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The Soul of Dynamic Programming Formulations and Implementations

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March 1, 16

Special Topic for Algorithmic Problem Solving

What is DP all about

- Advanced technique? ☺

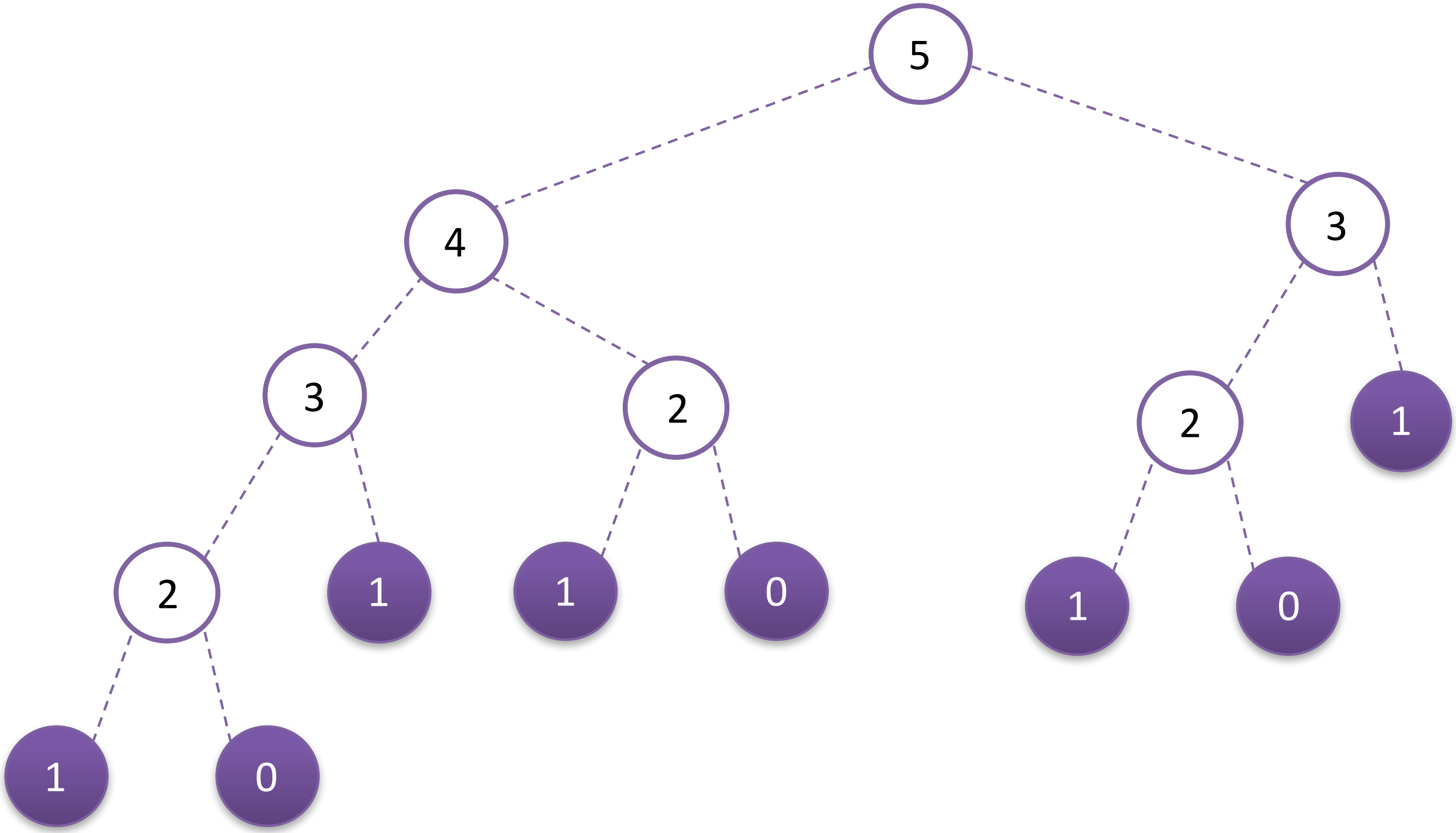
- Transition Function
- Memoization

An example that everyone uses

- Calculate the Fibonacci numbers
- F(i) = F(i-1) + F(i-2)

```
int fib(int i) {
    if (i <= 1) return i;
    return fib(i - 1) + fib(i - 2);
}
```

We know this is slow



Optimization?

- No need to compute a same value more than once

```
int memo[MAXN];
int fib(int i) {
    if (memo[i] != -1) return memo[i];
    if (i <= 1) return memo[i] = i;
    return memo[i] = fib(i - 1) + fib(i - 2);
}
```

But...

- Why the -1’s?

```
int fib[MAXN];
void allFibs() {
  fib[0] = 0;
  fib[1] = 1;
  for (int i = 2; i < MAXN; i++) {
    fib[i] = fib[i - 1] + fib[i - 2];
  }
}
```

So-called top-down and bottom-up

- Top-down:
  - Derive the DP entry dependencies
  - Memoize and compute recursively

```
int memo[MAXN];
int fib(int i) {
  if (memo[i] != -1) return memo[i];
  if (i <= 1) return memo[i] = i;
  return memo[i] = fib(i - 1) + fib(i - 2);
}
```

- Bottom-up:
  - Compute values in specific order
  - No memoization involved

```
int fib[MAXN];
void allFibs() {
  fib[0] = 0;
  fib[1] = 1;
  for (int i = 2; i < MAXN; i++) {
    fib[i] = fib[i - 1] + fib[i - 2];
  }
}
```

Two different DPs?

- Not really.

- They are actually the same computation, as they have exactly the same transitions $F(i) = F(i-1) + F(i-2)$.

- We need to get to the soul of DP to actually see them unified.

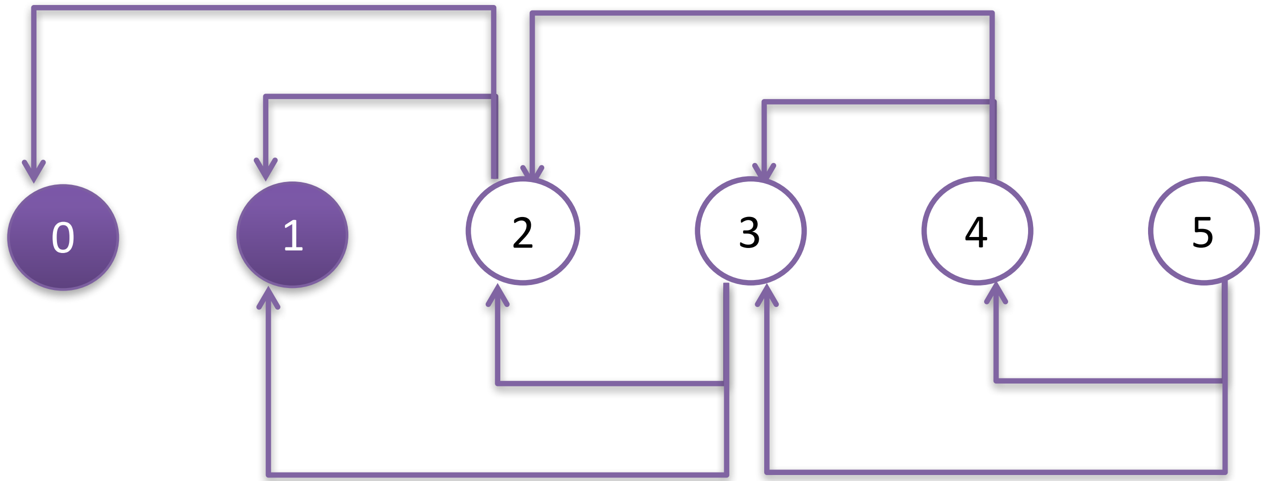
Revisiting DP

- ***State*** – an entry of some value of your interest
- ***Transition*** – relations between the states
- ***Order*** – how states get computed

Fibonacci Sequence

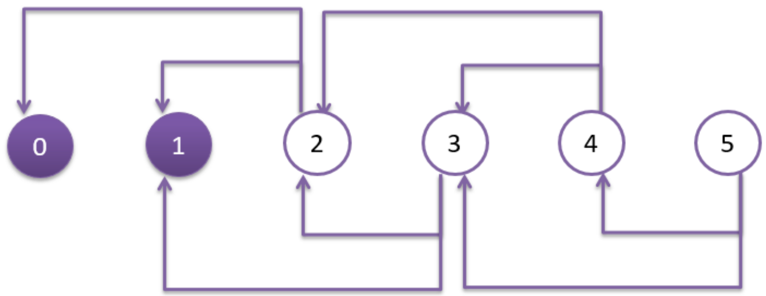
- ***State***
  - Fib(i)
- ***Transition***
  - Fib(i) = Fib(i – 1) + Fib(i – 2)
- ***Order***?
  - We always have larger Fibs depending on smaller Fibs.
  - Therefore, we can compute smaller Fibs first before computing larger Fibs.

Thinking on a graph



- This is a ***Directed Acyclic Graph (DAG)***

How DAG works?



- We can always find a ***topological order*** of a DAG

- The states on the graph can be computed in this topo order, so that when a state is being computed, its dependencies would have already been computed before.

- A DP must have a corresponding DAG (most of the time implicit), otherwise we cannot find a valid order for computation.

Counting Problem

- Given a set of **N distinct positive** integers: v_1, v_2, … v_N
- How many ways can we compose an **ordered list** with sum S, by choosing the values from the set?

