3$ in the static model and for $\lambda < \lambda_* \simeq 3.81$ in the configuration model.

Using Eq. (11), we obtain

$$g_1'(1) = \frac{k_0^2 - k_0 + 2{x_0}^{\lambda-1}\{\zeta_\infty(\lambda-2,k_0+1) - \zeta_\infty(\lambda-1,k_0+1)\}}{k_0 + {x_0}^{\lambda-1}\zeta_\infty(\lambda-1,k_0+1)}, \tag{12}$$

where the relation $(\partial/\partial z)\Phi(z,s,a) = z^{-1}\{\Phi(z,s-1,a) - a\Phi(z,s,a)\}$ is used. To derive the condition for $g_1'(1) > 1$, we introduce $Q(\lambda, x_0)$ and $B(\lambda, n)$ for integer n defined as

$$\begin{aligned} Q(\lambda, x_0) &\equiv {k_0}^2 - 2k_0 + {x_0}^{\lambda-1}B(\lambda, k_0), \\ B(\lambda, n) &\equiv 2\zeta_\infty(\lambda-2, n+1) - 3\zeta_\infty(\lambda-1, n+1), \end{aligned} \tag{13}$$

which help us see better how $g_1'(1)$ in Eq. (12) depends on λ and x_0. As $Q = \langle k\rangle\{g_1'(1) - 1\} = \langle k^2\rangle - 2\langle k\rangle$, the giant component appears if $Q > 0$, and the critical point is x_{0c} such that $Q > 0$ for $x_0 > x_{0c}$. We now analyze the conditions of giant component formation in the plane of λ and x_0, eventually leading to the phase diagram in Fig. 3 as shown below.

For $2 < \lambda \leq 3$, the function $B(\lambda, n)$ diverges, and Q is positive for any $x_0 > 0$, giving the vanishing critical point $x_{0c} = 0$. In the region $\lambda > 3$, $B(\lambda, n)$ decreases with increasing λ, asymptotically approaching -1 for $n = 0$ and 0 for $n \geq 1$ [see Appendix D]. Therefore Q can be negative only when $x_0 < 2$ ($n < 1$), and the critical point x_{0c} for $\lambda > 3$ should be between 0 and 2, if it exists.

To further understand the behavior of Q in the case of $\lambda > 3$, let us first look into the region $0 < x_0 < 1 (k_0 = 0)$. In this region, $Q = {x_0}^{\lambda-1}B(\lambda, 0)$ is positive if $B(\lambda, 0) > 0$, which holds for $\lambda < \lambda_*$ where λ_* is the value of λ satisfying the relation

$$B(\lambda_*, 0) = 2\zeta(\lambda_* - 2) - 3\zeta(\lambda_* - 1) = 0, \tag{14}$$

and is found to be

$$\lambda_* = 3.810639333567\ldots. \tag{15}$$

This means that, if $0 < x_0 < 1$, Q is positive ($\langle k^2\rangle/\langle k\rangle > 2$) in the region $3 < \lambda < \lambda_*$ and negative where $\lambda > \lambda_*$. Next we examine the region $1 \le x_0 < 2$ and $\lambda > 3$. One can see that $Q = -1 + {x_0}^{\lambda-1}B(\lambda, 1)$ is positive as long as $x_0 > x_{0c}(\lambda)$ with

$$\begin{aligned} x_{0c}(\lambda) &= B(\lambda, 1)^{-\frac{1}{\lambda-1}} \\ &= \{2\zeta(\lambda-2) - 3\zeta(\lambda-1) + 1\}^{-\frac{1}{\lambda-1}} \end{aligned} \tag{16}$$

for $\lambda \ge \lambda_*$. Notice that $x_{0c} = 1$ at $\lambda = \lambda_*$ and approaches 2 as λ goes to infinity [Fig. 8].

Using $x_{0c} = 0$ for $2 < \lambda < \lambda_*$ and Eq. (16) for $\lambda \ge \lambda_*$ and the relation between x_0 and K in Eq. (5), we find that the giant component emerges for $K > K_c(\lambda)$ with

$$K_c = \begin{cases} 0 & \text{for } 2 < \lambda < \lambda_*, \\ \frac{1}{2}\left[1 + x_{0c}^{\lambda-1}\zeta(\lambda-1, 2)\right] & \\ = \frac{\zeta(\lambda-2)-\zeta(\lambda-1)}{2\zeta(\lambda-2)-3\zeta(\lambda-1)+1} & \text{for } \lambda \ge \lambda_*, \end{cases} \tag{17}$$

where λ_* and x_{0c} are given in Eqs. (15) and (16), respectively. The critical point is $K_c(\lambda_*) = (1/2)[1+\zeta(\lambda_*-1, 2)] \simeq 0.62217$ at $\lambda = \lambda_*$ and approaches 1 for $\lambda \to \infty$, which is shown along with the critical point Eq. (B2) of the static model in Fig. 3. The critical points for selected values of λ in the simulation results in Fig. 2 are consistent with Eq. (17).

