M2AA3 Project 2

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1

We use the definition of the inner product on V

$$\left\langle \frac{df}{dx}, \sin\left(\frac{m\pi\cdot}{d}\right)\right\rangle = \int_{-d}^{d} \frac{df}{dx} \sin\left(\frac{m\pi x}{d}\right) dx$$

We integrate by parts, setting $u' = \frac{df}{dx}$ and $v = \sin\left(\frac{m\pi x}{d}\right)$ to obtain

$$\left[f(x)\sin\left(\frac{m\pi x}{d}\right)\right]_{-d}^{d} - \int_{-d}^{d} \frac{m\pi}{d} f(x)\cos\left(\frac{m\pi x}{d}\right) dx$$

which is valid for $m \neq 0$
The left limit evaluates to zero since the inner terms of sin are integer multiples of π.
We are left with the integrand on the right which is

$$-\frac{m\pi}{d}\int_{-d}^{d} f(x)\cos\left(\frac{m\pi x}{d}\right) dx = -\frac{m\pi}{d}\left\langle f(x), \cos\left(\frac{m\pi x}{d}\right)\right\rangle$$

for $m = 1 \to n$ this gives the result $-m\pi a_m$.
The second inner product follows simillarly:

$$\left\langle \frac{df}{dx}, \cos\left(\frac{m\pi\cdot}{d}\right)\right\rangle = \int_{-d}^{d} \frac{df}{dx} \cos\left(\frac{m\pi x}{d}\right) dx$$

We integrate by parts, setting $u' = \frac{df}{dx}$ and $v = \cos\left(\frac{m\pi x}{d}\right)$ to obtain

$$\left[f(x)\cos\left(\frac{m\pi x}{d}\right)\right]_{-d}^{d} + \int_{-d}^{d} \frac{m\pi}{d} f(x)\sin\left(\frac{m\pi x}{d}\right) dx$$

The left limit evaluates to zero since $f(d) = f(-d)$, and cos is even about zero.
We are left with the integrand on the right which is

$$\frac{m\pi}{d}\int_{-d}^{d} f(x)\sin\left(\frac{m\pi x}{d}\right)dx = \frac{m\pi}{d}\left\langle f(x), \sin\left(\frac{m\pi x}{d}\right)\right\rangle$$

which is $m\pi b_m$ for $m = 1 \to n$.

2

Let $\bar{f}_n^*$ be the best approximation to $\frac{df}{dx}$ in $\|\cdot\|$ from V_n. Then we have

$$\bar{f}_n^* = \frac{\bar{a}_0}{2} + \sum_{m=1}^{n}\left[\bar{a}_m\cos\left(\frac{m\pi x}{d}\right) + \bar{b}_m\sin\left(\frac{m\pi x}{d}\right)\right]$$

where

$$\bar{a}_m = \frac{1}{d}\left\langle\frac{df}{dx}, \cos\left(\frac{m\pi\cdot}{d}\right)\right\rangle, m = 0 \to n$$
$$\bar{b}_m = \frac{1}{d}\left\langle\frac{df}{dx}, \sin\left(\frac{m\pi\cdot}{d}\right)\right\rangle, m = 1 \to n$$

Note that for $m = 1 \to n$, $\bar{a}_m = \frac{m\pi b_m}{d}$ and $\bar{b}_m = -\frac{m\pi a_m}{d}$
Moreover,

$$\bar{a}_0 = \frac{1}{d}\left\langle\frac{df}{dx}, 1\right\rangle = \frac{1}{d}\int_{-d}^{d}\frac{df}{dx}dx = \frac{1}{d}\left[f(x)\right]_{-d}^{d} = 0$$

Now consider

$$\begin{aligned}\frac{df_n^*}{dx} &= \frac{d}{dx}\left(\frac{a_0}{2} + \sum_{m=1}^{n}\left[a_m\cos\left(\frac{m\pi x}{d}\right) + b_m\sin\left(\frac{m\pi x}{d}\right)\right]\right)\\ &= \sum_{m=1}^{n}\left[-\frac{m\pi a_m}{d}\sin\left(\frac{m\pi x}{d}\right) + \frac{m\pi b_m}{d}\cos\left(\frac{m\pi x}{d}\right)\right]\\ &= \sum_{m=1}^{n}\left[\bar{b}_m\sin\left(\frac{m\pi x}{d}\right) + \bar{a}_m\cos\left(\frac{m\pi x}{d}\right)\right]\\ &= \bar{f}_n^*\end{aligned}$$