  (Each value can be chosen multiple times)

- For the discussion purpose, today we:
  - assume all integers values in this lecture are reasonably small so that they fit well in the memory, e.g. in this case $N <= 20$, $v_i <= 100$.
  - ignore integer overflows, which are typically handled by modulo arithmetic

Counting Problem

- ***State***
  - f(S): the number of ways we can compose sum S
- ***Transition***
  - $f(\mathrm{s}) = \sum_i f(s - v_i)$
- ***Order***
  - In increasing S

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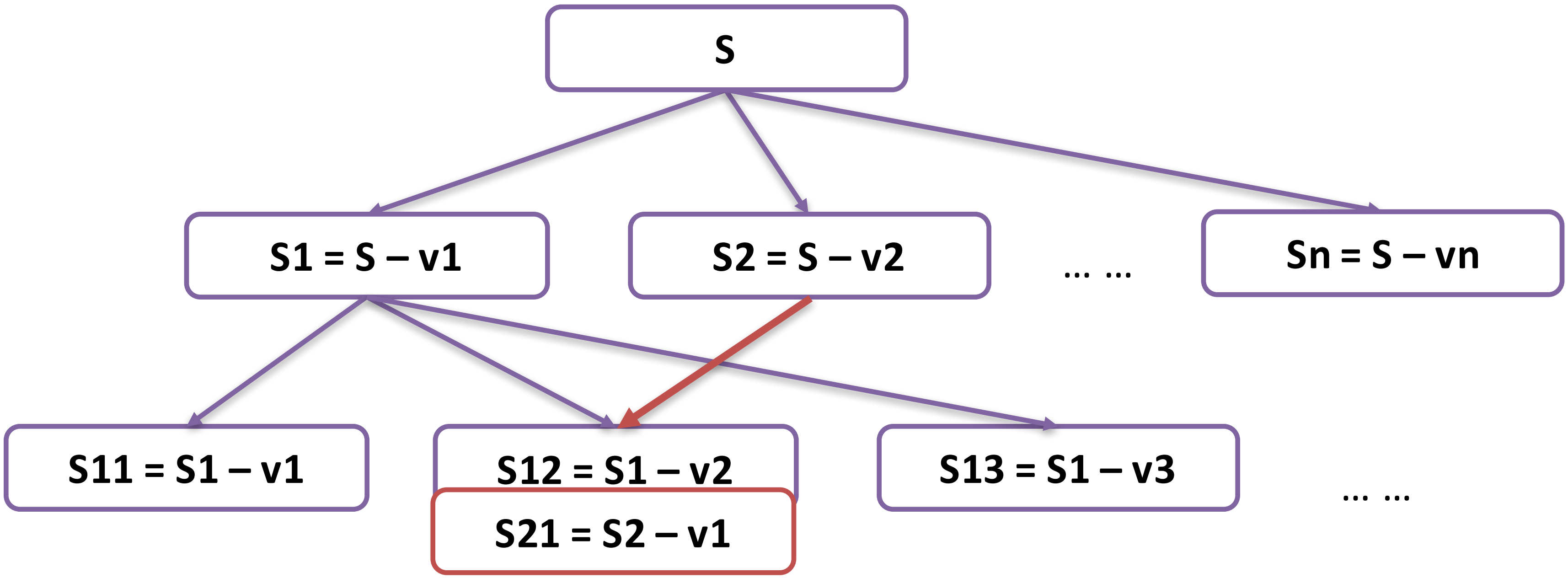
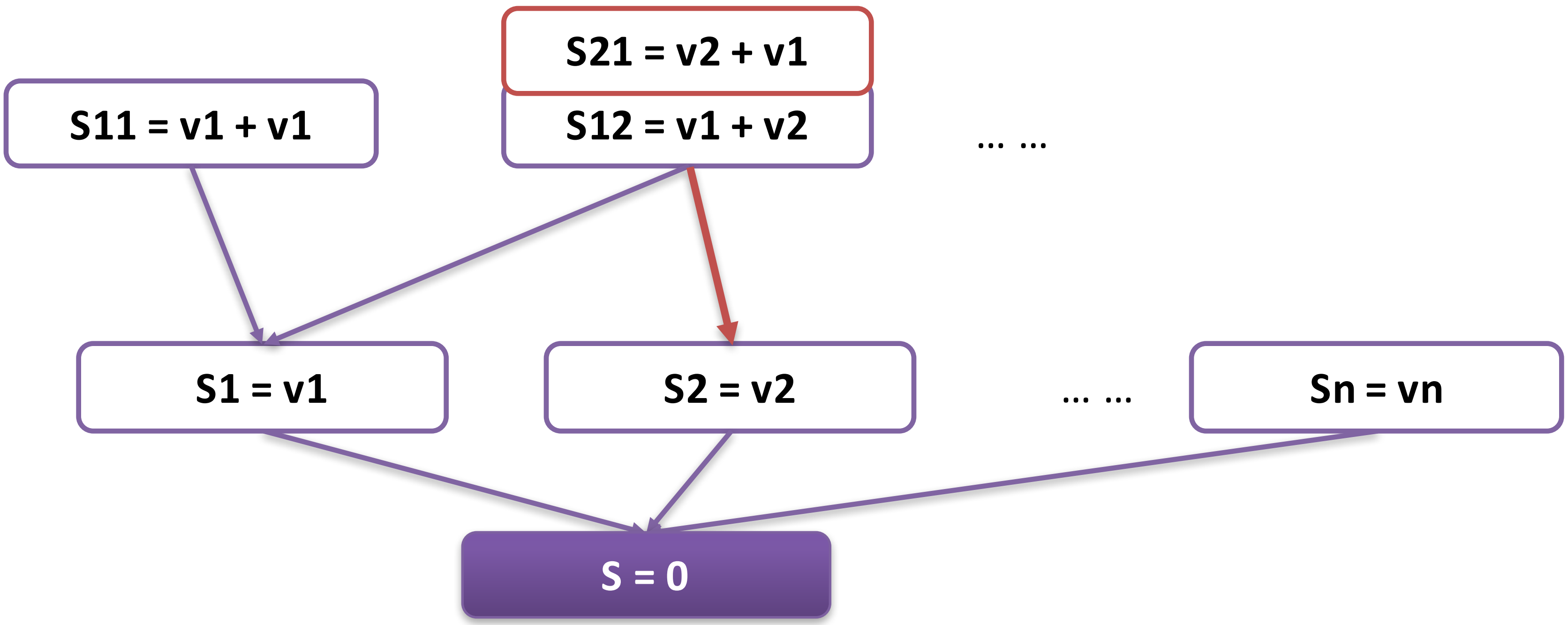
Counting Problem – DAG (top)



Counting Problem – DAG (bottom)



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Two implementations

- Top-down

```cpp
int N, v[MAXN], dp[MAXS];
int count(int S) {
  if (dp[S] != -1) return dp[S];
  int ans = 0;
  for (int i = 0; i < N; i++) {
    if (S >= v[i]) ans += count(S - v[i]);
  }
  return dp[S] = ans;
}
```

- Bottom-up

```cpp
int N, v[MAXN], dp[MAXS];
void count() {
  dp[0] = 1;
  for (int S = 1; S < MAXS; S++) {
    for (int i = 0; i < N; i++) {
      if (S >= v[i]) dp[S] += dp[S - v[i]];
    }
  }
}
```

Not much difference ☺

Two topo-sort algorithms

* Assuming the graph is DAG

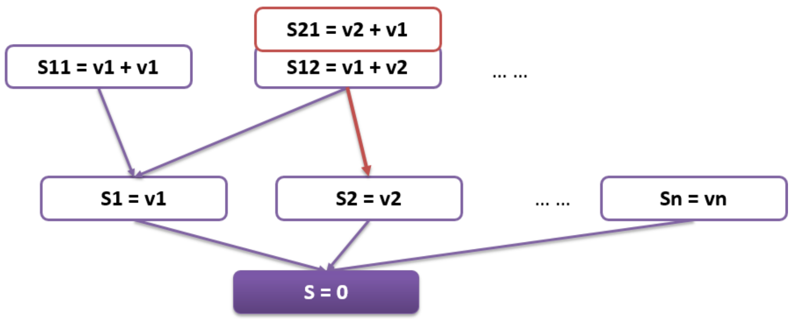
- Queue (Kahn's)
  - Add nodes with in-degree = 0 to the queue
  - Pop nodes out of the queue, decrement their neighbors' in-degrees. Add those new nodes with in-degree = 0 to the queue.

- DFS
  - The current node that completes its DFS has no more outgoing edges and therefore comes first in topo order.
  - Do DP computation in post-order traversal.

Topo-sort and DP

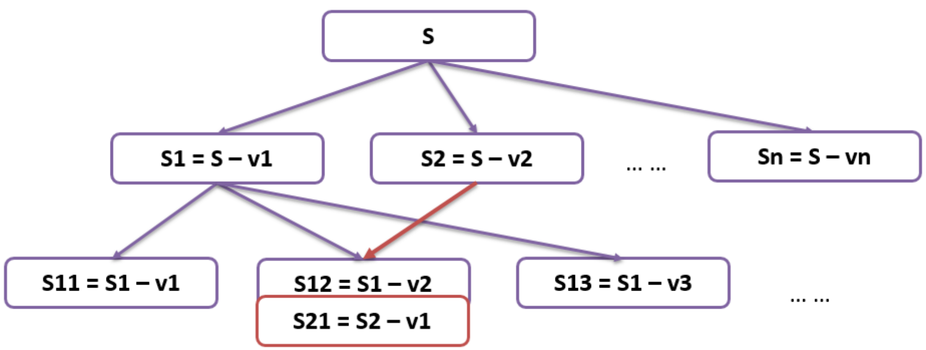
Which one is better?

- Use Kahn's algorithm for topo-order: bottom-up DP




```
int N, v[MAXN], dp[MAXS];
void count() {
  dp[0] = 1;
  for (int S = 1; S < MAXS; S++) {
    for (int i = 0; i < N; i++) {
      if (S >= v[i]) dp[S] += dp[S - v[i]];
    }
  }
}
```

- Use DFS for topo-order: top-down DP




```
int N, v[MAXN], dp[MAXS];
int count(int S) {
  if (dp[S] != -1) return dp[S];
  int ans = 0;
  for (int i = 0; i < N; i++) {
    if (S >= v[i]) ans += count(S - v[i]);
  }
  return dp[S] = ans;
}
```

Which one is better?