D. Critical exponent β

The relative size m of the giant component near the critical point can be obtained analytically by solving Eq. (8). Let us consider the giant component size for K near $K_c = 0$ and $2 < \lambda < \lambda_*$. When $0 < x_0 < 1$ or $k_0 = 0$, corresponding to the number of links per node being in the range $0 < K < K_1(\lambda)$ with

$$K_1(\lambda) \equiv \frac{1}{2}\zeta(\lambda-1), \tag{18}$$

the function $g_1(z)$ is given by

$$g_1(z) = \frac{z^{-2}\left[\mathrm{Li}_{\lambda-1}(z) - (1-z)\mathrm{Li}_{\lambda-2}(z)\right]}{\zeta(\lambda-1)}, \tag{19}$$

independent of x_0 or of K, which turns out to be responsible for the linear growth of m for small K and $2 < \lambda < \lambda_*$ as shown below. Let $u_1(\lambda)$ denote the solution to $u = g_1(u)$ with Eq. (19) used. The solution u to Eq. (8) remains fixed at $u_1(\lambda)$ as K increases up to K_1. Then the relative size m of the giant component is found to be proportional to K as

$$m = 1 - g_0(u_1) = 2a_{(\mathrm{I})}K \tag{20}$$

with the coefficient $a_{(\mathrm{I})} = (1-u_1)\frac{\mathrm{Li}_{\lambda-1}(u_1)}{u_1\zeta(\lambda-1)}$. Therefore the size of the giant component grows linearly with K for given λ. If we define the critical exponent β in the relation $m \sim K^\beta$ in case of $K_c = 0$, we find $\beta = 1$ for $2 < \lambda < \lambda_*$.

For $\lambda \ge \lambda_*$, the critical point K_c increases from $K_c(\lambda_*) \simeq 0.62217$ towards 1 as λ increases from λ_* to ∞, corresponding to $1 < x_{0c} < 2$. Let us assume that K is larger than $K_c(\lambda)$ but staying around it such that $K_c(\lambda) < K < 1$. The generating function $g_1(z)$ with $k_0 = 1$ depends on x_0 or on K, in contrast to the case of $2 < \lambda < \lambda_*$, and is given by

$$g_1(z) = \frac{1 + {x_0}^{\lambda-1}\left\{z^{-2}\mathrm{Li}_{\lambda-1}(z) + z^{-1}(1-z^{-1})\mathrm{Li}_{\lambda-2}(z) - 1\right\}}{1 + {x_0}^{\lambda-1}\zeta(\lambda-1, 2)}. \tag{21}$$

Recall that its derivative at $z = 1$ is larger than 1 only for $x_0 > x_{0c}(\lambda)$ in Eq. (16) or equivalently $K > K_c$ in Eq. (17). Let us expand Eq. (21) in terms of $\alpha = -\ln z$, small around $z = 1$, as

$$\begin{aligned} &g_1(z = e^{-\alpha}) \\ &\quad = 1 + c_1\alpha + c_2\frac{\alpha^2}{2} + \cdots + c_{\lambda-2}\frac{\alpha^{\lambda-2}}{(\lambda-2)!}\left(1 + O(\alpha)\right), \end{aligned} \tag{22}$$

where the coefficients are

$$\begin{aligned} c_1 &= -1 - \frac{K - K_c}{K_c\,(2K_c - 1)}, \\ c_2 &= 1 + \frac{3}{2}\frac{\zeta(\lambda-3) - 3\zeta(\lambda-2) + 2\zeta(\lambda-1)}{\zeta(\lambda-2) - \zeta(\lambda-1)}, \\ c_{\lambda-2} &= \frac{\Gamma(\lambda)\Gamma(2-\lambda)}{2\{\zeta(\lambda-2) - \zeta(\lambda-1)\}}, \end{aligned}$$

with $\Gamma(s)$ the gamma function. Using Eq. (22) in Eq. (8), we obtain the solution $u = e^{-\alpha}$ with α given by

$$\alpha \simeq \begin{cases} \left\{-\frac{c_1+1}{c_{\lambda-2}}(\lambda-2)!\right\}^{\frac{1}{\lambda-3}} = a_{(\mathrm{II})}\Delta^{\frac{1}{\lambda-3}} & \text{for } \lambda_* \le \lambda < 4, \\ -2\frac{c_1+1}{c_2-1} = a_{(\mathrm{III})}\Delta & \text{for } \lambda > 4, \end{cases} \tag{23}$$

with $\Delta \equiv (K/K_c - 1)$, and the coefficients $a_{(\mathrm{II})} = \left[\frac{2\{\zeta(\lambda-2)-\zeta(\lambda-1)\}}{(2K_c-1)(\lambda-1)\Gamma(2-\lambda)}\right]^{\frac{1}{\lambda-3}}$ and $a_{(\mathrm{III})} = \frac{4}{3}\frac{\zeta(\lambda-2)-\zeta(\lambda-1)}{(2K_c-1)\{\zeta(\lambda-3)-3\zeta(\lambda-2)+2\zeta(\lambda-1)\}}$. Inserting Eq. (23) into $m = 1 - g_0(u = e^{-\alpha}) \simeq 2K_c\alpha$ with $k_0 = 1$, we obtain the relative size m of the giant component around the critical point K_c for $\lambda \ge \lambda_*$. Using these results and Eq. (20), we find that m near the critical point K_c follows

$$m \simeq \begin{cases} 2\,a_{(\mathrm{I})}\,K & \text{for } 2 < \lambda < \lambda_*, \\ 2\,K_c\,a_{(\mathrm{II})}\,\Delta^{\frac{1}{\lambda-3}} & \text{for } \lambda_* \le \lambda < 4, \\ 2\,K_c\,a_{(\mathrm{III})}\,\Delta & \text{for} \lambda > 4. \end{cases} \tag{24}$$

