CS1200: Intro. to Algorithms and their Limitations Prof. Salil Vadhan

Lecture 16: Logic

Harvard SEAS - Fall 2025 *2025-10-28*

1 Announcements

- Midterm grades released. Median is $42/64 \to 2.875$ on 4.0 scale (low B), lower than last year, despite being curved more generously.
- Conjecture: too much reliance on collaborators on psets (whether human or AI). There's no substitute for doing the hard work of problem-solving yourself for developing your own understanding.
- Deadline for us to approve pass/fail postponed to next Monday 11/3.

Recommended Reading:

- Hesterberg–Vadhan 17
- Lewis–Zax 9–10
- Roughgarden IV, Sec. 21.5, Ch. 24

2 Loose Ends: Modelling the Prioritarian Objectives

- Assign each patient v to have *two* weights $w_0(v)$ and $w_1(v)$, with $w_0(v)$ giving what v's expected welfare (e.g. as measured by QALYs) would be if they do not receive a kidney donation and $w_1(v)$ giving what v's expected welfare would be if they do receive a kidney donation.
- *Maximin* objective: Applied to kidney exchange, we can mathematically model the maximin principle by seeking a matching M that first attempts to maximize:

$$\min\left(\{w_0(v) : v \text{ unmatched by } M\} \cup \{w_1(v) : v \text{ matched by } M\}\right).$$

  If two matchings M and M' have the same value of the above objective function, i.e. the worst-off patients v in each of the matchings have the same expected welfare, then we turn to comparing the second-worst-off patients in the two matchings, and if that turns out to be a tie, we proceed to the third-worst-off patients and so on.

- *Prioritarian* objective: For some concave, non-decreasing function f (e.g. $f(x) = \sqrt{x}$ or $f(x) = \log x$, maximizes

$$\sum_{v:v \text{ unmatched by } M} f(w_0(v)) + \sum_{v:v \text{ matched by } M} f(w_1(v)).$$

This is equivalent to solving the Maximum Vertex-Weighted Matching problem with vertex weights $w(v) = f(w_1(v)) - f(w_0(v))$.

3 Loose Ends: Maximum Matching Algorithm

`MaxMatchingAugPaths`(G):
Input : A bipartite graph $G = (V, E)$
Output : A maximum-size matching $M \subseteq E$

Remove isolated vertices from G;
Let V_0, V_1 be the bipartition (i.e. 2-coloring) of V;
$M = \emptyset$;
repeat
  Let U be the vertices unmatched by M, $U_0 = V_0 \cap U$, $U_1 = V_1 \cap U$;
  Use BFS to find a shortest alternating walk P that starts in U_0 and ends in U_1;
  if $P \neq \perp$ **then** augment M using P via Lemma 3.3;
until $P = \perp$;
return M

Algorithm 3.1: `MaxMatchingAugPaths()`

Lemma 3.1 ("Certain shortest alternating walks are augmenting paths")**.** *Let G be bipartite, with bipartition (V_0, V_1),[1] and let M be a matching in G that is not of maximum size. Let U be the vertices that are not matched by M, and $U_0 = V_0 \cap U$ and $U_1 = V_1 \cap U$. Then:*

1. *G has an alternating walk with respect to M that starts in U_0 and ends in U_1.*
2. *Every shortest alternating walk from U_0 to U_1 is an augmenting path.*

Lemma 3.2 ("Shortest alternating walks are easy to find")**.** *Finding shortest alternating walks in bipartite graphs reduces to finding shortest paths in directed graphs in time $O(n + m)$, where $n = |V|$ and $m = |E|$.*

Lemma 3.3 ("Augmenting paths can be used to efficiently grow matchings")**.** *Given a graph $G = (V, E)$, a matching M, and an augmenting path P with respect to M, we can construct a matching M' with $|M'| = |M| + 1$ in time $O(n)$.*

[1] Recall that a *bipartite* graph is a graph that is 2-colorable, and a *bipartition* of a bipartite graph is a partition of the vertex set $V = V_0 \cup V_1$ into the 2 color classes given by a 2-coloring. Thus all of the edges in the graph have one endpoint in V_0 and one endpoint in V_1.