- It depends:
  - If the problem has a clear implicit topo-order (e.g. the Fibonacci Sequence, the counting problem), then bottom-up could be simpler to implement.
  - If the problem has a clear sub-dividing strategy, then top-down could be more intuitive.

Cutting Sticks

- Given a stick of integer length **L**, cut it into multiple segments of integer lengths.
- You are given a list of (length, value) pairs. For every possible length there is an associated value.
- Your task is to maximize your total value after cutting.

- Example:

Length	Value
1	2
2	4
3	7
4	1

Answer with L = 4 is 9.

Cut into 2 segments with lengths 1 and 3.

Cutting Sticks

- ***State***
  - Cut(L): maximum total value we can get for a stick of length L
- ***Transition***
  - $cut(L) = \max_{i}\{value(i) + cut(L-i)\}$
- ***Order***
  - Larger L depends on smaller L's

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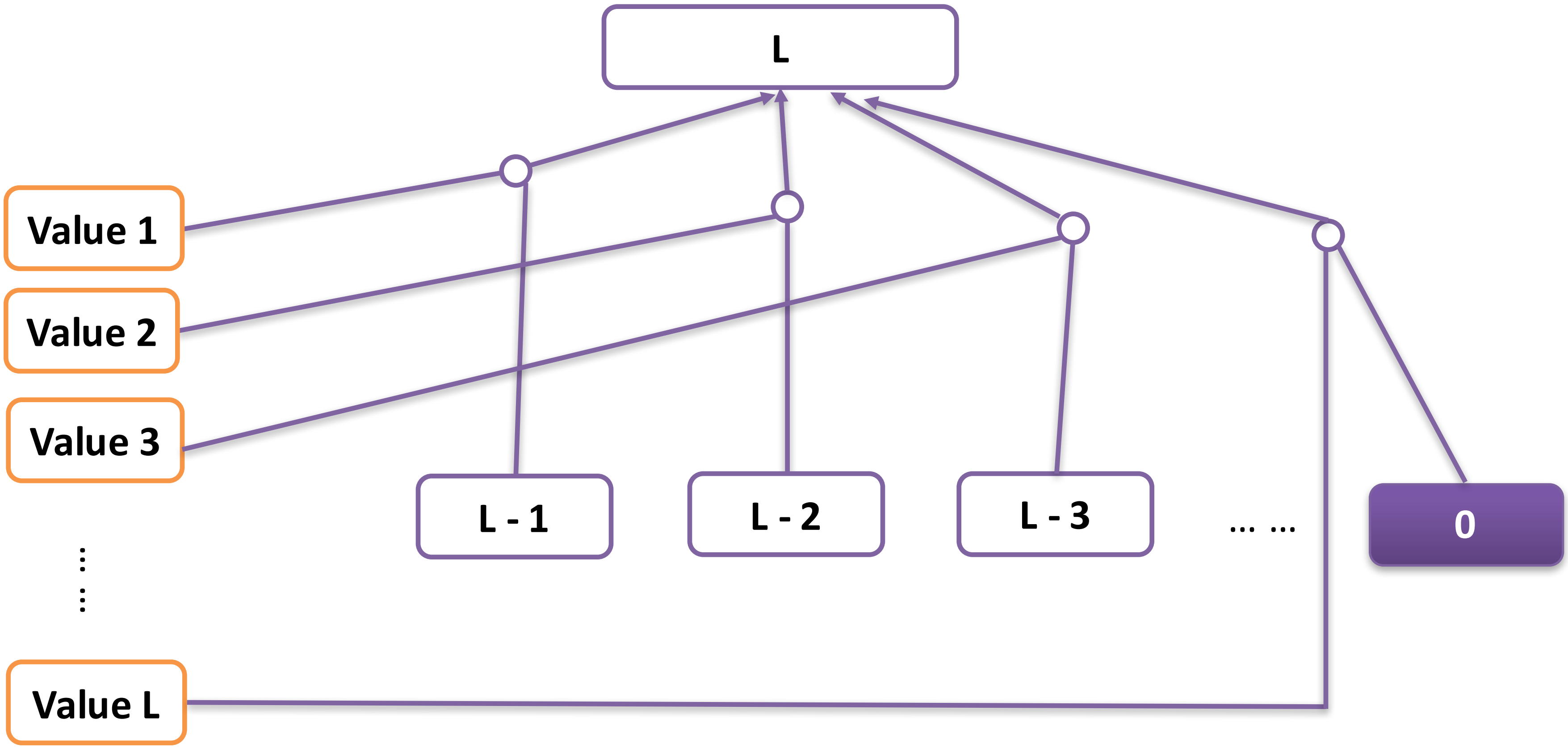
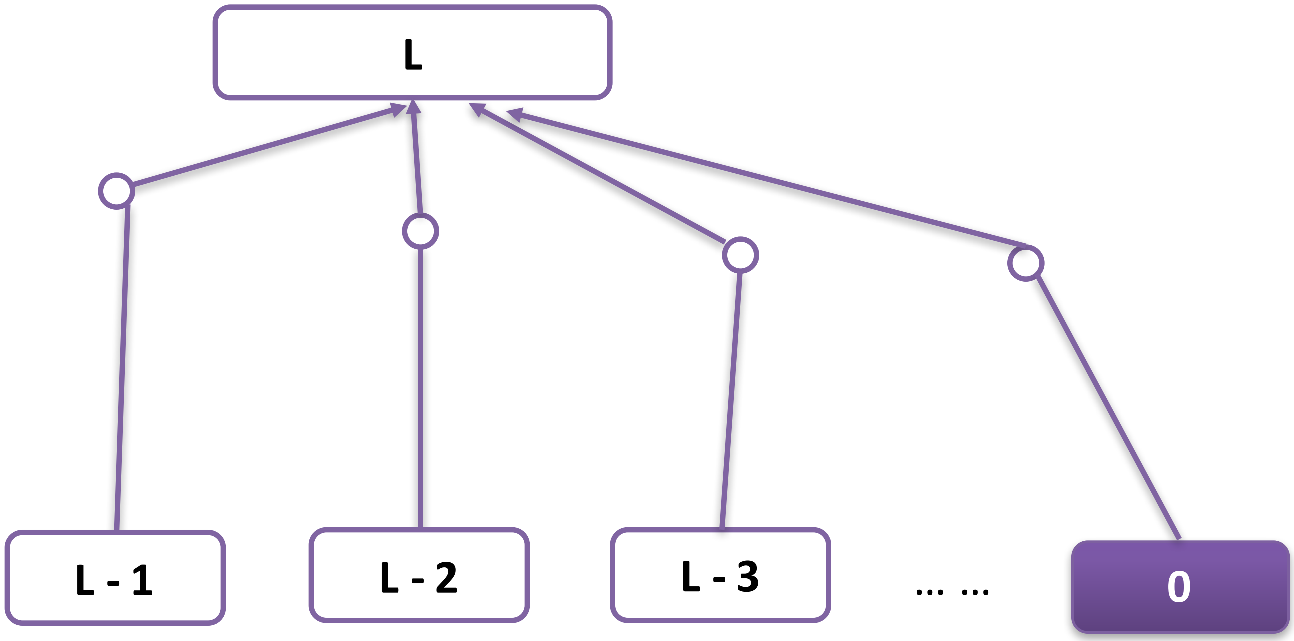
Cutting Sticks – DAG (top)



* Note that values are **NOT** states

Cutting Sticks – DAG (top)



* Note that values are **NOT** states

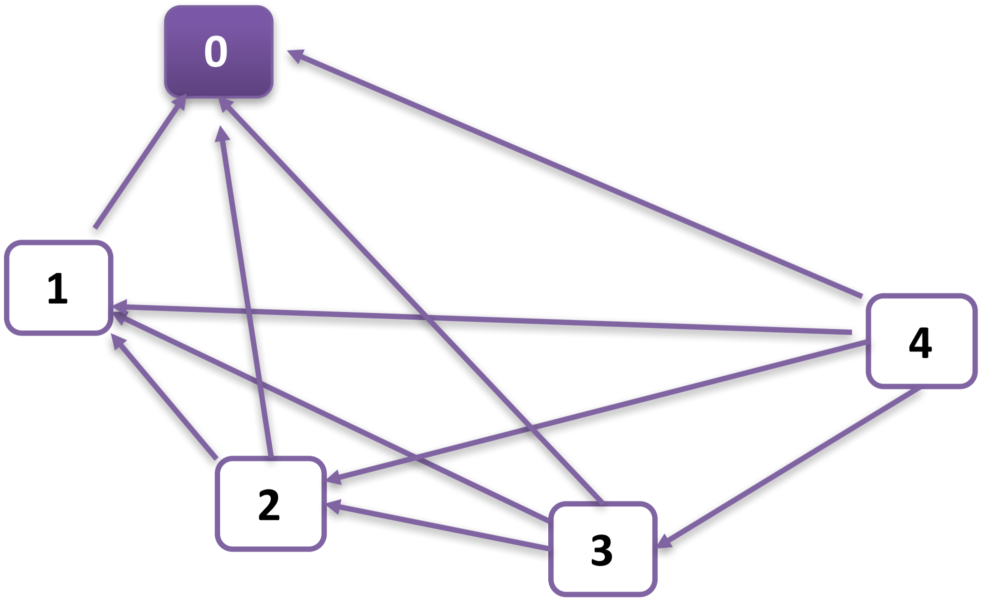
Cutting sticks – top-down implementation

- Because of the action of cutting, it seems that using top-down is more intuitive.

```
int value[MAXN], dp[MAXN];
int cut(int L) {
  if (dp[L] != -1) return dp[L];
  if (L == 0) return dp[L] = 0;
  int ans = 0;
  for (int i = 1; i <= L; i++) {
    ans = max(ans, value[i] + cut(L - i));
  }
  return dp[L] = ans;
}
```

Cutting Sticks - DAG

- Actually the DAG is a complete DAG.



