FVE3011

Backpropagation, activation function

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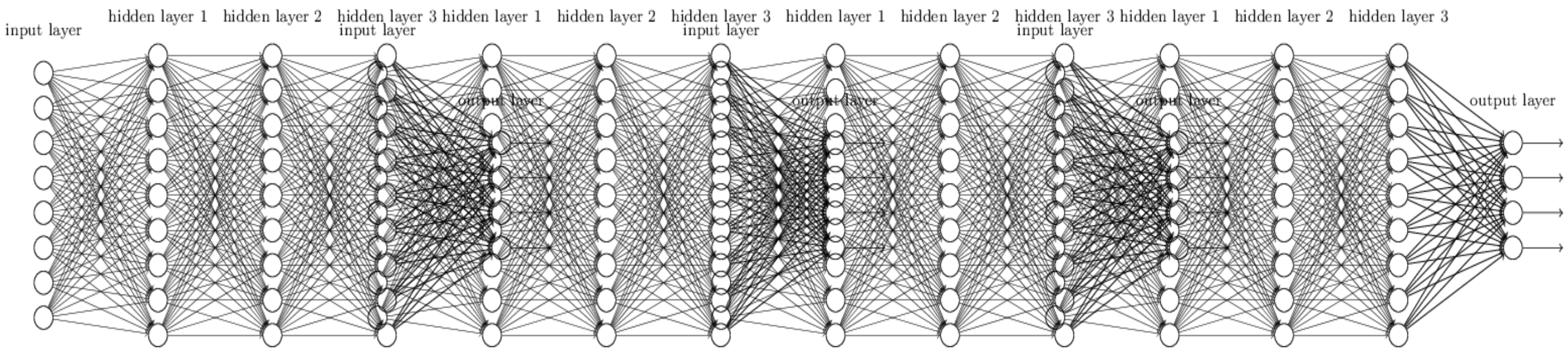
인하대학교

Contents

- Backpropagation

- Activation function

How to train?



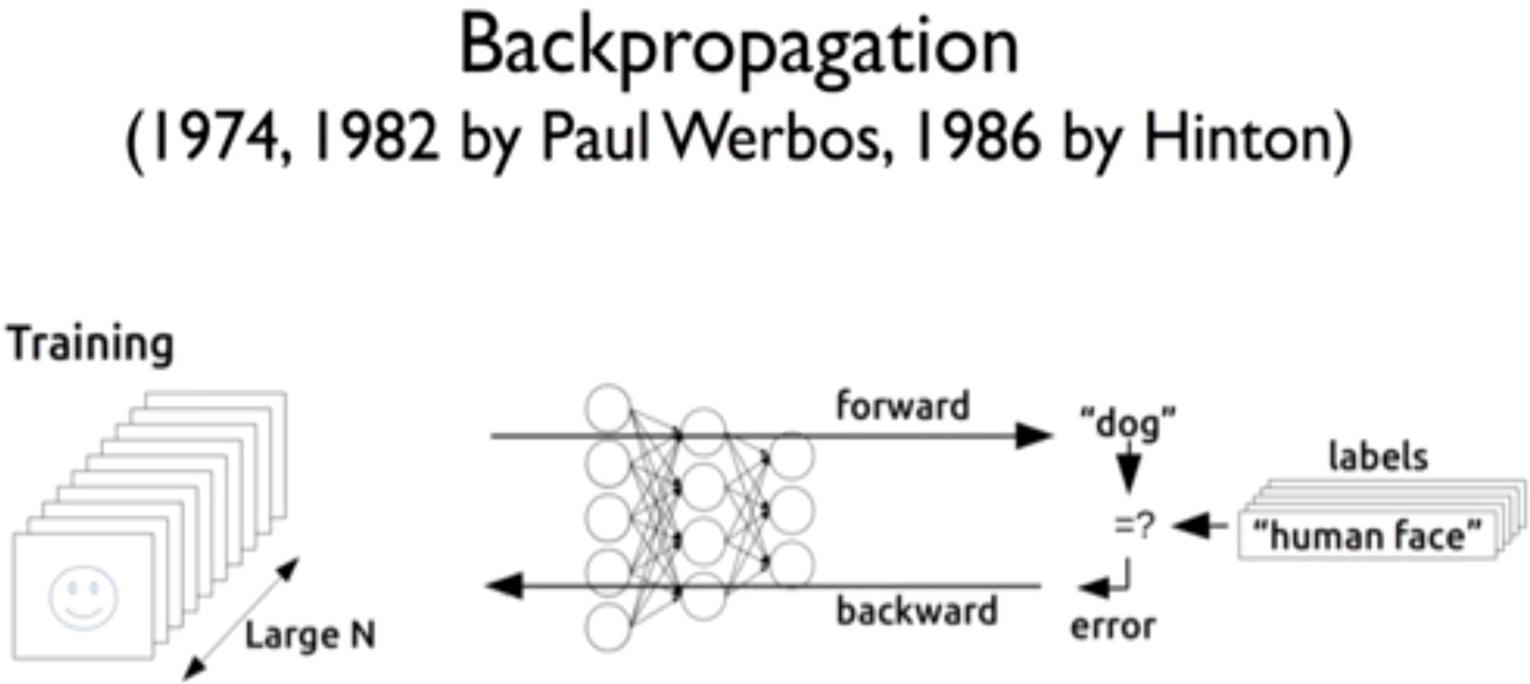
- GDA(Gradient descent algorithm)

$$W := W - \alpha \frac{1}{m} \sum_{i=1}^{m} (Wx^{(i)} - y^{(i)}) x^{(i)}$$

$$w = w - \alpha \frac{\partial E}{\partial w}$$

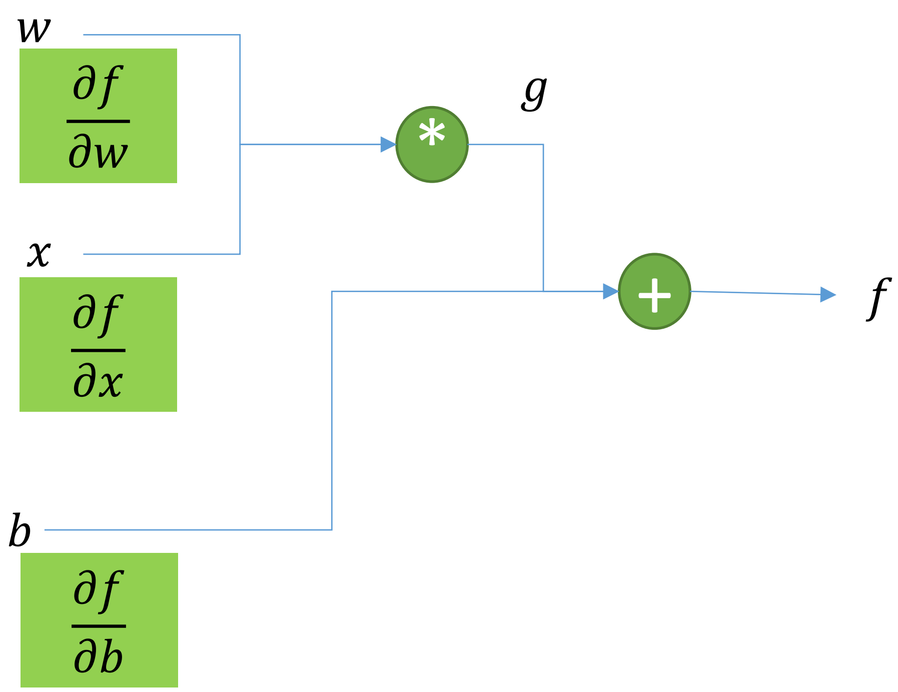
Backpropagation

- Solve the XOR problem using backpropagation



Back propagation (chain rule)

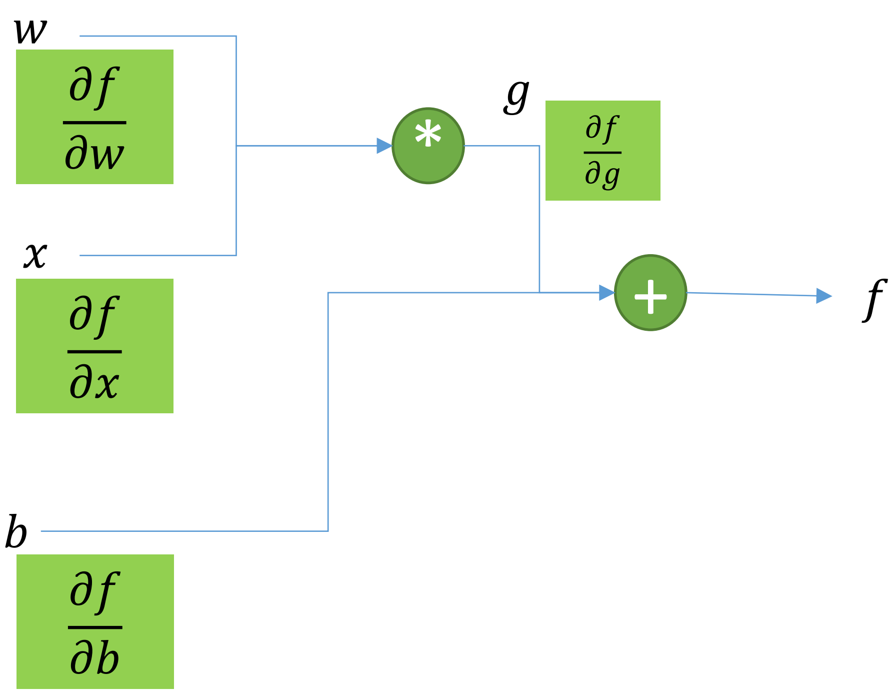
$f = wx + b$ $\quad\quad$ $g = wx$ $\quad\quad$ $f = g + b$



Back propagation (chain rule)

$\frac{\partial g}{\partial w} = x$ $\frac{\partial g}{\partial x} = w$ $\frac{\partial f}{\partial g} = 1$ $\frac{\partial f}{\partial b} = 1$

$f = wx + b$ $g = wx$ $f = g + b$



(1) Forward : w=-2,x=5,b=3

(2) backward :

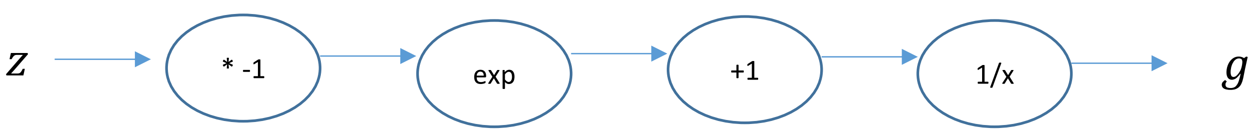
$$f(g(x)) = y$$

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial x}$$

Sigmoid

Sigmoid: $g(z) = \frac{1}{1+e^{-z}}$

$$\boxed{\frac{\partial g(z)}{\partial z} = ?}$$



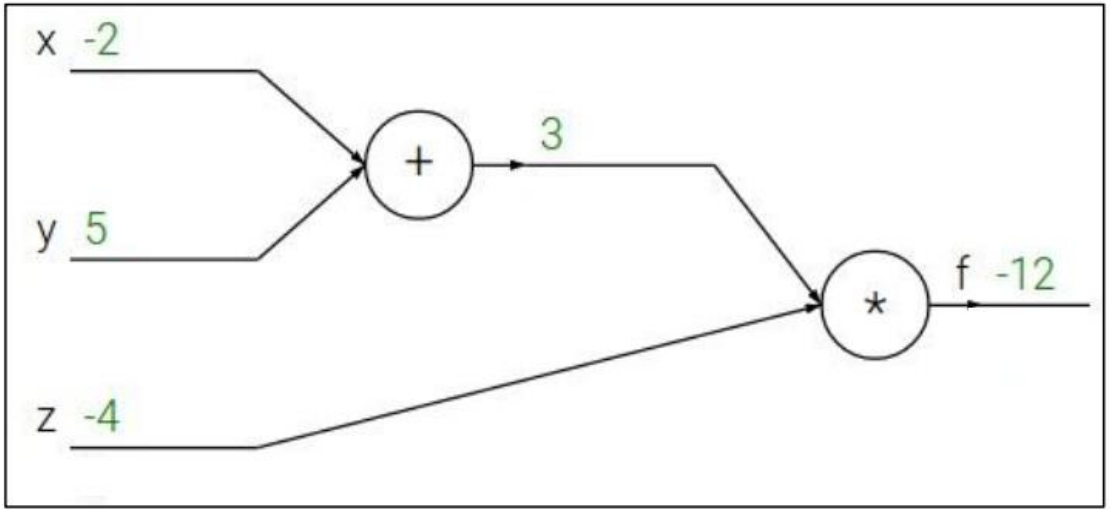
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sigma(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1+e^{-x}}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(1+e^{-x}\right)^{-1} \\
&= -1 * (1+e^{-x})^{-1-1} * -1 * (e^{-x}) \\
&= -1 * (1+e^{-x})^{-2} * -(e^{-x}) \\
&= \frac{e^{-x}}{(1+e^{-x})^2} \\
&= \frac{1}{(1+e^{-x})} * \frac{e^{-x}}{(1+e^{-x})} \\
&= \frac{1}{(1+e^{-x})} * \frac{1+e^{-x}-1}{(1+e^{-x})} \\
&= \frac{1}{(1+e^{-x})} * \left(\frac{(1+e^{-x})}{(1+e^{-x})} - \frac{1}{(1+e^{-x})}\right) \\
&= \frac{1}{(1+e^{-x})} * \left(1 - \frac{1}{(1+e^{-x})}\right)
\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{dx}}\sigma(x) = \sigma(x)(1-\sigma(x))$$

Start from Simple

$f(x, y, z) = (x + y)z$

e.g. x = -2, y=5, z = -4



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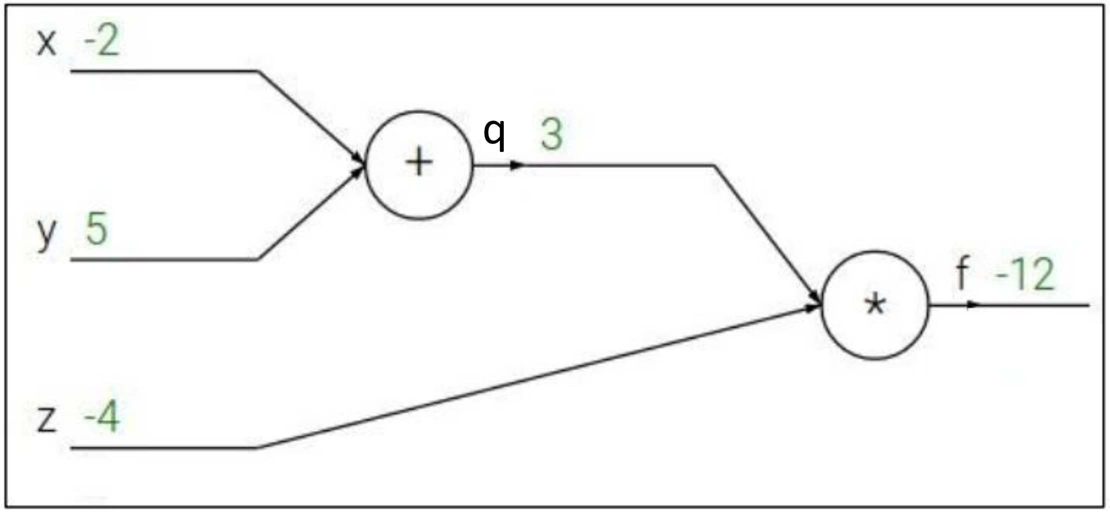

$$q = x + y \qquad \frac{\partial q}{\partial x} = 1, \frac{\partial q}{\partial y} = 1$$

$$f = qz \qquad \frac{\partial f}{\partial q} = z, \frac{\partial f}{\partial z} = q$$

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

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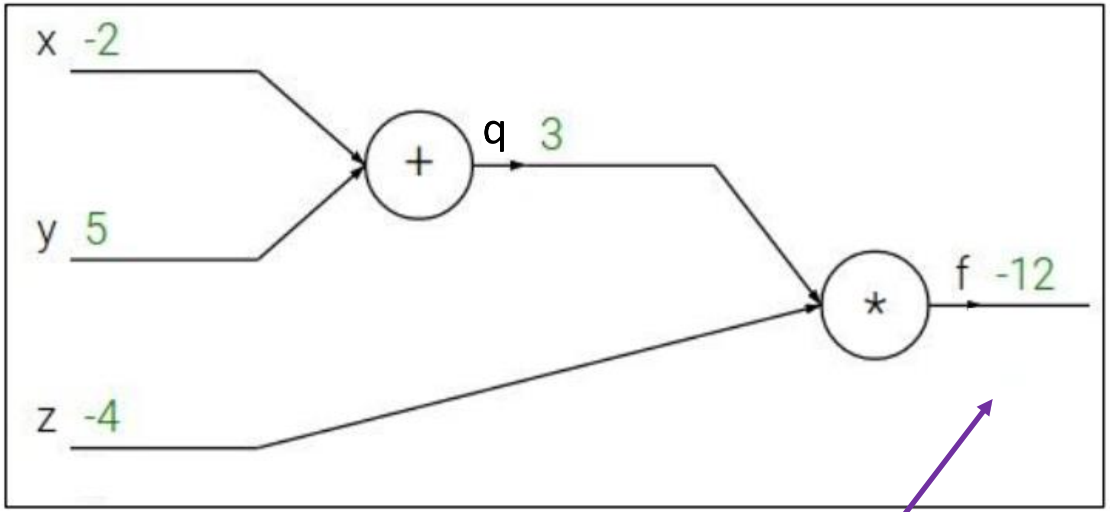

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$$\frac{\partial f}{\partial f}$$

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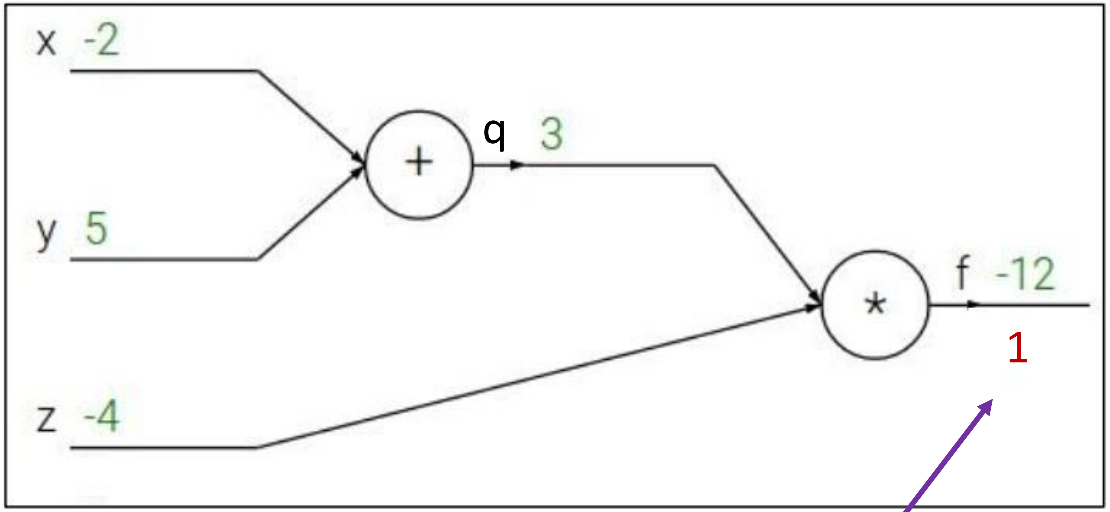

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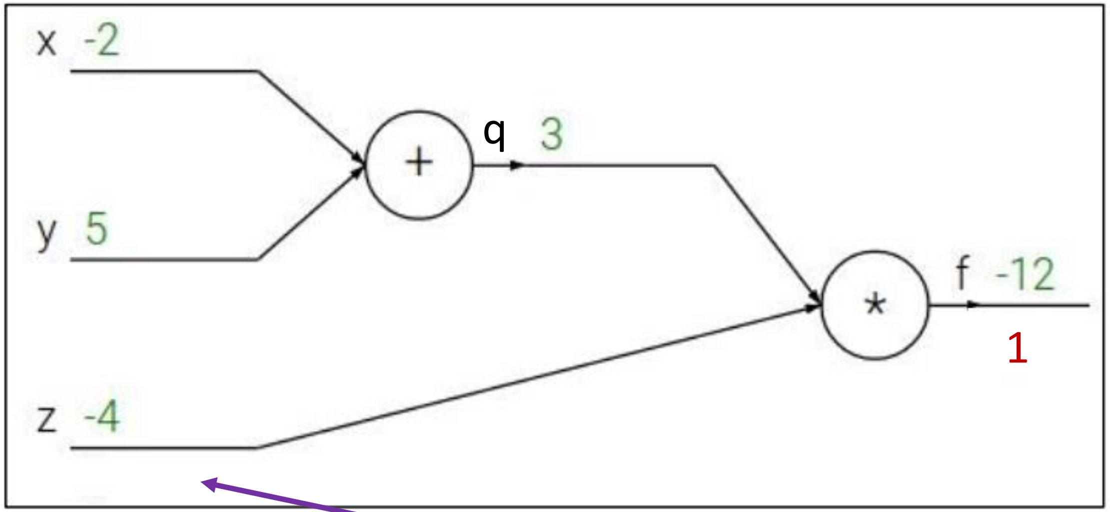

$\frac{\partial f}{\partial z}$

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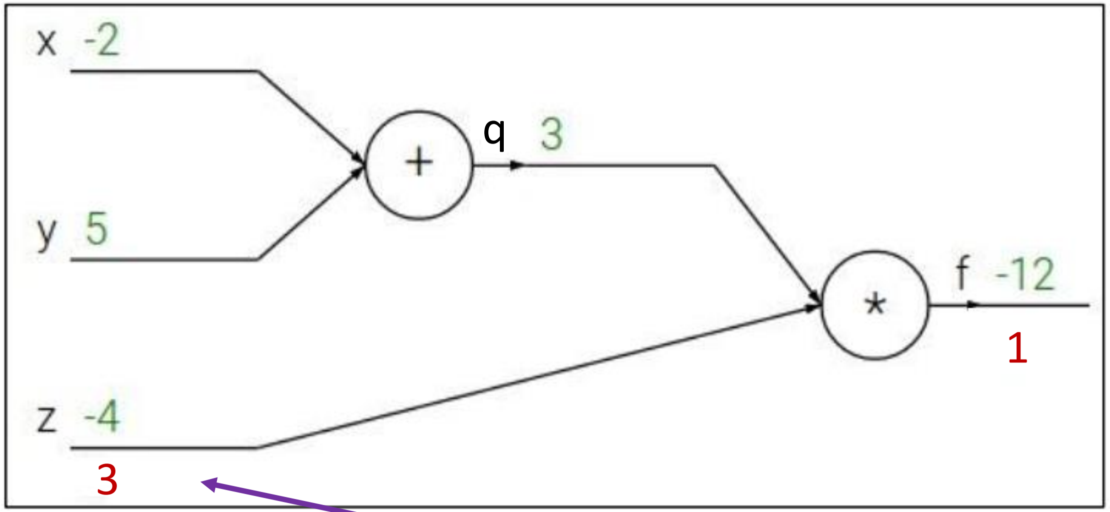