Therefore
$$\left\|\frac{df}{dx}-\frac{df_n^*}{dx}\right\| \leq \left\|\frac{df}{dx}-v_n\right\|, \forall v_n \in V_n$$

3

We have

$$\begin{aligned}\frac{\pi^2}{d}\sum_{m=1}^{n} m^2\left[a_m^2+b_m^2\right] &= d\sum_{m=1}^{n}\left[\left(\frac{m\pi b_m}{d}\right)^2+\left(-\frac{m\pi a_m}{d}\right)^2\right]\\ &= d\left(\frac{\bar{a}_0^2}{2}+\sum_{m=1}^{n}\left[\bar{a}_m^2+\bar{b}_m^2\right]\right)=\left\|\bar{f}_n^*\right\|^2\\ &= \left\|\frac{df_n^*}{dx}\right\|^2=\left\|\frac{df}{dx}\right\|^2-\left\|\frac{df}{dx}-\frac{df_n^*}{dx}\right\|^2 \leq \left\|\frac{df}{dx}\right\|^2\end{aligned}$$

By Bessel's inequality.

4

Consider
$$\frac{\pi^2}{d}(n+1)^2\left(a_{n+1}^2+b_{n+1}^2\right)=\left\|\frac{df_{n+1}^*}{dx}\right\|^2-\left\|\frac{df_n^*}{dx}\right\|^2$$

Since $\left\{\frac{df_n^*}{dx}\right\}_{n=1}^{\infty}$ converges to $\frac{df}{dx}$, we have

$$\lim_{m\to\infty}\left[m^2\left(a_m^2+b_m^2\right)\right]=\lim_{m\to\infty} m^2a_m^2+\lim_{m\to\infty} m^2b_m^2=0$$

Moreover since the terms are non-negative, we can say each limit tends to zero, and also take square roots to obtain the result.

5

First we find the coefficients $\{a_m\}_{m=0}^{\infty}$:

$$a_m=\frac{1}{d}\left\langle f(x),\cos\left(\frac{m\pi\cdot}{d}\right)\right\rangle=\left\langle\frac{1}{2}x^2,\cos\left(m\pi\cdot\right)\right\rangle=\int_{-1}^{1}\frac{1}{2}x^2\cos\left(m\pi x\right)dx$$

For $m = 0$ we have

$$a_0 = \int_{-1}^{1} \frac{1}{2}x^2 dx = 2\left[\frac{1}{6}x^3\right]_0^1 = \frac{1}{3}$$

For all other m, we integrate by parts, setting $u' = \cos(m\pi x)$ and $v = \frac{1}{2}x^2$

$$a_m = \left[\frac{1}{2}x^2\frac{\sin(m\pi x)}{m\pi}\right]_{-1}^1 - \int_{-1}^{1} x\frac{\sin(m\pi x)}{m\pi}dx$$

Integrate by parts again, with $u' = \sin(m\pi x)$ and $v = x$

$$\begin{aligned}
a_m &= 2\left[\frac{1}{2}x^2\frac{\sin(m\pi x)}{m\pi}\right]_0^1 + \left[x\frac{\cos(m\pi x)}{m^2\pi^2}\right]_{-1}^1 - \int_{-1}^{1}\frac{\cos(m\pi x)}{m^2\pi^2}dx \\
&= \frac{\sin(m\pi)}{m\pi} + 2\left[x\frac{\cos(m\pi x)}{m^2\pi^2}\right]_0^1 - \left[\frac{\sin(m\pi x)}{m^3\pi^3}\right]_{-1}^1 \\
&= \frac{2\cos(m\pi)}{m^2\pi^2} + 2\left[\frac{\sin(m\pi x)}{m^3\pi^3}\right]_0^1 \\
&= \frac{2(-1)^m}{m^2\pi^2} + \frac{\sin(m\pi)}{m^3\pi^3} \\
&= \frac{2(-1)^m}{m^2\pi^2}
\end{aligned}$$