Cutting sticks – bottom-up implementation

- So it could be intuitive too to compute bottom up ☺

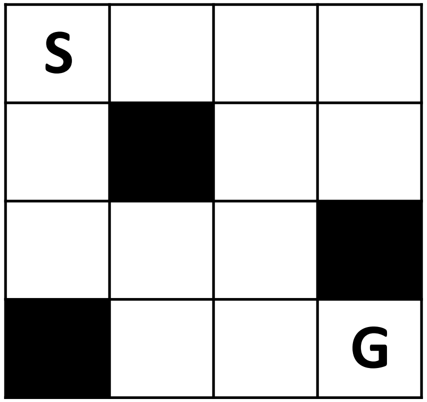
```
int value[MAXN], dp[MAXN];
void cut() {
  dp[0] = 0;
  for (int L = 1; L < MAXL; L++) {
    for (int i = 1; i <= L; i++) {
      dp[L] = max(dp[L], value[i] + dp[L - i]);
    }
  }
}
```

Why DAG is important

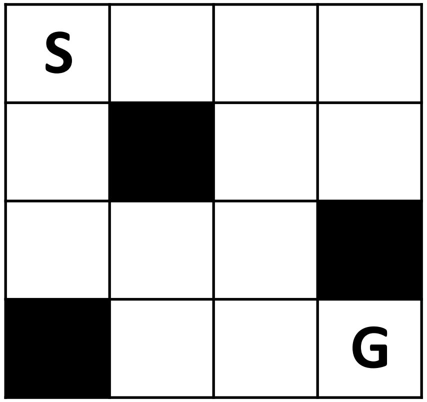
- It helps you ***perceive*** the problem better.
- It helps you find the simplest way to ***implement*** the same DP algorithm.
- It helps you ***determine*** if a problem can be solved by DP.

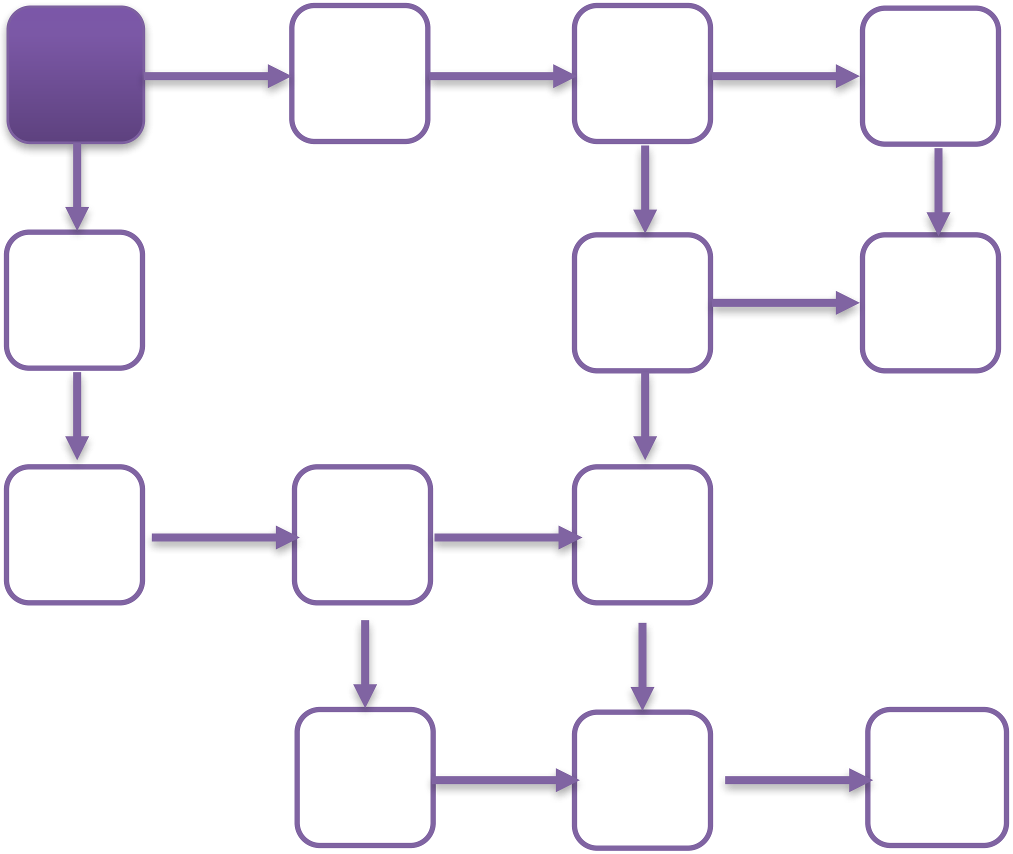
Grid Walking

- Given a grid map with blocks, determine the number of different ways to walk from (top, left) to (bottom, right) if you can only walk east or south.



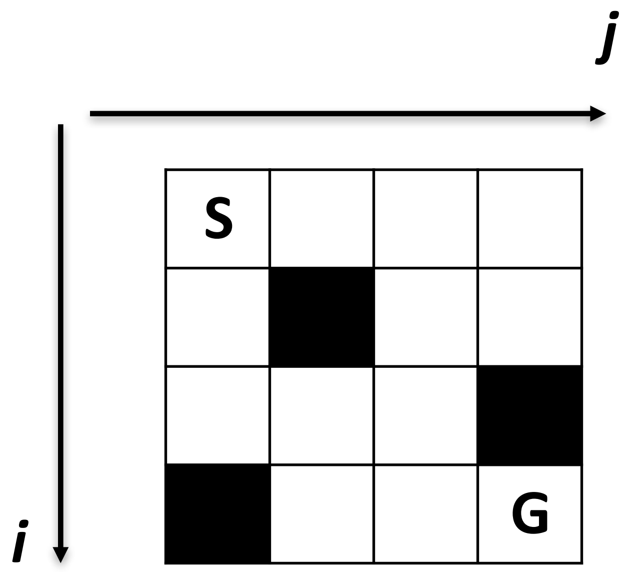
Perceive – Grid Walking





Perceive - Grid Walking (cont'd)

- ***State***
  - ways(i, j): number of ways to reach cell(i, j) from the starting cell
- ***Transition***
  - $ways(i,j) = ways(i-1,j) + ways(i,j-1)$
  - $ways(i,j) = 0$ if (i, j) is a blocked cell
- ***Order***
  - Right depends on left
  - Bottom depends on top



Implement – Grid Walking

- The grid map has implicit topology that allows us to get a topo-order simply by iterating i and j.

```cpp
bool block[MAXN][MAXN];
int N, dp[MAXN][MAXN] = {};
int walk() {
  dp[1][1] = 1;
  for (int i = 1; i <= N; i++) {
    for (int j = 1; j <= N; j++) {
      if (block[i][j]) continue;
      dp[i][j] += dp[i - 1][j];
      dp[i][j] += dp[i][j - 1];
    }
  }
  return dp[N][N];
}
```

Implement – Grid Walking (cont'd)

- Won't be much work to do top-down either? ☺

```
bool block[MAXN][MAXN];
int N, dp[MAXN][MAXN];
bool out(int i, int j) {
  return i < 1 || j < 1 || i > N || j > N;
}
int walk(int i, int j) {
  if (dp[i][j] != -1) return dp[i][j];
  if (i == 1 && j == 1) return dp[i][j] == 1;
  if (block[i][j] || out(i, j)) return dp[i][j] = 0;
  return dp[i][j] = walk(i - 1, j) + walk(i + 1, j);
}
```

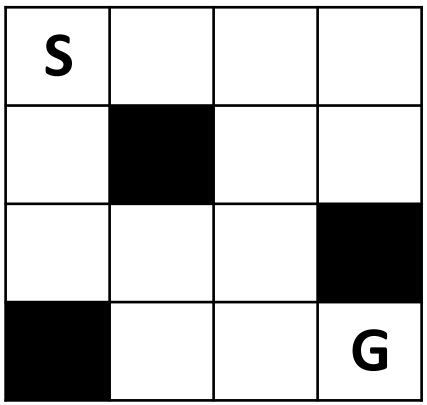
Determine – Grid Walking

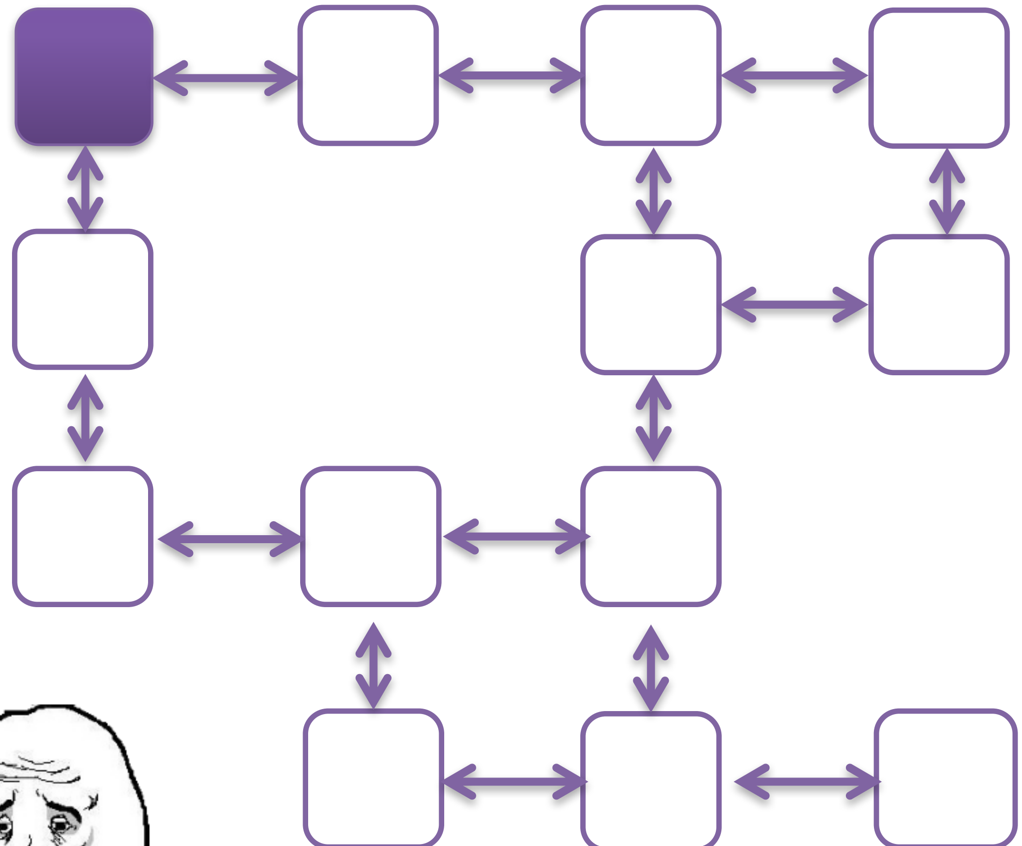
- The previous Grid Walking problem clearly **has** a DAG.