Hence, the critical exponent β is

$$\beta = \begin{cases} 1 & \text{for } 2 < \lambda < \lambda_*, \\ \frac{1}{\lambda-3} & \text{for } \lambda_* \le \lambda < 4, \\ 1 & \text{for } \lambda > 4. \end{cases} \tag{25}$$

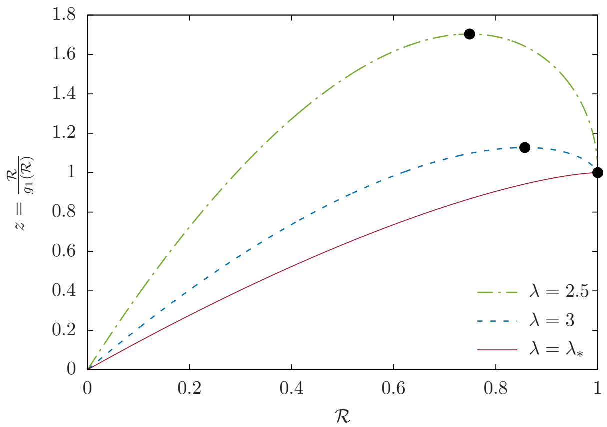

FIG. 4. Plots of the inverse of $\mathcal{R}(z)$, $z(\mathcal{R}) = \mathcal{R}/g_1(\mathcal{R})$ with $k_0 = 0$ for selected values of the degree exponent λ. The dots are at $(\mathcal{R}_0, z_0)$, where the derivative $dz/d\mathcal{R}$ is zero. For example, $(\mathcal{R}_0, z_0) = (0.749, 1.70)$ and $(0.857, 1.13)$ for $\lambda = 2.5$ and 3.0, respectively. It converges to $(1, 1)$ as λ approaches λ_*. Note that only the branch in the region $\mathcal{R} < \mathcal{R}_0$ and $0 \leq z < z_0$ corresponds to the generating function $\mathcal{R}(z)$.

This is different from previous results obtained for the static-model SF networks, Eqs. (B4) and (B5). The most striking deviation is the linear growth of m with K even for λ larger than 3, up to λ_*, whereas in the static model, the giant component appears only for $K > K_c > 0$ in this range of degree exponents.

E. Cluster-size distribution at or near the critical point

While the solution to Eq. (6) in the closed form may be hard to obtain, the leading singularity of the generating function $\mathcal{P}(z)$ can be often identified, revealing the tail behavior of the cluster-size distribution $P(s)$ [12]. According to Eq. (7), the inverse of the generating function $\mathcal{R}(z)$ is represented as

$$z(\mathcal{R}) = \frac{\mathcal{R}}{g_1(\mathcal{R})}, \tag{26}$$

and some examples are shown in Fig. 4. Once the singularity of $\mathcal{R}(z)$ is identified, one can obtain that of $\mathcal{P}(z)$ by using Eq. (6).

We are interested in the asymptotic behavior of the cluster-size distribution near and at the critical point for $\lambda < \lambda_*$ and $\lambda > \lambda_*$, respectively. Let us first consider the cluster-size distribution for small K and $\lambda < \lambda_*$, for which the critical point is zero, i.e., $K_c = 0$. When K is smaller than $K_1(\lambda)$ in Eq. (18), $g_1(z)$ is given by Eq. (19). As shown in Fig. 4, there is a point $(\mathcal{R}_0, z_0 = \mathcal{R}_0/g_1(\mathcal{R}_0))$ where the inverse function $z(\mathcal{R})$ has zero derivative. Around the point, it is expanded as

$$z = z_0 - \frac{1}{2}\frac{\mathcal{R}_0 g_1''(\mathcal{R}_0)}{g_1(\mathcal{R}_0)^2}(\mathcal{R} - \mathcal{R}_0)^2 + \cdots, \tag{27}$$

where $g_1(x)$ is given in Eq. (19) and $g_1''(x) = (d^2/dx^2)g_1(x > 0)$. Therefore $\mathcal{R}(z)$ possesses a square-root singularity around z_0 as

$$\mathcal{R}(z) \simeq \mathcal{R}_0 - \sqrt{\frac{2g_1(\mathcal{R}_0)^2}{\mathcal{R}_0 g_1''(\mathcal{R}_0)}}\sqrt{z_0 - z} + \cdots. \tag{28}$$

Expanding $\mathcal{P}(z) = zg_0(\mathcal{R}(z))$, as in Eq. (6), around z_0, we find that

$$\mathcal{P}(z) \simeq z_0 g_0(\mathcal{R}_0) - 2K\sqrt{\frac{2\mathcal{R}_0 g_1(\mathcal{R}_0)^2}{g_1''(\mathcal{R}_0)}}\sqrt{z_0 - z} + \cdots. \tag{29}$$

Recalling that $\mathcal{P}(z) = \sum_s P(s)z^s$ and using the relation

$$\begin{aligned}(1-x)^\theta &= \sum_{s=0}^{\infty}\frac{(-x)^s}{s!}\frac{\Gamma(\theta+1)}{\Gamma(\theta-s+1)}\\ &= -\sum_{s=0}^{\infty}\frac{x^s}{s!}\frac{\theta!(s-\theta-1)!\sin\pi\theta}{\pi},\end{aligned} \tag{30}$$

which allows us to use the Stirling's formula $s! \simeq s^s e^{-s}\sqrt{2\pi s}$ for large s, we obtain the tail behavior of $P(s)$ as

$$P(s) \simeq 2K\, p_{(\mathrm{I})} s^{-\frac{3}{2}} e^{-\frac{s}{s_0}}, \tag{31}$$

where $p_{(\mathrm{I})} = \sqrt{\frac{z_0 \mathcal{R}_0 g_1(\mathcal{R}_0)^2}{2\pi g_1''(\mathcal{R}_0)}}$ and $s_0 = \frac{1}{\ln z_0} = \frac{1}{\ln\left(\frac{\mathcal{R}_0}{g_1(\mathcal{R}_0)}\right)}$ are constants depending on λ.

