CS1200: Intro. to Algorithms and their Limitations	Prof. Salil Vadhan

Sender–Receiver Exercise 4: Reading for Receivers

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1 Sender–Receiver Exercise: Beating Exhaustive Search for Coloring

If you haven't already done so, read the SRE instructions.

The goals of this exercise are:

- to develop your skills at understanding, distilling, and communicating proofs and the conceptual ideas in them, especially for proofs in graph theory
- to reinforce the definition and algorithms we have seen for Graph Coloring and the related concept of independent sets
- to expose you to a nontrivial exponential-time algorithm

To prepare for this exercise as a Receiver, you should try to understand the theorem statement and definition in Section 1.1 below, and review the material covered in Chapter 13 and Section 14.1 of the Hesterberg-Vadhan textbook. Your partner Sender will communicate the proof of Theorem 1.1 to you.

1.1 The Result

In Corollary 13.5 of the Hesterberg-Vadhan textbook, we saw that Graph 2-Coloring can be solved in time $O(n+m)$ via BFS. However, for Graph 3-Coloring we have not seen any algorithm but exhaustive search, which can take time $O(m \cdot 3^n)$: there are 3^n ways to pick a color for each of the n vertices, and m edges whose endpoints must be checked to have different colors. Here you will see an algorithm with a better running time than exhaustive search:

Theorem 1.1. Graph 3-Coloring *can be solved in time* $O((1.89)^n)$.

Even though this is still exponential, the improvement over 3^n is significant and allows for solving noticeably larger problem sizes. The best known running time for 3-coloring is approximately $O((1.33)^n)$.

The proof will exploit the following basic relationship between coloring and independent sets:

> *A k-coloring of a graph $G = (V, E)$ is equivalent to a partition of V into k independent sets (one corresponding to each color class).*

Algorithm.

Correctness Lemma.

Proof of Lemma.

Runtime.