- What if we change the possible walking directions to all four? (up, down, left, right)

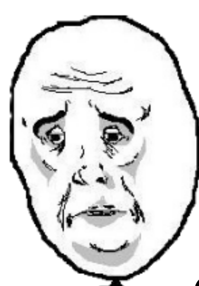
- Of course we would have infinite number of ways to reach the goal. Therefore we restrict our walk to take exactly **K** steps.
  - Count how many ways to reach cell (i, j) from (1, 1) with exactly K moves

Grid Walking 2 – DAG?





Unfortunately this is no longer a DAG and there is NO way to do DP on this.



Okay

What should we do if it is not a DAG

- Try to make a DAG if you are not given a DAG directly.

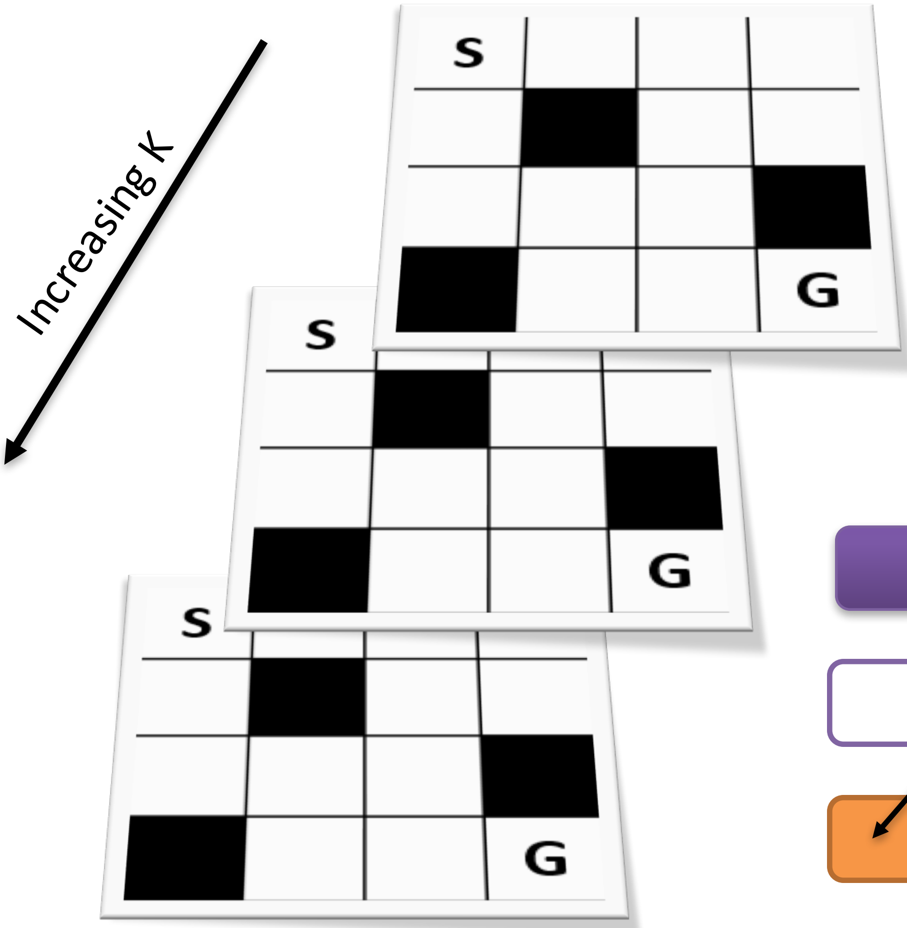
- How?
  - Observe where we cannot go back to previous states
  - In other words, observe the monotonicity based on which we can create a DAG

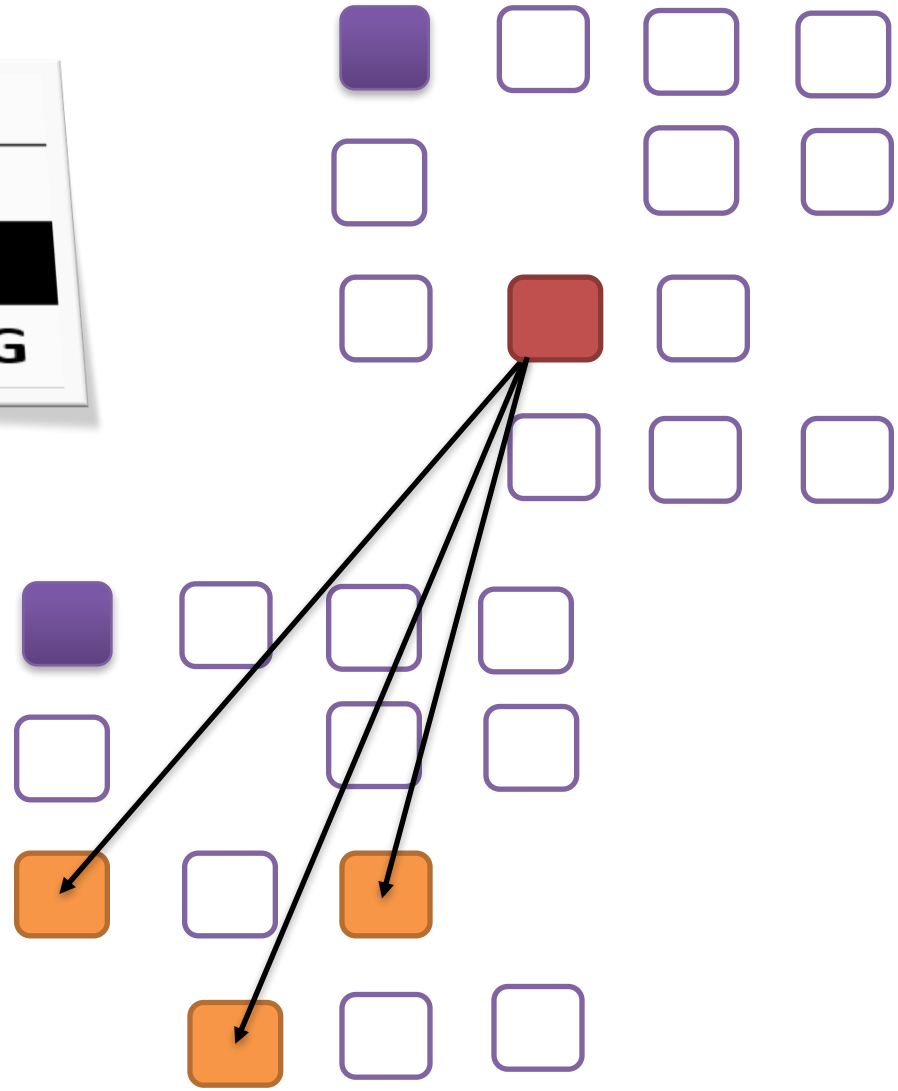
Grid Walking 2 – DAG creation

- Where we cannot go back
  - After taking K steps, we can never go back to the states where we have taken k < K steps.
  - In other words, K is monotonically increasing.

- Idea: Build a DAG with states ways(i, j, k): the number of ways we can reach cell (i, j) using **exactly** k steps.

Grid Walking 2 - DAG





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How to implement

- Typically, one more dimension in your DP state corresponds to:
  - Top-down: a new argument of your recursive function
  - Bottom-up: a new dimension of your multi-dimensional array

Grid Walking 2 – Top-down

```
bool block[MAXN][MAXN];
int N, dp[MAXN][MAXN][MAXK];
bool out(int i, int j) {
  return i < 1 || j < 1 || i > N || j > N;
}
int walk(int i, int j, int k) {
  if (dp[i][j][k] != -1) return dp[i][j][k];
  if (k == 0) return i == 1 && j == 1;
  if (block[i][j] || out(i, j)) return dp[i][j][k] == 0;
  return dp[i][j][k] =
    walk(i + 1, j, k - 1) +
    walk(i - 1, j, k - 1) +
    walk(i, j + 1, k - 1) +
    walk(i, j - 1, k - 1);
}
```

Grid Walking 2 – Bottom-up

```
bool block[MAXN][MAXN];
int N, dp[MAXN][MAXN][MAXK];
void walk() {
  dp[1][1][0] = 1;
  for (int k = 1; k < MAXK; k++) {
    for (int i = 1; i <= N; i++) {
      for (int j = 1; j <= N; j++) {
        if (block[i][j]) continue;
        dp[i][j][k] += dp[i + 1][j][k - 1];
        dp[i][j][k] += dp[i - 1][j][k - 1];
        dp[i][j][k] += dp[i][j + 1][k - 1];
        dp[i][j][k] += dp[i][j - 1][k - 1];
      }
    }
  }
}
```

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Which one is better? ☺

```
bool block[MAXN][MAXN];
int N, dp[MAXN][MAXN][MAXK];
bool out(int i, int j) {
  return i < 1 || j < 1 || i > N || j > N;
}
int walk(int i, int j, int k) {
  if (dp[i][j][k] != -1) return dp[i][j][k];
  if (k == 0) return i == 1 && j == 1;
  if (block[i][j] || out(i, j)) return dp[i][j][k] == 0;
  return dp[i][j][k] =
    walk(i + 1, j, k - 1) +
    walk(i - 1, j, k - 1) +
    walk(i, j + 1, k - 1) +
    walk(i, j - 1, k - 1);
}
```

```
bool block[MAXN][MAXN];
int N, dp[MAXN][MAXN][MAXK];
void walk() {
  dp[1][1][0] = 1;
  for (int k = 1; k < MAXK; k++) {
    for (int i = 1; i <= N; i++) {
      for (int j = 1; j <= N; j++) {
        if (block[i][j]) continue;
        dp[i][j][k] += dp[i + 1][j][k - 1];
        dp[i][j][k] += dp[i - 1][j][k - 1];
        dp[i][j][k] += dp[i][j + 1][k - 1];
        dp[i][j][k] += dp[i][j - 1][k - 1];
      }
    }
  }
}
```

Subtle differences between bottom-up and top-down

- Top-down requires significant stack space while bottom-up uses little.
- Top-down doesn't support the *memory saving trick*.
- It is observed that when a problem has implicit simple DAG, it is faster and neater to code bottom-up.
- Bottom-up may not neatly compute only the necessary states. Sometimes redundant states are involved, resulting in additional computation time.