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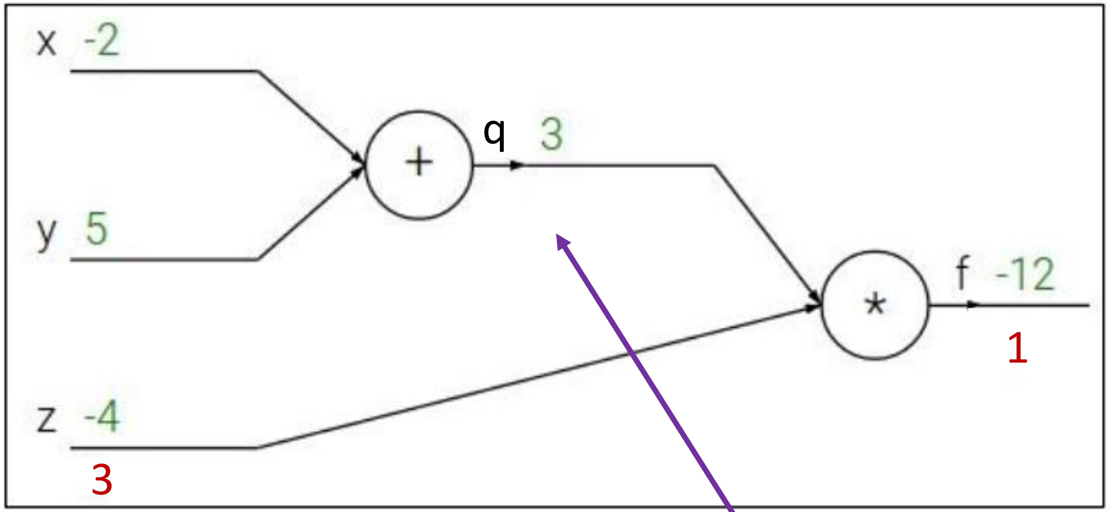

$$\frac{\partial f}{\partial q}$$

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

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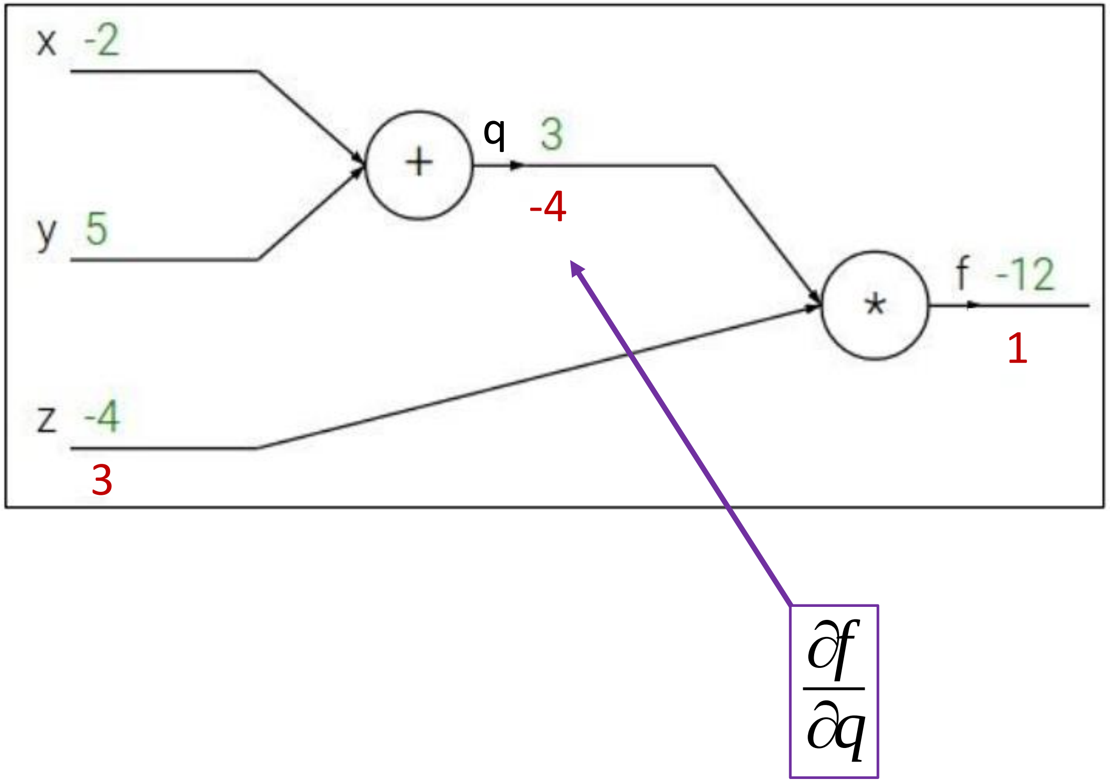

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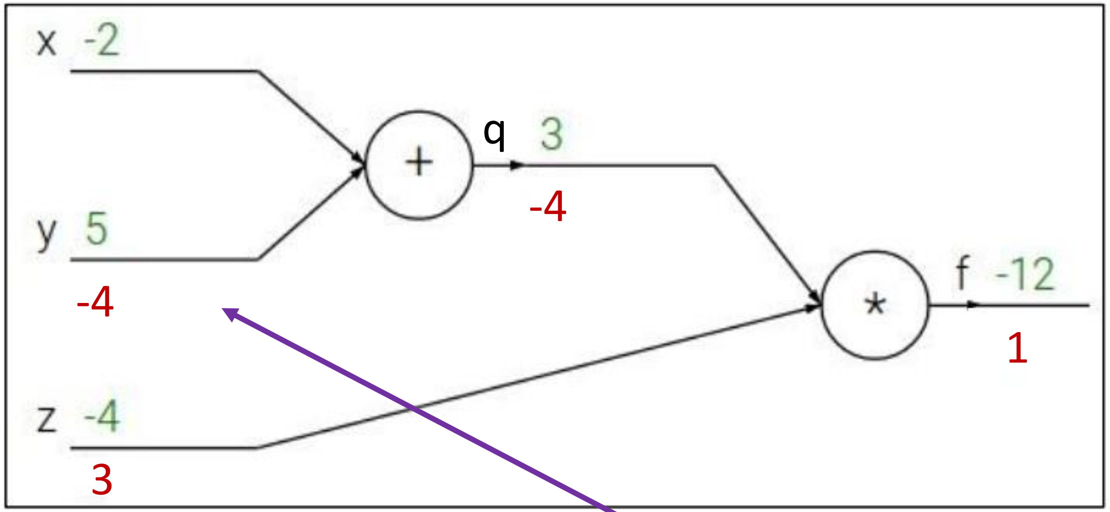

$$\frac{\partial f}{\partial y}$$

Chain rule:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q}\frac{\partial q}{\partial y}$$

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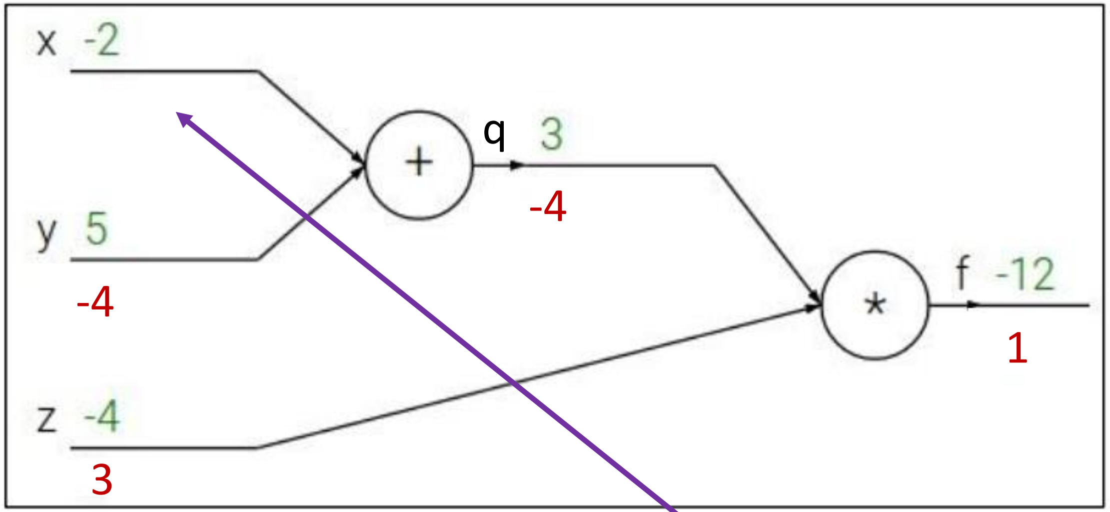

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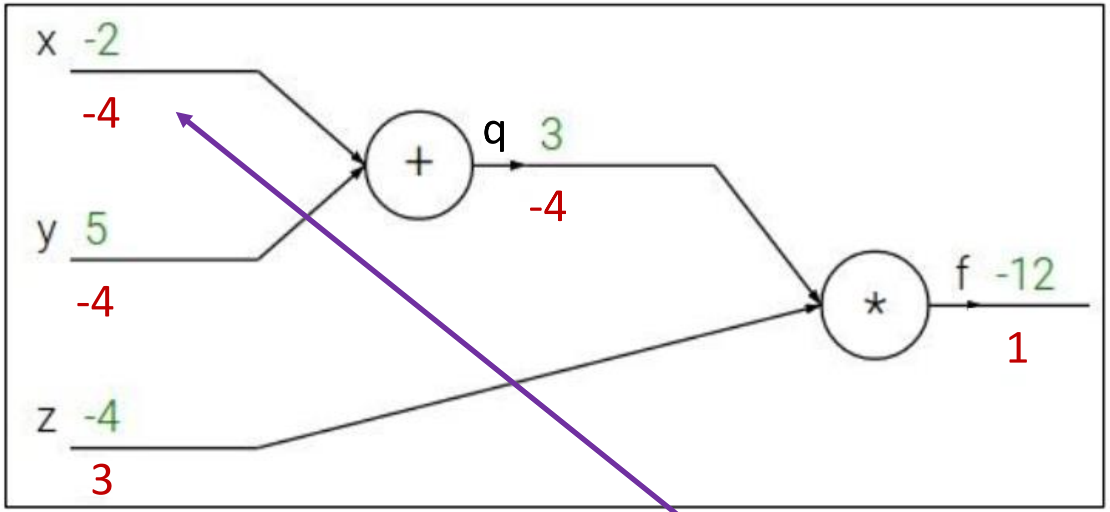

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Chain rule:

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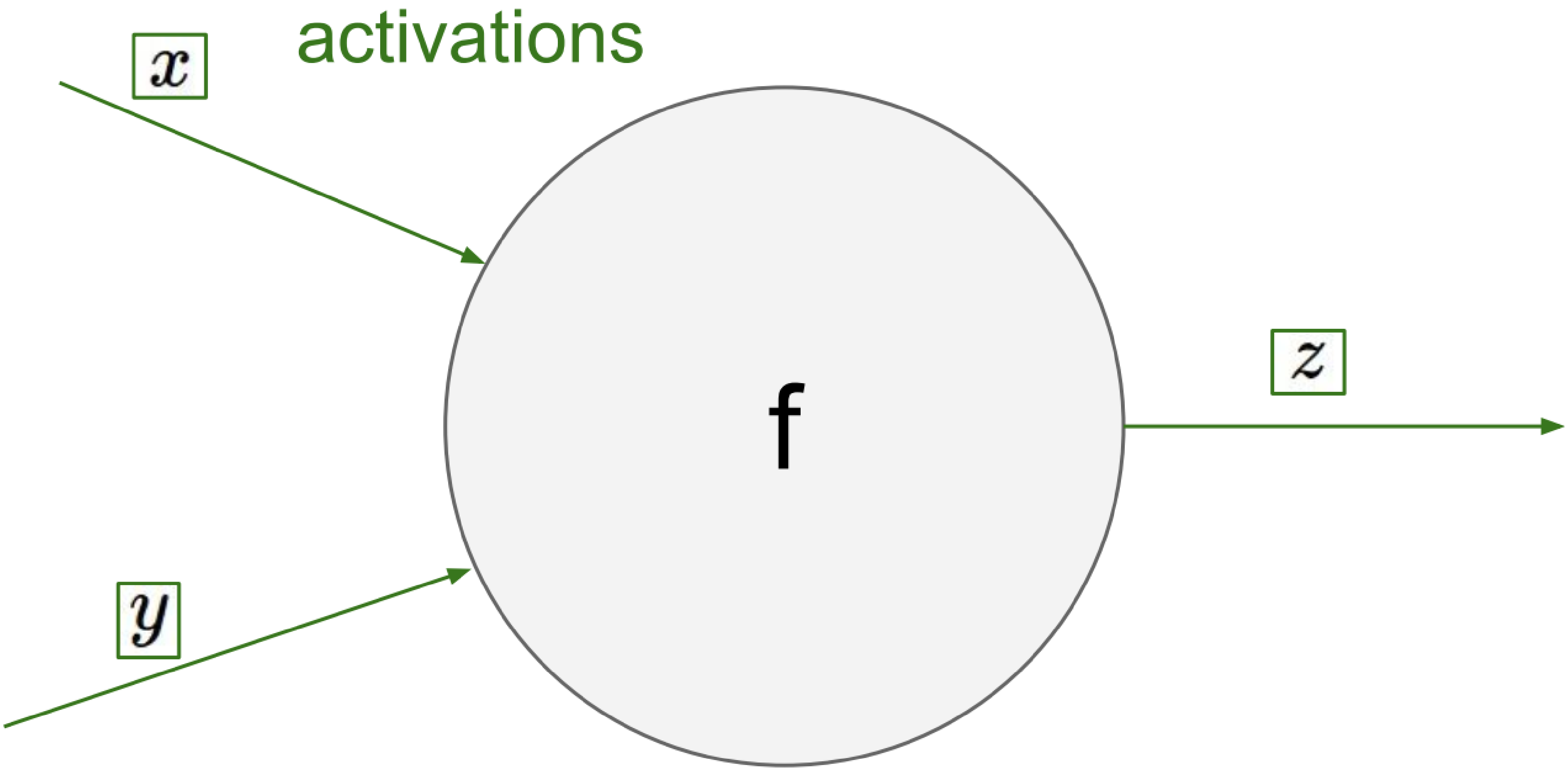
Backpropagation



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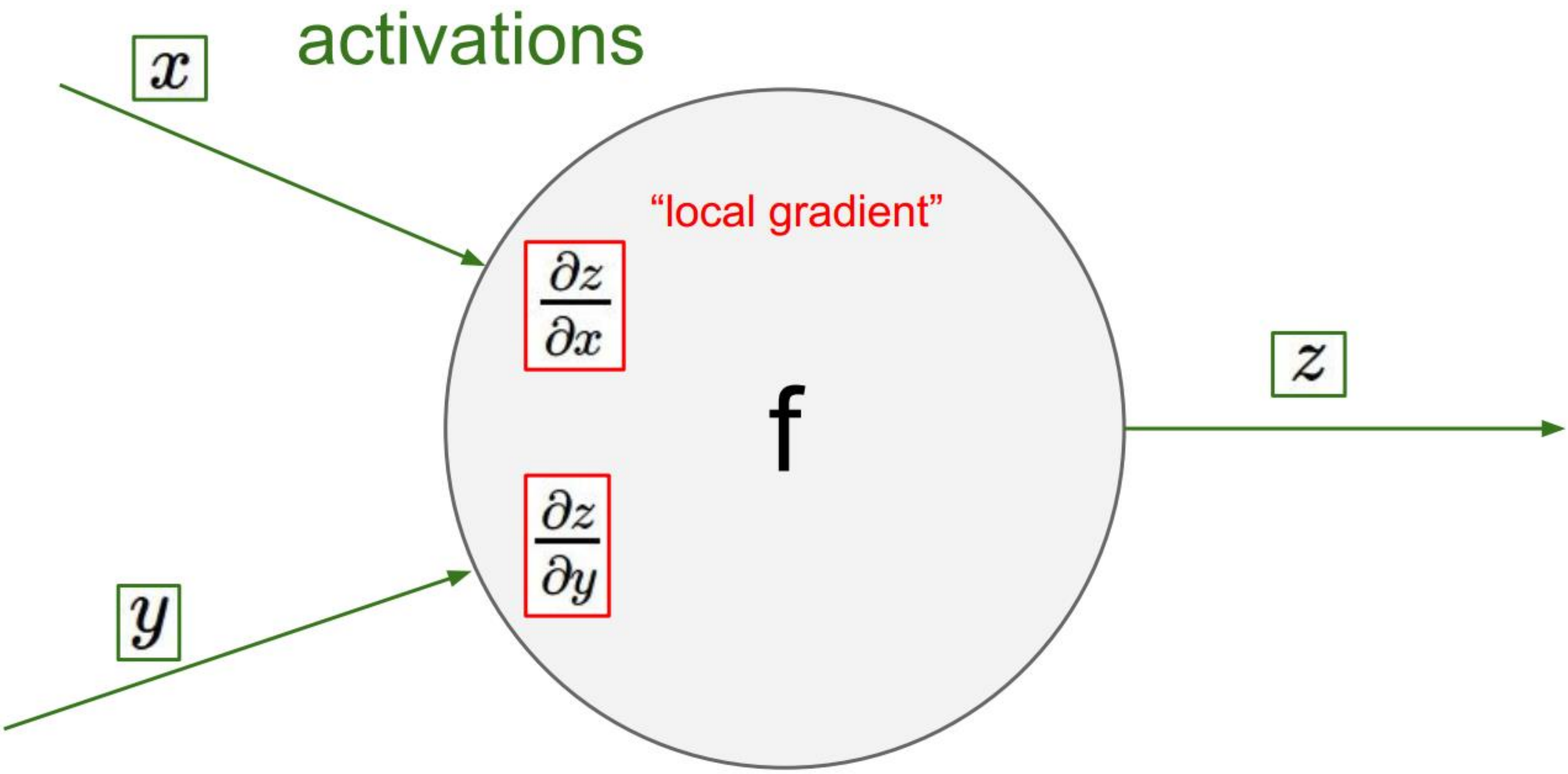

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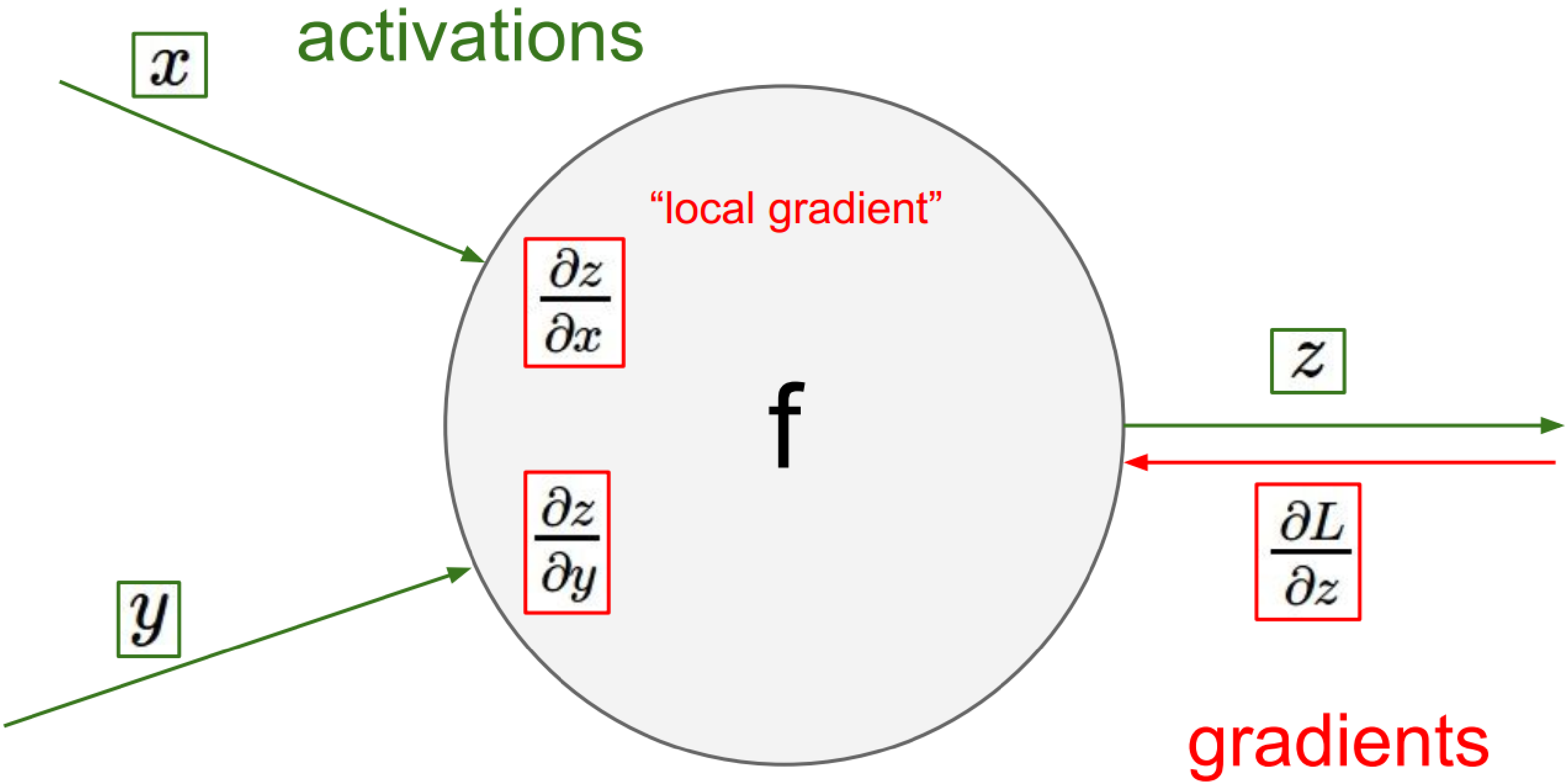

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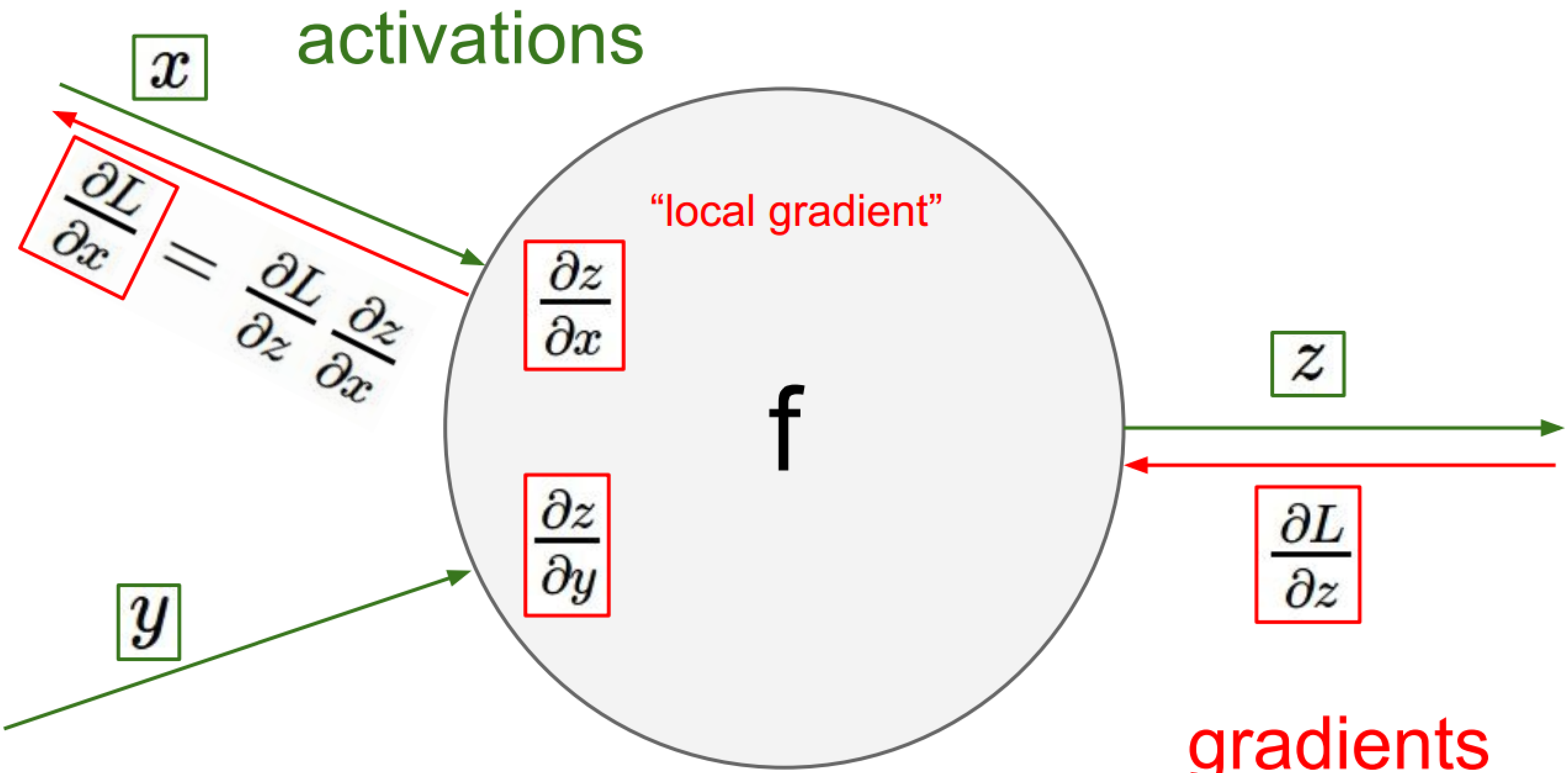

