Homework1 (Probability and Linear Algebra Review)

1 Probability(47)

1.1 Expectation(10)

Suppose X, Y are two random variables.

- Show $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.
- Show $Var(X) = \mathbb{E}[var(X|Y)] + var(\mathbb{E}[X|Y])$

1.2 Function of random variables(10)

Suppose $X_1, X_2, ..., X_n$ are iid random variables. find the probability density function of $Y_1 = max\{X_1, X_2, ..., X_n\}$, $Y_2 = min\{X_1, X_2, ..., X_n\}$.

1.3 Independence(10)

Assume X, Y are two independent random variables such that:

$$\begin{aligned} &P(X = k) = P(Y = k) = pq^k, k = 0, 1, 2, ... \\ &q = p - 1 \end{aligned} \tag{1}$$

Show that $X - Y$ and $min(X, Y)$ are independent random variables.

1.4 Nice Problem(17)

We had a coin that it's head probability is $\frac{1}{2}$. Can you design a procedure with this coin to make probability 0.55? what about $\frac{1}{3}$?

2 Linear Algebra(38)

2.1 Derivative(15)

Assume that x, a are vectors and A is a square matrix. show that:

- $\frac{d}{dx} aTx = aT$
- $\frac{d}{dx} xTAx = x^T(A + A^T)$
- $\frac{d}{dx} x^T A = A^T$

2.2 Eigenvalue and Eigenvectors(18)

1. (8) Prove below statements about eigenvalues:

   - Determinant of square matrix A is product of it's eigenvalues.
   - Trace of square matrix A is sum of it's eigenvalues.
   - Eigenvalues of A and A^T are same.

2. (10) Assume A is a linear transform which is defined on vector space **V**. Show if A is a diagonalizable matrix:

$$V = Null(A) \oplus Range(A) \tag{2}$$

2.3 SVD(5)

Suppose A is a square matrix and we have $A = QR$. Q is orthonormal matrix. In this case SVD of matrix A will be look like SVD of matrix R. which one of these three matrix Σ, U, V will be different for A and R?

3 Estimation(15)

Let $x_1, x_2, ..., x_n$ be iid samples from $N(\mu, \sigma^2)$. Where σ^2 is known but μ is unknown.

- Find MLE of μ.
- Assume $\mu \sim \mathcal{N}(v, \beta^2)$. Find MAP estimation of μ.
- Compare the result when N goes to infinity.