Note that the cut-off constant s_0 is finite for $\lambda < \lambda_*$ and diverges at $\lambda = \lambda_*$. Therefore one cannot observe a power-law decay of $P(s)$ even for small K unless $\lambda = \lambda_*$. The exponential decay of $P(s)$ for any non-zero K and $\lambda < \lambda_*$ implies that the configuration-model network is in the supercritical (percolating) phase. Moreover, $P(s)$ is independent of K as long as $0 < K < K_1(\lambda)$ for given λ. All these features are contrasted to $P(s)$ in the static model, which decays as a power-law for a wide range of s, depending on K, if K is small and $2 < \lambda < 3$ [12]. The invariance of $P(s)/(2K)$ in Eq. (31) against the variation of K in the range $0 < K < K_1(\lambda)$, as shown also in Fig. 5 (a), originates in the specific form of the degree distribution for $0 < x_0 < 1$:

$$D(k) = \begin{cases} 1 - \frac{\langle k\rangle}{\zeta(\lambda-1)} & \text{for } k = 0,\\ \langle k\rangle\frac{k^{1-\lambda}-(k+1)^{1-\lambda}}{\zeta(\lambda-1)} & \text{for } k \geq 1,\end{cases} \tag{32}$$

which leads $(k+1)D(k+1)/\langle k\rangle$ for $k \geq 0$ and its generating function $g_1(z)$ to be independent of K or x_0.

At the critical point K_c for $\lambda \geq \lambda_*$, the inverse function $z(\mathcal{R})$ in Eq. (26) should be computed with $g_1(x)$ given in Eq. (21) since $k_0 = 1$ at the critical point. Then one finds $z(\mathcal{R})$ has zero derivative at $\mathcal{R}_0 = 1$. Using Eq. (22) at $K = K_c$, where $c_1 = -1$, we find that $z(\mathcal{R})$ is expanded around $(\mathcal{R}_0, z_0) = (1, 1)$ as

$$z = 1 - \frac{c_2 - 1}{2}(1-\mathcal{R})^2 - \frac{c_{\lambda-2}}{(\lambda-2)!}(1-\mathcal{R})^{\lambda-2} + \cdots. \tag{33}$$

In the right-hand side of Eq. (33), the $(1-\mathcal{R})^2$ term is dominant over $(1-\mathcal{R})^{\lambda-2}$ for $\lambda > 4$ and the latter is dominant for $\lambda_* \leq$

$\lambda < 4$. By the relation between $\mathcal{P}(z)$ and $\mathcal{R}(z)$ in Eq. (6) and the expansion $g_0(z) = 1 - 2K(1-z) + O((1-z)^2, (1-z)^{\lambda-1})$, we find that $\mathcal{P}(z)$ behaves around $z = 1$ as

$$\mathcal{P}(z) \simeq \begin{cases} 1 - 2K\left(\frac{(\lambda-2)!}{c_{\lambda-2}}\right)^{\frac{1}{\lambda-2}}(1-z)^{\frac{1}{\lambda-2}} & \text{for } \lambda_* \le \lambda < 4, \\ 1 - 2K\sqrt{\frac{2}{c_2-1}}\sqrt{1-z} + \cdots & \text{for } \lambda > 4. \end{cases} \tag{34}$$

Finally we obtain the tail behavior of the cluster-size distribution by using Eq. (30), which is characterized by the exponent $\frac{\lambda-1}{\lambda-2}$ and $3/2$ for $\lambda_* \le \lambda < 4$ and $\lambda > 4$, respectively, as

$$P(s) \simeq \begin{cases} 2K_c\, p_{(\mathrm{II})} s^{-\frac{\lambda-1}{\lambda-2}} & \text{for } \lambda_* \le \lambda < 4, \\ 2K_c\, p_{(\mathrm{III})} s^{-\frac{3}{2}} & \text{for } \lambda > 4, \end{cases} \tag{35}$$

with the coefficients given by $p_{(\mathrm{II})} = \left(\frac{(\lambda-2)!}{c_{\lambda-2}}\right)^{\frac{1}{\lambda-2}} \frac{\sin\left(\frac{\pi}{\lambda-2}\right)}{\pi}\left(\frac{1}{\lambda-2}\right)!$ and $p_{(\mathrm{III})} = \sqrt{\frac{1}{2\pi(c_2-1)}}$. While the exponent depends on λ for $3 < \lambda < 4$ in the static model, it does only for $\lambda_* \le \lambda < 4$ in the configuration model.

In Fig. 5, we present the theoretical results in Eqs. (31) and (35) along with the simulation results for the cluster-size distributions for selected values of the degree exponent, which are in good agreement regarding their tail behaviors.