Now for $\{b_m\}_{m=0}^{\infty}$:

$$b_m = \left\langle \frac{1}{2}x^2, \sin(m\pi\cdot)\right\rangle = \int_{-1}^{1}\frac{1}{2}x^2\sin(m\pi x)\,dx = 0$$

since the integrand is even about zero.
Therefore we have

$$f_n^*(x) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{m=1}^{n}\frac{(-1)^m}{m^2}\cos(m\pi x)$$

and

$$\frac{df_n^*}{dx} = -\frac{2}{\pi}\sum_{m=1}^{n}\frac{(-1)^m}{m}\sin(m\pi x)$$

6

To find $\|f - f_n^*\|^2$ we must first find the integrals $\int_{-1}^1 f^2 dx$, $\int_{-1}^1 f f_n^* dx$ and $\int_{-1}^1 f_n^{*2} dx$

$$\int_{-1}^1 f^2 dx = \int_{-1}^1 \frac{1}{4} x^4 dx = 2\left[\frac{1}{20}x^5\right]_0^1 = \frac{1}{10}$$

$$\begin{aligned}
\int_{-1}^1 f f_n^* dx &= \int_{-1}^1 \frac{1}{12}x^2 + \frac{2}{\pi^2}\sum_{m=1}^n \frac{(-1)^m}{m^2}\frac{1}{2}x^2\cos(m\pi x)dx \\
&= 2\left[\frac{1}{36}x^3\right]_0^1 + \frac{2}{\pi^2}\sum_{m=1}^n \frac{(-1)^m}{m^2} a_m \\
&= \frac{1}{18} + \frac{4}{\pi^4}\sum_{m=1}^n \frac{1}{m^4}
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^1 f_n^{*2} dx = &\int_{-1}^1 \frac{1}{36}dx + \int_{-1}^1 \frac{1}{3\pi^2}\sum_{m=1}^n \frac{(-1)^m}{m^2}\cos(m\pi x)dx \\
&+ \int_{-1}^1 \frac{2}{\pi^2}\sum_{m=1}^n \frac{(-1)^m}{m^2}\cos(m\pi x)\cdot\frac{2}{\pi^2}\sum_{m=1}^n \frac{(-1)^m}{m^2}\cos(m\pi x)dx \\
&= \frac{1}{18} + \frac{1}{3\pi^2}\sum_{m=1}^n \frac{(-1)^m)}{m^2}\cdot 2\left[\frac{\sin(m\pi x)}{m\pi}\right]_0^1 \\
&+ \sum_{m=1}^n\sum_{p=1}^n \frac{4(-1)^{m+p}}{m^2p^2\pi^4}\int_{-1}^1 \cos(m\pi x)\cos(p\pi x)dx \\
&= \frac{1}{18} + \sum_{m=1}^n\sum_{p=1}^n \frac{2(-1)^{m+p}}{m^2p^2\pi^4}\int_{-1}^1 \cos((m+p)\pi x) + \cos((m-p)\pi x)\,dx \\
&= \frac{1}{18} + \sum_{m=1}^n\sum_{p=1}^n \frac{4(-1)^{m+p}}{m^2p^2\pi^4}\left[\frac{\sin((m+p)\pi x}{(m+p)\pi} + \frac{\sin((m-p)\pi x}{(m-p)\pi}\right]_0^1 \\
&= \frac{1}{18}
\end{aligned}$$

Now, by definition

$$\begin{aligned}
\|f - f_n^*\|^2 = \langle f - f_n^*, f - f_n^* \rangle = \int_{-1}^{1} [f - f_n^*]^2 dx &= \int_{-1}^{1} f^2 - 2ff_n^* + f_n^{*2} dx \\
&= \frac{1}{10} - 2\left(\frac{1}{18} + \frac{4}{\pi^4}\sum_{m=1}^{n}\frac{1}{m^4}\right) + \frac{1}{18} \\
&= \frac{2}{45} - \frac{8}{\pi^4}\sum_{m=1}^{n}\frac{1}{m^4}
\end{aligned}$$

To find $\left\|\frac{df}{dx} - \frac{df_n^*}{dx}\right\|^2$ we must find the integrals $\int_{-1}^{1}(\frac{df}{dx})^2 dx$, $\int_{-1}^{1}\frac{df}{dx}\frac{df_n^*}{dx}dx$ and $\int_{-1}^{1}\left(\frac{df_n^*}{dx}\right)^2 dx$

$$\int_{-1}^{1}\left(\frac{df}{dx}\right)^2 dx = \int_{-1}^{1} x^2 dx = 2\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3}$$

$$\begin{aligned}
\int_{-1}^{1}\frac{df}{dx}\frac{df_n^*}{dx}dx &= \int_{-1}^{1} -\frac{2}{\pi}\sum_{m=1}^{n}\frac{(-1)^m}{m^2} x\sin(m\pi x)dx \\
&= -\frac{2}{\pi}\sum_{m=1}^{n}\frac{(-1)^m}{m^2}\int_{-1}^{1} x\sin(m\pi x)dx \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{1}\left(\frac{df_n^*}{dx}\right)^2 dx &= \int_{-1}^{1}\left(-\frac{2}{\pi}\sum_{m=1}^{n}\frac{(-1)^m}{m}\sin(m\pi x)\right)^2 dx \\
&= \sum_{m=1}^{n}\sum_{p=1}^{n}\frac{4(-1)^{m+p}}{mp\pi^2}\int_{-1}^{1}\sin(m\pi x)\sin(p\pi x)dx \\
&= 0
\end{aligned}$$