Using these lemmas we will prove:

Theorem 3.4. Maximum Matching *can be solved in time* $O(mn)$ *on bipartite graphs with* m *edges and* n *vertices.*

We defer the proof of Lemmas 3.1 to the textbook.

Proof sketch of Lemma 3.2.

$\square$

Proof of Lemma 3.3.

$\square$

4 Propositional Logic

Motivation: Logic is a fundamental building block for computation (e.g. digital circuits) and a very expressive language for encoding computational problems we want to solve.

Definition 4.1 (boolean formulas, informal)**.** A *boolean formula* φ is a formula built up from a finite set of variables, say $x_0, \ldots, x_{n-1}$, using the logical operators $\wedge$ (AND), $\vee$ (OR), and $\neg$ (NOT) and parentheses.

Every boolean formula φ on n variables defines a boolean function, which we'll also denote by $\varphi : \{0,1\}^n \to \{0,1\}$, where we interpret 0 as false and 1 as true, and give $\wedge, \vee, \neg$ their usual semantics (meaning).

The Lewis–Zax text (textbook for CS 20) contains formal, inductive definitions of boolean formulas and the corresponding boolean functions.

Example 4.2.

$$\varphi_{maj}(x_0, x_1, x_2) = (x_0 \wedge x_1) \vee (x_1 \wedge x_2) \vee (x_2 \wedge x_0)$$

is a boolean formula. It evaluates to 1 if

$$\varphi_{pal}(x_0, x_1, x_2, x_3) = ((x_0 \wedge x_3) \vee (\neg x_0 \wedge \neg x_3)) \wedge ((x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2))$$

is a boolean formula. It evaluates to 1 if

We now turn to two important special cases of boolean formulas.

Definition 4.3 (DNF and CNF formulas)**.**

- A *literal* is a variable (e.g. x_i) or its negation ($\neg x_i$).
- A *term* is an AND of a sequence of literals.
- A *clause* is an OR of a sequence of literals.
- A boolean formula is in *disjunctive normal form (DNF)* if it is the OR of a sequence of terms.
- A boolean formula is in *conjunctive normal form (CNF)* if it is the AND of a sequence of clauses.

Q: What truth value should an empty term (an AND of no literals) have?

Q: What truth value should an empty clause have?

Q: For each of the examples φ_{maj} and φ_{pal} above, is it in DNF, CNF, both, or neither?

Simplifying clauses. We define a function `Simplify` which takes a clause C and simplifies it as follows:

1.
2.
3.

One reason that DNF and CNF are commonly used is that they can express all boolean functions:

Lemma 4.4. *For every boolean function $f : \{0,1\}^n \to \{0,1\}$, there are boolean formulas φ and ψ in DNF and CNF, respectively, such that $f \equiv \varphi$ and $f \equiv \psi$, where we use $\equiv$ to indicate equivalence as functions, i.e. $f \equiv g$ iff $\forall x : f(x) = g(x)$.*

Proof.

□

Example 4.5. The majority function on 3 bits can be written in CNF as follows:

$\psi(x_0, x_1, x_2) =$

This example shows that the DNF and CNF given by the general construction are not necessarily the smallest ones possible for a given function, as the majority function can also be expressed by the simpler CNF formula

5 Computational Problems in Propositional Logic

Here are three natural computational problems about boolean formulas:

Input: A boolean formula φ on n variables **Output:** An $\alpha \in \{0,1\}^n$ such that $\varphi(\alpha) = 1$ (if one exists)

Computational Problem SATISFIABILITY

Input: A CNF formula φ on n variables **Output:** An $\alpha \in \{0,1\}^n$ such that $\varphi(\alpha) = 1$ (if one exists)

Computational Problem CNF-SATISFIABILITY

Input: A DNF formula φ on n variables **Output:** An $\alpha \in \{0,1\}^n$ such that $\varphi(\alpha) = 1$ (if one exists)

Computational Problem DNF-SATISFIABILITY

Q: One of these problems is algorithmically very easy. Which one?