- Anyway, most of the time it is your call.

How to get a DP procedure

- Formulate its DAG (either explicitly or implicitly)
  - Determine the states (DAG nodes)
  - Determine the transitions (DAG edges)
  - Determine the order (implicit or explicit DAG topo-sort)

- Think about the computation costs
  - Affordable Memory
  - Affordable Time

Affordable Memory

- The maximum number of states during computation must fit in the memory limit
  - DP space most commonly is the number of total states. DAG size must not be too large.
  - Memory saving trick: reduce the memory requirement by one more dimension.

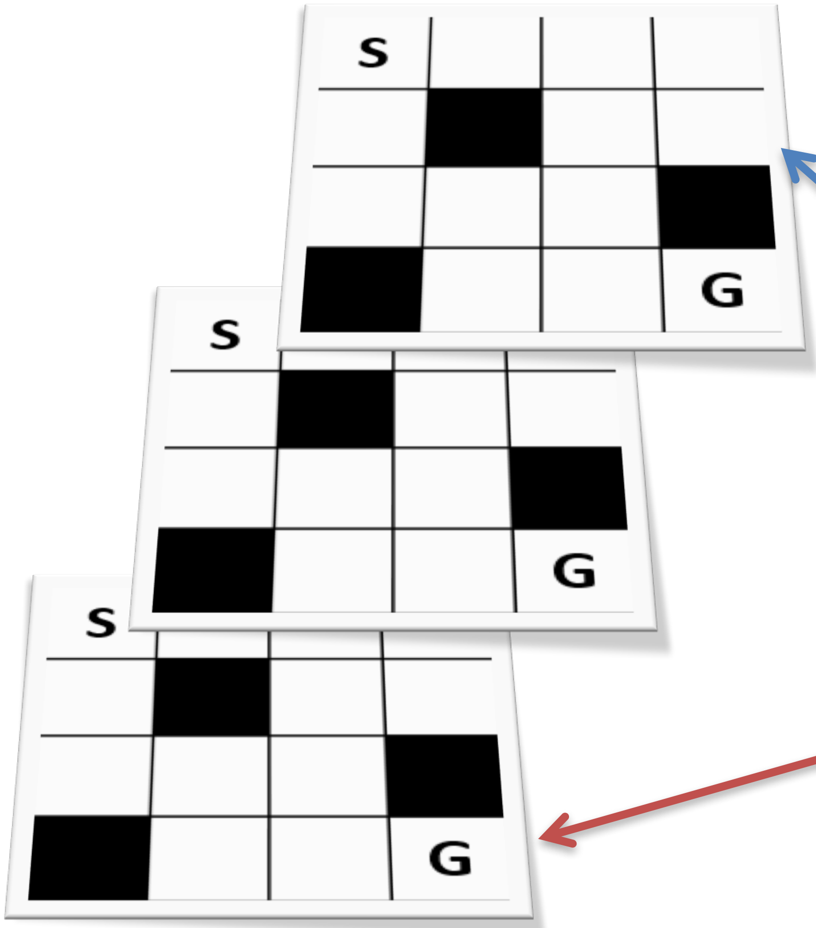
- Fibonacci Sequence:
  - We can only work up to Fib(N) where O(N) fits in the memory

- Grid Walking 2:
  - O(NMK) space, as there are N * M *K nodes in the DAG
  - or O(NM) if you use the memory saving trick

Memory saving trick

- Simply put: forget about the states if they will no longer be depended on.
  - In other words, let those states be gone forever after they are processed in the topo order.

Memory saving trick – Grid Walking 2



this layer will never be used again,

once we reach **this layer**

Implementation – memory saving trick

```
bool block[MAXN][MAXN];
int N, dp[2][MAXN][MAXN];
void walk() {
  int now = 0, pre;
  dp[now][1][1] = 1;
  for (int k = 1; k <= MAXK; k++) {
    pre = now; now ^= 1;
    memset(dp[now], 0, sizeof(dp[now]));
    for (int i = 1; i <= N; i++) {
      for (int j = 1; j <= N; j++) {
        if (block[i][j]) continue;
        dp[now][i][j] += dp[pre][i - 1][j];
        dp[now][i][j] += dp[pre][i + 1][j];
        dp[now][i][j] += dp[pre][i][j + 1];
        dp[now][i][j] += dp[pre][i][j - 1];
      }
    }
  }
}
```

Affordable Time

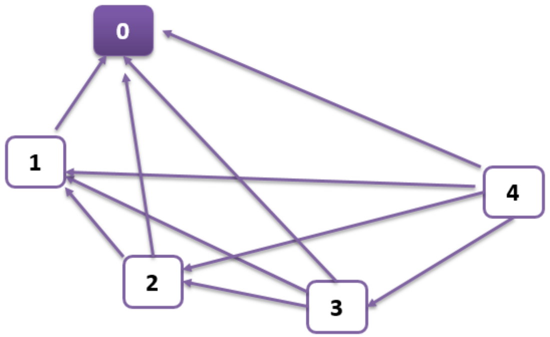
- The total computation cost must fit in the Time Limit.

- If the total number of states is C, then the time complexity would be $\Omega(C)$, as every state shall be computed.

- However, it could be more than that, depending on how much time is needed for each state.

Time needed to compute each state

- Recall the Cutting Sticks problem.
  - We have a complete DAG where each state *Cut(i)* depends on every *Cut(j)*, $j < i$



- In order to compute *Cut(i)*, we need to traverse i - 1 states. Therefore we need on average linear time to compute each state. The total time complexity is thus $O(L^2)$.

Online judge scenarios

Problem	Time	Space	Small	Large
Fibonacci Sequence	O(N)	O(N)	N <= 100	N <= 10^7
Counting Problem	O(NS)	O(S)	N <= 20 S <= 1000	N <= 100 S <= 10^6
Cutting Sticks	O(L^2)	O(L)	L <= 100	L <= 5000
Grid Walking	O(NM)	O(NM)	N,M <= 100	N, M <= 5000
Grid Walking 2	O(NMK)	O(NM)	N,M,K <= 100	N,M,K <= 500 * Memory saving trick required

Finding DP states

- A DP problem typically contains important variables that can be used as a dimension of the DP state.
  - Counting Problem: sum **S** given
  - Cutting Sticks: stick length **L** given
  - Grid Walking 2: number of steps **K** given

- More challenging DP questions demand more insights into the computation process, where we may have DP states corresponding not to the problem variables, but to our computation.

Minimum Balance

- Given N positive integers $\{a_1, a_2 \dots a_N\}$, as a multiset (allowing duplicates), split them into two multisets so that the two multisets have the smallest possible absolute difference.

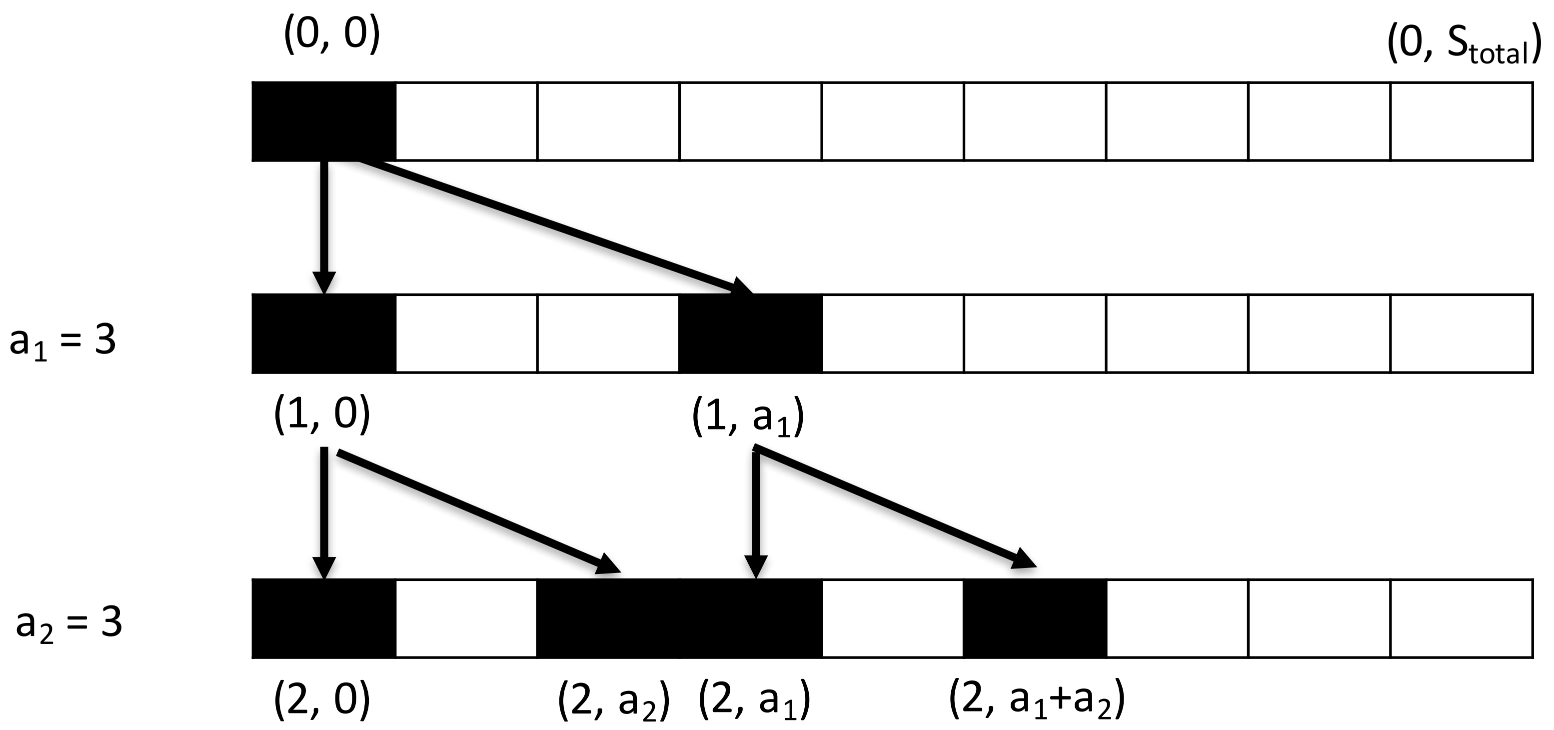
- Example:
  - {1,2,3,4} Answer: minDiff = 0, e.g. {1,4}, {2,3}
  - {2,4,5,6} Answer: minDiff = 1, e.g. {2,6}, {4,5}
  - {1,1,10} Answer: minDiff = 8, e.g. {1,1}, {10}