IV. EFFECTS OF THE FRACTION OF LOW-DEGREE NODES ON THE GIANT COMPONENT

As critical phenomena are in general independent of microscopic details but display universality, one would expect that the percolation transition in SF networks should exhibit universal features, regardless of the concrete form of the degree distribution, as long as its power-law decay is characterized by a given degree exponent. In this light, the fact that our results go against this expectation is rather surprising. Moreover, it is remarkable that two major models for SF networks, the static and the configuration model, exhibit different critical phenomena for the same degree exponent.

The excellent agreement between simulations and analytic results, derived based on the degree distribution only, means that such different behaviors of the percolation transition between the two models stem from their different degree distributions. Examining $D(k)$ and $D_{\mathrm{static}}(k)$ given in Eqs. (3) and (B1) and exemplified in Fig. 1, one sees their difference particularly in the small-k region suggesting the relevance of the small-k behavior of the degree distribution to forming connected components in networks.

The difference of the giant component size m as a function of the number of links per node K between the two models is the most dramatic when the degree exponent λ is between 3 and λ_*; The critical point K_c is non-zero for the static model while it is zero for the configuration model. In Fig. 6, it is shown that m is larger in the configuration-model network if K is either very small or large, while it is larger in the static model in the intermediate range of K. While such difference can be traced back to the full functional form of their degree

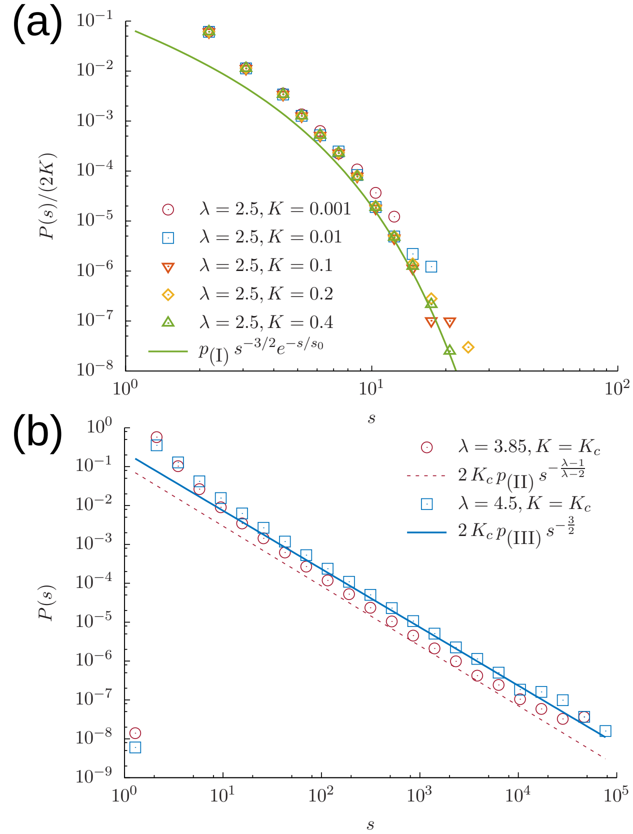

FIG. 5. Cluster-size distribution $P(s)$ near or at the critical point in the configuration-model SF networks of $N = 10^7$ nodes and several degree exponents. (a) Plots of $P(s)/(2K)$ versus s for $\lambda = 2.5$ and $K = 0.0001, 0.01, 0.1, 0.2$ and 0.4. The theoretical prediction from Eq. (31) is also shown (line). (b) Plots of $P(s)$ versus s at the critical point $K_c = 0.628$ and $K_c = 0.709$ for $\lambda = 3.85$ and $\lambda = 4.5$, respectively. The lines are the theoretical predictions from Eq. (35).

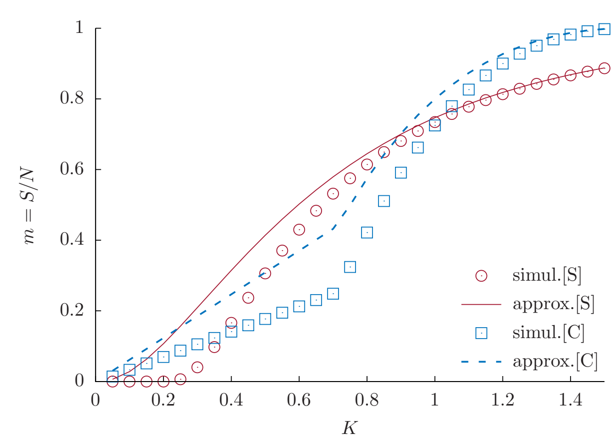

FIG. 6. Plots of the relative size of the giant component m versus the number of links per node K in the static model ([S]) and the configuration model ([C]) with $N = 10^6$ and $\lambda = 3.3$. Points are from the exact solution to Eq. (8) and lines are from the approximation in Eq. (37).

