CS 577 # Assignment 1 – Discrete Review Spring 2024

Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

Related Readings: `http://pages.cs.wisc.edu/~hasti/cs240/readings/`

Name: _____ Wisc id: _____

Logic

1. Using a truth table, show the equivalence of the following statements.
   (a) $P \vee (\neg P \wedge Q) \equiv P \vee Q$

   Solution:

   (b) $\neg P \vee \neg Q \equiv \neg(P \wedge Q)$

   Solution:

(c) $\neg P \vee P \equiv \text{true}$

Solution:

(d) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Solution:

Sets

2. Based on the definitions of the sets A and B, calculate the following: $|A|$, $|B|$, $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$.

   (a) $A = \{1, 2, 6, 10\}$ and $B = \{2, 4, 9, 10\}$

   Solution:

   (b) $A = \{x \mid x \in \mathbb{N}\}$ and $B = \{x \in \mathbb{N} \mid x \text{ is even}\}$

   Solution:

Relations and Functions

3. For each of the following relations, indicate if it is reflexive, antireflexive, symmetric, antisymmetric, or transitive.

   (a) $\{(x, y) : x \leq y\}$

   Solution:

   (b) $\{(x, y) : x > y\}$

   Solution:

(c) $\{(x, y) : x < y\}$

Solution:

(d) $\{(x, y) : x = y\}$

Solution:

4. For each of the following functions (assume that they are all $f : \mathbb{Z} \to \mathbb{Z}$), indicate if it is surjective (onto), injective (one-to-one), or bijective.

(a) $f(x) = x$

Solution:

(b) $f(x) = 2x - 3$

Solution:

(c) $f(x) = x^2$

Solution:

5. Show that $h(x) = g(f(x))$ is a bijection if $g(x)$ and $f(x)$ are bijections.

Solution:

Induction

6. Prove the following by induction.
   (a) $\sum_{i=1}^{n} i = n(n+1)/2$

Solution:

   (b) $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

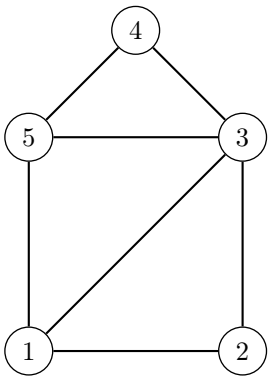
Solution:

   (c) $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$

Solution:

Graphs and Trees

7. Give the adjacency matrix, adjacency list, edge list, and incidence matrix for the following graph.



Solution:

8. How many edges are there is a complete graph of size n? Prove by induction.

Solution:

9. Draw all possible (unlabelled) trees with 4 nodes.

Solution:

10. Show by induction that, for all trees, $|E| = |V| - 1$.

Solution:

Counting

11. How many 3 digit pin codes are there?

Solution:

12. What is the expression for the sum of the ith line (indexing starts at 1) of the following:

1
2 3
4 5 6
7 8 9 10
⋮

Solution:

13. A standard deck of 52 cards has 4 suits, and each suit has card number 1 (ace) to 10, a jack, a queen, and a king. A standard poker hand has 5 cards. For the following, how many ways can the described hand be drawn from a standard deck.
    (a) A royal flush: all 5 cards have the same suit and are 10, jack, queen, king, ace.

    Solution:

    (b) A straight flush: all 5 cards have the same suit and are in sequence, but not a royal flush.

    Solution:

    (c) A flush: all 5 cards have the same suit, but not a royal or straight flush.

    Solution:

    (d) Only one pair (2 of the 5 cards have the same number/rank, while the remaining 3 cards all have different numbers/ranks):

    Solution:

Proofs

14. Show that $2x$ is even for all $x \in \mathbb{N}$.

    (a) By direct proof.

    Solution:

    (b) By contradiction.

    Solution:

15. For all $x, y \in \mathbb{R}$, show that $|x + y| \leq |x| + |y|$. (Hint: use proof by cases.)

    Solution:

Program Correctness (and Invariants)

16. For the following algorithms, describe the loop invariant(s) and prove that they are sound and complete.

(a)

Algorithm 1: findMin

Input: a: A non-empty array of integers (indexed starting at 1)
Output: The smallest element in the array

```
begin
    min ← ∞
    for i ← 1 to len(a) do
        if a[i] < min then
            min ← a[i]
        end
    end
    return min
end
```

Solution:

(b)

Algorithm 2: InsertionSort

Input: a: A non-empty array of integers (indexed starting at 1)
Output: a sorted from largest to smallest

```
begin
    for i ← 2 to len(a) do
        val ← a[i]
        for j ← 1 to i − 1 do
            if val > a[j] then
                shift a[j..i − 1] to a[j + 1..i]
                a[j] ← val
                break
            end
        end
    end
    return a
end
```

Solution:

Recurrences

17. Solve the following recurrences. Show work and do not use the master theorem.

(a) $c_0 = 1; c_n = c_{n-1} + 4$

Solution:

(b) $d_0 = 4; d_n = 3 \cdot d_{n-1}$

Solution:

(c) $T(1) = 1; T(n) = 2T(n/2) + n$ (An upper bound is sufficient.)

Solution:

(d) $f(1) = 1; f(n) = \sum_1^{n-1} (i \cdot f(i))$
(Hint: compute $f(n+1) - f(n)$ for $n > 1$)

Solution:

Coding Question: Hello World

Most assignments will have a coding question. You can code in C, C++, C#, Java, Python, or Rust. You will submit a Makefile and a source code file.

Makefile: In the Makefile, there needs to be a build command and a run command. Below is a sample Makefile for a C++ program. You will find this Makefile in assignment details. Download the sample Makefile and edit it for your chosen programming language and code.

```
#Build commands to copy:
#Replace g++ -o HelloWorld HelloWord.cpp below with the appropriate command.
#Java:
#       javac source_file.java
#Python:
#       echo "Nothing to compile."
#C#:
#       mcs -out:exec_name source_file.cs
#C:
#       gcc -o exec_name source_file.c
#C++:
#       g++ -o exec_name source_file.cpp
#Rust:
#       rustc source_file.rs

build:
        g++ -o HelloWorld HelloWord.cpp

#Run commands to copy:
#Replace ./HelloWorld below with the appropriate command.
#Java:
#       java source_file
#Python 3:
#       python3 source_file.py
#C#:
#       mono exec_name
#C/C++:
#       ./exec_name
#Rust:
#       ./source_file

run:
        ./HelloWorld
```

18. **HelloWorld Program Details**

The input will start with a positive integer, giving the number of instances that follow. For each instance, there will be a string. For each string s, the program should output Hello, s! on its own line.

A sample input is the following:

```
3
World
Marc
Owen
```

The output for the sample input should be the following:

```
Hello, World!
Hello, Marc!
Hello, Owen!
```