Chain Rule

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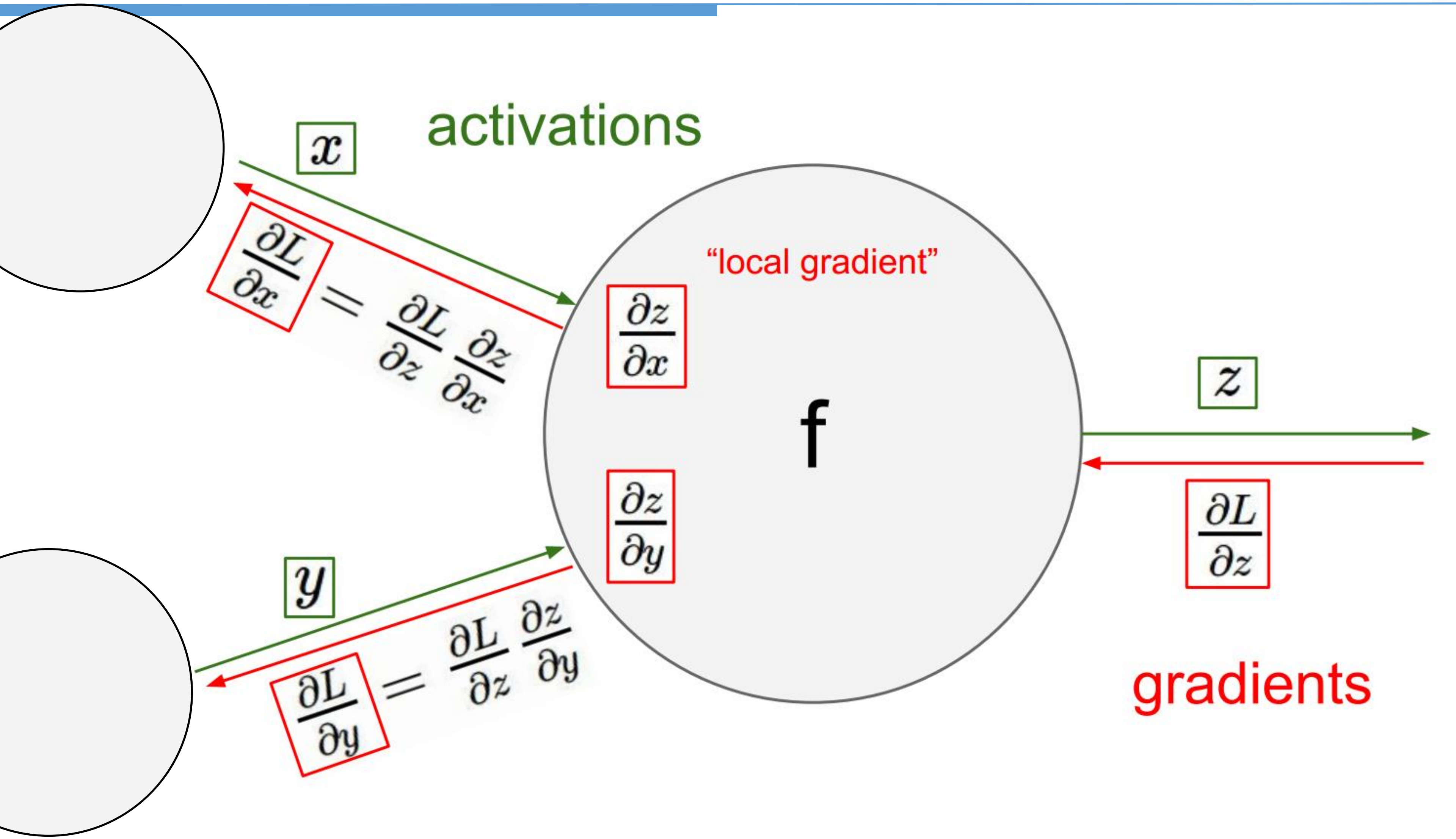
Backpropagation



Chain rule

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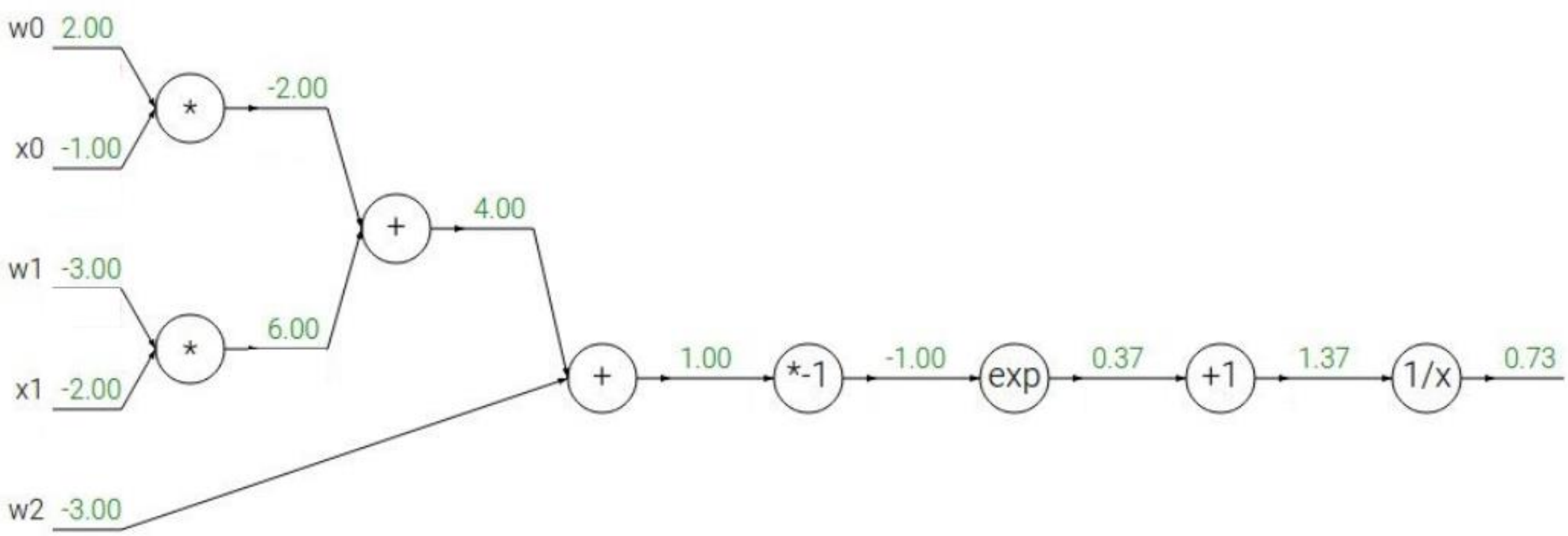
Another Example

Sigmoid Neuron: $$f(w,x)=\frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$



Computation graph of sigmoid neuron

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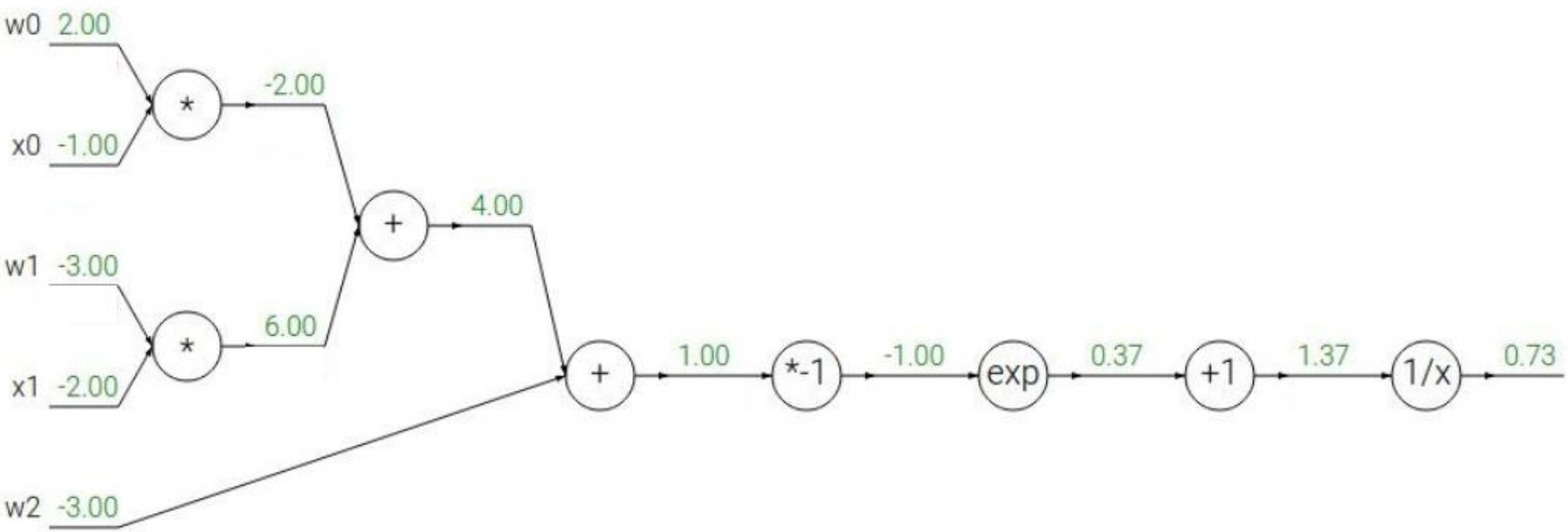

$f(x)=e^x \quad \rightarrow \quad \frac{df}{dx}=e^x$	$f(x)=\frac{1}{x} \quad \rightarrow \quad \frac{df}{dx}=-1/x^2$
$f_a(x)=ax \quad \rightarrow \quad \frac{df}{dx}=a$	$f_c(x)=c+x \quad \rightarrow \quad \frac{df}{dx}=1$

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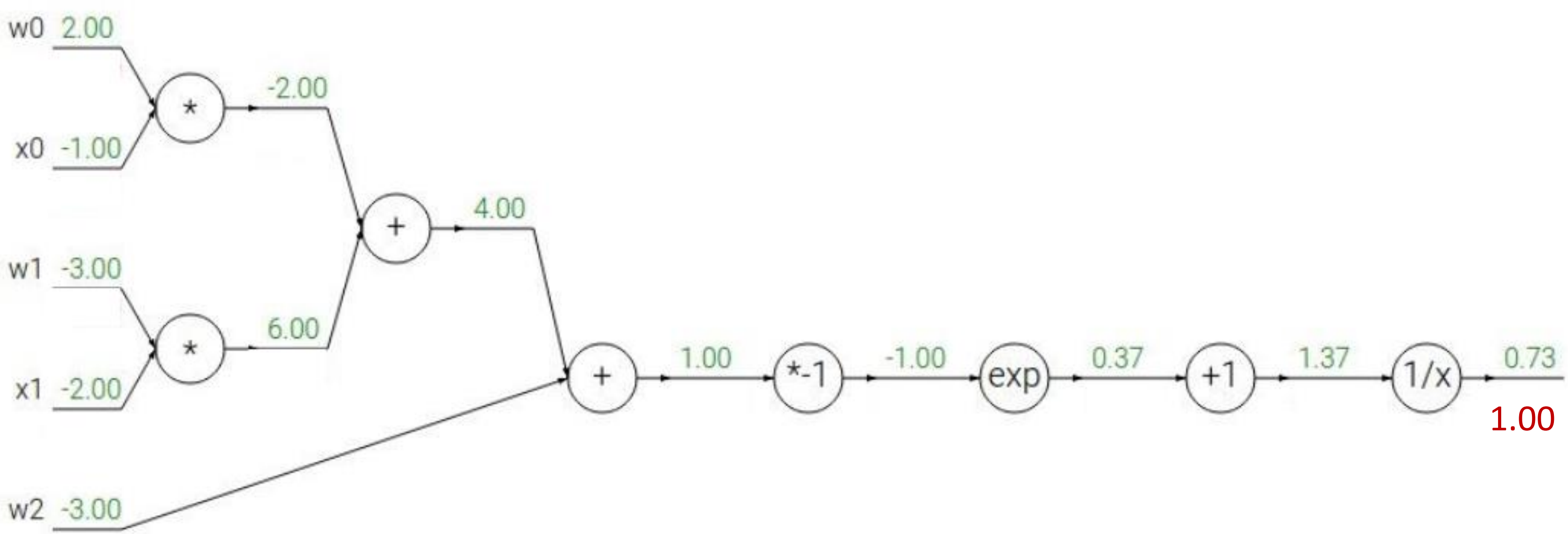
Another Example

Sigmoid Neuron: $$f(w,x) = \frac{1}{1+e^{-(w_0 x_0 + w_1 x_1 + w_2)}}$$



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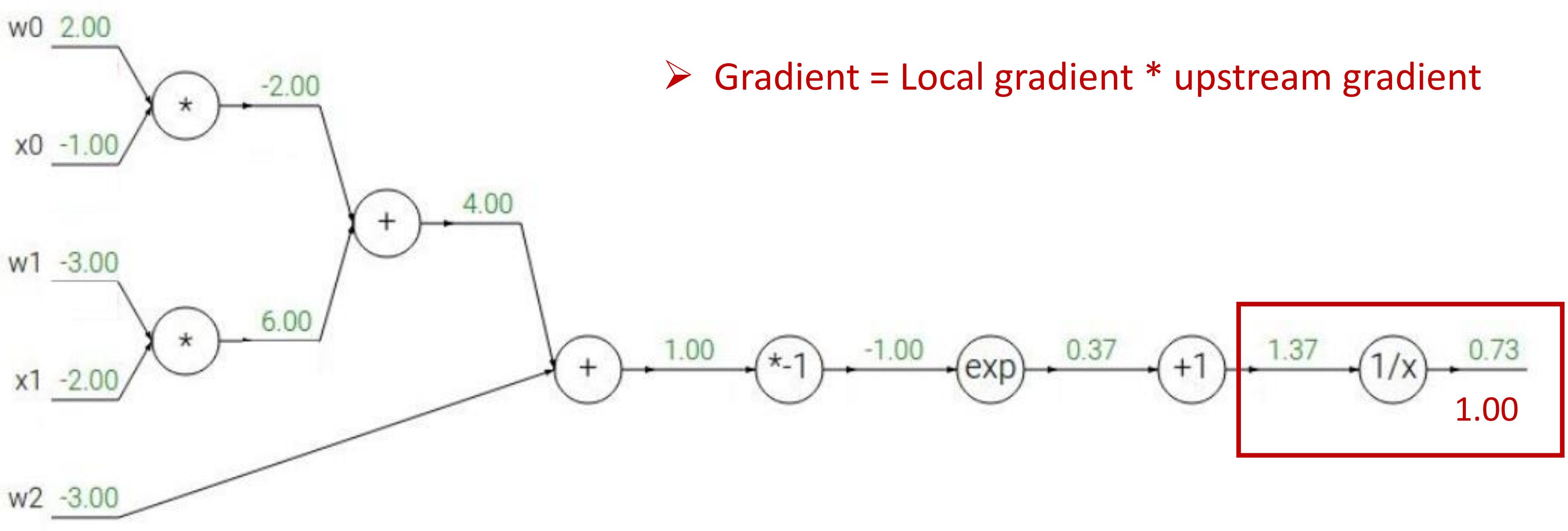

- Gradient = Local gradient * upstream gradient

$$f(x)=e^x \quad \rightarrow \quad \frac{df}{dx}=e^x$$

$$f_a(x)=ax \quad \rightarrow \quad \frac{df}{dx}=a$$

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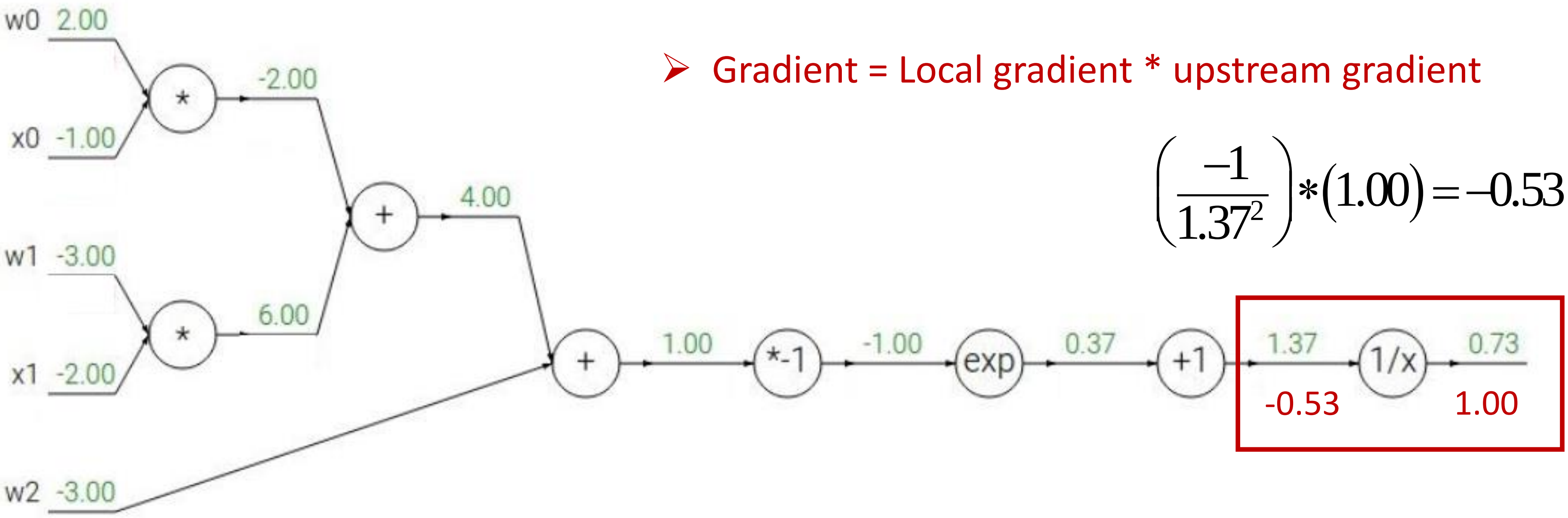
- Gradient = Local gradient * upstream gradient

$$\left(\frac{-1}{1.37^2}\right)*(1.00)=-0.53$$



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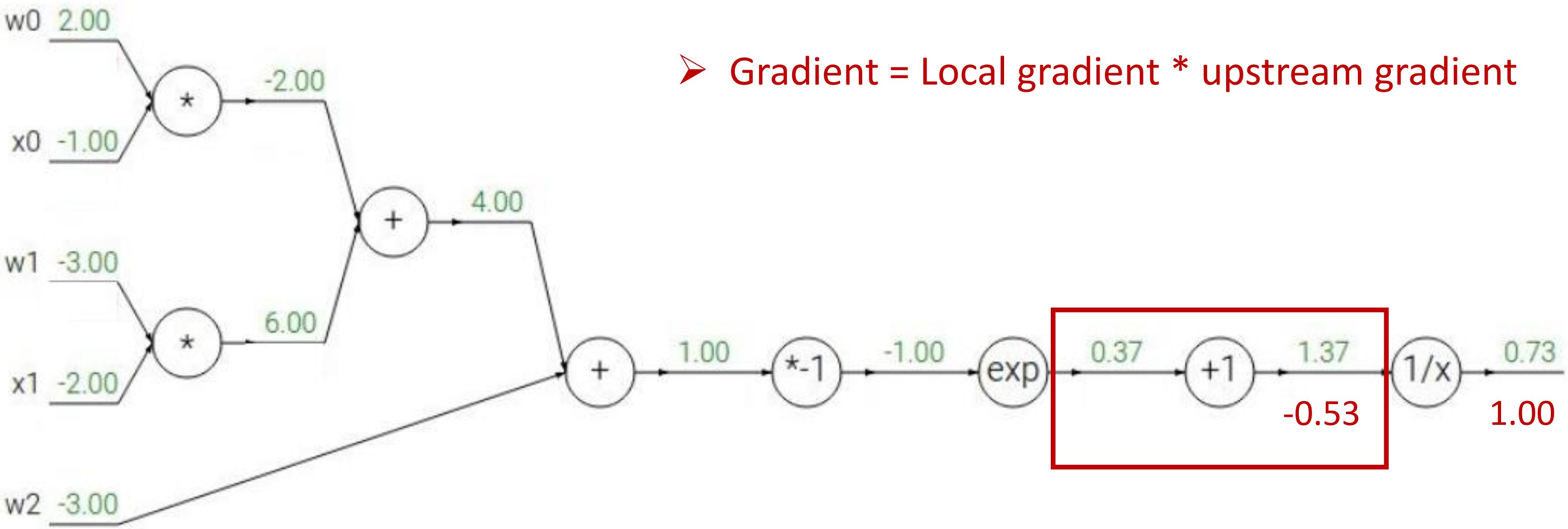