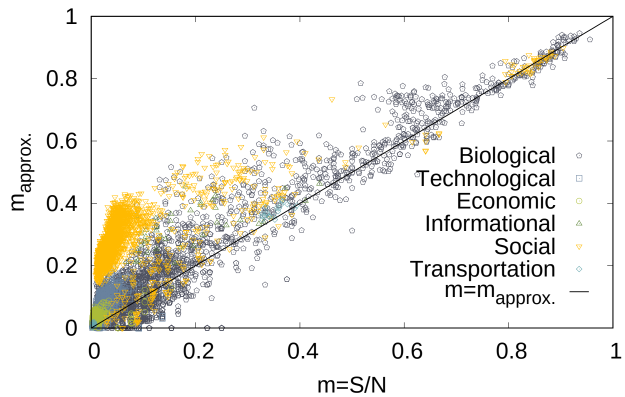

FIG. 7. Scatter plots of the approximation $m_{\rm approx.}$ from Eq. (37) versus the true value m of the relative size of the giant component in real-world networks after 90% of links are removed randomly. 20 such damaged networks are generated for each of 401 real networks.

distributions, we can clarify analytically the importance of the fraction of low-degree nodes when K is large. Assuming that K is large, we find u from Eq. (8) expanded as

$$u = \frac{D(1)}{2K} + \frac{D(1)D(2)}{2K^2} + O(K^{-3}), \tag{36}$$

and the giant component has relative size m given by

$$\begin{aligned} m_{\rm approx.} &\simeq 1 - D(0) - D(1)u - D(2)u^2 \\ &\simeq 1 - D(0) - \frac{D(1)^2}{2K} - \frac{3D(1)^2 D(2)}{4K^2}. \end{aligned} \tag{37}$$

While obtained for K large, this approximation works very well for $K \gtrsim 1$, and reasonably well even for K small, both in the static and the configuration model [Fig. 6]. Moreover, as noted above, the static model will form a larger giant component than the configuration model for intermediate values of K, for example, $0.4 \lesssim K \lesssim 1$ for $\lambda = 3.3$ in Fig. 6, and we observe the same phenomenon for $m_{\rm approx.}$ in the range $0.3 \lesssim K \lesssim 0.9$. This suggests that the difference of $D(k)$ for small k such as $k = 0, 1$, or 2 between the two models is partly responsible for their difference in m.

The approximation in Eq. (37) allows us to estimate the size of the giant component in terms of the fraction of low-degree nodes, which can be of practical use. Suppose that a real network is being attacked, losing a significant fraction of links. In such emergency, it is important to know the size of the giant component. But the full adjacency matrix, necessary to identify the giant component and its size, may be unavailable due to insufficient time or resources. Rather than struggling to collect information of the full adjacency matrix, one can instead count just the number of significantly damaged nodes such as those having zero, one or just two connected neighbors and use them in Eq. (37) to approximate the size of the giant component. To test this idea, we generate the damaged networks by removing randomly 90 % of links in various real-world networks, available in [19], and compute the relative sizes of the giant components as well as the fractions of low-degree nodes. In Fig. 7, the approximation $m_{\rm approx.}$ made by Eq. (37) shows a good agreement with m, suggesting the usefulness of Eq. (37) in practical applications.

V. DISCUSSION

We have shown that the full functional form of the degree distribution can control the percolation transition and critical phenomena on scale-free (SF) networks. The exponent characterizing the power-law decay of the degree distribution, which has received most attention in theoretical and empirical settings, may not be sufficient to predict such behaviors, even those previously thought to be universal for SF networks. We have demonstrated this point on SF networks generated with the configuration model, given a degree distribution where number of links and power law exponent can be tuned separately. By studying the percolation transition in these networks numerically and analytically, and comparing its outcomes to known results illustrated by the static model [12], we have shown that important deviations originated in different shapes in the *lower* range of the degree distribution, i.e. for nodes with few links.

In previous studies, the role of diverging moments was shown to be important across models and dynamics, from percolation to other phenomena on SF networks such as disease spreading and synchronization. Likewise, we propose that nodes with low degree may also wield a general influence on critical behaviors, which should be explored. A better understanding of whether and when the lower range of the degree distribution controls critical and general dynamical properties would prove beneficial for a wide range of studies and applications.

Our proposed degree distribution exhibits, for parameter values $0 < x_0 < 1$ in Eq. (32), a power-law shape across the largest range of degrees k. It possesses a special property of invariance: the probability of being connected to a node with k links (computed as $kD(k)/\langle k\rangle$) does not depend on the average degree $\langle k\rangle$. Therefore, critical behaviors become independent of the number of links in the network, and we expect this property to translate to similarly robust phenomena in other dynamical processes.

ACKNOWLEDGMENTS

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Appendix A: The largest degree in the configuration-model SF networks

In the configuration-model SF networks, the fraction of self or multiple connections is negligible if $\lambda > 3$ or the maximum degree cutoff $k_{\rm max} \sim \sqrt{N}$ is introduced for $2 < \lambda < 3$ [13]. In simulations, we restrict the range of x to $x \in [x_0, x_{\rm max}]$ in Eq. (1) so as to realize the upper cutoff $k_{\rm max} \sim \sqrt{N}$ for $2 < \lambda <$

3. However, the introduction of $k_{\max}$ does not significantly change any of the presented theoretical results in the limit $N \to \infty$, and so we will use Eq. (1) for simplicity in the theoretical analysis.