6 Modelling using Satisfiability

One reason that SAT is important is its richness in encoding other problems. Thus any improvements to algorithms for (CNF-)SAT (aka "SAT Solvers") can be easily be applied to many other problems we want to solve.

Theorem 6.1. GRAPH COLORING *with k colors on graphs with n nodes and m edges can be reduced in time $O(n + km)$ to* CNF-SATISFIABILITY *with kn variables and $n + km$ clauses.*

Proof. Given $G = (V, E)$ and $k \in \mathbb{N}$, we will construct a CNF formula ϕ_G that captures the constraints of the Graph Coloring problem. The variables of ϕ_G are *indicator variables* $x_{v,i}$ for each $v \in V$ and $i \in [k]$, which intuitively are meant to correspond to vertex v being assigned to color i.

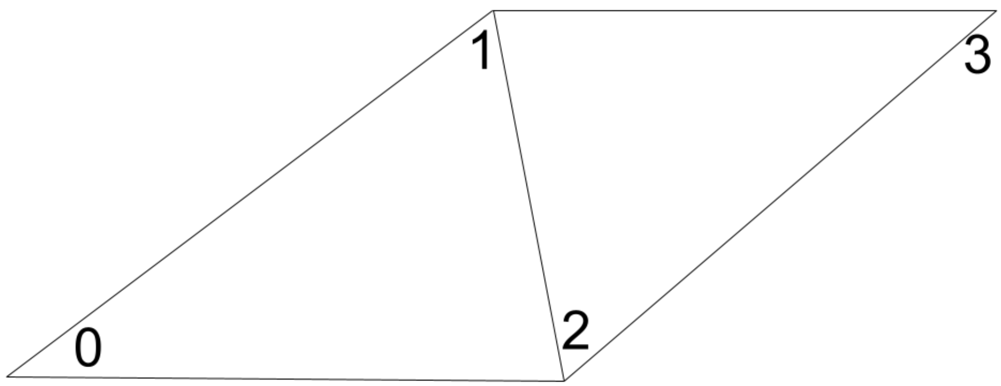
We then have a few types of clauses:

1.

2.

3.

For instance, if G is the graph below and $k = 3$,



then we make the following SAT instance ϕ_G:

We then call the SAT oracle on ϕ_G and get an assignment α. If $\alpha = \perp$, we say G is not k-colorable. Otherwise, we construct and output the coloring f_α given by:

$$f_\alpha(v) =$$

The runtime essentially follows from our description.

For correctness, we make two claims:

Claim 6.2. *If G has a valid k-coloring, then ϕ_G is satisfiable.*

Claim 6.3. *If α satisfies ϕ_G, then f_α is a proper k-coloring of G.*

Both of these claims are worth checking. Note that f_α is *well-defined* because α satisfies clauses of type 1 (so the definition of f_α doesn't try to take the minimum of an empty set) and is *proper* due to clauses of type 2. $\square$

Unfortunately, the fastest known algorithms for SAT have worst-case runtime exponential in n. However, enormous effort has gone into designing heuristics that complete much more quickly on many real-world instances. In particular, SAT Solvers—with many additional optimizations—were used to solve large-scale Graph Coloring instances arising in the 2016 US Federal Communications Commission (FCC) auction to reallocate wireless spectrum.

As we will see repeatedly in the rest of the course, SAT can encode a very wide variety of problems of interest, with the next example being Longest Path in the Sender–Receiver Exercise next week.

Thus motivated, in the next classwe will turn to algorithms for SAT, to get a taste of some of the ideas that go into SAT Solvers.