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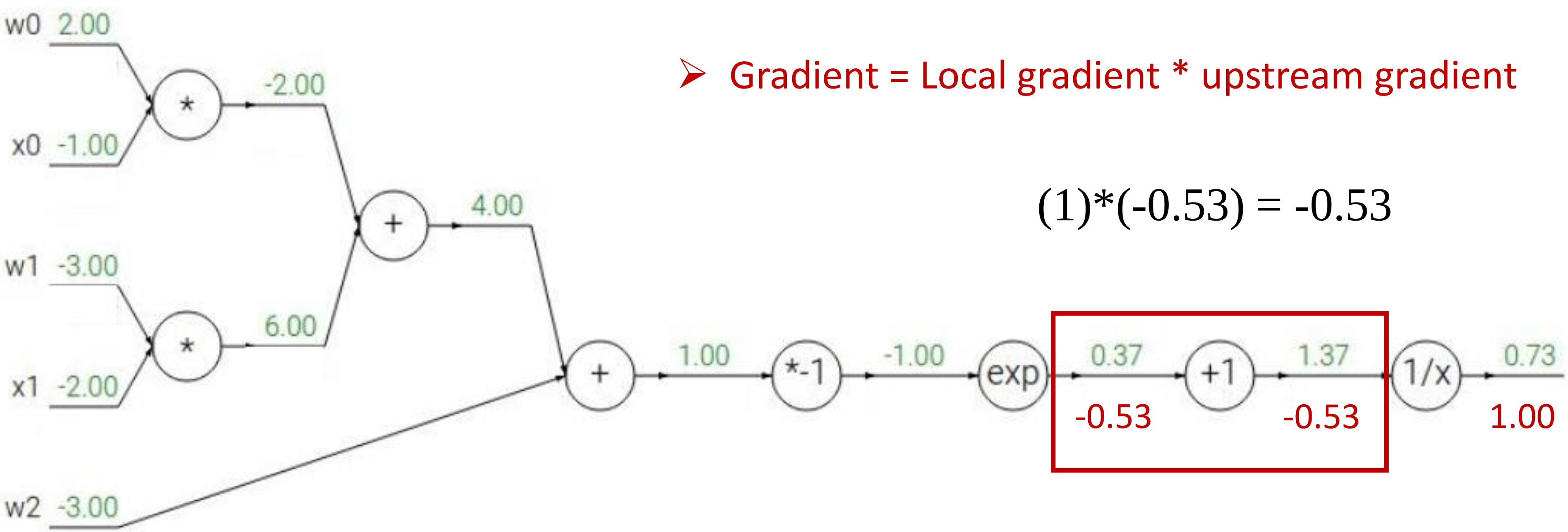

- Gradient = Local gradient * upstream gradient

(1)*(-0.53) = -0.53

$$f(x)=e^x \quad \rightarrow \quad \frac{df}{dx}=e^x$$

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Another Example

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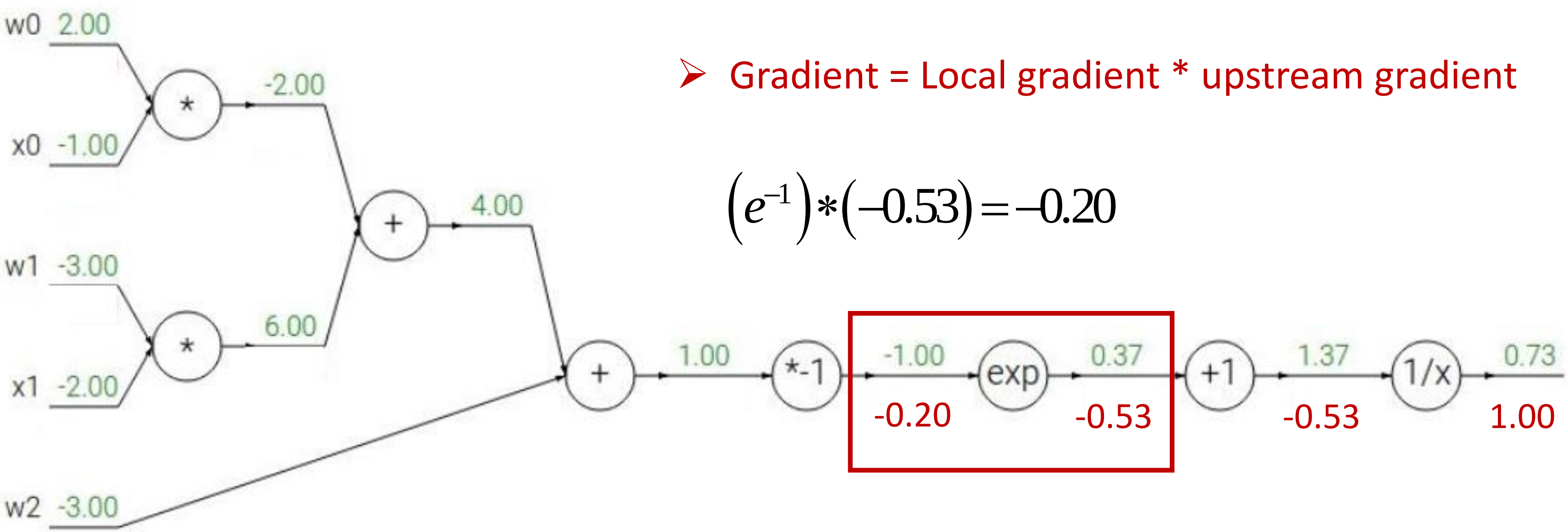
- Gradient = Local gradient * upstream gradient

$$\left(e^{-1}\right)*(-0.53)=-0.20$$



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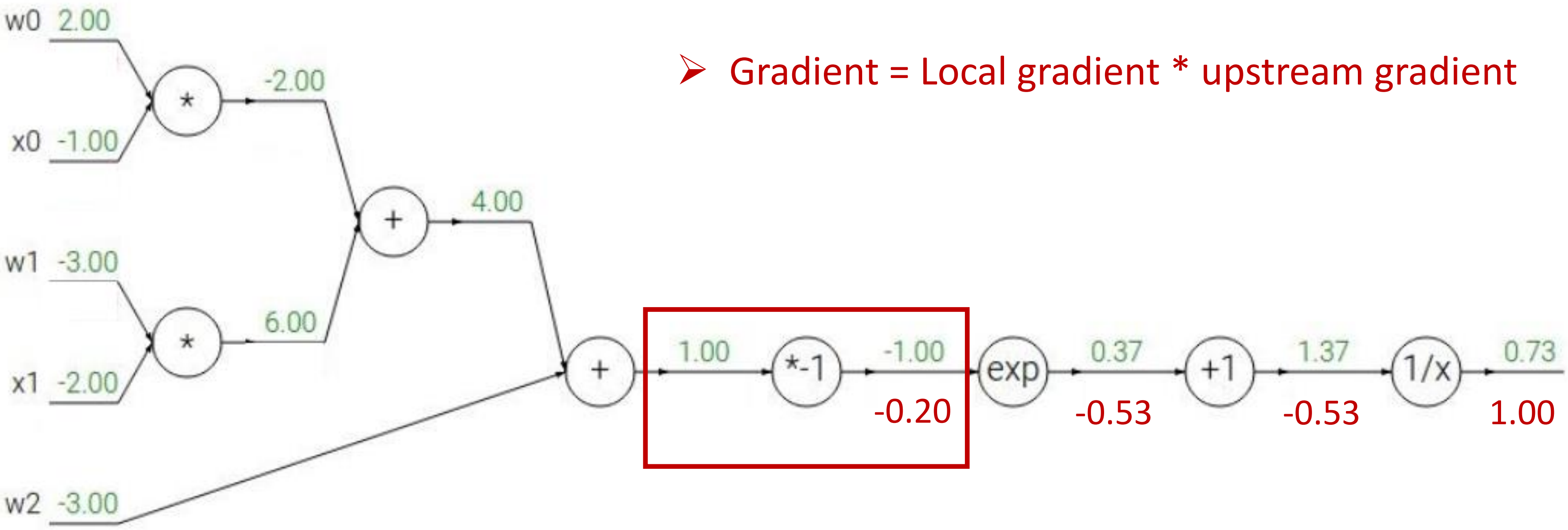

- Gradient = Local gradient * upstream gradient

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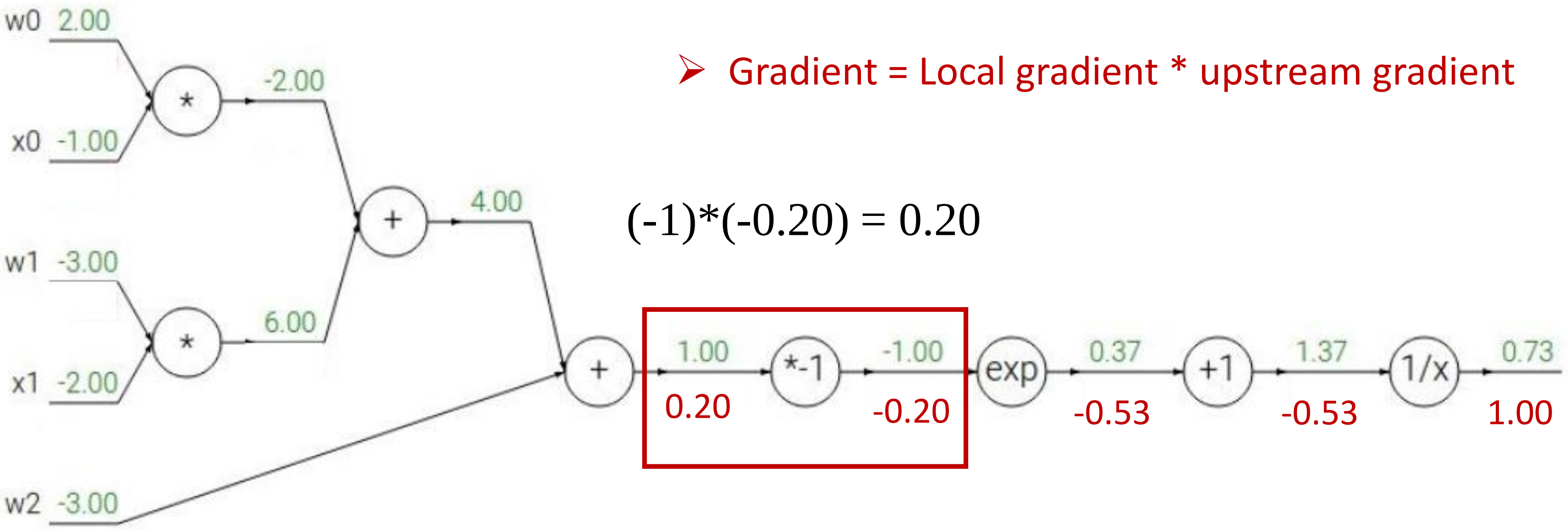

- Gradient = Local gradient * upstream gradient

(-1)*(-0.20) = 0.20

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- Gradient = Local gradient * upstream gradient

(1)*(0.20) = 0.20

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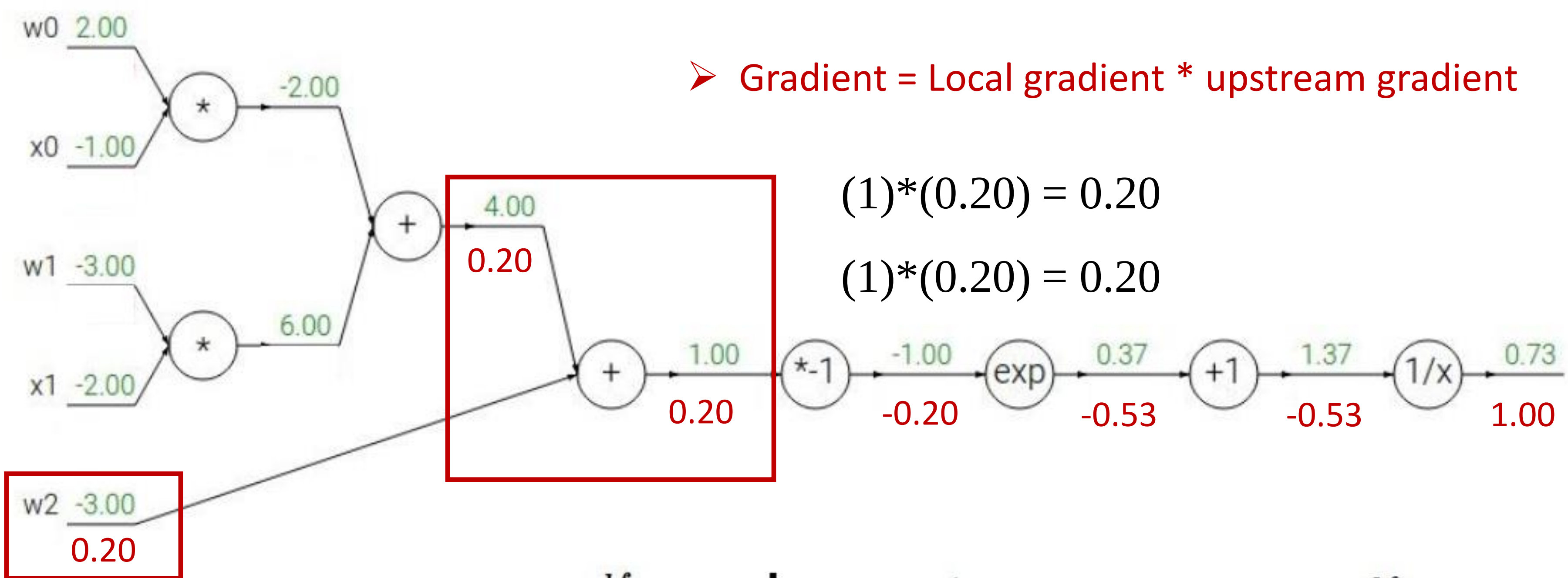

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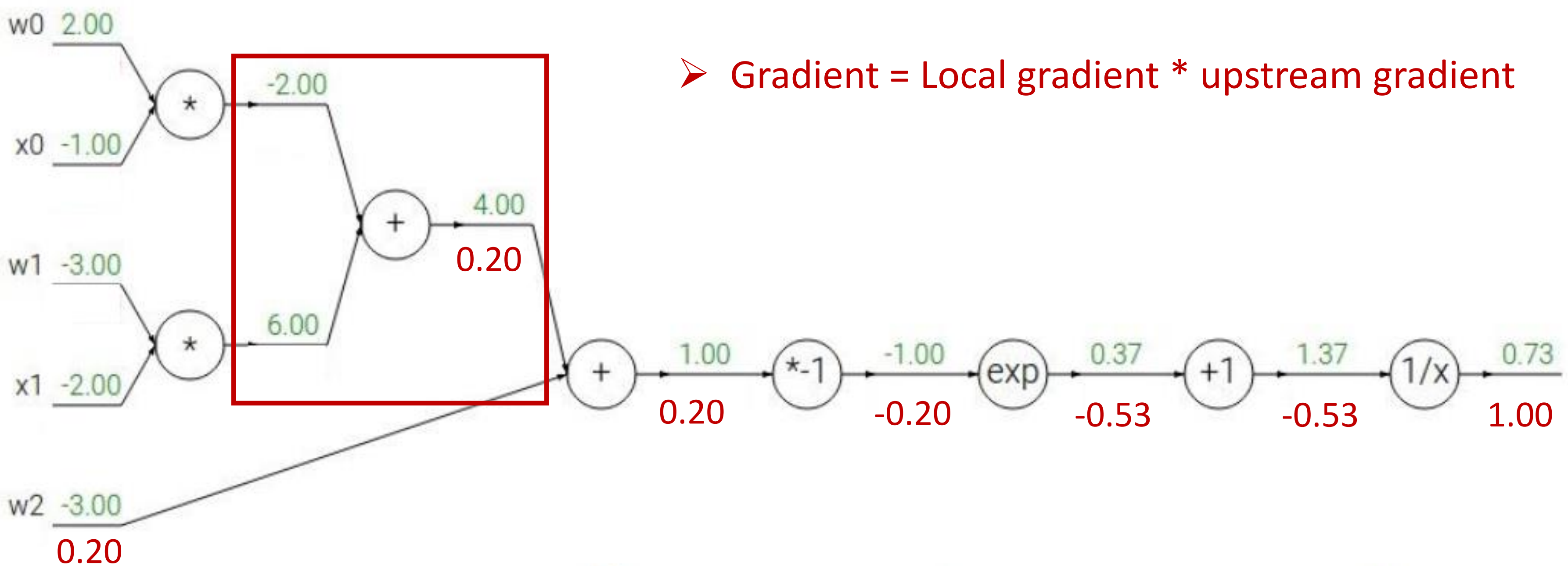

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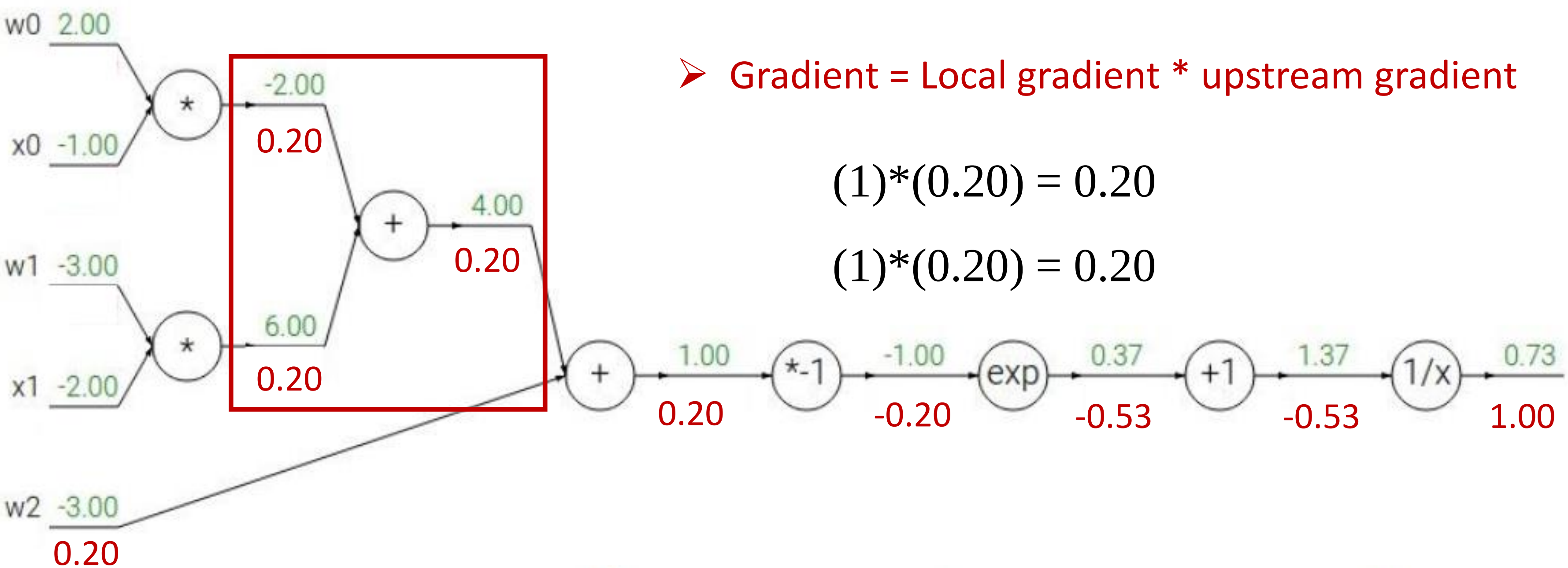

- Gradient = Local gradient * upstream gradient

(1)*(0.20) = 0.20

(1)*(0.20) = 0.20

$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x \qquad\qquad f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a \qquad\qquad \boxed{f_c(x) = c + x \quad \rightarrow \quad \frac{df}{dx} = 1}$$

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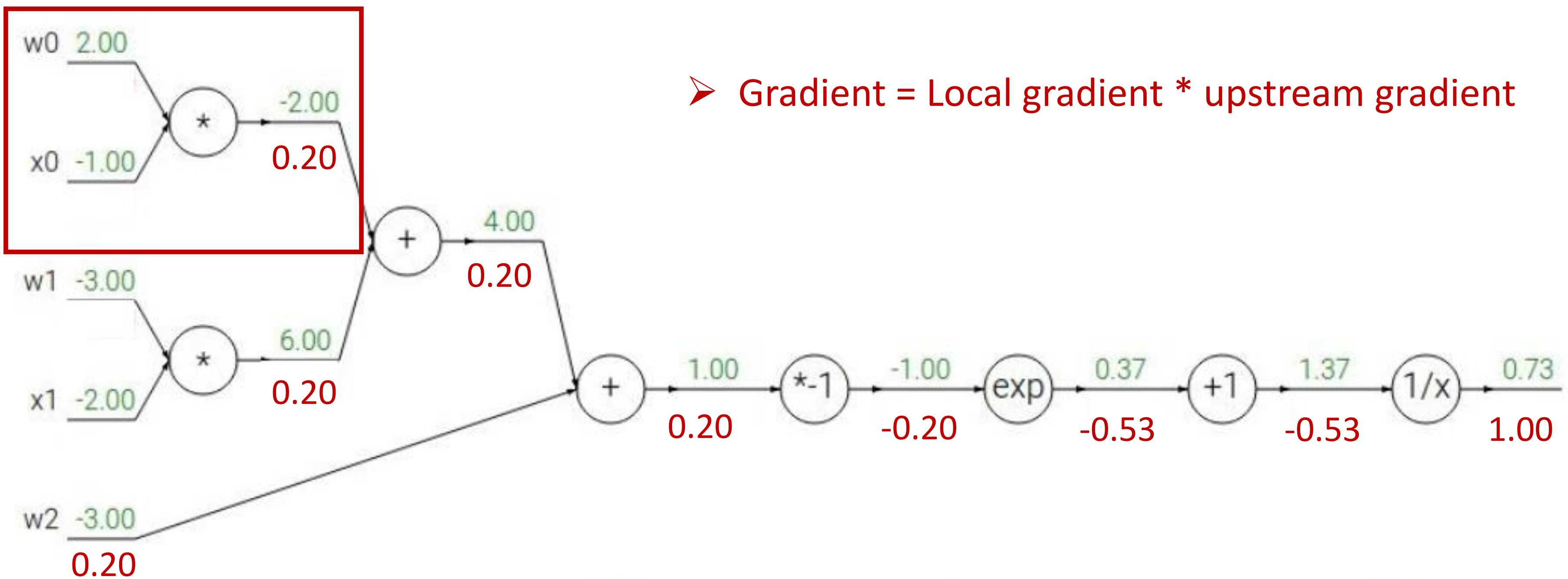
Another Example

Sigmoid Neuron: $$f(w,x)=\frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$



- Gradient = Local gradient * upstream gradient

$f(x)=e^x \quad \rightarrow \quad \frac{df}{dx}=e^x$

$\boxed{f_a(x)=ax \quad \rightarrow \quad \frac{df}{dx}=a}$

$f(x)=\frac{1}{x} \quad \rightarrow \quad \frac{df}{dx}=-1/x^2$

$f_c(x)=c+x \quad \rightarrow \quad \frac{df}{dx}=1$

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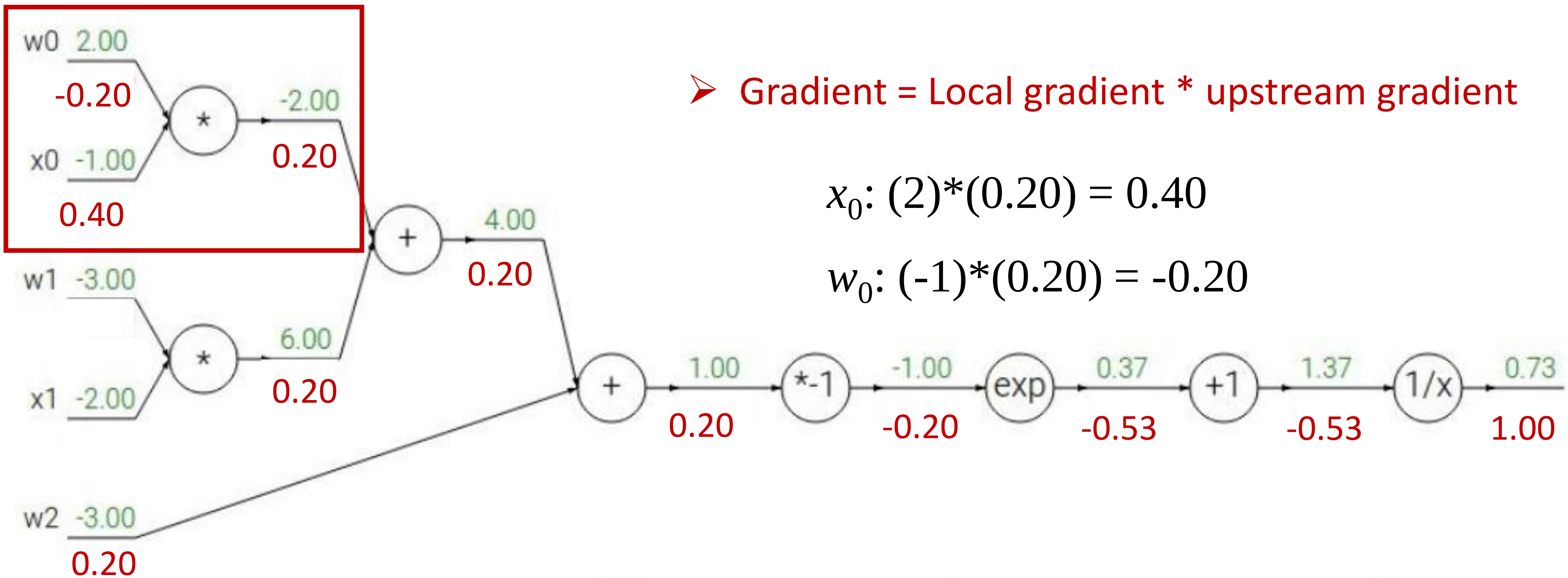
Another Example