Appendix B: Percolation transition in the static model

In the static model, to construct a SF network with N nodes, $L = NK$ links, and degree exponent λ, each node i is assigned a selection probability $w_i = i^{-\alpha}/[\zeta(\alpha) - \zeta(\alpha, N+1)]$ with $\alpha = 1/(\lambda - 1)$, which is used to connect a pair of nodes i and j with probability $w_i w_j$. As a result, the degree distribution is given by

$$D_{\text{static}}(k) = \frac{1}{k!}\frac{d^k}{d\omega^k}\tilde{\Gamma}(2K(1-\omega)) \text{ with}$$
$$\tilde{\Gamma}(y) \equiv (\lambda - 1)\left(\frac{\lambda-2}{\lambda-1}y\right)^{\lambda-1}\Gamma\left(1-\lambda, \frac{\lambda-2}{\lambda-1}y\right), \quad \text{(B1)}$$

where $\Gamma(s,x)$ the incomplete Gamma function $\Gamma(s,x) \equiv \int_x^\infty dt\, t^{s-1}e^{-t}$, and takes a power-law form $D_{\text{static}}(k) \sim k^{-\lambda}$ for large k. As shown in Fig. 1 (b), the behavior of the degree distribution of the static model is different from that of the configuration model particularly in the small-k range, even if they share the same number of links per node K and the same large-k behavior with identical degree exponent λ. The static model is a generalization of the Erdös-Rényi network [20]. The independence of the probabilities of connecting different pairs of nodes allows one to obtain exactly the giant component and the size distribution of the finite-size clusters, with the help of the Potts model formulation [12]. Due to this independence, networks obtained by removing links randomly, leaving only K' links per node (while preserving the asymptotic power-law behavior of the degree distribution), are similar to networks generated directly with K' links per node. This similarity holds notably for the degree distribution and the critical phenomena associated with the percolation transition. Here we summarize the important properties of the static-model SF networks, as their comparison with the results for the configuration model is our main concern in the present work.

The relative size m of the LCC in the static model exhibits a transition, from zero to a non-zero value as a function of K at a threshold K_c if the degree exponent λ is larger than 3 [2, 12]. On the other hand, there is no such transition for $2 < \lambda < 3$, and m grows superlinearly with K in the small-K regime. Such distinct critical behaviors below and above $\lambda = 3$ have been proposed as the most remarkable feature of critical phenomena on SF networks, originating in the diverging second moment of the power-law degree distribution for $\lambda \leq 3$.

To be more specific, the percolation threshold K_c in the static model is given by [12]

$$K_c = \begin{cases} 0 & \text{for } 2 < \lambda \leq 3, \\ \frac{(\lambda-1)(\lambda-3)}{2(\lambda-2)^2} & \text{for } \lambda > 3. \end{cases} \quad \text{(B2)}$$

The relative size m of the LCC is zero for $K < K_c$ and

$$m \sim \left(\frac{K}{K_c} - 1\right)^\beta \quad \text{(B3)}$$

for $K \geq K_c$ if the degree exponent is as large as $\lambda > 3$. Here the critical exponent β is given by

$$\beta = \begin{cases} \frac{1}{\lambda-3} & \text{for } 3 < \lambda < 4, \\ 1 & \text{for } \lambda > 4. \end{cases} \quad \text{(B4)}$$

For $2 < \lambda < 3$, the LCC size behaves as

$$m \sim K^{\frac{1}{3-\lambda}} \text{ for } K \ll 1. \quad \text{(B5)}$$

At $K = K_c$ for $\lambda > 3$, the cluster-size distribution $P(s)$ takes a power-law form as

$$P(s) \sim s^{1-\tau} \quad \text{(B6)}$$

with the critical exponent

$$\tau = \begin{cases} \frac{2\lambda-3}{\lambda-2} & \text{for } 3 < \lambda < 4, \\ \frac{5}{2} & \text{for } \lambda > 4. \end{cases} \quad \text{(B7)}$$

For $2 < \lambda < 3$, $P(s)$ for small K (near the zero critical point) behaves as

$$P(s) \sim \left(\frac{s}{K}\right)^{1-\lambda}. \quad \text{(B8)}$$

In the removal of randomly-selected links reducing K in a SF network, the degree distribution maintains its asymptotic power-law behavior, and thus the shrinkage and extinction of the giant component is also expected to be characterized by the above results, as different pairs of nodes are connected independently, based only on K and the degree exponent λ. It is worthy to note that the degree distribution of the static-model SF networks with f fraction of links removed is equal to that of the static model with $K' = K(1-f)$, as shown in Appendix C. Therefore one might expect that the critical behaviors shown in this section hold generally in the percolation transition in SF networks.

The absence of a critical threshold for $2 < \lambda < 3$ and the critical exponents continuously varying with the degree exponent λ are observed in a wide range of dynamical processes including epidemic spreading [4], Ising model [3], synchronization [5, 21], order-disorder transition in the boolean dynamics [22], etc. For instance, the critical exponent β for the Ising model and the synchronization order parameter in the Kuramoto model on SF networks is also given by $\beta = 1/2$ for $\gamma > 5$ and $1/(\lambda - 3)$ for $3 < \lambda < 5$.

Appendix C: Degree distribution of the static model after links are removed

The generating function of the degree distribution of the static model is a function of $2K(1-z)$, that is, $g_0(z) \equiv$

$\sum_k D(k)z^k = \tilde{\Gamma}(2K(1-z))$ with $\Sigma_1(x)$ in Eq. (B1). When a fraction f of links are removed randomly, the degree distribution is changed to $D(k;f) = \sum_{k'} D(k)\binom{k'}{k}(1-f)^{k'-k}f^k$, the generating function of which is $g(z;f) = g(f+(1-f)z)$. For the static model network with K links per node before removal, it follows that $g(z;f) = \tilde{\Gamma}(2K(1-f)(1-z)) = \tilde{\Gamma}(2K'(1-z))$ with $K' = K(1-f)$ the very number of links per node after the removal of a fraction f of links.