We have

$$\begin{aligned}
\left\|\frac{df}{dx} - \frac{df_n^*}{dx}\right\|^2 &= \int_{-1}^{1}\left(\frac{df}{dx}\right)^2 dx - 2\int_{-1}^{1}\frac{df}{dx}\frac{df_n^*}{dx}dx + \int_{-1}^{1}\left(\frac{df_n^*}{dx}\right)^2 dx \\
&= \frac{2}{3}
\end{aligned}$$

7

To show:

$$\frac{d^2[f-f_n^*]}{dx^2}(x) = \frac{(-1^n)\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\cos\left(\frac{\pi x}{2}\right)}$$

From our evaluations we obtain:

$$f - f_n^* = \frac{1}{2}x^2 - \frac{1}{6} - \frac{2}{\pi^2}\sum_{m=1}^{n}\frac{(-1)^m}{m^2}\cos(m\pi x)$$

$$\frac{df}{dx} - \frac{df_n^*}{dx} = x + \frac{2}{\pi}\sum_{m=1}^{n}\frac{(-1)^m}{m}\sin(m\pi x)$$

$$\begin{aligned}\implies \frac{d^2[f-f_n^*]}{dx^2}(x) &= 1 + \frac{d}{dx}\left(\frac{2}{\pi}\sum_{m=1}^{n}\frac{(-1)^m}{m}\sin(m\pi x)\right)\\ &= 1 + 2\sum_{m=1}^{n}(-1)^m\cos(m\pi x)\end{aligned}$$

Now we must evaluate the RHS of the given formula to check it equals our evaluation of the LHS. We will use the following trigonometric identity.

$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

We also use the cosine half-angle formula.

$$\begin{aligned}\cos((n+\frac{1}{2})\pi x) &= \cos(n\pi x)\cos(\frac{\pi x}{2}) - \sin(n\pi x)\sin(\frac{\pi x}{2})\\ &= \cos(n\pi x)\cos(\frac{\pi x}{2}) - \frac{1}{2}(\cos((n-\frac{1}{2})\pi x) - \cos((n+\frac{1}{2})\pi x))\end{aligned}$$

$$\implies \cos((n+\frac{1}{2})\pi x) = 2\cos(n\pi x)\cos(\frac{\pi x}{2}) - \cos((n-\frac{1}{2})\pi x)$$

$$\implies \frac{(-1^n)\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\cos\left(\frac{\pi x}{2}\right)} = (-1)^n 2\cos(n\pi x) + \frac{(-1^{n-1})\cos\left(\left(n-\frac{1}{2}\right)\pi x\right)}{\cos\left(\frac{\pi x}{2}\right)}$$

We have now have a recursive relation that we can unfold to obtain our result.

$$\begin{aligned}\frac{(-1^n)\cos\left(\left(n+\frac{1}{2}\right)\pi x\right)}{\cos\left(\frac{\pi x}{2}\right)} &= 2\sum_{m=1}^{n}(-1)^m\cos(m\pi x) + \frac{(-1)^0\cos\left(\frac{\pi x}{2}\right)}{\cos\left(\frac{\pi x}{2}\right)} \\ &= 1 + 2\sum_{m=1}^{n}(-1)^m\cos(m\pi x)\end{aligned}$$

which holds for $x \in (-1, 1)$.

8

In order for $\frac{d^2[f-f_n^*]}{dx^2}(x) = 0$, we need to choose $x \in (-1, 1)$ such that $\cos((n+\frac{1}{2})\pi x) = 0$ and $\cos(\frac{\pi x}{2}) \neq 0$.
Note the second condition implies that that we must have $|x| < 1$.
To satisfy these conditions, we can immediately see that we need to choose $x = \frac{y}{2n+1}$,
where $y \in (-2n+1, 2n-1)$ is an odd integer.