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1 1D Wave Equation

The given second order leapfrog scheme can be rearranged as

$$U_n^{j+1} = 2U_n^j - U_n^{j-1} + q^2\left(U_{n+1}^j - 2U_n^j + U_{n-1}^j\right) \tag{1}$$

where $q = h_t/h_x$ is the ratio of the step sizes, and the Courant number for this equation. This solves solves for the new time level $j+1$.

At $t=0$ we define U^0 as specified by the initial conditions, however to find U^1 we require the ghost points U^{-1} to apply the above leapfrog scheme. Here we apply the initial condition $u_t = 0$ to give $U^1 = U^{-1}$, and the leapfrog equation is modified as follows

$$U_n^1 = U_n^0 + \frac{q^2}{2}\left(U_{n+1}^0 - 2U_n^0 + U_{n-1}^0\right)$$

1.1 Non Reflective boundary

At a non reflective boundary we negate the reflecting waves as much as possible. For this problem at the $x=-2$ boundary we want to minimise the right-ward waves, so we impose the condition $u_x = u_t$. Similarly at the $x=2$ boundary, we want to reduce the left-ward waves so we have $u_x = -u_t$. So given

$$u_x = \mp u_t$$

at $x = \pm 2$, we have

$$\mp q\left(U_{n+1}^j - U_{n-1}^j\right) = U_n^{j+1} - U_n^{j-1} \tag{2}$$

Then with $(1) + q(2)$ we get

$$(1+q)U_n^{j+1} - qU_n^{j-1} = 2U_n^j - U_n^{j-1} + 2q^2(\tilde{U}^j - U_n^j)$$

where $\tilde{U}^j = U_{n\pm1}^j$ at $x = \pm 2$. This can be rearranged to give the boundary condition

$$U_n^{j+1} = 2(1-q)U_n^j - \frac{1-q}{1+q}U_n^{j-1} + \frac{2q^2}{1+q}\tilde{U}^j$$

which is accurate to $O(h_x^2 h_t^2)$.

1.2 Solid Wall

At a reflective boundary we apply the Neumann condition $u_x = 0$ which gives $U_{n+1} = U_{n-1}$ and we have the boundary condition

$$U_n^{j+1} = 2U_n^j - U_n^{j-1} + 2q^2(\tilde{U}^j - U_n^j)$$

where $\tilde{U}^j$ is defined as above. Again this scheme is accurate to $O(h_x^2 h_t^2)$.

1.3 Analysis

I fixed $h_x = 0.01$ for computational speed, and made plots of q through time. The plots are provided in the directory "part_a_results/".

Generally speaking as q decreased, the amount of noise increased, particularly for lower δ (namely due to the value of h_x), and for lower q. Due to my formulation and the definition of $q = ch_t/h_x$ where $c = 1$, decreasing q would decrease the wave speed as well, so for $q = 0.5$ say, the waves would take 4 seconds to reach the boundary whereas at $q = 1$ they take 2 seconds.

2 2D Wave Equation