

Sender–Receiver Exercise 3: Reading for Receivers

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1 Sender–Receiver Exercise: Connected Components

You should review Sections 12.5.3 in the Hesterberg-Vadhan textbook and the non-optional sections below before doing this exercise.

1.1 Connected Components

We begin by defining the *connected components* of an undirected graph. To gain intuition, you may find it useful to draw some pictures of graphs with multiple connected components and use them to help you follow along the proof.

Theorem .1. *Every undirected graph $G = (V, E)$ can be partitioned into connected components. That is, there are sets $V_0, \dots, V_{c-1} \subseteq V$ of vertices such that:*

1. V_0, \dots, V_{c-1} are disjoint, nonempty, and $V_0 \cup V_1 \cup \dots \cup V_{c-1} = V$. (This is what it means for V_0, \dots, V_{c-1} to be a partition of V .)
2. For every two vertices $u, v \in V$, u and v are in the same component V_i if and only if there is a path from u to v .

Moreover the sets V_0, \dots, V_{c-1} are unique (up to ordering), and are called the connected components of V .

In case you are interested, we include a proof of this theorem in the Hesterberg-Vadhan textbook (Section 12.5.3), but studying that proof is not required for this exercise.

We remark that for *directed* graphs, one can consider *weakly connected components*, where we ignore the directions of edges, and *strongly connected components*, where two vertices u, v are the same component if and only if there is a directed path from u to v and a directed path from v to u . Strongly connected components are more useful, but more complicated. In particular, unlike in undirected graphs (or weakly connected components), there can be edges crossing between strongly connected components.

1.2 Finding Connected Components via BFS

The main result of this exercise is an efficient algorithm for finding connected components:

Theorem .2. *There is an algorithm that, given an undirected graph $G = (V, E)$ with n vertices and m edges, partitions V into connected components in time $O(n + m)$.*

Proof.

Proof outline:

Modification of BFS:

Runtime of modified BFS:

Algorithm to Find Connected Components:

Correctness of Algorithm:

Runtime of Algorithm:

□

1.3 (Optional) Proof of Theorem .1

Proof. For every vertex u , define

$$\llbracket u \rrbracket = \{v : \text{there is a path from } u \text{ to } v \text{ in } G\}.$$

Observe that $u \in \llbracket u \rrbracket$; in particular, the set $\llbracket u \rrbracket$ is nonempty.

Now let's show that for every two vertices u and w , we have either that $\llbracket u \rrbracket$ and $\llbracket w \rrbracket$ are disjoint or equal. Suppose they are not disjoint, i.e. there is a vertex $v \in \llbracket u \rrbracket \cap \llbracket w \rrbracket$. This means that there is a path p_{uv} from u to v and a path p_{wv} from w to v . Now we argue that $\llbracket u \rrbracket \subseteq \llbracket w \rrbracket$. Let a be any vertex in $\llbracket u \rrbracket$, so there is a path p_{ua} from u to a . Then we can get a path from w to a by first following the path p_{wv} to get from w to v , then reversing the edges in p_{uv} to get from v to u , and then following the path p_{ua} to get from u to a . Thus, $a \in \llbracket w \rrbracket$. Since we showed that this holds for every $a \in \llbracket u \rrbracket$, we conclude that $\llbracket u \rrbracket \subseteq \llbracket w \rrbracket$. The reverse inclusion $\llbracket w \rrbracket \subseteq \llbracket u \rrbracket$ is proved in a similar manner.

So now we take V_0, \dots, V_{c-1} to be all of the distinct sets that occur among those of the form $\llbracket u \rrbracket$. Since every vertex $u \in V$ is in the set $\llbracket u \rrbracket$, the sets V_0, \dots, V_{c-1} will cover all of V , and by what we just showed, any two distinct sets will be disjoint from each other. This establishes Item 1. Now if a vertex u is in component V_i , this means that $\llbracket u \rrbracket = V_i$ (else $\llbracket u \rrbracket$ and V_i would be distinct but not disjoint, contradicting what we showed above). So V_i contains exactly the vertices v that are reachable from u , establishing Item 2.

We omit the proof of uniqueness of the set of connected components. □

If you have seen equivalence relations, you may recognize some similarity with the above proof. Indeed, the above proof amounts to showing that “ v is reachable from u ” is an equivalence relation on V , and then taking the connected components to be the equivalence classes under that relation.