DP formulation

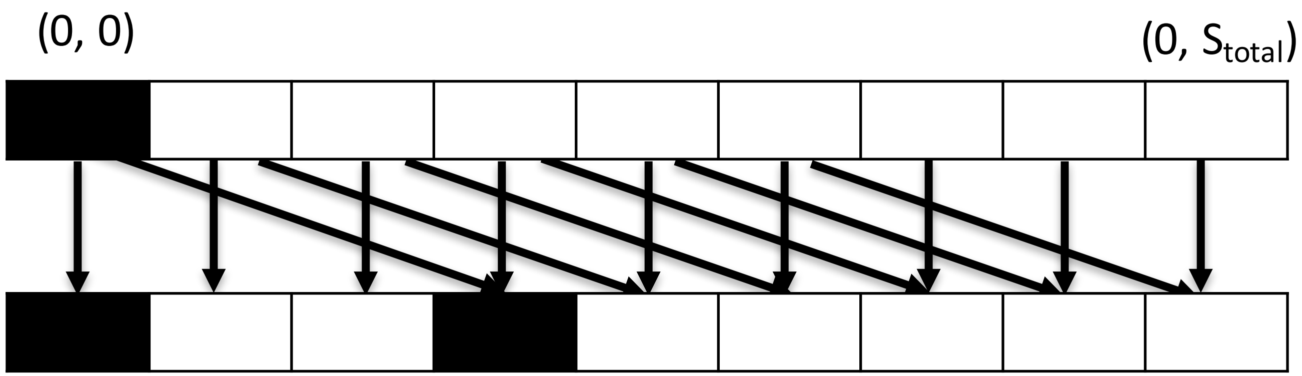
- ***State:*** possible(i, S), whether we can find a multiset of integers with sum S, among the first i integers.

- Note that S is not directly given in the problem. But it is closely related to the balance that is asked for.
  - If we know possible(N, S) is true, then we can split the integers into two multisets with S and S_{total} - S, and their difference is $|S - (S_{total} - S|)$.
  - The problem is equivalent to finding the value S closest to $S_{total}/2$ so that possible(N, S) is true.

Minimum Balance - DAG



Minimum Balance - DAG



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Minimum Balance - Implementation

```
bool dp[MAXN][MAXS];
int N, Stotal, a[MAXN]; // a[1..N] (1-based)
int minBalance() {
  dp[0][0] = true;
  for (int i = 1; i <= N; i++) {
    for (int S = 0; S < MAXS; S++) {
      dp[i][S] |= dp[i - 1][S];
      if (S >= a[i])
        dp[i][S] |= dp[i - 1][S - a[i]];
    }
  }
  int ans = INF;
  for (int S = 0; S < MAXS; S++) {
    if (dp[N][S])
      ans = min(ans, abs(S - (Stotal - S)));
  }
  return ans;
}
```

Lucky Strings

- A string **X** is *lucky* if it contains a lucky character **t**.
- Given the lucky character **t**, count how many lucky strings X of length **N** exist (|X| = **N**).
- Strings contain lowercase English letters. For simplicity, we only use the first 3 letters: *'a', 'b', 'c'*.

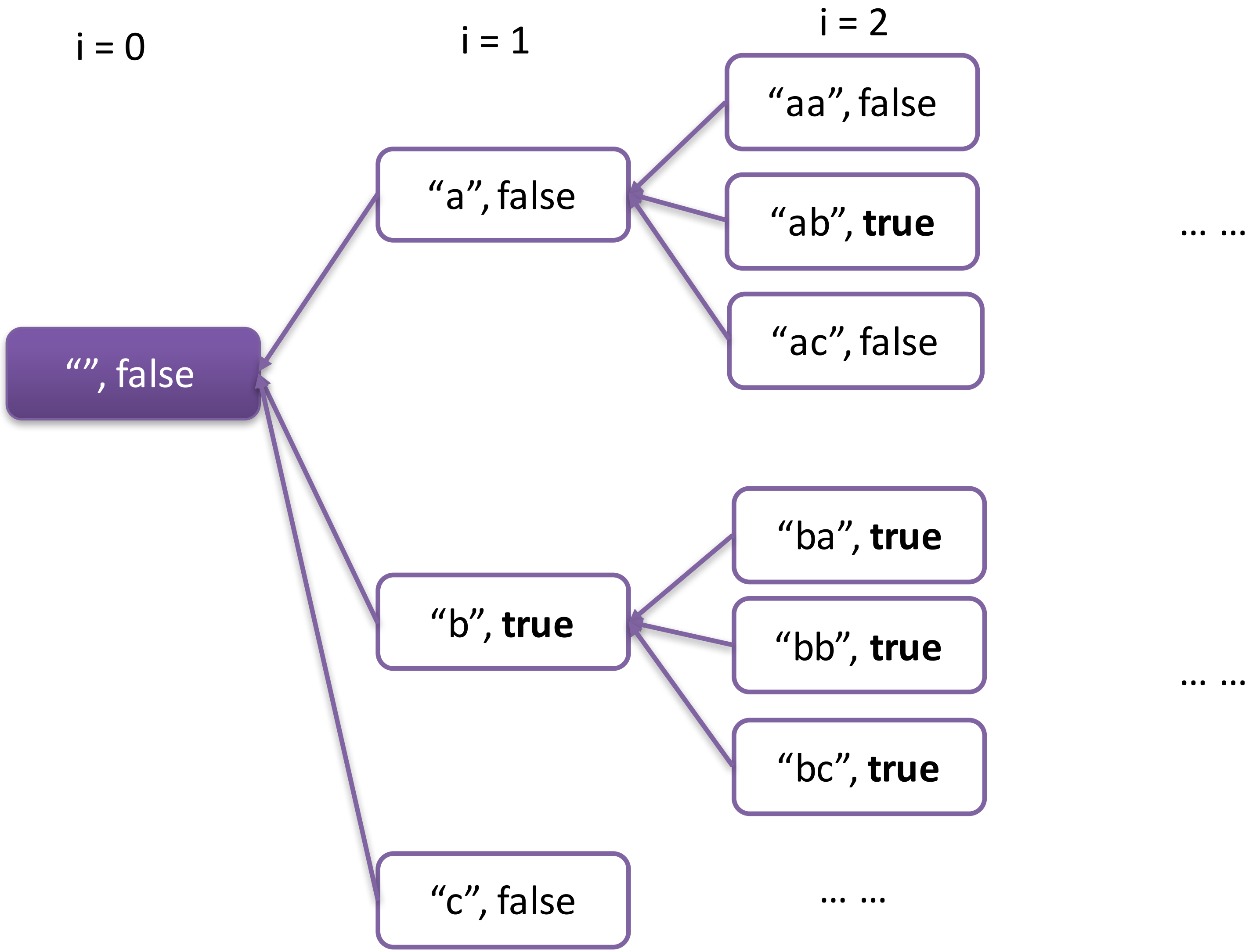
Lucky Strings – DP Formulation

- ***State:*** count(*i, exist*)
  - *i*: We have written *i* letters (*1 <= i < = N*)
  - *exist*: we have already written at least once the lucky character c

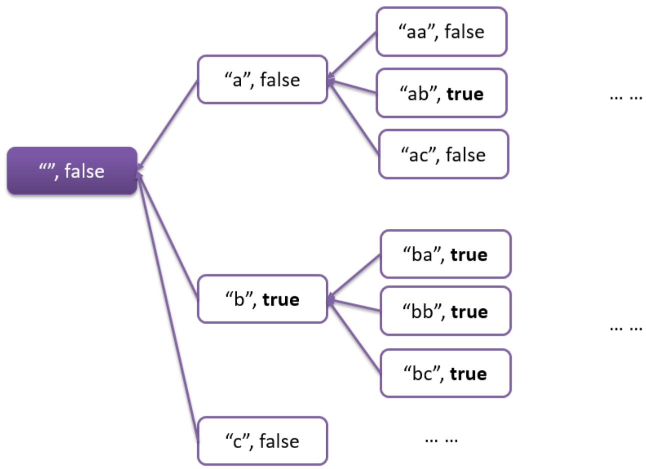
- Note that ***exist*** is not directly retrieved from the variables defined by the problem. We come up with it for our computation.