Sigmoid Neuron: $$f(w,x)=\frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$



- Gradient = Local gradient * upstream gradient

x_0: (2)*(0.20) = 0.40

w_0: (-1)*(0.20) = -0.20

$$f(x)=e^x \quad \rightarrow \quad \frac{df}{dx}=e^x$$

$$f_a(x)=ax \quad \rightarrow \quad \frac{df}{dx}=a$$

$$f(x)=\frac{1}{x} \quad \rightarrow \quad \frac{df}{dx}=-1/x^2$$

$$f_c(x)=c+x \quad \rightarrow \quad \frac{df}{dx}=1$$

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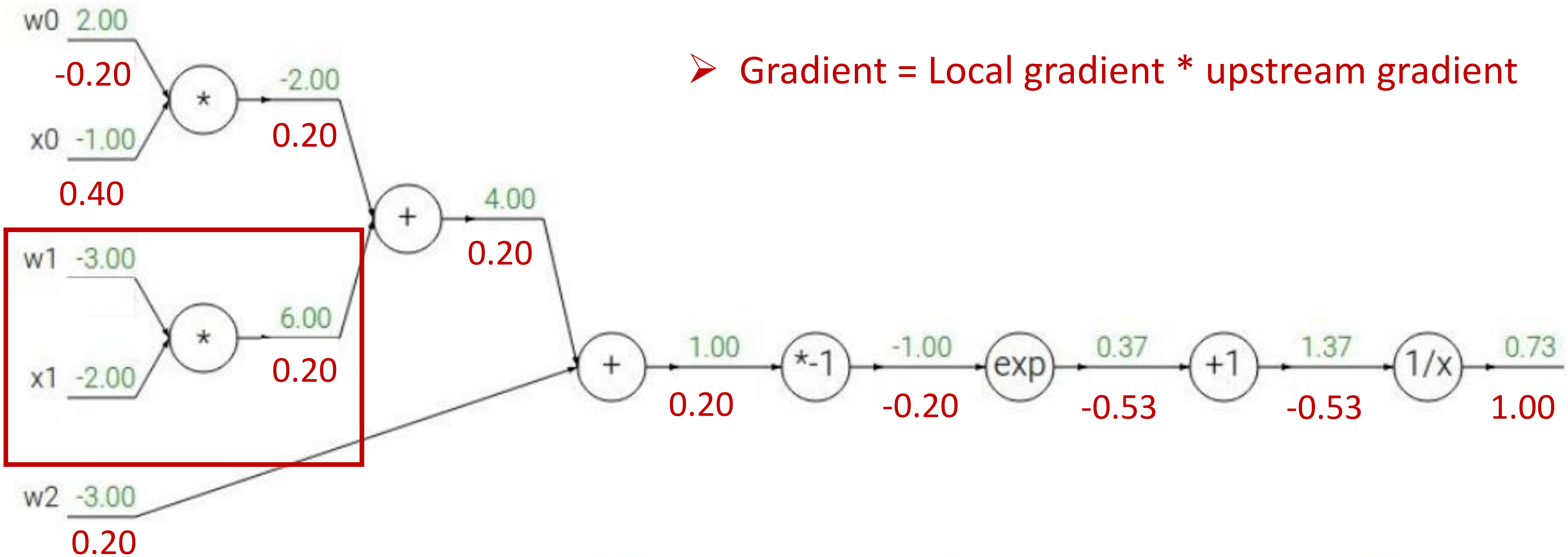
Another Example

Sigmoid Neuron: $$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

- Gradient = Local gradient * upstream gradient



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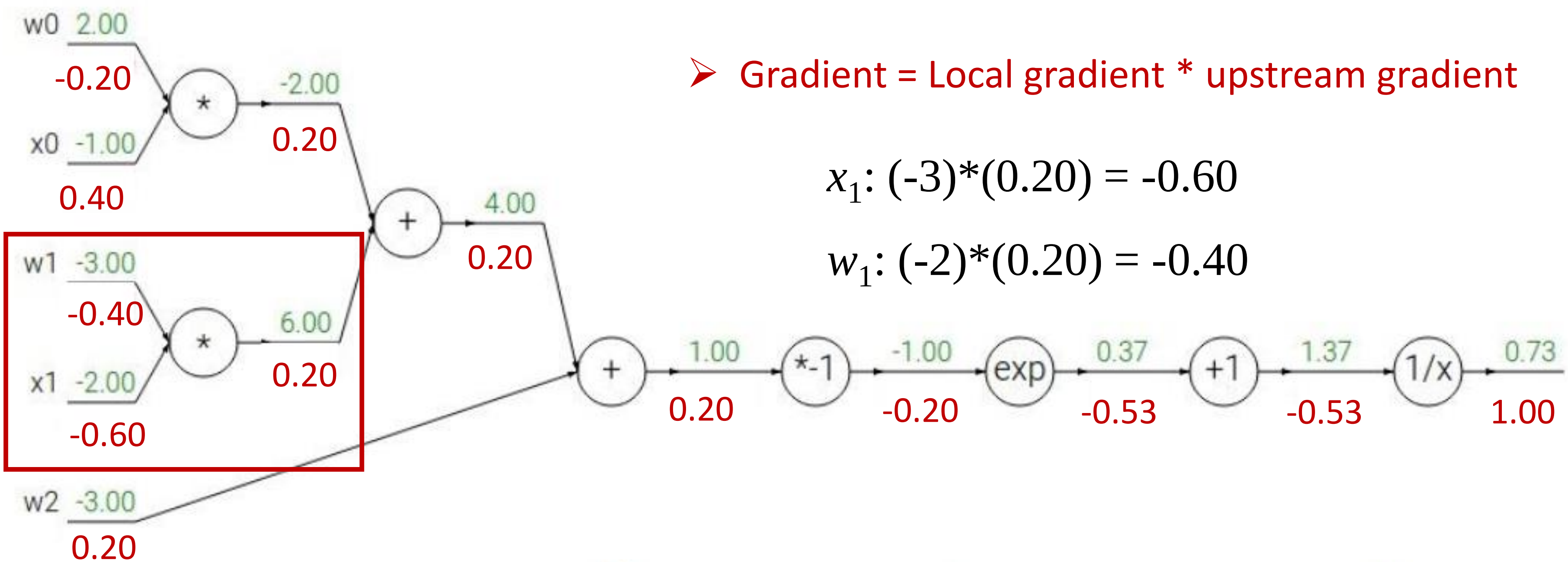
Another Example

Sigmoid Neuron: $$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$



- Gradient = Local gradient * upstream gradient

x_1: (-3)*(0.20) = -0.60

w_1: (-2)*(0.20) = -0.40

$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x$$

$$\boxed{f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a}$$

$$f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

$$f_c(x) = c + x \quad \rightarrow \quad \frac{df}{dx} = 1$$

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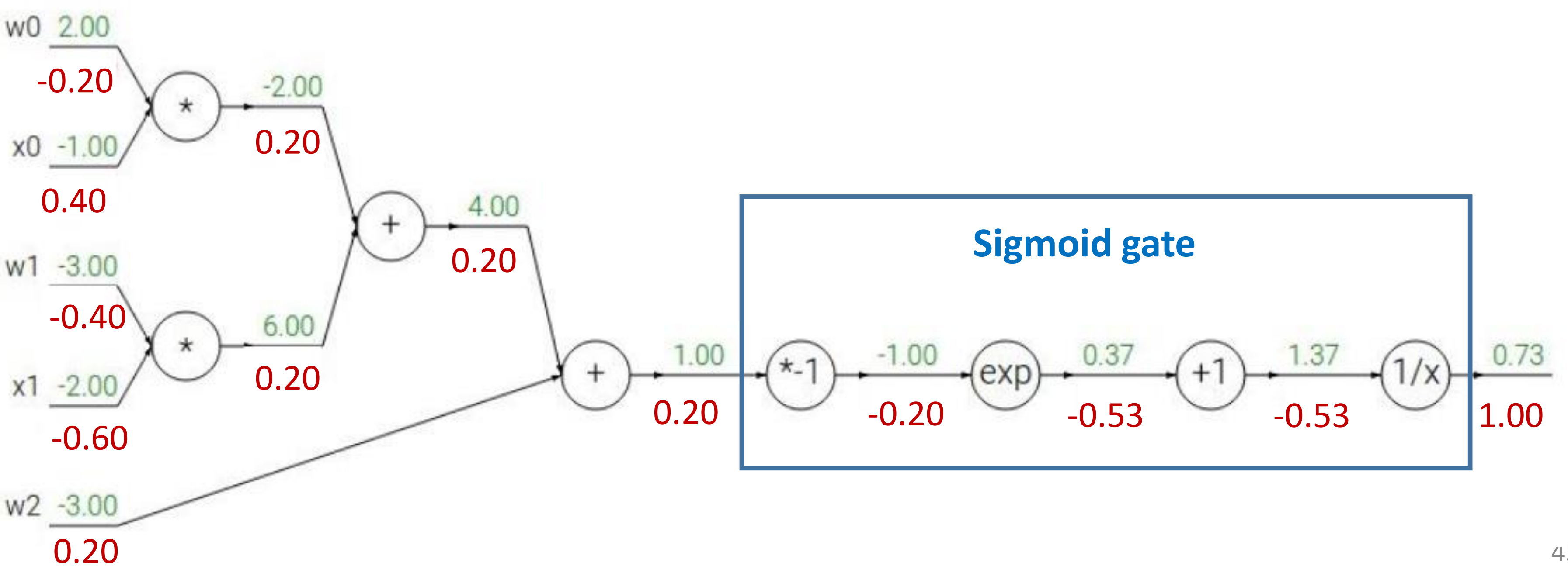
Sigmoid Function

$$f(\mathbf{w},\mathbf{x}) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

$$\boxed{\sigma(x) = \frac{1}{1+e^{-x}}}$$

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} = \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right) = (1-\sigma(x))\sigma(x)$$



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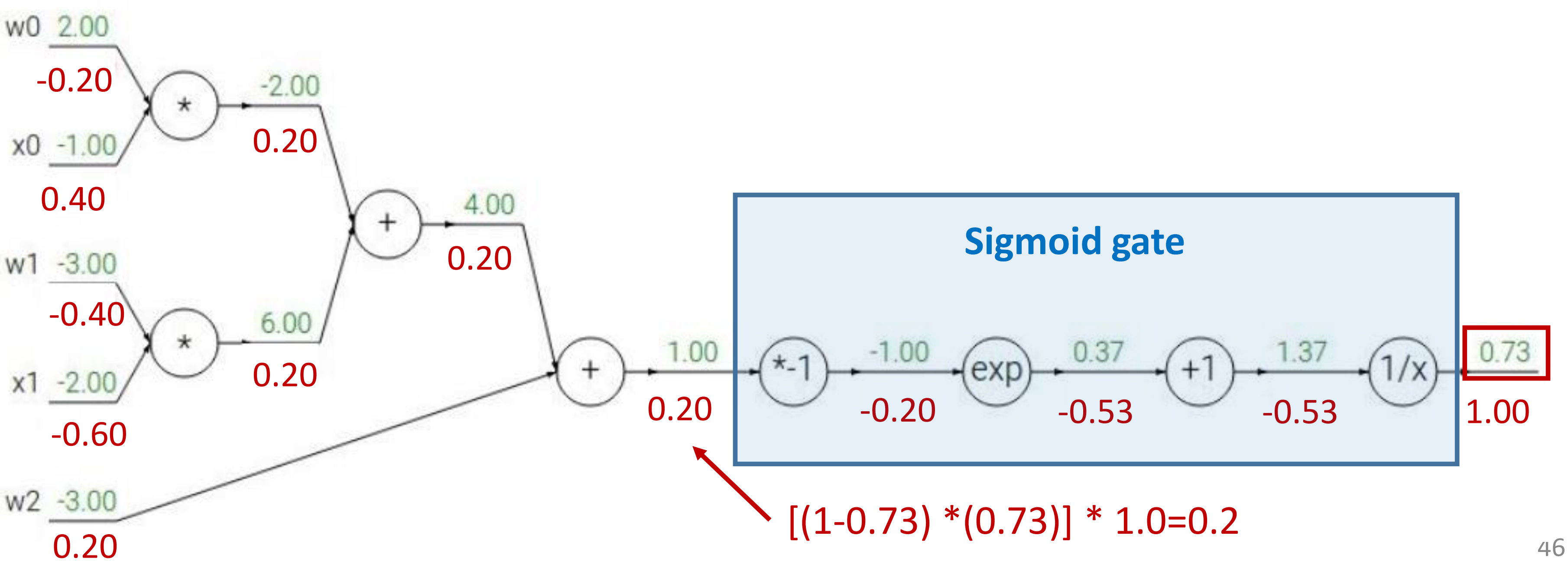

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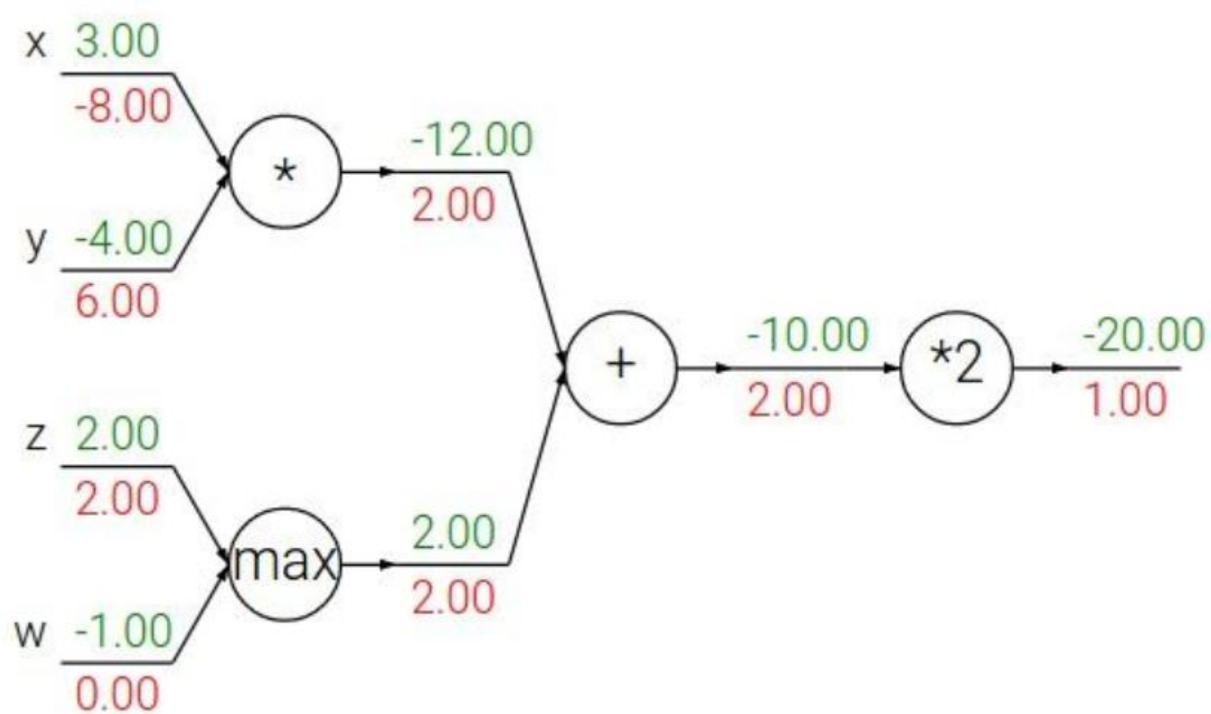
Patterns in Backward Flow

add gate: gradient distributor

max gate: gradient router

mul gate: gradient “switcher”



max(-1, 2) = 2

max(-1+δ, 2) = 2

max(-1, 2+δ) = 2+δ

Question #1: what will happen in max gate when two inputs are the same?

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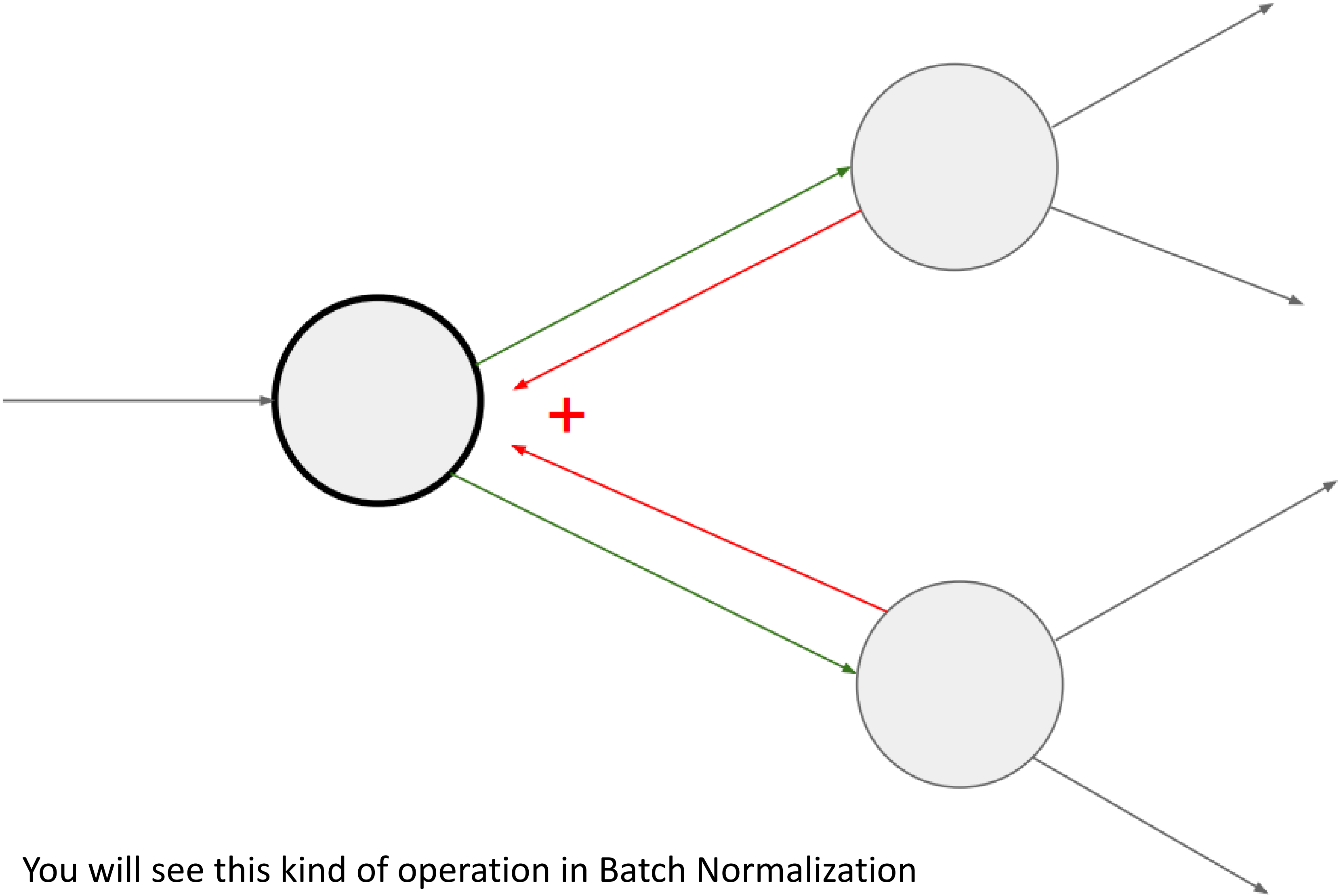
Gradients Add at Branches