Appendix D: Behaviors of $Q(\lambda, k_0)$ and $B(\lambda, n)$

Here we investigate the functional behaviors of $B(\lambda, n)$ and $Q(\lambda, x_0)$ defined in Eq. (13), which are used to obtain the phase diagram. For very large λ, the function $B(\lambda, n)$ in Eq. (13) can be approximated as $B(\lambda, n) \simeq 2(n+1)^{2-\lambda} - 3(n+1)^{1-\lambda} = (2n-1)(n+1)^{1-\lambda}$, which converges to

$$\lim_{\lambda\to\infty} B(\lambda, n) = \begin{cases} -1 & \text{for } n = 0, \\ 0 & \text{for } n \geq 1. \end{cases} \tag{D1}$$

For given n, $B(\lambda, n)$ decreases monotonically with λ, as its derivative is negative for all λ and $n \geq 0$;

$$\frac{\partial B(\lambda, n)}{\partial \lambda} = -\sum_{k=n+1}^{\infty} \frac{2k-3}{k^{\lambda-1}} \ln k < 0. \tag{D2}$$

Note that $B(\lambda, n)$ diverges for $\lambda \leq 3$. For $\lambda > 3$, we can refer to Eqs. (D1) and (D2) to find that $B(\lambda, n)$ is positive if n is positive. For $n = 0$, $B(\lambda, n)$ becomes negative for $\lambda > \lambda_*$ with λ_* in Eq. (15). These behaviors of $B(\lambda, n)$ are shown in Fig. 8 (a), which leads $Q(\lambda, x_0)$ to behave as in Fig. 8 (b) by Eq. (13). One finds that the value of λ at which $Q(\lambda, x_0) = 0$ is fixed at λ_* if $0 < x_0 \leq 1$ and increases from λ_* to infinity as x_0 increases from 1 to 2. This boundary between $Q > 0$ and $Q < 0$ can be best represented by the critical line $x_{0c}(\lambda)$ as a function of λ for $\lambda \geq \lambda_*$ given in Eq. (16) and another line $0 < x_0 \leq 1$ at $\lambda = \lambda_*$, which are shown in Fig. 8 (c) and give the phase diagram in the plane (λ, K) in Fig. 3.

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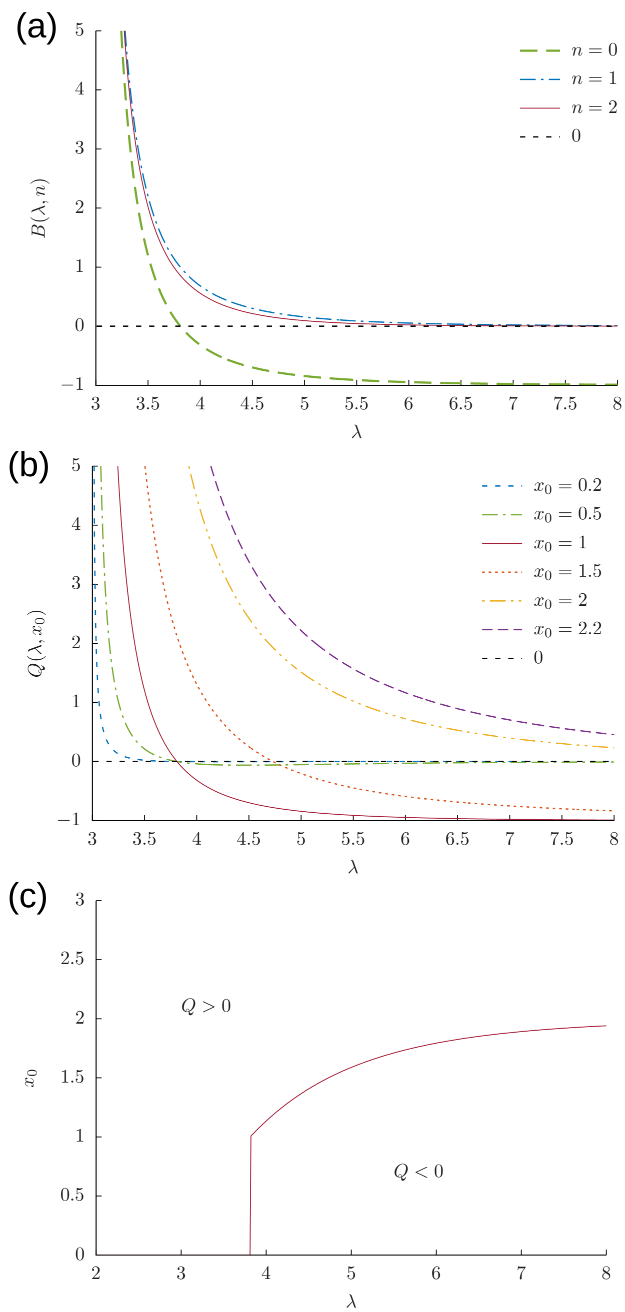

FIG. 8. Conditions for $Q > 0$. (a) The behavior of $B(\lambda, n)$ as a function of λ for different n's. (b) $Q(\lambda, x_0)$ versus λ for different x_0's. (c) The boundary between $Q > 0$ and $Q < 0$ in the (λ, x_0) plane drawn by Eq. (16).