Lucky String - DAG

Lucky character *t* = *‘b’*



Lucky String - Turn it to code




```
int N, dp[MAXN][2];
int count() {
  dp[0][0] = 1;
  for (int i = 1; i <= N; i++) {
    for (int exist = 0; exist < 2; exist++) {
      dp[i][1] += dp[i - 1][exist]; // write 'b'
      dp[i][exist] += 2 * dp[i - 1][exist]; // write 'a' or 'c'
    }
  }
  return dp[N][1];
}
```

Lucky Strings 2

- A string **X** is *lucky* if it contains a lucky substring **Y**.
- Given the lucky substring **Y**, count how many lucky strings **X** of length **N** exist (|**X**| = **N**). |**Y**| <= **N**.
- All letters in Y are distinct.

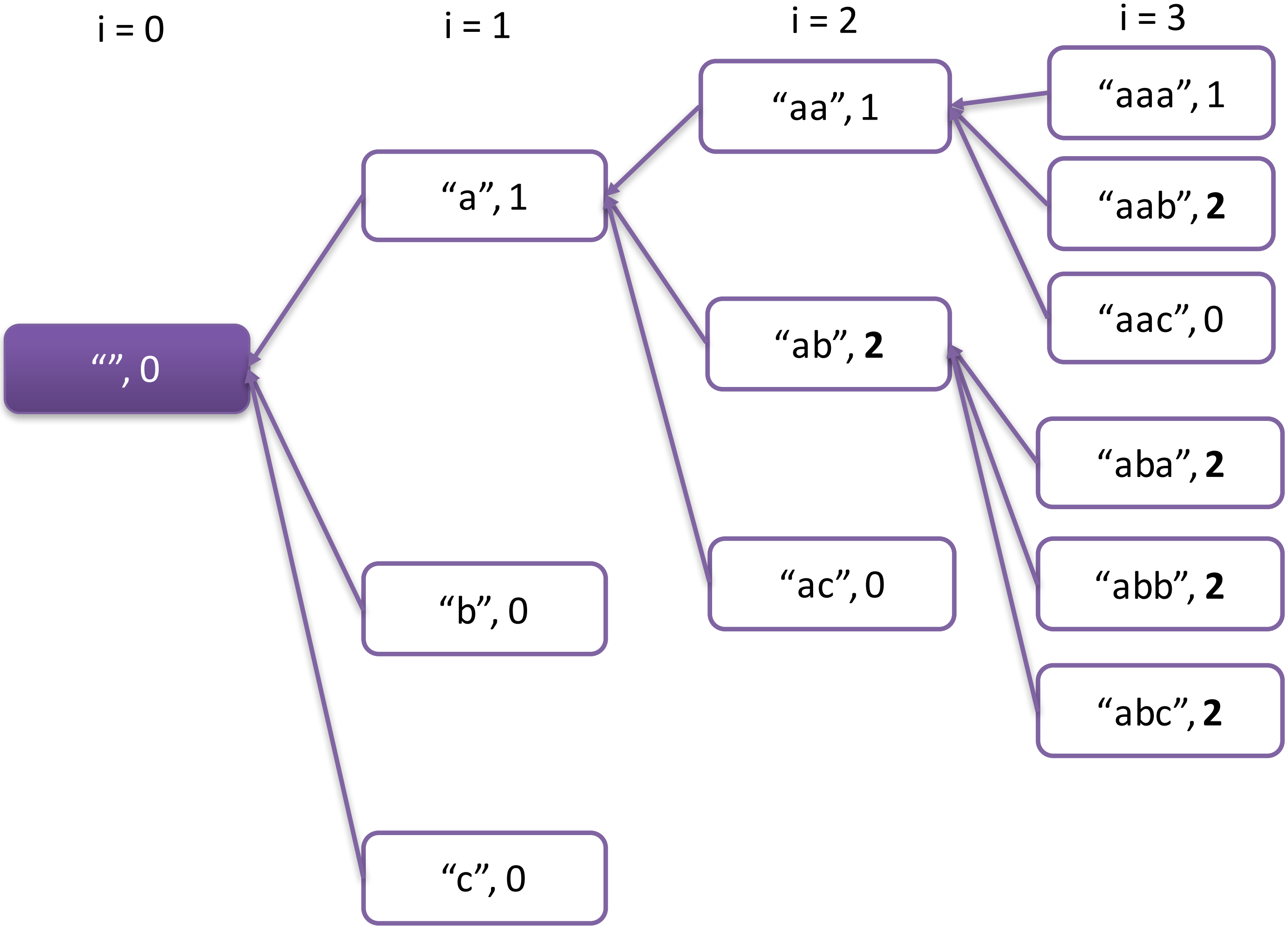
Lucky Strings 2 – DP Formulation

- ***State:*** count(i, m)
  - i: We have written i letters ($1 <= i < = N$)
  - m: the last m letters we wrote match the first m letters of Y.
    - If $m = |Y|$, then it means we have at least written Y once previously.

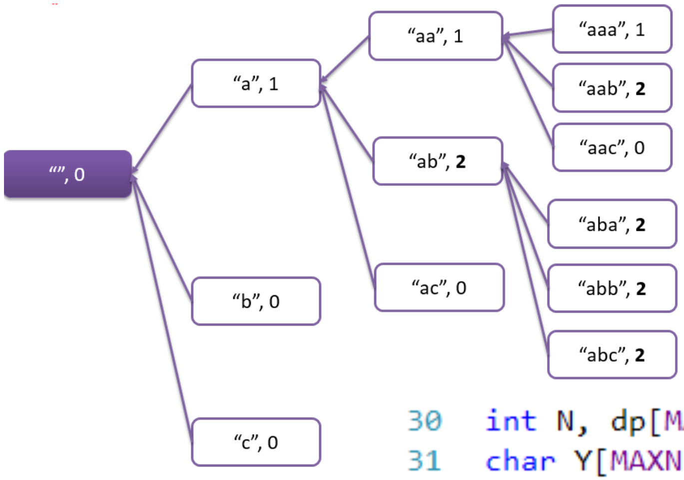
- Note that ***m*** is not directly retrieved from the variables defined by the problem either. We come up with it to track the substring matching process.

Lucky Strings 2 - DAG

Lucky substring *Y = “ab”*



Lucky String 2 - Turn it to code




```
int N, dp[MAXN][MAXN];
char Y[MAXN];
int count() {
  dp[0][0] = 1;
  int M = strlen(Y);
  for (int i = 1; i <= N; i++) {
    for (int m = 0; m < M; m++) {
      // write Y[m]
      dp[i][m + 1] = dp[i - 1][m];
      // write anything other than Y[0] and Y[m], since we only
      // have 'a', 'b', 'c', and Y[m] != Y[0], there leaves
      // one choice left
      dp[i][0] += dp[i - 1][m];
      // write Y[0]
      dp[i][1] += dp[i - 1][m];
    }
    dp[i][M] += dp[i - 1][M] * 3; // anything works
  }
  return dp[N][M];
}
```

Lucky String 2 - Notice

- Y has to contain distinct letters. Otherwise our DP has to be changed.

```
int N, dp[MAXN][MAXN];
char Y[MAXN];
int matchHead(int m, int c) {
  // return the longest length 'len' so that
  // Y[m-(len-1)..m-1] + c = Y[0..len-1]
}
int count() {
  dp[0][0] = 1;
  int M = strlen(Y);
  for (int i = 1; i <= N; i++) {
    for (int m = 0; m < M; m++) {
      dp[i][m + 1] += dp[i - 1][m]; // write Y[m]
      for (int c = 0; c < 3; c++) {
        if (c == Y[m]) continue;
        // write every possible char other than Y[m]
        dp[i][matchHead(m, c)] += dp[i - 1][m];
      }
    }
    dp[i][M] += dp[i - 1][M] * 3; // anything works
  }
  return dp[N][M];
}
```

- Consider *Y = "abac".*

When *i = 4* and *m = 3*, if we write 'b', it would be a mismatch.

But the new *m* would be 2, instead of 0.

This is because the m letters we've just written could match some prefix of Y even when the next letter we are writing doesn't match Y[m].

Counting in DP – modulo arithmetic

- Answer could grow exponentially.

- (a + b) % M = ((a % M) + (b % M)) % M
  - Ensure that ~2M fits in an 32-bit integer.
  - Typically when M ~= 10^9, the above is satisfied.

- a*b % M = ((a % M) * (b % M)) % M
  - Ensure M^2 fits in an integer
  - Typically when M ~= 10^9, we must use 64-bit integer for the multiplication: ((long long)a * b) % M

Summary

- Thinking about the DAG behind DP helps you
  - **Perceive** the problem more clearly
  - See how to **implement** the DP procedure neatly
  - **Determine** if a problem is solvable by DP, or how to solve it by DP.
- Formulate your DP
  - **Think about the DAG:** find states from problem variables, derived dimensions; get transitions
  - **Affordable memory:** check number of states, memory saving trick
  - **Affordable time:** check number of states, computation cost for each state

Exercises

- Cutting Sticks
  - Compute the maximum sum of values after cutting the original sticks into exactly **K** segments.
  - Suppose the stick values are in dollars. Each cut costs you **D** dollars. Find the maximum dollars you can achieve after cutting.

- Minimum Balance
  - Suppose **N** is even. Find a split into two multisets so that not only the difference between two sums are minimum, but also the two multisets have the same number of integers.

- Lucky String
  - Analyze the complexity requirement for Lucky String and Lucky String 2.
  - Note that the time complexity for Lucky String 2 may additionally include the alphabet size, why?

Challenges

- Lucky String 2:
  - Remove the constraint of Y's letters being distinct.
  - Naive implementation of *matchHead(m, c)* could work but would give larger time complexity. Use an efficient algorithm that performs *matchHead(m,c)* in O(1) time.
  - Hint: string matching algorithm

- Grid Walking 2: What if $N, M <= 10, K <= 10^9$?
  - We cannot afford any solution that needs $O(10^9)$ time **OR** space.
  - Still solvable, as an advanced exercise ☺
  - Hint: why should N, M become smaller, i.e. <= 10?