- You will see this kind of operation in Batch Normalization

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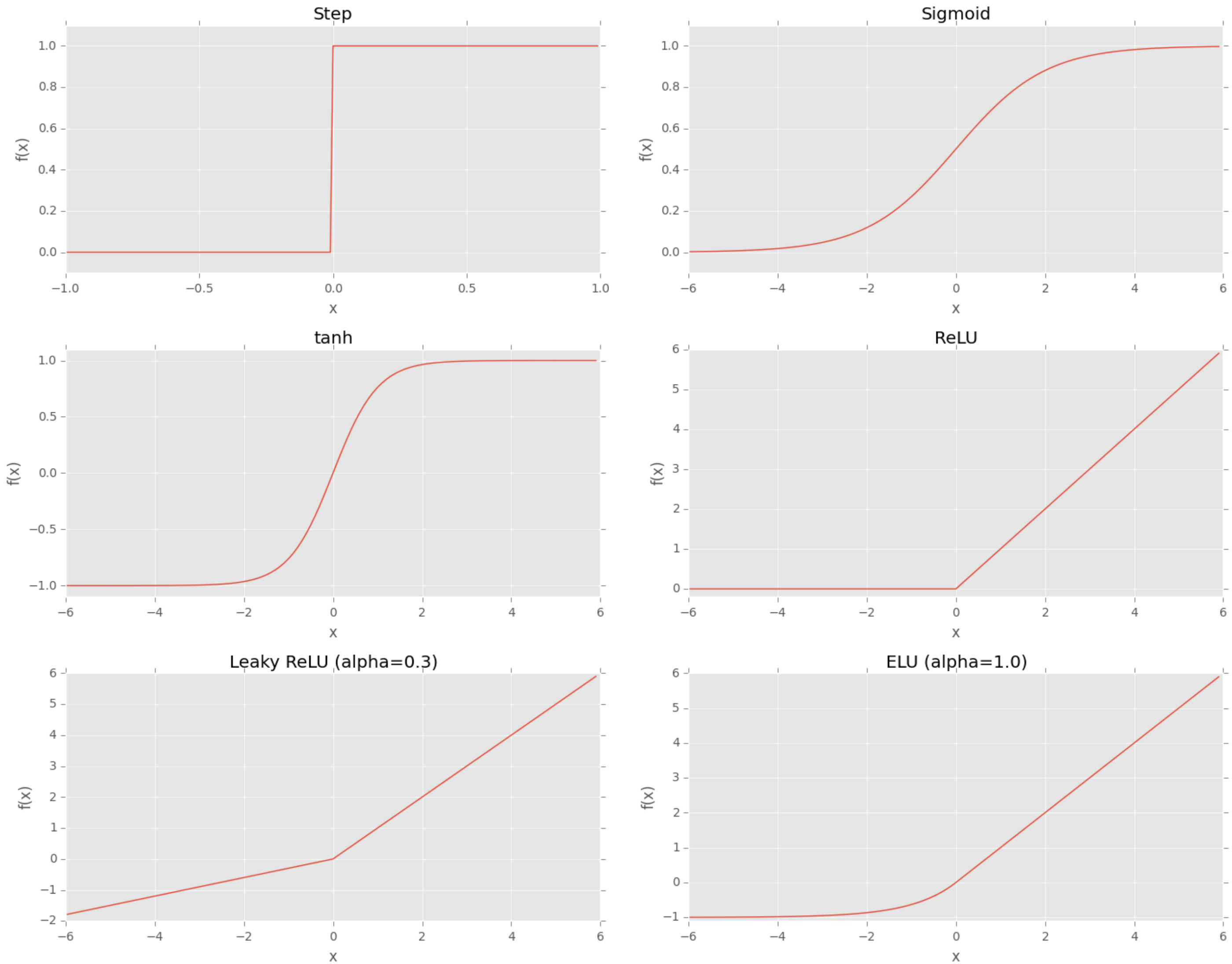
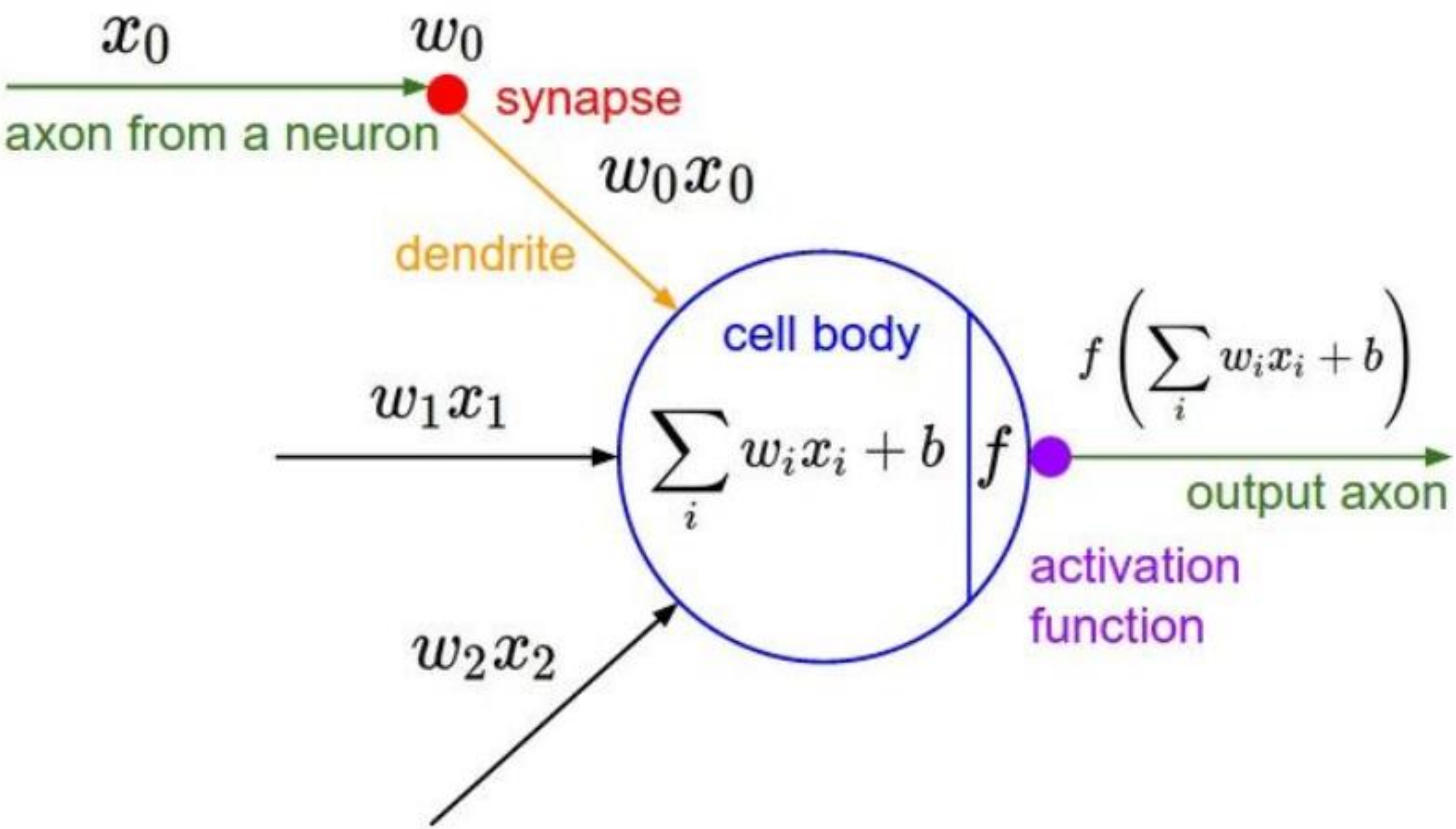
Activation functions



Activation Functions

Activation Functions



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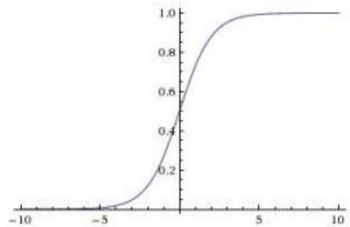
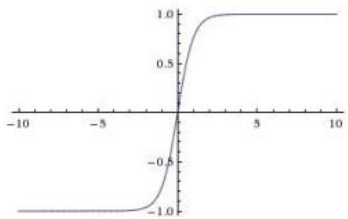
Activation Functions

Sigmoid

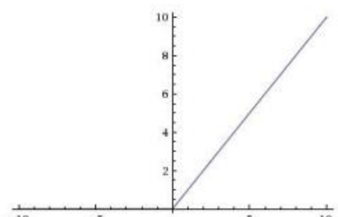
$$\sigma(x) = 1/(1+e^{-x})$$





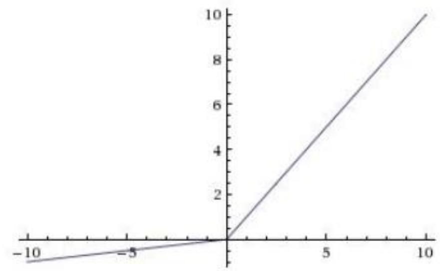
tanh tanh(x)

ReLU max(0,x)



Leaky ReLU

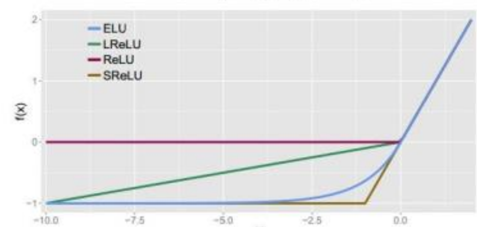
max(0.1x, x)



Maxout $\max(w_1^T x + b_1, w_2^T x + b_2)$

ELU

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ \alpha\,(\exp(x) - 1) & \text{if } x \le 0 \end{cases}$$



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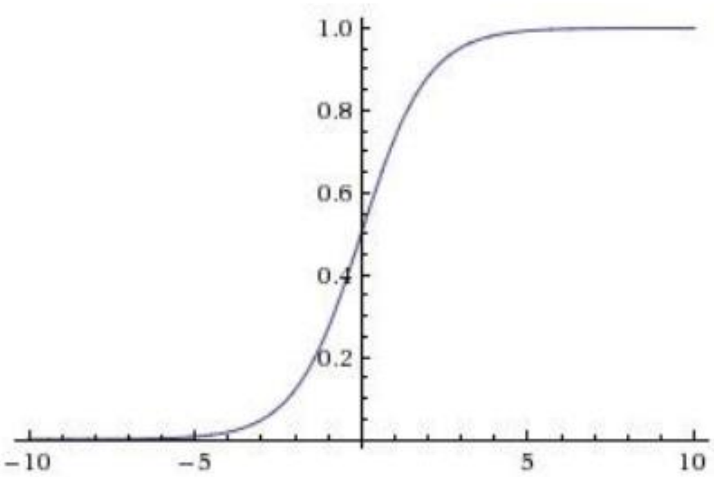
Activation Functions - Sigmoid

$$\sigma(x) = \frac{1}{\left(1+e^{-x}\right)}$$



Sigmoid

- Squashes numbers to range [0, 1]
- Historically popular since they have nice interpretation as a saturating "firing rate" of a neuron

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Activation Functions - Sigmoid

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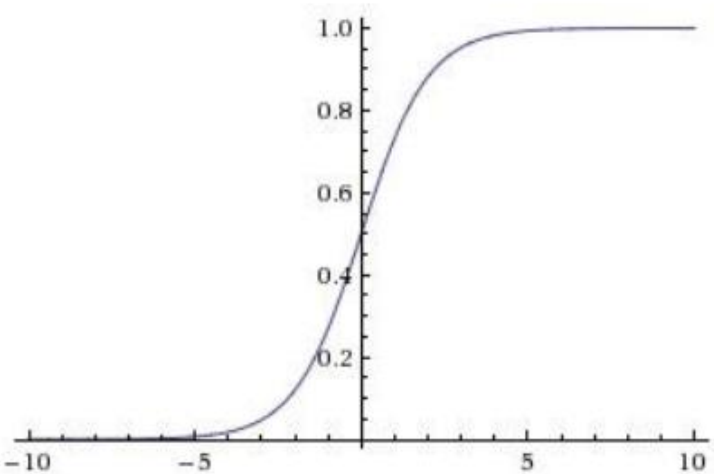

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3 problems:

1. Saturated neurons "kill" the gradients

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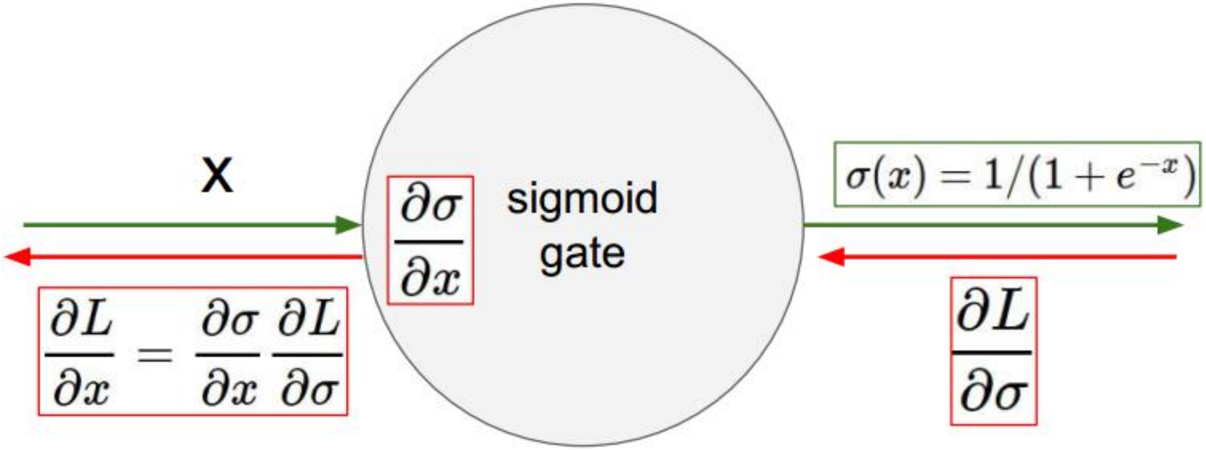
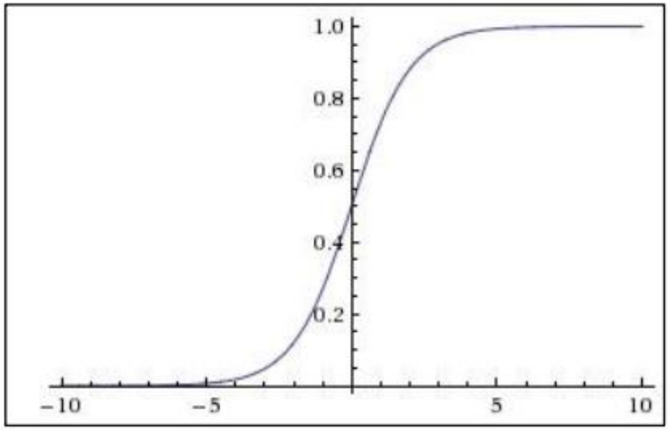
Activation Functions - Sigmoid





What happens when x = -10?

What happens when x = 0?

What happens when x = 10?

Vanishing gradient problem

SGD:

W += -learning_rate * d**W**

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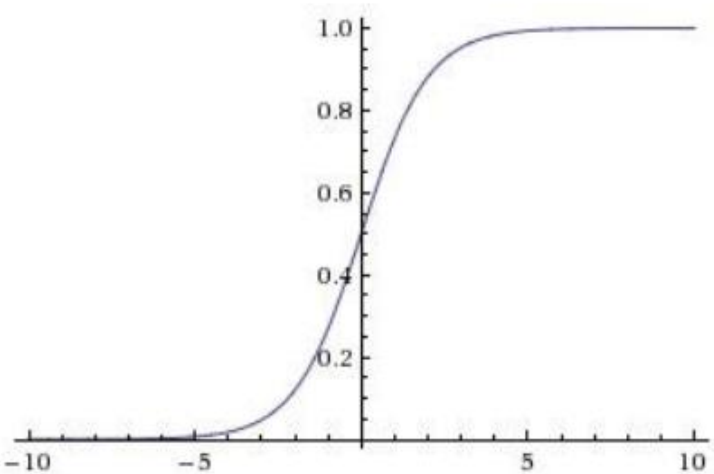
Activation Functions - Sigmoid



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3 problems:

1. Saturated neurons “kill” the gradients
2. Sigmoid outputs are not zero-centered

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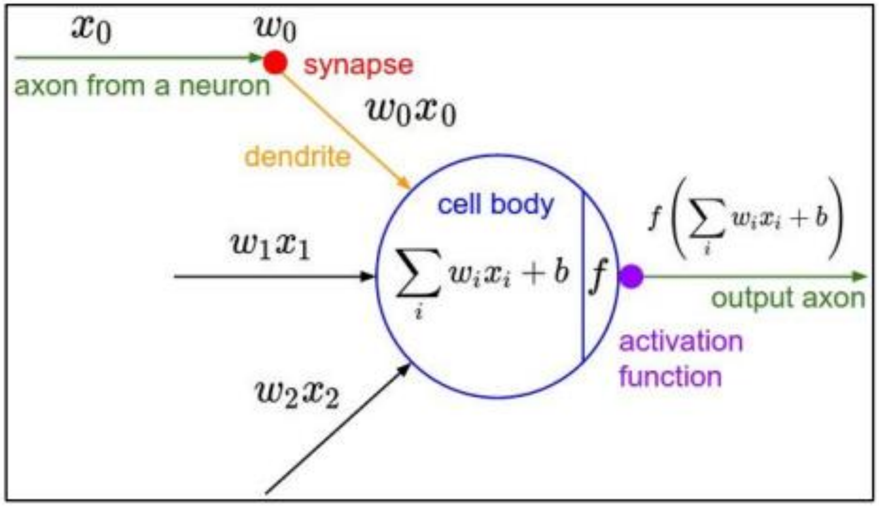
Activation Functions - Sigmoid

- <u>Consider what happens when the input to a neuron (x) is always positive</u>:



$$f\left(\sum_i w_i x_i + b\right)$$

- **Question #1:** what can we say about the gradients on **w**?

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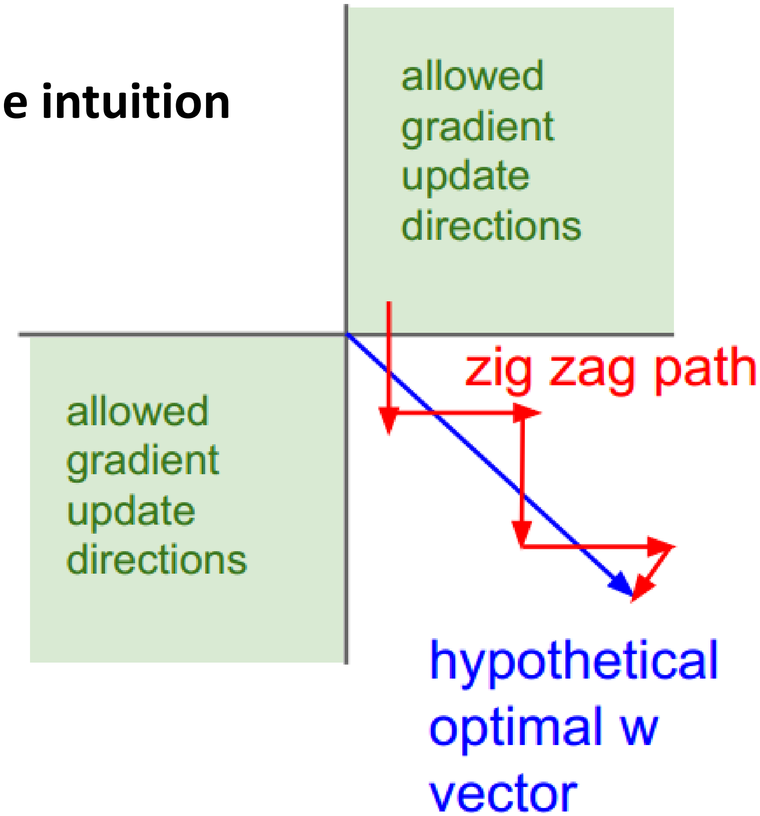
Activation Functions - Sigmoid

- <u>Consider what happens when the input to a neuron (x) is always positive:</u>

$$f\left(\sum_i w_i x_i + b\right)$$

Simple intuition



(this is also why we want to zero-mean data!)

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Activation Functions - Sigmoid

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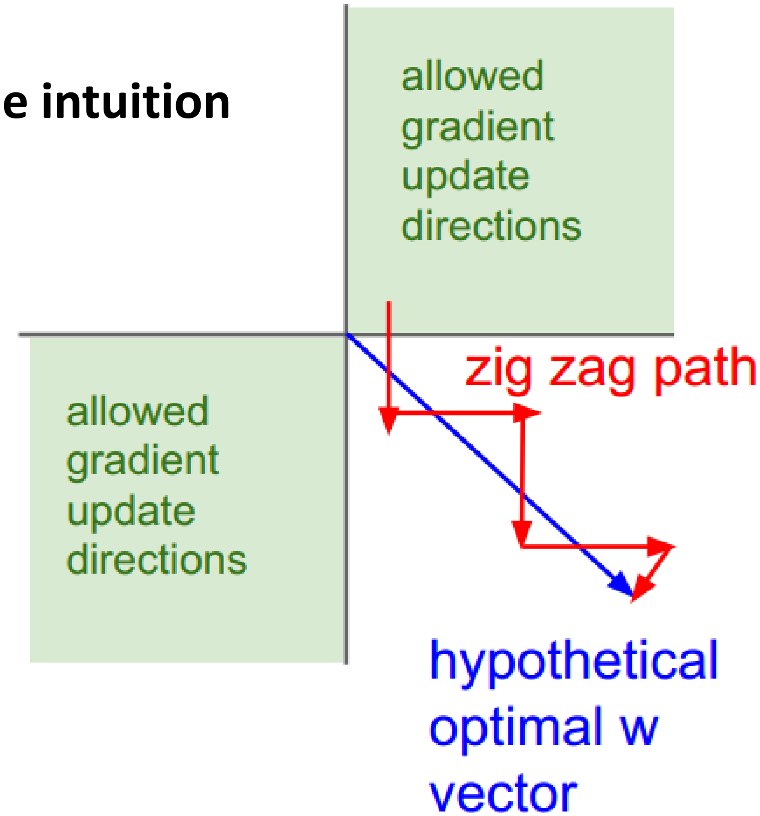

- **Not zero centered data have slower convergence**

(this is also why we want to zero-mean data!)

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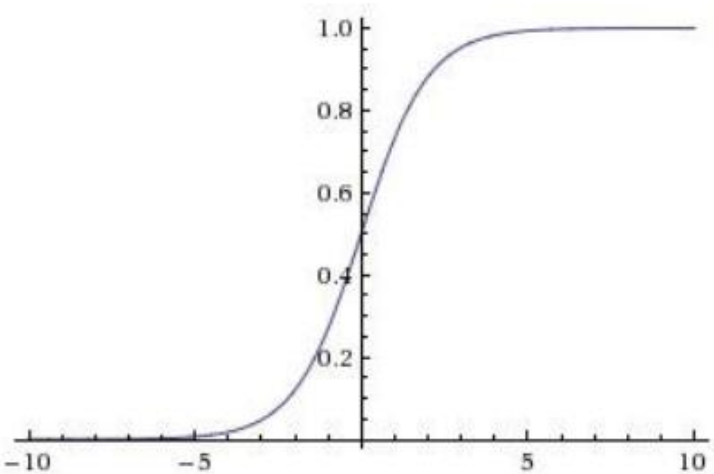
Activation Functions - Sigmoid

$$\sigma(x) = \frac{1}{\left(1+e^{-x}\right)}$$



Sigmoid

- Squashes numbers to range [0, 1]
- Historically popular since they have nice interpretation as a saturating "firing rate" of a neuron

3 problems:

1. Saturated neurons "kill" the gradients
2. Sigmoid outputs are not zero-centered
3. Exp() is a bit compute expensive

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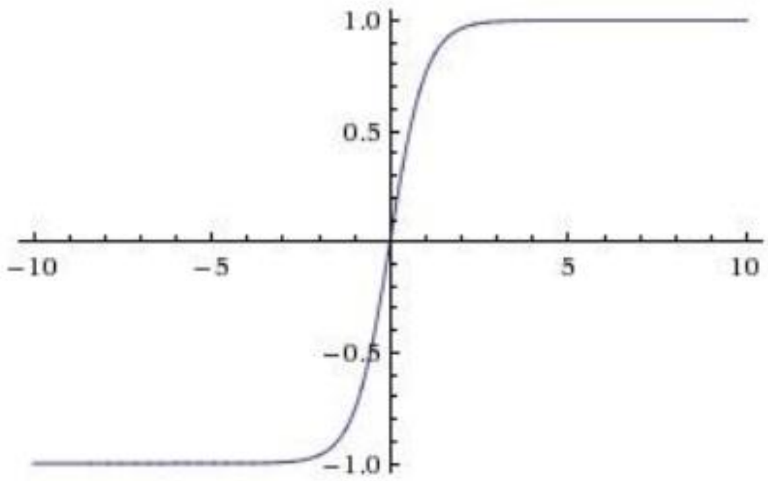
Activation Functions - Tanh



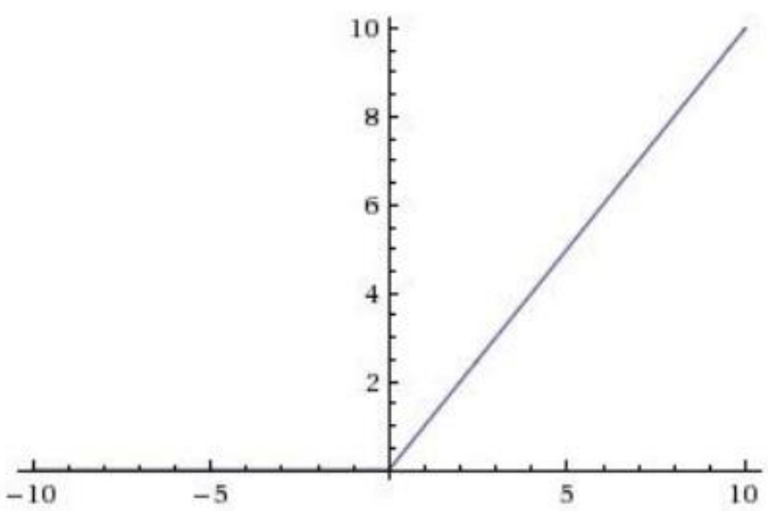
Tanh(x)

- Squashes numbers to range [-1,1]
- Zero centered (nice)
- Still kills gradients when saturated

[LeCun et al., 1991]

Activation Functions - ReLU

- ReLU (Rectified Linear Unit)



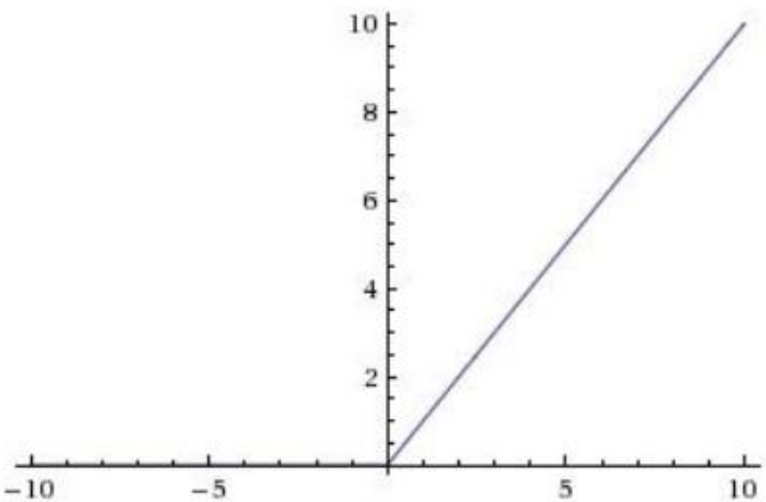
- Computes **f(x) = max(0, x)**
- Does not saturate (in +region)
- Very computationally efficient
- Converges much faster than sigmoid/tanh in practice (e.g. 6x)

- ReLU is the **default recommendation** what you should use.

[Krizhevsky et al., 2012]

Activation Functions - ReLU

- ReLU (Rectified Linear Unit)



- Computes **f(x) = max(0, x)**

- Does not saturate (in +region)
- Very computationally efficient
- Converges much faster than sigmoid/tanh in practice (e.g. 6x)

Problems:

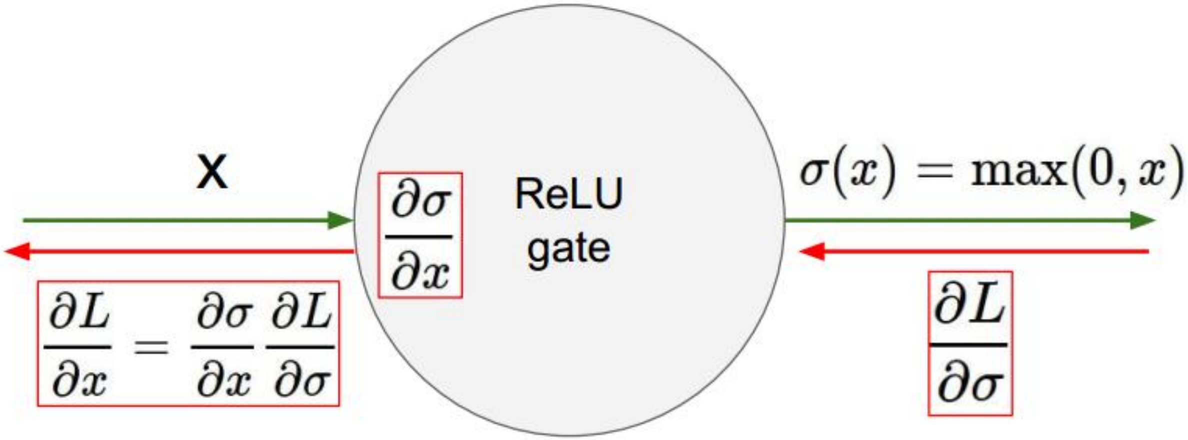
- Not zero-centered output
- An annoyance:

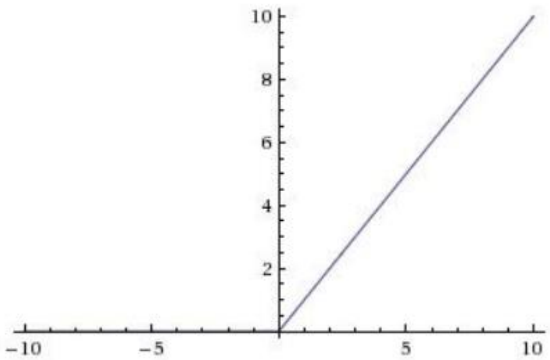
hint: what is the gradient when x < 0?

- ReLU is the **default recommendation** what you should use.

[Krizhevsky et al., 2012]

Activation Functions - ReLU





- What happens when x = -10? => **Dead ReLU**
- What happens when x = 0?
- What happens when x = 10?

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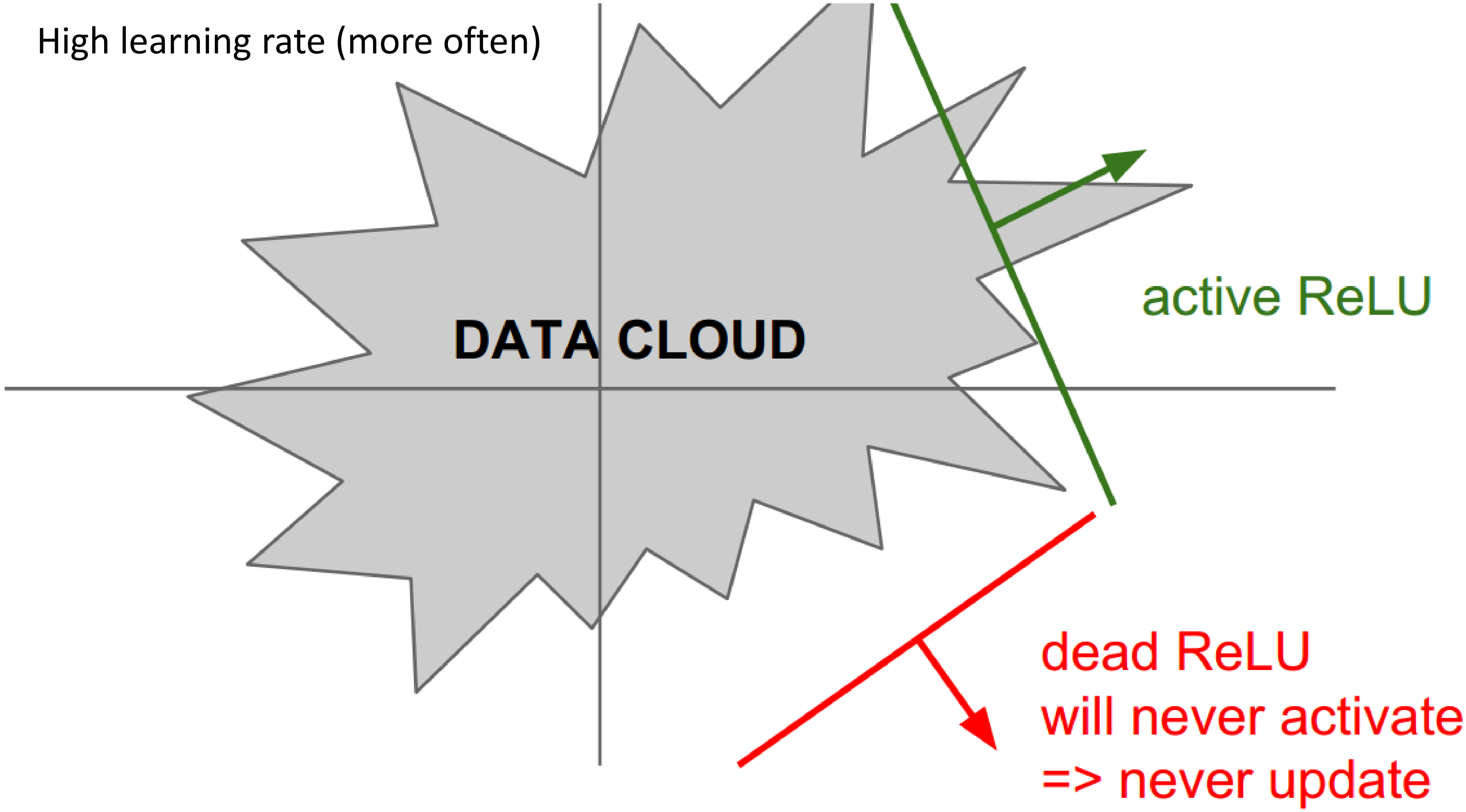
Activation Functions - ReLU

Dead ReLU:

- Initialization (unlucky)
- High learning rate (more often)



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Activation Functions - ReLU

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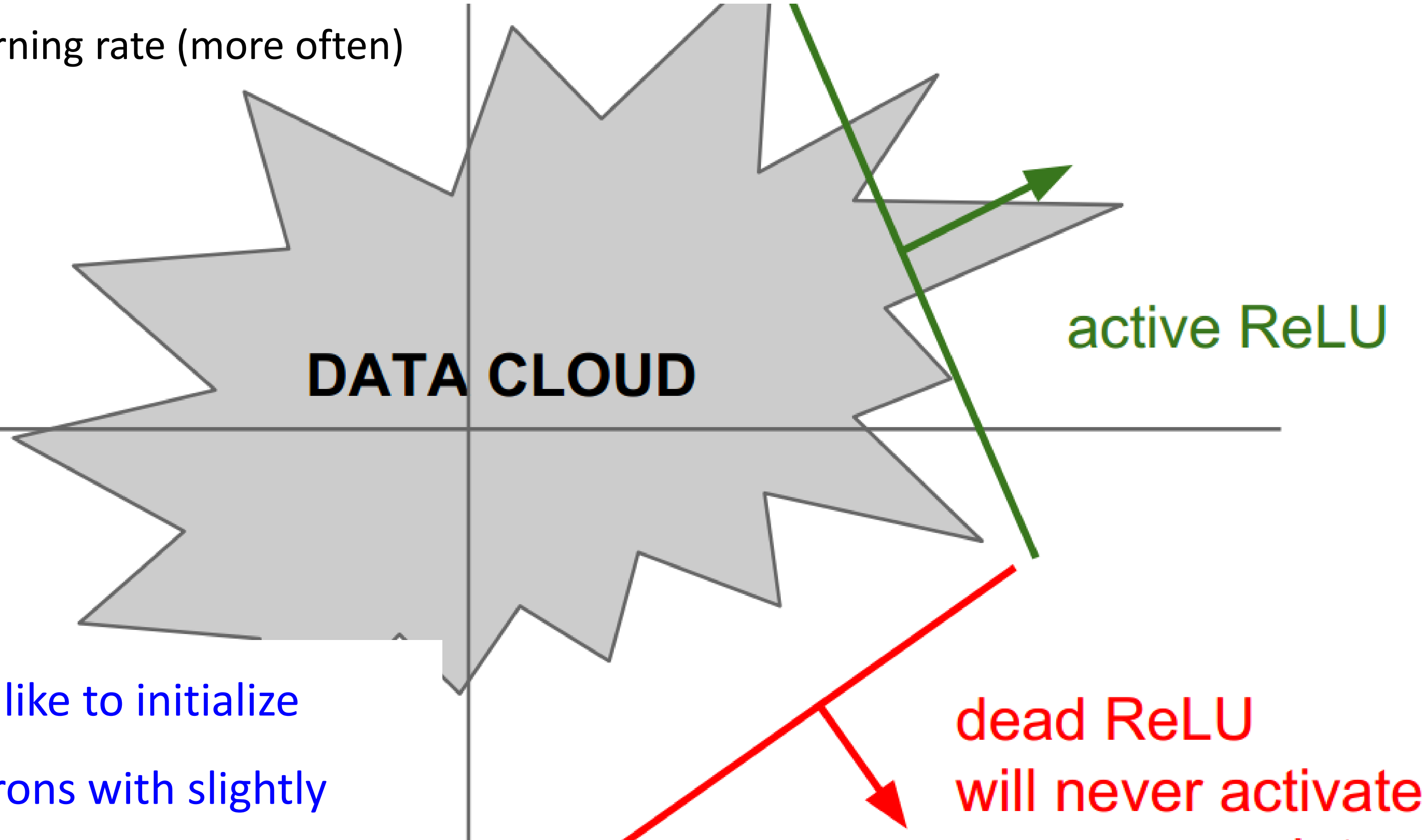

$\Rightarrow$ People like to initialize ReLU neurons with slightly positive biases (e.g. 0.01)

$f(\mathbf{x} + \mathbf{b})$???

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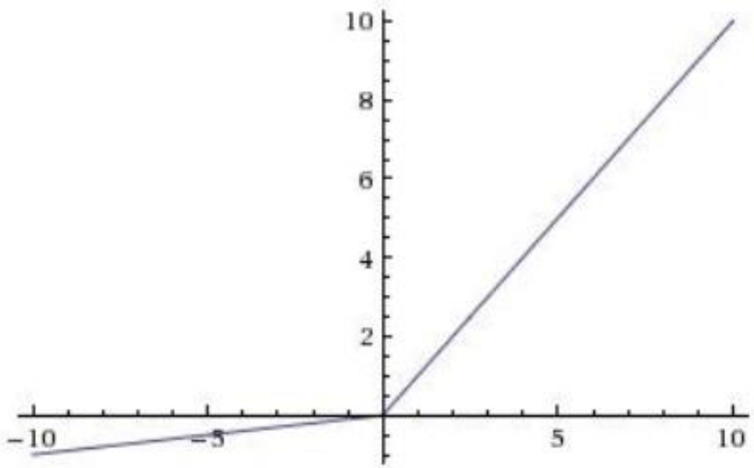
Activation Functions – Leaky ReLU



- Does not saturate
- Computationally efficient
- Converges much faster than sigmoid/tanh in practice! (e.g. 6x)
- **Will not "die"**

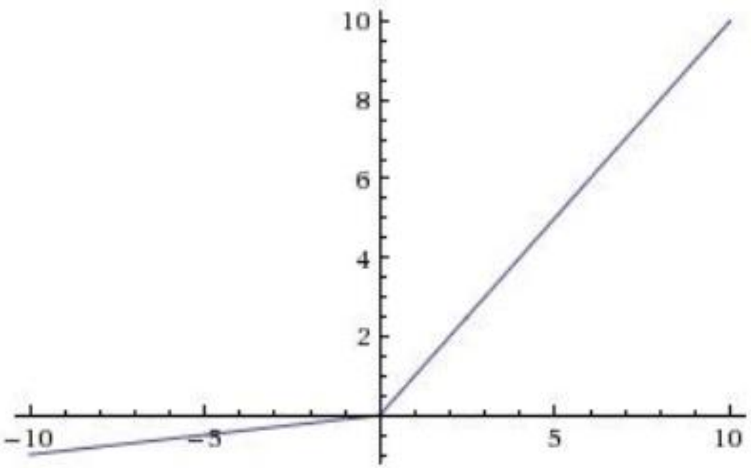
- **Leaky ReLU**

$$f(x) = \max(0.01x, x)$$

[Mass et al., 2013]
[He et al., 2015]

Activation Functions – Leaky ReLU

- Does not saturate
- Computationally efficient
- Converges much faster than sigmoid/tanh in practice! (e.g. 6x)
- **Will not "die"**



- **Leaky ReLU**

$$f(x) = \max(0.01x, x)$$

- **Parametric Rectifier (PReLU)**

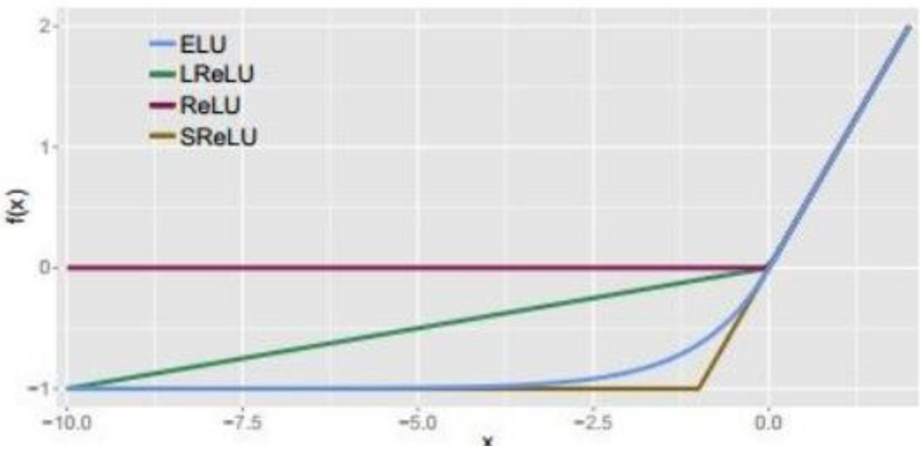
$$f(x) = \max(\alpha x, x)$$

Backprop into α to learn!

[Mass et al., 2013]
[He et al., 2015]

Activation Functions – ELU

- **Exponential Linear Units (ELU)**



- All benefits of ReLU
- Does not die
- Closer to zero mean outputs

- Computation requires exp()

$$f(x)=\begin{cases} x & \text{if } x>0 \\ \alpha(\exp(x)-1) & \text{if } x\le 0 \end{cases}$$

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Activation Functions – Maxout "Neuron"

- Very different form of the neuron, it's not just an activation function looks diff erent
- It changes with the neuro compute and how it computes

- Generalizes **ReLU** and **Leaky ReLU**
- Linear Regime! Does not saturate! Does not die!

$$\max\left(w_1^T x + b_1, w_2^T x + b_2\right)$$

- **Problem:** doubles the number of parameters/neuron ☹

[Goodfellow et al., 2013]

Summary

In practice:

- Use **ReLU**. Be careful with your learning rates
- Try out **Leaky ReLU / Maxout / ELU**
- Try out **tanh** but don't expect much
- Don't use **sigmoid,** tanh is better than it

- RNN / LSTM still uses sigmoid, but there are specific reason…

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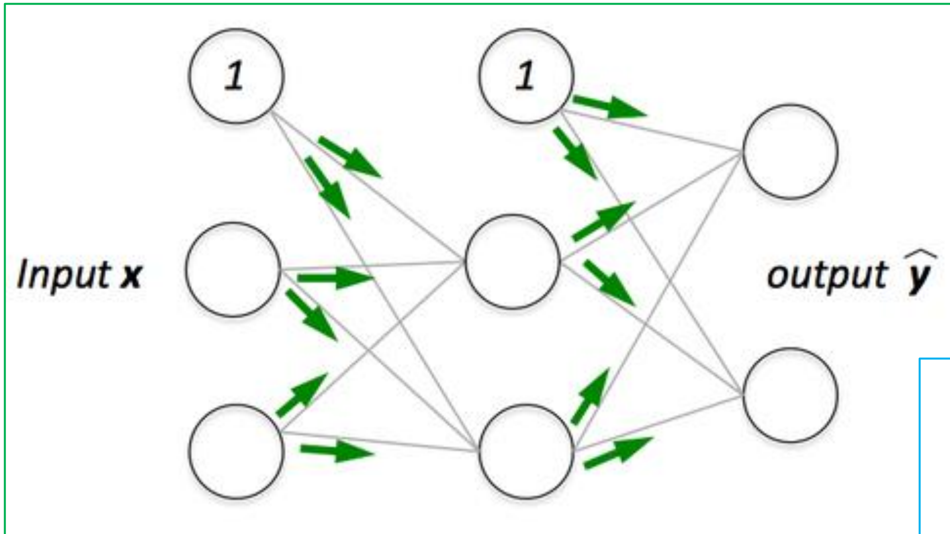
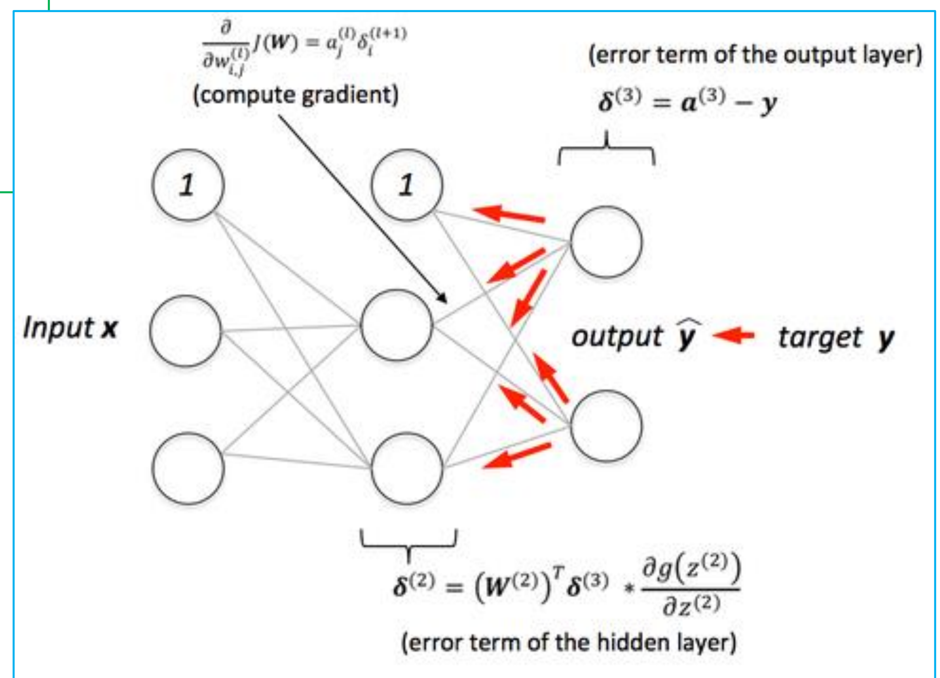
참고자료

Backpropagation

- Apply the back propagation to update the NN





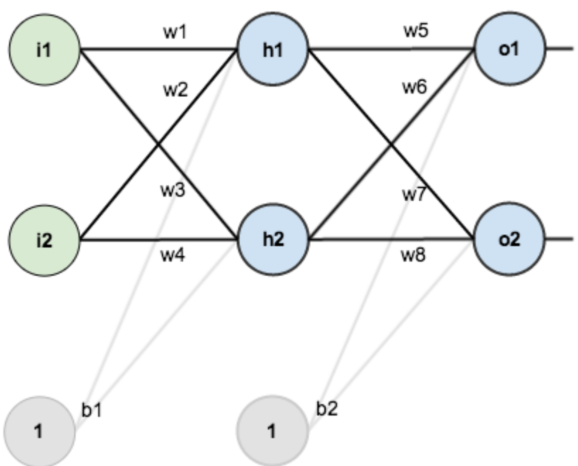
Chain rule:

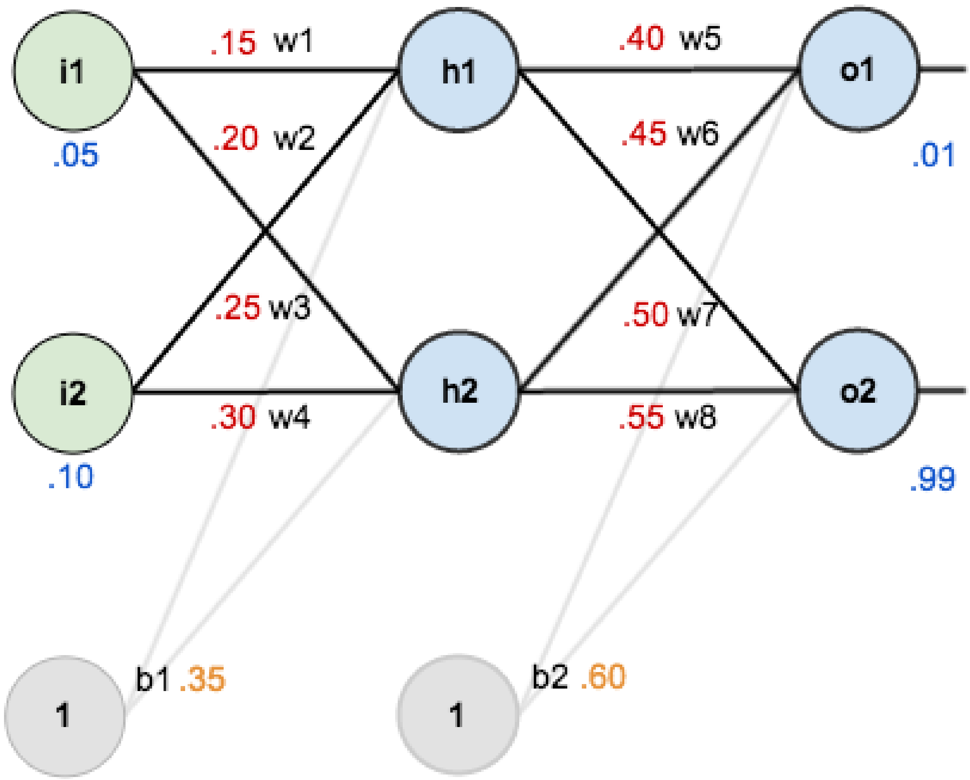
$$f(g(x)) = y$$

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial g} \frac{\partial g}{\partial x}$$

Basic Structure

In order to have some numbers to work with, here's are the initial weights, the biases, and training inputs/outputs:





The Forward Pass

We figure out the *total net input* to each hidden layer neuron, *squash* the total net input using an *activation function* (here we use the *logistic function*), then repeat the process with the output layer neurons.

Here's how we calculate the total net input for h_1:

$$net_{h1} = w_1 * i_1 + w_2 * i_2 + b_1 * 1$$

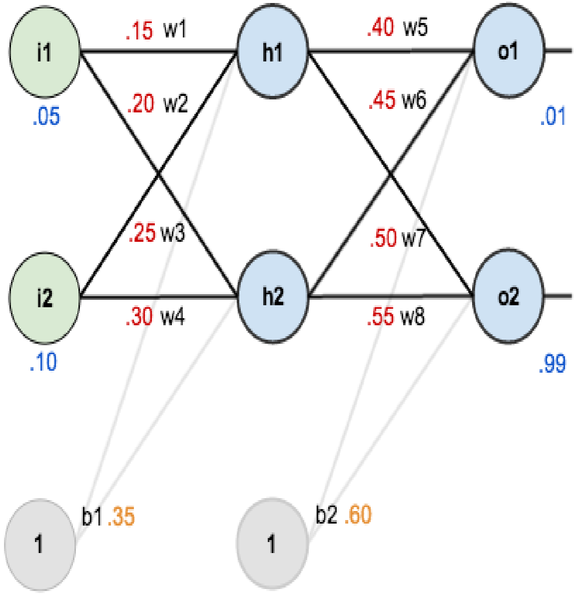
$$net_{h1} = 0.15 * 0.05 + 0.2 * 0.1 + 0.35 * 1 = 0.3775$$

We then squash it using the logistic function to get the output of h_1:

$$out_{h1} = \frac{1}{1+e^{-net_{h1}}} = \frac{1}{1+e^{-0.3775}} = 0.593269992$$

Carrying out the same process for h_2 we get:

$$out_{h2} = 0.596884378$$



The Forward Pass

We repeat this process for the output layer neurons, using the output from the hidden layer neurons as inputs.

Here's the output for o_1:

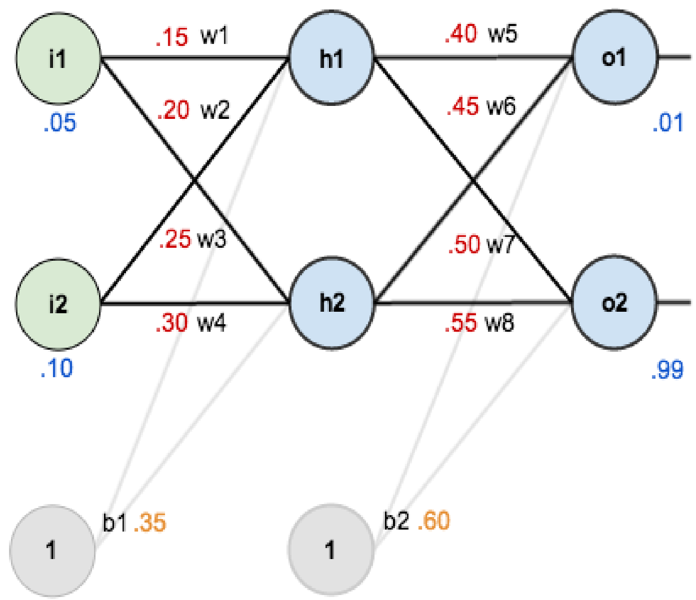
$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$net_{o1} = 0.4 * 0.593269992 + 0.45 * 0.596884378 + 0.6 * 1 = 1.105905967$$

$$out_{o1} = \frac{1}{1+e^{-net_{h1}}} = \frac{1}{1+e^{-1.105905967}} = 0.75136507$$

And carrying out the same process for o_2 we get:

$$out_{o2} = 0.772928465$$



The Forward Pass

Calculating the Total Error

We can now calculate the error for each output neuron using the squared error function and sum them to get the total error:

$$E_{total} = \sum \frac{1}{2}(target - output)^2$$

Some sources refer to the target as the *ideal* and the output as the *actual*.

The $\frac{1}{2}$ is included so that exponent is cancelled when we differentiate later on. The result is eventually multiplied by a learning rate anyway so it doesn't matter that we introduce a constant here [1].

The Forward Pass

For example, the target output for o_1 is 0.01 but the neural network output 0.75136507, therefore its error is:

$$E_{o1} = \frac{1}{2}(target_{o1} - out_{o1})^2 = \frac{1}{2}(0.01 - 0.75136507)^2 = 0.274811083$$

Repeating this process for o_2 (remembering that the target is 0.99) we get:

$$E_{o2} = 0.023560026$$

The total error for the neural network is the sum of these errors:

$$E_{total} = E_{o1} + E_{o2} = 0.274811083 + 0.023560026 = 0.298371109$$

The Backwards Pass

Output Layer

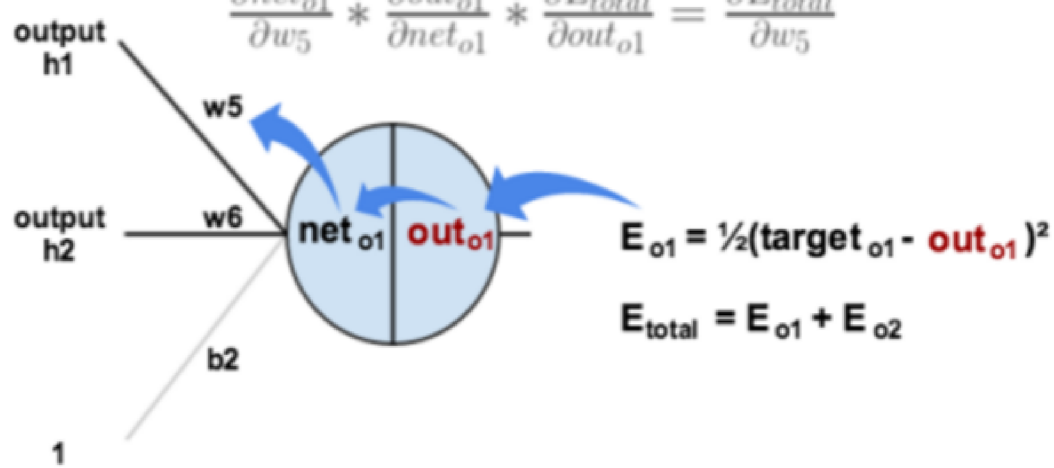
Consider w_5. We want to know how much a change in w_5 affects the total error, aka $\frac{\partial E_{total}}{\partial w_5}$.

> $\frac{\partial E_{total}}{\partial w_5}$ is read as "the partial derivative of E_{total} with respect to w_5". You can also say "the gradient with respect to w_5".

By applying the chain rule we know that:

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

$$\frac{\partial net_{o1}}{\partial w_5} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial E_{total}}{\partial out_{o1}} = \frac{\partial E_{total}}{\partial w_5}$$



The Backwards Pass

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

First, how much does the total error change with respect to the output?

$$E_{total} = \frac{1}{2}(target_{o1} - out_{o1})^2 + \frac{1}{2}(target_{o2} - out_{o2})^2$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = 2 * \frac{1}{2}(target_{o1} - out_{o1})^{2-1} * -1 + 0$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = -(target_{o1} - out_{o1}) = -(0.01 - 0.75136507) = 0.74136507$$

$-(target - out)$ is sometimes expressed as $out - target$

When we take the partial derivative of the total error with respect to out_{o1}, the quantity $\frac{1}{2}(target_{o2} - out_{o2})^2$ becomes zero because out_{o1} does not affect it which means we're taking the derivative of a constant which is zero.

The Backwards Pass

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

The partial derivative of the logistic function is the output multiplied by 1 minus the output:

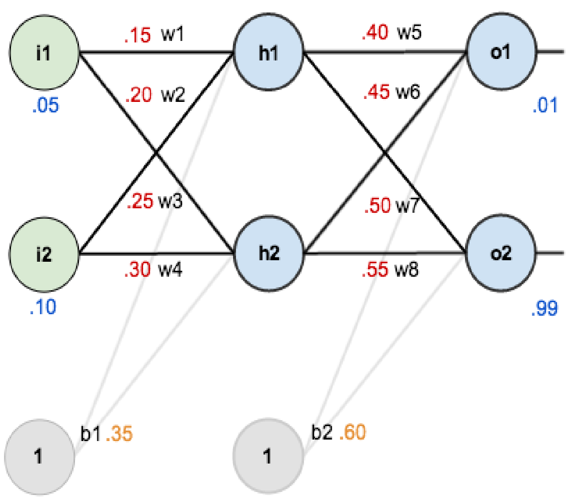
$$out_{o1} = \frac{1}{1+e^{-net_{o1}}}$$

$$\frac{\partial out_{o1}}{\partial net_{o1}} = out_{o1}(1 - out_{o1}) = 0.75136507(1 - 0.75136507) = 0.186815602$$

Finally, how much does the total net input of $o1$ change with respect to w_5?

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$\frac{\partial net_{o1}}{\partial w_5} = 1 * out_{h1} * w_5^{(1-1)} + 0 + 0 = out_{h1} = 0.593269992$$



Putting it all together:

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

$$\frac{\partial E_{total}}{\partial w_5} = 0.74136507 * 0.186815602 * 0.593269992 = 0.082167041$$

The Backwards Pass

$$\frac{\partial E_{total}}{\partial w_5} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial w_5}$$

You'll often see this calculation combined in the form of the delta rule:

$$\frac{\partial E_{total}}{\partial w_5} = -(target_{o1} - out_{o1}) * out_{o1}(1 - out_{o1}) * out_{h1}$$

Alternatively, we have $\frac{\partial E_{total}}{\partial out_{o1}}$ and $\frac{\partial out_{o1}}{\partial net_{o1}}$ which can be written as $\frac{\partial E_{total}}{\partial net_{o1}}$, aka δ_{o1} (the Greek letter delta) aka the *node delta*. We can use this to rewrite the calculation above:

$$\delta_{o1} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} = \frac{\partial E_{total}}{\partial net_{o1}}$$

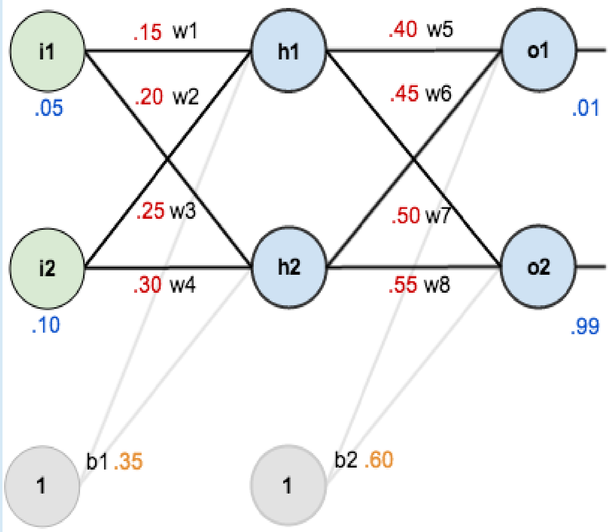
$$\delta_{o1} = -(target_{o1} - out_{o1}) * out_{o1}(1 - out_{o1})$$

Therefore:

$$\frac{\partial E_{total}}{\partial w_5} = \delta_{o1} out_{h1}$$

Some sources extract the negative sign from δ so it would be written as:

$$\frac{\partial E_{total}}{\partial w_5} = -\delta_{o1} out_{h1}$$



The Backwards Pass

To decrease the error, we then subtract this value from the current weight (optionally multiplied by some learning rate, eta, which we'll set to 0.5):

$$w_5^+ = w_5 - \eta * \frac{\partial E_{total}}{\partial w_5} = 0.4 - 0.5 * 0.082167041 = 0.35891648$$

> Some sources use α (alpha) to represent the learning rate, others use η (eta), and others even use ϵ (epsilon).

We can repeat this process to get the new weights w_6, w_7, and w_8:

$$w_6^+ = 0.408666186$$

$$w_7^+ = 0.511301270$$

$$w_8^+ = 0.561370121$$

The Backwards Pass

Hidden Layer

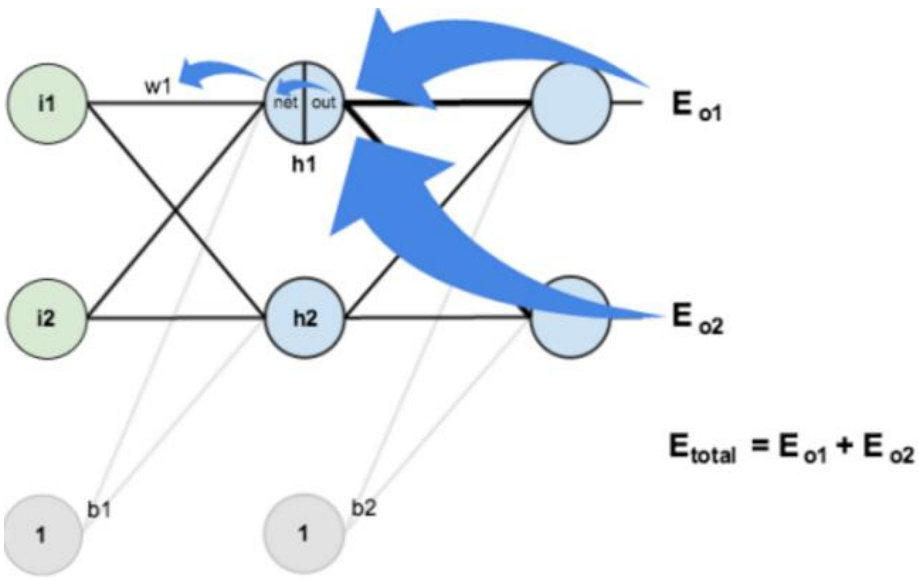
Next, we'll continue the backwards pass by calculating new values for w_1, w_2, w_3, and w_4.

Big picture, here's what we need to figure out:

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}}$$



The Backwards Pass

$$\delta_{o1} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} = \frac{\partial E_{total}}{\partial net_{o1}}$$

$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}}$$

Starting with $\frac{\partial E_{o1}}{\partial out_{h1}}$:

$$\frac{\partial E_{o1}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial out_{h1}}$$

$$E_{total} = \frac{1}{2}(target_{o1} - out_{o1})^2 + \frac{1}{2}(target_{o2} - out_{o2})^2$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = 2 * \frac{1}{2}(target_{o1} - out_{o1})^{2-1} * -1 + 0$$

$$\frac{\partial E_{total}}{\partial out_{o1}} = -(target_{o1} - out_{o1}) = -(0.01 - 0.75136507) = 0.74136507$$

$$out_{o1} = \frac{1}{1+e^{-net_{o1}}}$$

$$\frac{\partial out_{o1}}{\partial net_{o1}} = out_{o1}(1 - out_{o1}) = 0.75136507(1 - 0.75136507) = 0.186815602$$

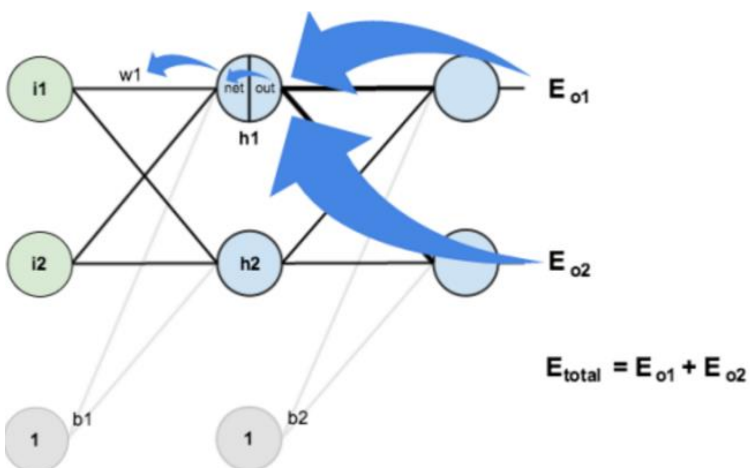
We can calculate $\frac{\partial E_{o1}}{\partial net_{o1}}$ using values we calculated earlier:

$$\frac{\partial E_{o1}}{\partial net_{o1}} = \frac{\partial E_{o1}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} = 0.74136507 * 0.186815602 = 0.138498562$$

And $\frac{\partial net_{o1}}{\partial out_{h1}}$ is equal to w_5:

$$net_{o1} = w_5 * out_{h1} + w_6 * out_{h2} + b_2 * 1$$

$$\frac{\partial net_{o1}}{\partial out_{h1}} = w_5 = 0.40$$



Plugging them in:

$$\frac{\partial E_{o1}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial net_{o1}} * \frac{\partial net_{o1}}{\partial out_{h1}} = 0.138498562 * 0.40 = 0.055399425$$

The Backwards Pass

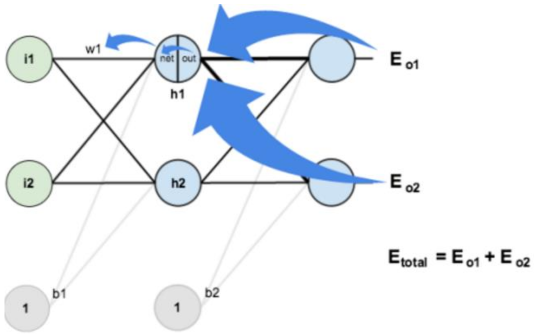
$$\delta_{o1} = \frac{\partial E_{total}}{\partial out_{o1}} * \frac{\partial out_{o1}}{\partial net_{o1}} = \frac{\partial E_{total}}{\partial net_{o1}}$$

Following the same process for $\frac{\partial E_{o2}}{\partial out_{o1}}$, we get:

$$\frac{\partial E_{o2}}{\partial out_{h1}} = -0.019049119$$

Therefore:

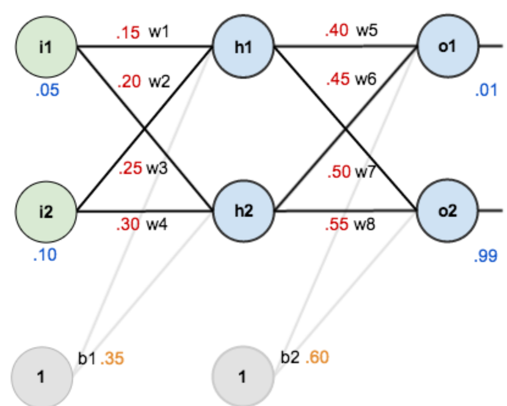
$$\frac{\partial E_{total}}{\partial out_{h1}} = \frac{\partial E_{o1}}{\partial out_{h1}} + \frac{\partial E_{o2}}{\partial out_{h1}} = 0.055399425 + -0.019049119 = 0.036350306$$



Now that we have $\frac{\partial E_{total}}{\partial out_{h1}}$, we need to figure out $\frac{\partial out_{h1}}{\partial net_{h1}}$ and then $\frac{\partial net_{h1}}{\partial w}$ for each weight:

$$out_{h1} = \frac{1}{1+e^{-net_{h1}}}$$

$$\frac{\partial out_{h1}}{\partial net_{h1}} = out_{h1}(1 - out_{h1}) = 0.59326999(1 - 0.59326999) = 0.241300709$$



We calculate the partial derivative of the total net input to h_1 with respect to w_1 the same as we did for the output neuron:

$$net_{h1} = w_1 * i_1 + w_2 * i_2 + b_1 * 1$$

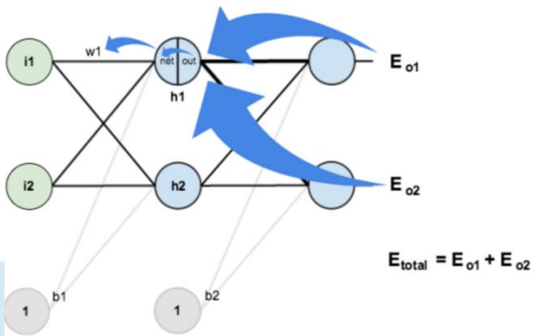
$$\frac{\partial net_{h1}}{\partial w_1} = i_1 = 0.05$$

The Backwards Pass

Putting it all together:

$$\frac{\partial E_{total}}{\partial w_1} = \frac{\partial E_{total}}{\partial out_{h1}} * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial w_1} = 0.036350306 * 0.241300709 * 0.05 = 0.000438568$$

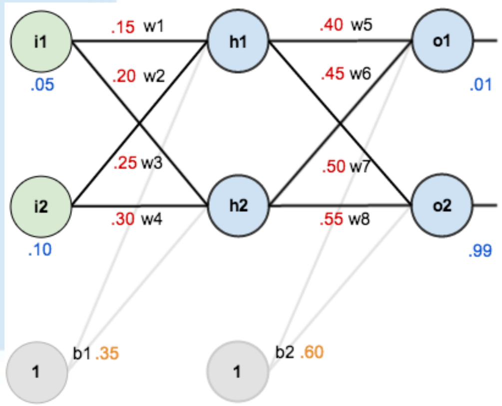


You might also see this written as:

$$\frac{\partial E_{total}}{\partial w_1} = \left(\sum_o \frac{\partial E_{total}}{\partial out_o} * \frac{\partial out_o}{\partial net_o} * \frac{\partial net_o}{\partial out_{h1}}\right) * \frac{\partial out_{h1}}{\partial net_{h1}} * \frac{\partial net_{h1}}{\partial w_1}$$

$$\frac{\partial E_{total}}{\partial w_1} = \left(\sum_o \delta_o * w_{ho}\right) * out_{h1}(1 - out_{h1}) * i_1$$

$$\frac{\partial E_{total}}{\partial w_1} = \delta_{h1} i_1$$



We can now update w_1:

$$w_1^+ = w_1 - \eta * \frac{\partial E_{total}}{\partial w_1} = 0.15 - 0.5 * 0.000438568 = 0.149780716$$

The Backwards Pass

Repeating this for w_2, w_3, and w_4

$$w_2^+ = 0.19956143$$

$$w_3^+ = 0.24975114$$

$$w_4^+ = 0.29950229$$